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ON THE DISTRIBUTION OF k-TH POWER FREE INTEGERS, II

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Abstract

The indicator function of the set of k-th power free integers is naturally extended to a random variable $X^{(k)}(\,\cdot\,)$ on $(\hat{\mathbb{Z}},\lambda)$, where $\hat{\mathbb{Z}}$ is the ring of finite integral adeles and λ is the Haar probability measure. In the previous paper, the first author noted the strong law of large numbers for $\{X^{(k)}(\,\cdot\,+n)\}_{n=1}^{\infty}$, and showed the asymptotics: $E^{\lambda}[(Y_N^{(k)})^2] \asymp 1$ as $N\to\infty$, where $Y_N^{(k)}(x) := N^{-1/2k}\sum_{n=1}^{N}\bigl(X^{(k)}(x+n)-1/\zeta(k)\bigr)$. In the present paper, we prove the convergence of $E^{\lambda}[(Y_N^{(k)})^2]$. For this, we present a general proposition of analytic number theory, and give a proof to this.

1. Introduction

Let $\hat{\mathbb{Z}}$ be the ring of finite integral adeles; $\mathcal{B}$ the Borel σ-field of $\hat{\mathbb{Z}}$; λ the Haar probability measure on $(\hat{\mathbb{Z}},\mathcal{B})$. In [4, 1], the triplet $(\hat{\mathbb{Z}},\mathcal{B},\lambda)$ is introduced in the following way: For a prime number p, the p-adic metric d_p on $\mathbb{Z}$ is defined by

$$d_p(x,y) := \inf\{p^{-l};\ p^l \mid (x-y)\},\quad x, y\in\mathbb{Z}.$$

The completion of $\mathbb{Z}$ by d_p is denoted by $\mathbb{Z}_p$. By extending the algebraic operations '$+$' and '$\times$' in $\mathbb{Z}$ continuously to those in $\mathbb{Z}_p$, the compact metric space $(\mathbb{Z}_p, d_p)$ becomes a ring. In particular, $(\mathbb{Z}_p,d_p)$ is a compact abelian group with respect to '$+$'. Thus, there is a unique Haar probability measure λ_p with respect to '$+$' on $(\mathbb{Z}_p,\mathcal{B}(\mathbb{Z}_p))$, where $\mathcal{B}(\mathbb{Z}_p)$ is the Borel σ-field of $\mathbb{Z}_p$.

Putting $p_i = i$-th prime number $(i=1,2,\dots)$, we set

$$\hat{\mathbb{Z}} := \prod_{i=1}^{\infty}\mathbb{Z}_{p_i},\quad \lambda := \prod_{i=1}^{\infty}\lambda_{p_i}.$$

For $x=(x_i)$, $y=(y_i)\in\hat{\mathbb{Z}}$, we define

$$d(x,y) := \sum_{i=1}^{\infty}\frac{1}{2^i}d_{p_i}(x_i,y_i),$$

$$x+y := (x_i+y_i),\quad xy := (x_iy_i).$$

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By these definitions, $\hat{\mathbb{Z}}$ becomes a ring, which is just the ring of finite integral adeles stated above. $(\hat{\mathbb{Z}}, d)$ is again a compact metric space, and both '$+$' and '$\times$' are continuous. In particular, this is a compact abelian group with respect to '$+$', and its Haar probability measure is nothing but λ. By identifying $\mathbb{Z}$ with the diagonal set $\{(n, n, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots;\ n \in \mathbb{Z}\} \subset \hat{\mathbb{Z}}$, it is seen that $\mathbb{Z}$ is a dense subring of $\hat{\mathbb{Z}}$. Thus $\hat{\mathbb{Z}}$ is a compactification of $\mathbb{Z}$.

Let k be an integer, ≥ 2. Let $B^{(k)}$ be the set of all elements in $\hat{\mathbb{Z}}$ having no k-th power factors, i.e.,

$$B^{(k)} := \{x \in \hat{\mathbb{Z}};\ p^k \nmid x\ \ (\forall p\text{: prime})\},$$

where $d \mid x \Leftrightarrow x \in d\hat{\mathbb{Z}}$ (, so $d \nmid x \Leftrightarrow x \in \hat{\mathbb{Z}} \setminus d\hat{\mathbb{Z}}$), and $X^{(k)} := \mathbf{1}_{B^{(k)}}$ ($=$ the indicator function of $B^{(k)}$).

The following are results of Duy [1]:

Fact 1 (Strong law of large numbers). $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N} X^{(k)}(x+n) = 1/\zeta(k)$, *$\lambda$-a.e. x. Here $\zeta(\cdot)$ is the Riemann zeta function.*

For each $N \in \mathbb{N}$, we set

$$Y_N^{(k)}(x) := \frac{1}{N^{1/(2k)}} \sum_{n=1}^{N} \left(X^{(k)}(x+n) - \frac{1}{\zeta(k)} \right). \tag{1}$$

Fact 2. $E^{\lambda}[(Y_N^{(k)})^2] \asymp 1$ *as* $N \to \infty$.

Fact 3. *A sequence $\{Y_N^{(k)}\}_{N=1}^{\infty}$ in $L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$ has no limit point. Namely, for any subsequence $\{N_i\}_{i=1}^{\infty}$, $\{Y_{N_i}^{(k)}\}_{i=1}^{\infty}$ is not convergent in L^2 as $i \to \infty$.*

Fact 1 follows at once from the ergodicity of the shift $x \mapsto x+1$ and $E^{\lambda}[X^{(k)}] = 1/\zeta(k)$[1]. From this fact, we have the following question: *When $\sum_{n=1}^{N}(X^{(k)}(x+n) - 1/\zeta(k))$ is normalized appropriately, is its distribution weakly convergent as $N \to \infty$?* Fact 2 tells us that a normalizing constant must be $N^{1/(2k)}$, and that a sequence $\{\lambda(Y_N^{(k)} \in *)\}_{N=1}^{\infty}$ of distributions on $\mathbb{R}$ is tight. Fact 3 is a functional analytical result and brings no news for the behavior of $Y_N^{(k)}$ as $N \to \infty$. But, for this, we expect to have a limit theorem in probability theory. (Unfortunately, we still have no information on this limit theorem.)

In this paper, we make some remark about Fact 2 and Fact 3.

[1] Cf. 1° in the proof of Claim 1.

Theorem 1.

$$\lim_{N\to\infty} E^{\lambda}[(Y_N^{(k)})^2] = \left(\prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{2}{p^k}\right)\right)\frac{\zeta(2-1/k)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/(2k))}.$$

Theorem 2. (i) $\lim_{N\to\infty}\lim_{M\to\infty} E^{\lambda}[(Y_M^{(k)}-Y_N^{(k)})^2] = 2\bigl(\prod_p(1-1/p)(1+1/p-2/p^k)\bigr)\zeta(2-1/k)/((2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/2k)) > 0$. Fact 3 *above is a consequence of this.*
(ii) *But, a whole sequence* $\{Y_N^{(k)}\}_{N=1}^{\infty}$ *in* $L^2(\hat{\mathbb{Z}},\mathcal{B},\lambda)$ *is weakly convergent to* 0 *as* $N\to\infty$.

Throughout this paper, the letter p denotes a prime number, and the symbols $\prod_p$ and $\sum_p$ are a product and a summation extended over all prime numbers, respectively.

Theorems above will be proved in Section 4. In Section 2, an another computation of $E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$, which is different from one in Duy [1], is given. And, in Section 3, to prove Theorem 1, we prepare Proposition 1. This is a general proposition of analytic number theory, and will be proved in Section 5.

The authors would like to thank the referee for good advice which enabled us to make proofs clear and considerably short.

2. Computation of $E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$

By a different approach[2] from Duy [1], we compute $E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$.

Claim 1. *For* $M\geq N\geq 1$,

$$E^{\lambda}[Y_M^{(k)}Y_N^{(k)}] = \frac{1}{M^{1/(2k)}}\frac{1}{N^{1/(2k)}}\sum_{c=1}^{\infty}|\mu(c)|\left(\prod_{p\nmid c}\left(1-\frac{2}{p^k}\right)\right)\left\{\frac{M}{c^k}\right\}\wedge\left\{\frac{N}{c^k}\right\}\left(1-\left\{\frac{M}{c^k}\right\}\vee\left\{\frac{N}{c^k}\right\}\right).$$

Here $\mu(\cdot)$ *is the Möbius function and* $\{a\}$ *is the fractional part of the real number* a.

Proof. Fix $M\geq N\geq 1$. We divide the proof into three steps:
<u>1°</u> First

$$E^{\lambda}[Y_M^{(k)}Y_N^{(k)}] = E^{\lambda}\left[\frac{1}{M^{1/(2k)}}\frac{1}{N^{1/(2k)}}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(X^{(k)}(x+m)-\frac{1}{\zeta(k)}\right)\left(X^{(k)}(x+n)-\frac{1}{\zeta(k)}\right)\right]$$

[2] Duy's method is originally due to [4]. The same kind of computation in the proof of Claim 1 appears in early study of [4]. So, a phrase 'different approach' may be too much to say.

$$= \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \Bigg(E^{\lambda}[X^{(k)}(x+m) X^{(k)}(x+n)] - \frac{1}{\zeta(k)} (E^{\lambda}[X^{(k)}(x+m)] + E^{\lambda}[X^{(k)}(x+n)]) + \left(\frac{1}{\zeta(k)}\right)^2 \Bigg).$$

Noting that

$$X^{(k)}(y) = \prod_p (1 - \rho_{p^k}(y)), \tag{2}$$

where, for $d \in \mathbb{N}$, $\rho_d(y) := \begin{cases} 1, & d \mid y \ (\Leftrightarrow y \in d\hat{\mathbb{Z}}), \\ 0, & \text{otherwise}, \end{cases}$

$$\{\rho_{p^k}\}_p \quad \text{is independent,} \tag{3}$$

$$\lambda(\rho_d = 1) = \frac{1}{d}, \quad \lambda(\rho_d = 0) = 1 - \frac{1}{d}, \tag{4}$$

we have

$$\begin{aligned} E^{\lambda}[X^{(k)}(x+m)] &= E^{\lambda}[X^{(k)}(x)] \quad \text{(by the shift invariance of } \lambda) \\ &= \prod_p \left(1 - \frac{1}{p^k}\right) \\ &= \frac{1}{\zeta(k)} \quad \text{(by Euler's product of } \zeta(\,\cdot\,)), \end{aligned}$$

and thus

$$\begin{aligned} &E^{\lambda}[Y_M^{(k)} Y_N^{(k)}] \\ &= \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \left(E^{\lambda}[X^{(k)}(x+m) X^{(k)}(x+n)] - \left(\frac{1}{\zeta(k)}\right)^2 \right). \end{aligned}$$

Since, by (2)

$$\begin{aligned} &X^{(k)}(x+m) X^{(k)}(x+n) \\ &= \prod_p (1 - \rho_{p^k}(x+m)) \cdot (1 - \rho_{p^k}(x+n)) \\ &= \prod_p (1 - \rho_{p^k}(x+n) - \rho_{p^k}(x+m) + \rho_{p^k}(x+m) \rho_{p^k}(x+n)) \\ &= \prod_p (1 - \rho_{p^k}(x+n) - \rho_{p^k}(x+m) + \rho_{p^k}(m-n) \rho_{p^k}(x+n)) \\ &\qquad \text{(by an identity: } \rho_d(x+m)\rho_d(x+n) = \rho_d(m-n)\rho_d(x+n)), \end{aligned}$$

we see from (3) and (4) that

$$E^{\lambda}[Y_M^{(k)} Y_N^{(k)}] = \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \left(\prod_p \left(1 - \frac{2}{p^k} + \rho_{p^k}(m-n) \frac{1}{p^k} \right) - \left(\frac{1}{\zeta(k)} \right)^2 \right).$$

2° By Euler's product of $\zeta(\cdot)$

$$\begin{aligned}\left(\frac{1}{\zeta(k)} \right)^2 &= \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{2k}} \right) \\ &= \prod_p \left(1 + \frac{1}{p^k} \left(-2 + \frac{1}{p^k} \right) \right) \\ &= \sum_d \frac{|\mu(d)|}{d^k} \prod_{p|d} \left(-2 + \frac{1}{p^k} \right) \\ &= \sum_d \frac{|\mu(d)|}{d^k} \sum_{c|d} (-2)^{\omega(d/c)} \prod_{p|c} \frac{1}{p^k}\end{aligned}$$

(where $\omega(n) := \#\{p;\, p \mid n\}$ = the number of different prime factors of n)

$$\begin{aligned}&= \sum_d \frac{|\mu(d)|}{d^k} \sum_{c|d} (-2)^{\omega(d/c)} \frac{1}{c^k} \\ &= \sum_{c_1, d_1} \frac{|\mu(c_1 d_1)|}{(c_1 d_1)^k} (-2)^{\omega(d_1)} \frac{1}{c_1^k}\end{aligned}$$

(there exists a one-to-one correspondence between the set $\{(c, d);\, d$ is square free and $c \mid d\}$ and the set $\{(c_1, d_1);\, c_1 d_1$ is square free$\}$; a correspondence from the former to the latter is $(c, d) \mapsto (c, d/c)$ and one from the latter to the former is $(c_1, d_1) \mapsto (c_1, c_1 d_1)$. Here (c, d) and (c_1, d_1) denote a pair of c and d, and that of c_1 and d_1, respectively)

$$= \sum_{c_1, d_1} \frac{|\mu(c_1 d_1)|}{c_1^{2k}} \mathbf{1}_{(c_1, d_1) = 1} \frac{(-2)^{\omega(d_1)}}{d_1^k}$$

(where (c_1, d_1) is the greatest common divisor of c_1 and d_1. Note that $\mu(c_1 d_1) = 0$ if $(c_1, d_1) > 1$)

$$= \sum_{c_1, d_1} \frac{|\mu(c_1)|\, |\mu(d_1)|}{c_1^{2k}} \mathbf{1}_{(c_1, d_1) = 1} \frac{(-2)^{\omega(d_1)}}{d_1^k}$$

(by the multiplicativity of μ)

$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^{2k}} \sum_{d_1} \frac{|\mu(d_1)|}{d_1^k} \mathbf{1}_{(c_1,d_1)=1} (-2)^{\omega(d_1)}$$
$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^{2k}} \prod_{p} \left(1 + \frac{|\mu(p)|}{p^k} \mathbf{1}_{(c_1,p)=1} (-2)^{\omega(p)}\right)$$

(by the multiplicativity of $d_1 \mapsto |\mu(d_1)| \mathbf{1}_{(c_1,d_1)=1} (-2)^{\omega(d_1)}$)

$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^{2k}} \prod_{p \nmid c_1} \left(1 - \frac{2}{p^k}\right).$$

Similarly, since

$$\prod_{p} \left(1 - \frac{2}{p^k} + \rho_{p^k}(m-n) \frac{1}{p^k}\right)$$
$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^k} \prod_{p \nmid c_1} \left(1 - \frac{2}{p^k}\right) \rho_{c_1^k}(m-n),$$

we have

$$\prod_{p} \left(1 - \frac{2}{p^k} + \rho_{p^k}(m-n) \frac{1}{p^k}\right) - \left(\frac{1}{\zeta(k)}\right)^2$$
$$= \sum_{c} \frac{|\mu(c)|}{c^k} \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right) \left(\rho_{c^k}(m-n) - \frac{1}{c^k}\right).$$

Thus, by 1° and 3° below

$$E^\lambda[Y_M^{(k)} Y_N^{(k)}]$$
$$= \frac{1}{M^{1/(2k)} N^{1/(2k)}} \sum_{\substack{1 \le m \le M,\\ 1 \le n \le N}} \sum_{c} \frac{|\mu(c)|}{c^k} \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right) \left(\rho_{c^k}(m-n) - \frac{1}{c^k}\right)$$
$$= \frac{1}{M^{1/(2k)} N^{1/(2k)}} \sum_{c} |\mu(c)| \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right) \frac{1}{c^k} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\rho_{c^k}(m-n) - \frac{1}{c^k}\right)$$
$$= \frac{1}{M^{1/(2k)} N^{1/(2k)}} \sum_{c} |\mu(c)| \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right) \left\{\frac{M}{c^k}\right\} \wedge \left\{\frac{N}{c^k}\right\} \left(1 - \left\{\frac{M}{c^k}\right\} \vee \left\{\frac{N}{c^k}\right\}\right).$$

This is the assertion of the claim.

3° Fix $u \in \mathbb{N}$. Let Q and s be a quotient and a remainder of N divided by u, respectively. Thus $N = Qu + s$, where $Q = \lfloor N/u \rfloor$[3], $s = \{N/u\}u \in \{0, 1, \ldots, u-1\}$. Then

$$\begin{aligned}
&\sum_{n=1}^{N} \rho_u(m-n) \\
&= \sum_{q=1}^{Q} \sum_{j=1}^{u} \rho_u(m - ((q-1)u + j)) + \sum_{j=1}^{s} \rho_u(m - (Qu + j)) \\
&= \sum_{q=1}^{Q} \sum_{j=1}^{u} \rho_u(m - j - (q-1)u) + \sum_{j=1}^{s} \rho_u(m - j - Qu) \\
&= \sum_{q=1}^{Q} \sum_{j=1}^{u} \rho_u(m - j) + \sum_{j=1}^{s} \rho_u(m - j) \quad \text{(by an identity: } \rho_u(y+u) = \rho_u(y)) \\
&= Q \sum_{j=1}^{u} \rho_u(m - j) + \sum_{j=1}^{s} \rho_u(m - j) \\
&= Q + \sum_{j=1}^{s} \rho_u(m - j)
\end{aligned}$$

(first $\sum_{j=1}^{u} \rho_u(m-j) = \sum_{0 \le j < u} \rho_u(m-j) = \sum_{0 \le j < u} \rho_u(m \bmod u - j)$, where $m \bmod u :=$ the remainder of m divided by u. Secondly, noting that for $0 \le j < u$, $\rho_u(m \bmod u - j) = 1 \Leftrightarrow j \equiv m \bmod u \pmod u \Leftrightarrow j = m \pmod u$, we see $\sum_{j=1}^{u} \rho_u(m - j) = 1$)

$$= \left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s} \rho_u(m-j).$$

Therefore

$$\begin{aligned}
&\frac{1}{u} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\rho_u(m-n) - \frac{1}{u} \right) \\
&= \frac{1}{u} \sum_{m=1}^{M} \sum_{n=1}^{N} \rho_u(m-n) - \frac{MN}{u^2} \\
&= \frac{1}{u} \sum_{m=1}^{M} \left(\left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s} \rho_u(m-j) \right) - \frac{MN}{u^2}
\end{aligned}$$

[3]For $a \in \mathbb{R}$, $\lfloor a \rfloor := \max\{n \in \mathbb{Z};\ n \le a\}$ and $\lceil a \rceil := \min\{n \in \mathbb{Z};\ a \le n\}$. We call $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ and $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ the floor function and the ceiling function, respectively. Note that $\{a\} = a - \lfloor a \rfloor \in [0, 1)$.

$$
\begin{aligned}
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s}\sum_{m=1}^{M} \rho_u(j-m) \right) - \frac{MN}{u^2} \\
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s}\left(\left\lfloor \frac{M}{u} \right\rfloor + \sum_{i=1}^{r} \rho_u(j-i) \right) \right) - \frac{MN}{u^2} \quad (\text{where } r = \{M/u\}u) \\
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + s\left\lfloor \frac{M}{u} \right\rfloor + \sum_{i=1}^{r}\sum_{j=1}^{s} \rho_u(i-j) \right) - \frac{MN}{u^2} \\
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + s\left\lfloor \frac{M}{u} \right\rfloor + r \wedge s \right) - \frac{MN}{u^2}
\end{aligned}
$$

(for $0 < i, j < u$, $-u < i - j < u$. Also $\rho_u(i-j) = 1 \Leftrightarrow i - j \equiv 0 \pmod{u}$. Thus $\rho_u(i-j) = 1 \Leftrightarrow i = j$)

$$
\begin{aligned}
&= \frac{r}{u} \wedge \frac{s}{u} - \left(\frac{M}{u}\frac{N}{u} - \frac{M}{u}\left\lfloor \frac{N}{u} \right\rfloor - \frac{s}{u}\left\lfloor \frac{M}{u} \right\rfloor \right) \\
&= \frac{r}{u} \wedge \frac{s}{u} - \frac{r}{u} \cdot \frac{s}{u} \quad (\text{because } \{M/u\} = r/u,\ \{N/u\} = s/u) \\
&= \left\{ \frac{M}{u} \right\} \wedge \left\{ \frac{N}{u} \right\}\left(1 - \left\{ \frac{M}{u} \right\} \vee \left\{ \frac{N}{u} \right\} \right) \quad (\text{by an identity: } ab = (a \wedge b)(a \vee b)). \ \square
\end{aligned}
$$

Claim 2. *For each* $N \in \mathbb{N}$, $\lim_{M\to\infty} E^\lambda[Y_M^{(k)} Y_N^{(k)}] = 0$.

Proof. Let $M \geq N \geq 1$. Since $0 \leq \{M/c^k\}, \{N/c^k\} < 1$,

$$
\begin{aligned}
0 &\leq \left\{ \frac{M}{c^k} \right\} \wedge \left\{ \frac{N}{c^k} \right\}\left(1 - \left\{ \frac{M}{c^k} \right\} \vee \left\{ \frac{N}{c^k} \right\} \right) \\
&\leq \left\{ \frac{N}{c^k} \right\}\left(1 - \left\{ \frac{N}{c^k} \right\} \right).
\end{aligned}
$$

Multiplying both sides by $(1/M^{1/(2k)})(1/N^{1/(2k)})|\mu(c)| \prod_{p \nmid c}(1 - 2/p^k)$, and then adding them over $c \in \mathbb{N}$ yield that

$$
\begin{aligned}
0 &\leq E^\lambda[Y_M^{(k)} Y_N^{(k)}] \\
&\leq \frac{1}{M^{1/(2k)}}\frac{1}{N^{1/(2k)}} \sum_{c=1}^{\infty} |\mu(c)| \left(\prod_{p \nmid c}\left(1 - \frac{2}{p^k} \right) \right) \left\{ \frac{N}{c^k} \right\}\left(1 - \left\{ \frac{N}{c^k} \right\} \right) \\
&= \left(\frac{N}{M} \right)^{1/(2k)} E^\lambda[(Y_N^{(k)})^2].
\end{aligned}
$$

From this, the assertion of the claim follows. $\square$

3. Presentation of Proposition 1

By Claim 1

$$E^{\lambda}[(Y_N^{(k)})^2] = \left(\prod_p \left(1 - \frac{2}{p^k}\right)\right) \frac{1}{N^{1/k}} \sum_{c=1}^{\infty} \frac{|\mu(c)|}{\prod_{p|c}(1-2/p^k)} \left\{\frac{N}{c^k}\right\}\left(1 - \left\{\frac{N}{c^k}\right\}\right)$$

(5)

$$= \left(\prod_p \left(1 - \frac{2}{p^k}\right)\right) \frac{1}{N^{1/k}} \sum_{c=1}^{\infty} f(c) \left\{\frac{N}{c^k}\right\}\left(1 - \left\{\frac{N}{c^k}\right\}\right),$$

where

(6) $$f(n) := \frac{|\mu(n)|}{\prod_{p|n}(1-2/p^k)}, \quad n \in \mathbb{N}.$$

To show the convergence of $E^{\lambda}[(Y_N^{(k)})^2]$ as $N \to \infty$ and to find the value of this limit, we present a general proposition:

Proposition 1. *Let an arithmetic function f, i.e., $f\colon \mathbb{N} \to \mathbb{C}$ satisfy the following condition* (7) *or* (8):

(7) $$\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) \right| < \infty,$$

(8)
- $\sup_{n \in \mathbb{N}} |f(n)| < \infty$,
- *f has the mean-value $M(f)$, i.e., $\lim_{x\to\infty}(1/x)\sum_{n\le x} f(n)$ is convergent to a finite limit $M(f)$.*

Then, it holds that for $\forall k \in (1, \infty)$[4] *and $\forall h \in C^1[0,1]$ with $h(0) = 0$*

(9) $$\lim_{N\to\infty} N^{-1/k} \sum_{n=1}^{\infty} f(n) h\left(\left\{\frac{N}{n^k}\right\}\right) = M(f) \frac{1}{k} \int_0^{\infty} \frac{h(\{x\})}{x^{1/k+1}}\, dx.$$

Before proving this proposition, we give some comments on the conditions (7) and (8):

Claim 3. *If $f : \mathbb{N} \to \mathbb{C}$ satisfies the condition* (7), *then f has the mean-value*

$$M(f) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right).$$

[4]Here k may be a real number, > 1, though k was an integer, ≥ 2 at the beginning of this paper.

Proof. For simplicity, we define $f' : \mathbb{N} \to \mathbb{C}$ by

$$f'(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right), \quad n \in \mathbb{N}. \tag{10}$$

Since, by the Möbius inversion formula

$$f(n) = \sum_{d|n} f'(d), \tag{11}$$

we have for $x, y \in [1, \infty)$

$$\begin{aligned}
\frac{1}{x}\sum_{n\le x} f(n) &= \frac{1}{x}\sum_{d\le x}\left(\sum_{n\le x; d|n} 1\right) f'(d) \\
&= \frac{1}{x}\sum_{d\le x}\left\lfloor \frac{x}{d}\right\rfloor f'(d) \\
&= \sum_{d\le x}\frac{f'(d)}{d} - \frac{1}{x}\sum_{d\le x}\left\{\frac{x}{d}\right\} f'(d) \\
&= \sum_{d=1}^{\infty}\frac{f'(d)}{d} - \sum_{d>x}\frac{f'(d)}{d} - \sum_{d\le x/y}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{f'(d)}{d} \\
&\quad - \sum_{x/y<d\le x}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{f'(d)}{d}.
\end{aligned}$$

Transposing the first term of the last right-hand side, and then taking the absolute value, we see that

$$\begin{aligned}
&\left|\frac{1}{x}\sum_{n\le x} f(n) - \sum_{d=1}^{\infty}\frac{f'(d)}{d}\right| \\
&\le \sum_{d>x}\frac{|f'(d)|}{d} + \sum_{d\le x/y}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{|f'(d)|}{d} + \sum_{x/y<d\le x}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{|f'(d)|}{d} \\
&\le \sum_{d>x}\frac{|f'(d)|}{d} + \frac{1}{y}\sum_{d\le x/y}\frac{|f'(d)|}{d} + \sum_{x/y<d\le x}\frac{|f'(d)|}{d} \\
&\le 2\sum_{d>x/y}\frac{|f'(d)|}{d} + \frac{1}{y}\sum_{d\le x/y}\frac{|f'(d)|}{d}.
\end{aligned} \tag{12}$$

By letting $x \to \infty$ and $y \to \infty$, the assertion of the claim follows. □

REMARK 1. Schwarz–Spilker [3] calls Claim 3 Wintner's theorem.

We give an example of f satisfying the condition (7):

EXAMPLE 1. Let $f\colon \mathbb{N}\to\mathbb{C}$ be multiplicative, i.e., $f\neq 0$ and $f(mn)=f(m)f(n)$ provided that $(m,n)=1$. If, in addition,

$$\sum_p \frac{|f(p)-1|}{p}<\infty,\quad \sum_p\sum_{l\geq 2}\frac{|f(p^l)|}{p^l}<\infty, \tag{13}$$

then f satisfies the condition (7).

Proof. Multiplicativity of μ and f is inherited to f', and so $|f'|$. In general, multiplicativity of an arithmetic function implies a product representation of Dirichlet series associated with the function. Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}|f'(n)| &= \prod_p\left(1+\frac{|f'(p)|}{p}+\frac{|f'(p^2)|}{p^2}+\cdots\right)\\ &\leq \exp\left\{\sum_p\frac{|f'(p)|}{p}+\sum_p\sum_{l\geq 2}\frac{|f'(p^l)|}{p^l}\right\}\\ &\qquad\text{(by an inequality: } 1+x\leq e^x\ (\forall x\in\mathbb{R})).\end{aligned}$$

Since, by (10)

$$\begin{aligned}f'(p) &= \mu(1)f(p)+\mu(p)f(1)f\left(\frac{p}{d}\right)\\ &= f(p)-1 \quad \text{(note that } f(1)=1),\end{aligned} \tag{14}$$

$$\begin{aligned}f'(p^l) &= \mu(1)f(p^l)+\mu(p)f(p^{l-1}) \quad \text{(note that } \mu(p^j)=0\ (j\geq 2))\\ &= f(p^l)-f(p^{l-1}) \quad (l\geq 2),\end{aligned} \tag{15}$$

we have

$$\begin{aligned}&\sum_p\frac{|f'(p)|}{p}+\sum_p\sum_{l\geq 2}\frac{|f'(p^l)|}{p^l}\\ &=\sum_p\frac{|f(p)-1|}{p}+\sum_p\sum_{l\geq 2}\frac{|f(p^l)-f(p^{l-1})|}{p^l}\\ &\leq\sum_p\frac{|f(p)-1|}{p}+\sum_p\sum_{l\geq 2}\frac{|f(p^l)|}{p^l}+\sum_p\sum_{l\geq 2}\frac{|f(p^l)|}{p^{l+1}}+\sum_p\frac{|f(p)|}{p^2}\\ &\leq\sum_p\left(1+\frac{1}{p}\right)\frac{|f(p)-1|}{p}+\sum_p\frac{1}{p^2}+\sum_p\sum_{l\geq 2}\left(1+\frac{1}{p}\right)\frac{|f(p^l)|}{p^l}\end{aligned}$$

$$\leq \frac{3}{2}\left(\sum_p \frac{|f(p)-1|}{p} + \sum_p \sum_{l\geq 2} \frac{|f(p^l)|}{p^l}\right) + \sum_p \frac{1}{p^2}$$
$$< \infty \quad \text{(by (13))}.$$

Therefore f satisfies the condition (7). □

The condition (7) does not always imply the condition (8).

EXAMPLE 2. Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative, and satisfy

$$f(p) = 1 + \frac{1}{p^\alpha}, \quad f(p^l) = 0 \quad (l \geq 2)$$

for each prime p, where $\alpha \in (0, \infty)$. Since

$$\sum_p \frac{|f(p)-1|}{p} = \sum_p \frac{1}{p^{\alpha+1}} < \infty,$$

$f(\cdot)$ satisfies the condition (7) from Example 1. Also, since

$$f(p_1 \cdots p_m) = f(p_1)\cdots f(p_m)$$
$$= \prod_{i=1}^m \left(1 + \frac{1}{p_i^\alpha}\right) \begin{cases} \leq \prod_{i=1}^m e^{1/p_i^\alpha} = e^{\sum_{i=1}^m 1/p_i^\alpha}, \\ \geq \prod_{i=1}^m e^{(1/p_i^\alpha)/(1+1/p_i^\alpha)} \\ \text{(by an inequality: } \log(1+x) \geq x/(1+x)\ (x \geq 0)) \\ \qquad = e^{\sum_{i=1}^m (1/p_i^\alpha)/(1+1/p_i^\alpha)} \\ \qquad \geq e^{(2^\alpha/(2^\alpha+1)) \sum_{i=1}^m 1/p_i^\alpha}, \end{cases}$$

we see that

$$\lim_{m\to\infty} f(p_1 \cdots p_m) \begin{cases} < \infty & \text{if } \alpha > 1, \\ = \infty & \text{if } 0 < \alpha \leq 1. \end{cases}$$

This implies that

$$\sup_{n\geq 1} |f(n)| \begin{cases} < \infty & \text{if } \alpha > 1, \\ = \infty & \text{if } 0 < \alpha \leq 1. \end{cases}$$

4. Proof of two theorems

Proof of Theorem 1. $f:\mathbb{N}\to\mathbb{C}$, defined by (6), is clearly multiplicative, and satisfies

$$\sum_p \frac{|f(p)-1|}{p} = \sum_p \frac{1}{p^{k+1}}\frac{2}{1-2/p^k} \le \frac{2^k}{2^{k-1}-1}\sum_p \frac{1}{p^{k+1}} < \infty,$$

$$\sum_p \sum_{l\ge 2} \frac{|f(p^l)|}{p^l} = 0 < \infty.$$

Also, note that

$$0 \le f(n) \le e^{\sum_p 4/p^k} \quad (n\in\mathbb{N})$$

(because, by $1/(1-2/p^k) \le 1+4/p^k$, $\prod_{p|n} 1/(1-2/p^k) \le \prod_{p|n}(1+4/p^k) \le \prod_{p|n} e^{4/p^k} = e^{\sum_{p|n} 4/p^k} \le e^{\sum_p 4/p^k}$). Hence, this $f(\cdot)$ satisfies both the condition (7) and the condition (8), so that applying Proposition 1, we see

$$\lim_{N\to\infty} \frac{1}{N^{1/k}} \sum_{c=1}^{\infty} f(c)\left\{\frac{N}{c^k}\right\}\left(1-\left\{\frac{N}{c^k}\right\}\right) = M(f)\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx. \tag{16}$$

Let f' be a multiplicative function defined by (10). By (14) and (15)

$$f'(p) = \frac{2/p^k}{1-2/p^k},$$

$$f'(p^l) = \begin{cases} -\dfrac{1}{1-2/p^k}, & l=2,\\ 0, & l\ge 3 \end{cases}$$

for prime p and integer $l, \ge 2$. Claim 3 then implies that

$$\begin{aligned} M(f) &= \sum_{n=1}^{\infty} \frac{f'(n)}{n} \\ &= \prod_p \left(1+\frac{f'(p)}{p}+\sum_{l\ge 2}\frac{f'(p^l)}{p^l}\right) \\ &= \prod_p \left(1+\frac{1}{p}\frac{2/p^k}{1-2/p^k}-\frac{1}{p^2}\frac{1}{1-2/p^k}\right) \\ &= \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\frac{1}{1-2/p^k}\right). \end{aligned}$$

Collecting (5), (16) and this, we have

$$\lim_{N\to\infty} E^{\lambda}[(Y_N^{(k)})^2] = \left(\prod_p \left(1-\frac{1}{p}\right)\left(1-\frac{2}{p^k}+\frac{1}{p}\right)\right)\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx.$$

Let us find the value of an integral on the right-hand side. The Fourier expansion of a function $\{x\}(1-\{x\})$ is as follows:

$$\begin{aligned}\{x\}(1-\{x\}) &= \frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^\infty \frac{\cos 2n\pi x}{n^2}\\ &= \frac{1}{\pi^2}\sum_{n=1}^\infty \frac{1-\cos 2n\pi x}{n^2} \qquad \left(\text{because } \textstyle\sum_{n=1}^\infty 1/n^2 = \pi^2/6\right)\\ &= \frac{2}{\pi^2}\sum_{n=1}^\infty \frac{\sin^2 n\pi x}{n^2}.\end{aligned}$$

Termwise integration yields that

$$\begin{aligned}\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx &= \frac{1}{k}\frac{2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{\sin^2 n\pi x}{x^{1/k+1}}\,dx\\ &= \frac{1}{k}\frac{2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{\sin^2 y}{(y/n\pi)^{1/k+1}}\frac{dy}{n\pi}\\ &= \frac{1}{k}\frac{2}{\pi^{2-1/k}}\left(\sum_{n=1}^\infty \frac{1}{n^{2-1/k}}\right)\int_0^\infty \frac{\sin^2 y}{y^{1/k+1}}\,dy\\ &= \frac{2}{\pi^{2-1/k}}\left(\frac{1}{k}\int_0^\infty \frac{\sin^2 y}{y^{1/k+1}}\,dy\right)\zeta\left(2-\frac{1}{k}\right).\end{aligned}$$

We here note that from a formula: $\int_0^\infty (\sin vx)/x^u\,dx = \pi v^{u-1}/(2\Gamma(u)\sin(u\pi/2))$ $(0 < u < 2,\ v > 0)$

$$\begin{aligned}\frac{1}{k}\int_0^\infty \frac{\sin^2 y}{y^{1/k+1}}\,dy &= \int_0^\infty (-y^{-1/k})' \sin^2 y\,dy\\ &= [-y^{-1/k}\sin^2 y]_0^\infty - \int_0^\infty (-y^{-1/k}) 2\sin y\cos y\,dy\\ &= \int_0^\infty \frac{\sin 2y}{y^{1/k}}\,dy\end{aligned}$$

(because $\lim_{y\to 0} y^{-1/k}\sin^2 y = \lim_{y\to 0} y^{2-1/k}((\sin y)/y)^2 = 0$, $\lim_{y\to\infty} y^{-1/k}\sin^2 y = \lim_{y\to\infty}\sin^2 y/y^{1/k} = 0$)

$$= \frac{\pi 2^{1/k-1}}{2\Gamma(1/k)\sin(\pi/2k)}.$$

Substituting this into the above, we have

$$\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx = \frac{2}{\pi^{2-1/k}}\frac{\pi 2^{1/k-1}}{2\Gamma(1/k)\sin(\pi/2k)}\zeta\left(2-\frac{1}{k}\right)$$
$$= \frac{\zeta(2-1/k)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/2k)}.$$

Consequently, the assertion of the theorem follows at once. □

REMARK 2. Since, by the functional equation

$$\zeta(s) = 2\Gamma(1-s)\sin\left(\frac{\pi}{2}s\right)(2\pi)^{s-1}\zeta(1-s)$$

of the Riemann zeta function,

$$\zeta\left(2-\frac{1}{k}\right) = 2\Gamma\left(\frac{1}{k}-1\right)\left(\sin\frac{\pi}{2k}\right)(2\pi)^{1-1/k}\zeta\left(\frac{1}{k}-1\right),$$

we see

$$\frac{\zeta(2-1/k)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/(2k))} = \frac{2\Gamma(1/k-1)(\sin(\pi/(2k)))(2\pi)^{1-1/k}\zeta(1/k-1)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/(2k))}$$
$$= 2\frac{\zeta(1/k-1)}{1/k-1}.$$

Then the appearance of Theorem 1 becomes good as

$$\lim_{N\to\infty} E^\lambda[(Y_N^{(k)})^2] = \left(\prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{2}{p^k}\right)\right)2\frac{\zeta(1/k-1)}{1/k-1}.$$

Proof of Theorem 2. (i) For $M \geq N \geq 1$

$$E^\lambda[(Y_M^{(k)}-Y_N^{(k)})^2] = E^\lambda[(Y_M^{(k)})^2] - 2E^\lambda[Y_M^{(k)}Y_N^{(k)}] + E^\lambda[(Y_N^{(k)})^2].$$

The assertion of (i) is obvious from Claim 2 and Theorem 1.

(ii) By Theorem 1, $\{Y_N^{(k)}\}_{N=1}^\infty$ is L^2-bounded, and thus for any subsequence $\{N_i\}_{i=1}^\infty$

$$\exists\{i_m\}_{m=1}^\infty\text{: subsequence, } \exists Y \in L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda) \quad \text{s.t.} \quad \text{w-lim}_{m\to\infty} Y_{N_{i_m}}^{(k)} = Y.$$

Then

$$\lim_{m\to\infty} E^{\lambda}[Y_{N_{i_m}}^{(k)} Y_{N_{i_n}}^{(k)}] = E^{\lambda}[Y Y_{N_{i_n}}^{(k)}], \quad \forall n \in \mathbb{N}.$$

But, by Claim 2

$$E^{\lambda}[Y Y_{N_{i_n}}^{(k)}] = 0 \quad (\forall n \in \mathbb{N}).$$

Letting $n \to \infty$ yields that $E^{\lambda}[Y^2] = 0$. This implies that $\text{w-lim}_{N\to\infty} Y_N^{(k)} = 0$. □

5. Proof of Proposition 1

We now take up the proof of Proposition 1.

Suppose $f(\cdot)$ satisfies the condition (7) or (8). Fix $k \in (1, \infty)$ and $h \in C^1[0, 1]$ with $h(0) = 0$. We divide $N^{-1/k} \sum_{n=1}^{\infty} f(n)h(\{N/n^k\})$ into three terms as

$$\begin{aligned} N^{-1/k} \sum_{n=1}^{\infty} f(n) h\left(\left\{\frac{N}{n^k}\right\}\right) &= M(f) N^{-1/k} \sum_{n \le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) \\ &\quad + N^{-1/k} \sum_{n \le N^{1/k}} (f(n) - M(f)) h\left(\left\{\frac{N}{n^k}\right\}\right) \\ &\quad + N^{-1/k} \sum_{n > N^{1/k}} f(n) h\left(\frac{N}{n^k}\right). \end{aligned} \tag{17}$$

To find a limit of each term as $N \to \infty$, we present the following lemma:

Lemma 1. *Let $1 \le a < b < \infty$ and $\varphi \in C^1[a, b]$.*

(i) *Given a sequence $\{a_n\}_{n=1}^{\infty}$, set $S(t) = \sum_{n \le t} a_n$ $(t \in \mathbb{R})$. Then, for $a \le \forall x < \forall y \le b$*

$$\sum_{x<n\le y} a_n \varphi(n) = -\int_x^y S(t)\varphi'(t)\,dt + S(y)\varphi(y) - S(x)\varphi(x).$$

(ii) *For $a \le \forall x < \forall y \le b$*

$$\begin{aligned} \sum_{x<n\le y} \varphi(n) &= \int_x^y \varphi(t)\,dt \\ &\quad - \left(\left(\{y\} - \frac{1}{2}\right)\varphi(y) - \left(\{x\} - \frac{1}{2}\right)\varphi(x)\right) + \int_x^y \left(\{t\} - \frac{1}{2}\right)\varphi'(t)\,dt. \end{aligned}$$

Proof. Let $1 \le a < b < \infty$, $\varphi \in C^1[a, b]$ and $a \le x < y \le b$.

(i) In case $\lfloor x \rfloor < \lfloor y \rfloor$, noting that $a \le x < \lfloor x \rfloor + 1 \le \lfloor y \rfloor \le y \le b$, we have

$$\begin{aligned}
&\text{the left-hand side}\\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor} a_n \varphi(n)\\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor} (S(n) - S(n-1))\varphi(n)\\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor} S(n)\varphi(n) - \sum_{\lfloor x \rfloor \le n \le \lfloor y \rfloor - 1} S(n)\varphi(n+1)\\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor - 1} S(n)(\varphi(n) - \varphi(n+1)) + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1)\\
&= - \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor - 1} \int_n^{n+1} S(n)\varphi'(t)\,dt + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1)\\
&= - \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor - 1} \int_n^{n+1} S(t)\varphi'(t)\,dt + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1)
\end{aligned}$$

(because $S(t) = S(\lfloor t \rfloor)$)

$$\begin{aligned}
&= - \int_{\lfloor x \rfloor + 1}^{\lfloor y \rfloor} S(t)\varphi'(t)\,dt + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1)\\
&= - \int_x^y S(t)\varphi'(t)\,dt + \int_x^{\lfloor x \rfloor + 1} S(t)\varphi'(t)\,dt + \int_{\lfloor y \rfloor}^y S(t)\varphi'(t)\,dt\\
&\quad + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1)\\
&= - \int_x^y S(t)\varphi'(t)\,dt + S(y)\varphi(y) - S(x)\varphi(x)\\
&= \text{the right-hand side.}
\end{aligned}$$

In case $\lfloor x \rfloor = \lfloor y \rfloor$, since $\lfloor x \rfloor \le x < y < \lfloor y \rfloor + 1 = \lfloor x \rfloor + 1$,

$$\text{the left-hand side} = \sum_{\lfloor x \rfloor < n \le \lfloor x \rfloor} a_n \varphi(n) = 0,$$

$$\text{the right-hand side} = -S(\lfloor x \rfloor)(\varphi(y) - \varphi(x)) + S(\lfloor x \rfloor)(\varphi(y) - \varphi(x)) = 0.$$

Thus, we obtain the assertion of (i).

(ii) Let $a_n = 1$ ($n \in \mathbb{N}$). In this case, $S(t) = \lfloor t \rfloor$ ($t \geq 0$), so by (i)

$$\begin{aligned}\sum_{x<n\leq y} \varphi(n) &= -\int_x^y \lfloor t \rfloor \varphi'(t)\,dt + \lfloor y \rfloor \varphi(y) - \lfloor x \rfloor \varphi(x) \\ &= -\int_x^y t\varphi'(t)\,dt + \int_x^y \{t\}\varphi'(t)\,dt \\ &\quad + y\varphi(y) - x\varphi(x) - \{y\}\varphi(y) + \{x\}\varphi(x) \\ &= -[t\varphi(t)]_x^y + \int_x^y \varphi(t)\,dt + \int_x^y \left(\{t\} - \frac{1}{2}\right)\varphi'(t)\,dt \\ &\quad + \frac{1}{2}(\varphi(y) - \varphi(x)) + [t\varphi(t)]_x^y - \{y\}\varphi(y) + \{x\}\varphi(x) \\ &= \int_x^y \varphi(t)\,dt - \left(\left(\{y\} - \frac{1}{2}\right)\varphi(y) - \left(\{x\} - \frac{1}{2}\right)\varphi(x)\right) \\ &\quad + \int_x^y \left(\{t\} - \frac{1}{2}\right)\varphi'(t)\,dt. \end{aligned}$$

□

REMARK 3. This identity is called the Euler summation formula (cf. [3, Theorem 1.2 in Chapter I]) or the Euler–Maclaurin summation formula (cf. [2, Lemma 2.1]).

Proof of Proposition 1 under the condition (7).
1° The first term of (17).
1°-1 For $L \in \mathbb{N}$ with $L + 1 \leq N$,

$$\sum_{(N/(L+1))^{1/k}<n\leq N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= \sum_{l=1}^{L} \sum_{n;\lfloor N/n^k \rfloor = l} h\left(\left\{\frac{N}{n^k}\right\}\right)$$

(note that $(N/(L+1))^{1/k} < n \leq N^{1/k} \Leftrightarrow 1 \leq \lfloor N/n^k \rfloor \leq L$)

$$= \sum_{l=1}^{L} \sum_{l\leq N/n^k<l+1} h\left(\frac{N}{n^k} - l\right)$$

(when $\lfloor N/n^k \rfloor = l$, $\{N/n^k\} = N/n^k - l$. Also $\lfloor N/n^k \rfloor = l \Leftrightarrow l \leq N/n^k < l + 1$)

$$= \sum_{l=1}^{L} \sum_{(N/(l+1))^{1/k}<n\leq(N/l)^{1/k}} h\left(\frac{N}{n^k} - l\right)$$

$$= \sum_{l=1}^{L}\left(\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h\left(\frac{N}{t^k}-l\right)dt + h(1)\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)\right.$$
$$\left. - k\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h'\left(\frac{N}{t^k}-l\right)\frac{\{t\}-1/2}{t^{k+1}}N\,dt\right)$$

(apply Lemma 1 (ii) for $\varphi(t) = h(N/t^k - l)$ $((N/(l+1))^{1/k} \le t \le (N/l)^{1/k})$)

$$= \sum_{l=1}^{L}\left(\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h\left(\left\{\frac{N}{t^k}\right\}\right)dt + h(1)\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)\right.$$
$$\left. - k\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h'\left(\left\{\frac{N}{t^k}\right\}\right)\frac{\{t\}-1/2}{t^{k+1}}N\,dt\right)$$

(note that $(N/(l+1))^{1/k} < t \le (N/l)^{1/k} \Leftrightarrow \lfloor N/t^k \rfloor = l$)

$$= \int_{(N/(L+1))^{1/k}}^{N^{1/k}} h\left(\left\{\frac{N}{t^k}\right\}\right)dt + h(1)\sum_{l=1}^{L}\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)$$
$$- k\int_{(N/(L+1))^{1/k}}^{N^{1/k}} h'\left(\left\{\frac{N}{t^k}\right\}\right)\frac{\{t\}-1/2}{t^{k+1}}N\,dt$$
$$= N^{1/k}\frac{1}{k}\int_1^{L+1}\frac{h(\{x\})}{x^{1/k+1}}\,dx + h(1)\sum_{l=1}^{L}\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)$$
$$- \int_1^{L+1} h'(\{x\})\left(\left\{\left(\frac{N}{x}\right)^{1/k}\right\}-\frac{1}{2}\right)dx \quad \text{(by change of variable: } x = N/t^k\text{).}$$

1°-2 Let $N \gg 1$ and $L = L(N) = \lfloor N^{1/(k+1)} \rfloor$. Then $L(N) \le N^{1/(k+1)} < L(N)+1$, and so $L(N)+1 < N$, $1/(L(N)+1) < (1/N)^{1/(k+1)}$. Since, by 1°-1

$$N^{-1/k}\sum_{n\le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= N^{-1/k}\sum_{n\le (N/(L(N)+1))^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) + N^{-1/k}\sum_{(N/(L(N)+1))^{1/k}<n\le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= N^{-1/k}\sum_{n\le (N/(L(N)+1))^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$+ \frac{1}{k}\int_1^{L(N)+1}\frac{h(\{x\})}{x^{1/k+1}}\,dx + h(1)N^{-1/k}\sum_{l=1}^{L(N)}\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)$$
$$- N^{-1/k}\int_1^{L(N)+1} h'(\{x\})\left(\left\{\left(\frac{N}{x}\right)^{1/k}\right\}-\frac{1}{2}\right)dx,$$

we see

$$\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) - \frac{1}{k}\int_1^{L(N)+1} \frac{h(\{x\})}{x^{1/k+1}}\,dx \right| \\
&\le N^{-1/k} \sum_{n \le (N/(L(N)+1))^{1/k}} \left| h\left(\left\{\frac{N}{n^k}\right\}\right)\right| \\
&\quad + |h(1)| N^{-1/k} \sum_{l=1}^{L(N)} \left| \left\{\left(\frac{N}{l+1}\right)^{1/k}\right\} - \frac{1}{2}\right| \\
&\quad + N^{-1/k}\int_1^{L(N)+1} |h'(\{x\})| \left| \left\{\left(\frac{N}{x}\right)^{1/k}\right\} - \frac{1}{2}\right| dx \\
&\le N^{-1/k}\left(\frac{N}{L(N)+1}\right)^{1/k}\left(\max_{0\le x\le 1}|h(x)|\right) + |h(1)| N^{-1/k} L(N)\cdot\frac{1}{2} \\
&\quad + N^{-1/k}L(N)\left(\max_{0\le x\le 1}|h'(x)|\right)\cdot \frac{1}{2} \\
&\le \left(\left(\frac{1}{L(N)+1}\right)^{1/k} + \left(\frac{1}{N}\right)^{1/k} L(N)\right)\left(\max_{0\le x\le 1}|h'(x)|\right)
\end{aligned}$$

(note that $\max_{0\le x\le 1}|h(x)| \le \max_{0\le x\le 1}|h'(x)|$)

$$\begin{aligned}
&\le \left(\left(\frac{1}{N}\right)^{(1/k)\cdot(1/(k+1))} + \left(\frac{1}{N}\right)^{1/k-1/(k+1)}\right)\left(\max_{0\le x\le 1}|h'(x)|\right) \\
&= 2\left(\frac{1}{N}\right)^{1/(k(k+1))}\left(\max_{0\le x\le 1}|h'(x)|\right) \to 0 \quad \text{as} \quad N\to\infty.
\end{aligned}$$

This shows that

$$\text{(18)} \qquad \text{the first term of (17)} \to M(f)\frac{1}{k}\int_1^\infty \frac{h(\{x\})}{x^{1/k+1}}\,dx \quad \text{as} \quad N\to\infty.$$

2° The second term of (17).

For simplicity, set $a_n = f(n) - M(f)$, $S(x) = \sum_{n\le x} a_n$.

2°-1 First

$$\begin{aligned}
\frac{1}{y}|S(y)| &= \frac{1}{y}\left|\sum_{n\le y} f(n) - \lfloor y\rfloor M(f)\right| \\
&= \left|\frac{1}{y}\sum_{n\le y} f(n) - M(f) + \frac{\{y\}}{y}M(f)\right|
\end{aligned}$$

$$\le \left| \frac{1}{y} \sum_{n \le y} f(n) - M(f) \right| + \frac{1}{y}|M(f)|$$
$$\to 0 \quad \text{as} \quad y \to \infty.$$

2°-2 In the same way as in 1°-1, we have that for $L \in \mathbb{N}$ with $L+1 \le N$

$$\sum_{(N/(L+1))^{1/k} < n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= \sum_{l=1}^{L} \sum_{(N/(l+1))^{1/k} < n \le (N/l)^{1/k}} a_n h\left(\frac{N}{n^k} - l\right)$$
$$= \sum_{l=1}^{L} \left(k \int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h'\left(\frac{N}{t^k} - l\right) \frac{S(t)}{t^{k+1}} N\, dt - h(1) S\left(\left(\frac{N}{l+1}\right)^{1/k}\right)\right)$$

(apply Lemma 1 (i) for $\varphi(t) = h(N/t^k - l)$ $((N/(l+1))^{1/k} \le t \le (N/l)^{1/k})$)

$$= k \int_{(N/(L+1))^{1/k}}^{N^{1/k}} h'\left(\left\{\frac{N}{t^k}\right\}\right) \frac{S(t)}{t^{k+1}} N\, dt - \sum_{l=1}^{L} h(1) S\left(\left(\frac{N}{l+1}\right)^{1/k}\right)$$
$$= \int_1^{L+1} h'(\{x\}) S\left(\left(\frac{N}{x}\right)^{1/k}\right) dx - \sum_{l=1}^{L} \int_l^{l+1} h(1) S\left(\left(\frac{N}{l+1}\right)^{1/k}\right) dx$$
$$= \int_1^{L+1} \left(h'(\{x\}) S\left(\left(\frac{N}{x}\right)^{1/k}\right) - h(1) S\left(\left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right)\right) dx.$$

2°-3 Fix $L \in \mathbb{N}$ with $L+1 \le N$. By 2°-2

$$\left| N^{-1/k} \sum_{(N/(L+1))^{1/k} < n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right|$$
$$\le N^{-1/k} \int_1^{L+1} \left(|h'(\{x\})| \left| S\left(\left(\frac{N}{x}\right)^{1/k}\right)\right| + |h(1)| \left| S\left(\left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right)\right|\right) dx$$
$$\le \left(\max_{0 \le x \le 1} |h'(x)|\right) N^{-1/k} \int_1^{L+1} \left(\frac{|S((N/x)^{1/k})|}{(N/x)^{1/k}} \left(\frac{N}{x}\right)^{1/k} + \frac{|S((N/\lceil x \rceil)^{1/k})|}{(N/\lceil x \rceil)^{1/k}} \left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right) dx$$
$$\le \left(\max_{0 \le x \le 1} |h'(x)|\right) N^{-1/k} \int_1^{L+1} \left(\frac{|S((N/x)^{1/k})|}{(N/x)^{1/k}} + \frac{|S((N/\lceil x \rceil)^{1/k})|}{(N/\lceil x \rceil)^{1/k}}\right) \left(\frac{N}{x}\right)^{1/k} dx$$

(note that $1 \le x \le L+1 \Rightarrow 1 \le x \le \lceil x \rceil \le L+1 \Rightarrow (N/(L+1))^{1/k} \le (N/\lceil x \rceil)^{1/k} \le (N/x)^{1/k}$)

$$\le \left(\max_{0 \le x \le 1} |h'(x)|\right) \left(2 \sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) \int_1^{L+1} x^{-1/k}\, dx$$

$$= \left(\max_{0\le x\le 1}|h'(x)|\right)\left(2\sup_{y\ge (N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)\left[\frac{x^{1-1/k}}{1-1/k}\right]_1^{L+1}$$
$$\le \left(\max_{0\le x\le 1}|h'(x)|\right)\frac{k}{k-1}\left(2\sup_{y\ge (N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)(L+1)^{1-1/k}$$
$$\le \left(\left(\max_{0\le x\le 1}|h'(x)|\right)\frac{2k}{k-1}+1\right)\left(\sup_{y\ge (N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)(L+1)^{1-1/k}.$$

Also

$$\left|N^{-1/k}\sum_{n\le (N/(L+1))^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right|$$
$$= \left|N^{-1/k}\sum_{n\le (N/(L+1))^{1/k}} f(n)h\left(\left\{\frac{N}{n^k}\right\}\right) - M(f)N^{-1/k}\sum_{n\le (N/(L+1))^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)\right|$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\sum_{n\le (N/(L+1))^{1/k}}|f(n)| + |M(f)|N^{-1/k}\sum_{n\le (N/(L+1))^{1/k}}1\right)$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\sum_{n\le (N/(L+1))^{1/k}}\sum_{d|n}|f'(d)| + |M(f)|N^{-1/k}\left(\frac{N}{L+1}\right)^{1/k}\right)$$
$$= \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\sum_{d\le (N/(L+1))^{1/k}}\left\lfloor\frac{1}{d}\left(\frac{N}{L+1}\right)^{1/k}\right\rfloor|f'(d)| + |M(f)|\left(\frac{1}{L+1}\right)^{1/k}\right)$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\left(\frac{N}{L+1}\right)^{1/k}\sum_{d\le (N/(L+1))^{1/k}}\frac{|f'(d)|}{d} + |M(f)|\left(\frac{1}{L+1}\right)^{1/k}\right)$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(\frac{1}{L+1}\right)^{1/k}2\sum_{d=1}^{\infty}\frac{|f'(d)|}{d}$$
$$\le \left(2\left(\max_{0\le x\le 1}|h(x)|\right)\sum_{d=1}^{\infty}\frac{|f'(d)|}{d}+1\right)(L+1)^{-1/k}.$$

Combining two estimates above, we have

$$\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right| \\
&\le \left(\left(\max_{0\le x\le 1}|h'(x)|\right)\frac{2k}{k-1}+1\right)\left(\sup_{y\ge (N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)(L+1)^{1-1/k} \\
&\quad + \left(2\left(\max_{0\le x\le 1}|h(x)|\right)\sum_{d=1}^{\infty}\frac{|f'(d)|}{d}+1\right)(L+1)^{-1/k} \\
&=: A\left(\sup_{y\ge (N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)(L+1)^{1-1/k}+B(L+1)^{-1/k}.
\end{aligned} \tag{19}$$

<u>2°-4</u> Take $\varepsilon > 0$ so that $B/((k-1)A\varepsilon) > 2$. By 2°-1

$$\exists y_0 > 1 \text{ s.t. } \frac{|S(y)|}{y} < \varepsilon \quad (\forall y \ge y_0).$$

Let $L = \lfloor B/((k-1)A\varepsilon)\rfloor - 1 \in \mathbb{N}$. For $N \ge y_0^k B/((k-1)A\varepsilon)$,

$$\frac{N}{y_0^k} \ge \left\lfloor \frac{N}{y_0^k}\right\rfloor \ge \left\lfloor \frac{B}{(k-1)A\varepsilon}\right\rfloor = L+1.$$

Since $(N/(L+1))^{1/k} \ge y_0$, $\sup_{y\ge (N/(L+1))^{1/k}}|S(y)|/y \le \varepsilon$. Using this in (19), we have

$$\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right| \\
&\le A\varepsilon(L+1)^{1-1/k} + B(L+1)^{-1/k} \\
&= A\varepsilon\left\lfloor \frac{B}{(k-1)A\varepsilon}\right\rfloor^{1-1/k} + B\left\lfloor \frac{B}{(k-1)A\varepsilon}\right\rfloor^{-1/k} \\
&\le A\varepsilon\left(\frac{B}{(k-1)A\varepsilon}\right)^{1-1/k} + B\left(\frac{B}{(k-1)A\varepsilon}-1\right)^{-1/k} \\
&\le A\varepsilon\left(\frac{B}{(k-1)A\varepsilon}\right)^{1-1/k} + B\left(\frac{1}{2}\frac{B}{(k-1)A\varepsilon}\right)^{-1/k} \\
&= (k-1)^{1/k-1}A^{1/k}B^{1-1/k}\varepsilon^{1/k} + 2^{1/k}(k-1)^{1/k}A^{1/k}B^{1-1/k}\varepsilon^{1/k} \\
&= ((k-1)^{1/k-1} + 2^{1/k}(k-1)^{1/k})A^{1/k}B^{1-1/k}\varepsilon^{1/k}.
\end{aligned}$$

Letting $N \to \infty$ yields

$$\varlimsup_{N\to\infty} \left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right|$$
$$\le ((k-1)^{1/k-1} + 2^{1/k}(k-1)^{1/k})A^{1/k}B^{1-1/k}\varepsilon^{1/k} \to 0 \quad \text{as} \quad \varepsilon \searrow 0.$$

This shows that

$$\text{the second term of (17)} \to 0 \quad \text{as} \quad N \to \infty. \tag{20}$$

3° The third term of (17).

3°-1 First, we check the convergence of a series $\sum_{n>N^{1/k}} f(n)h(N/n^k)$. Let $L, M \in \mathbb{N}$, $N \le L < M$. Lemma 1 (i) for $\varphi(t) = h(N/t^k)$ $(L^{1/k} \le t \le M^{1/k})$ tells us that

$$\sum_{L^{1/k}<n\le M^{1/k}} f(n)h\left(\frac{N}{n^k}\right)$$
$$= \int_{L^{1/k}}^{M^{1/k}} \left(\frac{1}{t}\sum_{n\le t} f(n)\right) h'\left(\frac{N}{t^k}\right)\frac{Nk}{t^k}\,dt$$
$$+ \left(\frac{1}{M^{1/k}}\sum_{n\le M^{1/k}} f(n)\right)\frac{h(N/M)}{N/M}\frac{N}{M^{1-1/k}} - \left(\frac{1}{L^{1/k}}\sum_{n\le L^{1/k}} f(n)\right)\frac{h(N/L)}{N/L}\frac{N}{L^{1-1/k}}.$$

Here, noting that since $k > 1$, $\int_1^\infty dt/t^k < \infty$, $\lim_{M\to\infty} 1/M^{1-1/k} = \lim_{L\to\infty} 1/L^{1-1/k} = 0$ and since $h(0) = 0$, $\lim_{x\to 0} h(x)/x = h'(0)$, we see the convergence of this series.

3°-2 Next, letting $L = N$ and $M \to \infty$ in the above yields that

$$\sum_{n>N^{1/k}} f(n)h\left(\frac{N}{n^k}\right)$$
$$= \int_{N^{1/k}}^{\infty} \left(\frac{1}{t}\sum_{n\le t} f(n)\right) h'\left(\frac{N}{t^k}\right)\frac{Nk}{t^k}\,dt - N^{1/k}\left(\frac{1}{N^{1/k}}\sum_{n\le N^{1/k}} f(n)\right)h(1)$$
$$= \int_1^\infty \left(\frac{1}{N^{1/k}\tau}\sum_{n\le N^{1/k}\tau} f(n)\right) h'(\tau^{-k})\frac{k}{\tau^k}N^{1/k}\,d\tau$$
$$- N^{1/k}\left(\frac{1}{N^{1/k}}\sum_{n\le N^{1/k}} f(n)\right)h(1) \quad \text{(by change of variable: } \tau = t/N^{1/k}\text{)}.$$

By multiplying both sides by $N^{-1/k}$, it turns out that

$$\begin{aligned}&\text{the third term of (17)}\\&=\int_1^\infty\left(\frac{1}{N^{1/k}\tau}\sum_{n\le N^{1/k}\tau}f(n)\right)h'(\tau^{-k})\frac{k}{\tau^k}\,d\tau-\left(\frac{1}{N^{1/k}}\sum_{n\le N^{1/k}}f(n)\right)h(1).\end{aligned}$$

Thus, by the Lebesgue convergence theorem,

(21)

$$\begin{aligned}\text{the third term of (17)}&\to\int_1^\infty M(f)h'(\tau^{-k})k\tau^{-k}\,d\tau-M(f)h(1)\\&=M(f)\left(\int_1^\infty(h'(\tau^{-k})k\tau^{-k-1})\tau\,d\tau-h(1)\right)\\&=M(f)\left(\int_1^\infty(-h(\tau^{-k}))'\tau\,d\tau-h(1)\right)\\&=M(f)\left([-h(\tau^{-k})\tau]_1^\infty+\int_1^\infty h(\tau^{-k})\,d\tau-h(1)\right)\\&=M(f)\int_1^\infty h(\tau^{-k})\,d\tau\end{aligned}$$

(because $h(\tau^{-k})\tau=(h(\tau^{-k})/\tau^{-k})(1/\tau^{k-1})\to 0$ as $\tau\to\infty$)

$$=M(f)\frac{1}{k}\int_0^1\frac{h(x)}{x^{1/k+1}}\,dx$$

(by change of variable: $x=\tau^{-k}$)

$$=M(f)\frac{1}{k}\int_0^1\frac{h(\{x\})}{x^{1/k+1}}\,dx\quad\text{as}\quad N\to\infty.$$

4° Collecting (18), (20) and (20), we have

$$\begin{aligned}&\lim_{N\to\infty}N^{-1/k}\sum_{n=1}^\infty f(n)h\left(\left\{\frac{N}{n^k}\right\}\right)\\&=M(f)\frac{1}{k}\int_1^\infty\frac{h(\{x\})}{x^{1/k+1}}\,dx+M(f)\frac{1}{k}\int_0^1\frac{h(\{x\})}{x^{1/k+1}}\,dx\\&=M(f)\frac{1}{k}\int_0^\infty\frac{h(\{x\})}{x^{1/k+1}}\,dx.\end{aligned}$$ □

Proof of Proposition 1 under the condition (8). The argument of 1° and 3° in the previous proof is also valid in this case. Thus we have the convergences (18) and (21).

In the following, we slightly modify the argument of 2° in the previous proof. Let $a_n = f(n) - M(f)$, $S(x) = \sum_{n \le x} a_n$.

<u>2°-1</u> First

$$\frac{1}{y}|S(y)| \to 0 \quad \text{as} \quad y \to \infty.$$

<u>2°-2</u> For $L \in \mathbb{N}$ with $L + 1 \le N$

$$\begin{aligned}&\sum_{(N/(L+1))^{1/k} < n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \\ &= \int_1^{L+1} \left(h'(\{x\}) S\left(\left(\frac{N}{x}\right)^{1/k}\right) - h(1) S\left(\left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right)\right) dx.\end{aligned}$$

<u>2°-3</u> Fix $L \in \mathbb{N}$ with $L + 1 \le N$. By 2°-2

$$\begin{aligned}&\left| N^{-1/k} \sum_{(N/(L+1))^{1/k} < n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\ &\le \left(\left(\max_{0 \le x \le 1} |h'(x)|\right) \frac{2k}{k-1} + 1\right) \left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) (L+1)^{1-1/k}.\end{aligned}$$

Clearly

$$\begin{aligned}\left| N^{-1/k} \sum_{n \le (N/(L+1))^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| &\le N^{-1/k} \sum_{n \le (N/(L+1))^{1/k}} |a_n| \left| h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\ &\le \left(\sup_{n \ge 1} |a_n|\right) \left(\max_{0 \le x \le 1} |h(x)|\right) (L+1)^{-1/k} \\ &\le \left(\left(\sup_{n \ge 1} |a_n|\right) \left(\max_{0 \le x \le 1} |h(x)|\right) + 1\right) (L+1)^{-1/k}.\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned}&\left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\ &\le \left(\left(\max_{0 \le x \le 1} |h'(x)|\right) \frac{2k}{k-1} + 1\right) \left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) (L+1)^{1-1/k} \\ &\quad + \left(\left(\sup_{n \ge 1} |a_n|\right) \left(\max_{0 \le x \le 1} |h(x)|\right) + 1\right) (L+1)^{-1/k} \\ &=: A \left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) (L+1)^{1-1/k} + C (L+1)^{-1/k}.\end{aligned} \tag{22}$$

<u>2°-4</u> Take $\varepsilon > 0$ so that $C/((k-1)A\varepsilon) > 2$ and choose $y_0 > 1$ such that $|S(y)|/y < \varepsilon$ ($\forall y \geq y_0$). Let $L = \lfloor C/((k-1)A\varepsilon) \rfloor - 1 \in \mathbb{N}$. For $N \geq y_0^k C/((k-1)A\varepsilon)$, $\sup_{y \geq (N/(L+1))^{1/k}} |S(y)|/y \leq \varepsilon$. Using this in (22)

$$\left| N^{-1/k} \sum_{n \leq N^{1/k}} a_n h\left(\left\{ \frac{N}{n^k} \right\} \right) \right| \leq A\varepsilon(L+1)^{1-1/k} + C(L+1)^{-1/k}$$
$$\leq ((k-1)^{1/k-1} + 2^{1/k}(k-1)^{1/k}) A^{1/k} C^{1-1/k} \varepsilon^{1/k}.$$

Letting $N \to \infty$, and then $\varepsilon \searrow 0$, we have the convergence (20).

Consequently, the assertion of Proposition 1 under the condition (8) follows. □

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