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# THE MAGNETIC FLOW ON THE MANIFOLD OF ORIENTED GEODESICS OF A THREE DIMENSIONAL SPACE FORM

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**Abstract**

Let $M$ be the three dimensional complete simply connected manifold of constant sectional curvature 0, 1 or $-1$. Let $\mathcal{L}$ be the manifold of all (unparametrized) complete oriented geodesics of $M$, endowed with its canonical pseudo-Riemannian metric of signature (2, 2) and Kähler structure $J$. A smooth curve in $\mathcal{L}$ determines a ruled surface in $M$.

We characterize the ruled surfaces of $M$ associated with the magnetic geodesics of $\mathcal{L}$, that is, those curves $\sigma$ in $\mathcal{L}$ satisfying $\nabla_{\dot{\sigma}}\dot{\sigma} = J\dot{\sigma}$. More precisely: a time-like (space-like) magnetic geodesic determines the ruled surface in $M$ given by the binormal vector field along a helix with positive (negative) torsion. Null magnetic geodesics describe cones, cylinders or, in the hyperbolic case, also cones with vertices at infinity. This provides a relationship between the geometries of $\mathcal{L}$ and $M$.

## 1. Introduction

For $\kappa = 0, 1, -1$, let $M_\kappa$ be the three dimensional complete simply connected manifold of constant sectional curvature $\kappa$, that is, $\mathbb{R}^3$, $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$. Let $\mathcal{L}_\kappa$ be the manifold of all (unparametrized) complete oriented geodesics of $M_\kappa$. We may think of an element $c$ in $\mathcal{L}_\kappa$ as the equivalence class of unit speed geodesics $\gamma\colon \mathbb{R} \to M_\kappa$ with image $c$ such that $\{\dot{\gamma}(s)\}$ is a positive basis of $T_{\gamma(s)}c$ for all $s$.

Let $\gamma$ be a complete unit speed geodesic of $M_\kappa$ and let $\mathcal{J}_\gamma$ be the space of all Jacobi fields along $\gamma$ which are orthogonal to $\gamma$. There exists a well-defined canonical isomorphism

$$T_\gamma\colon \mathcal{J}_\gamma \to T_{[\gamma]}\mathcal{L}_\kappa, \quad T_\gamma(J) = \left.\frac{d}{dt}\right|_0 [\gamma_t], \tag{1}$$

where $\gamma_t$ is any variation of $\gamma$ by unit speed geodesics associated with $J$ (see [11]).

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A pseudo-Riemannian metric of signature (2, 2) can be defined on $\mathcal{L}_\kappa$ as follows [12]: For $X \in T_{[\gamma]}\mathcal{L}_\kappa$, the square norm $\|X\| = \langle X, X\rangle$ is well defined by

$$\|X\| = \langle \dot\gamma \times J, J'\rangle, \tag{2}$$

where $X = T_\gamma(J)$, the cross product $\times$ is induced by a fixed orientation of $M_\kappa$ and $J'$ denotes the covariant derivative of $J$ along $\gamma$. Indeed, the right hand side of (2) is a constant function. In the following, for any vector $X$, we will denote $\|X\| = \langle X,X\rangle$ and $|X| = \sqrt{|\langle X, X\rangle|}$. Recall that $X$ is null, time-like or space-like if $\|X\| = 0$, $\|X\| < 0$ or $\|X\| > 0$, respectively.

Let $[\gamma] \in \mathcal{L}_\kappa$ and let $R_\gamma$ be the rotation in $M_\kappa$ fixing $\gamma$ through an angle of $\pi/2$. This rotation induces an isometry $\tilde R_\gamma$ of $\mathcal{L}_\kappa$ whose differential at $[\gamma]$ is a linear isometry of $T_{[\gamma]}\mathcal{L}_\kappa$ squaring to $-\,\mathrm{id}$. This yields a complex structure $J$ on $\mathcal{L}_\kappa$. With the metric defined above, $\mathcal{L}_\kappa$ is Kähler. Recent generalizations of this fact can be found in [1, 3].

A *magnetic geodesic* $\sigma$ of $\mathcal{L}_\kappa$ is a curve satisfying $\nabla_{\dot\sigma}\dot\sigma = J\dot\sigma$. These curves have constant speed, so they will be null, time-like or space-like.

A smooth curve in $\mathcal{L}_\kappa$ determines a ruled surface in $M_\kappa$. For $\kappa = 0, -1$, a generic geodesic of $\mathcal{L}_\kappa$ describes a helicoid in $M_\kappa$ [7, 6, respectively]. Our purpose is to characterize the ruled surfaces in $M_\kappa$ associated with the magnetic geodesics of $\mathcal{L}_\kappa$. For $v \in TM_\kappa$, $\gamma_v$ denotes the geodesic of $M_\kappa$ with initial velocity $v$.

**Theorem 1.** *A generic magnetic geodesic $\sigma$ of $\mathcal{L}_\kappa$ describes the ruled surface in $M_\kappa$ given by the binormal vector field of a helix. More precisely, $\sigma$ is a time-like* (*space-like*) *magnetic geodesic of $\mathcal{L}_\kappa$ if and only if $\sigma$ has the form*

$$\sigma(t) = [\gamma_{B(t)}], \tag{3}$$

*where $B$ is the binormal vector field of a helix in $M_\kappa$ with curvature $k$, speed $1/k$ and positive* (*negative*) *torsion, for some $k > 0$.*

Now we study null magnetic geodesics in $\mathcal{L}_{-1} = \mathcal{L}(\mathbb{H}^3)$. We recall some concepts related with the hyperbolic space (see for instance [5]).

Two unit speed geodesics $\gamma$ and $\alpha$ of $\mathbb{H}^3$ are said to be asymptotic if there exists a positive constant $C$ such that $d(\gamma(s), \sigma(s)) \leq C$, $\forall s \geq 0$. Two unit vectors $v, w \in T^1\mathbb{H}^3$ are said to be asymptotic if the corresponding geodesics $\gamma_v$ and $\gamma_w$ have this property.

A point at infinity for $\mathbb{H}^3$ is an equivalence class of asymptotic geodesics of $\mathbb{H}^3$. The set of all points at infinity for $\mathbb{H}^3$ is denoted by $\mathbb{H}^3(\infty)$ and has a canonical differentiable structure diffeomorphic to the 2-sphere. The equivalence class represented by a geodesic $\gamma$ is denoted by $\gamma(\infty)$, and the equivalence class represented by the oppositely oriented geodesic $s \mapsto \gamma(-s)$ is denoted by $\gamma(-\infty)$.

Given $v \in T^1\mathbb{H}^3$, the *horosphere* $H(v)$ is the limit of metric spheres $\{S_n\}$ in $\mathbb{H}^3$ that pass through the foot point of $v$ as the centers $\{p_n\}$ of $\{S_n\}$ converge to $\gamma_v(\infty)$. Below we present a more precise definition.

Let $\psi^\pm \colon \mathcal{L}(\mathbb{H}^3) \to \mathbb{H}^3(\infty)$ be the smooth functions given by $\psi^\pm([\gamma]) = \gamma(\pm\infty)$ and let $\mathcal{D}^\pm$ be the distributions on $\mathcal{L}(\mathbb{H}^3)$ given by $\mathcal{D}^\pm_{[\gamma]} = \operatorname{Ker}(d\psi^\pm_{[\gamma]})$. These distributions are called *the horospherical distributions* on $\mathcal{L}(\mathbb{H}^3)$.

*Cones with vertices at infinity*: Let $x \in \mathbb{H}^3(\infty)$ and let $v_o \in T^1\mathbb{H}^3$ such that $\gamma_{v_o}(\pm\infty) \in x$. Let $t \mapsto v(t)$ be a curve in $T^1\mathbb{H}^3$ such that $v(0) = \pm v_o$, $v(t)$ is asymptotic to $\pm v_o$ for all $t \in \mathbb{R}$ and the foot points of $v(t)$ lie on a circle of geodesic curvature $\pm k$ (with $k > 0$) and speed $1/k$ in the horosphere determined by $\pm v_o$. Under these conditions we say that the curve in $\mathcal{L}(\mathbb{H}^3)$ given by $t \mapsto [\gamma_{\pm v(t)}]$ describes a *forward cone with vertex at $x$ (for $+$) or a backward cone with vertex at $x$ (for $-$)*. These cones can be better visualized in the upper half space model of $\mathbb{H}^3$ (in particular $\mathbb{H}^3(\infty) = \{z = 0\} \cup \{\infty\}$): Let $\gamma_t^\pm(s) = ((1/k)\cos(t), \pm(1/k)\sin(t), e^{\pm s})$. A curve $\sigma$ in $\mathcal{L}(\mathbb{H}^3)$ describes a cone with forward (respectively, backward) vertex at $\infty$ if it is $Sl(2,\mathbb{C})$-congruent to $t \mapsto [\gamma_t^+]$ (respectively, to $t \mapsto [\gamma_t^-]$).

**Theorem 2.** *A null magnetic geodesic of $\mathcal{L}(\mathbb{H}^3)$ describes in $\mathbb{H}^3$ a cylinder, a cone with vertex at $p \in \mathbb{H}^3$ or a cone with vertex at infinity. More precisely, if $\sigma$ is a curve in $\mathcal{L}(\mathbb{H}^3)$, then*

a) *$\sigma$ is a null magnetic geodesic with $\dot\sigma(0) \in \mathcal{D}^\pm_{\sigma(0)}$ if and only if $\sigma$ describes a cone with vertex at $\sigma(0)(\pm\infty)$ (forward for $+$ and backward for $-$);*

b) *$\sigma$ is a null magnetic geodesic with $\dot\sigma(0) \notin \mathcal{D}^\pm_{\sigma(0)}$ if and only if $\sigma$ either has the form*

$$\sigma(t) = [\gamma_{B(t)}], \tag{4}$$

*where $B$ is the binormal vector field of a helix $h$ in $\mathbb{H}^3$ with curvature $k$, speed $1/k$ and zero torsion (in particular, $h$ is contained in a totally geodesic surface $S$ and $B$ is normal to $S$ and parallel along $h$), or $\sigma$ has the form*

$$\sigma(t) = [\gamma_{v(t)}], \tag{5}$$

*where $v$ is a curve with geodesic curvature $k$ and speed $1/k$ in $T^1_p\mathbb{H}^3$, for some $p \in \mathbb{H}^3$, for certain $k > 0$.*

**Theorem 3.** *The ruled surfaces associated with null magnetic geodesics of $\mathcal{L}_\kappa$ for $\kappa = 0, 1$ are described in an analogous manner as in the previous theorem, except that case* a) *is empty. Besides, for $\kappa = 1$, a null magnetic geodesic has simultaneously the forms* (4) *and* (5).

## 2. Preliminaries

For the simultaneous analysis of the three cases $\kappa = 0,1,-1$, we consider the standard presentation of $M_\kappa$ as a submanifold of $\mathbb{R}^4$. That is, $\mathbb{R}^3 = \{(1, x) \in \mathbb{R}^4 \mid x \in \mathbb{R}^3\}$, $\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid |x|^2 = 1\}$ and $\mathbb{H}^3 = \{x \in \mathbb{R}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0\}$.

Let $G_\kappa$ be the identity component of the isometry group of $M_\kappa$, that is, $G_0 = SO_3 \ltimes \mathbb{R}^3$, $G_1 = SO_4$ and $G_{-1} = O_o(1, 3)$. We consider the usual presentation of $G_0$ as a subgroup of $Gl_4(\mathbb{R})$. The group $G_\kappa$ acts on $\mathcal{L}_\kappa$ as follows: $g \cdot [\gamma] = [g \circ \gamma]$. This action is transitive and smooth.

If we denote by $\mathfrak{g}_\kappa$ the Lie algebra of $G_\kappa$ we have that

$$\mathfrak{g}_\kappa = \left\{ \begin{pmatrix} 0 & -\kappa x^t \\ x & B \end{pmatrix} \,\middle|\, x \in \mathbb{R}^3,\ B \in so_3 \right\}.$$

Let $\gamma_o$ be the geodesic in $M_\kappa$ with $\gamma_o(0) = e_0$ and initial velocity $e_1 \in T_{e_0} M_\kappa$, where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$. For $A,\ B \in \mathbb{R}^{2\times 2}$, let $\operatorname{diag}(A, B) = \begin{pmatrix} A & 0_2 \\ 0_2 & B \end{pmatrix}$, where $0_2$ denotes the $2 \times 2$ zero matrix. Then the isotropy subgroup of $G_\kappa$ at $[\gamma_o]$ is

$$H_\kappa = \{\operatorname{diag}(R_\kappa(t), B) \mid t \in \mathbb{R},\ B \in SO_2\},$$

where

$$R_0(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad R_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad R_{-1}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \tag{6}$$

Let $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The Lie algebra of $H_\kappa$ is

$$\mathfrak{h}_\kappa = \{\operatorname{diag}(r_\kappa(t), sj) \mid s, t \in \mathbb{R}\},$$

where $r_\kappa(t) = \begin{pmatrix} 0 & -\kappa t \\ t & 0 \end{pmatrix}$. We may identify $\mathcal{L}_\kappa$ with $G_\kappa / H_\kappa$ via the diffeomorphism

$$\phi \colon G_\kappa / H_\kappa \to \mathcal{L}_\kappa, \quad \phi(gH_\kappa) = g \cdot [\gamma_o]. \tag{7}$$

For $x, y \in \mathbb{R}^2$ we denote $Z(x, y) = \begin{pmatrix} 0_2 & (-\kappa x, -y)^t \\ (x, y) & 0_2 \end{pmatrix}$. Let

$$\mathfrak{p}_\kappa = \{Z(x, y) \in \mathfrak{g}_\kappa \mid x, y \in \mathbb{R}^2\},$$

which is an $\operatorname{Ad}(H_\kappa)$-invariant complement of $\mathfrak{h}_\kappa$.

For $\kappa = 0, 1$, we consider on $\mathfrak{g}_\kappa$ the inner product such that $\mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa$, $\|Z(x, y)\| = \det(x, y)$ and

$$\|\operatorname{diag}(r_\kappa(t), sj)\| = -ts,$$

(for $\kappa = 0$, we have learnt of this inner product from [9, p. 499]). On $\mathfrak{g}_{-1}$ we consider the Killing form ($\mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa$ also holds). For $\kappa = 0,1,-1$, this inner product on $\mathfrak{g}_\kappa$ induces on $G_\kappa$ a bi-invariant metric. Thus, there exists an unique pseudo-Riemannian metric on $\mathcal{L}_\kappa \simeq G_\kappa/H_\kappa$ such that $\pi\colon G_\kappa \to G_\kappa/H_\kappa$ is a pseudo-Riemannian submersion. For $\kappa = 0, 1$, this metric on $\mathcal{L}_\kappa$ coincides with the given in (2), see Lemma 5 b). For $\kappa = -1$, the metric on $\mathcal{L}_{-1}$ associated with the Killing form is different from the one in (2). However, the magnetic geodesics of either metric on $\mathcal{L}_{-1}$ are the same. This follows since the geodesics are the same (see [11]), so the Levi-Civita connections coincide.

Let us call $A = \operatorname{diag}(0_2, j)$, which is in the center of $\mathfrak{h}_\kappa$. We have that $\operatorname{ad}_A$ is orthogonal and $\operatorname{ad}_A^2 = -\operatorname{id}$ in $\mathfrak{p}_\kappa$. Hence, $\operatorname{ad}_A$ induces a complex structure on $G_\kappa/H_\kappa$. A straightforward computation shows that it coincides, via $\phi$ in (7), with the complex structure given in the introduction. With the metric above and this complex structure, $\mathcal{L}_\kappa$ is a Hermitian symmetric space.

As a direct application of a result by Adachi, Maeda and Udagawa in [2] (see also [8] and Remark 1 in [4]) we have

**Theorem 4.** *Let $\sigma$ be a magnetic geodesic of $G_\kappa/H_\kappa$ with initial conditions $\sigma(0) = H_\kappa$ and $\dot\sigma(0) = X \in \mathfrak{p}_\kappa$. Then $\sigma(t) = \pi(\exp t(X + A))$.*

As we saw in (1), $\mathcal{J}_{\gamma_o}$ is isomorphic to $T_{[\gamma_o]}\mathcal{L}_\kappa \cong \mathfrak{p}_\kappa$. In the next Lemma we relate $\mathfrak{p}_\kappa$ and $\mathcal{J}_{\gamma_o}$ explicitly, involving the matrix $A$.

**Lemma 5.** *Let $Z = Z(x, y) \in \mathfrak{p}_\kappa$.*

a) *The Jacobi field $J(s) = (d/dt)|_0 \exp t(Z + A) \cdot \gamma_o(s)$ in $\mathcal{J}_{\gamma_o}$ is the unique one that satisfies $J(0) = (0, 0, x)^t$ and $J'(0) = (0, 0, y)^t$.*

b) *$T_{\gamma_o}(J) = d(\phi \circ \pi)Z$ and its norm is $\|d(\phi \circ \pi)Z\| = \det(x, y)$.*

Proof. For each $\kappa$, we consider the following parameterization of $\gamma_o$:

$$\gamma_o(s) = (1, s, 0, 0), \quad \text{if} \quad \kappa = 0;$$

$$\gamma_o(s) = (\cos s, \sin s, 0, 0), \quad \text{if} \quad \kappa = 1;$$

$$\gamma_o(s) = (\cosh s, \sinh s, 0, 0), \quad \text{if} \quad \kappa = -1.$$

Given $Z = Z(x, y) \in \mathfrak{p}_\kappa$, the Jacobi field along $\gamma_o$ defined by $J(s) = (d/dt)|_0 \exp t(Z + A) \cdot \gamma_o(s)$ belongs to $\mathcal{J}_{\gamma_o}$, because for all $s \in \mathbb{R}$,

$$\langle J(s), \dot\gamma_o(s)\rangle = \langle (Z + A)(\gamma_o(s)), \dot\gamma_o(s)\rangle = 0,$$

since $(Z + A)(\gamma_o(s))$ is orthogonal to $e_0$ and $e_1$, while $\dot\gamma_o(s)$ has non zero components only in these two directions.

One verifies easily that $J(0) = (Z + A)(e_0) = (0, 0, x)^t$. On the other hand,

$$J'(0) = \left.\frac{D}{\partial s}\right|_0 \left.\frac{\partial}{\partial t}\right|_0 \exp t(Z + A) \cdot \gamma_o(s)$$
$$= \left.\frac{D}{\partial t}\right|_0 \exp t(Z + A)(e_1) = (Z + A)(e_1) = (0, 0, y)^t.$$

Besides,

$$T_{\gamma_o}(J) = \left.\frac{d}{dt}\right|_0 [\exp t(Z + A) \cdot \gamma_o] = \left.\frac{d}{dt}\right|_0 \phi(\exp t(Z + A)H_\kappa)$$
$$= \left.\frac{d}{dt}\right|_0 \phi(\pi(\exp t(Z + A))) = d\phi \circ d\pi\, Z,$$

where the last equality holds since $A \in \mathfrak{h}_\kappa$. Finally, the norm (2) of $d(\phi \circ \pi)Z$ equals

$$\|d(\phi \circ \pi)Z\| = \langle \dot{\gamma}_o(0) \times J(0), J'(0)\rangle = \det(x, y)$$

and the assertions of b) are verified. $\square$

Let $Z(x, y) \in \mathfrak{p}_\kappa$ and let $h = \operatorname{diag}(R_\kappa(t), B) \in H_\kappa$, where $B \in SO_2$ and

$$R_\kappa(t) = \begin{pmatrix} c_\kappa(t) & -\kappa s_\kappa(t) \\ s_\kappa(t) & c_\kappa(t) \end{pmatrix}$$

is as in (6). Then $\operatorname{Ad}(h)Z(x, y) = Z(Bx_t, By_t)$, where

$$x_t = c_\kappa(t)x - s_\kappa(t)y, \quad y_t = \kappa s_\kappa(t)x + c_\kappa(t)y.$$

We denote by $\epsilon_1$ and $\epsilon_2$ the vectors of the canonical basis of $\mathbb{R}^2$.

**Lemma 6.** *Let $Z(x, y) \neq 0$ in $\mathfrak{p}_\kappa$.*
a) *If $\{x, y\}$ is a linearly independent set of $\mathbb{R}^2$, then there exists $h \in H_\kappa$ such that $\operatorname{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$, for $\kappa = 0, \pm 1$.*
b) *If $\kappa = 0, 1$ and $\{x, y\}$ is a linearly dependent set of $\mathbb{R}^2$, then there exists $h \in H_\kappa$ such that either $\operatorname{Ad}(h)Z(x, y) = Z(0, b\epsilon_2)$, with $b \neq 0$, or $\operatorname{Ad}(h)Z(x, y) = Z(a\epsilon_1, 0)$, with $a > 0$. This is true for $\kappa = -1$ if in addition $|x| \neq |y|$.*
c) *For $\kappa = 1$, there exists $h \in H_\kappa$ such that $\operatorname{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$.*

Proof. For the proof of a), as $\{x, y\}$ is a linearly independent set, then for $\kappa = 0, \pm 1$ there exists $t \in \mathbb{R}$ such that $\langle x_t, y_t\rangle = 0$. Indeed, for each $\kappa$, this is equivalent to the fact that the equation

$$
\begin{aligned}
&c_3 - c_2 t = 0 \quad \text{if} \quad \kappa = 0;\\
&\frac{1}{2}(c_1 - c_2)\sin(2t) + c_3\cos(2t) = 0 \quad \text{if} \quad \kappa = 1;\\
&-\frac{1}{2}(c_1 + c_2)\sinh(2t) + c_3\cosh(2t) = 0 \quad \text{if} \quad \kappa = -1
\end{aligned}
$$

has a real solution, where $c_1 = \langle x, x\rangle$, $c_2 = \langle y, y\rangle$ and $c_3 = \langle x, y\rangle$. But the linear independence of $x$ and $y$ determines the existence of the solution in each case. Then, we can take $B \in SO_2$ such that $Bx_t = a\epsilon_1$, with $a > 0$ and $By_t = b\epsilon_2$, with $b \neq 0$. Therefore the isometry $h = \mathrm{diag}(R_\kappa(t), B) \in H_\kappa$ satisfies $\mathrm{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2)$.

For the proof of b), first we suppose that $x = 0$ or $y = 0$ (but not both zero since $Z(x, y) \neq 0$). Let $B \in SO_2$ such that $Bx = a\epsilon_1$ with $a > 0$, if $x \neq 0$, and in the case that $y \neq 0$, let $B \in SO_2$ such that $By = b\epsilon_2$, with $b \neq 0$. Then we can take $h = \mathrm{diag}(I, B) \in H_\kappa$.

Now, let $x \neq 0$ and $y \neq 0$. So $x = \lambda y$ or $y = \lambda x$, with $\lambda \neq 0$. We suppose that $y = \lambda x$ (for $x = \lambda y$ the argument is similar). In the cases $\kappa = 0, 1$ there exists $t \in \mathbb{R}$ such that $x_t = 0$. In fact, from the hypothesis and some computations, $t \in \mathbb{R}$ is obtained by solving

$$
1 - \lambda t = 0, \quad \text{if} \quad \kappa = 0 \quad \text{and} \quad \cos t - \lambda \sin t = 0, \quad \text{if} \quad \kappa = 1.
$$

Thus, taking $B \in SO_2$ such that $By_t = b\epsilon_2$ (with $b \neq 0$ as $y_t \neq 0$), we have that $h = \mathrm{diag}(R_\kappa(t), B) \in H_\kappa$ satisfies $\mathrm{Ad}(h)Z(x, y) = Z(0, b\epsilon_2)$.

For $\kappa = -1$, as in the cases $\kappa = 0, 1$, we find $t \in \mathbb{R}$ such that either $x_t = 0$ or $y_t = 0$ by solving

$$
\cosh t - \lambda \sinh t = 0, \quad \text{and} \quad -\sinh t + \lambda \cosh t = 0,
$$

respectively. But these equations have a solution if and only if $\lambda \neq \pm 1$. That is, if and only if $|x| \neq |y|$. Hence, taking $B \in SO_2$ such that either $By_t = b\epsilon_2$ or $Bx_t = a\epsilon_1$ (with $a > 0$; here again we have that $x_t \neq 0$), as appropriate. Then $h = \mathrm{diag}(R_{-1}(t), B) \in H_{-1}$ is as desired in this case.

For part c), we observe that $h = \mathrm{diag}(R_1(\pi/2), B) \in H_1$, where $B \in SO_2$ takes $\epsilon_1$ to $\epsilon_2$, satisfies $\mathrm{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$. □

REMARK. The previous lemma corresponds, geometrically, with the fact of finding $s \in \mathbb{R}$ at which the Jacobi field associated with $Z(x, y)$ (given by Lemma 5) and its covariant derivative are orthogonal.

Recall that if $h$ is a regular curve in $M_\kappa$ of constant speed $a$, then the Frenet frame of $h$ is

$$T(t) = \frac{1}{a}\dot{h}(t), \quad N(t) = \frac{\dot{h}'(t)}{|\dot{h}'(t)|}, \quad B(t) = T(t) \times N(t) \tag{8}$$

(here the prime denotes the covariant derivative along $h$), and its curvature and torsion are given by

$$k(t) = \frac{1}{a^2}|\dot{h}'(t)|, \quad \tau(t) = -\frac{1}{a}\langle B'(t), N(t)\rangle. \tag{9}$$

For each $g \in G_\kappa$ we have that $g$ is an isometry of $\mathcal{L}_\kappa$ and preserves the Hermitian structure. Hence, $g$ takes magnetic geodesics to magnetic geodesics.

## 3. Time- and space-like magnetic geodesics

Proof of Theorem 1. Let $Z \in \mathfrak{p}_\kappa$ be the initial velocity of $\sigma$, with $\|Z\| \neq 0$. First, we consider the case $Z = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$.

For each $t \in \mathbb{R}$, let $\alpha(t) = \exp t(Z + A)$. By Theorem 4 and the diffeomorphism $\phi$ in (7), we know that $\sigma(t) = \alpha(t) \cdot [\gamma_o]$, that is, $\sigma(t) = [\alpha(t) \cdot \gamma_o]$.

Let $h$ be the curve in $M_\kappa$ given by $h(t) = \alpha(t)(e_0)$. As $\alpha$ is a one-parameter subgroup of isometries of $M_\kappa$, we have that $h$ is a curve with constant curvature and torsion, thus $h$ is a helix in $M_\kappa$.

Let us see that $\sigma(t) = [\gamma_{B(t)}]$, where $B(t)$ is the binormal field of $h$. For each $t \in \mathbb{R}$, the initial velocity of the geodesic $\alpha(t) \cdot \gamma_o$ is $d(\alpha(t))(e_1)$, hence $\sigma(t) = [\gamma_{d(\alpha(t))(e_1)}]$. Then, we have to verify that $B(t) = d(\alpha(t))(e_1)$, for all $t \in \mathbb{R}$. Since $\alpha(t)$ is an isometry that preserves the helix and takes the Frenet frame at $t = 0$ to the Frenet frame at $t$, is suffices to show that $B(0) = e_1$.

By the usual identifications, since $\alpha(t)$ is a linear transformation, we can write $d(\alpha(t))(e_1) = \alpha(t)(e_1)$, so

$$\dot{h}(t) = \alpha(t)((Z + A)e_0) \quad \text{and} \quad \dot{h}'(t) = [\alpha(t)((Z + A)^2 e_0)]^{\mathrm{T}},$$

where T denotes the tangent projection. Since

$$\begin{aligned} \dot{h}(0) &= (Z + A)e_0 = ae_2, \\ \dot{h}'(0) &= [(Z + A)^2 e_0]^T = [-\kappa a^2 e_0 + ae_3]^T = ae_3 \end{aligned}$$

and $\alpha(t)$ is an isometry, we have $|\dot{h}(t)| = a = |\dot{h}'(t)|$. By the computation before and (8) we obtain

$$B(0) = \frac{1}{a^2}\dot{h}(0) \times \dot{h}'(0) = e_1.$$

Consequently, $B(t) = \alpha(t)(e_1)$. Then $B'(t) = [\alpha(t)((Z + A)e_1)]^T$ and $B'(0) = be_3$. Besides, using (8) and the previous computations, it follows that $N(0) = e_3$. Therefore, by (9) we have that the curvature and torsion of $h$ are equal to

$$k = \frac{1}{a}, \quad \tau = -\frac{b}{a}. \tag{10}$$

The assertion regarding the sign of the torsion is immediate from Lemma 5 b) and (10). Thus, the theorem is proved in this particular case.

Now, let $\sigma$ be a magnetic geodesic with $\sigma(0) = [\gamma]$ and initial velocity with non zero norm. Since $G_\kappa$ acts transitively on $\mathcal{L}_\kappa$, there is an isometry $g$ such that $g \cdot [\gamma] = [\gamma_o]$. So, the magnetic geodesic $g \cdot \sigma$ also has initial velocity with non zero norm and $g \cdot \sigma(0) = [\gamma_o]$. By Lemma 5 b), if $d(\phi \circ \pi)Z(x, y)$ is the initial velocity of $g \cdot \sigma$, we have that the vectors $\{x, y\}$ are linearly independent. Then, by Lemma 6 a), there exists $h \in H_\kappa$ such that $\mathrm{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$. Since $((h \circ g) \cdot \sigma)'(0) = d(\phi \circ \pi)(\mathrm{Ad}(h)Z(x, y))$, the curve $(h \circ g) \cdot \sigma$ is a magnetic geodesic of the type studied above. Therefore, $\sigma$ has the form (3).

Conversely, let $h$ be a helix in $M_\kappa$ with curvature $k > 0$, non zero torsion $\tau$ and speed $1/k$. Let $\{T, B, N\}$ be the Frenet frame of $h$. As $M_\kappa$ is a simply connected manifold of constant curvature, we have that there exists an isometry $g$ of $M_\kappa$ preserving the orientation such that $g(h(0)) = e_0$ and its differential at $h(0)$ takes $B(0)$ to $e_1$, $T(0)$ to $e_2$ and $N(0)$ to $e_3$.

Let $a = 1/k$ and $b = -\tau/k$. Let $Z = Z(a\epsilon_1, b\epsilon_2) \in \mathfrak{p}_\kappa$. We consider, for each $t \in \mathbb{R}$, $\alpha(t) = \exp t(Z + A)$. According to computations from the first part of the proof, both helices have initial position $e_0$, curvature $k$, torsion $\tau$, speed $1/k$ and the same Frenet frame at $t = 0$. Hence $(g \circ h)(t) = \alpha(t)e_0$. So, if we call $\bar{B}$ the binormal field of $g \circ h$, we have that $\bar{B}(t) = d(\alpha(t))e_1$, for all $t$. Finally, since the curve $[\gamma_{\bar{B}(t)}]$ is a magnetic geodesic in $\mathcal{L}_\kappa$ and

$$[\gamma_{B(t)}] = [\gamma_{dg^{-1}\bar{B}(t)}] = g^{-1} \cdot [\gamma_{\bar{B}(t)}],$$

we obtain that $[\gamma_{B(t)}]$ is a magnetic geodesic. □

## 4. Null magnetic geodesics

We deal first with the hyperbolic case. We use the notation given in the introduction and we recall from [5] certain properties of horospheres and related concepts. To simplify the notation we omit the subindex $\kappa = -1$.

Let $\gamma$ be a geodesic of $\mathbb{H}^3$. Then, for each $p \in \mathbb{H}^3$ there exists a unique unit speed geodesic $\alpha$ of $\mathbb{H}^3$ such that $\alpha(0) = p$ and $\alpha$ is asymptotic to $\gamma$. Let $v \in T^1\mathbb{H}^3$. If $p$ is any point of $\mathbb{H}^3$, then $v(p)$ denotes the unique unit tangent vector at $p$ that is asymptotic to $v$. The Busemann function $f_v \colon \mathbb{H}^3 \to \mathbb{R}$ is defined by

$$f_v(p) = \lim_{s \to +\infty} d(p, \gamma_v(s)) - s,$$

and satisfies $\operatorname{grad}_p(f_v) = -v(p)$. The *horosphere* determined by $v$ is given by

$$H(v) = \{q \in M \mid f_v(q) = 0\}.$$

The Jacobi vector fields orthogonal to $\dot{\gamma}_o$ have the form

$$J(s) = e^s U(s) + e^{-s} V(s), \tag{11}$$

where $U$ and $V$ are parallel vector fields along $\gamma_o$ and orthogonal to $\dot{\gamma}_o$.

A Jacobi vector field $Y$ along a geodesic $\gamma$ of $\mathbb{H}^3$ is said to be *stable* (*unstable*) if there exists a constant $c > 0$ such that

$$|Y(s)| \leq c \quad \forall s \geq 0 \quad (\forall s \leq 0).$$

In what follows we shall denote by $\hat{\pi}$ the canonical projection from $T\mathbb{H}^3$ onto $\mathbb{H}^3$. We recall that in the introduction we have defined the smooth maps $\psi^{\pm}$: $\mathcal{L}(\mathbb{H}^3) \to \mathbb{H}^3(\infty)$ by $\psi^{\pm}[\gamma] = \gamma(\pm\infty)$ and the distributions $\mathcal{D}^{\pm}$ in $\mathcal{L}(\mathbb{H}^3)$ given by $\mathcal{D}^{\pm}_{[\gamma]} = \operatorname{Ker}(d\psi^{\pm}_{[\gamma]})$. We need to relate the distributions $\mathcal{D}^{\pm}$ with distributions $\bar{\mathcal{E}}^{\pm}$ and $\mathcal{E}^{\pm}$ on $G$ and $T^1\mathbb{H}^3$, respectively.

Let $\bar{\mathcal{E}}^{\pm}$ be the left invariant distribution on $G$ defined at $I \in G$ by

$$\bar{\mathcal{E}}^{\pm}_I = \{Z(u, \mp u) \in \mathfrak{p} \mid u \in \mathbb{R}^2\}.$$

As the canonical action of $G$ on $T^1\mathbb{H}^3$ is transitive, the projection $\bar{p}$: $G \to T^1\mathbb{H}^3$ given by $\bar{p}(g) = dg_{e_0} e_1$ is a submersion. Since given $v \in T^1\mathbb{H}^3$ there exists $g \in G$ such that $\bar{p}(g) = v$, we define:

$$\mathcal{E}^{\pm}(v) = (d\bar{p}\ \ \bar{\mathcal{E}}^{\pm})(\bar{p}(g)) = d\bar{p}_g(\bar{\mathcal{E}}^{\pm}_g).$$

We have that $\mathcal{E}^{\pm}$ determines a well defined distribution on $T^1\mathbb{H}^3$, which is called the *horospherical distribution* on $T^1\mathbb{H}^3$. This distribution has the following property: if $t \mapsto v(t)$ is a curve in $T^1\mathbb{H}^3$ tangent to the distribution $\mathcal{E}^{\pm}$, then $\hat{\pi}(v(t))$ is in the horosphere $H(\pm v(0))$.

**Lemma 7.** *Let $Z \in \bar{\mathcal{E}}^{\pm}_I$. For each $t \in \mathbb{R}$, let $\gamma_t^{\pm}(s) = \exp t(Z + A) \cdot \gamma_o(\pm s)$. Then the geodesics $\gamma_t^{\pm}$ are asymptotic to each other for all $t \in \mathbb{R}$.*

Proof. Let $J$ be the Jacobi vector field associated with the variation by geodesics $t \mapsto \gamma_t^{\pm}$. By Lemma 5 a), $J(0) = -J'(0)$. Hence, by (11) we have that $J(s) = e^{-s}U(s)$, where $U$ is a parallel vector field along $\gamma_o$ orthogonal to $\dot{\gamma}_o$. Thus, $J$ is a stable vector field, that is, there exists $c > 0$ such that $|J(s)| \leq c\ \forall s \geq 0$.

We have to show that given $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, there exists $N > 0$ such that

$$d(\gamma_{t_0}^{\pm}(s), \gamma_{t_1}^{\pm}(s)) \leq N \quad \forall s \geq 0.$$

For fixed $s$,

$$d(\gamma_{t_0}^{\pm}(s), \gamma_{t_1}^{\pm}(s)) \leq \text{length}([t_0, t_1] \ni t \mapsto \gamma_t^{\pm}(s)) = \int_{t_0}^{t_1} \left| \frac{d}{dt} \gamma_t^{\pm}(s) \right| dt.$$

For each $t \in \mathbb{R}$, let $J_t(s) = (d/dt)\gamma_t^{\pm}(s)$. We observe that $J_{t'+t}(s) = d \exp(t'Z) J_t(s)$ for all $t, t'$. Since $\exp(t'Z)$ is an isometry, we have $|J_t(s)| = |J(s)|$. Therefore,

$$\int_{t_0}^{t_1} |J_t(s)|\, dt = \int_{t_0}^{t_1} |J(s)|\, dt \leq c(t_1 - t_0)$$

for all $s \geq 0$. Then, we may take $N = c(t_1 - t_0) > 0$. $\square$

We consider the projection $p \colon T^1\mathbb{H}^3 \to \mathcal{L}(\mathbb{H}^3)$, $p(v) = [\gamma_v]$. We call $\bar{\mathcal{D}}^{\pm}$ the distribution on $\mathcal{L}(\mathbb{H}^3)$ $p$-related with $\mathcal{E}^{\pm}$ (well defined). More specifically, given $[\gamma] \in \mathcal{L}(\mathbb{H}^3)$ and $v \in T^1\mathbb{H}^3$ such that $p(v) = [\gamma]$,

$$\bar{\mathcal{D}}^{\pm}([\gamma]) = dp_v \mathcal{E}_v^{\pm}.$$

**Proposition 8.** *Let $\mathcal{D}^{\pm}$ and $\bar{\mathcal{D}}^{\pm}$ be the distributions on $\mathcal{L}(\mathbb{H}^3)$ defined above. Then $\mathcal{D}^{\pm} = \bar{\mathcal{D}}^{\pm}$.*

Proof. Since $\mathcal{D}^{\pm}$ and $\bar{\mathcal{D}}^{\pm}$ are $G$-invariant, it is enough to show $\mathcal{D}^{\pm}_{[\gamma_o]} = dp_{(e_0,e_1)}(\mathcal{E}^{\pm}_{(e_0,e_1)})$ (we observe that $\bar{p}(I) = (e_0, e_1)$ and $p(e_0, e_1) = [\gamma_o]$).

Let $Z \in \bar{\mathcal{E}}^{\pm}_I$. We take the curve in $\mathcal{L}(\mathbb{H}^3)$ given by $\alpha(t) = \exp tZ \cdot [\gamma_o]$. As $\alpha(t) = p \circ \bar{p}(\exp tZ)$, we have that $\alpha(0) = [\gamma_o]$ and $\dot{\alpha}(0) = d(p \circ \bar{p})_I Z$. That is, $\dot{\alpha}(0) \in dp_{(e_0,e_1)}(\mathcal{E}^{\pm}_{(e_0,e_1)})$. Besides,

$$\left.\frac{d}{dt}\right|_0 \exp tZ \cdot \gamma_o(s) = \left.\frac{d}{dt}\right|_0 \exp t(Z + A) \cdot \gamma_o(s), \tag{12}$$

since both Jacobi fields have the same initial conditions. Hence, Lemma 7 applies to the geodesics $\gamma_t^{\pm}(s) = \exp tZ \cdot \gamma_o(\pm s)$. Thus, $\psi^{\pm} \circ \alpha$ is constant. Then $(d\psi^{\pm})_{[\gamma_o]}(\dot{\alpha}(0)) = 0$, that is, $\dot{\alpha}(0) \in \mathcal{D}^{\pm}_{[\gamma_o]}$.

On the other hand, let $\varphi \colon T^1_{e_0}\mathbb{H}^3 \to \mathcal{L}(\mathbb{H}^3)$, $\varphi(v) = [\gamma_v]$, be the submanifold whose image $\mathcal{L}_{e_0}(\mathbb{H}^3)$ consists of all the oriented geodesics passing through $e_0$. Besides, $H(\infty)$ is a manifold with the differentiable structure (well defined) such that $F_{e_0} \colon T^1_{e_0}\mathbb{H}^3 \to H(\infty)$ given by $F_{e_0}(v) = \gamma_v(\infty)$ is a diffeomorphism. Then, since $\psi^+|_{\mathcal{L}_{e_0}(\mathbb{H}^3)} \circ \varphi = F_{e_0}$, we have that $(d\psi^+)_{[\gamma_o]}$ is surjective. Now, $(d\psi^-)_{[\gamma_o]}$ is also surjective because $\psi^-$ is the composition of $\psi^+$ with the diffeomorphism of $\mathcal{L}(\mathbb{H}^3)$ assigning $[\gamma^{-1}]$ to $[\gamma]$. Therefore, $\dim \mathcal{D}^{\pm}_{[\gamma_o]} = \dim \bar{\mathcal{D}}^{\pm}_{[\gamma_o]}$ and equality follows. $\square$

The word *cylinder* in the statement of Theorem 2 refers to a ruled surface determined by a parallel vector field along a curve $c$ of constant geodesic curvature $k$ contained in a totally geodesic surface in $M_\kappa$ (and normal to it), as explained. For $\kappa = -1$, this ruled surface is diffeomorphic to $S^1 \times \mathbb{R}$ if $|k| > 1$; otherwise it is diffeomorphic to a plane.

Proof of Theorem 2 a). By Lemma 5 b), we have that every element of $\mathcal{D}^{\pm}_{[\gamma]}$ is null. As $G$ acts transitively on $\mathcal{L}(\mathbb{H}^3)$ and by the $G$-invariance of the horospherical distributions, we may suppose without loss of generality that $\sigma(0) = [\gamma_o]$, hence $\dot\sigma(0) \in \mathcal{D}^{\pm}_{[\gamma_o]}$. By Proposition 8, there exists $Z \in \bar{\mathcal{E}}^{\pm}_I$ such that $\dot\sigma(0) = (dp)_{(e_0,e_1)}(d\bar p)_I Z$. Thus, by Theorem 4, $\sigma(t) = [\exp t(Z + A) \cdot \gamma_o]$.

We assume that $Z \in \bar{\mathcal{E}}^{+}_I$. Let us show that $\sigma$ describes a forward cone with vertex at $\gamma_o(+\infty)$. In a similar way, if $Z \in \bar{\mathcal{E}}^{-}_I$, then $\sigma$ describes a backward cone with vertex at $\gamma_o(-\infty)$.

We consider the geodesics $\gamma_t(s) = \exp t(Z + A) \cdot \gamma_o(s)$ of $\mathbb{H}^3$. As $Z \in \bar{\mathcal{E}}^{+}_I$, by Lemma 7, we have that the geodesics $\gamma_t$ are asymptotic to each other for all $t$. Hence, $z(t) = \dot\gamma_t(0)$ is a curve in $T^1\mathbb{H}^3$ of asymptotic vectors to $e_1$.

Let $c(t) = \hat\pi(z(t)) = \exp t(Z + A)(e_0)$. In order to see that $c(t) \in H(e_1)$ for all $t$, we observe that

$$\frac{d}{dt} f_{e_1}(c(t)) = (d f_{e_1})_{c(t)}\dot c(t) = \langle \mathrm{grad}_{c(t)}(f_{e_1}), \dot c(t)\rangle. \tag{13}$$

Since $\mathrm{grad}_p(f_v) = -v(p)$ we have that

$$\mathrm{grad}_{c(t)}(f_{e_1}) = -z(t) = -d(\exp t(Z + A))e_1.$$

On the other hand,

$$\dot c(t) = d(\exp t(Z + A))(Z + A)e_0.$$

Since $\exp t(Z+A)$ is an isometry and observing that $(Z+A)e_0$ and $e_1$ are perpendicular ($Z \in \bar{\mathcal{E}}^{+}_I$), it follows that the expression in (13) is equal to $-\langle e_1, (Z+A)(e_0)\rangle = 0$. Then, $f_{e_1}(c(t)) = f_{e_1}(e_0) = 0$ for all $t$, that is, $c(t) \in H(e_1)$ for all $t$.

Now, as $c$ is the orbit through $e_0$ of a one-parameter subgroup of isometries of $G$ preserving $H(e_1)$, its geodesic curvature and speed are constant. If $Z = Z(u, -u)$ for certain $0 \neq u \in \mathbb{R}^2$, we obtain that the speed of $c$ is $|u|$. For each $v \in T^1H^3$ we consider on $H(v)$ the orientation given by $-\,\mathrm{grad}\, f_v$. The geodesic curvature of $c$ is then

$$k = \frac{\langle -\mathrm{grad}_{e_0}(f_{e_1}), \dot c(0) \times \dot c'(0)\rangle}{|u|^3} = \frac{1}{|u|},$$

since $\dot c(0) = (Z + A)e_0$ and $\dot c'(0) = ((Z + A)^2 e_0)^{\mathrm{T}}$. As for each $v \in T^1\mathbb{H}^3$, $H(v)$, with the induced metric of $\mathbb{H}^3$, is isometric to $\mathbb{R}^2$, we have that $c(t)$ runs along a circle on $H(e_1)$ of geodesic curvature $k = 1/|u| > 0$ and speed $1/k = |u|$.

Besides, $\sigma(t) = [\gamma_{z(t)}]$. Thus we have that all conditions are satisfied in order to assert that $\sigma$ describes a forward cone with vertex at $\gamma_o(+\infty)$.

Conversely, let $\sigma$ be a curve in $\mathcal{L}(\mathbb{H}^3)$ that describes a forward cone with vertex at infinity. As $G$ acts transitively on the positively oriented frame bundle, and also each element of $G$ takes horospheres to horospheres, preserving their orientation, we may suppose that $\sigma(t) = [\gamma_{v(t)}]$, where $v(t)$ is a curve in $T^1\mathbb{H}^3$ of asymptotic vectors to $v(0) = e_1$ and $c(t) = \hat{\pi}(v(t))$ is a curve of geodesic curvature $k$ and speed $1/k$ in $H(e_1)$ with $\dot{c}(0) = (1/k)e_2$, for some $k > 0$. Let $Z = Z((1/k)\epsilon_1, -(1/k)\epsilon_1) \in \bar{\mathcal{E}}_I^+$. We define

$$\bar{c}(t) = \exp t(Z + A)(e_0) \quad \text{and} \quad \bar{v}(t) = d(\exp t(Z + A))(e_1).$$

We showed above that $\bar{c}(t)$ is a curve of geodesic curvature $k$ and speed $1/k$ in $H(e_1)$. Moreover, $\bar{c}(0) = e_0$ and the initial velocity of $\bar{c}$ is $(1/k)e_2$. So, we obtain that $\bar{c} = c$. This implies, together with the identities $\hat{\pi} \circ \bar{v} = \bar{c}$ and $\hat{\pi} \circ v = c$, that $\hat{\pi} \circ \bar{v} = \hat{\pi} \circ v$.

According to the first part of the proof, $\bar{v}$ and $v$ are curves of asymptotic vectors to $e_1$. Hence, $-\bar{v}(t) = \text{grad}_{\bar{c}(t)}(f_{e_1}) = -v(t)$. Therefore, $[\gamma_{\bar{v}(t)}] = [\gamma_{v(t)}]$, which is a null magnetic geodesic with initial velocity in the horospherical distribution since $[\gamma_{v(t)}] = [\exp t(Z + A) \cdot \gamma_o]$. $\square$

*Proof of Theorem 2 b).* We suppose first that $\sigma$ is a null magnetic geodesic such that $\sigma(0) = [\gamma_o]$ and $\dot{\sigma}(0) = d(\phi \circ \pi)Z(a\epsilon_1, 0)$, with $a > 0$. The expression (4) and the relation between the speed and curvature of $h$ are obtained as in the prove of Theorem 1. By (10) we know that the torsion of $h$ is $\tau = -b/a = 0$ (since $b = 0$). Thus $h$ is contained in a totally geodesic surface $S$ of $\mathbb{H}^3$ and $B$ is normal to $S$.

Now, we suppose that $\dot{\sigma}(0) = d(\phi \circ \pi)Z$, where $Z = Z(0, b\epsilon_2)$ with $b \neq 0$. By Theorem 4 we have that $\sigma(t) = [\alpha(t) \cdot \gamma_o]$, where $\alpha(t) = \exp t(Z + A)$. Since $Z + A$ is in the Lie algebra of the isotropy subgroup of $G$ at $e_0 \in \mathbb{H}^3$, we get that $\alpha(t)$ fixes $e_0$. Moreover, if $v$ is the curve in $T^1_{e_0}\mathbb{H}^3$ given by $v(t) = d(\alpha(t))e_1$, then

$$\sigma(t) = [\alpha(t) \cdot \gamma_o] = [\gamma_{v(t)}],$$

since the initial velocity of the geodesic $\alpha(t) \cdot \gamma_o$ is $v(t)$, for each $t \in \mathbb{R}$.

Furthermore, as $v$ is the orbit through $e_1$ of a one-parameter subgroup of $H$ (the canonical differential action of $G$ on $T^1_{e_0}\mathbb{H}^3$), then $v$ has constant speed and constant geodesic curvature in $T^1_{e_0}\mathbb{H}^3 \cong \mathbb{S}^2$. Easy computations yield

$$\dot{v}(0) = (0, 0, b)^t \quad \text{and} \quad \ddot{v}(0) = (-b^2, -b, 0)^t.$$

So, the speed of $v$ is $|b|$ and its geodesic curvature is

$$k = \frac{\langle v(0), \dot{v}(0) \times \ddot{v}(0) \rangle}{|b|^3} = \frac{1}{|b|}$$

(we consider the orientation of the sphere given by the unit normal field pointing outwards). Thus, $v$ is a curve in $T^1_{e_0}\mathbb{H}^3$ of geodesic curvature $k > 0$ and speed $1/k$. Consequently, $\sigma$ has the form (5).

Now, let $\sigma$ be a null magnetic geodesic such that $\sigma(0) = [\gamma]$ and $\dot\sigma(0) \notin \mathcal{D}^{\pm}_{[\gamma]}$. As $G$ acts transitively on $\mathcal{L}(\mathbb{H}^3)$ and by the $G$-invariance of the horospherical distributions, we may suppose that $\sigma(0) = [\gamma_o]$ and $\dot\sigma(0) \notin \mathcal{D}^{\pm}_{[\gamma_o]}$. Let $Z = Z(x, y) \in \mathfrak{p}$ such that $\dot\sigma(0) = d(\phi \circ \pi)Z$. By Lemma 5 b), as the norm of the initial velocity of $\sigma$ is zero, we have that $x$ and $y$ are linearly dependent, and since $d(\phi \circ \pi)Z \notin \mathcal{D}^{\pm}_{[\gamma_o]}$, we also have $|x| \neq |y|$. Now, the isometries in Lemma 6 b) take $\sigma$ to magnetic geodesics of the particular types studied above. Therefore, $\sigma$ has the form (4) or has the form (5), as desired.

Conversely, given a helix $h$ in $\mathbb{H}^3$ with curvature $k$, speed $1/k$ and torsion $\tau = 0$, the proof that the expression (4) is a magnetic geodesic is identical to the proof of the converse of Theorem 1. As $h$ has zero torsion, the initial velocity of the magnetic geodesic in (4) is not in the distributions $\mathcal{D}^{\pm}$.

Let $v$ be a curve in $T^1_p\mathbb{H}^3$ with geodesic curvature $k > 0$ and speed $1/k$. Let $g$ be the isometry of $\mathbb{H}^3$ preserving the orientation such that $g(p) = e_0$, $dg(v(0)) = e_1$ and $dg(\dot v(0)) = be_3$, for certain $b > 0$. Hence, $g \cdot v$ is a curve in $T^1_{e_0}\mathbb{H}^3$ having the same geodesic curvature and the same speed as $v$, and also $b = 1/k$. As we showed above, $\bar v$ is a curve in $T^1_{e_0}\mathbb{H}^3$ with $\bar v(0) = g \cdot v(0)$ and with the same initial velocity and geodesic curvature that $g \cdot v$. By uniqueness, we have that $\bar v = g \cdot v$. To complete the proof we observe that $g \cdot [\gamma_{v(t)}] = [\gamma_{g\cdot v(t)}] = [\gamma_{\bar v(t)}]$. $\square$

Proof of Theorem 3. Lemma 6 b) implies that the analogue of Theorem 2 a) is empty for the cases $\kappa = 0, 1$. The proof of the fact that every curve $\sigma$ in $\mathcal{L}_\kappa$ is a null magnetic geodesic if and only if $\sigma$ has the form (4) or (5) is similar to that of Theorem 2 b).

We check the last statement of the theorem. Without lost of generality, we consider only null magnetic geodesics passing through $[\gamma_o]$ at $t = 0$. We observe that if, in particular, $\sigma$ is a magnetic geodesic with initial velocity $d(\phi \circ \pi)Z(a\epsilon_1, 0)$, with $a > 0$, (that is, $\sigma$ has the form (4)), then by Lemma 6 c) there exists $h \in H_1$ such that $\mathrm{Ad}(h)Z(a\epsilon_1,0) = Z(0,a\epsilon_2)$. Hence, $h\cdot\sigma$ is a null magnetic geodesic with initial velocity $d(\phi \circ \pi)Z(0, a\epsilon_2)$, and then it has the form (5). So, $\sigma$ also has this form. $\square$

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