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ON SURFACES WITH $p_g = q = 2$, $K^2 = 5$ AND ALBANESE MAP OF DEGREE 3

To Professors F. Catanese and C. Ciliberto on the occasion of their 60th birthday

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Abstract

We construct a connected, irreducible component of the moduli space of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 5$, which contains both examples given by Chen–Hacon and the first author. This component is generically smooth of dimension 4, and all its points parametrize surfaces whose Albanese map is a generically finite triple cover.

0. Introduction

The classification of minimal, complex surfaces S of general type with small birational invariants is still far from being achieved; nevertheless, the study of such surfaces has produced in the last years a considerable amount of results, see for instance the survey paper [9]. If we assume $1 = \chi(\mathcal{O}_S) = 1 - q + p_g$, that is $p_g = q$, and S *irregular*, that is $q > 0$, then the inequalities of Bogomolov–Miyaoka–Yau and Debarre imply $1 \leq p_g \leq 4$. If $p_g = q = 4$ then S is a product of curves of genus 2, as shown by Beauville in the appendix to [18], while the case $p_g = q = 3$ was understood through the work of several authors, see [14], [22], [35]. The classification becomes more and more complicated as the value of p_g decreases; indeed already for $p_g = 2$ one has only a partial understanding of the situation.

Let us summarize what is known for surfaces with $p_g = q = 2$ in terms of K_S^2; in this case the inequalities mentioned above yield $4 \leq K_S^2 \leq 9$. The case $K_S^2 = 4$ was investigated by the first author, who constructed three families of surfaces which admit an isotrivial fibration, see [32]. Previously, surfaces with these invariants were also studied by Ciliberto and Mendes Lopes (in connection with the problem of birationality of the bicanonical map, see [17]) and Manetti (in his work on the Severi conjecture, see [25]). For $K_S^2 = 5$ there were so far only two examples, see [16] and [32]. As the title suggests, the present work deals with this case. For $K_S^2 = 6$ there is only one example, see [32], [33], [34]. The study of the case $K_S^2 = 8$ was started by Zucconi in [45] and continued by the first author in [32]. They produced a complete classification of

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surfaces with $p_g = q = 2$ and $K_S^2 = 8$ which are isogenous to a product of curves; as a by-product, they obtained the classification of all surfaces with these invariant which are not of Albanese general type, i.e., such that the image of the Albanese map is a curve. Finally, for $K_S^2 = 7$ and $K_S^2 = 9$ there are hitherto no examples known.

In this article we consider surfaces with $p_g = q = 2$ and $K_S^2 = 5$. Our work started when we noticed that the surfaces constructed in [16] and [32] have many features in common. More precisely, in both cases the Albanese map $\alpha\colon S \to \mathrm{Alb}(S)$ is a generically finite triple cover, and the Albanese variety $\mathrm{Alb}(S)$ is an abelian surface with a polarization of type (1, 2). Moreover, S contains a (-3)-curve, which is obviously contracted by α. We shall prove that Penegini's and Chen–Hacon's examples actually belong to the same connected component of the moduli space of surfaces of general type with $p_g = q = 2$ and $K_S^2 = 5$.

In order to formulate our results, let us introduce some terminology. Let S be a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 5$, such that its Albanese map $\alpha\colon S \to \mathrm{Alb}(S)$ is a generically finite morphism of degree 3. If one considers the Stein factorization of α, i.e.,

$$S \xrightarrow{p} \hat{X} \xrightarrow{\hat{f}} \mathrm{Alb}(S),$$

then the map $\hat{f}\colon \hat{X} \to \mathrm{Alb}(S)$ is a flat triple cover, which can be studied by applying the techniques developed in [27]. In particular, $\hat{f}$ is determined by a rank 2 vector bundle $\mathcal{E}$ on $\mathrm{Alb}(S)$, called the *Tschirnhausen bundle* of the cover, and by a global section $\eta \in H^0\big(\mathrm{Alb}(S), S^3\mathcal{E}^\vee \otimes \bigwedge^2 \mathcal{E}\big)$. In the examples of [16] and [32] the surface $\hat{X}$ is singular; nevertheless in both cases the numerical invariants of $\mathcal{E}$ are the same predicted by the formulae of [27], as if $\hat{X}$ were smooth. This leads us to introduce the definition of *negligible singularity* for a triple cover, see Definition 1.5 and Remark 1.9. Then, inspired by the construction in [16], we say that S is a *Chen–Hacon surface* if there exists a polarization $\mathcal{L}$ of type (1, 2) on $\mathrm{Pic}^0(S) = \widehat{\mathrm{Alb}(S)}$ such that $\mathcal{E}^\vee$ is the Fourier–Mukai transform of the line bundle $\mathcal{L}^{-1}$, see Definition 4.1.

Our first main result is the following characterization of Chen–Hacon surfaces, see Proposition 4.11 and Theorem 5.1.

Theorem A. *Let S be a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 5$ such that the Albanese map $\alpha\colon S \to \mathrm{Alb}(S)$ is a generically finite morphism of degree 3. Let*

$$S \xrightarrow{p} \hat{X} \xrightarrow{\hat{f}} \mathrm{Alb}(S)$$

be the Stein factorization of α. Then S is a Chen–Hacon surface if and only if $\hat{X}$ has only negligible singularities.

Moreover, we can completely describe all the possibilities for the singular locus of $\hat{X}$, see Proposition 4.9. It follows that $\hat{X}$ is never smooth, since it always contains a cyclic quotient singularity of type (1/3)(1, 1). Therefore S always contains a (−3)-curve, which turns out to be the fixed part of the canonical system $|K_S|$, see Proposition 5.13.

Now let $\mathcal{M}$ be the moduli space of surfaces with $p_g = q = 2$ and let $\mathcal{M}^{\mathrm{CH}} \subset \mathcal{M}$ be the subset whose points parametrize (isomorphism classes of) Chen–Hacon surfaces. Our second main result is the following, see Theorem 6.6.

Theorem B. *$\mathcal{M}^{\mathrm{CH}}$ is an irreducible, connected, generically smooth component of $\mathcal{M}$ of dimension* 4.

Since Chen and Hacon constructed in [16] only the *general* surface in $\mathcal{M}^{\mathrm{CH}}$, we need considerable work in order to establish Theorem B. Our proof uses in an essential way the fact that the degree of the Albanese map is a topological invariant of S, see [13]. As a by-product, we obtain some results of independent interest about the embedded deformations of S in the projective bundle $\mathbb{P}(\mathcal{E}^\vee)$, see Proposition 6.2.

We believe that the interest of our paper is twofold. First of all, it provides the first construction of a connected component of the moduli space of surfaces of general type with $p_g = q = 2$, $K_S^2 = 5$. Secondly, Theorem B shows that every small deformation of a Chen–Hacon surface is still a Chen–Hacon surface; in particular, no small deformation of S makes the (−3)-curve disappear. Moreover, since $\mathcal{M}^{\mathrm{CH}}$ is generically smooth, the same is true for the first-order deformations. By contrast, Burns and Wahl proved in [10] that first-order deformations always smooth all the (−2)-curves, and Catanese used this fact in [11] in order to produce examples of surfaces of general type with everywhere non-reduced moduli spaces. Theorem B demonstrates rather strikingly that the results of Burns–Wahl and Catanese cannot be extended to the case of (−3)-curves and, as far as we know, provides the first explicit example of this situation.

Although Theorems A and B shed some light on the structure of surfaces with $p_g = q = 2$ and $K_S^2 = 5$, many questions still remain unanswered. For instance:

- Are there surfaces with these invariants whose Albanese map has degree different from 3?
- Are there surfaces with these invariants whose Albanese map has degree 3, but which are not Chen–Hacon surfaces? Because of Theorem A, this is the same to ask whether $\hat{X}$ may have non-negligible singularities.

And, more generally:

- How many connected components of the moduli space of surfaces with $p_g = q = 2$ and $K_S^2 = 5$ are there?

In order to answer the last question, it would be desirable to find an effective bound for the degree of $\alpha\colon S \to \mathrm{Alb}(S)$, but so far we have not been able to do this.

Another problem that arises quite naturally and which is at present unsolved is the following.

- What are the possible degenerations of Chen–Hacon surfaces?

An answer to this question would be a major step toward a compactification of $\mathcal{M}^{\mathrm{CH}}$. In Proposition 5.11 we give a partial result, analyzing some degenerations of the triple cover $\hat{f}\colon \hat{X} \to \mathrm{Alb}(S)$ which provide reducible, non-normal surfaces.

Now let us describe how this paper is organized. In Section 1 we present some preliminaries, and we set up notation and terminology. In particular we recall Miranda's theory of triple covers, introducing the definition of negligible singularity, and we discuss the geometry of (1, 2)-polarized abelian surfaces. For the reader's convenience, we recall the relevant material from [27] and [4] without proofs, thus making our exposition self-contained.

In Section 2, which is the technical core of the paper, we describe all possibilities for the Tschirnhausen bundle of the triple cover $\hat{f}\colon \hat{X} \to \hat{A}$. The analysis is particularly subtle in the case where the (1, 2)-polarization is of product type; eventually, we are able to rule out this case, showing that it gives rise to a surface $\hat{X}$ which is not of general type (see Corollaries 2.8 and 2.11).

In Section 3 we briefly explain the two examples from [16] and [32], which motivate our definition of Chen–Hacon surfaces. The properties of such surfaces are then investigated in detail in Section 4.

Finally, in Section 5 we prove Theorem A, whereas Section 6 deals with the proof of Theorem B.

Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers.

If A is an abelian variety and $\hat{A} := \mathrm{Pic}^0(A)$ its dual, we denote by o and $\hat{o}$ the zero point of A and $\hat{A}$, respectively.

If $\mathcal{L}$ is a line bundle on A we denote by $\phi_{\mathcal{L}}$ the morphism $\phi_{\mathcal{L}}\colon A \to \hat{A}$ given by $x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. If $c_1(\mathcal{L})$ is non-degenerate then $\phi_{\mathcal{L}}$ is an isogeny, and we denote by $K(\mathcal{L})$ its kernel.

A coherent sheaf $\mathcal{F}$ on A is called a *IT-sheaf of index i* if

$$H^j(A, \mathcal{F} \otimes \mathcal{Q}) = 0 \quad \text{for all} \quad \mathcal{Q} \in \mathrm{Pic}^0(A) \quad \text{and} \quad j \neq i.$$

If $\mathcal{F}$ is an IT-sheaf of index i and $\mathcal{P}$ it the normalized Poincaré bundle on $A \times \hat{A}$, the coherent sheaf

$$\hat{\mathcal{F}} := R^i \pi_{\hat{A}*}(\mathcal{P} \otimes \pi_A^* \mathcal{F})$$

is a vector bundle of rank $h^i(A, \mathcal{F})$, called the *Fourier–Mukai transform* of $\mathcal{F}$.

By "surface" we mean a projective, non-singular surface S, and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler–Poincaré characteristic*. If $q(S) > 0$, we denote by $\alpha\colon S \to \mathrm{Alb}(S)$ the Albanese map of S.

If $|D|$ is any linear system of curves on a surface, its base locus will be denoted by $\mathrm{Bs}|D|$. If D is any divisor, D_{red} stands for its support.

If Z is a zero-dimensional scheme, we denote its length by $l(Z)$.

If X is any scheme, by “first-order deformation” of X we mean a deformation over $\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$, whereas by “small deformation” we mean a deformation over a disk $\mathcal{B}_r = \{t \in \mathbb{C} \mid |t| < r\}$.

1. Preliminaries

1.1. Triple covers of surfaces. The theory of triple covers in algebraic geometry was developed by R. Miranda in his paper [27], whose main result is the following.

Theorem 1.1 ([27, Theorem 1.1])**.** *A triple cover $f\colon X \to Y$ of an algebraic variety Y is determined by a rank 2 vector bundle $\mathcal{E}$ on Y and by a global section $\eta \in H^0\big(Y, S^3\mathcal{E}^\vee \otimes \bigwedge^2 \mathcal{E}\big)$, and conversely.*

The vector bundle $\mathcal{E}$ is called the *Tschirnhausen bundle* of the cover, and it satisfies

$$f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}.$$

In the case of smooth surfaces, one has the following formulae.

Proposition 1.2 ([27, Proposition 10.3])**.** *Let $f\colon S \to Y$ be a triple cover of smooth surfaces with Tschirnhausen bundle $\mathcal{E}$. Then*
(i) $h^i(S, \mathcal{O}_S) = h^i(Y, \mathcal{O}_Y) + h^i(Y, \mathcal{E})$ *for all* $i \geq 0$;
(ii) $K_S^2 = 3K_Y^2 - 4c_1(\mathcal{E})K_Y + 2c_1^2(\mathcal{E}) - 3c_2(\mathcal{E})$.

Let $f\colon X \to Y$ be a triple cover, and let us denote by $D \subset Y$ and by $R \subset X$ the branch locus and the ramification locus of f, respectively. By [27, Proposition 4.7], D is a divisor whose associated line bundle is $\bigwedge^2 \mathcal{E}^\vee$. If Y is smooth, then f is smooth over $Y - D$, in other words all the singularities of X come from the singularities of the branch locus. More precisely, we have

Proposition 1.3 ([30, Proposition 5.4])**.** *Let $y \in \operatorname{Sing}(D)$. Then X is singular over y if and only if one of the following conditions holds*:
(i) *f in not totally ramified over y*;
(ii) *f is totally ramified over y and* $\operatorname{mult}_y(D) \geq 3$.

Proposition 1.4 ([42, Theorem 4.1])**.** *Let $f\colon X \to Y$ be a triple cover of a smooth surface Y, with X normal. Then there are a finite number of blow-ups $\sigma\colon \tilde{Y} \to Y$ of Y*

and a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "\sigma"] & Y, \end{tikzcd} \tag{1}$$

where $\tilde{X}$ *is the normalization of* $\tilde{Y} \times_Y X$, *such that* $\tilde{f}$ *is a triple cover with smooth branch locus. In particular,* $\tilde{X}$ *is a resolution of the singularities of* X.

We shall call $\tilde{X}$ the *canonical resolution* of the singularities of $f \colon X \to Y$. In general, it does not coincide with the *minimal resolution* of the singularities of X, which will be denoted instead by S.

DEFINITION 1.5. Let $f \colon X \to Y$ be a triple cover of a smooth algebraic surface Y, with Tschirnhausen bundle $\mathcal{E}$. We say that X has only *negligible* (or *non essential*) singularities if the invariants of the minimal resolution S are given by the formulae in Proposition 1.2.

In other words, negligible singularities have no effect on the computation of invariants. Let us give some examples.

EXAMPLE 1.6. Assume that the branch locus $D = D_{\mathrm{red}}$ contains an ordinary quadruple point p over which f is totally ramified. In this case $\tilde{Y}$ is the blow-up of Y at p, and one sees that the exceptional divisor is not in the branch locus of $\tilde{f}$. We have $S = \tilde{X}$ and the inverse image of the exceptional divisor on $\tilde{X}$ is a (-3)-curve. Therefore X has a singular point of type $(1/3)(1, 1)$ over p, and by straightforward computations (see [42, Section 6]) one checks that it is a negligible singularity.

EXAMPLE 1.7. Assume that the branch locus $D = D_{\mathrm{red}}$ contains an ordinary double point p. A standard topological argument shows that X cannot be smooth over p, so Proposition 1.3 implies that p is not a point of total ramification for f. Again, $\tilde{Y}$ is the blow-up of Y at p and the exceptional divisor is not in the branch locus of $\tilde{f}$. The inverse image of the exceptional divisor on $\tilde{X}$ consists of the disjoint union of a (-1)-curve and a (-2)-curve; then the canonical resolution $\tilde{X}$ does not coincide with the minimal resolution S, which is obtained by contracting the (-1)-curve. It follows that X has both a smooth point and a singular point of type $(1/2)(1, 1)$ over p, and as in the previous case one checks that this is a negligible singularity for X.

EXAMPLE 1.8. Assume that $D = 2D_{\mathrm{red}}$ and suppose in addition that $D_{\mathrm{red}} = D_1 + D_2$, where D_1 and D_2 are smooth curves intersecting transversally in precisely two points. We will provide examples where f is totally ramified and *non-Galois*, the singularities of X are a point of type $(1/3)(1,1)$ and a point of type $(1/3)(1,2)$, and moreover both of them are negligible.

REMARK 1.9. The definitions of canonical resolution for a triple cover is similar to the corresponding definition for double covers, that can be found for instance in [7, Chapter V]. However, in contrast with the double cover case, with our definition negligible singularities for triple covers are not necessarily rational double points, see for instance Example 1.6.

1.2. Abelian surfaces with (1,2) polarization. Let A be an abelian surface and L an ample divisor in A with $L^2 = 4$. Then L defines a polarization $\mathcal{L} := \mathcal{O}_A(L)$ of type $(1,2)$, in particular $h^0(A, \mathcal{L}) = 2$ so the linear system $|L|$ is a pencil. Such surfaces have been investigated by several authors, see for instance [4], [24], [6, Chapter 10] and [8]. Here we just recall the results we need.

Proposition 1.10 ([4, p. 46]). *Let $(A, \mathcal{L})$ be a $(1,2)$-polarized abelian surface, with $\mathcal{L} = \mathcal{O}_A(L)$, and let $C \in |L|$. Then we are in one of the following cases*:
(a) *C is a smooth, connected curve of genus* 3;
(b) *C is an irreducible curve of geometric genus* 2, *with an ordinary double point*;
(c) *$C = E + F$, where E and F are elliptic curves and $EF = 2$*;
(d) *$C = E + F_1 + F_2$, with E, F_1, F_2 elliptic curves such that $EF_1 = 1$, $EF_2 = 1$, $F_1 F_2 = 0$.*
Moreover, in case (c) *the surface A is isogenous to a product of two elliptic curves, and the polarization of A is the pull-back of the principal product polarization, whereas in case* (d) *the surface A itself is a product $E \times F$ and $\mathcal{L} = \mathcal{O}_A(E + 2F)$.*

Let us denote by $\mathcal{W}(1,2)$ the moduli space of $(1,2)$-polarized abelian surfaces; then there exists a Zariski dense open set $\mathcal{U} \subset \mathcal{W}(1,2)$ such that, given any $(A, \mathcal{L}) \in \mathcal{U}$, all divisors in $|L|$ are irreducible, i.e., of type (a) or (b), see [8, Section 3].

DEFINITION 1.11. If $(A, \mathcal{L}) \in \mathcal{U}$, we say that $\mathcal{L}$ is a *general* $(1,2)$-polarization. If $|L|$ contains some divisor of type (c), we say that $\mathcal{L}$ is a *special* $(1,2)$-polarization. Finally, if the divisors in $|L|$ are of type (d), we say that $\mathcal{L}$ is a *product* $(1,2)$-polarization.

If $\mathcal{L}$ is not a product polarization, then $|L|$ has four distinct base points $\{e_0, e_1, e_2, e_3\}$, which form an orbit for the action of $K(\mathcal{L}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ on A. Moreover all curves in $|L|$ are smooth at each of these base points, see [4, Section 1]. There is also a natural action of $K(\mathcal{L})$ on $|L|$, given by translation.

Let us denote by $(-1)_A$ the involution $x \to -x$ on A. Then we say that a divisor C on A is *symmetric* if $(-1)_A^* C = C$. Analogously, we say that a vector bundle $\mathcal{F}$ on A is symmetric if $(-1)_A^* \mathcal{F} = \mathcal{F}$.

Since $\mathcal{L}$ is ample, [6, Section 4.6] implies that, up to translation, it satisfies the following

ASSUMPTION 1.12. $\mathcal{L}$ is symmetric and the base locus of $|L|$ coincides with $K(\mathcal{L})$.

In the sequel we will tacitly suppose that Assumption 1.12 is satisfied.

Proposition 1.13. *The following holds*:
(i) *for all sections* $s \in H^0(A, \mathcal{L})$ *we have* $(-1)_A^* s = s$. *In particular, all divisors in* $|L|$ *are symmetric*;
(ii) *we may assume* $e_0 = o$ *and that* e_1, e_2, e_3 *are* 2*-division points, satisfying* $e_1 + e_2 = e_3$.

Proof. The first part of the statement follows from [6, Corollary 4.6.6], whereas the second part follows from Assumption 1.12. □

Proposition 1.14. *Let* $\mathcal{Q} = \mathcal{O}_A(Q) \in \mathrm{Pic}^0(A)$ *be a non-trivial, degree* 0 *line bundle. Then we have* $o \notin Bs|L + Q|$, *and moreover*

$$h^0(A, \mathcal{L} \otimes \mathcal{Q} \otimes \mathcal{I}_o) = 1, \quad h^1(A, \mathcal{L} \otimes \mathcal{Q} \otimes \mathcal{I}_o) = 0, \quad h^2(A, \mathcal{L} \otimes \mathcal{Q} \otimes \mathcal{I}_o) = 0.$$

Proof. Since $\mathcal{L}$ is ample, the line bundle $\mathcal{L} \otimes \mathcal{Q}$ is equal to $t_x^* \mathcal{L}$ for some $x \in A$. Then $o \in \mathrm{Bs}|L + Q|$ if and only if $x \in K(\mathcal{L})$, that is $\mathcal{L} \otimes \mathcal{Q} = \mathcal{L}$, which is impossible since $\mathcal{Q}$ is non-trivial. The rest of the proof follows by tensoring with $\mathcal{Q}$ the short exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{I}_o \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_o \to 0$$

and by taking cohomology. □

In the rest of this section we assume that $\mathcal{L}$ is not a product polarization. We denote by $e_4, \ldots, e_{15}$ the twelve 2-division points of A distinct from e_0, e_1, e_2, e_3. Some of the following results are probably known to the experts; however, since we have not been able to find a comprehensive reference, for the reader's convenience we give all the proofs.

Proposition 1.15. *The following holds.*
(a) *Assume that* $\mathcal{L}$ *is a general* $(1,2)$*-polarization. Then* $|L|$ *contains exactly* 12 *singular curves* $L_5, \ldots, L_{16}$. *Every* L_i *has an ordinary double point at* e_i, *and the set* $\{L_i\}_{i=4,\ldots,15}$ *consists of three orbits for the action of* $K(\mathcal{L})$ *on* $|L|$.

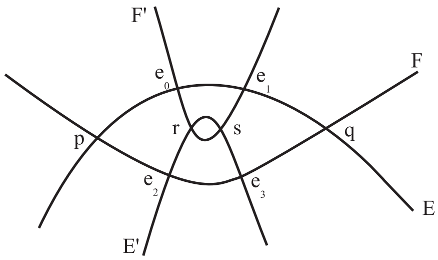

Fig. 1. The reducible curves $E + F$ and $E' + F'$ in the linear system $|L|$.

(b) *Assume that $\mathcal{L}$ is a special* $(1, 2)$*-polarization, and let $E + F \in |L|$ be a reducible divisor. Then the $K(\mathcal{L})$-orbit of $E + F$ consists of two curves $E + F$, $E' + F'$ which intersect as in* Fig. 1. *Referring to this figure, the set $\{p, q, r, s\}$ is contained in $\{e_4, \dots, e_{15}\}$, and it is an orbit for the action of $K(\mathcal{L})$ on A.*

Proof. (a) If a curve of $|L|$ contains any of the points $e_4, \dots, e_{15}$ then it must have a node there, see [4, Section 1.7] and [44, Remark 11]. In order to prove that there are no more singular curves, we blow-up the base points of $|L|$ obtaining a genus 3 fibration $\tau \colon \tilde{A} \to \mathbb{P}^1$. By the Zeuthen–Segre formula, see [5, Lemma 6.4], we have

$$c_2(\tilde{A}) = e(\mathbb{P}^1)e(L) + \sum (e(L_s) - e(L)), \tag{2}$$

where the sum is taken on all the singular curves L_s of $|L|$. Since $e(L_s) = e(L) + 1$ for a nodal curve, relation (2) implies that $|L|$ contains precisely 12 singular elements. This proves our first statement. The second statement is clear since the twelve points $e_4, \dots, e_{15}$ consist of three orbits for the action of $K(\mathcal{L})$ on A.

(b) Both curves E and F are fixed by the involution $(-1)_A$, so they must both contain exactly four 2-division points. In particular the two intersection points of E and F must be 2-division points, say $E \cap F = \{p, q\}$. Since we have

$$t^*_{e_0}E = t^*_{e_1}E = E, \quad t^*_{e_0}F = t^*_{e_1}F = F,$$

it follows that the orbit of $E + F$ contains exactly two elements, namely $E + F$ and $E' + F'$ where

$$E' := t^*_{e_2}E = t^*_{e_3}E, \quad F' := t^*_{e_2}F = t^*_{e_3}F.$$

Setting $E' \cap F' = \{r, s\}$, it is straightforward to check that the set of 2-division points $\{p, q, r, s\}$ is an orbit for the action of $K(\mathcal{L})$ on A. $\square$

REMARK 1.16. In case (b) of Proposition 1.15, if one makes the further assumption that A is not isomorphic to the product of two elliptic curves, it is not difficult to see that $E+F$ and $E'+F'$ are the unique reducible curves in $|L|$, and that the singular elements of $|L|$ distinct from $E+F$ and $E'+F'$ are eight irreducible curves L_i which have an ordinary double point at the 2-division points of A distinct from e_0, e_1, e_2, e_3, p, q, r, s. Moreover, these curves form two orbits for the action of $K(\mathcal{L})$ on $|L|$.

There exist examples of abelian surfaces which are isomorphic to the product of two elliptic curves and which admit also a special (1, 2)-polarization $\mathcal{L}$ besides the product polarization, see [44]. For such surfaces, the linear system $|L|$ could possibly contain more than two reducible curves (hence, less than eight irreducible nodal curves).

The other special elements of the pencil $|L|$ are smooth hyperelliptic curves; let us compute their number.

Proposition 1.17. *The following holds.*
(a) *Assume that $\mathcal{L}$ is a general* $(1,2)$*-polarization. Then $|L|$ contains exactly six smooth hyperelliptic curves.*
(b) *Assume that $\mathcal{L}$ is a special* (1, 2)*-polarization. Then $|L|$ contains at most four smooth hyperelliptic curves. More precisely, the number of such curves is given by* $6-\nu$*, where ν is the number of reducible curves in $|L|$.*
In any case, the set of hyperelliptic curves is union of orbits for the action of $K(\mathcal{L})$ on $|L|$, and each of these orbits has cardinality 2.

Proof. (a) We borrow the following argument from [8, Proposition 3.3]. Let us consider again the blow-up $\tilde{A}$ of A at the four base points of $|L|$ and the induced genus 3 fibration $\tau\colon \tilde{A} \to \mathbb{P}^1$. By [37, Sections 3.2 and 3.3] there is an equality

$$K_{\tilde{A}}^2 = 3\chi(\mathcal{O}_{\tilde{A}}) - 10 + \deg \mathcal{T}, \tag{3}$$

where $\mathcal{T}$ is a torsion sheaf on $\mathbb{P}^1$ supported over the points corresponding to the hyperelliptic fibres of τ. Since $\mathcal{L}$ is a general polarization, we can have only smooth hyperelliptic fibres and the contribution of each of them to $\deg \mathcal{T}$, which is usually called the Horikawa number, is equal to 1. So (3) implies that τ has exactly six smooth hyperelliptic fibres. On the other hand $K(\mathcal{L})$ acts on the set of hyperelliptic curves of $|L|$, so have three orbits of cardinality 2.

(b) The Horikawa number of a reducible curve in $|L|$ is equal to 1, see [1], so (3) implies that $|L|$ contains precisely $6-\nu$ smooth hyperelliptic curves. In particular, by Remark 1.16, $|L|$ contains exactly six hyperelliptic curves if A is not isomorphic to the product of two elliptic curves. Since the hyperelliptic curves have non-trivial stabilizer for the action of $K(\mathcal{L})$ on $|L|$ when $\mathcal{L}$ is a general polarization (see part (a)), by a limit argument we deduce that this is also true when $\mathcal{L}$ is a special polarization. It follows that the orbit of each hyperelliptic curve consists again of exactly two curves. $\square$

Proposition 1.18. *Let* $(A,\mathcal{L})$ *be a* $(1,2)$*-polarized abelian surface and let* $C \in |L|$*. Then the stabilizer of* C *for the action of* $K(\mathcal{L})$ *on* $|L|$ *is non-trivial if and only if either* C *is a smooth hyperelliptic curve or* C *is a reducible curve* (*in the latter case,* $\mathcal{L}$ *is necessarily a special polarization*).

Proof. The action of $K(\mathcal{L})$ on $|L| \cong \mathbb{P}^1$ induces a $(\mathbb{Z}/2\mathbb{Z})^2$-cover $\mathbb{P}^1 \to \mathbb{P}^1$, which is branched in three points by the Riemann–Hurwitz formula. This implies that there are exactly six elements of $|L|$ having non-trivial stabilizer. Our claim is now an immediate consequence of Proposition 1.17 and Proposition 1.15, part (b). $\square$

Let us consider the line bundle $\mathcal{L}^2 = \mathcal{O}_A(2L)$. It is a polarization of type $(2,4)$ on A, hence $h^0(A,\mathcal{L}^2) = 8$. Moreover, since $\mathcal{L}$ satisfies Assumption 1.12, the same is true for $\mathcal{L}^2$. Let $H^0(A,\mathcal{L}^2)^+$ and $H^0(A,\mathcal{L}^2)^-$ be the subspaces of invariant and anti-invariant sections for $(-1)_A$, respectively. One proves that

$$\dim H^0(A, \mathcal{L}^2)^+ = 6, \quad \dim H^0(A, \mathcal{L}^2)^- = 2,$$

see [4, Section 2].

Proposition 1.19 ([4, Section 5])**.** *The pencil* $\mathbb{P}H^0(A, \mathcal{L}^2)^-$ *of anti-invariant sections has precisely* 16 *distinct base points, namely* $e_0, e_1, \ldots, e_{15}$*. Moreover all the corresponding divisors are smooth at these base points.*

The 12 points $e_4, \ldots, e_{15}$ form three orbits for the action of $K(\mathcal{L})$ on A; without loss of generality, we may assume that these orbits are

$$\{e_4, e_5, e_6, e_7\}, \quad \{e_8, e_9, e_{10}, e_{11}\}, \quad \{e_{12}, e_{13}, e_{14}, e_{15}\}.$$

Now let us take the 2-torsion line bundles $\mathcal{Q}_i := \mathcal{O}_A(Q_i)$, $i = 1, 2, 3$ such that

$$t_{e_4}^*\mathcal{L} = \mathcal{L} \otimes \mathcal{Q}_1, \quad t_{e_8}^*\mathcal{L} = \mathcal{L} \otimes \mathcal{Q}_2, \quad t_{e_{12}}^*\mathcal{L} = \mathcal{L} \otimes \mathcal{Q}_3. \tag{4}$$

Then

$$\begin{aligned} \mathrm{Bs}|L + Q_1| &= \{e_4, e_5, e_6, e_7\}, \\ \mathrm{Bs}|L + Q_2| &= \{e_8, e_9, e_{10}, e_{11}\}, \\ \mathrm{Bs}|L + Q_3| &= \{e_{12}, e_{13}, e_{14}, e_{15}\}. \end{aligned}$$

Moreover, for all $i = 1, 2, 3$,

$$h^0(\hat{A}, \mathcal{L} \otimes \mathcal{Q} \otimes \mathcal{I}_o) = h^0(\hat{A}, \mathcal{L} \otimes \mathcal{Q} \otimes \mathcal{I}_o^2) = 1. \tag{5}$$

Let us call N_i, $i = 1, 2, 3$, the unique curve in the pencil $|L + Q_i|$ containing o (and having a node there, see (5)). If $\mathcal{L}$ is a general $(1,2)$-polarization then the N_i are all irreducible, in particular they are smooth outside o.

DEFINITION 1.20. We denote by $\mathfrak{D}$ the linear system $\mathbb{P}H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$. Geometrically speaking, $\mathfrak{D}$ consists of the curves in $|2L|$ having a point of multiplicity at least 4 at o.

Proposition 1.21. *The linear system $\mathfrak{D} \subset |2L|$ is a pencil whose general element is irreducible, with an ordinary quadruple point at o and no other singularities.*

Proof. Since the sections corresponding to the three curves $2N_i$ obviously belongs to $H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$, by Bertini theorem it follows that the general element of $\mathfrak{D}$ is irreducible, and smooth outside o. On the other hand, $(2L)^2 = 16$, so the singularity at o is actually an ordinary quadruple point. Blowing up this point, the strict transform of the general curve in $\mathfrak{D}$ has self-intersection 0, so $\mathfrak{D}$ is a pencil. $\square$

The following classification of the curves in $\mathfrak{D}$ will be needed in the proof of Theorem 6.6.

Proposition 1.22. *Let $(A, \mathcal{L})$ be a $(1,2)$-polarized abelian surface, and let $C \in \mathfrak{D}$. Then we are in one of the following cases*:
(a) *C is an irreducible curve of geometric genus 3, with an ordinary quadruple point*;
(b) *C is an irreducible curve of geometric genus 2, with an ordinary quadruple point and an ordinary double point*;
(c) *$C = 2C'$, where C' is an irreducible curve of geometric genus 2 with an ordinary double point*;
(d) *$\mathcal{L}$ is a special $(1,2)$-polarization and $C = 2C'$, where C' is the union of two elliptic curves intersecting in two points.*

Proof. By Proposition 1.21 the general element of $\mathfrak{D}$ is as in case (a). Now assume first that $\mathcal{L}$ is a general polarization. Then $\mathfrak{D}$ contains the following distinguished elements:
- three reduced, irreducible curves B_1, B_2, B_3 such that B_i has an ordinary quadruple point at o, an ordinary double point at e_i and no other singularities (see [6, Corollary 4.7.6]). These curves are as in case (b);
- three non-reduced elements, namely $2N_1$, $2N_2$, $2N_3$. These curves are as in case (c).

Moreover, all the other elements of $\mathfrak{D}$ are smooth outside o; one can see this by blowing-up o and applying Zeuthen–Segre formula as in the proof of Proposition 1.15.

Finally, assume that $\mathcal{L}$ is a special polarization. Then there is just one more possibility, namely $C = 2C'$, where C' is the translate of a reducible curve $E + F \in |L|$ by a suitable 2-division point. This yields case (d). $\square$

Proposition 1.23. *Every $s \in H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$ satisfies $(-1)_A^* s = s$.*

Proof. Let $v_1 \in H^0(A, \mathcal{L} \otimes \mathcal{Q}_1)$, $v_2 \in H^0(A, \mathcal{L} \otimes \mathcal{Q}_2)$ be sections corresponding to the curves N_1 and N_2, respectively. Since N_1 and N_2 are invariant divisors, it follows $(-1)_A^* v_1 = \pm v_1$ and $(-1)_A^* v_2 = \pm v_2$. Therefore $(-1)_A^* v_1^2 = v_1^2$ and $(-1)_A^* v_2^2 = v_2^2$. But v_1^2, v_2^2 form a basis for $H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_0^4)$, so we are done. □

Proposition 1.24. *We have*

$$h^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3) = h^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4) = 2.$$

Geometrically speaking, every curve in $|2L|$*, having multiplicity at least* 3 *at* o*, actually has multiplicity* 4*.*

Proof. By contradiction, suppose that $H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$ is strictly contained in $H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3)$. Then there exists $w \in H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3)$, $w \notin H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$ such that the three sections $v_1^2, v_2^2, w \in H^0(A, \mathcal{L}^2)$ are linearly independent; let us write

$$w = w^+ + w^-, \quad \text{where} \quad w^+ \in H^0(A, \mathcal{L}^2)^+ \quad \text{and} \quad w^- \in H^0(A, \mathcal{L}^2)^-.$$

Consider the sum

$$s = v_1^2 + v_2^2 + w = v_1^2 + v_2^2 + w^+ + w^- \in H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3);$$

then Proposition 1.23 implies

$$(-1)_A^* s = v_1^2 + v_2^2 + w^+ - w^-.$$

On the other hand, $(-1)_A$ fixes the tangent cone at o of the curve corresponding to s; hence $(-1)_A^* s$ also vanishes of order at least 3 in o, that is $(-1)_A^* s \in H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3)$. This implies

$$w^+, w^- \in H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3).$$

Since by assumption $w = w^+ + w^- \notin H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$, it follows that either $w^+ \notin H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$ or $w^- \notin H^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4)$. In the former case, the curve $W^+ := \mathrm{div}(w^+)$ is an *even* divisor (i.e., corresponding to an invariant section) in $|2L|$ which has multiplicity exactly 3 at o; but this is impossible, since every even divisor in $|2L|$ has even multiplicity at the 2-division points of A, see [6, Corollary 4.7.6]. In the latter case, the curve $W^- := \mathrm{div}(w^-)$ is an *odd* divisor (i.e., corresponding to an anti-invariant section) in $|2L|$ which has multiplicity exactly 3 at o; but this is again a contradiction, since all the odd divisors in $|2L|$ are smooth at the 2-division points of A, see Proposition 1.19. □

2. Computations on vector bundles

Let $(A, \mathcal{L})$ be a $(1, 2)$-polarized abelian surface. Throughout this section, $\mathcal{F}$ will denote a rank 2 vector bundle on A such that

$$h^0(A, \mathcal{F}) = 1, \quad h^1(A, \mathcal{F}) = 0, \quad h^2(A, \mathcal{F}) = 0, \quad \det \mathcal{F} = \mathcal{L}; \tag{6}$$

note that (6) together with Hirzebruch–Riemann–Roch implies $c_2(\mathcal{F}) = 1$. These results will be needed in Section 5.

Proposition 2.1. *If $\mathcal{F}$ is the direct sum of two line bundles, then it cannot be strictly $\mathcal{L}$-semistable.*

Proof. Set $\mathcal{F} = \mathcal{O}_A(C_1) \oplus \mathcal{O}_A(C_2)$, where C_1, C_2 are divisors in A, and suppose by contradiction that $\mathcal{F}$ is $\mathcal{L}$-semistable. Since $L = C_1 + C_2$, we obtain

$$C_1(C_1 + C_2) = C_2(C_1 + C_2) = 2.$$

On the other hand $1 = c_2(\mathcal{F}) = C_1C_2$ and so $C_1^2 = C_2^2 = 1$, which is absurd. $\square$

From now on, we assume that $\mathcal{F}$ is *indecomposable*. We divide the rest of the section into three subsections according to the properties of $\mathcal{L}$ and $\mathcal{F}$.

2.1. The case where $\mathcal{L}$ is not a product polarization.

Proposition 2.2. *If $\mathcal{L}$ is not a product polarization, then $\mathcal{F}$ is isomorphic to the unique locally free extension*

$$0 \to \mathcal{O}_A \to \mathcal{F} \to \mathcal{L} \otimes \mathcal{I}_x \to 0,$$

with $x \in K(\mathcal{L})$. Moreover, $\mathcal{F}$ is $\mathcal{H}$-stable for any ample line bundle $\mathcal{H}$ on A.

Proof. Since $h^0(A, \mathcal{F}) = 1$, there exists an injective morphism of sheaves $\mathcal{O}_A \hookrightarrow \mathcal{F}$. By [19, Proposition 5 p. 33] we can find an effective divisor C and a zero-dimensional subscheme Z such that $\mathcal{F}$ fits into a short exact sequence

$$0 \to \mathcal{O}_A(C) \to \mathcal{F} \to \mathcal{I}_Z(L - C) \to 0. \tag{7}$$

Then $h^0(A, \mathcal{O}_A(C)) = 1$ and

$$1 = c_2(\mathcal{F}) = C(L - C) + l(Z). \tag{8}$$

Now there are three possibilities:

(i) C is an elliptic curve;
(ii) C is a principal polarization;
(iii) $C = 0$.

In case (i) we have $C^2 = 0$, then by (8) we obtain $CL = 1$ and $l(Z) = 0$. Thus [6, Lemma 10.4.6] implies that $\mathcal{L}$ is a product polarization, contradiction.

In case (ii), the index theorem yields $(CL)^2 \geq C^2L^2 = 8$, so using (8) we deduce $CL = 3$, $l(Z) = 0$. Setting $\mathcal{C} := \mathcal{O}_A(C)$, sequence (7) becomes

$$0 \to \mathcal{C} \to \mathcal{F} \to \mathcal{C}^{-1} \otimes \mathcal{L} \to 0.$$

Being $\mathcal{F}$ indecomposable by assumption, we have

$$H^1(A, \mathcal{C}^2 \otimes \mathcal{L}^{-1}) = \mathrm{Ext}^1(\mathcal{C}^{-1} \otimes \mathcal{L}, \mathcal{C}) \neq 0. \tag{9}$$

Moreover, since $(-2C + L)L = -2$, the divisor $-2C + L$ is not effective, that is

$$H^2(A, \mathcal{C}^2 \otimes \mathcal{L}^{-1}) = H^0(A, \mathcal{C}^{-2} \otimes \mathcal{L}) = 0. \tag{10}$$

On the other hand, by Riemann–Roch we have

$$\chi(A, \mathcal{C}^2 \otimes \mathcal{L}^{-1}) = \frac{1}{2}(2C - L)^2 = 0,$$

so (9) and (10) yield $H^0(A, \mathcal{C}^2 \otimes \mathcal{L}^{-1}) \neq 0$. This implies that $2C - L$ is effective, so by using [4, Lemma 1.1] and the equality $(2C - L)C = 1$ one concludes that there exists an elliptic curve E on A such that $2C - L = E$. Thus [6, Lemma 10.4.6] implies that A is a product of elliptic curves and that C is a principal product polarization. In other words $A = E \times F$ and C is algebraically equivalent to $E + F$. But then L is algebraically equivalent to $E + 2F$, contradicting the fact that $\mathcal{L}$ is not a product polarization.

Therefore the only possibility is (iii), namely $C = 0$. It follows that Z consists of a single point $x \in A$ and, since $\mathcal{F}$ is locally free, x is a base point of $|L|$, i.e., $x \in K(\mathcal{L})$.

Therefore (7) becomes

$$0 \to \mathcal{O}_A \to \mathcal{F} \to \mathcal{L} \otimes \mathcal{I}_x \to 0. \tag{11}$$

Tensoring (11) with $\mathcal{F}^\vee$ and taking cohomology, we obtain

$$\begin{aligned} 1 \leq h^0(A, \mathcal{F} \otimes \mathcal{F}^\vee) = h^0(A, \mathcal{F}^\vee \otimes \mathcal{L} \otimes \mathcal{I}_x) = h^0\left(A, \mathcal{F}^\vee \otimes \bigwedge^2 \mathcal{F} \otimes \mathcal{I}_x\right) \\ = h^0(A, \mathcal{F} \otimes \mathcal{I}_x) \leq h^0(A, \mathcal{F}) = 1. \end{aligned}$$

Therefore $H^0(A, \mathcal{F} \otimes \mathcal{F}^\vee) = \mathbb{C}$, that is $\mathcal{F}$ is simple. Since $c_1^2(\mathcal{F}) - 4c_2(\mathcal{F}) = 0$, by [40, Proposition 5.1] and [41, Proposition 2.1] it follows that $\mathcal{F}$ is $\mathcal{H}$-ample for any ample line bundle $\mathcal{H}$ on A.

It remains to show that (11) defines a unique locally free extension. By applying the functor $\mathrm{Hom}(-, \mathcal{O}_A)$ to

$$0 \to \mathcal{L} \otimes \mathcal{I}_x \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_x \to 0 \tag{12}$$

and using Serre duality, we get

$$\begin{aligned} 0 \to \mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{I}_x, \mathcal{O}_A) \to \mathrm{Ext}^2(\mathcal{L} \otimes \mathcal{O}_x, \mathcal{O}_A) &\cong H^0(A, \mathcal{L} \otimes \mathcal{O}_x)^\vee \\ \xrightarrow{\varphi} \mathrm{Ext}^2(\mathcal{L}, \mathcal{O}_A) &\cong H^0(A, \mathcal{L})^\vee. \end{aligned}$$

Being $x \in \mathrm{Bs}|L|$, it follows that φ is the zero map (see [12, Theorem 1.4]), so

$$\mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{I}_x, \mathcal{O}_A) = \mathbb{C}. \tag{13}$$

This completes the proof. $\square$

REMARK 2.3. Up to replacing $\mathcal{L}$ by $t_x^*\mathcal{L}$, which is still a symmetric $(1, 2)$-polarization, we may assume $x = o$. So $\mathcal{F}$ will be isomorphic to the unique locally free extension

$$0 \to \mathcal{O}_A \to \mathcal{F} \to \mathcal{L} \otimes \mathcal{I}_o \to 0. \tag{14}$$

Proposition 2.4. *If $\mathcal{L}$ is not a product polarization, $\mathcal{F}$ is a symmetric* IT-*sheaf of index* 0*.*

Proof. Since $\mathcal{L}$ is a symmetric polarization, by applying $(-1)_A^*$ to (14) we get

$$0 \to \mathcal{O}_A \to (-1)_A^*\mathcal{F} \to \mathcal{L} \otimes \mathcal{I}_o \to 0.$$

But (13) implies that $\mathcal{F}$ is the unique locally free extension of $\mathcal{L} \otimes \mathcal{I}_o$ by $\mathcal{O}_A$, so we obtain $(-1)_A^*\mathcal{F} = \mathcal{F}$, that is $\mathcal{F}$ is symmetric.

In order to prove that $\mathcal{F}$ satisfies IT of index 0, we must show that

$$\begin{aligned} V^1(A, \mathcal{F}) &:= \{\mathcal{Q} \in \mathrm{Pic}^0(A) \mid h^1(A, \mathcal{F} \otimes \mathcal{Q}) > 0\} = \emptyset, \\ V^2(A, \mathcal{F}) &:= \{\mathcal{Q} \in \mathrm{Pic}^0(A) \mid h^2(A, \mathcal{F} \otimes \mathcal{Q}) > 0\} = \emptyset. \end{aligned} \tag{15}$$

First, notice that $\mathcal{O}_A \notin V^1(A, \mathcal{F})$ and $\mathcal{O}_A \notin V^2(A, \mathcal{F})$, since $h^1(A, \mathcal{F}) = h^2(A, \mathcal{F}) = 0$. Now take $\mathcal{Q} \in \mathrm{Pic}^0(A)$ such that $\mathcal{Q} \neq \mathcal{O}_A$. Tensoring (14) with $\mathcal{Q}$ and using Proposition 1.14, we obtain

$$h^0(A, \mathcal{F} \otimes \mathcal{Q}) = 1, \quad h^1(A, \mathcal{F} \otimes \mathcal{Q}) = 0, \quad h^2(A, \mathcal{F} \otimes \mathcal{Q}) = 0.$$

Hence (15) is satisfied, and the proof is complete. $\square$

Since $\mathcal{F}$ is simple and $\chi(A, \mathcal{F} \otimes \mathcal{F}^\vee) = 0$, we have

$$h^0(A, \mathcal{F} \otimes \mathcal{F}^\vee) = 1, \quad h^1(A, \mathcal{F} \otimes \mathcal{F}^\vee) = 2, \quad h^2(A, \mathcal{F} \otimes \mathcal{F}^\vee) = 1. \tag{16}$$

On the other hand, the Clebsch–Gordan formula for the tensor product ([3, p. 438]) gives an isomorphism

$$\mathcal{O}_A \oplus \left(S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = \mathcal{F} \otimes \mathcal{F}^\vee,$$

so by using (16) we obtain

$$\begin{aligned} &h^0\left(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0, \quad h^1\left(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0,\\ &h^2\left(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0. \end{aligned} \tag{17}$$

Proposition 2.5. *If $\mathcal{L}$ is not a product polarization, we have*

$$h^0\left(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = h^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^3) = 2. \tag{18}$$

Proof. The Eagon–Northcott complex applied to (14) yields

$$0 \to S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \to S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \to \mathcal{L}^2 \otimes \mathcal{I}_o^3 \to 0,$$

so our assertion is an immediate consequence of (17) and Proposition 1.24. $\square$

2.2. The case where $\mathcal{L}$ is a product polarization and $\mathcal{F}$ is not simple. Now let us assume that $\mathcal{L}$ is a product $(1, 2)$-polarization. Then $A = E \times F$, where E and F are two elliptic curves, whose zero elements are both denoted by o. Let $\pi_E \colon E \times F \to E$ and $\pi_F \colon E \times F \to F$ be the natural projections. For any $p \in F$ and $q \in E$, we will write E_p and F_q instead of $\pi_F^{-1}(p)$ and $\pi_E^{-1}(q)$.

Furthermore, up to translations we may assume $\mathcal{L} = \mathcal{O}_A(E_o + 2F_o)$.

Following the terminology of [29], we say that $\mathcal{F}$ is *of Schwarzenberger type* if it is indecomposable but not simple.

Proposition 2.6. *Suppose that $\mathcal{L}$ is a product* $(1, 2)$*-polarization. Then $\mathcal{F}$ is of Schwarzenberger type if and only if it is a non-trivial extension of the form*

$$0 \to \mathcal{C} \to \mathcal{F} \to \mathcal{L} \otimes \mathcal{C}^{-1} \to 0, \tag{19}$$

where $\mathcal{C} := \mathcal{O}_A(E_p + F_q)$, with $p \in F$ different from o and $q \in E$ a 2*-division point.*

Proof. If $\mathcal{F}$ is a non-trivial extension of type (19), then [29, Lemma p. 251] shows that $\mathcal{F}$ is indecomposable but $h^0(A, \mathcal{F} \otimes \mathcal{F}^\vee) = 2$, so $\mathcal{F}$ is not simple.

Conversely, assume that $\mathcal{F}$ is of Schwarzenberger type. Being $\mathcal{F}$ not simple, it is not $\mathcal{H}$-stable with respect to any ample line bundle $\mathcal{H}$ on A. In particular, $\mathcal{F}$ is not $\mathcal{L}$-stable. An argument similar to the one used in the proof of Proposition 2.1 shows that $\mathcal{F}$ is not strictly $\mathcal{L}$-semistable, so it must be $\mathcal{L}$-unstable. This implies that there exists a unique sub-line bundle $\mathcal{C} := \mathcal{O}_A(C)$ of $\mathcal{F}$ with torsion-free quotient such that

$$2CL > L^2 = 4. \tag{20}$$

Now let us write

$$0 \to \mathcal{O}_A(C) \to \mathcal{F} \to \mathcal{I}_Z(L - C) \to 0, \tag{21}$$

where $Z \subset A$ is a zero-dimensional subscheme. Then by using (20) we obtain

$$1 = c_2(\mathcal{F}) = C(L - C) + l(Z) > 2 - C^2 + l(Z), \tag{22}$$

that is $C^2 > 1 + l(Z)$. On the other hand, since $h^0(A, \mathcal{C}) = 1$, the only possibility is $l(Z) = 0$ and $C^2 = 2$, in particular $\mathcal{C}$ is a principal polarization. But (22) also gives $3 = CL = C(E_o + 2F_o)$, so C is numerically equivalent to $E_o + F_o$. Therefore we can write $C = E_p + F_q$ for some $p \in F$, $q \in E$ and (21) becomes

$$0 \to \mathcal{O}_A(E_p + F_q) \to \mathcal{F} \to \mathcal{O}_A(E_o - E_p + 2F_o - F_q) \to 0. \tag{23}$$

Since $h^0(A, \mathcal{F}) = 1$, we have $p \neq o$. On the other hand, since $\mathcal{F}$ is indecomposable, (23) must be non-split, so

$$H^1(A, \mathcal{O}_A(2E_p - E_o + 2F_q - 2F_o)) \neq 0.$$

This implies that $2F_q$ is linearly equivalent to $2F_o$, that is $q \in E$ is a 2-division point. □

Proposition 2.7. *If $\mathcal{L}$ is a product polarization and $\mathcal{F}$ is of Schwarzenberger type, we have*

$$h^0\left(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = h^0\left(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}\right) = h^0\left(A, \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}^2\right)$$
$$= 3.$$

Proof. The Eagon–Northcott complex applied to (19) gives

$$0 \to \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}^2 \to S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C} \to \mathcal{L} \otimes \mathcal{C}^{-1} \to 0,$$
$$0 \to S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C} \to S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \to \mathcal{L}^2 \otimes \mathcal{C}^{-3} \to 0.$$

On the other hand, we have

$$H^0(A, \mathcal{L} \otimes \mathcal{C}^{-1}) = H^0(A, \mathcal{O}_A(E_o - E_p + F_q)) = 0,$$
$$H^0(A, \mathcal{L}^2 \otimes \mathcal{C}^{-3}) = H^0(A, \mathcal{O}_A(2E_o - 3E_p + F_q)) = 0.$$

Tensoring (19) with $\bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}^2$ we obtain $h^0\big(A, \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}^2\big) = 3$, so the claim follows. □

Corollary 2.8. *If $\mathcal{L}$ is a product polarization and $\mathcal{F}$ is of Schwarzenberger type, then the natural product map*

$$H^0\left(A, \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}^2\right) \otimes H^0(A, \mathcal{F} \otimes \mathcal{C}^{-1})^{\otimes 2} \to H^0\left(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right)$$

is bijective. Therefore, if $f\colon X \to A$ is the triple cover corresponding to a non-zero section $\eta \in H^0\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big)$, the surface X is reducible and non-reduced.

Proof. The first statement follows from Proposition 2.7 and from $H^0(A, \mathcal{F} \otimes \mathcal{C}^{-1}) = \mathbb{C}$. The second statement is an immediate consequence of the first one, since η can be written as $\eta = \eta_1\eta_2^2$, where $\eta_1 \in H^0\big(A, \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{C}^2\big)$ and η_2 is a generator of $H^0(A, \mathcal{F} \otimes \mathcal{C}^{-1})$. □

2.3. The case where $\mathcal{L}$ is a product polarization and $\mathcal{F}$ is simple.

Proposition 2.9. *Suppose that $\mathcal{L}$ is a product* (1,2)*-polarization. Then the following are equivalent*:

(i) *$\mathcal{F}$ is simple*;

(ii) *$\mathcal{F}$ is $\mathcal{H}$-stable for any ample line bundle $\mathcal{H}$ on A*;

(iii) *there exists a* 2*-division point $q \in E$ such that $\mathcal{F}$ is isomorphic to the unique non-trivial extension*

$$0 \to \mathcal{O}_A(F_q) \to \mathcal{F} \to \mathcal{O}_A(E_o + F_q) \to 0;$$

(iv) *there exists a* 2*-division point $q \in E$ such that $\mathcal{F}(-F_q) = \pi_F^*\mathcal{G}$, where $\mathcal{G}$ is the unique non-trivial extension*

$$0 \to \mathcal{O}_F \to \mathcal{G} \to \mathcal{O}_F(o) \to 0.$$

Proof. (i) $\Rightarrow$ (ii) See [40, Proposition 5.1] and [41, Proposition 2.1].

(ii) $\Rightarrow$ (iii) If $\mathcal{F}$ is $\mathcal{H}$-stable, then it is simple. By [29, Corollary p. 249], there exists an abelian surface B, a degree 2 isogeny $\varphi\colon B \to A$ and a line bundle $\mathcal{N} := \mathcal{O}_B(N)$ on B such that

$$\varphi_* \mathcal{N} = \mathcal{F}. \tag{24}$$

Let $\mathcal{Q} := \mathcal{O}_A(Q) \in \mathrm{Pic}(A)$ be the 2-torsion line bundle defining the double cover φ; then the following equality holds in $\mathrm{Pic}(A)$:

$$\mathcal{O}_A(E_o + 2F_o) = c_1(\mathcal{F}) = \mathcal{O}_A(\varphi_* N + Q),$$

see [19, Proposition 27 p. 47]. This implies

- $B = E \times \tilde{F}$ and

$$\varphi = id \times \tilde{\varphi}\colon E \times \tilde{F} \to E \times F,$$

where $\tilde{\varphi}\colon \tilde{F} \to F$ is a degree 2 isogeny. Note that $Q = E_p - E_o$, where $p \in F$ is a 2-division point.

- N is a principal product polarization of the form $N = E_a + \tilde{F}_q$, where $a \in \tilde{F}$ is such that $\tilde{\varphi}(a) = p$ and $q \in E$ is a 2-division point.

Since $\tilde{F}_q = \varphi^* F_q$, by using (24) and projection formula we obtain

$$\varphi_* \mathcal{O}_B(E_a) = \varphi_*(\mathcal{N}(-\tilde{F}_q)) = \mathcal{F}(-F_q).$$

Thus $h^0(A, \mathcal{F}(-F_q)) = 1$, and so there exists an injective morphism of sheaves $\mathcal{O}_A(F_q) \hookrightarrow \mathcal{F}$. Then we can find an effective divisor D on A and a zero-dimensional subscheme $Z \subset A$ such that $\mathcal{F}$ fits into a short exact sequence

$$0 \to \mathcal{O}_A(F_q + D) \to \mathcal{F} \to \mathcal{I}_Z(E_o + F_q - D) \to 0.$$

Since $h^0(A, \mathcal{O}_A(F_q + D)) = H^0(A, \mathcal{F}) = 1$, either $D = 0$ or $F_q + D$ is a principal product polarization. The latter possibility cannot occur, otherwise $\mathcal{F}$ would be of Schwarzenberger type (Proposition 2.6). Then $D = 0$ and $l(Z) = c_2(\mathcal{F}) - (E_o + F_q)F_q = 0$, so Z is empty and we are done.

(iii) $\Rightarrow$ (iv) We have $\mathcal{O}_A(E_o) = \pi_F^* \mathcal{O}_F(o)$. By [29, Footnote $***$, p. 257] the map

$$\mathrm{Ext}^1(\mathcal{O}_F(o), \mathcal{O}_F) \to \mathrm{Ext}^1(\mathcal{O}_A(E_o), \mathcal{O}_A)$$

is an isomorphism. Since the unique not-trivial extension of $\mathcal{O}_F(o)$ with $\mathcal{O}_F$ is $\mathcal{G}$, we get (iv).

(iv) $\Rightarrow$ (i) Again, [29, p. 257] gives $\mathrm{End}(\mathcal{F}) = \mathrm{End}(\mathcal{G}) = \mathbb{C}$. $\square$

Proposition 2.10. *If $\mathcal{L}$ is a product polarization and $\mathcal{F}$ is simple, we have*

$$h^0\left(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{O}_A(-F_q)\right) = h^0\left(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 2.$$

Proof. By Proposition 2.9, we have $\mathcal{F}(-F_q) = \pi_F^*\mathcal{G}$, where $\mathcal{G}$ is the unique non-trivial extension of $\mathcal{O}_F(o)$ by $\mathcal{O}_F$. Therefore

$$h^0(A, \mathcal{F}(-F_q)) = h^0(F, \pi_{F*}\mathcal{F}(-F_q)) = h^0(F, \mathcal{G}) = 1. \tag{25}$$

By [3, pp. 438–439] we have

$$S^2\mathcal{G}(-o) \oplus \mathcal{O}_A = \mathcal{G} \otimes \mathcal{G}^\vee = \mathcal{O}_A \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3$$

where the $\mathcal{Q}_i$ are the non trivial 2-torsion line bundles on A. Since the decomposition of a vector bundle in indecomposable summands is unique ([2]) we get

$$S^2\mathcal{G} = \mathcal{Q}_1(o) \oplus \mathcal{Q}_2(o) \oplus \mathcal{Q}_3(o),$$

hence

$$\begin{aligned} S^3\mathcal{G} \oplus \mathcal{G}(o) = S^2\mathcal{G} \otimes \mathcal{G} &= \mathcal{G} \otimes \mathcal{Q}_1(o) \oplus \mathcal{G} \otimes \mathcal{Q}_2(o) \oplus \mathcal{G} \otimes \mathcal{Q}_3(o) \\ &= \mathcal{G}(o) \oplus \mathcal{G}(o) \oplus \mathcal{G}(o). \end{aligned}$$

Therefore $S^3\mathcal{G} = \mathcal{G}(o) \oplus \mathcal{G}(o)$ and by straightforward computations one obtains

$$S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee = \mathcal{F} \oplus \mathcal{F}. \tag{26}$$

Now the claim follows from (25) and (26). □

Corollary 2.11. *Assume that $\mathcal{L}$ is a product polarization and that $\mathcal{F}$ is simple, and let $f\colon X \to A$ be the triple cover defined by a general section $\eta \in H^0\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big)$. Then the variety X is non-normal, and its normalization X^ν is a properly elliptic surface with $p_g(X^\nu) = 2$, $q(X^\nu) = 3$.*

Proof. Proposition 2.10 shows that every section of $S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee$ vanishes along the curve F_q; this implies that X is singular along $f^{-1}(F_q)$, in particular X is non-normal. The composition of $f\colon X \to A$ with the normalization map is a triple cover $f^\nu\colon X^\nu \to A$, whose Tschirnhausen bundle is $\mathcal{E}^\nu := \mathcal{F}(-F_q)^\vee$. Since $\bigwedge^2 \mathcal{E}^\nu = \mathcal{O}_A(-E_o)$, the morphism f^ν is branched over a divisor belonging to the linear system $|2E_o|$, hence X^ν contains an elliptic fibration. Moreover $c_1^2(\mathcal{E}^\nu) = 0$, $c_2(\mathcal{E}^\nu) = 0$ and a straightforward computation using $\mathcal{F}(-F_q) = \pi_F^*\mathcal{G}$ and Leray spectral sequence yields

$$h^0(A, \mathcal{E}^\nu) = 0, \quad h^1(A, \mathcal{E}^\nu) = 1, \quad h^2(A, \mathcal{E}^\nu) = 1.$$

Therefore Proposition 1.2 implies $p_g(X^\nu) = 2$, $q(X^\nu) = 3$ and $K^2_{X^\nu} = 0$, hence X^ν is a properly elliptic surface. □

3. Surfaces with $p_g = q = 2$, $K_S^2 = 5$ and Albanese map of degree 3

3.1. The triple cover construction. The first example of a surface S of general type with $p_g = q = 2$ and $K_S^2 = 5$ was given by Chen and Hacon in [16], as a triple cover of an abelian surface. In order to fix our notation, let us recall their construction.

Let $(A, \mathcal{L})$ be a $(1, 2)$-polarized abelian surface, and assume that $\mathcal{L}$ is a general, symmetric polarization. Since

$$h^0(A, \mathcal{L} \otimes \mathcal{Q}) = 2, \quad h^1(A, \mathcal{L} \otimes \mathcal{Q}) = 0, \quad h^2(A, \mathcal{L} \otimes \mathcal{Q}) = 0$$

for all $\mathcal{Q} \in \mathrm{Pic}^0(A)$, the line bundle $\mathcal{L}^{-1}$ satisfies IT of index 2. Then its Fourier–Mukai transform $\mathcal{F} := \widehat{\mathcal{L}^{-1}}$ is a rank 2 vector bundle on $\hat{A}$ which satisfies IT of index 0, see [6, Theorem 14.2.2]. Let us consider the isogeny

$$\phi := \phi_{\mathcal{L}^{-1}} \colon A \to \hat{A},$$

whose kernel is $K(\mathcal{L}^{-1}) = K(\mathcal{L})$; then by [28, Proposition 3.11] we have

$$\phi^* \mathcal{F} = \mathcal{L} \oplus \mathcal{L}. \tag{27}$$

Proposition 3.1. *The vector bundle $S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee$ satisfies*

$$h^0\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 2, \quad h^1\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0,$$
$$h^2\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0.$$

Proof. We could use Proposition 2.5, but we prefer a different argument exploiting the isogeny ϕ. Since $\chi\big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big) = 2$, it is sufficient to show that $h^1\big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big) = h^2\big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big) = 0$. Since ϕ is a finite map, we obtain

$$H^i\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) \cong \phi^* H^i\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) \subseteq H^i\left(A, \phi^*\left(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right)\right)$$

for all $i = 0, 1, 2$. On the other hand, (27) yields

$$H^i\left(A, \phi^*\left(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right)\right) = H^i(A, \mathcal{L})^{\oplus 4},$$

so the claim follows. □

By Theorem 1.1 there is a 2-dimensional family of triple covers $\hat{f}\colon \hat{X} \to \hat{A}$ with Tschirnhausen bundle $\mathcal{E} = \mathcal{F}^\vee$. We have the commutative diagram

$$\begin{tikzcd} X = A \times_{\hat{A}} X \arrow[r, "\psi"] \arrow[d, "f"] & \hat{X} \arrow[d, "\hat{f}"] \\ A \arrow[r, "\phi"] & \hat{A} \end{tikzcd} \tag{28}$$

where $\psi\colon X \to \hat{X}$ is a quadruple étale cover and $f\colon X \to A$ is a triple cover determined by a section of

$$\phi^* H^0\Big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\Big) \subset H^0\Big(A, \phi^*\Big(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\Big)\Big) = H^0(A, \mathcal{L})^{\oplus 4}.$$

By [6, Chapter 6] there exists a canonical Schrödinger representation of the Heisenberg group $\mathcal{H}_2$ on $H^0(A, \mathcal{L})$, where the latter space is identified with the vector space $\mathbb{C}(\mathbb{Z}/2\mathbb{Z})$ of all complex valued function on the finite group $\mathbb{Z}/2\mathbb{Z}$.

Following [16, Section 2] we can identify the 2-dimensional subspace of $H^0\big(A, \phi^*\big(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big)\big)$ corresponding to $\phi^* H^0\big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big)$ with

$$\{(sx, ty, -tx, -sy) \mid s, t \in \mathbb{C}\} \subset H^0(A, \mathcal{L})^{\oplus 4}, \tag{29}$$

where $x, y \in H^0(A, \mathcal{L})$ form the canonical basis induced by the characteristic functions of 0 and 1 in $\mathbb{C}(\mathbb{Z}/2\mathbb{Z})$. By [27], we can construct the triple cover $f\colon X \to A$ using the data

$$a = sx, \quad b = ty, \quad c = -tx, \quad d = -sy. \tag{30}$$

Over an affine open subset U of A the surface X is defined in $U \times \mathbb{A}^2$ by the determinantal equations

$$\operatorname{rank}\begin{pmatrix} z + a & w - 2d & c \\ b & z - 2a & w + d \end{pmatrix} \leq 1, \tag{31}$$

where w, z are coordinates in $\mathbb{A}^2$. Moreover, the branch locus D of $f\colon X \to A$ is given by

$$D = (t^2 - s^2)^2 x^2 y^2 - 4(s^2x^2 + sty^2)(s^2y^2 + stx^2) \in H^0(A, \mathcal{L}^4). \tag{32}$$

This corresponds to a divisor $D_1 + D_2 + D_3 + D_4$ with $D_i \in |L|$; moreover the set $\{D_1, \ldots, D_4\}$ is an orbit for the action of $K(\mathcal{L})$ on $|L|$. For a general choice of s, t, the D_i are all smooth, so the singularity of D are four ordinary quadruple points at e_0, e_1,

e_2, e_3. Over these points $f: X \to A$ is totally ramified and X has four singularities of type $(1/3)(1,1)$. Blowing up these points and the base points of $|L|$ we obtain a smooth triple cover $\tilde{f}: \tilde{X} \to \tilde{A}$, which is actually the canonical resolution of singularities of X, see Proposition 1.4. Let $\{E_i\}_{i=1,\dots,4}$ be the exceptional divisor in $\tilde{X}$ and $\{R_i\}_{i=1,\dots,4}$ be the proper transform of the D_i in $\tilde{X}$. Then $E_i^2 = -3$, $E_iE_j = 0$ for $i \neq j$, $R_iR_j = 0$ and $R_iE_j = 1$ for all i, j. Since

$$K_{\tilde{X}} = \sum_{i=1}^{4} R_i + \sum_{i=1}^{4} E_i,$$

we obtain $K_{\tilde{X}}^2 = 20$. Moreover X has only rational singularities, so if $\tilde{\sigma}: \tilde{X} \to X$ is the resolution map we have $R^1\tilde{\sigma}_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$; therefore

$$p_g(\tilde{X}) = h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^2(X, \mathcal{O}_X) = h^2(A, \mathcal{O}_A) + 2h^2(A, \mathcal{L}^{-1}) = 5,$$
$$q(\tilde{X}) = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^1(X, \mathcal{O}_X) = h^1(A, \mathcal{O}_A) + 2h^1(A, \mathcal{L}^{-1}) = 2.$$

This shows that $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 4$. Now let S be the canonical resolution of singularities of $\hat{X}$; then K_S is ample and $\hat{A} = \mathrm{Alb}(S)$. Since there is a quadruple, étale cover $\tilde{\psi}: \tilde{X} \to S$ induced by $\psi: X \to \hat{X}$, the invariants of S are

$$p_g(S) = q(S) = 2, \quad K_S^2 = 5.$$

REMARK 3.2. Both X and $\hat{X}$ only contain singular points of type (1/3)(1, 1), which are negligible singularities, see Example 1.6. Hence we could compute the invariants of both $\tilde{X}$ and S by directly using Proposition 1.2.

3.2. The product-quotient construction. In [32] it is shown that there exists precisely one family of surfaces with $p_g = q = 2$ and $K_S^2 = 5$ which contain an isotrivial fibration. Now we briefly explain how this family is obtained, referring the reader to [32] for further details.

By using the Riemann existence theorem, one can construct two smooth curves C_1, C_2 of genus 3 which admit an action of the finite group S_3, such that the 2-cycles act without fixed points, whereas the cyclic subgroup generated by the 3-cycles has exactly two fixed points. Then $E_i := C_i/S_3$ is a smooth elliptic curve and the Galois cover $C_i \to E_i$ is branched in exactly one point with branching number 3. Now let us consider the quotient $\hat{X} := (C_1 \times C_2)/S_3$, where S_3 acts diagonally on the product. Then $\hat{X}$ contains precisely two cyclic quotient singularities and, since the 3-cycles are conjugated in S_3, it is not difficult to show that one singularity is of type $(1/3)(1,1)$ whereas the other is of type (1/3)(1, 2). Let $S \to \hat{X}$ be the minimal resolution of singularities of $\hat{X}$; then S is a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 5$; notice

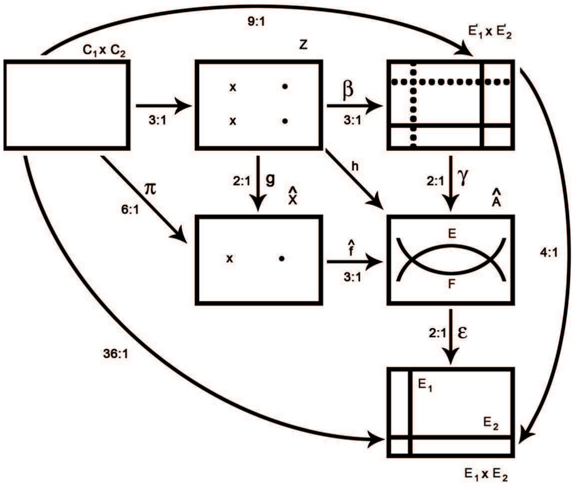

Fig. 2. The product-quotient construction.

that K_S is *not* ample. The surface S admits two isotrivial fibrations $S \to E_i$, which are induced by the two natural projections of $C_1 \times C_2$.

The Albanese variety $\hat{A}$ of S is an étale double cover of $E_1 \times E_2$; it is actually a $(1, 2)$-polarized abelian variety, whose polarization $\mathcal{L}$ is of special type. The Albanese map $\alpha\colon S \to \hat{A}$ is totally ramified, and its reduced branch locus $\Delta_{\mathrm{red}} = E + F$ is a curve of type (c) in Proposition 1.10, having one of its nodes in $\hat{o}$. It is clear that the two singular points of $\hat{X}$ lie precisely over the two nodes of Δ_{red}. In particular $\hat{X}$ has only negligible singularities, see Example 1.8. This construction is summarized in Fig. 2.

There $\pi : C_1 \times C_2 \to \hat{X}$ is induced by the the diagonal action of S_3 on $C_1 \times C_2$, while $\hat{f}\colon \hat{X} \to \hat{A}$ is the Stein factorization of the Albanese map $\alpha\colon S \to \hat{A}$. Since the diagonal subgroup is not normal in $S_3 \times S_3$, it follows that $\hat{f}$ is not a Galois cover; let $h\colon Z \to \hat{A}$ be its Galois closure, which has Galois group S_3. The surface Z is isomorphic to the diagonal quotient $(C_1 \times C_2)/(\mathbb{Z}/3\mathbb{Z})$, where $\mathbb{Z}/3\mathbb{Z}$ is the subgroup of S_3 generated by the 3-cycles; therefore Z has four singular points coming from the four points with non-trivial stabilizer on $C_1 \times C_2$. More precisely,

$$\mathrm{Sing}(Z) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2).$$

In addition, the cover $C_i \to E_i$ factors through the cover $C_i \to E_i' := C_i/(\mathbb{Z}/3\mathbb{Z})$, where E_i' is an elliptic curve isogenous to E_i; this induces the cover $C_1 \times C_2 \to (C_1 \times C_2)/(\mathbb{Z}/3\mathbb{Z})^2 = E_1' \times E_2'$, which clearly factors through Z. Observe that also the cover $\pi\colon C_1 \times C_2 \to \hat{X}$ factors through Z. Finally the composition $\epsilon \circ \gamma\colon E_1 \times E_2 \to E_1' \times E_2'$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover, which factors through $\hat{A}$. Using the commutativity of the diagrams in Fig. 2 and the theory of abelian covers developed in [31], one can check, looking at the building data of $\beta \colon Z \to E_1' \times E_2'$ and $\gamma \colon E_1' \times E_2' \to \hat{A}$, that the Tschirnhausen bundle $\mathcal{E}$ of $f \colon \hat{X} \to \hat{A}$ satisfies $\bigwedge^2 \mathcal{E}^\vee = \mathcal{L} \otimes \mathcal{Q}$, where $\mathcal{Q}$ is a non-trivial, 2-torsion line bundle. This is a particular case of a more general situation, see Proposition 5.8.

4. Chen–Hacon surfaces

In this section we will generalize the triple cover construction described in Subsection 3.1. In fact, since we want to be able to "take the limit" of a 1-parameter family of surfaces obtained in that way, we shall drop the assumptions that $\mathcal{L}$ is a general polarization and that s and t are general complex numbers. Among other results, we will show that the product-quotient surface described in Subsection 3.2 can be also obtained as a specialization of Chen–Hacon's example, see Corollary 5.6.

Let us start with the following

DEFINITION 4.1. Let S be a minimal surface of general type with $p_g = q = 2$ such that its Albanese map $\alpha \colon S \to \hat{A} := \mathrm{Alb}(S)$ is a generically finite morphism of degree 3 onto an abelian surface $\hat{A}$. Let

$$S \xrightarrow{p} \hat{X} \xrightarrow{\hat{f}} \hat{A} \tag{33}$$

be the Stein factorization of α, and $\mathcal{F}^\vee$ be the Tschirnhausen bundle associated with the triple cover $\hat{f}$. We say that S a *Chen–Hacon surface* if there exist a polarization $\mathcal{L}$ of type (1, 2) on $A = \mathrm{Pic}^0(\hat{A})$ such that $\mathcal{F} = \widehat{\mathcal{L}^{-1}}$.

REMARK 4.2. Since $\hat{A}$ is an abelian variety and $\hat{f}$ is a finite map, it follows that p contracts all rational curves in S. The surface S is the minimal resolution of singularities of $\hat{X}$ but it is, in general, different from the canonical resolution $\tilde{X}$ described in Proposition 1.4. For instance, in Example 1.7 the surface $\tilde{X}$ contains a (-1)-curve.

The line bundle $\mathcal{L}$ is a IT-sheaf of index 0, so by [6, Theorem 14.2.2] and [6, Proposition 14.4.3] we have

$$h^0(\hat{A}, \mathcal{F}) = 1, \quad h^1(\hat{A}, \mathcal{F}) = 0, \quad h^2(\hat{A}, \mathcal{F}) = 0, \quad \det \mathcal{F} = \mathcal{L}_\delta, \tag{34}$$

where $\mathcal{L}_\delta := \mathcal{O}_{\hat{A}}(L_\delta)$ is the dual polarization of $\mathcal{L}$. Therefore $\mathcal{F}$ belongs to the family of bundles studied in Section 2.

Proposition 4.3. *Let S be a Chen–Hacon surface. Then $\mathcal{F}$ is indecomposable.*

Proof. Since $\mathcal{L}$ is a non-degenerate line bundle, by [6, Corollary 14.3.10] it follows that $\mathcal{F}$ is $\mathcal{H}$-semistable with respect to any polarization $\mathcal{H}$. Now the claim follows from

Proposition 2.1. Alternatively, one could also remark that since $\mathcal{L}^{-1}$ is indecomposable the same must be true for its Fourier–Mukai transform $\mathcal{F}$. □

Proposition 4.4. *Let S be a Chen–Hacon surface. Then $\mathcal{L}$ is not a product polarization.*

Proof. $\mathcal{L}$ is a product polarization if and only if $\mathcal{L}_\delta$ is a product polarization. If $\mathcal{L}_\delta$ were of product type, then $\hat{X}$ would not be a surface of general type (see Corollaries 2.8 and 2.11), contradiction. □

Since $\mathcal{L}$ is not a product polarization, we may use the results of Subsection 1.2. Moreover, for any Chen–Hacon surface S we can consider its associated diagram (28). Being the morphism ψ étale, X is nonsingular in codimension one if and only if the same holds for $\hat{X}$. Similarly, f is totally ramified if and only if $\hat{f}$ is totally ramified.

Proposition 4.5. *The following holds*:
(i) *X has only isolated singularities unless $t = 0$ or $t^2 - 9s^2 = 0$.*
(ii) *If $t = 0$ or $t^2 - 9s^2 = 0$, then X has non-isolated singularities. Moreover, if $\nu\colon X^\nu \to X$ is the normalization map, then the composition $f \circ \nu\colon X^\nu \to A$ is an étale triple cover. Therefore, in this case X is not a surface of general type.*

Proof. (i) A local computation as in [16, Claim 2] shows that, if $t \neq 0$ and $t^2 \neq 9s^2$, above a neighborhood of any of the base points of $|L|$ the equations (31) define a cone over a twisted cubic, hence an isolated singularity of type $(1/3)(1, 1)$.

(ii) We can assume $t = 0$, since the proof in the other cases is the same. Looking at (31), we see that in a neighborhood of any of the base points e_0, e_1, e_2, e_3, the surface X is defined in $\mathbb{A}^4$ by

$$(x + z)(2x - z) = 0, \quad (2y + w)(y - w) = 0, \quad (x + z)(y - w) = 0,$$

and it is straightforward to see that these equations define the union of three 2-planes intersecting along two lines. This shows that X contains non-isolated singularities. The normalization map $\nu\colon X^\nu \to X$ can be computed by using the computer algebra system `Singular`, see [39]. It turns out that X^ν is locally given by three mutually disjoint 2-planes in $\mathbb{A}^5$; moreover, for each of these planes the projection onto the first two coordinates of $\mathbb{A}^5$ is an isomorphism. In the global picture this means that X^ν is smooth and $f \circ \nu\colon X^\nu \to A$ is an étale triple cover. □

REMARK 4.6. In Proposition 5.11 we will show that if $t = 0$ or $t^2 - 9s^2 = 0$ then $\hat{X}$ (and hence X) is a reducible surface.

Proposition 4.7. *Assume that X has only isolated singularities. Then the following holds*:

(i) *$f\colon X \to A$ is totally ramified if and only if $s = t$, $s = -t$ or $s = 0$.*

(ii) *$f\colon X \to A$ is totally ramified if and only if*

(iia) *either $D = 2D_1 + 2D_2$, where $D_1, D_2 \in |L|$ are distinct, smooth hyperelliptic curves belonging to the same $K(\mathcal{L})$-orbit, or*

(iib) *$\mathcal{L}$ is a special polarization and $D = 2(E + F) + 2(E' + F')$, where $E + F$ and $E' + F'$ are as in* Proposition 1.15 (b).

Proof. (i) The triple cover $f\colon X \to A$ is totally ramified if and only if the discriminant of the polynomial defining D in (32) vanishes. This happens exactly for $s = 0$, $t = 0$, $s = t$, $s = -t$, $t = 3s$, $t = -3s$. Since we are assuming that X has isolated singularities, the only acceptable values are $s = t$, $s = -t$ and $s = 0$ (see Proposition 4.5).

(ii) The triple cover $f\colon X \to A$ is totally ramified if and only if $D = 2D'$ for some effective divisor D'. Since the four curves D_i form an orbit for the action of $K(\mathcal{L})$ on $|L|$, this is equivalent to say that the D_i have non-trivial stabilizer. Now the assertion follows from Proposition 1.18. □

Proposition 4.8. *Assume that $\hat{X}$ has only isolated singularities. Then $\hat{X}$ always contains a singular point of type* (1/3)(1, 1), *lying over $\hat{o} \in \hat{A}$. Moreover, this point is the unique singular point of $\hat{X}$, unless*:

(i) *one of the D_i is an irreducible, nodal curve*; *in this case $\hat{X}$ also contains a singular point of type* (1/2)(1, 1);

(ii) *$\mathcal{L}$ is a special polarization and we are in case* (iib) *of* Proposition 4.7. *Then $\hat{f}\colon \hat{X} \to \hat{A}$ is totally ramified over the image in $\hat{A}$ of the divisor $E + F + E' + F'$, which is a curve isomorphic to $E + F$ and having a node at $\hat{o}$. In this case $\hat{X}$ also contains a singular point of type* (1/3)(1, 2).

Proof. Since there exists an étale morphism $\psi\colon X \to \hat{X}$, it is sufficient to analyze the triple cover $f\colon X \to A$. If all divisors D_i are smooth, then the only singularities of X are the four points of type (1/3)(1, 1) lying over the base points of $|L|$. If one of the D_i is an irreducible, nodal curve, then all the D_i are so, because they form a single $K(\mathcal{L})$-orbit, see Proposition 1.15 and Remark 1.16. In this case X also contains four points of type (1/2)(1, 1), which are identified by ψ to a unique point of type (1/2)(1, 1) in $\hat{X}$; this yields (i). Finally, if $\mathcal{L}$ is a special polarization and $D = 2(E + F) + 2(E' + F')$, then locally around any of the four points p, q, r, s the equation of X can be written as $z^3 = xy$, so they give singularities of type $(1/3)(1, 2)$. The morphism ψ identifies E with E' and F with F'. Then $\hat{f}\colon \hat{X} \to \hat{A}$ is totally ramified and its reduced branch locus is isomorphic to $E + F$, in particular it has two nodes. One of these nodes is at $\hat{o}$ and it gives the singular point of type (1/3)(1, 1); the second one gives instead a singular point of type (1/3)(1, 2). This is case (ii). □

In the sequel we will denote by Δ the branch locus of $\hat{f}\colon \hat{X} \to \hat{A}$. By construction, it is precisely the image of D via $\phi\colon A \to \hat{A}$. It follows that Δ always has a point of multiplicity 4 at $\hat{o} \in \hat{A}$. More precisely, we have the following

Proposition 4.9. *The branch locus Δ belongs precisely to one of the following types*:
(a) Δ *is reduced and its only singularity is an ordinary quadruple point at $\hat{o}$*; *in this case* $\mathrm{Sing}(\hat{X}) = (1/3)(1, 1)$.
(b) Δ *is reduced and its only singularities are an ordinary quadruple point at $\hat{o}$ and an ordinary double point*; *in this case* $\mathrm{Sing}(\hat{X}) = (1/3)(1, 1) + (1/2)(1, 1)$.
(c) $\Delta = 2\Delta_{\mathrm{red}}$, *where Δ_{red} is an irreducible curve whose unique singularity is an ordinary double point at $\hat{o}$*; *in this case* $\mathrm{Sing}(\hat{X}) = (1/3)(1, 1)$.
(d) $\Delta = 2\Delta_{\mathrm{red}}$ *and $\Delta_{\mathrm{red}} = E + F$, where E, F are elliptic curves such that $EF = 2$ and $\hat{o} \in E \cap F$*; *in this case* $\mathrm{Sing}(\hat{X}) = (1/3)(1, 1) + (1/3)(1, 2)$.
The canonical divisor K_S is ample if and only if we are either in case (a) *or in case* (c).

Proof. Case (a) corresponds to the general situation. Case (b) corresponds to Proposition 4.8, (i). Case (c) corresponds to Proposition 4.7, (iia). Finally, Case (d) corresponds to Proposition 4.8, (ii) or, equivalently, to Proposition 4.7, (iib). □

REMARK 4.10. The equation of Δ is given by a non-zero element in $H^0(\hat{A}, \mathcal{L}_\delta^2 \otimes \mathcal{I}_{\hat{o}}^4)$, where $\mathcal{L}_\delta$ is a (1, 2)-polarization on $\hat{A}$ which coincides, up to translations, with the dual polarization of $\mathcal{L}$, see [6, Chapter 14] (we cannot denote the dual polarization by $\widehat{\mathcal{L}}$, since this is the Fourier–Mukai transform of $\mathcal{L}$). Notice that the four cases in Proposition 4.9 correspond exactly to the ones in Proposition 1.22.

Summarizing the results obtained in this section, we have

Proposition 4.11. *If S is a Chen–Hacon surface, then it is a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 5$. Moreover $\hat{X}$ contains at least one and at most two isolated, negligible singularities, which belong to the the types described in* Examples 1.6, 1.7, 1.8. *In particular, $\hat{X}$ is never smooth.*

5. Characterization of Chen–Hacon surfaces

In this section we prove one of the key results of the paper, namely the following converse of Proposition 4.11.

Theorem 5.1. *Let S be a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 5$ such that the Albanese map $\alpha\colon S \to \hat{A} := \mathrm{Alb}(S)$ is a generically finite morphism of degree 3. Let*

$$S \xrightarrow{p} \hat{X} \xrightarrow{\hat{f}} \hat{A}$$

be the Stein factorization of α. If $\hat{X}$ has at most negligible singularities, then S is a Chen–Hacon surface.

The proof will be a consequence of Propositions 5.2 and 5.4 below. Let $\mathcal{E}$ be the Tschirnhausen bundle of the triple cover $\hat{f}\colon \hat{X} \to \hat{A}$. Since by assumption $\hat{X}$ has at most negligible singularities, Proposition 1.2 implies

$$\begin{aligned} &h^0(\hat{A}, \mathcal{E}) = 0, \quad h^1(\hat{A}, \mathcal{E}) = 0, \quad h^2(\hat{A}, \mathcal{E}) = 1; \\ &c_1^2(\mathcal{E}) = 4, \quad c_2(\mathcal{E}) = 1. \end{aligned} \tag{35}$$

In particular, $\bigwedge^2 \mathcal{E}^\vee$ yields a polarization of type $(1, 2)$ on $\hat{A}$; let us denote it by $\mathcal{L}_\delta = \mathcal{O}_A(L_\delta)$. Setting $\mathcal{F} := \mathcal{E}^\vee$, we have

$$h^0(\hat{A}, \mathcal{F}) = 1, \quad h^1(\hat{A}, \mathcal{F}) = 0, \quad h^2(\hat{A}, \mathcal{F}) = 0, \quad \det \mathcal{F} = \mathcal{L}_\delta,$$

that is $\mathcal{F}$ belongs to the family of vector bundles studied in Section 2.

Proposition 5.2. *$\mathcal{F}$ is an indecomposable vector bundle.*

Proof. Assume that $\mathcal{F}$ is decomposable. Then there exists a line bundle $\mathcal{C} = \mathcal{O}_{\hat{A}}(C)$ such that

$$\mathcal{F} = \mathcal{C} \oplus (\mathcal{C}^{-1} \otimes \mathcal{L}_\delta).$$

Following [27, Section 6], we can construct $\hat{f}\colon \hat{X} \to \hat{A}$ by using the data

$$\begin{aligned} &a \in H^0(\hat{A}, \mathcal{C}), \\ &b \in H^0(\hat{A}, \mathcal{C}^3 \otimes \mathcal{L}_\delta^{-1}), \\ &c \in H^0(\hat{A}, \mathcal{C}^{-3} \otimes \mathcal{L}_\delta^2), \\ &d \in H^0(\hat{A}, \mathcal{C}^{-1} \otimes \mathcal{L}_\delta). \end{aligned}$$

Moreover, being $\hat{X}$ irreducible, b and c are both non-zero.

Since $h^0(\hat{A}, \mathcal{F}) = 1$, we may assume $h^0(\hat{A}, \mathcal{C}) = 1$ and $h^0(\hat{A}, \mathcal{C}^{-1} \otimes \mathcal{L}_\delta) = 0$. Therefore there are two possibilities:

(i) C is an elliptic curve;

(ii) C is a principal polarization.

In case (i), we have $1 = C(L_\delta - C) = CL_\delta$. Then $(3C - L_\delta)L_\delta = -1$, so $3C - L_\delta$ cannot be effective. This implies $b = 0$, contradiction.

In case (ii), the index theorem yields $8 = C^2 L_\delta^2 \leq (CL_\delta)^2$, so $CL_\delta \geq 3$. It follows

$$(-3C + 2L_\delta)L_\delta = -3CL_\delta + 8 \leq -1,$$

hence $c = 0$, contradiction. □

Proposition 5.3. *$\mathcal{L}_\delta$ is not a product polarization.*

Proof. By the results of Section 2, especially Corollaries 2.8 and 2.11, if $\mathcal{L}_\delta$ were a product polarization then $\hat{X}$ would be either a reducible surface or a non-normal surface birational to a properly elliptic surface, in particular it would not be a surface of general type. □

Proposition 5.4. *There exists a symmetric* $(1, 2)$*-polarization $\mathcal{L}$ on A such that*

$$\widehat{\mathcal{L}^{-1}} = \mathcal{F}.$$

Proof. Since $\mathcal{L}_\delta$ is not a product polarization (Proposition 5.3), it follows that $\mathcal{F}$ is the unique non-trivial extension

$$0 \to \mathcal{O}_{\hat{A}} \to \mathcal{F} \to \mathcal{L}_\delta \otimes \mathcal{I}_{\hat{o}} \to 0, \tag{36}$$

see Proposition 2.2. Moreover, $(-1)^*_{\hat{A}}\mathcal{F} = \mathcal{F}$ and $\mathcal{F}$ satisfies IT of index 0 (Proposition 2.4). Thus $\hat{\mathcal{F}}$ is a line bundle on A that we denote by $\mathcal{L}^{-1}$; the sheaf $\mathcal{L}$ satisfies IT of index 0 too, see [6, Theorem 14.2.2]. Therefore by [28] we get

$$\widehat{\mathcal{L}^{-1}} = \widehat{(\hat{\mathcal{F}})} = (-1)^*_{\hat{A}}\mathcal{F} = \mathcal{F}.$$

Since $h^0(A, \mathcal{L}) = \text{rank}(\mathcal{F}) = 2$, it follows that $\mathcal{L}$ is a $(1, 2)$-polarization. Notice that $\mathcal{L}$ coincides with the dual polarization of $\mathcal{L}_\delta$, in particular it is not a product polarization (see also Remark 4.10). □

This completes the proof of Theorem 5.1.

REMARK 5.5. It is interesting to compare Proposition 2.5 with Proposition 1.24. In fact, an explicit isomorphism $H^0(\hat{A}, \mathcal{L}_\delta^2 \otimes \mathcal{I}_{\hat{o}}^3) \xrightarrow{\cong} H^0(\hat{A}, \mathcal{L}_\delta^2 \otimes \mathcal{I}_{\hat{o}}^4)$ can be given by associating to every section $\eta \in H^0(\hat{A}, \mathcal{L}_\delta^2 \otimes \mathcal{I}_{\hat{o}}^3) \cong H^0\big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big)$ the equation defining the branch locus Δ of the triple cover given by η, see again Remark 4.10.

An immediate consequence of Theorem 5.1 is

Corollary 5.6. *The isotrivially fibred surface constructed in* [32], *i.e., the product-quotient surface of* Subsection 3.2, *is a Chen–Hacon surface. More precisely, it corresponds to case* (ii) *of* Proposition 4.8 *or, equivalently, to case* (d) *of* Proposition 4.9.

Proof. The product-quotient surface contains only negligible singularities, see Example 1.8, so Theorem 5.1 implies that it is a Chen–Hacon surface. Since $\hat{X}$ has one singularity of type $(1/3)(1, 1)$ and one singularity of type $(1/3)(1, 2)$, looking at Proposition 4.8 we see that it corresponds to case (ii). □

The remainder of this section deals with some further properties of Chen–Hacon surfaces.

Proposition 5.7. *Let S be a Chen–Hacon surface. Then $\alpha\colon S \to \hat{A}$ is never a finite morphism.*

Proof. By Proposition 4.9, S always contains a (-3)-curve, which is contracted by α. □

Proposition 5.8. *Let S be a Chen–Hacon surface, and assume that $\hat{f}\colon \hat{X} \to \hat{A}$ is totally ramified. Then Δ_{red} is linearly equivalent to $L_\delta + Q$, where Q is a non-trivial, 2-torsion divisor.*

Proof. By [27, Proposition 4.7] the divisor $\Delta = 2\Delta_{\mathrm{red}}$ is linearly equivalent to $2L_\delta$, hence Δ_{red} is linearly equivalent to $L_\delta + Q$, where Q is a 2-torsion divisor. On the other hand, Δ_{red} is singular at $\hat{o}$ (Proposition 4.9), so Q is not trivial. □

Proposition 5.9. *Let S be a Chen–Hacon surface. Then $\hat{f}\colon \hat{X} \to \hat{A}$ is never a Galois cover.*

Proof. By [43, Theorem 5.5] it follows that $\hat{f}$ is a Galois cover if and only if it is totally ramified and the line bundle $\bigwedge^2 \mathcal{F}$ is isomorphic to $\mathcal{O}_{\hat{A}}(\Delta_{\mathrm{red}})$. Since $\bigwedge^2 \mathcal{F} = \mathcal{L}_\delta$, this is excluded by Proposition 5.8. □

Proposition 5.10. *Let S be a Chen–Hacon surface, and assume that A is a simple abelian surface. Then S does not contain any pencil $p\colon S \to B$ over a curve B with $g(B) \geq 1$.*

Proof. Since A is simple, the same is true for $\hat{A}$. Then the set $V^1(S) := \{\mathcal{Q} \in \mathrm{Pic}^0(S) \mid h^1(S, \mathcal{Q}^\vee) > 0\}$ cannot contain any component of positive dimension, and S does not admit any pencil over a curve B with $g(B) \geq 2$, see [22, Theorem 2.6]. If instead $g(B) = 1$, the universal property of the Albanese map yields a surjective morphism $\hat{A} \to B$, contradicting again the fact that $\hat{A}$ is simple. This concludes the proof. $\square$

It would be very interesting to classify the possible degenerations of Chen–Hacon surfaces; however, this problem is at present far from being solved. The following result describes some natural degenerations obtained by taking reducible triple covers.

Proposition 5.11. *Let $\hat{f}\colon \hat{X} \to \hat{A}$ be the non-normal triple cover corresponding to either $t = 0$ or $t^2 = 9s^2$ (see* Proposition 4.5*). Then $\hat{X}$ is a reducible surface. More precisely, there exists $i \in \{1, 2, 3\}$ such that the section defining $\hat{f}$ is in the image of the multiplication map*

$$H^0\left(\hat{A}, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i\right) \otimes H^0(\hat{A}, \mathcal{F} \otimes \mathcal{Q}_i) \to H^0\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right),$$

where the $\mathcal{Q}_i$ are the non-trivial, 2-torsion line bundles on $\hat{A}$ defined as in (4).

Proof. It is sufficient to show that

$$h^0\left(\hat{A}, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i\right) \neq 0 \quad \text{and} \quad h^0(\hat{A}, \mathcal{F} \otimes \mathcal{Q}_i) \neq 0$$

for $i = 1, 2, 3$. Tensoring (14) with $\mathcal{Q}_i$ and using (5) we obtain

$$h^0(\hat{A}, \mathcal{F} \otimes \mathcal{Q}_i) = h^0(\hat{A}, \mathcal{L}_\delta \otimes \mathcal{Q}_i \otimes \mathcal{I}_{\hat{o}}) = 1.$$

On the other hand, Eagon–Northcott complex applied to (36) gives

$$0 \to \mathcal{F} \to S^2\mathcal{F} \to \mathcal{L}_\delta^2 \otimes \mathcal{I}_{\hat{o}}^2 \to 0,$$

hence we obtain

$$0 \to \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i \to S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i \to \mathcal{L}_\delta \otimes \mathcal{Q}_i \otimes \mathcal{I}_{\hat{o}}^2 \to 0. \tag{37}$$

Using $\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee = \mathcal{F}^\vee$, Serre duality and (15) we deduce

$$h^0\left(\hat{A}, \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i\right) = h^0(\hat{A}, \mathcal{F}^\vee \otimes \mathcal{Q}_i) = h^2(\hat{A}, \mathcal{F} \otimes \mathcal{Q}_i) = 0,$$

$$h^1\left(\hat{A}, \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i\right) = h^1(\hat{A}, \mathcal{F}^\vee \otimes \mathcal{Q}_i) = h^1(\hat{A}, \mathcal{F} \otimes \mathcal{Q}_i) = 0,$$

so by (5) we have

$$h^0\left(\hat{A},\, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}_i\right) = h^0(\hat{A},\, \mathcal{L}_\delta \otimes \mathcal{Q}_i \otimes \mathcal{I}_{\hat{o}}^2) = 1.$$

This completes the proof. □

REMARK 5.12. Further degenerations of Chen–Hacon surfaces could be obtained by looking at the case where $\mathcal{L}_\delta$ becomes a product polarization, see Corollaries 2.8 and 2.11.

We will now describe the canonical system $|K_S|$ of a Chen–Hacon surface S, showing that it is composed with a rational pencil of curves of genus 3.

For the sake of simplicity, we will assume that A is a simple abelian surface. Let $\alpha\colon S \to \hat{A}$ be the Albanese map of S, let $\sigma\colon \hat{A}^\sharp \to \hat{A}$ be the blow-up of $\hat{A}$ at $\hat{o}$, and let $\Lambda \subset \hat{A}$ be the exceptional divisor. Then there is an induced map $\beta\colon S \to \hat{A}^\sharp$, which is a flat triple cover. The branch locus of β coincides with the strict transform of the branch locus Δ of $\hat{f}$, so it belongs to the strict transform of the pencil $\mathfrak{D}_\delta \subset |2L_\delta|$ given by $\mathbb{P}H^0(\hat{A},\, \mathcal{L}_\delta^2 \otimes \mathcal{I}_{\hat{o}}^4)$. The general element in this pencil is a smooth curve of genus 3 and self-intersection 0, meeting Λ in precisely four distinct points; so we have a base-point free pencil $\hat{\varphi}\colon \hat{A}^\sharp \to \mathbb{P}^1$. The exceptional divisor Λ is not in the branch locus of β and $\Xi := \beta^*(\Lambda)$ is the unique (-3)-curve in S. Considering the Stein factorization of the composed map $S \xrightarrow{\beta} \hat{A}^\sharp \xrightarrow{\hat{\varphi}} \mathbb{P}^1$, and using Proposition 5.10, we obtain a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[d, "\varphi"'] & \hat{A}^\sharp \arrow[d, "\hat{\varphi}"] \\ \mathbb{P}^1 \arrow[r, "b"] & \mathbb{P}^1, \end{tikzcd} \tag{38}$$

where $b\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a triple cover simply branched on four points, corresponding to the branch curve of β and to the three double curves in $\mathfrak{D}_\delta$, see Proposition 1.22.

Proposition 5.13. *Let S be a Chen–Hacon surface. Then $|K_S| = \Xi + |\Phi|$, where Φ is a smooth curve of genus 3 which satisfies $h^0(S, \mathcal{O}_S(\Phi)) = 2$, $\Phi^2 = 0$, $\Xi\Phi = 4$. It follows that $\varphi\colon S \to \mathbb{P}^1$ coincides with the canonical map $\varphi_{|K_S|}$ of S.*

Proof. The canonical divisor of S is given by

$$K_S = \beta^* K_{\hat{A}^\sharp} + R = \Xi + R,$$

where R is the ramification divisor of β. By diagram (38) it follows that $R \in |\Phi|$, where $|\Phi|$ is the pencil induced by φ. The general element of $|\Phi|$ is a smooth curve of genus 3, isomorphic to the strict transform of the general element of $\mathfrak{D}_\delta$. Since

$$2 = h^0(S, \mathcal{O}_S(K_S)) = h^0(S, \mathcal{O}_S(\Phi)) = h^0(S, \mathcal{O}_S(K_S - \Xi)),$$

it follows that Ξ is contained in the fixed part of $|K_S|$. The rest of the proof is now clear. $\square$

6. The moduli space

The aim of this section is to investigate the deformations of Chen–Hacon surfaces. The first step is to embed S in the projective bundle $\mathbb{P}(\mathcal{F})$ as a divisor *containing a fibre*.

Proposition 6.1. *Let S be a Chen–Hacon surface with ample canonical class; then there is an embedding $i\colon S \hookrightarrow \mathbb{P}(\mathcal{F})$ whose image is a smooth divisor in the linear system $|3\xi - \pi^* L_\delta|$, where ξ is the divisor class of $\mathcal{O}_{\mathbb{P}}(1)$ and $\pi : \mathbb{P} \to \hat{A}$ is the natural projection. Moreover $i(S)$ contains the fibre $\pi^{-1}(\hat{o})$ of $\mathbb{P}$; more precisely, $\pi^{-1}(\hat{o})$ coincides with the unique (-3)-curve Ξ of S.*

Proof. Let us consider again the blow-up $\sigma : \hat{A}^\sharp \to \hat{A}$, with exceptional divisor $\Lambda \subset \hat{A}^\sharp$, and the flat triple cover $\beta\colon S \to \hat{A}^\sharp$ described in the previous section. A straightforward calculation shows that the Tschirnhausen bundle associated to β is

$$\mathcal{F}^\sharp = \sigma^*\mathcal{F} \otimes \mathcal{O}_{\hat{A}^\sharp}(-\Lambda)$$

and that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{F}^\sharp) & \xrightarrow{\tau} & \mathbb{P}(\mathcal{F}) \\ {\scriptstyle \pi^\sharp}\downarrow & & \downarrow{\scriptstyle \pi} \\ \hat{A}^\sharp & \xrightarrow{\sigma} & \hat{A}. \end{array} \tag{39}$$

Since S is smooth and β is flat, by [15] there is an embedding $i^\sharp\colon S \hookrightarrow \mathbb{P}(\mathcal{F}^\sharp)$. Its image is a divisor in the linear system $|3\xi^\sharp - (\pi^\sharp)^* \det \mathcal{F}^\sharp|$, where $\xi^\sharp$ is the divisor class of $\mathcal{O}_{\mathbb{P}(\mathcal{F}^\sharp)}(1)$ and $\pi^\sharp\colon \mathbb{P}(\mathcal{F}^\sharp) \to \hat{A}^\sharp$ is the projection. We have natural identifications

$$\begin{aligned} H^0(\mathbb{P}(\mathcal{F}^\sharp), 3\xi^\sharp - (\pi^\sharp)^* \det \mathcal{F}^\sharp) &\cong H^0\left(\hat{A}^\sharp, S^3\mathcal{F}^\sharp \otimes \bigwedge^2 (\mathcal{F}^\sharp)^\vee\right) \\ &\cong H^0\left(\hat{A}^\sharp, \sigma^*\left(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) \otimes \mathcal{O}_{\hat{A}^\sharp}(-\Lambda)\right) \end{aligned}$$

$$\cong H^0\left(\hat{A}, \sigma_*\left(\sigma^*\left(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) \otimes \mathcal{O}_{\hat{A}^\sharp}(-\Lambda)\right)\right)$$
$$\cong H^0\left(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{I}_{\hat{o}}\right),$$

hence $i = \tau \circ i^\sharp \colon S \hookrightarrow \mathbb{P}(\mathcal{F})$ provides an embedding of S as a divisor in the linear system $|3\xi - \pi^* L_\delta|$ containing the fibre $\pi^{-1}(\hat{o})$. Such a fibre must coincide with Ξ, because Ξ is the unique rational curve in S. $\square$

Given the embedding $i \colon S \hookrightarrow \mathbb{P}$, the Albanese map $\alpha \colon S \to \hat{A}$ of S factors as $\alpha = \pi \circ i$, as in the following diagram:

(40)
$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "\alpha"'] & \mathbb{P} \arrow[d, "\pi"] \\ & \hat{A}. \end{tikzcd}$$

Since $K_\mathbb{P} = -2\xi + \pi^* L_\delta$, the adjunction formula implies that the canonical line bundle of S is the restriction of $\mathcal{O}_\mathbb{P}(1)$ to S, that is

(41)
$$\omega_S = \mathcal{O}_S(\xi).$$

In the sequel we shall exploit the following short exact sequences:

- the normal bundle sequence of $i \colon S \hookrightarrow \mathbb{P}$, i.e.,

(42)
$$0 \to \mathcal{O}_\mathbb{P} \to \mathcal{O}_\mathbb{P}(S) \to N_{S/\mathbb{P}} \to 0;$$

- the tangent bundle sequence of $i \colon S \hookrightarrow \mathbb{P}$, i.e.,

(43)
$$0 \to T_S \to T_\mathbb{P} \otimes \mathcal{O}_S \to N_{S/\mathbb{P}} \to 0;$$

- the tangent bundle sequence of $\pi \colon \mathbb{P} \to \hat{A}$, i.e.

(44)
$$0 \to T_{\mathbb{P}/\hat{A}} \to T_\mathbb{P} \to \pi^* T_{\hat{A}} \to 0.$$

Recalling that $S \in |3\xi - \pi^* L_\delta|$, we get

$$\pi_*\mathcal{O}_\mathbb{P} = \mathcal{O}_{\hat{A}}, \quad R^1\pi_*\mathcal{O}_\mathbb{P} = 0, \quad \pi_*\mathcal{O}_\mathbb{P}(S) = S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee, \quad R^1\pi_*\mathcal{O}_\mathbb{P}(S) = 0,$$

so by the Leray spectral sequence we obtain

$$H^i(\mathbb{P}, \mathcal{O}_\mathbb{P}) = H^i(\hat{A}, \mathcal{O}_{\hat{A}}), \quad H^i(\mathbb{P}, \mathcal{O}_\mathbb{P}(S)) = H^i(\hat{A}, S^3\mathcal{F} \otimes \Lambda^2\mathcal{F}^\vee), \quad i \geq 0.$$

Hence, considering the long exact sequence associated with (42) and using Proposition 3.1, we deduce

$$h^0(S, N_{S/\mathbb{P}}) = 3, \quad h^1(S, N_{S/\mathbb{P}}) = 1, \quad h^2(S, N_{S/\mathbb{P}}) = 0. \tag{45}$$

Let us denote by $H_S^{\mathbb{P}}$ the complex analytic space representing the functor of embedded deformations of S inside $\mathbb{P}$ (with $\mathbb{P}$ fixed), see [38, p. 123]. An immediate consequence of (45) is

Proposition 6.2. *$H_S^{\mathbb{P}}$ is generically smooth, of dimension* 3*.*

Proof. Since $h^0(S, N_{S/\mathbb{P}}) = 3$, the dimension of the tangent space of $H_S^{\mathbb{P}}$ at the point corresponding to S is 3. On the other hand, the family of embedded deformations of S in $\mathbb{P}$ is at least 3-dimensional: indeed, we can move S into the 1-dimensional linear system $|\mathcal{O}_{\mathbb{P}}(S)|$ and we can translate it by using the 2-dimensional family of translations of $\hat{A}$. Therefore $H_S^{\mathbb{P}}$ is smooth at S, hence generically smooth of dimension 3. In particular, the obstructions in $H^1(S, N_{S/\mathbb{P}}) = \mathbb{C}$ actually vanish. □

Now let us consider the long cohomology sequence associated with (43). Since S is a surface of general type, we have $H^0(S, T_S) = 0$ and we get

$$\begin{aligned} &0 \to H^0(S, T_{\mathbb{P}} \otimes \mathcal{O}_S) \to H^0(S, N_{S/\mathbb{P}}) \xrightarrow{\delta^0} H^1(S, T_S) \\ &\quad \to H^1(S, T_{\mathbb{P}} \otimes \mathcal{O}_S) \to H^1(S, N_{S/\mathbb{P}}) \xrightarrow{\delta^1} H^2(S, T_S) \to H^2(S, T_{\mathbb{P}} \otimes \mathcal{O}_S) \to 0. \end{aligned} \tag{46}$$

By standard deformation theory, see for instance [38, p. 132], the map $\delta^0\colon H^0(S, N_{S/\mathbb{P}}) \to H^1(S, T_S)$ is precisely the map induced on tangent spaces by the "forgetful morphism" $H_S^{\mathbb{P}} \to \mathrm{Def}(S)$, where $\mathrm{Def}(S)$ is the base of the Kuranishi family of S.

Now we look at (44). Since $T_{\hat{A}}$ is trivial, we obtain

$$T_{\mathbb{P}/\hat{A}} = \mathcal{O}_{\mathbb{P}}(-K_{\mathbb{P}}) = \mathcal{O}_{\mathbb{P}}(2\xi - \pi^* L_\delta).$$

Then $R^1\pi_* T_{\mathbb{P}/\hat{A}} = 0$, and Leray spectral sequence together with (17) yields

$$H^i(\mathbb{P}, T_{\mathbb{P}/\hat{A}}) = H^i\left(\hat{A}, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0, \quad i \geq 0. \tag{47}$$

Therefore $\mathrm{Ext}^1(\pi^* T_{\hat{A}}, T_{\mathbb{P}/\hat{A}}) = H^1(\mathbb{P}, T_{\mathbb{P}/\hat{A}})^{\oplus 2} = 0$, so (44) actually splits and we have

$$T_{\mathbb{P}} = T_{\mathbb{P}/\hat{A}} \oplus \pi^* T_{\hat{A}} = \mathcal{O}_{\mathbb{P}}(2\xi - \pi^* L_\delta) \oplus \pi^* T_{\hat{A}}. \tag{48}$$

Since $N_{S/\mathbb{P}} = \mathcal{O}_S(3\xi - \pi^* L_\delta)$, by restricting (48) to S and using (41), we obtain

$$T_{\mathbb{P}} \otimes \mathcal{O}_S = (T_{\mathbb{P}/\hat{A}} \otimes \mathcal{O}_S) \oplus (\pi^* T_{\hat{A}} \otimes \mathcal{O}_S) = (N_{S/\mathbb{P}} \otimes \omega_S^{-1}) \oplus \alpha^* T_{\hat{A}}. \tag{49}$$

Let us now compute the cohomology groups of $N_{S/\mathbb{P}} \otimes \omega_S^{-1} = \mathcal{O}_S(2\xi - \pi^* L_\delta)$.

Lemma 6.3. *We have*

$$h^0(S, N_{S/\mathbb{P}} \otimes \omega_S^{-1}) = 0, \quad h^1(S, N_{S/\mathbb{P}} \otimes \omega_S^{-1}) = 0, \quad h^2(S, N_{S/\mathbb{P}} \otimes \omega_S^{-1}) = 0.$$

Proof. Let us consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-\xi) \to \mathcal{O}_{\mathbb{P}}(2\xi - \pi^* L_\delta) \to \mathcal{O}_S(2\xi - \pi^* L_\delta) \to 0.$$

By [20, p. 253] we have $\pi_* \mathcal{O}_{\mathbb{P}}(-\xi) = R^1 \pi_* \mathcal{O}_{\mathbb{P}}(-\xi) = 0$, so by Leray spectral sequence we deduce $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-\xi)) = H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-\xi)) = 0$. It follows

$$H^i(S, N_{S/\mathbb{P}} \otimes \omega_S^{-1}) = H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2\xi - \pi^* L_\delta)) = H^i\left(\hat{A}, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\right) = 0$$

for $i = 0, 1, 2$, see (17). □

By using (46), (49) and Lemma 6.3 we obtain the exact sequence

$$\begin{aligned} 0 \to H^0(S, \alpha^* T_{\hat{A}}) \to H^0(S, N_{S/\mathbb{P}}) &\xrightarrow{\delta^0} H^1(S, T_S) \\ \xrightarrow{\gamma} H^1(S, \alpha^* T_{\hat{A}}) \to H^1(S, N_{S/\mathbb{P}}) &\xrightarrow{\delta^1} H^2(S, T_S) \to H^2(S, \alpha^* T_{\hat{A}}) \to 0. \end{aligned} \tag{50}$$

The key remark is now contained in the following

Proposition 6.4. *The image of* $\gamma \colon H^1(S, T_S) \to H^1(S, \alpha^* T_{\hat{A}})$ *has dimension* 3.

Proof. Since $T_{\hat{A}}$ is trivial and there is a natural isomorphism

$$H^1(S, \mathcal{O}_S) \cong H^1(\hat{A}, \mathcal{O}_{\hat{A}}),$$

we can see the map γ as a map

$$\gamma \colon H^1(S, T_S) \to H^1(\hat{A}, T_{\hat{A}}).$$

Take a positive integer $m \geq 2$ such that there exists a smooth pluricanonical divisor $C \in |mK_S|$ and let C' be the image of C in $\hat{A}$; then we have a commutative diagram

$$\begin{tikzcd} H^1(S, T_S\langle C\rangle) \arrow[r, "\gamma'"] \arrow[d, "\epsilon"'] & H^1(\hat{A}, T_{\hat{A}}\langle C'\rangle) \arrow[d, "\epsilon'"] \\ H^1(S, T_S) \arrow[r, "\gamma"] & H^1(\hat{A}, T_{\hat{A}}). \end{tikzcd}$$

Here, following [38, Section 3.4.4 p. 177], for each closed subscheme $\mathcal{X}$ of a projective scheme $\mathcal{Y}$ we denote by $T_{\mathcal{Y}}\langle\mathcal{X}\rangle$ the sheaf of germs of tangent vectors to $\mathcal{Y}$ which are tangent to $\mathcal{X}$. When $\mathcal{Y}$ is smooth, the vector space $H^1(\mathcal{Y}, T_{\mathcal{Y}}\langle\mathcal{X}\rangle)$ parameterizes the first-order deformations of the pair $(\mathcal{Y}, \mathcal{X})$. Notice that $T_{\mathcal{Y}}\langle\mathcal{X}\rangle$ is usually denoted by $T_{\mathcal{Y}}(-\log\mathcal{X})$ when $\mathcal{X}$ is a normal crossing divisor with smooth components.

Let us observe now the following facts.

- Since S is smooth, the line bundle ω_S^m extends along any first-order deformation of S, because the relative dualizing sheaf is locally free for any smooth morphism of schemes, see [26, p. 182]. Moreover, since S is minimal of general type, we have $h^1(S, \omega_S^m) = 0$, so every section of ω_S^m extends as well, see [38, Section 3.3.4]. This means that no first-order deformation of S makes C disappear, in other words ϵ is surjective. Therefore $im\,\gamma \subseteq im\,\epsilon'$.
- Since $(C')^2 > 0$, the line bundle $\mathcal{O}_{\hat{A}}(C')$ is ample on $\hat{A}$; therefore it deforms along a subspace of $H^1(\hat{A}, T_{\hat{A}})$ of dimension 3, see [38, p. 152]. Since every first-order deformation of the pair $(\hat{A}, C')$ induces a first-order deformation of the pair $(\hat{A}, \mathcal{O}_{\hat{A}}(C'))$, it follows that the image of ϵ' is at most 3-dimensional.

According to the above remarks, we obtain

$$\dim(im\,\gamma) \leq \dim(im\,\epsilon') \leq 3.$$

On the other hand, given any abelian surface $\hat{A}$ with a $(1, 2)$-polarization which is not of product type we may construct a Chen–Hacon surface S such that $\mathrm{Alb}(S) = \hat{A}$. Hence the dimension of $im\,\gamma$ is exactly 3. □

Corollary 6.5. *We have*

$$h^1(S, T_S) = 4, \quad h^2(S, T_S) = 4.$$

Proof. By Riemann–Roch theorem we obtain $h^1(S, T_S) - h^2(S, T_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 = 0$. On the other hand, Proposition 6.4 together with (45) implies $h^1(S, T_S) = 4$, so we are done. □

Now let us denote by $\mathcal{M}$ the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K_S^2 = 5$ and by $\mathcal{M}^{\mathrm{CH}}$ the subset of $\mathcal{M}$ given by isomorphism classes of Chen–Hacon surfaces.

Theorem 6.6. *$\mathcal{M}^{\mathrm{CH}}$ is a connected, irreducible, generically smooth component of $\mathcal{M}$ of dimension* 4.

Proof. The construction of Chen–Hacon surfaces depends on four parameters: in fact, the moduli space $\mathcal{W}(1, 2)$ of $(1, 2)$-polarized abelian surfaces has dimension 3, whereas $\mathbb{P}H^0\big(\hat{A}, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee\big)$ is 1-dimensional (note that $\mathcal{F}$ does not give any contribution to the number of parameters because of (13)). This argument also shows that one has a generically finite, dominant map

$$\mathcal{P} \to \mathcal{M}^{\mathrm{CH}},$$

where $\mathcal{P}$ is a suitable $\mathbb{P}^1$-bundle over $\mathcal{W}(1,2)$. Therefore $\mathcal{M}^{\mathrm{CH}}$ is an irreducible, algebraic subset of $\mathcal{M}$ and $\dim \mathcal{M}^{\mathrm{CH}} = 4$. On the other hand, if K_S is ample Corollary 6.5 implies

$$\dim T_{[S]}\mathcal{M}^{\mathrm{CH}} = H^1(S, T_S) = 4,$$

so $\mathcal{M}^{\mathrm{CH}}$ is generically smooth.

It remains to show that $\mathcal{M}^{\mathrm{CH}}$ is a connected component of $\mathcal{M}$, i.e. that it is both open and closed therein.

$\mathcal{M}^{\mathrm{CH}}$ is open in $\mathcal{M}$.

Let $\mathcal{S} \xrightarrow{\pi} \mathcal{B}$ be a deformation over a small disk such that $S_0 := \pi^{-1}(0)$ is a Chen–Hacon surface. We want to show that the same is true for $S_t := \pi^{-1}(t)$. By Ehresmann's theorem, S_t is diffeomorphic to S_0, so it is a minimal surface of general type with $p_g = q = 2$, $K^2_{S_t} = 5$. Moreover, by [13, p. 267], the differentiable structure of the general fibre of the Albanese map of S_t is completely determined by the differentiable structure of S_t; it follows that the Albanese map $\alpha_t \colon S_t \to \mathrm{Alb}(S_t)$ is a generically finite triple cover. Let

$$S_t \xrightarrow{p_t} X_t \xrightarrow{f_t} \mathrm{Alb}(S_t)$$

be the Stein factorization of α_t, and let $\mathcal{E}_t$ be the Tschirnhausen bundle associated with the flat triple cover $f_t \colon X_t \to \mathrm{Alb}(S_t)$, that is

$$f_{t*}\mathcal{O}_{X_t} = \mathcal{O}_{\mathrm{Alb}(S_t)} \oplus \mathcal{E}_t. \tag{51}$$

By Proposition 4.9, X_0 has only rational singularities, so the same holds for X_t if $\mathcal{B}$ is small enough.

The branch locus Δ_t of α_t is a deformation of Δ_0, in particular $p_a(\Delta_t) = p_a(\Delta_0) = 9$; moreover, by [27, Proposition 4.7] the class of Δ_t must be 2-divisible in the Picard group of $\mathrm{Alb}(S_t)$. It follows that $\mathrm{Alb}(S_t)$ is a (1, 2)-polarized abelian surface and $\Delta_t \in |2L_t|$. Moreover $\bigwedge^2 \mathcal{E}_t^\vee$ is numerically equivalent to $\mathcal{L}_t$, in particular $c_1^2(\mathcal{E}_t) = 4$. Since f_t is a finite map and X_t has only rational singularities, we obtain

$$h^1(\mathrm{Alb}(S_t),\ f_{t*}\mathcal{O}_{X_t}) = h^1(X_t, \mathcal{O}_{X_t}) = h^1(S_t, \mathcal{O}_{S_t}) = 2,$$
$$h^2(\mathrm{Alb}(S_t),\ f_{t*}\mathcal{O}_{X_t}) = h^2(X_t, \mathcal{O}_{X_t}) = h^2(S_t, \mathcal{O}_{S_t}) = 2,$$

so by using (51) we deduce

$$h^0(\mathrm{Alb}(S_t), \mathcal{E}_t) = 0, \quad h^1(\mathrm{Alb}(S_t), \mathcal{E}_t) = 0, \quad h^2(\mathrm{Alb}(S_t), \mathcal{E}_t) = 1.$$

Now Hirzebruch–Riemann–Roch theorem yields

$$1 = \chi(\mathrm{Alb}(S_t), \mathcal{E}_t) = \frac{1}{2}c_1^2(\mathcal{E}_t) - c_2(\mathcal{E}_t),$$

hence $c_2(\mathcal{E}_t) = 1$. It follows that the invariant of S_t are computed by formulae in Proposition 1.2, in other words X_t contains only negligible singularities. By Theorem 5.1, S_t is a Chen–Hacon surface.

$\mathcal{M}^{\mathrm{CH}}$ is closed in $\mathcal{M}$.

Let $\mathcal{S} \xrightarrow{\pi} \mathcal{B}$ be a small deformation and assume that, for $t \neq 0$, S_t is a Chen–Hacon surface. We want to show that S_0 is a Chen–Hacon surface. Arguing as before, we see that $\alpha_0 \colon S_0 \to \mathrm{Alb}(S_0)$ is a generically finite triple cover, and that $\mathrm{Alb}(S_0)$ is a $(1, 2)$-polarized abelian surface. Let $\mathfrak{D}_t \subset |2L_t|$ be the linear system $\mathbb{P}H^0(\mathrm{Alb}(S_t), \mathcal{L}_t^2 \otimes \mathcal{I}_o^4)$. Since $\Delta_t \in \mathfrak{D}_t$ for all $t \neq 0$, we have $\Delta_0 \in \mathfrak{D}_0$. The possible curves in $|\mathfrak{D}_0|$ are classified in Proposition 1.22; in all cases the corresponding triple cover contains only negligible singularities (see Examples 1.6, 1.7, 1.8), so S_0 is a Chen–Hacon surface and we are done.

This concludes the proof of Theorem 6.6. $\square$

Theorem 6.6 shows that every small deformation of a Chen–Hacon surface is still a Chen–Hacon surface; in particular, no small deformation of S makes the (-3)-curve disappear. Moreover, since $\mathcal{M}^{\mathrm{CH}}$ is generically smooth, the same is true for the first-order deformations. By contrast, Burns and Wahl proved in [10] that first-order deformations always smooth all the (-2)-curves, and Catanese used this fact in [11] in order to produce examples of surfaces of general type with everywhere non-reduced moduli spaces. Theorem 6.6 demonstrates rather strikingly that the results of Burns–Wahl and Catanese cannot be extended to the case of (-3)-curves and, as far as we know, it also provides the first explicit example of this situation.

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