Koiso, N.
Osaka J. Math.
17 (1980), 51–73

RIGIDITY AND STABILITY OF EINSTEIN METRICS —THE CASE OF COMPACT SYMMETRIC SPACES

NORIHITO KOISO

(Received October 23, 1978)
(Revised May 14, 1979)

1. Introduction and results

Let M be a compact connected manifold of $\dim M \geqq 2$ and g an Einstein metric on M. If (M, g) is the standard sphere, then all Einstein metrics g' on M near g are of constant sectional curvature, and so (M, g') are homothetic with (M, g) (Berger [2] Proposition 6.4, Muto [23] p457 Theorem). Such an Einstein metric g is said to be *rigid*. We know that some of Einstein metrics with vanishing Ricci tensors are not rigid. For example, flat torus and the $K3$-surfaces are not rigid (Bourguignon [6] 08). But we know few Einstein metrics with negative definite Ricci tensors which are not rigid. In fact, in this paper we prove the rigidity of Einstein metrics g such that the universal riemannian covering manifold of (M, g) is a symmetric space of non-compact type without 2-dimensional factors (Corollary 3.4). Furthermore, for irreducible locally symmetric spaces of compact type, we show the following.

Theorem 1.1. *The following simply connected symmetric spaces are infinitesimally deformable.* (*For the definition of the infinitesimal deformability, see Definition* 2.4.)
$SU(n+1)$ $(n \geqq 2)$, $SU(n)/SO(n)$ $(n \geqq 3)$, $SU(2n)/Sp(n)$ $(n \geqq 3)$, E_6/F_4.

Theorem 1.2. *Let* (M, g) *be an irreducible locally symmetric space of compact type. If the universal riemannian covering manifold of* (M, g) *is neither one of the types in Theorem* 1.1 *nor of the type* $U(p+q)/U(p)\times U(q)$ $(p \geqq q \geqq 2)$, *then* g *is rigid.*

Moreover we study the stability of Einstein metrics. It is well-known that Einstein metrics g are nothing but critical metrics with respect to the total scalar curvature T (Hilbert [12]). In general, this critical point is neither maximal nor minimal (Berger [1] p290, Muto [24] p 521 Theorem). But if we consider only metrics of constant scalar curvature, then some critical points are maximal. That is, if we denote by C the set of all riemannian metrics on M of constant scalar curvature and with volume 1, then some Einstein metrics are maximal in $\mathcal{C}$. Such an Einstein metric g is said to be *stable*. For example, all Einstein metrics of compact locally symmetric spaces of non-compact type without 2-dimensional irreducible factors are stable (Koiso [19] Remark 2.6). If an Einstein metric g is a saddle point of the total scalar curvature T in $\mathcal{C}$, then g is said to be *unstable*. We show the following theorems on the stability of locally symmetric spaces of compact type.

Theorem 1.3. *The following simply connected symmetric spaces are unstable.*

$Spin(5)$, $Sp(n)$ $(n\geqq 3)$, $Sp(n)/U(n)$ $(n\geqq 3)$.

Theorem 1.4. *Let (M, g) be an irreducible locally symmetric space of compact type but not the standard sphere. If the universal riemannian covering manifold of (M, g) is neither one of the types in Theorem 1.3 nor one of the following types, then g is stable.*

$SU(n+1)$ $(n\geqq 2)$, $U(p+q)/U(p)\times U(q)$ $(p\geqq q\geqq 2)$, $Sp(p+q)/Sp(p)\times Sp(q)$ $(p=2,\ q=1$ *or* $p\geqq q\geqq 2)$, $F_4/Spin(9)$, $SU(n)/SO(n)$ $(n\geqq 3)$, $SU(2n)/Sp(n)$ $(n\geqq 3)$, E_6/F_4.

The above results are obtained from evaluations of infinitesimal Einstein deformations and the second differential of T by means of the representation theory, which are made from the tables at the end of this paper.

The author would like to express his sincere gratitude to the referee for his kind suggestions.

2. Preliminaries

In this section, we recall some fundamental definitions and some known facts concerning the space of riemannian metrics. Let M be a compact connected C^∞-manifold with $n=\dim M\geqq 2$. Riemannian metrics on M, etc. are all to be in C^∞-category, unless otherwise stated. For a fibre bundle F over M, we *denote by* $H^s(F)$ the set of all H^s-cross sections of F. Here and throughout in this paper H^s means an object which has derivatives defined almost everywhere up to order s and such that each partial derivative is square integrable. $H^s(F)$ becomes a Hilbert manifold. We *denote by* $\mathcal{M}^s$, $\mathcal{D}^s$, $\mathcal{F}^s$ the Hilbert manifold of all H^s-riemannian metrics on M, the group of all H^s-diffeomorphisms of M, the Hilbert manifold of all positive H^s-functions on M, respectively. (Here, we assume that s is sufficiently large.)

Let g be a riemannian metric on M. We denote by $(\ ,\ )$ the inner product on tensors on M and by $\langle\ ,\ \rangle$ the global inner product for tensor fields, i.e., $\langle\ ,\ \rangle=\int_M(\ ,\ )v_g$, where v_g is the volume element of g.

Lemma 2.1 (Ebin [8] 8.20 Theorem). *Let g be a riemannian metric on M*

and I_g *the isometry group of* (M, g). *If* $s>n/2+2$, *then there is a canonically defined submanifold* $\mathcal{S}_g^s$ *of* $\mathcal{M}^s$ *containing* g *with the following properties.*

$S1)$ *If* $\gamma\in I_g$, *then* $\gamma^*(\mathcal{S}_g^s)=\mathcal{S}_g^s$.

$S2)$ *If* $\gamma\in\mathcal{D}^{s+1}$ *and* $\gamma^*(\mathcal{S}_g^s)\cap\mathcal{S}_g^s\neq\phi$, then $\gamma\in I_g$.

$S3)$ *There is a* C^∞*-local section* $\chi\colon \mathcal{D}^{s+1}/I_g\to\mathcal{D}^{s+1}$ *defined on an open neighbourhood* U *of* I_g *such that if* $F\colon U\times\mathcal{S}_g^s\to\mathcal{M}^s$ *is defined by* $F(u, g')=\chi(u)^*g'$ *then* F *is a homeomorphism onto an open neighbourhood of* g. *Note here that the quotient space* $\mathcal{D}^{s+1}/I_g$ *is a Hilbert manifold.*

Moreover the orbit $(\mathcal{D}^{s+1})^*g$ becomes a closed Hilbert submanifold of $\mathcal{M}^s$. The tangent space of $\mathcal{M}^s$ at g is decomposed into the sum of the tangent space of $(\mathcal{D}^{s+1})^*g$ at g and the tangent space of $\mathcal{S}_g^s$ at g in the following way.

$$
\begin{aligned}
(2.1.1)\qquad & H^s(S^2M) = \delta^*(H^{s+1}(S^1M))\oplus\mathrm{Ker}\,\delta \text{ (orthogonal direct sum)}\,,\\
& T_g(\mathcal{M}^s) = H^s(S^2M),\quad T_g((\mathcal{D}^{s+1})^*g) = \delta^*(H^{s+1}(S^1M))\,,\\
& T_g(\mathcal{S}_g^s) = \mathrm{Ker}\,\delta\,,
\end{aligned}
$$

where S^pM is the vector bundle of covariant symmetric p-tensors on M, δ^* and δ are differential operators defined by

$$
\begin{aligned}
& 2(\delta^*\xi)_{ij} = \nabla_i\xi_j+\nabla_j\xi_i \qquad \text{for } \xi\in H^{s+1}(S^1M)\,,\\
& (\delta h)_i = -\nabla^l h_{li} \qquad \text{for } h\in H^s(S^2M)\,.
\end{aligned}
$$

Denote by $\mathcal{M}_c^s$ the space of all H^s-riemannian metrics on M with volume c. Then $\mathcal{M}_c^s$ and $\mathcal{S}_g^s\cap\mathcal{M}_c^s$ become a closed submanifold of $\mathcal{M}^s$ and a submanifold of $\mathcal{M}_c^s$ respectively, and the above lemma holds also replacing $\mathcal{M}^s$, $\mathcal{S}_g^s$ by $\mathcal{M}_c^s$, $\mathcal{S}_g^s\cap\mathcal{M}_c^s$. In this situation the above decomposition (2.1.1) turns out to

$$
(2.1.2)\qquad \mathrm{Ker}\,\int=\mathrm{Im}\,\delta^*\oplus\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int \text{ (orthogonal direct sum)},
$$

where $\int$ is defined by $\int h=\langle h, g\rangle$ for $h\in H^s(S^2M)$, and $T_g\mathcal{M}_c^s=\mathrm{Ker}\,\int$, $T_g(\mathcal{S}_g^r\cap\mathcal{M}_c^s)=\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int$.

DEFINITION 2.2. Let g be an Einstein metric on M with volume c. If there is a $\mathcal{D}^{s+1}$-invariant open set N of $\mathcal{M}_c^s$ containing $(\mathcal{D}^{s+1})^*g$ such that every H^s-Einstein metric in N is an element of $(\mathcal{D}^{s+1})^*g$, then g is said to be *rigid*.

REMARK (1). Let g be a rigid Einstein metric on M in the sense of the above definition. Then g is rigid in the sense of the Introduction. In fact, let $\mathcal{M}_c^\infty$ and $\mathcal{D}^\infty$ be the space of C^∞-riemannian metrics on M with volume c and the group of C^∞-diffeomorphisms of M with C^∞-topologies, respectively. Then $N\cap\mathcal{M}_c^\infty$ is open in $\mathcal{M}_c^\infty$ and invariant under the action of $\mathcal{D}^\infty$. If g' is an Einstein metric in $N\cap\mathcal{M}_c^\infty$, then there is $\gamma\in\mathcal{D}^{s+1}$ such that $\gamma^*g=g'$. Then $\gamma\in\mathcal{D}^\infty$ by Palais [26] and so $g'\in(\mathcal{D}^\infty)^*g$. This implies the rigidity of g in the sense of the Introduction.

Remark (2). Let g be an Einstein metric on M with volume c. If all 1-parameter families $g(t)$ of H^s-Einstein metrics in $\mathcal{M}_c^s$ such that $g(0)=g$ are contained in $(\mathcal{D}^{s+1})^*g$, then g is said to be *non-deformable*. We easily see from the closedness of $(\mathcal{D}^{s+1})^*g$ in $\mathcal{M}_c^s$ that if g is rigid, then g is non-deformable.

Note that the defining equation of Einstein metric is given as follows: If we define a C^∞-map E: $\mathcal{M}^s \to H^{s-2}(S^2M)$ by

$$E(g) = S_g - (\langle S_g, g\rangle / n \cdot \mathrm{Vol}\,(M, g)) \cdot g \qquad \text{for } g \in M^s\,,$$

where S_g is the Ricci tensor of g, then g is Einstein if and only if $E(g)=0$.

Lemma 2.3. *Let s be an integer $> n/2+2$ and g an Einstein metric on M with volume c. We restrict the C^∞-map E: $\mathcal{M}^s \to H^{s-2}(S^2M)$ to $\mathcal{S}_g^s \cap \mathcal{M}_c^s$. Then the differential dE of E at g is given by*

$$(dE)(h) = \frac{1}{2}(\bar{\Delta} + 2L - \mathrm{Hess}\ \mathrm{tr})h \qquad \text{for } h \in \mathrm{Ker}\ \delta \cap \mathrm{Ker} \textstyle\int .$$

Moreover, we have

$$\mathrm{Ker}\,(dE) \cap \mathrm{Ker}\ \delta \cap \mathrm{Ker} \textstyle\int = \mathrm{Ker}\,(\bar{\Delta}+2L) \cap \mathrm{Ker}\ \delta \cap \mathrm{Ker}\ \mathrm{tr}\,.$$

where

$$(\bar{\Delta}h)_{ij} = -\nabla^l \nabla_l h_{ij}\,,$$

$$(Lh)_{ij} = R_i{}^k{}_j{}^l h_{kl} \qquad \text{for } h \in H^s(S^2M)\,,$$

$$\mathrm{tr}\, h = g^{kl} h_{kl} \qquad \text{for } h \in H^s(S^2M)\,,$$

$$(\mathrm{Hess} f)_{ij} = \nabla_i \nabla_j f \qquad \text{for } f \in H^s(M) = H^s(S^0M)\,,$$

and the sign of the curvature tensor R is given as $R_{ijij} \leqq 0$ for the standard sphere.

Proof. Similar to Berger and Ebin [3] Lemma 7.1.

Definition 2.4. Let g be an Einstein metric on M with volume c. If the space $\mathrm{Ker}\,(dE) \cap T_g(\mathcal{S}_g^s \cap \mathcal{M}_c^s)$ vanishes, then g is said to be *infinitesimally non-deformable*. Otherwise g is said to be *infinitesimally deformable*.

For $s > n/2+4$, we denote by C_c^s the space of all H^s-riemannian metrics g on M with volume c and of constant scalar curvature.

Lemma 2.5 (Fischer and Marsden [9] Theorem 3, Koiso [18] Theorem 2.5, [19] Proposition 2.1). *Let s be an integer $> n/2+4$ and g an Einstein metric on M with volume c, but (M,g) is not the standard sphere. Then there is a neighbourhood U of g in M^s such that $U \cap C_c^s$ becomes a closed submanifold of U. If we define a map χ: $\mathcal{F}^s \times (U \cap C^s) \to M^s$ by $\chi(f, g') = f \cdot g'$, then χ is a diffeomorphism onto an open set of $\mathcal{M}^s$. Moreover the decomposition $T_g\mathcal{M}^s = T_g(\mathcal{F}^s \cdot g) \oplus T_g(C_c^s \cap U)$ of the tengent space $T_g\mathcal{M}^s = H^2(S^2M)$ is given by*

$$T_g(\mathscr{F}^s\cdot g) = \boldsymbol{R}\cdot g\oplus \mathrm{Ker}\, \int \cap H^s(M)\cdot g\,,$$
$$T_g(C_c^s\cap U) = \mathrm{Im}\,\delta^*\oplus \mathrm{Ker}\,\alpha\cap \mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int .$$

Here α: $H^s(S^2M)\to H^{s-4}(M)$ *is a differential operator given by*

$$\alpha(h) = \Delta(\Delta-\varepsilon)\,\mathrm{tr}\, h \qquad \textit{for } h\in H^s(S^2M)\,,$$

where Δ *is the Laplacian of* (M, g) *and* ε *is the constant defined by* $S=\varepsilon\cdot g$.

If we denote by $T(g)$ the total scalar curvature of H^s-riemannian metric g on M, i.e., $T(g)=\langle\tau_g, 1\rangle$ where τ_g is the scalar curvature of g, then a riemannian metric g on M with volume c is an Einstein metric if and only if g is a critical point of C^∞-function T on $\mathcal{M}_c^s$ (Hilbert [12]). Therefore, for an Einstein metric g, we see

$$(dT)_g(\mathrm{Ker}\,\int) = 0\,.$$

As for the Hessian $(\mathrm{Hess}\ T)_g$ on $(\mathrm{Ker}\,\int)\times(\mathrm{Ker}\,\int)$, we know the following

Lemma 2.6 (Koiso [19] Theorem 2.4, Theorem 2.5). *Let g be an Einstein metric on M. If (M, g) is not the standard sphere, then* $(\mathrm{Hess}\ T)_g\,|\,(\mathrm{Ker}\,\int\cap H^s(M)\cdot g)\times(\mathrm{Ker}\,\int\cap H^s(M)\cdot g)$ *is positive definite,* $(\mathrm{Hess}\ T)_g\,|\,\mathrm{Im}\,\delta^*\times\mathrm{Im}\,\delta^*=0$, *and* $(\mathrm{Hess}\ T)_g\,|\,(\mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)\times(\mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)$ *is ginven by*

$$(\mathrm{Hess}\ T)_g(h, h) = -\frac{1}{2}\langle\bar{\Delta}h+2Lh, h\rangle.$$

DEFINITION 2.7. Let g be an Einstein metric on M but (M, g) be not the standard sphere. If $(\mathrm{Hess}\ T)_g\,|\,(\mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)\times(\mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)$ is negative definite, then the Einstein metric g is said to be *stable*. If there is an element h of $\mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int$ such that $(\mathrm{Hess}\ T)_g(h, h)>0$, then g is said to be *unstable*.

REMARK (1). Definition 2.4 (infinitesimal deformability) and Definition 2.7 (stability) are independent of the choice of s, since $\bar{\Delta}+2L$ is an elliptic operator and hence its eigentensor fields are C^∞.

REMARK (2). If g is a stable Einstein metric, then g is infinitesimally non-deformable. This follows from $\mathrm{Ker}\,(\bar{\Delta}+2L)\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\mathrm{tr}\subset\mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int$ and the above formula for Hess T in Lemma 2.6.

REMARK (3). Let g be an Einstein metric on M and $(\tilde{M}, \tilde{g})$ a compact riemannian covering manifold of (M, g). If $\tilde{g}$ is stable then g is also stable. In particular, the stability of an Einstein metric of a locally symmetric space of compact type reduces to the stability of an Einstein metric of a simply connected symmetric space of the same type.

Lemma 2.8 (Koiso [19] Theorem 2.5). *Let g be an Einstein metric on M. Then the space* $\operatorname{Ker}\alpha \cap \operatorname{Ker}\delta \cap \operatorname{Ker} \int$ *coinsides with* $\operatorname{Ker}\delta \cap \operatorname{Ker}\operatorname{tr}$. *Moreover, if* $h \in \operatorname{Ker}\delta \cap \operatorname{Ker}\operatorname{tr}$ *and*

$$\langle Lh, h\rangle > -\frac{\varepsilon}{2}\langle h, h\rangle \quad or \quad \langle Lh, h\rangle > \varepsilon\langle h, h\rangle,$$

where $S_g = \varepsilon \cdot g$, *then* $(\operatorname{Hess} T)_g(h, h) < 0$.

Corollary 2.9. *Let g be an Einstein metric on M. If the universal riemannian covering manifold of (M, g) is a symmetric space of non-compact type without 2-dimensional factors, then g is stable.*

Proof. In this case, the inequality $\langle Lh, h\rangle > \varepsilon\langle h, h\rangle$ holds for all non-zero $h \in \operatorname{Ker}\operatorname{tr}$ (Koiso [19] Remark 2.6). Thus Lemma 2.8 implies our Corollary. Q.E.D.

3. Infinitesimal non-deformability and rigidity

Lemma 3.1. *Let s be an integer $> n/2+2$ and g an Einstein metric on M with volume c. If there is an open neighbourhood V of g in $\mathcal{S}^s_g \cap \mathcal{M}^s_c$ such that g is the only one H^s-Einstein metric in V, then g is rigid.*

Proof. We use the notation in Lemma 2.1. We wee $(\mathcal{D}^{s+1})^*(V) = (\mathcal{D}^{s+1})^*(F(U \times V))$ and so $(\mathcal{D}^{s+1})^*(V)$ is a $\mathcal{D}^{s+1}$-invariant open set of $\mathcal{M}^s_c$. If g' is an H^s-Einstein metric in $(\mathcal{D}^{s+1})^*(V)$, then g' is isometric with an H^s-Einstein metric in V, which is nothing but g. Therefore $g' \in (\mathcal{D}^{s+1})^* g$, which means the rigidity of g. Q.E.D.

Lemma 3.2. *Let g be an Einstein metric on M with $S = \varepsilon \cdot g$. We define operators β: $H^s(S^2M) \to H^{s-2}(S^2M)$ and γ: $H^s(S^2M) \to H^{s-1}(S^1M)$ by*

$$\beta(h) = (\bar\Delta + 2L - \operatorname{Hess}\operatorname{tr})h,$$
$$\gamma(h) = \left(\delta + \frac{1}{2} d \operatorname{tr}\right)h.$$

Then $\quad \gamma\beta = (\bar\Delta - \varepsilon)\delta,$

where $\quad (\bar\Delta\xi)_i = -\nabla^l \nabla_l \xi_i, \quad$ *for* $\xi \in H^s(S^1M)$.

Proof. Remark that $\nabla^k R^l{}_k{}^m{}_i = 0$. In fact, by the second Bianchi identity,

$$\nabla^k R^l{}_k{}^m{}_i = -\nabla^m R^l{}_{ki}{}^k - \nabla_i R^l{}_k{}^{km} = \nabla^m S^l{}_i - \nabla_i S^{lm} = 0.$$

Now,

$$\begin{aligned}
&[\delta(\bar\Delta + 2L - \operatorname{Hess}\operatorname{tr})h]_i \\
&\quad = -\nabla^l(-\nabla^k \nabla_k h_{li} + 2R_l{}^k{}_i{}^m h_{km} - \nabla_i \nabla_l h^k{}_k) \\
&\quad = \nabla^l \nabla^k \nabla_k h_{li} - 2\nabla^l(R_l{}^k{}_i{}^m h_{km}) + \nabla^l \nabla_i \nabla_l h^k{}_k,
\end{aligned}$$

$$
\begin{aligned}
\nabla^l\nabla^k\nabla_k h_{li} &= R^{lkm}{}_k\nabla_m h_{li}+R^{lkm}{}_l\nabla_k h_{mi}+R^{lkm}{}_i\nabla_k h_{lm}+\nabla^k\nabla^l\nabla_k h_{li} \\
&= -S^{lm}\nabla_m h_{li}+S^{km}\nabla_k h_{mi}+R^l{}_k{}^m{}_i\nabla^k h_{lm} \\
&\qquad +\nabla^k(R^l{}_k{}^m{}_l h_{mi}+R^l{}_k{}^m{}_i h_{lm}+\nabla_k\nabla^l h_{li}) \\
&= R^l{}_k{}^m{}_i\nabla^k h_{lm}+S_k{}^m\nabla_k h_{mi}+R^l{}_k{}^m{}_i\nabla^k h_{lm}+(\bar\Delta\delta h)_i \\
&= 2R^l{}_k{}^m{}_i\nabla^k h_{lm}-\varepsilon(\delta h)_i+(\bar\Delta\delta h)_i\,, \\
\nabla^l(R_l{}^k{}_i{}^m h_{km}) &= R_l{}^k{}_i{}^m\nabla^l h_{km} = R^l{}_k{}^m{}_i\nabla^k h_{lm}\,, \\
\nabla^l\nabla_i\nabla_l h^k{}_k &= R^l{}_i{}^m{}_l\nabla_m h^k{}_k+\nabla_i\nabla^l\nabla_l h^k{}_k \\
&= S_i{}^m\nabla_m h^k{}_k-(d\Delta\operatorname{tr}h)_i \\
&= \varepsilon(d\operatorname{tr}h)_i-(d\Delta\operatorname{tr}h)_i\,.
\end{aligned}
$$

Hence $\delta(\bar\Delta+2L-\text{Hess tr})h=(\bar\Delta-\varepsilon)\delta h+d(\varepsilon-\Delta)\operatorname{tr}h$. On the other hand,

$$
\begin{aligned}
&\operatorname{tr}(\bar\Delta+2L-\text{Hess tr})h \\
&= -\nabla^l\nabla_l h^k{}_k+2R^{li}{}_l{}^j h_{ij}-\nabla^l\nabla_l h^k{}_k \\
&= 2\Delta\operatorname{tr}h-2\varepsilon\operatorname{tr}h\,.
\end{aligned}
$$

Therefore $\frac{1}{2}\,d\operatorname{tr}(\bar\Delta+2L-\text{Hess tr})h=d(\Delta-\varepsilon)\operatorname{tr}h$. Thus

$$
\gamma\beta h=(\bar\Delta-\varepsilon)\delta h\,. \qquad \text{Q.E.D.}
$$

Proposition 3.3. *Let g be an Einstein metric on M. If g is infinitesimally non-deformable, then g is rigid.*

REMARK. Let g be an Einstein metric on M and $(\tilde M,\tilde g)$ a compact riemannian covering manifold of (M,g). This proposition implies that if $\tilde g$ is infinitesimally non-deformable, then g is rigid. In particular, the rigidity of an Einstein metric of a locally symmetric space of compact type reduces to the infinitesimal non-deformability of an Einstein metric of a simply connected symmetric space of the same type.

Proof. First we show that $\beta(\operatorname{Ker}\delta\cap\operatorname{Ker}\int)$ is closed in $H^{s-2}(S^2M)$. Lemma 3.2 implies that $\beta(\operatorname{Ker}\delta)\subset\operatorname{Ker}\gamma$ and so $\beta(\operatorname{Ker}\delta)\subset\operatorname{Im}\beta\cap\operatorname{Ker}\gamma$, here the space $\operatorname{Im}\beta\cap\operatorname{Ker}\gamma$ is closed in $H^{s-2}(S^2H)$, since β is an elliptic operator. Let $h\in H^s(S^2M)$ satisfies $\beta h\in\operatorname{Ker}\gamma$. Decompose h by the formula (2.1.1) as $h=\psi+\delta^*\xi$; $\delta\psi=0$. Then by Lemma 3.2,

$$
0=\gamma\beta h=(\bar\Delta-\varepsilon)\delta h=(\bar\Delta-\varepsilon)\delta\delta^*\xi\,.
$$

This equation implies that such ξ is an element of the vector space $\operatorname{Ker}(\bar\Delta-\varepsilon)\delta\delta^*$, which is finite deminsional since $(\bar\Delta-\varepsilon)\delta\delta^*$ is elliptic. Let c be the volume of (M,g). Then $\delta(\psi-(\langle\psi,g\rangle/nc)\cdot g)=0$ and $\int(\psi-(\langle\psi,g\rangle/nc)\cdot g)=0$. Thus

$$
\beta(\operatorname{Ker}\delta\cap\operatorname{Ker}\textstyle\int)+\beta\delta^*(\operatorname{Ker}(\bar\Delta-\varepsilon)\delta\delta^*)+\boldsymbol{R}\cdot g\supset\operatorname{Im}\beta\cap\operatorname{Ker}\gamma\,.
$$

Therefore $\beta(\mathrm{Ker}\ \delta\cap\mathrm{Ker}\ f)$ is a finite codimensional subspace of the closed subspace $\mathrm{Im}\,\mathrm{Ker}\ \beta\cap\mathrm{Ker}\ \gamma$ of $H^{s-2}(S^2M)$, and so Palais [25] Chapter VII Proof of Theorem 1 implies that $\beta(\mathrm{Ker}\ \delta\cap\mathrm{Ker}\ f)$ is a closed subspace of $H^{s-2}(S^2M)$.

Next, we denote by p the orthogonal projection: $H^{s-2}(S^2M)\to\beta(\mathrm{Ker}\ \delta\cap\mathrm{Ker}\ f)$. Then we can apply the inverse function theorem to the C^∞-map $p\circ E$: $\mathcal{S}_g^s\cap\mathcal{M}_c^s\to\beta(\mathrm{Ker}\ \delta\cap\mathrm{Ker}\ f)$. In fact, Lemma 2.3 implies that $d(p\circ E\,|\,\mathcal{S}_g^s\cap\mathcal{M}_c^s)_g=\frac{1}{2}p\circ\beta\,|\,\mathrm{Ker}\ \delta\cap\mathrm{Ker}\ f$ and the assumption implies that this differential is bijective. Thus the assumption of Lemma 3.1 holds and hence we get our assertion. Q.E.D.

Corollary 3.4. *Let g be an Einstein metric on M but not the standard sphere. If g is stable, then g is rigid. In particular, if the inequality of Lemma 2.8 holds for all non-zero $h\in\mathrm{Ker}\ \delta\cap\mathrm{Ker}\ \mathrm{tr}$, then g is rigid. Moreover, any Einstein metric of a locally symmetric space of non-compact type without 2-dimensional factors is rigid.*

Proof. This is easily seen by Remark (2) following Definition 2.7, Lemma 2.8 and Corollary 2.9.

Remark. The author does not know whether the converse of Proposition 3.3 holds or not.

4. Fundamental formulae

In this section we assume that (M,g) is a locally symmetric Einstein manifold.

Lemma 4.1. *If $S=\varepsilon\cdot g$, then the following formulae hold.*

(4.1.1) $(\bar{\Delta}+2L)L=L(\bar{\Delta}+2L)$ *on* $C^\infty(S^2M)$,

(4.1.2) $Lg=-\varepsilon\cdot g$,

(4.1.3) $\delta(\bar{\Delta}+2L)=(\Delta-2\varepsilon)\delta$ *on* $C^\infty(S^2M)$,

(4.1.4) $(\bar{\Delta}+2L)\delta^*=\delta^*(\Delta-2\varepsilon)$ *on* $C^\infty(S^1M)$,

(4.1.5) $2\delta\delta^*=\Delta-2\varepsilon+d\delta$ *on* $C^\infty(S^1M)$.

Proof. These are easily seen, by $\nabla R=0$ and computations similar to Proof of Lemma 3.2.

Lemma 4.2. *Let (M,g) be a compact locally symmetric Einstein manifold. Let $h\in H^s(S^2M)$ satisfy $\bar{\Delta}h+2Lh=-\lambda h$ $(\lambda\geqq 0)$. Decompose h by* (2.1.1) *as $h=\delta^*\xi+\psi$; $\delta\psi=0$. If* $\mathrm{tr}\ \delta^*\xi=0$, *then* $\delta h=0$.

Proof. Note that $\delta\xi=-\mathrm{tr}\ \delta^*\xi=0$ and δ^* is the formal adjoint of δ.

$$\begin{aligned}
&\langle(\Delta-2\varepsilon)\xi,\ (\Delta-2\varepsilon)\xi\rangle \\
&\quad=\langle(\Delta-2\varepsilon)\xi,\ 2\delta\delta^*\xi\rangle \qquad \text{(by (4.1.5))} \\
&\quad=\langle\delta^*(\Delta-2\varepsilon)\xi,\ 2\delta^*\xi\rangle \\
&\quad=\langle(\bar{\Delta}+2L)\delta^*\xi,\ 2\delta^*\xi\rangle \qquad \text{(by (4.1.4))}\,.
\end{aligned}$$

Here the decomposition $\mathrm{Im}\,\delta^*\oplus\mathrm{Ker}\,\delta$ is invariant under $\bar{\Delta}+2L$ by (4.1.3) and (4.1.4), and hence $(\bar{\Delta}+2L)\delta^*\xi=-\lambda\delta^*\xi$. Thus

$$\langle(\Delta-2\varepsilon)\xi,\ (\Delta-2\varepsilon)\xi\rangle=-2\lambda\langle\delta^*\xi,\ \delta^*\xi\rangle\leqq 0. \qquad \text{(by } \lambda\geqq 0)$$

Therefore $(\Delta-2\varepsilon)\xi=0$, and

$$\begin{aligned}
\langle\delta^*\xi,\ \delta^*\xi\rangle=\langle\delta\delta^*\xi,\ \xi\rangle&=\frac{1}{2}\langle(\Delta-2\varepsilon)\xi,\ \xi\rangle \qquad \text{(by (4.1.5))} \\
&=0\,.
\end{aligned}$$

Thus $\delta^*\xi=0$ and so $h=\psi$. Hence $\delta h=0$. Q.E.D.

5. Lichnerowicz operator and Casimir operator

In this section, we assume that (M, g) is a compact symmetric space G/K, where G is a compact connected Lie group and (G, K) is a symmetric pair. As usual, let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{k}$ the Lie algebra of K, $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ the canonical decomposition. Then the tangent space T_oM of M at the origin o is identified with $\mathfrak{m}$. The metric of M is always induced by a G-invariant inner product B on $\mathfrak{g}$ with $B(\mathfrak{k}, \mathfrak{m})=0$. We fix such an inner product B once and for all. We extend B $\boldsymbol{C}$-bilinearly on $\mathfrak{g}^C\times\mathfrak{g}^C$ and often write as $B(X, Y)=(X, Y)$.

Since K acts on the complexification $\mathfrak{m}^C$ of $\mathfrak{m}$ by the adjoint action Ad, the tensor space $\otimes^p\mathfrak{m}^{*C}$ of degree p of the dual space $\mathfrak{m}^{*C}$ of $\mathfrak{m}^C$ is a K-module. Then the complex covariant p-tensor bundle T^pM^C of M is identified with the homogeneous vector bundle $G\times_K\otimes^p\mathfrak{m}^{*C}$ associated to the principal bundle $\pi\colon G\to G/K$, in the following way. Let a be a point in M and s an element of $T^p_aM^C$. For $x\in\pi^{-1}(a)$ we get $(x, (\pi^*s)_x\,|\,\mathfrak{m}^C\times\cdots\times\mathfrak{m}^C)\in G\times\otimes^p\mathfrak{m}^{*C}$, where we regard $X\in\mathfrak{m}^C$ as a left invariant vector field on G, and $\otimes^p\mathfrak{m}^{*C}$ the space of $\boldsymbol{C}$-multilinear forms on $\mathfrak{m}^C$. We identify s with the element $[(x, (\pi^*s)_x\,|\,\mathfrak{m}^C\times\cdots\times\mathfrak{m}^C)]\in G\times_K\otimes^p\mathfrak{m}^{*C}$, where we denote by $[*]$ the equivalence class of $*$

Generally, for a finite dimensional (complex) K-module U, a cross section s of the homogeneous vector bundle $G\times_KU$ over M may be identified with a U-valued function s on G such that $s(xy)=y^{-1}s(x)$ for all $x\in G$ and $y\in K$. Let $C^\infty(G, U)_K$ be the space of all such s. Then $C^\infty(G, U)_K$ becomes a G-module by the G-action $(xs)(y)=s(x^{-1}y)$ for $x, y\in G$. In particular, the vector space $C^\infty(T^pM^C)$ of all complex covariant p-tensor fields on M is identified with $C^\infty(G, \otimes^p\mathfrak{m}^{*C})_K$ as G-module. For a (differential) operator $\zeta\colon C^\infty(T^pM)\to$

$C^\infty(T^qM)$, we extend ζ C-linearly to the operator: $C^\infty(T^pM^C)\to C^\infty(T^qM^C)$ and denote it by the same symbol ζ.

Now, we define a linear map D: $C^\infty(G, \otimes^p\mathfrak{m}^{*C})_K\to C^\infty(G, \otimes^{p+1}\mathfrak{m}^{*C})_K$ by

$$(DS)(x)(X_0, \cdots, X_p) = (X_0[s(X_1, \cdots, X_p)])(x) \qquad (x\in G)$$

where $s\in C^\infty(G, \otimes^p\mathfrak{m}^{*C})_K$ and $X_i\in\mathfrak{m}$. It is easy to see that $Ds\in C^\infty(G, \otimes^{p+1}\mathfrak{m}^{*C})_K$ and D is a G-homomorphism.

Lemma 5.1. *The linear map D regarded as a linear map from $C^\infty(T^pM^C)$ to $C^\infty(T^{p+1}M^C)$ coincides with the covariant derivative ∇ of the symmetric space (M, g).*

Proof. Since ∇ and D are G-homomorphism, it is sufficient to prove that the equation holds at the identity $e\in G$. Let $s\in C^\infty(T^pM^C)=C^\infty(G, \otimes^p\mathfrak{m}^{*C})_K$. Then for $X_0, \cdots, X_p\in\mathfrak{m}^C$,

$$\begin{aligned}(\nabla s)(e)(X_0, \cdots, X_p) &= (\nabla s)_o(\pi_*X_0, \cdots, \pi_*X_p)\\ &= (\nabla_{\pi_*X_0}s)_o(\pi_*X_1, \cdots, \pi_*X_p)\,.\end{aligned}$$

Here we extend each $X_i\in T_e(G)^C$ to the right invariant vector field $\tilde{X}_i$. Then each $\pi_*\tilde{X}_i$ defines a vector field on M, and we get

$$\begin{aligned}&(\nabla s)(e)(X_0, \cdots, X_p)\\ &= (\nabla_{\pi_*\tilde{X}_0}s)_o(\pi_*\tilde{X}_1, \cdots, \pi_*\tilde{X}_p)\\ &= \{\pi_*\tilde{X}_0[s(\pi_*\tilde{X}_1, \cdots, \pi_*\tilde{X}_p)]-\sum_{i=1}^p s(\pi^*\tilde{X}_1, \cdots, \nabla_{\pi_*\tilde{X}_0}\pi_*\tilde{X}_i, \cdots, \pi_*\tilde{X}_p)\}_o\,.\end{aligned}$$

Since $X_0, X_i\in\mathfrak{m}^C$ and (M, g) is symmetric, we have $(\nabla_{\pi_*\tilde{X}_0}\pi_*\tilde{X}_i)_o=0$, and the right hand side$=X_0[(\pi^*s)(\tilde{X}_1, \cdots, \tilde{X}_p)]$. Moreover, if we regard X_i as left invariant vector fields on G, this is equal to

$$\begin{aligned}&(\mathcal{L}_{X_0}(\pi^*s))_e(\tilde{X}_1, \cdots, \tilde{X}_p)+\sum_{i=1}^p (\pi^*s)_e(\tilde{X}_1, \cdots, [X_0, \tilde{X}_i], \cdots, \tilde{X}_p)\\ &= (\mathcal{L}_{X_0}(\pi^*s))_e(X_1, \cdots, X_p)\\ &= (X_0[(\pi^*s)(X_1, \cdots, X_p)])(e)-\sum_{i=1}^p (\pi^*s)_e(X_1, \cdots, [X_0, X_i], \cdots, X_p)\\ &= (Ds)(e)(X_0, \cdots, X_p)\,.\end{aligned}$$

Q.E.D.

Let V be a $\mathfrak{g}$-module. We define an operator on V which is called the *Casimir operator*, by

$$C = -\sum_i Z_i\cdot Z_i\,,$$

where $\{Z_i\}$ are orthonomal basis of $\mathfrak{g}$ (with respect to the fixed inner product B). Note that if V is a finite dimensional G-module, then V is a $\mathfrak{g}$-module in the natural way, and so the Casimir operator on V is defined. If U is a finite dimensional K-module, then $\mathfrak{g}$ acts on $C^\infty(G, U)_K$ via the differentiation by left invariant vector fields, i.e.,

$$(Xs)(x) = \frac{d}{dt}\Big|_0 s(x \exp tX)$$

for $X \in \mathfrak{g}$, $s \in C^\infty(G, U)_K$, $x \in G$. Thus the Casimir operator C on $C^\infty(G, U)_K$ is defined.

DEFINITION 5.2. We define the operator Q, L, $\bar{\Delta}$ and the Lichenerowicz operator Δ on the vector space $C^\infty(T^p M^C)$ as follows.

$$p(Qs)_{i_1 \cdots i_p} = \sum_{a=1}^{p} S_{i_a}{}^k s_{i_1 \cdots \overset{(a)}{k} \cdots i_p},$$

$$(Ls)_{i_1 \cdots i_p} = \sum_{a<b} R_{i_a}{}^k{}_{i_b}{}^l s_{i_1 \cdots \overset{(a)}{k} \cdots \overset{(b)}{l} \cdots i_p},$$

$$(\bar{\Delta}s)_{i_1 \cdots i_p} = -\nabla^l \nabla_l s_{i_1 \cdots i_p},$$

$$\Delta s = \bar{\Delta}s + 2Ls + pQs.$$

REMARK. This definition does not contradict the previous definitions and the ordinary Laplace-Bertrami operator (Lichnerowicz [20] 10). But we shall not use these notations except in the following proposition.

Proposition 5.3. *The Lichnerowicz operator Δ regarded as an endomorphism of $C^\infty(G, \otimes^p \mathfrak{m}^{*C})_K$ coinsides with the Casimir operator C.*

Proof. Let S_i, T_j be orthonormal basis of $\mathfrak{k}$, $\mathfrak{m}$, respectively. It is sufficient to prove that the equation holds at the identity $e \in G$. For $s \in C^\infty(G, \otimes^p \mathfrak{m}^{*C})$ and $X_1, \cdots, X_p \in \mathfrak{m}^C$, which are regarded as left invariant vector fields on G, we have

$$\begin{aligned} &-(Cs)(X_1, \cdots, X_p) \\ &\quad = \sum_i S_i \cdot S_i[s(X_1, \cdots, X_p)] + \sum_j T_j \cdot T_j[s(X_1, \cdots, X_p)]. \\ &T_j \cdot T_j[s(X_1, \cdots, X_p)] \\ &\quad = T_j[(Ds)(T_j, X_1, \cdots, X_p)] \\ &\quad = T_j[(\nabla s)(T_j, X_1, \cdots, X_p)] \\ &\quad = (D\nabla s)(T_j, T_j, X_1, \cdots, T_p) \\ &\quad = (\nabla\nabla s)(T_j, T_j, X_1, \cdots, X_p). \end{aligned}$$

Therefore $\sum_j T_j \cdot T_j[s] = -\bar{\Delta}s$ at e. Moreover in virture of the equality $s(xy) = y^{-1}s(x)$ for $x \in G$, $y \in K$, we have

$$\begin{aligned} &S_i \cdot S_i[s(X_1, \cdots, X_p)] \\ &\quad = \sum_k S_i[s(X_1, \cdots, [S_i, X_k], \cdots, X_p)] \end{aligned}$$

$$= 2\sum_{k<l} s(X_1, \cdots, [S_i, X_k], \cdots, [S_i, X_l], \cdots, X_p)$$
$$+\sum_k s(X_1, \cdots, [S_i, [S_i, X_k]], \cdots, X_p)\,.$$

On the other hand, regarding an element of $\mathfrak{m}^C$ as a tangent vector of G at e, we have

$$\begin{aligned}
&(Ls)_o(\pi_*X_1, \cdots, \pi_*X_p)\\
&= \sum_{\substack{k<l\\ a,b}} (R(\pi_*X_k, \pi_*T_a)\pi_*X_l, \pi_*T_b)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \overset{(l)}{\pi_*T_b}, \cdots, \pi_*X_p)\\
&= -\sum_{\substack{k<l\\ a,b}} ([X_k, T_a], [X_l, T_b])s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \overset{(l)}{\pi_*T_b}, \cdots, \pi_*X_p)\\
&= -\sum_{\substack{k<l\\ a,b,i}} ([X_k, T_a], S_i)([X_l, T_b], S_i)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \overset{(l)}{\pi_*T_b}, \cdots, \pi_*X_p)\\
&= -\sum_{\substack{k<l\\ a,b,i}} ([S_i, X_k], T_a)([S_i, X_l], T_b)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \overset{(l)}{\pi_*T_b}, \cdots, \pi_*X_p)\\
&= -\sum_{\substack{k<l\\ i}} s(\pi_*X_1, \cdots, \pi_*[S_i, X_k], \cdots, \pi_*[S_i, X_l], \cdots, \pi_*X_p)\,.
\end{aligned}$$

$$\begin{aligned}
&p(Qs)_o(\pi_*X_1, \cdots, \pi_*X_p)\\
&= \sum_{k,a} S(\pi_*X_k, \pi_*T_a)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \pi_*X_p)\\
&= -\sum_{k,a,b} (R(\pi_*X_k, \pi_*T_b)\pi_*T_a, \pi_*T_b)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \pi_*X_p)\\
&= \sum_{k,a,b} ([X_k, T_b], [T_a, T_b])s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \pi_*X_p)\\
&= \sum_{k,a,b,i} ([X_k, T_b], S_i)([T_a, T_b], S_i)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \pi_*X_p)\\
&= \sum_{k,a,b,i} ([S_i, X_k), T_b)([S_i, T_a], T_b)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*X_a}, \cdots, \pi_*X_p)\\
&= \sum_{k,a,i} ([S_i, X_k], [S_i, T_a])s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \pi_*X_p)\\
&= -\sum_{k,a,i} ([S_i, [S_i, X_k]], T_a)s(\pi_*X_1, \cdots, \overset{(k)}{\pi_*T_a}, \cdots, \pi_*X_p)\\
&= -\sum_{k,i} s(\pi_*X_1, \cdots, \pi_*[S_i, [S_i, X_k]], \cdots, \pi_*X_p)\,.
\end{aligned}$$

Thus $\sum_i S_i \cdot S_i[s] = -(2Ls+pQs)$ at e Q.E.D.

Corollary 5.4. *If (M,g) is a symmetric Einstein manifold such that $S=\varepsilon\cdot g$, then for $p=2$ we have $\bar{\Delta}+2L=C-2\varepsilon$.*

Lemma 5.5 (Frobenius reciprocity, c.f. Wallach [29] Theorem 8.2). *Let V be a finite dimensional G-module and U a finite dimensional K-module. For a G-homomorphism ϕ: $V\to C^\infty(G, U)_K$, we define a K-homomorphism $\tilde{\phi}$: $V\to U$ by $\tilde{\phi}(v)=\phi(v)(e)$. Then the correspondance: $\phi\to\tilde{\phi}$ is an isomorphism as vector space from* $\mathrm{Hom}_G(V, C^\infty(G, U)_K)$ *to* $\mathrm{Hom}_K(V, U)$.

Remark. Let V be a finite dimensional G-submodule of $C^\infty(G, U)_K$.

Then C leaves V invariant and coinsides with the Casimir operator of the G-module V on V. This follows from $\sum Z_i \cdot Z_i = \sum \tilde{Z}_i \cdot \tilde{Z}_i$ and that for the inclusion ϕ: $V \rightarrow C^\infty(G, U)_K$ we have $\phi X = -\tilde{X}\phi$ on V for all $X \in g$.

Corollary 5.6. *Let $V \subset C^\infty(S^2M^C)$ be a finite dimensional G-submodule of the eigenspace of $\bar{\Delta}+2L$ with a non-positive eigenvalue. If* $\mathrm{Hom}_K(V, \boldsymbol{C})=0$ *or* $\mathrm{Hom}_K(V, \mathfrak{m}^{*C})=0$, *then* $\delta(V)=0$.

Proof. If $\mathrm{Hom}_K(V, \mathfrak{m}^{*C})=0$, then $\mathrm{Hom}_G(V, C^\infty(G, \mathfrak{m}^{*C})_K)=0$ and so $\delta(V)=0$. If $\mathrm{Hom}_K(V, \boldsymbol{C})=0$, then $\mathrm{Hom}_G(V, C^\infty(G, \boldsymbol{C})_K)=0$ and so the assumption of Lemma 4.2 holds for each $h \in V$. Thus $\delta(V)=0$. Q.E.D.

REMARK. The results in this section are true for any (not necessarily compact) symmetric space (M, g), except Lemma 5.5 and Corollary 5.6. This is pointed out by M. Takeuchi.

6. Fundamental lemmas for root systems

Let $(\mathfrak{g}, \mathfrak{k})$ be an effective symmetric pair of compact type with the inner product B fixed in **5**. Then $\mathfrak{g}$ is semi-simple. We introduce the following ordinary notation.

$\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$: the canonical decomposition,

θ: the involution of $\mathfrak{g}$ defined by $\theta|\mathfrak{k}=\mathrm{id}_{\mathfrak{k}}$, $\theta|\mathfrak{m}=-\mathrm{id}_{\mathfrak{m}}$,

$\mathfrak{a}$: a Cartan subalgebra of $\mathfrak{k}$,

$\mathfrak{t}$: a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$,

$\mathfrak{b}=\mathfrak{t}\cap\mathfrak{m}$,

$\sum(\mathfrak{g})\subset\mathfrak{t}$: the root system of $\mathfrak{g}$ with respect to B,

$\sum(k)\subset\mathfrak{a}$: the root system of $\mathfrak{k}$ with respect to $B|\mathfrak{k}\times\mathfrak{k}$, so that there are root vectors $\{X_\alpha \in \mathfrak{g}^C; \alpha \in \sum(\mathfrak{g})\}$ such that

$$
\begin{aligned}
&[H, X_\alpha] = \sqrt{-1}B(\alpha, H)X \qquad \text{for all } H \in \mathfrak{t}\,,\\
&[X_\alpha, X_{-\alpha}] = \sqrt{-1}\alpha\,,\\
&B(X_\alpha, X_{-\alpha}) = 1\,,\\
&[X_\alpha, X_\beta] = \sqrt{-1}N_{\alpha,\beta}X_{\alpha+\beta} \qquad (N_{\alpha\,\beta} \in \boldsymbol{R})\,,\\
&N_{-\alpha,-\beta} = -N_{\alpha,\beta}\,,\\
&(N_{\alpha,\beta})^2 = q(1+p)B(\alpha, \alpha)/2\,,
\end{aligned}
$$

where $\{\beta+n\alpha;\, n\in\boldsymbol{Z},\, -p\leqq n\leqq q\}$ is the maximal α-series containing β. Moreover X_α and $X_{-\alpha}$ are conjugate with respect to the complex conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ with one another. Let $\bar{\ }$ be the orthogonal projection: $\mathfrak{t}\rightarrow\mathfrak{a}$. Let $>$ be a linear order of $\mathfrak{t}$ such that if $H>0$ then $\theta H>0$ or $H\in\mathfrak{b}$. Let $\mathfrak{B}$ be the base with respect to the order $>$, i.e., $\mathfrak{B}$ is the set of all simple roots with respect to $>$. Let $\gg$ be the order defined by the base $\mathfrak{B}$, i.e., $x\gg y$

if and only if $x-y=\sum_{\alpha\in\mathfrak{B}}z^{\alpha}\cdot\alpha\neq 0$ and $z^{\alpha}\geqq 0$ for all $\alpha\in\mathfrak{B}$.

$h_{\mathfrak{g}}$: the highest root of $\sum(\mathfrak{g})$,

$2\delta_{\mathfrak{g}}$: the sum of positive roots of $\mathfrak{g}$,

$D(\mathfrak{g})$: the set of all dominant weights of $\sum(\mathfrak{g})$.

Lemma 6.1 (Murakami [21] (33)). $\sum(\mathfrak{g})=\sum_1\cup\sum_2\cup\sum_3$ *(disjoint union) where*

$$\begin{aligned}\textstyle\sum_1 &= \{\alpha\in\textstyle\sum(\mathfrak{g});\ \theta\alpha=\alpha,\ X_{\alpha}\in\mathfrak{k}\}\ ,\\ \textstyle\sum_2 &= \{\alpha\in\textstyle\sum(\mathfrak{g});\ \theta\alpha=\alpha,\ X_{\alpha}\in\mathfrak{m}\}\ ,\\ \textstyle\sum_3 &= \{\alpha\in\textstyle\sum(\mathfrak{g});\ \theta\alpha\neq\alpha\}\ .\end{aligned}$$

Lemma 6.2 (Murakami [21] Lemma 6). $\sum(\mathfrak{g})\cap\mathfrak{b}=\phi$.

Lemma 6.3 (c.f. Bourbaki [5] p148 Theoreme 1). *If* $\alpha,\beta\in\sum(\mathfrak{g})$, $\beta\neq\pm\alpha$, *then* α—sign $(\alpha,\beta)\cdot\beta\in\sum(\mathfrak{g})$.

Corollary 6.4. *Let* $\alpha,\beta\in\sum(\mathfrak{g})$. *If* $\bar{\alpha}=\bar{\beta}$, *then* $\beta=\alpha$ *or* $\beta=\theta\alpha$.

Proof. Assume that $\beta\neq\alpha,\theta\alpha$. If $(\alpha,\beta)>0$, then $\beta\neq-\alpha$ and Lemma 6.3 implies that $\alpha-\beta\in\sum(\mathfrak{g})$ which contradicts Lemma 6.2. Thus $(\alpha,\beta)\leqq 0$. Similarly $(\alpha,\theta\beta)\leqq 0$. Then $(\theta\alpha,\theta\beta)\leqq 0$, $(\theta\alpha,\beta)\leqq 0$ and

$$0\leqq(\bar{\alpha},\bar{\alpha})=(\bar{\alpha},\bar{\beta})=\frac{1}{4}(\alpha+\theta\alpha,\ \beta+\theta\beta)\leqq 0\,,$$

and so $\bar{\alpha}=0$, which contradicts Lemma 6.2. Q.E.D.

Corollary 6.5. $(\sum_1\cup\sum_2)\cap\overline{\sum_3}=\phi$.

Proposition 6.6. *Let* $\sum(\mathfrak{m})$ *be the set of all weights relative to* $\mathfrak{a}$, *with multiplicity counted, of the K-module* m^C. *Then* $\sum(\mathfrak{m})\backslash\{0\}=\overline{\sum(\mathfrak{g})\backslash\sum(\mathfrak{k})}$ *and the multiplicity of a non-zero weight in* $\sum(\mathfrak{m})$ *is one. The multiplicity of* 0 *in* $\sum(\mathfrak{m})$ *is* dim $\mathfrak{b}$.

Proof. $\sum(\mathfrak{m})\backslash\{0\}=\{\bar{\alpha};\ \alpha\in\sum(\mathfrak{g}),\ \theta X_{\alpha}\neq X_{\alpha}\}$ by Lemma 6.2. Thus the proof reduces to Corollary 6.5.

Lemma 6.7 (c.f. Bourbaki [5] p168 Proposition 29). *If* $\lambda\gg 0$, *then* $(\delta_{\mathfrak{g}},\lambda)>0$.

Proposition 6.8. *Let* $\lambda,\mu\in D(\mathfrak{g})$. *If* $\lambda\gg\mu$, *then* $(\lambda+2\delta_{\mathfrak{g}},\lambda)>(\mu+2\delta_{\mathfrak{g}},\mu)$.

Proof. We see $\lambda-\mu\gg 0$ and hence $(\lambda+\mu,\lambda-\mu)\geqq 0$, $(2\delta_{\mathfrak{g}},\lambda-\mu)>0$ by Lemma 6.7. Thus we get the assertion. Q.E.D.

7. Procedure of calculation

In this section we assume that (M, g) is a simply connected irreducible symmetric space G/K of compact type, with G compact simply connected and K connected. Thus (M, g) is an Einstein manifold and B is unique up to constant factor. We denote by $C^\infty(S^2M^C)$ the space of complex symmetric covariant 2-tensors on M. An element $s \in C^\infty(S^2M^C)$ is said to be G-finite if the smallest G-invariant subspace of $C^\infty(S^2M^C)$ containing s is of finite dimension. The space of all G-finite elements of $C^\infty(S^2M^C)$ is denoted by $C_f^\infty(S^2M^C)$. It is a G-submodule of $C^\infty(S^2M^C)$ containing all finite dimensional G-submodule of $C^\infty(S^2M^C)$. Since $\bar{\Delta}+2L$ is an elliptic G-invariant differential operator on $C^\infty(S^2M^C)$, each eigenspace of $\bar{\Delta}+2L$ is contained in $C_f^\infty(S^2M^C)$. Moreover, the self-adjointness of $\bar{\Delta}+2L$ and L and (4.1.1) imply that the G-module $C_f^\infty(S^2M^C)$ is decomposed into G-invariant simultaneous eigenspaces for $\bar{\Delta}+2L$ and L. We denote by $S^2\mathfrak{m}^{*C}$ and $S_0^2\mathfrak{m}^{*C}$ the second symmetric tensor product of $\mathfrak{m}^*$ and the subspace of $S^2\mathfrak{m}^{*C}$ consisting of all $s \in S^2\mathfrak{m}^{*C}$ with $\mathrm{tr}_B s=0$, respectively. Let

$$S^2\mathfrak{m}^{*C} = V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

be an irreducible decomposition of $S^2\mathfrak{m}^{*C}$ as K-module. In this decomposition we may assume that $V_0 = \boldsymbol{C}\cdot B$ and $\sum_{i=1}^r V_i = S_0^2\mathfrak{m}^{*C}$, and that L is a scalar operator l_i on V_i. The homogeneous vector bundle $G\times_K S^2\mathfrak{m}^{*C} = S^2M^C$ is decomposed into $\oplus_{i=0}^r G\times_K V_i$ and the G-module $C^\infty(G, S^2\mathfrak{m}^{*C})_K = C^\infty(S^2M^C)$ is decomposed into $\oplus_{i=0}^r C^\infty(G, V_i)_K$. Therefore the s-eigenspace E_s of the operator $\bar{\Delta}+2L = C-2\mathcal{E}$ (cf. Corollary 5.4) is decomposed as G-module in the following way.

$$E_s = \sum_{i=0}^r E_{s,i} \qquad \text{where } E_{s,i} = E_s \cap C^\infty(G_i, V)_K\,.$$

Note that L is the scalar operator l_i on $C^\infty(G, V_i)_K$.

Let V_c be a finite dimensional G-module such that the Casimir operator of V_c is the scalar operator c. If we apply Lemma 5.4 and Lemma 5.5 and Remark following Lemma 5.5 to V_c, we see that $\phi(V_c) \subset E_{c-2\mathcal{E},i}$ for the element ϕ of $\mathrm{Hom}_G(V_c, C^\infty(G, V_i)_K)$ corresponding to a non-zero $\tilde{\phi} \in \mathrm{Hom}_K(V_c, V_i)$.

Now, let us recall some facts on (finite dimensional) G-modules of a general compact connected Lie group G. In the same way as in previous section, we define an inner product B on $\mathfrak{g}$, a linear order $>$ on a Cartan subalgebra $\mathfrak{t}$, the order $\gg$ defined by the base, $D(\mathfrak{g})$, $\delta_\mathfrak{g}$ and so on. For $\lambda \in D(\mathfrak{g})$ we denote by $V_G(\lambda)$ the irreducible G-module whose highest weight relative to $\mathfrak{t}$ is λ, more precisely, the isomorphism class of such a G-module. For a G-module W we denote by $\Lambda_G(W)$ the set of all weights relative to $\mathfrak{t}$, with multiplicity counted. If W is an irreducible G-module, we denote by $\lambda_G(W)$ its highest weight.

Lemma 7.1 (Freudenthal [10] 43.1.9). *The Cassimir operator on the G-module $V_G(\lambda)$ is the scalar operator* $(\lambda+2\delta_{\mathfrak{g}}, \lambda)$.

Lemma 7.2 (Freudenthal [10] 48.3). *If $\mu\in D(g)$, then the multiplicity $m(\mu)$ of μ in $V_G(\lambda)$ is given recursively by the formula*;

$$(\lambda+\delta_{\mathfrak{g}}, \lambda+\delta_{\mathfrak{g}})-(\mu+\delta_{\mathfrak{g}}, \mu+\delta_{\mathfrak{g}}))m(\mu) = \sum_{\substack{\alpha\in\Sigma(\mathfrak{g}),\alpha\gg 0\\ i\geq 1}} 2m(\mu+i\alpha)(\mu+i\alpha, \alpha).$$

Remark. For the multiplicities $m(\mu)$ for small λ of the type E_8, F_4, G_2, see also Freudenthal [11] Table E, Veldkamp [28] Table I, Humphreys [13] p124 Table 2, respectively.

Let W be a G-module such that $\Lambda_G(W)$ is given concretely. Let λ be a $\gg$-maximal element of $\Lambda_G(W)$. We get $\Lambda_G(V_G(\lambda))$ concretely, using Lemma 7.2. Then W is decomposed as $W=W'\oplus V$, where $\Lambda_G(W')=\Lambda_G(W)\backslash\Lambda_G(V_G(\lambda))$ and $\Lambda_G(V)=\Lambda_G(V_G(\lambda))$. Thus, inducitvely, we get concretely the set of highest weights of irreducible components of W, with multiplicity counted. This set will be denoted by $H_G(W)$.

Now, we come back to our symmetric space G/K and give a procedure of calculation. The results are in the Table. We use Bourbaki [5] Planche I-IX, where all concrete tables of $\Sigma(\mathfrak{g})$ are given. We use also the inner product B given in the tables.

I. Murakami [22] p 297 and p 305 shows the relation between the basis of $\mathfrak{g}$ and $\mathfrak{k}$. Combining Planche I-IX in Bourbaki [5], we get $\Lambda_K(\mathfrak{m}^C)=\Sigma(\mathfrak{m})$ by Proposition 6.6.

II. We decompose the K-module $S^2\mathfrak{m}^{*C}$ into irreducible components. The weights $\Lambda_K(S^2\mathfrak{m}^{*C})$ is given as $\{\alpha+\beta;\ \{\alpha, \beta\}\subset\Sigma(\mathfrak{m})\}$. Thus we get $H_K(S^2\mathfrak{m}^{*C})$ in the above way. Let $H_K(S^2\mathfrak{m}^{*C})=\{\mu_0=0, \mu_1, \cdots, \mu_r\}$. As the result of calculation we know that μ_i are distinct each other (cf. Kaneyuki and Nagano [14], Takeuchi [27]). So we shall order them, in such way that $\mu_0<\mu_1<\cdots<\mu_r$. Denoting by V_i the irreducible K-submodule of $S^2\mathfrak{m}^{*C}$ with the highest weight μ_i, we get a decomposition of $S^2\mathfrak{m}^{*C}$ as in the beginning of this section.

III. We compute the eigenvalue l_i of L on V_i. Let Σ_3' be a subset of Σ_3 with the following property; for each $\alpha\in\Sigma_3$, either one and only one of α or $\theta\alpha$ belongs to Σ_3'. Set

$$T_\alpha = X_\alpha \qquad \text{if } \alpha\in\Sigma_2,$$

$$T_{\bar\alpha} = \frac{1}{\sqrt{2}}(X_\alpha-\theta X_\alpha) \qquad \text{if } \alpha\in\Sigma_3'.$$

Choose orthonormal basis $\{H_i\}$ of $\mathfrak{b}$. Then $\{T_\alpha, T_{\bar\beta}, H_i;\ \alpha\in\Sigma_2, \beta\in\Sigma_3'\}$ are basis of $\mathfrak{m}^C$. Let $\{T_\alpha^*, T_{\bar\beta}^*, H_i^*\}$ be the dual basis of $\mathfrak{m}^{*C}$. Then

$$B=\sum_{\alpha\in\Sigma_2} T_\alpha^*\cdot T_{-\alpha}^*+\sum_{\beta\in\Sigma_3'} T_{\bar\beta}^*\cdot T_{-\bar\beta}^*+\sum_i H_i^*\cdot H_i^*\cdot$$

By the formula $B(R(X, Y)Z,U)=-B([X, Y], [Z, U])$ for symmetric spaces (c.f. Kobayashi and Nomizu [16] p 231 Theorem 3.2), we can easily check that the $T_\alpha\cdot T_\beta$-coefficient of $L(T_\gamma\cdot T_\delta)$ is zero if $\alpha+\beta\neq\gamma+\delta$, where we write T_0 for H_i. Therefore, for $\mu\in\Lambda_K(S^2\mathfrak{m}^{*C})$, the subspace W_μ of $S^2\mathfrak{m}^{*C}$ generated by $\{T_\alpha^*\cdot T_\beta;\ \alpha+\beta=\mu\}$ is invariant by the operator L. We denote by $\mathrm{Tr}(\mu)$ the trace of L on W_μ. Then

$$\mathrm{Tr}(\mu_j)=\sum_{j=1}^{r} l_j\cdot m_j(\mu_i)\,,$$

where $m_j(\mu_i)$ is the multiplicity of μ_i in V_j (cf. Kaneyuki and Nagano [14] (2.1)). Thus if we know the value of $\mathrm{Tr}(\mu_i)$, then we get the value of l_j inductively. In particular, we know the value of ε by (4.1.2): $\varepsilon=-l_0$.

Now, we give the $T_\alpha\cdot T_\beta$-coefficient $L_{\alpha\beta}$ of $L(T_\alpha\cdot T_\beta)$. These will enable us to compute $\mathrm{Tr}(\mu_i)$ and to compute l_i by the above formula.

a) Group type. See Kaneyuki and Nagano [14] Lemma 2.2.

b) Inner type (that is, $\mathfrak{b}=0$). $L_{\alpha\alpha}=(\alpha,\alpha)$ and $L_{\alpha\beta}=(\alpha,\beta)+(N_{\alpha,-\beta})^2$ if $\alpha\neq\beta$.

c) Exterior type (that is, $\mathfrak{b}\neq 0$). We assume that $\alpha-\theta\beta\notin\sum(\mathfrak{g})$ for $\alpha\in\sum(\mathfrak{m})\setminus\{0\}$ and $\beta\in\sum_3'$. This assumption is satisfied in each case. In the following table, the symbol $\bar\alpha$ is used only if $\alpha\in\sum_3'$.

$$L_{\alpha\alpha}=(\alpha,\alpha)\,,$$
$$L_{\alpha\beta}=(\alpha,\beta)+(N_{\alpha,-\beta})^2 \qquad \text{if } \alpha\neq\beta\,,$$
$$L_{\bar\alpha\bar\alpha}=(\bar\alpha,\bar\alpha)\,,$$
$$L_{\bar\alpha\bar\beta}(\alpha\neq\beta)=\begin{cases}(\bar\alpha,\bar\beta)+(N_{\alpha,-\beta})^2/2 & \text{if } \alpha-\beta\notin\mathfrak{a}\,,\\ (\bar\alpha,\bar\beta) & \text{if } \alpha-\beta\in\mathfrak{a} \text{ and } \alpha-\beta\notin\sum(\mathfrak{k})\,,\\ (\bar\alpha,\bar\beta)+(N_{\alpha,-\beta})^2 & \text{if } \alpha-\beta\in\mathfrak{a} \text{ and } \alpha-\beta\notin\sum(\mathfrak{m})\,,\end{cases}$$
$$L_{\alpha\bar\beta}=(N_{\alpha,-\beta})^2/2\,,$$
$$\textstyle\sum_i L_{\alpha i}=(\alpha,\alpha)\,,$$
$$\textstyle\sum_i L_{\bar\alpha i}=(\alpha,\alpha)-(\bar\alpha,\bar\alpha)\,,$$
$$l_0=-\{\textstyle\sum_{\alpha\in\Sigma_2}(\alpha,\alpha)+\sum_{\beta\in\Sigma_3'}(\bar\beta,\bar\beta)\}/2\,.$$

REMARK. This method is developed by Borel [4], Kaneyuki and Nagano [14,15]. See also Calabi and Vesentini [7].

IV. Let $D_i=\{\lambda\in D(\mathfrak{g});\ \mu_i\leqq\bar\lambda\}$ for $1\leqq i\leqq r$ and put $\boldsymbol{D}=\cup_{i=1}^r D_i$. Then Lemma 5.5 implies that if $V_G(\lambda)$ is a G-submodule of $C^\infty(G, V_i)_K$ then $\lambda\in D_i$. Thus if $V_G(\lambda)$ is a G-submodule of Ker tr then $\lambda\in\boldsymbol{D}$.

Put $L_i=\min\{(\lambda+2\delta_{\mathfrak{g}},\lambda);\ \lambda\in D_i\}$ for $1\leqq i\leqq r$. In order to find $\lambda\in D_i$ which attains the minimum L_i, we have only to check $\gg$-minimal weights in D_i, by virture of Proposition 6.8. Note that if $L_i > 2\mathcal{E}$, then Lemma 7.1 implies $E_{s,i}=0$ for $s \leqq 0$.

Now assume that for each $1 \leqq i \leqq r$, $l_i > l_0/2$ or $L_i > 2\mathcal{E}$. Then $E_s \cap \operatorname{Ker} \delta \cap \operatorname{Ker} \operatorname{tr} = 0$ for $s \leqq 0$, which implies that this symmetric space is stable, unless it is the standard sphere. In fact, suppose that there exists a non-zero $h \in E_s \cap \operatorname{Ker} \delta \cap \operatorname{Ker} \operatorname{tr}$. Decompose h as

$$h = \sum_{i=0}^{r} h_i, \qquad h_i \in E_{s,i}.$$

Then from the above we have

$$h = \sum_{l_i > l_0/2} h_i$$

and hence the orthogonality $(V_i, V_j)=0$ $(i \neq j)$ implies $\langle Lh, h \rangle > -\mathcal{E}\langle h, h \rangle /2$. This contradicts Lemma 2.8. See Table.

V. Now, we consider a weight $\lambda \in \boldsymbol{D}$ such that $(\lambda + 2\delta_{\mathfrak{g}}, \lambda) \leqq 2\mathcal{E}$. If $H_K(S_0^2 \mathfrak{m}^{*C}) \cap H_K(V_G(\lambda)) = \phi$, then $V_G(\lambda)$ is not a G-submodule of Ker tr. If $H_K(S_0^2 \mathfrak{m}^{*C}) \cap H_K(V_G(\lambda)) \neq \phi$ and if $H_K(V_G(\lambda)) \not\ni 0$ or $h_{\mathfrak{m}}$, where $h_{\mathfrak{m}}$ is the highest weight of $\sum(\mathfrak{m})$, then Corollary 5.6 implies that $V_G(\lambda)$ is a G-submodule of Ker tr such that $\Delta h + 2Lh = \{(\lambda + 2\delta_{\mathfrak{g}}, \lambda) - 2\mathcal{E}\} h$ and $\delta h = 0$ for all $h \in V_G(\lambda)$.

Therefore if $(\lambda + 2\delta_{\mathfrak{g}}, \lambda) = 2\mathcal{E}$ for one of such λ, then g is infinitesimally deformable (A_n $(n \geqq 2)$, AI_1 $(n \geqq 1)$, AI_2 $(n \geqq 2)$, AII $(n \geqq 3)$, EIV). If $(\lambda + 2\delta_{\mathfrak{g}}, \lambda) < 2\mathcal{E}$ for one such λ, then g is unstable provided M is not the standard sphere (B_2, C_n $(n \geqq 3)$, CI $(n \geqq 3)$).

If $H_K(S_0^2 \mathfrak{m}^{*C}) \cap H_K(V_G(\lambda)) = \phi$ for all $\lambda \in \boldsymbol{D}$ such that $(\lambda + 2\delta_{\mathfrak{g}}, \lambda) = 2\mathcal{E}$, then g is infinitesimally non-deformable, and so it is rigid (B_2, C_n $(n \geqq 3)$, CII $(p=2$ or $q \geqq 3)$, FII).

VI. Let $M = G/K$ be of Hermitian type but not the standard sphere. Assume that $\lambda = h_{\mathfrak{g}}$ is the only one element of $\boldsymbol{D}$ such that $(\lambda + 2\delta_{\mathfrak{g}}, \lambda) = 2\mathcal{E}$ and $H_K(S_0^2 \mathfrak{m}^{*C}) \cap H_K(V_G(\lambda)) \neq \phi$, and that $H_K(S_0^2 \mathfrak{m}^{*C}) \cap H_K(V_G(h_{\mathfrak{g}}))$ consistes of exactly one element. Then $E_0 \cap \operatorname{Ker} \operatorname{tr}$ is the unique G-submodule of Ker tr which is isomorphic to $V_G(h_{\mathfrak{g}})$ as G-module. We denote by J the almost complex structure of M acting on $C^\infty(S^1 M^C)$. For a (complex) Killing vector field ξ on M, we define an element h of Ker tr by $h = \delta^* J\xi - ((\operatorname{tr} \delta^* J\xi)/n) \cdot g$ regarded as $\xi \in C^\infty(S^1 M)^C$. If $h=0$, then $\delta^* J\xi \in \operatorname{Im} \delta^* \cap C^\infty(M)^C \cdot g$. It follows from Lemma 2.5 that $\delta^* J\xi = 0$ and so $J\xi$ is also a Killing vector field. Thus we have $\xi = 0$. Therefore the correspondence $\xi \rightarrow h$ is injective. Moreover we see that if $\delta h = 0$, then $h \in \operatorname{Ker} \alpha \cap \operatorname{Ker} \delta \cap \operatorname{Ker} \int$, and hence $h=0$ by Lemma 2.5. Thus the space P of all h defined by Killing vector fields ξ is a non-trivial G-submodule of Ker tr such that $\operatorname{Ker} \delta \cap P = 0$. On the other hand, since the operators δ^*, J, tr are G-homomorphisms, P is isomorphic with $V_G(h_{\mathfrak{g}})$ as G-module. Thus $E_0 \cap \operatorname{Ker} \operatorname{tr} = P$. Therefore $E_0 \cap \operatorname{Ker} \delta \cap \operatorname{Ker} \operatorname{tr} = 0$ and hence M is rigid

$(B_2/A_1 \cdot T,\ CI\ (n \geqq 3))$. If, moreover, there is no element $\lambda \in \boldsymbol{D}$ such that $(\lambda+2\delta_{\mathfrak{g}}, \lambda) < 2\varepsilon$ and $H_K(S_0^2\mathfrak{m}^{*C}) \cap H_K(V_G(\lambda)) \neq \phi$, then we see G/K is stable $(B_1/A_2 \cdot T)$.

Table: Values of l_i and L_i

Type	G K condition	l_i $(i\geqq1)$ L_i $(i\geqq1)$ condition for existence of μ_i $(i\geqq1)$			ε condition for stability (*)
A_n	$A_n \times A_n$ A_n $n\geqq1$	$-(n+1)$ $2n(n+2)/(n+1)$ $n\geqq2$	-1 $4(n-1)(n+2)/(n+1)$ $n\geqq3$	1 $4(n+1)$	$n+1$ $n=1$
B_n	$B_n \times B_n$ B_n $n\geqq2$	-2 $4(2n-3)$	$-(n-3/2)$ $4n$	1 $4(2n-1)$	$2n-1$ $n\geqq3$
C_n	$C_n \times C_n$ C_n $n\geqq3$	$-(n+2)$ $4n-1$	-1 $8n$	2 $8(n+1)$	$2(n+1)$ ×
D_n	$D_n \times D_n$ D_n $n\geqq4$	-2 $8n-18$	$-(n-2)$ $2(2n-1)$	1 $8(n-1)$	$2(n-1)$ ○
E_6	$E_6 \times E_6$ E_6	-3 34	1 48		12 ○
E_7	$E_7 \times E_7$ E_7	-4 54	1 72		18 ○
E_8	$E_8 \times E_8$ E_8	-6 94	1 120		30 ○
F_4	$F_4 \times F_4$ F_4	$-5/2$ 24	1 36		9 ○
G_2	$G_2 \times G_2$ G_2	-5 24	3 48		12 ○

(*) ○ : stable.
× : the data do not lead the stability only by procedure I∼IV.

In this table we omit μ_j when $l_j=l_i$ and $L_j=L_i$ for some $i<j$.

AIII	A_{p+q-1} $A_{p-1}\cdot A_{q-1}\cdot T$ $p\geqq q\geqq 1$	$-p$ $2(l+1)$ $q\geqq 2$	$-q$ $2(l+1)$ $p\geqq 2$	0 $4l$ $q\geqq 2$	-2 $4l$ $q\geqq 2$	2 $4(l+2)$	$l+1$ $q=1$
BI *BII*	B_{p+q} $B_p\cdot D_q$ $p\geqq 0,\ q\geqq 1$	-2 $4(2l-3)$ (*1) $p\geqq 1$	$-2(q-1)$ $2(2l+1)$ $p\geqq 1$	$-(2p-1)$ $2(2l+1)$	2 8 $p\geqq 1$		$2l-1$ $p\neq 1$ or $q\geqq 2$
CI	C_n $A_{n-1}\cdot T$ $n\geqq 3$	$-(n+2)$ $4n$	0 $2(2n+1)$	-2 $2(2n+1)$	4 $8(n+2)$		$2(n+1)$ ×
CII	C_{p+q} $C_p\cdot C_q$ $p\geqq q\geqq 1$	$-2(q+1)$ $4l$ $p\geqq 2$	$-2(p+1)$ $4l$ $q\geqq 2$	-2 $8(l-1)$ $q\geqq 2$	2 $4(2l+1)$		$2(l+1)$ $p=1$ or $q=1,\ p\geqq 3$
DI$_1$	D_{p+q} $D_p\cdot D_q$ $p\geqq q\geqq 1$ $p+q\geqq 4$	-2 $8(l-2)$ (*2)	$-2(q-1)$ $4l$	$-2(p-1)$ $4l$	2 $4(2l-1)$		$2(l-1)$ ○
DIII	D_n $A_{n-1}\cdot T$ $n\geqq 3$	$-(n-2)$ $4(n-1)$	0 $8(n-2)$ $n\geqq 4$	-4 $8(n-2)$ $n\geqq 4$	2 $4(2n-1)$		$2(n-1)$ ○
EII	E_6 $A_1\cdot A_5$	-2 36	-4 36	2 52			12 ○
EIII	E_6 $D_5\cdot T$	-4 24	0 36	-6 36	2 52		12 ○
EV	E_7 A_7	-4 56	2 76				18 ○
EVI	E_7 $D_6\cdot A_1$	-2 56	-6 56	2 76			18 ○
EVII	E_7 $E_6\cdot T$	-6 36	0 56	-8 56	2 76		18 ○
EVIII	E_8 D_8	-6 96	2 124				30 ○
EIX	E_8 $E_7\cdot A_1$	-2 96	-10 96	2 124			30 ○

FI	F_4 $C_3 \cdot A_1$	-2 26	-3 26	2 40		9 ○
FII	F_4 B_4	-5 12	1 24			9 ×
G	G_2 $A_1 \cdot A_1$	-6 28	-4 28	6 60		12 ○
AI_1	A_{2n} B_n $n \geqq 1$	$-(n+1/2)$ $2(l+1)$	-1 $4l$ $n \geqq 2$	2 $4(l+2)$		$l+1$ ×
AI_2	A_{2n-1} D_n $n \geqq 2$	$-n$ $2(l+1)$	-1 $4l$	2 $4(l+2)$		$l+1$ ×
AII	A_{2n-1} C_n $n \geqq 2$	$-n$ $2(l+1)$ $n \geqq 3$	-2 $4l$ $n \geqq 4$	1 $4l$		$l+1$ $n=2$
DI_2 DII	D_{p+q+1} $B_p \cdot B_q$ $p \geqq q \geqq 0$ $p+q \geqq 3$	-2 $8(l-2)$ (*3) $q \geqq 1$	$-(2q-1)$ $4l$	$-2(p-1)$ $4l$ $q \geqq 1$	2 $4(2l-1)$ $q \geqq 1$	$2(l-1)$ ○
EI	E_6 C_4	-3 36	2 52			12 ○
EIV	E_6 F_4	-6 52/3	1 36			12 ×

(*1) This holds when $p, q \geqq 2$. The value is $6(l-1)$ if $p=1$, $q \geqq 2$, $2(2l-1)$ if $p \geqq 2$, $q=1$ and 4 if $p=q=1$.

(*2) This holds when $q \geqq 2$. The value is $4(l-1)$ if $q=1$.

(*3) This holds when $q \geqq 2$. The value is $3(2l-3)$ if $q=1$.

References

[1] M. Berger: *Quelques formules de variation pour une structure riemannienne,* Ann. Sci. Ecole Norm. Sup. 4e série **3** (1970), 285–294.

[2] M. Berger: *Sur les variétés d'Einstein compactes,* Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (1965), 35–55.

[3] M. Berger and D.G. Ebin: *Some decompositions of the space of symmetric tensors on a riemannian manifold,* J. Differential Geometry **3** (1969), 379–392.

[4] A. Borel: *On the curvature tensor of the hermitian symmetric manifolds,* Ann. of Math. **71** (1960), 508–521.
[5] N. Bourbaki: Groupes et algèbres de Lie chapitres 4, 5 et 6, Hermann, Paris, 1968.
[6] J.P. Bourguignon: *Premières formes de Chern des variétés Kählériennes compactes,* Séminaire Bourbaki **507** (1977), 01–18.
[7] E. Calabi and E. Vesentini: *On compact, locally symmetric Kähler manifolds,* Ann. of Math. **71** (1960), 472–507.
[8] D.G. Ebin: Espace des métriques riemanniennes et mouvement des fluides via les variétés d'applications, Centre de Mathématiques de l'Ecole Polytechnique et Université Paris VII, 1972.
[9] A.E. Fischer and J.E. Marsden: *Manifolds of riemannian metrics with prescribed scalar curvature,* Bull. Amer. Math. Soc. **80** (1974), 479–484.
[10] H. Freudenthal: Linear Lie groups, Academic Press, New York, 1969.
[11] H. Freudenthal: *Zur Berechnung der Characktere der halbeinfachen Lieschen Gruppen.* II, Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 487–491.
[12] D. Hilbert: *Die Grundlagen der Physik,* Nachr. Akad. Wiss. Göttingen Math. - Phys. Kl. II (1915), 395–407.
[13] J.E. Humphreys: Introduction to Lie algebras and representation theory, Springer-Verlag, New York, 1972.
[14] S. Kaneyuki and T. Nagano: *On the first Betti numbers of compact quotient spaces of complex semi-simple Lie groups by discrete subgroups,* Sci. Papers College Gen. Ed. Univ. Tokyo **12** (1962), 1–11.
[15] S. Kaneyuki and T. Nagano: *On certain quadratic forms related to symmetric riemannian spaces,* Osaka Math. J. **14** (1962), 241–252.
[16] S. Kobayashi and K. Nomizu: Foundations of differential geometry, Volume II, Interscience, New York, 1969.
[17] N. Koiso: *Non-deformability of Einstein metrics,* Osaka J. Math. **15** (1978), 419–433.
[18] N. Koiso: *A decomposition of the space M of riemannian metrics on a manifold,* Osaka J. Math. **16** (1979), 423–429.
[19] N. Koiso: *On the second derivative of the total scalar curvature,* Osaka J. Math. **16** (1979), 413–421.
[20] A. Lichnerowicz: *Propagateurs et commutateurs en relativité générale,* Inst. Hautes Etudes Sci. Publ. Math. **10** (1961), 293–344.
[21] S. Marakami: *On the automorphisms of a real semi-simple Lie algebra,* J. Math. Soc. Japan **4** (1952), 103–133; **5** (1953), 105–111.
[22] S. Murakami: *Sur la classification des algèbres de Lie réelles et simples,* Osaka J. Math. **2** (1965), 291–307.
[23] Y. Muto: *Einstein spaces of positive scalar curvature,* J. Diffenential Geometry **3** (1969), 457–459.
[24] Y. Muto: *On Einstein metrics,* J. Differential Geometry **9** (1974), 521–530.
[25] R.S. Palais: Seminar on the Atiyah-Singer index theorem, Princeton University Press, Princeton, 1975.
[26] R.S. Palais: *On the differentiability of isometries,* Proc. Amer. Math. Soc. **8** (1957), 805–807.

[27] M. Takeuchi: *Polynomial representations associated with symmetric bounded domains,* Osaka J. Math. **10** (1973), 441–475.

[28] F.D. Veldkamp: *Representations of algebraic groups of type F_4 in caracteristic* 2, J. Algebra **16** (1970), 326–339.

[29] N.A. Wallach: *Minimal immersions of symmetric spaces into spheres,* Symmetric spaces edited by W.M. Boothby and G.L. Weiss, Marcel Dekker, INC., New York, 1972.

Departmeot of Mathematics
Osaka University
Toyonaka, Osaka 560, Japan