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RATIONAL LAURENT SERIES WITH PURELY PERIODIC β-EXPANSIONS

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Abstract

The aim of this paper is to give families of Pisot and Salem elements β in $\mathbb{F}_q((x^{-1}))$ with the curious property that the β-expansion of any rational series in the unit disk $D(0, 1)$ is purely periodic. In contrast, the only known family of reals with the last property are quadratic Pisot numbers $\beta > 1$ that satisfy $\beta^2 = n\beta + 1$ for some integer $n \geq 1$.

1. Introduction

β-expansions of real numbers were introduced by A. Rényi [12]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors.

Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0,1]$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\{0, 1, \dots, [\beta]\}$ produced by the β-transformation $T_\beta\colon x \to \beta x \pmod 1$ as follows:

$$\forall i \geq 1, \quad x_i = [\beta T_\beta^{i-1}(x)], \quad \text{and thus} \quad x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

An expansion is finite if $(x_i)_{i\geq 1}$ is eventually 0. A β-expansion is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. We denote by $Per(\beta)$ the numbers in $[0,1)$ with periodic β-expansions, $Pur(\beta)$ the numbers in $[0, 1)$ with purely periodic β-expansions and $Fin(\beta)$ the numbers in $[0, 1)$ with finite β-expansions.

Let $\mathbb{Q}(\beta)$ be the smallest fields containing $\mathbb{Q}$ and β. An easy argument shows that $Per(\beta) \subseteq \mathbb{Q}(\beta) \cap [0, 1)$ for every real number $\beta > 1$. K. Schmidt [15] showed that if β is a Pisot number (an algebraic integer whose conjugates have modulus < 1), then $Per(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$.

The purely periodic β-expansions are also discussed by S. Ito and H. Rao in [7] when they characterize all reals in $[0,1[$ having purely periodic β-expansions with Pisot unit base. In [5], V. Berthé and A. Siegel completed the characterization in the Pisot non unit base.

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Set

$$\gamma(\beta) = \sup\{c \in [0,1) \colon \forall r \in \mathbb{Q} \cap [0,c],\ d_\beta(r) \text{ is purely periodic}\}.$$

S. Akiyama has proved in [3] that if β is a Pisot unit number satisfying the finiteness property ($Fin(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$), then $\gamma(\beta) > 0$.

In the quadratic case, K. Schmidt [15] has proved that if β satisfied $\beta^2 = n\beta + 1$ for some integer $n \geq 1$, then $\gamma(\beta) = 1$. Until now, it is the unique known family of reals for which $\gamma(\beta) = 1$. In [1] the authors has proved that if β is not Pisot unit, then $\gamma(\beta) = 0$, they also showed that if β is a cubic Pisot unit satisfying the finiteness property such that the number field $\mathbb{Q}(\beta)$ is not totally real, then $0 < \gamma(\beta) < 1$.

In this paper, we consider the analogue of this concept in the algebraic function over finite fields. We will show that the condition Pisot unit is not necessary to have $\gamma(\beta) > 0$. Especially, we give a sufficient condition for the conjugates of β to obtain $\gamma(\beta) = 1$.

2. $\boldsymbol{\beta}$-expansions in $\mathbb{F}_q((x^{-1}))$

Let $\mathbb{F}_q$ be a finite field of q elements, $\mathbb{F}_q[x]$ the ring of polynomials with coefficient in $\mathbb{F}_q$, $\mathbb{F}_q(x)$ the field of rational functions, $\mathbb{F}_q(x,\beta)$ the minimal extension of $\mathbb{F}_q$ containing x and β and $\mathbb{F}_q[x,\beta]$ the minimal ring containing x and β. Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series of the form:

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q$$

where

$$l = \deg f := \begin{cases} \max\{k \colon f_k \neq 0\} & \text{for} \quad f \neq 0; \\ -\infty & \text{for} \quad f = 0. \end{cases}$$

Define the absolute value

$$|f| = \begin{cases} q^{\deg f} & \text{for} \quad f \neq 0; \\ 0 & \text{for} \quad f = 0. \end{cases}$$

Since $|\,.\,|$ is not archimedean, $|\,.\,|$ fulfills the strict triangle inequality

$$\begin{aligned} |f+g| &\leq \max(|f|,|g|) \quad \text{and} \\ |f+g| &= \max(|f|,|g|) \quad \text{if} \quad |f| \neq |g|. \end{aligned}$$

Let $f \in \mathbb{F}_q((x^{-1}))$, define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k x^k$ where the empty sum, as usual, is defined to be zero. Therefore $[f] \in \mathbb{F}_q[x]$ and $(f - [f])$ is in the unit disk $D(0,1)$ for all $f \in \mathbb{F}_q((x^{-1}))$.

Proposition 2.1 ([11]). *Let K be complete field with respect to (a non archimedean absolute value $|\,.\,|$) and L/K ($K \subset L$) be an algebraic extension of degree m. Then $|.|$ has a unique extension to L defined by: $|a| = \sqrt[m]{|N_{L/K}(a)|}$ and L is complete with respect to this extension.*

We apply Proposition 2.1 to algebraic extensions of $\mathbb{F}_q((x^{-1}))$. Since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, every algebraic element over $\mathbb{F}_q[x]$ can be evaluated. However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed and uch an element do not necessarily expressed as a power series over x^{-1}. For a full characterization of the algebraic closure of $\mathbb{F}_q[x]$, we refer to Kedlaya [8].

An element $\beta = \beta_1 \in \mathbb{F}_q((x^{-1}))$ is called a Pisot (resp. Salem) element if it is an algebraic integer over $\mathbb{F}_q[x]$, $|\beta| > 1$ and $|\beta_j| < 1$ for all Galois conjugates β_j (resp. $|\beta_j| \leq 1$ and there exist at least one conjugate β_k such that $|\beta_k| = 1$).

P. Bateman and A.L. Duquette [4] had characterized the Pisot and Salem element in $\mathbb{F}_q((x^{-1}))$:

Theorem 2.1. *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and*

$$P(y) = y^n - A_1 y^{n-1} - \cdots - A_n, \quad A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then
(i) *β is a Pisot element if and only if $|A_1| > \max_{2\leq i\leq n}|A_i|$,*
(ii) *β is a Salem element if and only if $|A_1| = \max_{2\leq i\leq n}|A_i|$.*

Let $\beta,\ f \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. A representation in base β (or β-representation) of f is an infinite sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{F}_q[x]$, such that

$$f = \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f in base β, noted $d_\beta(f)$, which is obtained by using the β-transformation T_β in the unit disk which is given by $T_\beta(f) = \beta f - [\beta f]$. Then $d_\beta(f) = (a_i)_{i\geq 1}$ where $a_i = [\beta T_\beta^{i-1}(f)]$.

An equivalent definition of the β-expansion can be obtained by a greedy algorithm. This algorithm works as follows. Set $r_0 = f$ and let $a_i = [\beta r_{i-1}]$, $r_i = \beta r_{i-1} - a_i$ for all $i \geq 1$. The β-expansion of f will be noted as $d_\beta(f) = (a_i)_{i\geq 1}$.

Note that $d_\beta(f)$ is finite if and only if there is a $k \geq 0$ such that $T^k(f) = 0$, $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) for which $T_\beta^{p+s}(f) = T_\beta^p(f)$.

Now let $f \in \mathbb{F}_q((x^{-1}))$ be an element with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leq |f| < |\beta|^{k+1}$. Hence $|f/\beta^{k+1}| < 1$ and we can represent f by shifting

$d_\beta(f/\beta^{k+1})$ by k digits to the left. Therefore, if $d_\beta(f) = 0.d_1d_2d_3\cdots$, then $d_\beta(\beta f) = d_1.d_2d_3\cdots$.

If we have $d_\beta(f) = d_l d_{l-1}\cdots d_0.d_{-1}\cdots d_m$, then we put $\deg_\beta(f) = l$ and $\mathrm{ord}_\beta(f) = m$. In the sequal, we will use the following notations:

$$Fin(\beta) = \{f \in \mathbb{F}_q((x^{-1}))\colon d_\beta(f) \text{ is finite}\},$$
$$Per(\beta) = \{f \in \mathbb{F}_q((x^{-1}))\colon d_\beta(f) \text{ is eventually periodic}\},$$
$$Pur(\beta) = \{f \in \mathbb{F}_q((x^{-1})) \text{ and } |f| < 1\colon d_\beta(f) \text{ is purely periodic}\}.$$

REMARK 2.2. In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if z, $w \in \mathbb{F}_q((x^{-1}))$, we have $d_\beta(z + w) = d_\beta(z) + d_\beta(w)$ digitwise.

Theorem 2.2 ([6]). *A β-representation $(d_j)_{j\ge 1}$ is the β-expansion of f in the unit disk if and only if $|d_j| < |\beta|$ for $j \ge 1$.*

In the fields of formal series case, on the one hand, K. Scheicher, M. Jellali and M. Mkaouar [14] have studied the characterization of purely periodic β-expansions in the Pisot unit base. On the other hand, the following theorems are proved independently by Hbaib–Mkaouar and Scheicher.

Theorem 2.3 ([13]). *β is Pisot or Salem element if and only if $Per(\beta) = \mathbb{F}_q(x,\beta)$.*

Theorem 2.4 ([6]). *β is Pisot or Salem element if and only if $d_\beta(1)$ is periodic.*

In the papers [9] and [10], metric results are established and the relation to continued fractions is studied.

3. Results

By analogy with the real case, we define for each β such that $|\beta| > 1$ the quantity

$$\gamma(\beta) = \sup\{c \in [0, 1)\colon \forall\, f \in \mathbb{F}_q(x) \cap D(0, c),\ d_\beta(f) \text{ is purely periodic}\}.$$

In order to prove that $\gamma(\beta) > 0$ if β is a Pisot or Salem unit series, we need to introduce some basic notions: Let β be a Pisot or Salem unit series of minimal polynomial $\beta^d + A_{d-1}\beta^{n-1} + \cdots + A_0$ where $A_i \in \mathbb{F}_q[x]$ for $i \in \{1, \ldots, d-1\}$ and $A_0 \in \mathbb{F}_q^*$. Let $\beta^{(2)}, \ldots, \beta^{(d)}$ be the conjugates of β and we denote by $\overline{\beta}$ the vector conjugate of β given by $\overline{\beta} = \begin{pmatrix} \beta^{(2)} \\ \vdots \\ \beta^{(d)} \end{pmatrix}$. For $f = r_0 + r_1\beta + r_2\beta^2 + \cdots + r_{d-1}\beta^{d-1}$ with $r_i \in \mathbb{F}_q(x)$, the j-th conjugate of f in $\mathbb{F}_q(x,\beta)$ is given by $f^{(j)} = r_0 + r_1\beta^{(j)} + r_2(\beta^{(j)})^2 + \cdots + r_{d-1}(\beta^{(j)})^{d-1}$.

We define $\overline{f}$, the vector conjugate of f by $\overline{f} = \begin{pmatrix} f^{(2)} \\ \vdots \\ f^{(d)} \end{pmatrix}$ and $\|\overline{f}\| = \sup_{2\le k\le d}|f^{(k)}|$.

We begin with two lemmas which are essential for the development of the proof of Theorem 3.3.

Lemma 3.1 (Lemma 1, 2)**.** *Let β be an algebraic unit of degree n, and M be a positive number. Put*

$$X(p) = \{f \in Fin(\beta)\colon |f| \le M,\ \mathrm{ord}_\beta(f) = -p\}.$$

Then

$$\lim_{p\to\infty} \min_{f\in X(p)} \|\overline{f}\| = \infty.$$

Proof. Assume that there exist a constant B and an infinite sequence f_i $(i = 1, 2, \ldots)$ so that both

$$|f_i^{(j)}| \le B \quad \text{for} \quad j = 2, 3, \ldots, d \quad \text{and} \quad \lim_{i\to\infty} \mathrm{ord}_\beta(f_i) = -\infty$$

holds. As β is a unit, all f_i are in $\mathbb{F}_q[x, \beta]$ and $|f_i| \le M$, then these f_i's are finite. On the other hand, by the hypothesis $\lim_{i\to\infty} \mathrm{ord}_\beta(f_i) = -\infty$, the set $\{f_i,\ i \ge 1\}$ is infinite. This is absurd, which proves the lemma. □

Lemma 3.2. *Let β be a Pisot or Salem unit series. Then there exists $r > 0$ such that for every series h in $\mathbb{F}_q(x, \beta)$ satisfying $\mathrm{ord}_\beta(h) \le -1$, we have $\|\overline{h}\| > r$.*

Proof. According to Lemma 3.1, there exists $s > 0$ such that for every series f in $\mathbb{F}_q(x, \beta)$ satisfying $|f| < 1$ and $\mathrm{ord}_\beta(f) \le -s$, we have $\|\overline{f}\| > |\beta|$. Put $r = \inf_{j\in\{2,\ldots,d\}}|(\beta^{(j)})^{s-1}|\,|\beta|$, where $\beta^{(2)}, \ldots, \beta^{(d)}$ are the conjuguates of β.

Now, let h be a series in $\mathbb{F}_q(x, \beta)$ with $\mathrm{ord}_\beta(h) \le -1$. Then $h = \beta^{s-1}g$ where $\mathrm{ord}_\beta(g) \le -s$. Moreover h can be written such that $h = \beta^{s-1}(g_1 + g_2)$ where $\mathrm{ord}_\beta(g_1) \ge 0$, $\mathrm{ord}_\beta(g_2) = ord_\beta(g) \le -s$ and $|g_2| < 1$. Since $h = \beta^{s-1}(g_1 + g_2)$,

$$\overline{h} = \begin{pmatrix} (\beta^{(2)})^{s-1}(g_1^{(2)} + g_2^{(2)}) \\ (\beta^{(3)})^{s-1}(g_1^{(3)} + g_2^{(3)}) \\ \vdots \\ (\beta^{(d)})^{s-1}(g_1^{(d)} + g_2^{(d)}) \end{pmatrix}.$$

As β is a Pisot or Salem series and $g_1 = c_0 + c_1\beta + \cdots + c_{d-1}\beta^{d-1}$ with $c_i \in \mathbb{F}_q[x]$ and $|c_i| < |\beta|$, we have

$$|g_1^{(2)}| = |c_0 + c_1\beta^{(2)} + \cdots + c_{d-1}(\beta^{(2)})^{d-1}| \le |\beta|,$$

$$|g_1^{(3)}| = |c_0 + c_1\beta^{(3)} + \cdots + c_{d-1}(\beta^{(3)})^{d-1}| \leq |\beta|,$$
$$\vdots$$
$$|g_1^{(d)}| = |c_0 + c_1\beta^{(d)} + \cdots + c_{d-1}(\beta^{(d)})^{d-1}| \leq |\beta|.$$

Since $\mathrm{ord}_\beta(g_2) \leq -s$ and $|g_2| < 1$, we have $\|\overline{g}_2\| > |\beta|$. Thus, there exists $j_0 \in \{2,\ldots,n\}$ with $|g_2^{(j_0)}| > |\beta|$. So $|g_1^{(j_0)} + g_2^{(j_0)}| > |\beta|$, which implies that $|(\beta^{(j_0)})^{s-1}|\,|g_1^{(j_0)} + g_2^{(j_0)}| > \inf_{j\in\{2,\ldots,d\}} |(\beta^{(j)})^{s-1}|\,|\beta| = r$. Then we obtain $\|\overline{h}\| > r$. □

Theorem 3.3. *Let β be a Pisot or Salem unit series. Then $\gamma(\beta) > 0$.*

Proof. We will show that there exists a positive constant c such that every rational f with $|f| < c$ has a purely periodic β-expansion. Let $f \in \mathbb{F}_q(x,\beta) \cap D(0,1)$ and assume that f does not have a purely periodic β-expansion. Since β is a Pisot or Salem series, we know that $d_\beta(f)$ is periodic (by Theorem 2.3) and let m be the length of the period. So $d_\beta(f(\beta^m - 1))$ is finite because the β-expansion is closed under addition i.e.,

$$d_\beta(f(\beta^m - 1)) = d_\beta(f\beta^m) - d_\beta(f).$$

As $d_\beta(f)$ is not purely periodic, then $\mathrm{ord}_\beta(\beta^m f - f) < 0$. By Lemma 3.2, there exists $r > 0$ such that $\|\overline{\beta^m f - f}\| > r$.

Since β is a Pisot or Salem series, we have $\|\overline{f}\| \geq \|\overline{\beta^m f - f}\| \geq r$, with

$$\overline{\beta^m f - f} = \begin{pmatrix} (\beta^{(2)})^m f^{(2)} - f^{(2)} \\ (\beta^{(3)})^m f^{(3)} - f^{(3)} \\ \vdots \\ (\beta^{(d)})^m f^{(d)} - f^{(d)} \end{pmatrix}.$$

However $f \in \mathbb{F}_q(x)$, then for all $j \in \{2,\ldots,d\}$; $|f^{(j)}| = |f|$ and for this, we conclude that $|f| \geq r$. □

Theorem 3.4. *Let β be a Pisot or Salem element in $\mathbb{F}_q((x^{-1}))$ which has a conjugate $\tilde{\beta}$ satisfying $|\tilde{\beta}| \leq 1/|\beta|$. Then $\gamma(\beta) = 1$.*

Proof. Assume that β is a Pisot or Salem series, by Theorem 2.3 we can deduce that $d_\beta(f)$ is periodic. Let's suppose that f does not have a purely periodic β-expansion, so $d_\beta(f) = 0.a_1 \cdots a_p\overline{a_{p+1}\cdots a_{p+s}}$ and $a_p \neq a_{p+s}$. Hence

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s}\left(f - \frac{a_1}{\beta} - \cdots - \frac{a_p}{\beta^p}\right).$$

Since $a_1, \dots, a_{p+s} \in \mathbb{F}_q[x]$ and $f \in \mathbb{F}_q(x)$,

$$f = \frac{a_1}{\tilde{\beta}} + \cdots + \frac{a_p}{\tilde{\beta}^p} + \frac{a_{p+1}}{\tilde{\beta}^{p+1}} + \cdots + \frac{a_{p+s}}{\tilde{\beta}^{p+s}} + \frac{1}{\tilde{\beta}^s}\left(f - \frac{a_1}{\tilde{\beta}} - \cdots - \frac{a_p}{\tilde{\beta}^p}\right).$$

We get

$$f\left(1 - \frac{1}{\tilde{\beta}^s}\right) = \frac{a_1}{\tilde{\beta}} + \cdots + \frac{a_p}{\tilde{\beta}^p} + \frac{a_{p+1}}{\tilde{\beta}^{p+1}} + \cdots + \frac{a_{p+s}}{\tilde{\beta}^{p+s}} + \frac{1}{\tilde{\beta}^s}\left(- \frac{a_1}{\tilde{\beta}} - \cdots - \frac{a_p}{\tilde{\beta}^p}\right).$$

Therefore

$$f(\tilde{\beta}^{s+p} - \tilde{\beta}^p) = a_1\tilde{\beta}^{s+p-1} + \cdots + a_{p+s} - a_1\tilde{\beta}^{p-1} - \cdots - a_p.$$

Since $|\tilde{\beta}| \leq 1/|\beta|$, then we get

$$|f|\,|\tilde{\beta}^p| = |a_{p+s} - a_p|.$$

So

$$\frac{|f|}{|\beta|^p} \geq |a_{p+s} - a_p|.$$

Since $a_{p+s} - a_p \neq 0$, $|f| \geq |\beta|^p$. which is absurd because f is in the unit disk. $\square$

Proposition 3.1. *If β is a Pisot or Salem series which has a conjugate $\tilde{\beta}$ satisfying $|\tilde{\beta}| \leq 1/|\beta|$, then β is unit.*

Proof. Let β be a Salem series of degree d satisfying $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_1\beta + A_0 = 0$ where $A_i \in \mathbb{F}_q[x]$ $(A_0 \neq 0)$ and let $\beta_1 = \beta, \dots, \beta_d$ be the conjugates of β. So

$$|A_0| = |\beta\beta_2 \cdots \beta_d|.$$

If we have for example $|\beta_2| \leq 1/|\beta|$, so we get

$$|A_0| \leq |\beta_3 \cdots \beta_d|.$$

Therefore

$$|\beta_3| = |\beta_4| = \cdots = |\beta_d| = 1 \quad \text{and} \quad |A_0| = 1,$$

what gives that $A_0 \in \mathbb{F}_q^*$. $\square$

The “unit” condition is necessary in the Theorem 3.3. In fact, in the non unit base, we get $\gamma(\beta) = 0$. For that we will give the following result in an analogous way to the real case [3].

Proposition 3.2. *Let β be a series which is not a unit. Then $\gamma(\beta) = 0$.*

Proof. Let $P(f) = A_n f^n + A_{n-1} f^{n-1} + \cdots + A_0$ be the minimal polynomial of β with $A_i \in \mathbb{F}_q[x]$ for all $i \in \{1, \ldots, n\}$ and $A_0 \in \mathbb{F}_q[x] \setminus \mathbb{F}_q^*$. Let $f_n = 1/A_0^n$ with $n \in \mathbb{N}^*$, we will prove that f_n does not have purely periodic β-expansion. We see

$$\begin{aligned}
f_n &= \frac{a_1}{\beta} + \cdots + \frac{a_k}{\beta^k} + \frac{f}{\beta^k} \\
&= \left(\frac{a_1}{\beta} + \cdots + \frac{a_k}{\beta^k}\right)\left(1 + \frac{1}{\beta^k} + \frac{1}{\beta^{2k}} + \cdots\right) \\
&= \left(\sum_{i=1}^{k} a_i \beta^{-i}\right)\left(\sum_{i \geq 0} \frac{1}{\beta^{ik}}\right) \\
&= \frac{\sum_{i=1}^{k} a_i \beta^{-i}}{1 - \beta^{-k}} \\
&= \frac{\sum_{i=0}^{k-1} a_{k-i} \beta^{i}}{\beta^k - 1}.
\end{aligned}$$

So we have $f_n(1-\beta^k) = \sum_{i=0}^{k-1}(-a_{k-i})\beta^i = (1-\beta^k)/A_0^n \in \mathbb{F}_q[x, \beta]$, then $(1-\beta^k)/A_0^n = c_{n-1}\beta^{n-1} + c_{n-2}\beta^{n-2} + \cdots + c_0$ with $c_{n-1}, \ldots, c_0 \in \mathbb{F}_q[x]$. Consequently,

$$\begin{aligned}
1 - \beta^k &= {A_0}^n(c_{n-1}\beta^{n-1} + \cdots + c_0) \\
&= (-A_n\beta^n - A_{n-1}\beta^{n-1} - \cdots - A_1\underline{\beta})^n(c_{n-1}\beta^{n-1} + \cdots + c_0).
\end{aligned}$$

As a result $1 = \beta(z_t\beta^t + \cdots + z_0)$ and this contradicts the hypothesis that β is not unit. □

Theorem 3.5. *Let β be a quadratic Pisot unit series. Then $\gamma(\beta) = 1$.*

Proof. In this case β satisfies $\beta^2 + A\beta + c = 0$, where $|A| > 1$ and $c \in \mathbb{F}_q^*$ so, the unique conjugate of β is $\tilde{\beta}$ such that

$$\beta\tilde{\beta} = c, \quad \text{which} \quad |\tilde{\beta}| = \frac{1}{|\beta|}.$$

By Theorem 3.4, we obtain the result. □

REMARK 3.3. We remark that if β is a Pisot or Salem not unit series then β has not a conjugate $\tilde{\beta}$ such that $|\tilde{\beta}| = 1/|\beta|$ and the quadratic case is the only case where a Pisot unit series β has a conjugate $\tilde{\beta}$ such that $|\tilde{\beta}| = 1/|\beta|$.

However, if β is an algebraic integer of degree $d > 2$ over $\mathbb{F}_q[x]$ and $\beta_2, \ldots, \beta_d$ their $(d-1)$ conjugates, then we have $|\beta\beta_2 \cdots \beta_d| = 1$. If we suppose that for a certain

i with $|\beta_i| = 1/|\beta|$, then

$$\left|\prod_{j\neq i} \beta_i\right| = 1,$$

which is absurd because $|\beta_i| < 1$ for all i in $\{2, \dots, d\}$.

Theorem 3.6. *Let β be a Salem unit satisfying $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_1\beta + b = 0$, where $b \in \mathbb{F}_q^*$ and $|A_1| = |A_{d-1}|$. Then $\gamma(\beta) = 1$.*

Proof. Let $\beta_2, \dots, \beta_d$ be the $d-1$ conjugates of β and let's note that $\beta_1 = \beta$, so we have

$$\left|\prod_{1\leq i\leq d} \beta_i\right| = |b| = 1.$$

This implies that there exists at least one conjugate of absolute value less than 1.

In the other hand we have:

$$|\beta_1 + \beta_2 + \cdots + \beta_d| = |\beta| = |A_{d-1}|.$$

By the symmetrical relations between the roots, we get

$$\left|\sum_{1\leq i_1 < i_2 < \cdots < i_{d-1} \leq d} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_{d-1}}\right| = |A_1|.$$

So if we suppose that β has more then 2 conjugates of absolute value lower to 1 and the other of equal absolute value 1, then we obtain in this case $|A_1| < |\beta|$ which contradicts the hypothesis that $|\beta| = |A_{d-1}| = |A_1|$.

Finally we conclude that β has a unique conjugate $\tilde{\beta}$ such that $|\tilde{\beta}| < 1$ and the other conjugates of equal absolute value 1. So, $|\tilde{\beta}| = 1/|\beta|$ and by Theorem 3.4 every rational series in the unit disk have a purely periodic β-expansion. □

Corollary 3.7. *Let β be a cubic Salem unit series. Then $\gamma(\beta) = 1$.*

Proof. Let β be a cubic Salem unit series. In in this case the minimal polynomial of β is

$$P(y) = y^3 + A_2y^2 + A_1y + b \quad \text{where} \quad b \in \mathbb{F}_q^*,$$

and by Theorem 2.1, we have $|A_2| = |A_1|$. According to Theorem 3.4, we deduce that every rational series in the unit disk have a purely periodic β-expansion. □

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