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RATIONAL LAURENT SERIES WITH PURELY PERIODIC $\beta$-EXPANSIONS

FARAH ABBES and MOHAMED HBAIB

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Abstract
The aim of this paper is to give families of Pisot and Salem elements $\beta$ in $\mathbb{F}_p((x^{-1}))$ with the curious property that the $\beta$-expansion of any rational series in the unit disk $D(0, 1)$ is purely periodic. In contrast, the only known family of reals with the last property are quadratic Pisot numbers $\beta > 1$ that satisfy $\beta^2 = n\beta + 1$ for some integer $n \geq 1$.

1. Introduction

$\beta$-expansions of real numbers were introduced by A. Rényi [12]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors.

Let $\beta > 1$ be a real number. The $\beta$-expansion of a real number $x \in [0, 1]$ is defined as the sequence $(x_i)_{i \geq 1}$ with values in $\{0, 1, \ldots, [\beta]\}$ produced by the $\beta$-transformation $T_\beta : x \mapsto \beta x \pmod{1}$ as follows:

$$
\forall i \geq 1, \quad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \quad \text{and thus} \quad x = \sum_{i \geq 1} \frac{x_i}{\beta^i}.
$$

An expansion is finite if $(x_i)_{i \geq 1}$ is eventually 0. A $\beta$-expansion is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. We denote by $\text{Per}(\beta)$ the numbers in $[0, 1)$ with periodic $\beta$-expansions, $\text{Pur}(\beta)$ the numbers in $[0, 1)$ with purely periodic $\beta$-expansions and $\text{Fin}(\beta)$ the numbers in $[0, 1)$ with finite $\beta$-expansions.

Let $\mathbb{Q}(\beta)$ be the smallest fields containing $\mathbb{Q}$ and $\beta$. An easy argument shows that $\text{Per}(\beta) \subseteq \mathbb{Q}(\beta) \cap [0, 1)$ for every real number $\beta > 1$. K. Schmidt [15] showed that if $\beta$ is a Pisot number (an algebraic integer whose conjugates have modulus $< 1$), then $\text{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$.

The purely periodic $\beta$-expansions are also discussed by S. Ito and H. Rao in [7] when they characterize all reals in $[0,1]$ having purely periodic $\beta$-expansions with Pisot unit base. In [5], V. Berthé and A. Siegel completed the characterization in the Pisot non unit base.

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Set
\[ \gamma(\beta) = \sup\{c \in [0, 1) : \forall r \in \mathbb{Q} \cap [0, c], \ d_g(r) \text{ is purely periodic}\}. \]

S. Akiyama has proved in \cite{3} that if \( \beta \) is a Pisot unit number satisfying the finiteness property \( (\text{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+) \), then \( \gamma(\beta) > 0 \).

In the quadratic case, K. Schmidt \cite{15} has proved that if \( \beta \) satisfied \( \beta^2 = n\beta + 1 \) for some integer \( n \geq 1 \), then \( \gamma(\beta) = 1 \). Until now, it is the unique known family of reals for which \( \gamma(\beta) = 1 \). In \cite{1} the authors has proved that if \( \beta \) is not Pisot unit, then \( \gamma(\beta) = 0 \), they also showed that if \( \beta \) is a cubic Pisot unit satisfying the finiteness property such that the number field \( \mathbb{Q}(\beta) \) is not totally real, then \( 0 < \gamma(\beta) < 1 \).

In this paper, we consider the analogue of this concept in the algebraic function over finite fields. We will show that the condition Pisot unit is not necessary to have \( \gamma(\beta) > 0 \). Especially, we give a sufficient condition for the conjugates of \( \beta \) to obtain \( \gamma(\beta) = 1 \).

2. \( \beta \)-expansions in \( \mathbb{F}_q((x^{-1})) \)

Let \( \mathbb{F}_q \) be a finite field of \( q \) elements, \( \mathbb{F}_q[x] \) the ring of polynomials with coefficient in \( \mathbb{F}_q \), \( \mathbb{F}_q(x) \) the field of rational functions, \( \mathbb{F}_q(x, \beta) \) the minimal extension of \( \mathbb{F}_q \) containing \( x \) and \( \beta \) and \( \mathbb{F}_q[x, \beta] \) the minimal ring containing \( x \) and \( \beta \). Let \( \mathbb{F}_q((x^{-1})) \) be the field of formal power series of the form:

\[ f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q \]

where

\[ l = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{for } f \neq 0; \\ -\infty & \text{for } f = 0. \end{cases} \]

Define the absolute value

\[ |f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases} \]

Since \( |.| \) is not archimedean, \( |.| \) fulfills the strict triangle inequality

\[ |f + g| \leq \max(|f|, |g|) \quad \text{and} \]

\[ |f + g| = \max(|f|, |g|) \quad \text{if } |f| \neq |g|. \]

Let \( f \in \mathbb{F}_q((x^{-1})) \), define the integer (polynomial) part \( \lfloor f \rfloor = \sum_{k=0}^{l} f_k x^k \) where the empty sum, as usual, is defined to be zero. Therefore \( \lfloor f \rfloor \in \mathbb{F}_q[x] \) and \( (f - \lfloor f \rfloor) \) is in the unit disk \( D(0, 1) \) for all \( f \in \mathbb{F}_q((x^{-1})) \).
Proposition 2.1 (\cite{11}). Let \( K \) be complete field with respect to (a non archimedean absolute value \( | \cdot | \)) and \( L/K \) (\( K \subset L \)) be an algebraic extension of degree \( m \). Then \( | \cdot | \) has a unique extension to \( L \) defined by: \( |a| = \sqrt[m]{N_{L/K}(a)} \) and \( L \) is complete with respect to this extension.

We apply Proposition 2.1 to algebraic extensions of \( \mathbb{F}_q((x^{-1})) \). Since \( \mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1})) \), every algebraic element over \( \mathbb{F}_q[x] \) can be evaluated. However, since \( \mathbb{F}_q((x^{-1})) \) is not algebraically closed and such an element do not necessarily expressed as a power series over \( x^{-1} \). For a full characterization of the algebraic closure of \( \mathbb{F}_q[x] \), we refer to Kedlaya [8].

An element \( \beta = \beta_1 \in \mathbb{F}_q((x^{-1})) \) is called a Pisot (resp. Salem) element if it is an algebraic integer over \( \mathbb{F}_q[x] \), \( |\beta| > 1 \) and \( |\beta_j| < 1 \) for all Galois conjugates \( \beta_j \) (resp. \( |\beta_j| \leq 1 \) and there exist at least one conjugate \( \beta_k \) such that \( |\beta_k| = 1 \).

P. Bateman and A.L. Duquette [4] had characterized the Pisot and Salem element in \( \mathbb{F}_q((x^{-1})) \):

Theorem 2.1. Let \( \beta \in \mathbb{F}_q((x^{-1})) \) be an algebraic integer over \( \mathbb{F}_q[x] \) and

\[
P(y) = y^n - A_1 y^{n-1} - \cdots - A_n, \quad A_i \in \mathbb{F}_q[x],
\]

be its minimal polynomial. Then

(i) \( \beta \) is a Pisot element if and only if \( |A_1| > \max_{2 \leq i \leq n} |A_i| \),

(ii) \( \beta \) is a Salem element if and only if \( |A_1| = \max_{2 \leq i \leq n} |A_i| \).

Let \( \beta, f \in \mathbb{F}_q((x^{-1})) \) with \( |\beta| > 1 \). A representation in base \( \beta \) (or \( \beta \)-representation) of \( f \) is an infinite sequence \( (d_i)_{i \geq 1} \), \( d_i \in \mathbb{F}_q[x] \), such that

\[
f = \sum_{i \geq 1} \frac{d_i}{\beta^i}.
\]

A particular \( \beta \)-representation of \( f \) is called the \( \beta \)-expansion of \( f \) in base \( \beta \), noted \( d_\beta(f) \), which is obtained by using the \( \beta \)-transformation \( T_\beta \) in the unit disk which is given by \( T_\beta(f) = \beta f - \lfloor \beta f \rfloor \). Then \( d_\beta(f) = (a_i)_{i \geq 1} \) where \( a_i = \lfloor \beta T_\beta^{i-1}(f) \rfloor \).

An equivalent definition of the \( \beta \)-expansion can be obtained by a greedy algorithm. This algorithm works as follows. Set \( r_0 = f \) and let \( a_i = \lfloor \beta r_{i-1} \rfloor \), \( r_i = \beta r_{i-1} - a_i \) for all \( i \geq 1 \). The \( \beta \)-expansion of \( f \) will be noted as \( d_\beta(f) = (a_i)_{i \geq 1} \).

Note that \( d_\beta(f) \) is finite if and only if there is a \( k \geq 0 \) such that \( T^k(f) = 0 \). \( d_\beta(f) \) is ultimately periodic if and only if there is some smallest \( p \geq 0 \) (the pre-period length) and \( s \geq 1 \) (the period length) for which \( T_\beta^{p+s}(f) = T_\beta^p(f) \).

Now let \( f \in \mathbb{F}_q((x^{-1})) \) be an element with \( |f| \geq 1 \). Then there is a unique \( k \in \mathbb{N} \) such that \( |\beta|^k \leq |f| < |\beta|^{k+1} \). Hence \( |f/\beta^{k+1}| < 1 \) and we can represent \( f \) by shifting
$d_{\rho}(f/\beta^{k+1})$ by $k$ digits to the left. Therefore, if $d_{\rho}(f) = 0.d_1d_2d_3 \cdots$, then $d_{\rho}(\beta f) = d_1.d_2d_3 \cdots$.

If we have $d_{\rho}(f) = d_1d_2 \cdots d_{m-1}d_m$, then we put $\deg_{\rho}(f) = l$ and $\ord_{\rho}(f) = m$.

In the sequel, we will use the following notations:

$\text{Fin}(\beta) = \{ f \in \mathbb{F}_q((x^{-1})) : d_{\rho}(f) \text{ is finite} \}$,

$\text{Per}(\beta) = \{ f \in \mathbb{F}_q((x^{-1})) : d_{\rho}(f) \text{ is eventually periodic} \}$,

$\text{Pur}(\beta) = \{ f \in \mathbb{F}_q((x^{-1})) \text{ and } |f| < 1 : d_{\rho}(f) \text{ is purely periodic} \}$.

**Remark 2.2.** In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if $z, w \in \mathbb{F}_q((x^{-1}))$, we have $d_{\rho}(z + w) = d_{\rho}(z) + d_{\rho}(w)$ digitwise.

**Theorem 2.2 ([6]).** A $\beta$-representation $(d_j)_{j \geq 1}$ is the $\beta$-expansion of $f$ in the unit disk if and only if $|d_j| < |\beta|$ for $j \geq 1$.

In the fields of formal series case, on the one hand, K. Scheicher, M. Jellali and M. Mkaouar [14] have studied the characterization of purely periodic $\beta$-expansions in the Pisot unit base. On the other hand, the following theorems are proved independently by Hbaib-Mkaouar and Scheicher.

**Theorem 2.3 ([13]).** $\beta$ is Pisot or Salem element if and only if $\text{Per}(\beta) = \mathbb{F}_q(x, \beta)$.

**Theorem 2.4 ([6]).** $\beta$ is Pisot or Salem element if and only if $d_{\rho}(1)$ is periodic.

In the papers [9] and [10], metric results are established and the relation to continued fractions is studied.

3. Results

By analogy with the real case, we define for each $\beta$ such that $|\beta| > 1$ the quantity

$\gamma(\beta) = \sup\{c \in [0, 1) : \forall f \in \mathbb{F}_q(x) \cap D(0, c), d_{\rho}(f) \text{ is purely periodic} \}$.

In order to prove that $\gamma(\beta) > 0$ if $\beta$ is a Pisot or Salem unit series, we need to introduce some basic notions: Let $\beta$ be a Pisot or Salem unit series of minimal polynomial $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0$ where $A_i \in \mathbb{F}_q[x]$ for $i \in \{1, \ldots, d-1\}$ and $A_0 \in \mathbb{F}_q^*$. Let $\beta^{(2)}, \ldots, \beta^{(d)}$ be the conjugates of $\beta$ and we denote by $\overline{\beta}$ the vector conjugate of $\beta$ given by $\overline{\beta} = \begin{pmatrix} \beta^{(2)} \\ \vdots \\ \beta^{(d)} \end{pmatrix}$. For $f = r_0 + r_1\beta + r_2\beta^2 + \cdots + r_{d-1}\beta^{d-1}$ with $r_i \in \mathbb{F}_q(x)$, the $j$-th conjugate of $f$ in $\mathbb{F}_q(x, \beta)$ is given by $f^{(j)} = r_0 + r_1\beta^{(j)} + r_2(\beta^{(j)})^2 + \cdots + r_{d-1}(\beta^{(j)})^{d-1}$.
We define $\overline{f}$, the vector conjugate of $f$ by $\overline{f} = \begin{pmatrix} f^{(2)} \\ \vdots \\ f^{(d)} \end{pmatrix}$ and $\|\overline{f}\| = \sup_{2 \leq k \leq d} |f^{(k)}|.$

We begin with two lemmas which are essential for the development of the proof of Theorem 3.3.

**Lemma 3.1** (Lemma 1, 2). Let $\beta$ be an algebraic unit of degree $n$, and $M$ be a positive number. Put

$$X(p) = \{ f \in \text{Fin}(\beta) : |f| \leq M, \ ord_{\beta}(f) = -p \}.$$ 

Then

$$\lim_{p \to \infty} \min_{f \in X(p)} \|\overline{f}\| = \infty.$$ 

Proof. Assume that there exist a constant $B$ and an infinite sequence $f_i$ ($i = 1, 2, \ldots$) so that both

$$|f_i^{(j)}| \leq B \quad \text{for} \quad j = 2, 3, \ldots, d \quad \text{and} \quad \lim_{i \to \infty} \ord_{\beta}(f_i) = -\infty$$

holds. As $\beta$ is a unit, all $f_i$ are in $\mathbb{F}_q[x, \beta]$ and $|f_i| \leq M$, then these $f_i$’s are finite. On the other hand, by the hypothesis $\lim_{i \to \infty} \ord_{\beta}(f_i) = -\infty$, the set $\{ f_i, i \geq 1 \}$ is infinite. This is absurd, which proves the lemma. □

**Lemma 3.2.** Let $\beta$ be a Pisot or Salem unit series. Then there exists $r > 0$ such that for every series $h$ in $\mathbb{F}_q(x, \beta)$ satisfying $\ord_{\beta}(h) \leq -1$, we have $\|h\| > r$.

Proof. According to Lemma 3.1, there exists $s > 0$ such that for every series $f$ in $\mathbb{F}_q(x, \beta)$ satisfying $|f| < 1$ and $\ord_{\beta}(f) \leq -s$, we have $\|\overline{f}\| > |\beta|$. Put $r = \inf_{f \in [1, \ldots, d]} |\beta^{(j)}|^{s-1} |\beta|$, where $\beta^{(2)}, \ldots, \beta^{(d)}$ are the conjugates of $\beta$.

Now, let $h$ be a series in $\mathbb{F}_q(x, \beta)$ with $\ord_{\beta}(h) \leq -1$. Then $h = \beta^{t-1}g$ where $\ord_{\beta}(g) \leq -s$. Moreover $h$ can be written such that $h = \beta^{t-1}(g_1 + g_2)$ where $\ord_{\beta}(g_1) \geq 0$, $\ord_{\beta}(g_2) = \ord_{\beta}(g) \leq -s$ and $|g_2| < 1$. Since $h = \beta^{t-1}(g_1 + g_2),

$$\overline{h} = \begin{pmatrix} (\beta^{(2)})^{s-1}(g_1^{(2)} + g_2^{(2)}) \\ (\beta^{(3)})^{s-1}(g_1^{(3)} + g_2^{(3)}) \\ \vdots \\ (\beta^{(d)})^{s-1}(g_1^{(d)} + g_2^{(d)}) \end{pmatrix}.$$ 

As $\beta$ is a Pisot or Salem series and $g_1 = c_0 + c_1 \beta + \cdots + c_{d-1} \beta^{d-1} + c_d \beta^d$ with $c_i \in \mathbb{F}_q[x]$ and $|c_1| < |\beta|$, we have

$$|g_1^{(2)}| = |c_0 + c_1 \beta^{(2)} + \cdots + c_{d-1} \beta^{(d-1)}| \leq |\beta|,$$
\[ |g_1^{(3)}| = |c_0 + c_1 \beta^{(3)} + \cdots + c_{d-1}(\beta^{(3)})^{d-1}| \leq |\beta|, \]
\[ \vdots \]
\[ |g_1^{(d)}| = |c_0 + c_1 \beta^{(d)} + \cdots + c_{d-1}(\beta^{(d)})^{d-1}| \leq |\beta|. \]

Since \( \text{ord}_g(g_2) \leq -s \) and \( |g_2| < 1 \), we have \( \| g_2 \| > |\beta| \). Thus, there exists \( j_0 \in \{2, \ldots, n\} \) with \( |g_2^{(j_0)}| > |\beta| \). So \( |g_1^{(j_0)} + g_2^{(j_0)}| > |\beta| \), which implies that \( |(\beta^{(j_0)})^{i-1}| |g_1^{(j_0)} + g_2^{(j_0)}| > \inf_{j \in \{2, \ldots, d\}} |(\beta^{(j)})^{i-1}| |\beta| = r \). Then we obtain \( \| \beta \| > r. \)

**Theorem 3.3.** Let \( \beta \) be a Pisot or Salem unit series. Then \( \gamma(\beta) > 0. \)

**Proof.** We will show that there exists a positive constant \( c \) such that every rational \( f \) with \( |f| < c \) has a purely periodic \( \beta \)-expansion. Let \( f \in \mathbb{F}_q(x, \beta) \cap D(0, 1) \) and assume that \( f \) does not have a purely periodic \( \beta \)-expansion. Since \( \beta \) is a Pisot or Salem series, we know that \( d_\beta(f) \) is periodic (by Theorem 2.3) and let \( m \) be the length of the period. So \( d_\beta(f(\beta^m - 1)) \) is finite because the \( \beta \)-expansion is closed under addition i.e.,
\[ d_\beta(f(\beta^m - 1)) = d_\beta(f \beta^m) - d_\beta(f). \]
As \( d_\beta(f) \) is not purely periodic, then \( \text{ord}_\beta(\beta^m f - f) < 0 \). By Lemma 3.2, there exists \( r > 0 \) such that \( \| \beta^m f - f \| > r. \)
Since \( \beta \) is a Pisot or Salem series, we have \( \| f \| > \| \beta^m f - f \| \geq r \), with
\[ \beta^m f - f = \begin{pmatrix} (\beta^{(2)})^m f^{(2)} - f^{(2)} \\ (\beta^{(3)})^m f^{(3)} - f^{(3)} \\ \vdots \\ (\beta^{(d)})^m f^{(d)} - f^{(d)} \end{pmatrix}. \]
However \( f \in \mathbb{F}_q(x) \), then for all \( j \in \{2, \ldots, d\} \), \( |f^{(j)}| = |f| \) and for this, we conclude that \( |f| \geq r. \)

**Theorem 3.4.** Let \( \beta \) be a Pisot or Salem element in \( \mathbb{F}_q((x^{-1})) \) which has a conjugate \( \bar{\beta} \) satisfying \( |\bar{\beta}| \leq 1/|\beta|. \) Then \( \gamma(\beta) = 1. \)

**Proof.** Assume that \( \beta \) is a Pisot or Salem series, by Theorem 2.3 we can deduce that \( d_\beta(f) \) is periodic. Let’s suppose that \( f \) does not have a purely periodic \( \beta \)-expansion, so \( d_\beta(f) = 0.a_1 \cdots a_p a_{p+1} \cdots a_{p+s} \) and \( a_p \neq a_{p+s} \). Hence
\[ f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^p} \left( f - \frac{a_1}{\beta} - \cdots - \frac{a_p}{\beta^p} \right). \]
Since $a_1, \ldots, a_{p^s} \in \mathbb{F}_q[x]$ and $f \in \mathbb{F}_q(x)$,

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s} \left( f - \frac{a_1}{\beta} - \cdots - \frac{a_p}{\beta^p} \right).$$

We get

$$f \left( 1 - \frac{1}{\beta^s} \right) = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s} \left( f - \frac{a_1}{\beta} - \cdots - \frac{a_p}{\beta^p} \right).$$

Therefore

$$f (\beta^{s+p} - \tilde{\beta}^p) = a_1 \tilde{\beta}^{s+p-1} + \cdots + a_{p+s} - a_1 \tilde{\beta}^{p-1} - \cdots - a_p.$$

Since $|\tilde{\beta}| \leq 1/|\beta|$, then we get

$$|f| |\tilde{\beta}^p| = |a_{p+s} - a_p|.$$

So

$$\frac{|f|}{|\beta|^p} \geq |a_{p+s} - a_p|.$$

Since $a_{p+s} - a_p \neq 0$, $|f| \geq |\beta|^p$. which is absurd because $f$ is in the unit disk. \qed

**Proposition 3.1.** If $\beta$ is a Pisot or Salem series which has a conjugate $\tilde{\beta}$ satisfying $|\tilde{\beta}| \leq 1/|\beta|$, then $\beta$ is unit.

**Proof.** Let $\beta$ be a Salem series of degree $d$ satisfying $\beta^d + A_{d-1} \beta^{d-1} + \cdots + A_1 \beta + A_0 = 0$ where $A_i \in \mathbb{F}_q[x]$ ($A_0 \neq 0$) and let $\beta_1 = \beta, \ldots, \beta_d$ be the conjugates of $\beta$. So

$$|A_0| = |\beta \beta_2 \cdots \beta_d|.$$

If we have for example $|\beta_{2s}| \leq 1/|\beta|$, so we get

$$|A_0| \leq |\beta_3 \cdots \beta_d|.$$

Therefore

$$|\beta_3| = |\beta_4| = \cdots = |\beta_d| = 1 \quad \text{and} \quad |A_0| = 1,$$

what gives that $A_0 \in \mathbb{F}_q^*$. \qed

The “unit” condition is necessary in the Theorem 3.3. In fact, in the non unit base, we get $\gamma(\beta) = 0$. For that we will give the following result in an analogous way to the real case [3].
Proposition 3.2. Let \( \beta \) be a series which is not a unit. Then \( \gamma(\beta) = 0 \).

Proof. Let \( P(f) = A_n f^n + A_{n-1} f^{n-1} + \cdots + A_0 \) be the minimal polynomial of \( \beta \) with \( A_i \in \mathbb{F}_q[x] \) for all \( i \in \{1, \ldots, n\} \) and \( A_0 \in \mathbb{F}_q \setminus \mathbb{F}_q^* \). Let \( f_n = 1/A_0^n \) with \( n \in \mathbb{N}^* \), we will prove that \( f_n \) does not have purely periodic \( \beta \)-expansion. We see

\[
\begin{align*}
f_n &= \frac{a_1}{\beta} + \cdots + \frac{a_k}{\beta^k} + \frac{f}{\beta^k} \\
&= \left( \frac{a_1}{\beta} + \cdots + \frac{a_k}{\beta^k} \right) \left( 1 + \frac{1}{\beta^k} + \frac{1}{\beta^{2k}} + \cdots \right) \\
&= \left( \sum_{i=1}^k a_i \beta^{-i} \right) \left( \sum_{i=0}^{\infty} \frac{1}{\beta^i} \right) \\
&= \frac{\sum_{i=1}^k a_i \beta^{-i}}{1 - \beta^{-k}} \\
&= \frac{\sum_{i=0}^{k-1} a_{k-i} \beta^i}{\beta^k - 1}.
\end{align*}
\]

So we have \( f_n(1 - \beta^k) = \sum_{i=0}^{k-1} (-a_{k-i}) \beta^i = (1 - \beta^k)/A_0^n \in \mathbb{F}_q[x, \beta] \), then \( (1 - \beta^k)/A_0^n = c_{n-1} \beta^{n-1} + c_{n-2} \beta^{n-2} + \cdots + c_0 \) with \( c_{n-1}, \ldots, c_0 \in \mathbb{F}_q[x] \). Consequently,

\[
1 - \beta^k = A_0^n (c_{n-1} \beta^{n-1} + \cdots + c_0) \\
= (-A_n \beta^n - A_{n-1} \beta^{n-1} - \cdots - A_1 \beta) (c_{n-1} \beta^{n-1} + \cdots + c_0).
\]

As a result \( 1 = \beta(z_1 \beta^l + \cdots + z_0) \) and this contradicts the hypothesis that \( \beta \) is not unit. \( \square \)

Theorem 3.5. Let \( \beta \) be a quadratic Pisot unit series. Then \( \gamma(\beta) = 1 \).

Proof. In this case \( \beta \) satisfies \( \beta^2 + A \beta + c = 0 \), where \( |A| > 1 \) and \( c \in \mathbb{F}_q^* \) so, the unique conjugate of \( \beta \) is \( \tilde{\beta} \) such that

\[
\beta \tilde{\beta} = c, \quad \text{which} \quad |\tilde{\beta}| = \frac{1}{|\beta|}.
\]

By Theorem 3.4, we obtain the result. \( \square \)

Remark 3.3. We remark that if \( \beta \) is a Pisot or Salem not unit series then \( \beta \) has not a conjugate \( \tilde{\beta} \) such that \( |\tilde{\beta}| = 1/|\beta| \) and the quadratic case is the only case where a Pisot unit series \( \beta \) has a conjugate \( \tilde{\beta} \) such that \( |\tilde{\beta}| = 1/|\beta| \).

However, if \( \beta \) is an algebraic integer of degree \( d > 2 \) over \( \mathbb{F}_q[x] \) and \( \beta_2, \ldots, \beta_d \) their \( (d-1) \) conjugates, then we have \( |\beta \beta_2 \cdots \beta_d| = 1 \). If we suppose that for a certain
$i$ with $|\beta_i| = 1/|\beta|$, then

$$\left| \prod_{j \neq i} \beta_j \right| = 1,$$

which is absurd because $|\beta_i| < 1$ for all $i$ in $\{2, \ldots, d\}$.

**Theorem 3.6.** Let $\beta$ be a Salem unit satisfying $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_1\beta + b = 0$, where $b \in \mathbb{F}_q^*$ and $|A_1| = |A_{d-1}|$. Then $\gamma(\beta) = 1$.

**Proof.** Let $\beta_2, \ldots, \beta_d$ be the $d - 1$ conjugates of $\beta$ and let’s note that $\beta_1 = \beta$, so we have

$$\left| \prod_{1 \leq i \leq d} \beta_i \right| = |b| = 1.$$

This implies that there exists at least one conjugate of absolute value less than 1.

In the other hand we have:

$$|\beta_1 + \beta_2 + \cdots + \beta_d| = |\beta| = |A_{d-1}|.$$

By the symmetrical relations between the roots, we get

$$\left| \sum_{1 \leq i_1 < i_2 < \cdots < i_{d-1} \leq d} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_{d-1}} \right| = |A_1|.$$

So if we suppose that $\beta$ has more then 2 conjugates of absolute value lower to 1 and the other of equal absolute value 1, then we obtain in this case $|A_1| < |\beta|$ which contradicts the hypothesis that $|\beta| = |A_{d-1}| = |A_1|$.

Finally we conclude that $\beta$ has a unique conjugate $\tilde{\beta}$ such that $|\tilde{\beta}| < 1$ and the other conjugates of equal absolute value 1. So, $|\tilde{\beta}| = 1/|\beta|$ and by Theorem 3.4 every rational series in the unit disk have a purely periodic $\beta$-expansion.

**Corollary 3.7.** Let $\beta$ be a cubic Salem unit series. Then $\gamma(\beta) = 1$.

**Proof.** Let $\beta$ be a cubic Salem unit series. In in this case the minimal polynomial of $\beta$ is

$$P(y) = y^3 + A_2y^2 + A_1y + b \quad \text{where} \quad b \in \mathbb{F}_q^*,$$

and by Theorem 2.1, we have $|A_2| = |A_1|$. According to Theorem 3.4, we deduce that every rational series in the unit disk have a purely periodic $\beta$-expansion.
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References


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