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ON RELATIVE HEIGHT ZERO BRAUER CHARACTERS

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Abstract

Let $N \lhd G$ where G is a finite group and let B be a p-block of G, where p is a prime. A Brauer character $\psi \in \mathrm{IBr}_p(B)$ is said to be of relative height zero with respect to N provided that the height of ψ is equal to that of an irreducible constituent of ψ_N. Now assume G is p-solvable. In this paper, we count the number of relative height zero irreducible Brauer characters of B with respect to N that lie over any given $\varphi \in \mathrm{IBr}_p(N)$. As a consequence, we show that if D is a defect group of B and $\hat{B}$ is the unique p-block of $NN_G(D)$ with defect group D such that $\hat{B}^G = B$, then B and $\hat{B}$ have equal numbers of relative height zero irreducible Brauer characters with respect to N.

1. Introduction

Fix a prime p and let N be a normal subgroup of a finite group G. Let B be a p-block of G and let $\psi \in \mathrm{IBr}(B)$, the set of irreducible Brauer characters belonging to B. Suppose θ is an irreducible constituent of ψ_N, and write b for the p-block of N to which θ belongs. As in [8], a defect group D of B is called an inertial defect group of B (with respect to b) if it is a defect group of the Fong–Reynolds correspondent of B in the inertial group T of b in G.

By [8, Lemma 3.2], we have $\mathrm{ht}(\psi) \geq \mathrm{ht}(\theta)$. If θ' is any other irreducible constituent of ψ_N, then θ' is G-conjugate to θ and belongs to a G-conjugate of b. Since G-conjugate p-blocks of N have equal defects, the difference $\mathrm{ht}(\psi) - \mathrm{ht}(\theta)$ does not depend on the choice of the constituent θ.

The Brauer character ψ is said to be of *relative height zero* with respect to N provided that $\mathrm{ht}(\psi) = \mathrm{ht}(\theta)$. We denote by $\mathrm{IBr}_N^0(B)$, the set of irreducible Brauer characters belonging to B and having relative height zero with respect to N. If $\mathrm{ht}(\psi) = 0$, then $\mathrm{ht}(\theta) = 0$ as $\mathrm{ht}(\psi) \geq \mathrm{ht}(\theta)$. Hence every irreducible Brauer character in B of height zero lies in $\mathrm{IBr}_N^0(B)$, and so in particular $\mathrm{IBr}_N^0(B) \neq \emptyset$. Furthermore when $N = 1$, then $\mathrm{IBr}_N^0(B)$ is precisely the set of irreducible Brauer characters in B of height zero. (See also Corollary 2.2 below.)

Let $\varphi \in \mathrm{IBr}(b)$. We write $\mathrm{IBr}_N^0(B \mid \varphi)$ for the set of all those Brauer characters in $\mathrm{IBr}_N^0(B)$ that lie over φ. Theorem 3.3 in [8] implies that $\mathrm{IBr}_N^0(B \mid \varphi) \neq \emptyset$ if and only if

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φ is D-invariant for some inertial defect group D of B. So in particular, $\mathrm{IBr}^0_N(B \mid \varphi) \neq \emptyset$ when φ is G-invariant.

Assume now that G is p-solvable. In [6], we counted the number of height zero irreducible Brauer characters in B that lie over φ. If φ is not of height zero, then in view of the inequality above, there are no height zero irreducible Brauer characters in B over φ. So, in effect, the main result of [6] counts the number of elements in $\mathrm{IBr}^0_N(B \mid \varphi)$ in case φ is of height zero. In this paper, we count the number of elements in $\mathrm{IBr}^0_N(B \mid \varphi)$ for any $\varphi \in \mathrm{IBr}(b)$.

Theorem A. *Let G be a p-solvable group and let B be a p-block of G. Suppose N is a normal subgroup of G, b is a p-block of N covered by B, T is the inertial group of b in G and $\varphi \in \mathrm{IBr}(b)$. If D is an inertial defect group of B with respect to b and $\hat{B}$ is the unique p-block of $NN_G(D)$ with defect group D such that $\hat{B}^G = B$, then*

$$|\mathrm{IBr}^0_N(B \mid \varphi)| = \left| \bigcup_{t \in T} \mathrm{IBr}^0_N(\hat{B} \mid \varphi^t) \right|.$$

As a consequence of this result, we have the following.

Theorem B. *Let G be a p-solvable group and let B be a p-block of G with defect group D. Suppose N is a normal subgroup of G and let $\hat{B}$ be the unique p-block of $NN_G(D)$ with defect group D such that $\hat{B}^G = B$. Then*

$$|\mathrm{IBr}^0_N(B)| = |\mathrm{IBr}^0_N(\hat{B})|.$$

If we take $N = 1$, then Theorem B reduces to an equivalent form of [10, Theorem 4.1], which is a modular version of the Alperin–McKay conjecture for p-solvable groups.

We finally mention that Theorem B is not true if the group is not assumed to be p-solvable. For example, let $G = GL(3, 2)$. Let B be the principal 2-block of G and take $N = 1$. Then $|\mathrm{IBr}^0_N(B)| = 3$, as B has three irreducible Brauer characters, all of height zero. On the other hand, if D is a Sylow 2-subgroup of G, then $N_G(D) = D$ and so if $\hat{B}$ is the principal 2-block of $N_G(D)$, we have $|\mathrm{IBr}^0_N(\hat{B})| = 1$.

2. Proof of the results

In this section we prove Theorems A and B of the introduction. We begin with a number of preliminary results.

Let ψ be an irreducible p-Brauer character of an arbitrary finite group G. By a vertex of ψ we mean any vertex of the simple G-module (in characteristic p) that corresponds to ψ.

Lemma 2.1. *Let $N \lhd G$ where G is p-solvable and let B be a p-block of G with defect group D. Suppose $\psi \in \mathrm{IBr}(B)$.*
(i) $\psi \in \mathrm{IBr}_N^0(B)$ *if and only if* $|D|/|D \cap N| = |P|/|P \cap N|$ *where P is any vertex of ψ.*
(ii) *Suppose Q is a vertex of ψ with $Q \subseteq D$. Then $\psi \in \mathrm{IBr}_N^0(B)$ if and only if $QN = DN$.*

Proof. (i) Let φ be an irreducible constituent of ψ_N and let P be a vertex for ψ. Then [11, Corollary 3] implies that $|P \cap N|$ is the order of any vertex of φ. Furthermore, if b is the p-block of N to which φ belongs, $|D \cap N|$ is the order of any defect group of b from [3, Proposition 4.2]. Now $\psi \in \mathrm{IBr}_N^0(B)$ if and only if

$$\frac{|G|_p}{\psi(1)_p}\left(\frac{|N|_p}{\varphi(1)_p}\right)^{-1} = \frac{|D|}{|D \cap N|}.$$

The result is then immediate by [1, Theorem 2.1].

(ii) Part (i) implies that $\psi \in \mathrm{IBr}_N^0(B)$ if and only if $|QN| = |DN|$. Since $Q \subseteq D$, we have $QN \subseteq DN$ and the result immediately follows. □

Before stating our second preliminary result, we mention an easy corollary.

Corollary 2.2. *Let $M \subseteq N$ be normal subgroups of a p-solvable group G and let B be a p-block of G. Then $\mathrm{IBr}_M^0(B) \subseteq \mathrm{IBr}_N^0(B)$.*

Proof. Let D be a defect group of B and assume $\psi \in \mathrm{IBr}_M^0(B)$. Choose a vertex Q for ψ such that $Q \subseteq D$. By Lemma 2.1 (ii) we have $QM = DM$. So $QN = DN$ and again by Lemma 2.1 (ii), $\psi \in \mathrm{IBr}_N^0(B)$. □

Our next lemma involves the concept of a relative p-block introduced in [4, 5]. For the reader's convenience, we first give a brief review of this concept and other related notions needed for our purposes.

Let G be a p-solvable group. If χ is an ordinary character of G, we denote by χ^0 the restriction of χ to the set of p-regular elements of G.

In [2], Isaacs associates to every character $\chi \in \mathrm{Irr}(G)$ a unique (up to G-conjugacy) pair (W, γ) called nucleus of χ such that W is a subgroup of G, $\gamma \in \mathrm{Irr}(W)$ is p-factorable (i.e., γ is a product of a p-special and a p'-special characters of W), and $\chi = \gamma^G$. Let $\mathrm{B}_{p'}(G)$ be the set of all those $\chi \in \mathrm{Irr}(G)$ for which γ is p'-special. Then Isaacs in [2] proved that the restriction map $\chi \mapsto \chi^0$ defines a bijection of $\mathrm{B}_{p'}(G)$ onto $\mathrm{IBr}(G)$.

Now let $N \lhd G$ and $\mu \in \mathrm{B}_{p'}(N)$. We say that two characters $\chi, \chi' \in \mathrm{Irr}(G \mid \mu)$ (the set of irreducible characters of G lying over μ) are linked provided that there exist irreducible characters $\chi_0, \chi_1, \ldots, \chi_n$ of G lying over μ and irreducible Brauer characters $\psi_0, \ldots, \psi_{n-1}$ of G such that $\chi_0 = \chi$, $\chi_n = \chi'$ and $d_{\chi_i\psi_i} d_{\chi_{i+1}\psi_i} \neq 0$ for all

$0 \le i \le n-1$. This linking clearly defines an equivalence relation on $\mathrm{Irr}(G \mid \mu)$ and the resulting equivalence classes are called relative p-blocks of G with respect to (N, μ). Let $\mathrm{Bl}_p(G \mid \mu)$ be the set of all these relative p-blocks. If B is any (Brauer) p-block of G such that $\mathrm{Irr}(B) \cap \mathrm{Irr}(G \mid \mu) \neq \emptyset$, then it is not hard to see that $\mathrm{Irr}(B) \cap \mathrm{Irr}(G \mid \mu)$ is in fact a union of some relative p-blocks in $\mathrm{Bl}_p(G \mid \mu)$.

Let $\mathcal{B} \in \mathrm{Bl}_p(G \mid \mu)$. By $\mathrm{IBr}(\mathcal{B})$, we mean the set of all $\psi \in \mathrm{IBr}(G)$ such that $\psi = \chi^0$ where $\chi \in \mathcal{B}$. It turns out that $\mathrm{IBr}(\mathcal{B})$ is always nonempty.

Finally we should mention that a notion of the defect group of a relative p-block is defined in Section 4 of [5].

Lemma 2.3. *Let G be a p-solvable group and let B be a p-block of G with defect group D. Suppose $N \lhd G$ and let $\mu \in \mathrm{B}_{p'}(N)$ be G-invariant. If Q is a vertex of the unique irreducible Brauer character ψ of DN lying over the irreducible Brauer character μ^0, then*
(i) $QN = DN$;
(ii) *there exists a nucleus (W, γ) for μ such that Q is contained in the stabilizer S of (W, γ) in G and $Q \cap W$ is a Sylow p-subgroup of W;*
(iii) *assume $\mathcal{B} \in \mathrm{Bl}_p(G \mid \mu)$ has defect group Q and let $\hat{\mathcal{B}}$ be the relative p-block in $\mathrm{Bl}_p(N_G(QN) \mid \mu)$ with defect group Q corresponding to $\mathcal{B}$ via* Proposition 3.4 (c) *in* [6]. *Then if $\hat{B}$ is the p-block of $N_G(QN)$ $(= N_G(DN))$ with defect group D such that $\hat{B}^G = B$, we have $\mathcal{B} \subseteq \mathrm{Irr}(B)$ if and only if $\hat{\mathcal{B}} \subseteq \mathrm{Irr}(\hat{B})$.*

Proof. (i) Since μ is G-invariant, then so is μ^0. It follows by [9, Theorem 3.5.11 (ii)] that $\psi_N = \mu^0$, and so in particular $\psi(1)_p = \mu^0(1)_p$. Next as Q is a vertex for ψ, [11, Corollary 3] implies that $|Q \cap N|$ is the order of any vertex of μ^0. Then by [1, Theorem 2.1], $|DN|_p/|Q| = |N|_p/|Q \cap N|$, and hence $|D|/|D \cap N| = |Q|/|Q \cap N|$. Therefore $|DN| = |QN|$. But $QN \subseteq DN$ as $Q \subseteq DN$. It follows that $QN = DN$.

(ii) Since $QN = DN$ by (i), Lemma 2.4 (b) in [7] tells us that there exists a nucleus (W, γ) for μ such that Q is contained in the stabilizer S of (W, γ) in G.

Next by Lemma 2.3 (c) of [7], there is $\eta \in \mathrm{IBr}(S \cap (QN))$ lying over γ^0 and having vertex Q such that $\eta^{QN} = \psi$. Now as γ is an S-invariant p'-special character of W, we conclude by [6, Lemma 2.1 (b)] that $Q \cap W$ is a Sylow p-subgroup of W.

(iii) Let β (resp. $\tilde{\beta}$) be the p-block of G (resp. $N_G(QN)$) such that $\mathcal{B} \subseteq \mathrm{Irr}(\beta)$ (resp. $\hat{\mathcal{B}} \subseteq \mathrm{Irr}(\tilde{\beta})$). We claim that $\tilde{\beta}^G$ is defined and $\tilde{\beta}^G = \beta$.

Since $\hat{\mathcal{B}}$ has Q as a defect group, Corollary 2.5 in [6] says that there exists $\nu \in \mathrm{IBr}(\hat{\mathcal{B}})$ having vertex Q. Now $\nu \in \mathrm{IBr}(\tilde{\beta})$ and hence by [9, Theorem 5.1.9], there is some defect group P of $\tilde{\beta}$ such that $Q \subseteq P$. As $QC_G(Q) \subseteq N_G(QN)$, it follows by Corollary 5.3.7 and Theorem 5.3.6 of [9] that $\tilde{\beta}^G$ is defined. Next we show that $\tilde{\beta}^G = \beta$.

Choose $\alpha \in \hat{\mathcal{B}}$. Since α lies over μ, we may write $\alpha^G = \sum c_\delta \delta$ for $\delta \in \mathrm{Irr}(G \mid \mu)$. Let $\mathcal{B}_1 = \mathcal{B}, \ldots, \mathcal{B}_m$ be all the (distinct) relative p-blocks in $\mathrm{Bl}_p(G \mid \mu)$ that are contained in $\mathrm{Irr}(\beta)$. Then

$$\sum_{\delta\in\mathrm{Irr}(\beta)} c_\delta \delta = \sum_{i=1}^{m}\Bigl(\sum_{\delta\in\mathcal{B}_i} c_\delta \delta\Bigr).$$

Now as $\bigl[\sum_{\delta\in\mathcal{B}_1} c_\delta\delta(1)\bigr]_p = [\alpha^G(1)]_p$ and $\bigl[\sum_{\delta\in\mathcal{B}_i} c_\delta\delta(1)\bigr]_p > [\alpha^G(1)]_p$ for $i \neq 1$ by [6, Proposition 3.4 (a)], we get that $\bigl[\sum_{\delta\in\mathrm{Irr}(\beta)} c_\delta\delta(1)\bigr]_p = [\alpha^G(1)]_p$. Since $\alpha \in \hat{\mathcal{B}} \subseteq \mathrm{Irr}(\tilde{\beta})$, it follows by [9, Corollary 5.3.2] that $\tilde{\beta}^G = \beta$, as claimed.

Now if $\hat{\mathcal{B}} \subseteq \mathrm{Irr}(\hat{B})$, then $\mathcal{B} \subseteq \mathrm{Irr}(B)$ by the above. Next suppose $\mathcal{B} \subseteq \mathrm{Irr}(B)$. Then $\tilde{\beta}^G = B$ and so to finish the proof of (iii), by the uniqueness of the p-block $\hat{B}$, it suffices to show that $\tilde{\beta}$ has defect group D.

Recall that P is a defect group of $\tilde{\beta}$. Then in view of Lemma 5.3.3 of [9], there exists $g \in G$ for which

$$(*) \qquad P \subseteq D^g.$$

So in particular $QN \subseteq PN \subseteq D^gN$. Since $QN = DN$ by (i) and $|DN| = |D^gN|$, it follows that

$$(**) \qquad QN = PN = D^gN.$$

Next let b be the p-block of N such that $\mu \in \mathrm{Irr}(b)$. Since $\alpha \in \mathrm{Irr}(\tilde{\beta}) \cap \mathrm{Irr}(N_G(QN) \mid \mu)$, we have that $\tilde{\beta}$ covers b. Also B covers b, as $\mathcal{B} \subseteq \mathrm{Irr}(B)$. Proposition 4.2 in [3] now implies that $|P \cap N| = |D^g \cap N| = p^{d(b)}$, where $d(b)$ is the defect of b. In view of $(**)$, we are then forced to have $|P| = |D^g|$. Hence $P = D^g$ from $(*)$. Also, since $DN = QN = D^gN$, it is clear that $g \in N_G(DN)$. Then as P is a defect group of $\tilde{\beta}$ (a p-block of $N_G(DN)$), we deduce that D also is a defect group for $\tilde{\beta}$, as needed to be shown. □

Lemma 2.4. *Let $N \lhd G$ where G is p-solvable and let B be a p-block of G with defect group D. Assume $\mu \in \mathrm{B}_{p'}(N)$ is G-invariant and $\omega \in \mathrm{IBr}^0_N(B \mid \varphi)$ where $\varphi = \mu^0$. If Q is a vertex for the unique irreducible Brauer character ψ of DN lying over φ, then*

(i) *ω has vertex Q;*

(ii) *$\omega \in \mathrm{IBr}(\mathcal{B})$ for some relative p-block $\mathcal{B} \in \mathrm{Bl}_p(G \mid \mu)$ having defect group Q and such that $\mathcal{B} \subseteq \mathrm{Irr}(B)$.*

Proof. Let Q be a vertex of the unique irreducible Brauer character ψ of DN lying over φ. Then $QN = DN$ by Lemma 2.3 (i).

(i) Choose a vertex R for ω such that $R \subseteq D$. Then by Lemma 2.1 (ii), we have $RN = DN$. Now as φ is G-invariant, [7, Lemma 2.5] tells us that R is a vertex of ψ. So Q is DN-conjugate to R, and hence is a vertex for ω.

(ii) Let χ be the unique character in $\mathrm{B}_{p'}(G)$ such that $\chi^0 = \omega$. As ω lies over φ and all the irreducible constituents of χ_N are in $\mathrm{B}_{p'}(N)$ by [2, Corollary 7.5], χ must lie over μ. Let $\mathcal{B}$ be the relative p-block in $\mathrm{Bl}_p(G \mid \mu)$ in which χ lies. Then $\omega \in \mathrm{IBr}(\mathcal{B})$ and since $\chi \in \mathrm{Irr}(B)$, we have $\mathcal{B} \subseteq \mathrm{Irr}(B)$. Next we show that $\mathcal{B}$ has Q as a defect group.

By Corollary 2.5 in [6], $\mathcal{B}$ has a defect group P such that $Q \subseteq P$ (as $\omega \in \mathrm{IBr}(\mathcal{B})$), and there is $\xi \in \mathrm{IBr}(\mathcal{B})$ with P as a vertex. Now since $\xi \in \mathrm{IBr}(B)$, we have $P \subseteq D^g$ for some $g \in G$, and so in particular $PN \subseteq (DN)^g$. But $QN \subseteq PN$ and $QN = DN$. We deduce then that $QN = PN$. Also, as ω and ξ both lie over φ, [11, Corollary 3] implies that $|Q \cap N| = |P \cap N|$. It follows that $|Q| = |P|$ and therefore $Q = P$ as $Q \subseteq P$. We have thus shown that Q is a defect group for $\mathcal{B}$, as needed. □

The following is an essential step toward the proof of Theorem A.

Proposition 2.5. *Let $N \lhd G$ where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Suppose $\varphi \in \mathrm{IBr}(b)$ is G-invariant. Let D be a defect group of B and write $\hat{B}$ for the unique p-block of $N_G(DN)$ with defect group D such that $\hat{B}^G = B$, then*

$$|\mathrm{IBr}_N^0(B \mid \varphi)| = |\mathrm{IBr}_N^0(\hat{B} \mid \varphi)|.$$

Proof. First let μ be the character in $\mathrm{B}_{p'}(N)$ for which $\mu^0 = \varphi$. Since μ is uniquely determined by φ, we note that μ is G-invariant. Next fix a vertex Q for the unique irreducible Brauer character ψ of DN that lies over φ. Then, in view of Lemma 2.3 (i), we have $QN = DN$.

Next by [8, Lemma 3.2 (ii)], $\mathrm{IBr}_N^0(B \mid \varphi) \neq \emptyset$. Let $\omega \in \mathrm{IBr}_N^0(B \mid \varphi)$. Then Lemma 2.4 tells us that ω has vertex Q and that $\omega \in \mathrm{IBr}(\mathcal{B})$ for some relative p-block $\mathcal{B} \in \mathrm{Bl}_p(G \mid \mu)$ having Q as a defect group and such that $\mathcal{B} \subseteq \mathrm{Irr}(B)$.

Let now $\mathcal{B}_1, \ldots, \mathcal{B}_m$ be all the (distinct) relative p-blocks in $\mathrm{Bl}_p(G \mid \mu)$ having defect group Q and such that $\mathcal{B}_i \subseteq \mathrm{Irr}(B)$ for all $i \in \{1, \ldots, m\}$. Next write A for the set of all those $\eta \in \bigcup_{i=1}^m \mathrm{IBr}(\mathcal{B}_i)$ having Q as a vertex. Then $\mathrm{IBr}_N^0(B \mid \varphi) \subseteq A$ by the preceding paragraph. Since $QN = DN$, Lemma 2.1 implies that $A \subseteq \mathrm{IBr}_N^0(B \mid \varphi)$. Consequently, we have $\mathrm{IBr}_N^0(B \mid \varphi) = A$.

Next by Lemma 2.3 (ii), there exists a nucleus (W, γ) of μ such that Q is contained in the stabilizer of (W, γ) in G and $Q \cap W$ is a Sylow p-subgroup of W. Then for each i, let $\hat{\mathcal{B}}_i$ be the relative p-block in $\mathrm{Bl}_p(N_G(QN) \mid \mu)$ with defect group Q corresponding to $\mathcal{B}_i$ through Proposition 3.4 (c) of [6]. By Lemma 2.3 (iii), we have $\hat{\mathcal{B}}_i \subseteq \mathrm{Irr}(\hat{B})$.

Suppose $\hat{\beta}$ is a relative p-block in $\mathrm{Bl}_p(N_G(QN) \mid \mu)$ with defect group Q and such that $\hat{\beta} \subseteq \mathrm{Irr}(\hat{B})$. If β is the relative p-block in $\mathrm{Bl}_p(G \mid \mu)$ that corresponds to $\hat{\beta}$ via [6, Proposition 3.4 (c)], then Q is a defect group for β, and by Lemma 2.3 (iii) we have $\beta \subseteq \mathrm{Irr}(B)$. Therefore β is one of $\mathcal{B}_1, \ldots, \mathcal{B}_m$ and it follows that $\hat{\beta}$ is one of $\hat{\mathcal{B}}_1, \ldots, \hat{\mathcal{B}}_m$. This tells us that $\hat{\mathcal{B}}_1, \ldots, \hat{\mathcal{B}}_m$ are all the (distinct) relative p-blocks in $\mathrm{Bl}_p(N_G(QN) \mid \mu)$ having Q as a defect group and which are contained in $\mathrm{Irr}(\hat{B})$.

If $\rho \in \mathrm{Irr}(B)$, then the irreducible constituents of ρ_N all belong to b, as b is G-invariant. It follows from the remark following Lemma 3.1 in [6] that the p-block $\hat{B}$ covers b. Now let C be the set of all $\tau \in \bigcup_{i=1}^m \mathrm{IBr}(\hat{\mathcal{B}}_i)$ having vertex Q. Then by repeating the argument used above to show that $\mathrm{IBr}_N^0(B \mid \varphi) = A$, we get that $\mathrm{IBr}_N^0(\hat{B} \mid \varphi) = C$.

Next let $i \in \{1, \ldots, m\}$ and assume $\zeta \in \mathrm{IBr}(\hat{\mathcal{B}}_i)$. We claim that Q is a vertex of ζ. Since ζ lies over φ and ψ is the unique irreducible Brauer character of QN lying over φ, we have that ζ lies over ψ. Then as Q is a vertex for ψ, there exists a vertex P of ζ with $Q \subseteq P$ by [9, Lemma 4.3.4]. On the other hand, Corollary 2.5 of [6] implies that some $N_G(QN)$-conjugate of P is contained in the defect group Q of $\hat{\mathcal{B}}_i$. It follows that $P = Q$, which proves our claim.

We now have $\mathrm{IBr}_N^0(\hat{B} \mid \varphi) = C = \bigcup_{i=1}^m \mathrm{IBr}(\hat{\mathcal{B}}_i)$. By [6, Proposition 3.4 (b)], $|\mathrm{IBr}(\hat{\mathcal{B}}_i)|$ is equal to the number of all those Brauer characters in $\mathrm{IBr}(\mathcal{B}_i)$ that have Q as a vertex. It follows that $|\mathrm{IBr}_N^0(B \mid \varphi)| = |A| = \sum_{i=1}^m |\mathrm{IBr}(\hat{\mathcal{B}}_i)| = |\mathrm{IBr}_N^0(\hat{B} \mid \varphi)|$, and the proof of the proposition is complete. □

The stable case of our next proposition is the key for proving Theorem A. This result is analogous to Theorem A, where the subgroup $NN_G(D)$ is replaced by the possibly larger subgroup $N_G(DN)$ (note that $NN_G(D) \subseteq N_G(DN)$, always). We should mention that in the proof of this proposition, we will adapt some of the arguments used in the proof of the main theorem of [6].

Proposition 2.6. *Let $N \lhd G$ where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Let T be the inertial group of b in G and suppose $\varphi \in \mathrm{IBr}(b)$. If D is an inertial defect group of B with respect to b and $\hat{B}$ is the unique p-block of $N_G(DN)$ with defect group D such that $\hat{B}^G = B$, then*

$$|\mathrm{IBr}_N^0(B \mid \varphi)| = \left| \bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t) \right|.$$

Proof. First let I be the inertial group of φ in G.

CASE 1. Assume $\mathrm{IBr}_N^0(B \mid \varphi) = \emptyset$. We just need to show that $\mathrm{IBr}_N^0(\hat{B} \mid \varphi^t) = \emptyset$ for all $t \in T$. Suppose, on the contrary, that there is $t_0 \in T$ for which $\mathrm{IBr}_N^0(\hat{B} \mid \varphi^{t_0}) \neq \emptyset$. Then [8, Theorem 3.3] implies that an $N_G(DN)$-conjugate of D, hence DN, is contained in the inertial group of φ^{t_0} in G. So, in particular, $D^{t_0^{-1}} \subseteq I$. Now as D is an inertial defect group of B with respect to b and $t_0^{-1} \in T$, we have that $D^{t_0^{-1}}$ also is an inertial defect group of B with respect to b. Then in view of Theorem 3.3 in [8], it follows that $\mathrm{IBr}_N^0(B \mid \varphi) \neq \emptyset$, which contradicts our assumption.

CASE 2. Suppose that $\mathrm{IBr}_N^0(B \mid \varphi) \neq \emptyset$. Let $\omega \in \mathrm{IBr}_N^0(B \mid \varphi)$. Next denote by $\omega_{(I)}$ the unique irreducible Brauer character of I lying over φ such that $(\omega_{(I)})^G = \omega$ and let $\omega_{(T)}$ be the irreducible Brauer character $(\omega_{(I)})^T$. Then as $\varphi \in \mathrm{IBr}(b)$ and $\omega_{(T)}$ lies over φ, [9, Theorem 5.5.10] implies that $\omega_{(T)} \in \mathrm{IBr}(B')$, where B' is the Fong–Reynolds correspondent of B in T. Now let Q be a vertex of $\omega_{(I)}$. Then $\omega_{(T)}$ has vertex Q and so by [9, Theorem 5.1.9], $Q \subseteq D^t$ for some $t \in T$. Also, since Q is a vertex for ω and $\omega \in \mathrm{IBr}_N^0(B)$, we have $\omega_{(T)} \in \mathrm{IBr}_N^0(B')$ by Lemma 2.1 (i). Then part (ii) of Lemma 2.1 tells us that $QN = D^t N$.

Choose now a minimal subset U of T such that for each $\omega \in \mathrm{IBr}_N^0(B \mid \varphi)$, there exists a unique element $u_\omega \in U$ such that $D^{u_\omega} N = QN$ for some vertex Q of $\omega_{(I)}$.

STEP 1. Our objective in this step is to show that

$$\bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t) = \bigcup_{u \in U} \mathrm{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}}). \tag{1}$$

So let $t \in T$ and suppose $\theta \in \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t)$. Then as a vertex of $\theta^{t^{-1}}$ is conjugate to a vertex of θ and a defect group of $\hat{B}^{t^{-1}}$ is conjugate to a defect group of $\hat{B}$, we have $\theta^{t^{-1}} \in \mathrm{IBr}_N^0(\hat{B}^{t^{-1}} \mid \varphi)$ by Lemma 2.1 (i). Let now η be the unique element in $\mathrm{IBr}(N_G(D^{t^{-1}}N) \cap I)$ lying over φ such that $\eta^{N_G(D^{t^{-1}}N)} = \theta^{t^{-1}}$.

Since $\hat{B}^{t^{-1}}$ has defect group $D^{t^{-1}}$, the Brauer character $\theta^{t^{-1}}$ has some vertex $P \subseteq D^{t^{-1}}$ by [9, Theorem 5.1.9]. Then Lemma 2.1 (ii) says that $PN = D^{t^{-1}}N$. Next there is $h \in N_G(D^{t^{-1}}N)$ for which $R = P^h$ is a vertex for η. Now $RN = (PN)^h = D^{t^{-1}}N$. So in particular, as $RN \subseteq I$, we get $D^{t^{-1}}N \subseteq I$. Therefore $N_G(D^{t^{-1}}N) \cap I = N_I(D^{t^{-1}}N)$.

Now if β is the p-block of $N_I(D^{t^{-1}}N)$ for which $\eta \in \mathrm{IBr}(\beta)$, we have that $\beta^{N_G(D^{t^{-1}}N)}$ is defined and $\beta^{N_G(D^{t^{-1}}N)} = \hat{B}^{t^{-1}}$ by Lemma 3.1 in [8].

Next choose a defect group L for β such that $R \subseteq L$. By [9, Lemma 5.3.3], there exists $k \in N_G(D^{t^{-1}}N)$ such that $L \subseteq (D^{t^{-1}})^k$. Now

$$RN \subseteq LN \subseteq (D^{t^{-1}}N)^k = D^{t^{-1}}N.$$

Since $RN = D^{t^{-1}}N$, we get

$$LN = D^{t^{-1}}N. \tag{2}$$

Next as η lies over φ and $\varphi \in \mathrm{IBr}(b)$, the p-block β covers b. Then [3, Proposition 4.2] implies that $|L \cap N| = p^{d(b)}$ where $d(b)$ is the defect of b. Similarly, since $\hat{B}^{t^{-1}}$ covers b, we have $|D^{t^{-1}} \cap N| = p^{d(b)}$. Now from (2) $|L|/|L \cap N| = |D^{t^{-1}}|/|D^{t^{-1}} \cap N|$ and it follows that

$$|L| = |D^{t^{-1}}| = |D|. \tag{3}$$

Since $N_I(L) \subseteq N_I(LN) \subseteq I$ and β is a p-block of $N_I(LN)$ $(= N_I(D^{t^{-1}}N))$ with defect group L, Theorem 5.3.8 in [9] tells us that β^I is defined and has L as a defect group. Furthermore, as β covers b, note by the remark following Lemma 3.1 of [6] that β^I covers b as well.

Now since φ is I-invariant, there exists an element $\nu \in \mathrm{IBr}_N^0(\beta^I \mid \varphi)$ by [8, Lemma 3.2 (ii)]. Next we have $\beta^{N_G(D^{t^{-1}}N)} = \hat{B}^{t^{-1}}$ and $(\hat{B}^{t^{-1}})^G = B$. Then [9, Lemma 5.3.4] says that β^G is defined and equals B. By Lemma 5.3.4 in [9] again, we get that $(\beta^I)^G$ is defined and equals B. Now as ν is an irreducible Brauer character of β^I lying over φ, [8, Lemma 3.1] implies that ν^G is an irreducible Brauer character of B lying over φ. We claim that $\nu^G \in \mathrm{IBr}_N^0(B \mid \varphi)$.

As $\nu \in \mathrm{IBr}_N^0(\beta^I \mid \varphi)$ and the order of any defect group of β^I is $|D|$ by (3), we have $(\nu(1)_p|D|)/|I|_p = (\varphi(1)_p p^{d(b)})/|N|_p$. On the other hand, $(\nu(1)_p|D|)/|I|_p = (\nu^G(1)_p|D|)/|G|_p$. It follows that $\nu^G \in \mathrm{IBr}_N^0(B \mid \varphi)$, as claimed.

Next choose a vertex K of ν such that $K \subseteq L$ (recall that L is a defect group of β^I). Since $\nu \in \mathrm{IBr}_N^0(\beta^I)$, we have $KN = LN$ by Lemma 2.1 (ii). Further, by the choice of the subset U, as $\nu^G \in \mathrm{IBr}_N^0(B \mid \varphi)$, there exist $y \in I$ and $u_0 \in U$ such that $K^yN = D^{u_0}N$. Now in view of (2), we get that $(DN)^{t^{-1}y} = (D^{t^{-1}}N)^y = (DN)^{u_0}$. Hence $z = u_0y^{-1}t \in N_G(DN)$. Then $\varphi^t = \varphi^{yu_0^{-1}z} = (\varphi^{u_0^{-1}})^z$, which tells us that φ^t is $N_G(DN)$-conjugate to $\varphi^{u_0^{-1}}$. Now since $\theta \in \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t)$, we conclude that $\theta \in \mathrm{IBr}_N^0(\hat{B} \mid \varphi^{u_0^{-1}})$. This clearly proves (1).

STEP 2. For each $u \in U$, we let E_u be the set of $\omega \in \mathrm{IBr}_N^0(B \mid \varphi)$ such that $\omega_{(I)}$ has a vertex Q with $QN = D^uN$. By the minimality of U, it is clear that $E_u \neq \emptyset$. Our final goal in this step is to prove that

$$|E_u| = |\mathrm{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}})|, \tag{4}$$

for each $u \in U$.

Let $u \in U$. Suppose $\omega \in E_u$. If Δ is the p-block of I such that $\omega_{(I)} \in \mathrm{IBr}(\Delta)$, then Δ^G is defined and equals B by Lemma 3.1 of [8]. Let Q be a vertex for $\omega_{(I)}$ such that

$$QN = D^uN. \tag{5}$$

Then $Q \subseteq P$ for some defect group P of Δ. Next by [9, Lemma 5.3.3], there is $g \in G$ such that $P \subseteq (D^u)^g$. Now we have $QN \subseteq PN \subseteq (D^uN)^g$. Then from (5), it follows that

$$D^uN = QN = PN = (D^uN)^g. \tag{6}$$

Hence, by Lemma 2.1, we get that $\omega_{(I)} \in \mathrm{IBr}_N^0(\Delta \mid \varphi)$.

It is clear from (6) that $g \in N_G(D^uN)$. Also, as $PN = D^uN$ and $|P \cap N| = p^{d(b)} = |D^u \cap N|$ (as implied by [3, Proposition 4.2]), we have $|P| = |D^u|$. Since $P \subseteq (D^u)^g$, it follows that $P = (D^u)^g$. Hence Δ has some $N_G(D^uN)$-conjugate (namely P) of D^u as a defect group.

Let now $\Delta_{u,1}, \ldots, \Delta_{u,n_u}$ be all the (distinct) p-blocks of I covering b, each having some $N_G(D^u N)$-conjugate of D^u as a defect group and such that $(\Delta_{u,i})^G = B$ for all $i \in \{1, \ldots, n_u\}$.

Let $i \in \{1, \ldots, n_u\}$. Assume that $\tau \in \mathrm{IBr}^0_N(\Delta_{u,i} \mid \varphi)$. Then $\tau^G \in \mathrm{IBr}(B \mid \varphi)$ by [8, Lemma 3.1]. Next choose a defect group $R_{u,i}$ for $\Delta_{u,i}$ which is $N_G(D^u N)$-conjugate to D^u. Let L be a vertex of τ with $L \subseteq R_{u,i}$. Then by Lemma 2.1 (ii) we have $LN = R_{u,i}N = D^u N$. So, in particular, $|L|/|L \cap N| = |D|/|D \cap N|$. Since τ^G has vertex L, it follows by Lemma 2.1 (i) that $\tau^G \in E_u$.

Now the correspondence $\tau \mapsto \tau^G$ defines a map from $\bigcup_{i=1}^{n_u} \mathrm{IBr}^0_N(\Delta_{u,i} \mid \varphi)$ to E_u. This map is onto by the above discussion and 1-1 by Theorem 3.3.2 of [9]. Consequently we get $|E_u| = \sum_{i=1}^{n_u} |\mathrm{IBr}^0_N(\Delta_{u,i} \mid \varphi)|$.

For each i, we have seen above that $\Delta_{u,i}$ has a defect group $R_{u,i}$ such that $R_{u,i}N = D^u N$. Let $\hat{\Delta}_{u,i}$ be the unique p-block of $N_I(D^u N)$ having $R_{u,i}$ as a defect group and such that $(\hat{\Delta}_{u,i})^I = \Delta_{u,i}$. By Proposition 2.5, we have $|\mathrm{IBr}^0_N(\Delta_{u,i} \mid \varphi)| = |\mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi)|$. It follows that

$$|E_u| = \sum_{i=1}^{n_u} |\mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi)|. \tag{7}$$

Now let $i \in \{1, \ldots, n_u\}$. As φ is I-invariant, Lemma 3.2 (ii) in [8] tells us that $\mathrm{IBr}^0_N(\Delta_{u,i} \mid \varphi) \neq \emptyset$. So $\mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi) \neq \emptyset$. Suppose $\alpha \in \mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi)$. Then $\alpha^{N_G(D^u N)}$ is an irreducible Brauer character of $N_G(D^u N)$ lying over φ. Now by [8, Lemma 3.1], $(\hat{\Delta}_{u,i})^{N_G(D^u N)}$ is defined and $\alpha^{N_G(D^u N)} \in \mathrm{IBr}((\hat{\Delta}_{u,i})^{N_G(D^u N)})$. Next we have $(\hat{\Delta}_{u,i})^I = \Delta_{u,i}$ and $(\Delta_{u,i})^G = B$. So $(\hat{\Delta}_{u,i})^G$ is defined and equals B by [9, Lemma 5.3.4]. Applying Lemma 5.3.4 of [9] once again, we get that $((\hat{\Delta}_{u,i})^{N_G(D^u N)})^G$ is defined and equals B.

Next we know that $\hat{\Delta}_{u,i}$ has $R_{u,i}$ as a defect group. Then by Lemma 5.3.3 in [9], $(\hat{\Delta}_{u,i})^{N_G(D^u N)}$ has a defect group M for which $R_{u,i} \subseteq M \subseteq D^f$, where f is some element of G. But as $R_{u,i}$ is G-conjugate to D, we conclude that $M = R_{u,i}$. Further, since $R_{u,i}$ is $N_G(D^u N)$-conjugate to D^u, it follows that $(\hat{\Delta}_{u,i})^{N_G(D^u N)}$ has D^u as a defect group. However, $\hat{B}^u$ is the only p-block of $N_G(D^u N)$ that has D^u as a defect group and such that $(\hat{B}^u)^G = B$. We must then have that $(\hat{\Delta}_{u,i})^{N_G(D^u N)} = \hat{B}^u$. Then $\alpha^{N_G(D^u N)} \in \mathrm{IBr}(\hat{B}^u)$.

Let X be a vertex of α. As $\alpha \in \mathrm{IBr}^0_N(\hat{\Delta}_{u,i})$, we have $|R_{u,i}|/|R_{u,i} \cap N| = |X|/|X \cap N|$ by Lemma 2.1. Now since X is also a vertex for $\alpha^{N_G(D^u N)}$ and $R_{u,i}$ is a defect group of $\hat{B}^u$, it follows by the same lemma that $\alpha^{N_G(D^u N)} \in \mathrm{IBr}^0_N(\hat{B}^u \mid \varphi)$.

We may now define a map F from $\bigcup_{i=1}^{n_u} \mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi)$ to $\mathrm{IBr}^0_N(\hat{B}^u \mid \varphi)$ by $F(\alpha) = \alpha^{N_G(D^u N)}$. We claim that F is a bijection. First F is 1-1 by [9, Theorem 3.3.2]. Next we show that F is onto.

Let δ be an element of $\mathrm{IBr}^0_N(\hat{B}^u \mid \varphi)$. Next call γ, the element in $\mathrm{IBr}(N_I(D^u N))$ that lies over φ and such that $\gamma^{N_G(D^u N)} = \delta$. Now if Λ is the p-block of $N_I(D^u N)$

to which γ belongs, we have that $\Lambda^{N_G(D^u N)}$ is defined and equals $\hat{B}^u$ by Lemma 3.1 of [8].

Let Y be a defect group for Λ. Then by [9, Lemma 5.3.3], $Y \subseteq (D^u)^e$ for some $e \in N_G(D^u N)$. So, in particular, $YN \subseteq D^u N$. Now choose a vertex V for γ such that $V \subseteq Y$. Since $\delta \in \mathrm{IBr}^0_N(\hat{B}^u)$, V is a vertex of δ and $(D^u)^e$ is a defect group of $\hat{B}^u$, Lemma 2.1 (ii) tells us that $VN = (D^u)^e N = D^u N$. It follows that

$$VN = YN = D^u N. \tag{8}$$

Then, by Lemma 2.1 (ii) again, we have $\gamma \in \mathrm{IBr}^0_N(\Lambda \mid \varphi)$. Also by (8) we have $|Y|/|Y \cap N| = |D^u|/|D^u \cap N|$. As each of Λ and $\hat{B}^u$ covers the p-block b, Proposition 4.2 of [3] implies that $|Y \cap N| = |D^u \cap N|$. Hence $|Y| = |D^u|$ and so $Y = (D^u)^e$.

By [9, Theorem 5.3.8], we have that Λ^I is defined and has Y as a defect group. Next as $\Lambda^{N_G(D^u N)} = \hat{B}^u$ and $(\hat{B}^u)^G = B$, Lemma 5.3.4 in [9] tells us that Λ^G is defined and $\Lambda^G = B$. Another application of [9, Lemma 5.3.4] now gives us that $(\Lambda^I)^G$ is defined and $(\Lambda^I)^G = B$.

Since Λ covers b, then so does Λ^I. Furthermore, as Λ^I has defect group Y, which is $N_G(D^u N)$-conjugate to D^u, we deduce that $\Lambda^I = \Delta_{u,i_0}$ for some $i_0 \in \{1, \ldots, n_u\}$.

Recall that R_{u,i_0} is a defect group for Δ_{u,i_0} and that $R_{u,i_0}N = D^u N$. Since Y is also a defect group of Δ_{u,i_0}, we have $Y = (R_{u,i_0})^c$ with $c \in I$. Now as $YN = D^u N$ from (8), we get $R_{u,i_0}N = (R_{u,i_0}N)^c$. This tells us that $c \in N_I(R_{u,i_0}N) = N_I(D^u N)$. It follows that R_{u,i_0} $(= Y^{c^{-1}})$ is a defect group of Λ. Then we must have $\Lambda = \hat{\Delta}_{u,i_0}$.

Now $\gamma \in \mathrm{IBr}^0_N(\hat{\Delta}_{u,i_0} \mid \varphi)$ and $\gamma^{N_G(D^u N)} = \delta$. This proves that the map F is onto. Hence F is a bijection as claimed.

Now $|\mathrm{IBr}^0_N(\hat{B} \mid \varphi^{u^{-1}})| = |\mathrm{IBr}^0_N(\hat{B}^u \mid \varphi)| = \left|\bigcup_{i=1}^{n_u} \mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi)\right|$ (the second equality follows from the fact that F is a bijection). Since $\hat{\Delta}_{u,1}, \ldots, \hat{\Delta}_{u,n_u}$ are distinct p-blocks of $N_I(D^u N)$ (because $\Delta_{u,1}, \ldots, \Delta_{u,n_u}$ are distinct p-blocks) and $|E_u| = \sum_{i=1}^{n_u} |\mathrm{IBr}^0_N(\hat{\Delta}_{u,i} \mid \varphi)|$ by (7), equality (4) follows.

STEP 3. By our choice of the set U, since $\{E_u : u \in U\}$ is a partition of $\mathrm{IBr}^0_N(B \mid \varphi)$, then

$$\begin{aligned} |\mathrm{IBr}^0_N(B \mid \varphi)| &= \sum_{u \in U} |E_u| \\ &= \sum_{u \in U} |\mathrm{IBr}^0_N(\hat{B} \mid \varphi^{u^{-1}})| \quad \text{(by (4))}. \end{aligned}$$

Now to complete the proof of the proposition, in view of (1), it suffices to show that the sets $\mathrm{IBr}^0_N(\hat{B} \mid \varphi^{u^{-1}})$ are mutually disjoint.

So let $u, u' \in U$ and suppose $\mathrm{IBr}^0_N(\hat{B} \mid \varphi^{u^{-1}}) \cap \mathrm{IBr}^0_N(\hat{B} \mid \varphi^{u'^{-1}}) \neq \emptyset$. Then there is $x \in N_G(DN)$ for which $\varphi^{u^{-1}} = (\varphi^{u'^{-1}})^x$. So $u = x^{-1}u'w$ for some $w \in I$. Then $D^u N = (D^{u'} N)^w$ and by the choice of U, it follows that $u = u'$. The proof of the proposition is now complete. $\square$

We need two more lemmas to be able to prove the main theorems.

Lemma 2.7. *Let $N \lhd G$ where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Let T be the inertial group of b in G and let B' be the Fong–Reynolds correspondent of B in T. Suppose $\varphi \in \mathrm{IBr}(b)$. Then the map $\psi \mapsto \psi^G$ defines bijections from $\mathrm{IBr}^0_N(B')$ onto $\mathrm{IBr}^0_N(B)$ and from $\mathrm{IBr}^0_N(B' \mid \varphi)$ onto $\mathrm{IBr}^0_N(B \mid \varphi)$.*

Proof. In view of [9, Theorem 5.5.10 (ii)], the map $\psi \mapsto \psi^G$ defines a bijection of $\mathrm{IBr}(B')$ onto $\mathrm{IBr}(B)$. We shall see that this bijection restricts to a bijection from $\mathrm{IBr}^0_N(B')$ onto $\mathrm{IBr}^0_N(B)$. For that, it suffices to show that $\psi \in \mathrm{IBr}^0_N(B')$ if and only if $\psi^G \in \mathrm{IBr}^0_N(B)$. So let $\psi \in \mathrm{IBr}(B')$ and choose $\omega \in \mathrm{IBr}(b)$ under ψ. Then ψ^G lies over ω. Since $\mathrm{ht}(\psi) = \mathrm{ht}(\psi^G)$, we have $\mathrm{ht}(\psi) - \mathrm{ht}(\omega) = \mathrm{ht}(\psi^G) - \mathrm{ht}(\omega)$. The result then immediately follows.

By the above, the map $\psi \mapsto \psi^G$ defines an injection from $\mathrm{IBr}^0_N(B' \mid \varphi)$ to $\mathrm{IBr}^0_N(B \mid \varphi)$. Next suppose $\tau \in \mathrm{IBr}^0_N(B \mid \varphi)$. Since T contains the inertial group of φ (in G), then by Clifford's theorem ([9, Exercise 3.3.4]), there exists an irreducible Brauer character ν of T over φ with $\nu^G = \tau$. Let B_0 be the p-block of T to which ν belongs. Then B_0 covers b and by [9, Theorem 5.5.10 (ii)], we have $\tau \in \mathrm{IBr}(B_0^G)$. Hence $B_0^G = B$ and it follows (by Theorem 5.5.10 (i) of [9]) that $B_0 = B'$. So ν is an irreducible Brauer character of B' lying over φ. Since $\tau \in \mathrm{IBr}^0_N(B)$, we have $\nu \in \mathrm{IBr}^0_N(B')$ by the discussion in the first paragraph. This shows that our injection is, in fact, a bijection and the proof of the lemma is complete. □

Lemma 2.8. *Let $N \lhd G$ where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Let T be the inertial group of b in G and let B' be the Fong–Reynolds correspondent of B in T. Suppose D is a defect group of B'. Then B has defect group D. Moreover, if $\hat{B}$ (resp. $\hat{B}'$) is the unique p-block of $NN_G(D)$ (resp. $NN_T(D)$) with defect group D such that $\hat{B}^G = B$ (resp. ${\hat{B}'}^T = B'$), then $\hat{B}'$ covers b, ${\hat{B}'}^{NN_G(D)}$ is defined and ${\hat{B}'}^{NN_G(D)} = \hat{B}$.*

Proof. The fact that B has defect group D is immediate from [9, Theorem 5.5.10 (iv)]. Next, since b is T-stable and B' covers b, then by the remark following Lemma 3.1 in [6], it is easy to see that $\hat{B}'$ covers b. Also, as $NN_T(D)$ is the inertial group of b in $NN_G(D)$, we have that ${\hat{B}'}^{NN_G(D)}$ is defined and has defect group D by [9, Theorem 5.5.10]. Next ${\hat{B}'}^T = B'$ and $(B')^G = B$, and so in view of [9, Lemma 5.3.4], ${\hat{B}'}^G$ is defined and ${\hat{B}'}^G = B$. Using Lemma 5.3.4 of [9] once more, we get that $\left({\hat{B}'}^{NN_G(D)}\right)^G$ is defined and equals B. Now by the uniqueness of $\hat{B}$, we are forced to have ${\hat{B}'}^{NN_G(D)} = \hat{B}$. This ends the proof of the lemma. □

We are now ready to present a proof for Theorem A.

Proof of Theorem A. Let B' be the Fong–Reynolds correspondent of B in T. Then Lemma 2.7 implies that

$$(*) \qquad |\mathrm{IBr}^0_N(B' \mid \varphi)| = |\mathrm{IBr}^0_N(B \mid \varphi)|.$$

Next we have that D is a defect group for B' and we write $\hat{B}'$ for the unique p-block of $NN_T(D)$ with defect group D such that $\hat{B}'^T = B'$. Then Lemma 2.8 says that $\hat{B}'$ covers b and that, as $NN_T(D)$ is the inertial group of b in $NN_G(D)$, $\hat{B}'$ is the Fong–Reynolds correspondent of $\hat{B}$ in $NN_T(D)$.

Note that $\varphi^t \in \mathrm{IBr}(b)$ for every $t \in T$. Then by Lemma 2.7, the map $\psi \mapsto \psi^G$ defines a bijection from $\bigcup_{t\in T} \mathrm{IBr}^0_N(\hat{B}' \mid \varphi^t)$ onto $\bigcup_{t\in T} \mathrm{IBr}^0_N(\hat{B} \mid \varphi^t)$. It follows, in particular, that

$$(**) \qquad \left|\bigcup_{t\in T} \mathrm{IBr}^0_N(\hat{B}' \mid \varphi^t)\right| = \left|\bigcup_{t\in T} \mathrm{IBr}^0_N(\hat{B} \mid \varphi^t)\right|.$$

Next we claim that $NN_T(D) = N_T(DN)$.

First we easily see that $NN_T(D) \subseteq N_T(DN)$. Next, by [9, Corollary 5.5.6], there exists a unique p-block b_0 of DN covering b. Since b is $N_T(DN)$-stable, it follows that b_0 is $N_T(DN)$-stable. Also, as $\hat{B}'$ covers b, then $\hat{B}'$ must cover b_0 as well. Now in view of [3, Proposition 4.2], since $\hat{B}'$ has defect group D, there is $g \in N$ for which D^g is a defect group for b_0. Hence b_0 has D as a defect group. Let now x be any element of $N_T(DN)$. Then, as $b_0^x = b_0$, the subgroup D^x is a defect group of b_0. It follows that D^x is DN-conjugate to D. Hence $D^x = D^y$ for some $y \in N$. Therefore $x \in NN_T(D)$. We have thus shown that $N_T(DN) \subseteq NN_T(D)$, and our claim is valid.

By Proposition 2.6, we have $|\mathrm{IBr}^0_N(B' \mid \varphi)| = \left|\bigcup_{t\in T} \mathrm{IBr}^0_N(\hat{B}' \mid \varphi^t)\right|$. Then, taking into account $(*)$ and $(**)$, we get $|\mathrm{IBr}^0_N(B \mid \varphi)| = \left|\bigcup_{t\in T} \mathrm{IBr}^0_N(\hat{B} \mid \varphi^t)\right|$, as wanted. $\square$

Next, we take care of Theorem B.

Proof of Theorem B. Let b_0 be a p-block of N which is covered by B. Call T_0 the inertial group of b_0 in G and let B'_0 be the Fong–Reynolds correspondent of B in T_0. Then by [9, Theorem 5.5.10 (iv)], there is $g \in G$ for which D^g is a defect group for B'_0.

Let $b = (b_0)^{g^{-1}}$. Then B covers b, the inertial group of b in G is $T = (T_0)^{g^{-1}}$ and $B' = (B'_0)^{g^{-1}}$ is the Fong–Reynolds correspondent of B (with respect to b) in T. Note also that B' has D as a defect group. Let $\hat{B}'$ be the unique p-block of $NN_T(D)$ with defect group D such that $\hat{B}'^T = B'$. Then, by Lemma 2.8, we have that the p-block $\hat{B}'$ covers b and that it is the Fong–Reynolds correspondent of $\hat{B}$ in the inertial group $NN_T(D)$ of b in $NN_G(D)$.

Next T acts by conjugation on $\mathrm{IBr}(b)$. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the resulting orbits and for each $i \in \{1, \dots, r\}$, choose $\varphi_i \in \mathcal{O}_i$. By Theorem A, we have

$$|\mathrm{IBr}_N^0(B' \mid \varphi_i)| = \left| \bigcup_{\theta \in \mathcal{O}_i} \mathrm{IBr}_N^0(\hat{B}' \mid \theta) \right|.$$

It follows that

$$\begin{aligned} |\mathrm{IBr}_N^0(B')| &= \sum_{i=1}^{r} |\mathrm{IBr}_N^0(B' \mid \varphi_i)| \\ &= \sum_{i=1}^{r} \left| \bigcup_{\theta \in \mathcal{O}_i} \mathrm{IBr}_N^0(\hat{B}' \mid \theta) \right| \\ &= \left| \bigcup_{\theta \in \mathrm{IBr}(b)} \mathrm{IBr}_N^0(\hat{B}' \mid \theta) \right| \\ &= |\mathrm{IBr}_N^0(\hat{B}')|. \end{aligned}$$

Finally since $|\mathrm{IBr}_N^0(B)| = |\mathrm{IBr}_N^0(B')|$ and $|\mathrm{IBr}_N^0(\hat{B})| = |\mathrm{IBr}_N^0(\hat{B}')|$, as implied by Lemma 2.7, we get $|\mathrm{IBr}_N^0(B)| = |\mathrm{IBr}_N^0(\hat{B})|$, as required. $\square$

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