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Solution Concepts in Cooperative Fuzzy Games and Minimum Spanning Tree Games

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Solution Concepts in Cooperative Fuzzy Games and Minimum Spanning Tree Games
(協力ファジィゲームと最小スパニングツリーゲームにおける解概念)

by

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Abstract

Game theory formulates situations where some decision makers (players) conflict and/or cooperate with one another and enables us to analyze social behavior of players. Above all, a useful theory for the analysis of the situation where a cooperative relationship (a coalition) among players arises is cooperative game theory. In cooperative game theory, a function which assigns to each coalition the proceeds or the cost when the coalition is formed is called a game. Rational distribution of the proceeds or the cost can be derived from the corresponding game and is regarded as a solution concept. Such solution concepts enables objective analyses of power and rational decision making, so it is considered important in engineering to study solution concepts in cooperative game theory.

The first part of this thesis discusses a cooperative fuzzy game (a fuzzy game) and a minimum spanning tree game: the former is an extension of a conventional game, a cooperative crisp game (crisp game); and the latter is a cooperative game derived from one of realistic optimization problems, say the minimum spanning tree problem.

Fuzzy game theory is based on fuzzy coalitions defined by rates of players' participation to the coalition, while crisp game theory on crisp coalitions defined by whether each player cooperates or not. It follows that fuzzy game theory can deal with situations where some players cooperate partially. Thus, it can be applied to real situations more flexibly than crisp one.

This thesis discusses rational imputations of the proceeds when a fuzzy coalition is formed. The Shapley value, the core and the dominance core, which are important solution concepts in crisp game theory, are extended

to fuzzy game theory. The solution concepts are obtained through functions from a pair of a game and a fuzzy coalition. It is shown that the core and the dominance core coincide with each other if a given fuzzy game is superadditive and monotone nondecreasing with respect to rates of players' participation. Since there does not always exist some imputation which belongs to the core as in crisp games, a necessary and sufficient condition for the core to be nonempty is given in fuzzy games. Since it is not easy to obtain an explicit form of the Shapley value in an arbitrary fuzzy game, a class of fuzzy games is introduced. The class can be considered natural since any game in the class is monotone nondecreasing and continuous with respect to rates of players' participation. A function which derives the Shapley values of fuzzy games in the class is given in explicit form. Properties of the explicit form of the Shapley value are clarified. It is shown that the center of gravity of the core coincides with the Shapley value for a convex game in the class.

On the other hand, a problem such as to minimize the cost when an intelligence network is built in cooperation is a minimum spanning tree problem. When the optimal solution is obtained, a problem of how to allocate the cost may arise. A minimum spanning tree game and a monotonic cover game derived from minimum spanning tree problems will give some objective criteria to this problem. One of solution concepts proposed so far in the situation of the minimum spanning tree problem is not well-defined. Furthermore, this solution concept is not rational since it is not always included in the core of the corresponding monotone cover games and some players may pay negative cost in it.

The second part of this thesis gives a restriction of the definition of this solution concept to be well-defined. Furthermore, a new solution concept is introduced on the assumption that each player allows himself/herself to be a component of a network on behalf of other players as far as he/she need not pay the cost at all. It will be shown that any player will pay nonnegative cost in the solution concept and the solution concept is always included in the core of both minimum spanning tree games and monotonic cover game.

We introduce solution concepts in cooperative fuzzy games and minimum spanning tree games and discuss the rationality of them. Cooperative fuzzy game can deal with real situation flexibly and the situation which derives minimum spanning tree games is so realistic. We give realistic and appropriate decision making in distribution of the proceeds/cost by studying solution concepts in cooperative fuzzy games and minimum spanning tree games.

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Chapter 1

Introduction

1.1 General Introduction

Game theory was first introduced by J. von Neumann [27, 28] in 1928 and has been developed on the basis of Theory of Games and Economic Behavior by J. von Neumann et al. [29]. This theory enables us to analyze social behavior of human being objectively. It has been regarded as important for a long time and studied from so many angles. This theory can be classified into noncooperative game theory and cooperative game theory.

Cooperative game theory is so useful to understand social phenomena when some cooperative relationship (coalition) among decision makers (players) arises. Cooperative game theory is concerned with both matters: defining solution concepts and then investigating their properties, in general as well as in specific models coming from the various areas of application. This leads to mathematical theories that ultimately yield important and novel insights, quantitative as well as qualitative.

By forming a coalition, players may get benefit such as increasing the proceeds or decreasing the cost. The proceeds to be obtained or the cost to be paid when each coalition is formed is called the coalition value. In cooperative game theory, a function which assigns to each coalition its coalition value is called a game. Rational distribution of the proceeds or the cost can be derived from the corresponding game and is regarded as a solution concept. Such solution concepts enable objective analyses of power or rational decision making, so it is considered important in engineering to study solution concepts in cooperative game theory.

In this thesis, we introduce solution concepts in cooperative fuzzy games (fuzzy games) and minimum spanning tree games and investigate their properties. Fuzzy games enable us to represent rates of players' participation and to deal with real situation flexibly. In fuzzy games, we discuss solution concepts and investigate their properties in general. A minimum spanning tree game is a cooperative game derived from one of realistic optimization problems, say minimum spanning tree problem. It may be important since it can deal with the situation where players would like to reduce the cost of constructing a network, i.e., the situation which can be formulated as a minimum spanning tree problem. We give some rational cost allocations as solution concepts and investigate their properties in this game.

We give realistic, rational and appropriate means of decision making by discussing solution concepts in fuzzy games, which are flexible to deal with real situations, and minimum spanning tree games, which are realistic game-theoretical formulations.

1.1.1 Cooperative Fuzzy Games

The Shapley value [4, 22], the core and the dominance core [23, 24] are well-known solution concepts in cooperative game theory. The Shapley value is a vector and the other two solution concepts are sets of vectors whose elements show members' shares derived from several reasonable bases when a coalition is formed. Those solution concepts have been investigated by a number of researchers, most of whom treat games with conventional coalitions, which are called crisp coalitions and defined by whether each player participate or not. However, we may come across situations where some members do not fully participate in a coalition, but to a certain extent. For example, in a class of production games, partial participation in a coalition means to offer a part of resources while full participation means to offer all of resources. A coalition including some players who participate partially can be treated as a so-called fuzzy coalition [1, 2]. A fuzzy coalition is defined as a collection of players who transfer fractions of their representability [5] and identified with membership function on the set of players. A membership degree to which a player transfers his/her representability is called the rate of his/her participation [3]. Games with such fuzzy coalitions are usually called fuzzy games.

Butnariu [7] defined a Shapley function as a function which derives the Shapley value from a given pair of a fuzzy game and a fuzzy coalition. He showed the explicit form of the Shapley function on a limited class of fuzzy games. Most games in the class are neither monotone nondecreasing nor continuous with regard to rates of players' participation although crisp games are often considered monotone nondecreasing. Thus, the class cannot be regarded as quite natural. The core of a fuzzy game was also introduced [7]. The core is based on the unusual imputation set.

On the other hand, multi-choice cooperative games [15, 18, 19] and continuously-many-choice cooperative games [16] have been discussed. Those games deal with situations where each player has finite or infinite action levels, which may correspond to rates of his/her participation in coalitions. Several solution concepts for such games are considered [15, 16, 18, 19]. Each solution is represented by a matrix or a set of matrices whose element shows a player's rational gain in an action level. Each player's rational gain in an action level is independent of the other players' action levels, and not all n-vectors of the rational gains satisfy the efficiency of the corresponding (fuzzy) coalition. These two facts are essential differences from Butnariu's approach.

There also exists an approach similar to Butnariu's on such points. Sprumont [25] proposed a population monotonic allocation scheme (PMAS) as a reasonable solution concept. It as well as solution concepts by Butnariu meets the need for solution concepts which specify not only how to allocate $v(N)$ but also how to allocate the worth of every coalition $S \subseteq N$. But it was introduced only for crisp games.

In this thesis, we adopt Butnariu's approach and define the Shapley value, the core and the dominance core in cooperative fuzzy games. Solution concepts defined in this thesis are related to the solution concept by Sprumont. Rational properties of them is discussed.

1.1.2 Minimum Spanning Tree Games

It is well known that the mathematical modeling of various real-world decision making situations gives rise to optimization problem. However, for situations where plural decision makers (players) are involved, classical optimization theory does not suffice. Suppose that a group of decision makers decides to undertake a project together in order to increase the total revenue or decrease the total cost and finds an optimal way to execute the project by using classical optimization problem. Then they face the problem of how to allocate the revenue or the cost among the participants. It is quite easy to find examples where cooperation among several participants would have increased total revenue or decreased total cost but the cooperation did not work out because a reasonable allocation was not achieved. The solution concepts in cooperative game theory can be applied to arrive at some reasonable allocation and make an important contribution.

The problem of finding the most inexpensive way to build a distribution system such as an intelligence network servicing many cities is formulated as a minimum spanning tree problem. It is one of optimization problems and is equivalent to a problem to find a minimum spanning tree for the complete graph with edge cost. The problem of allocating the cost of a spanning tree in a graph among the users which are situated as the nodes of the graph, with one node reserved for a common supplier which is not to participate in the cost sharing, was first introduced by Claus and Kleitman [9]. Bird [6] was the first to suggest a game theoretic approach to the problem and gave a rational cost allocation, which is called a Bird tree allocation. The Bird tree allocation is always included in both the cores of the minimum spanning tree game (mst-game for short) and the monotone cover game (mc-game for short), which are game-theoretical formulations in this situation. However, the Bird tree allocation is not so realistic. It is because a player directly connected to the supplier does not benefit by forming the grand coalition although his/her cooperation is important for connecting the supplier and other players in many cases.

To improve this point, Granot and Huberman [14] proposed the weak demand operation (w.d.o.) by a player and the w.d.o. by a coalition. In their thesis, it was considered that both the allocations obtained through the w.d.o. by a player and that by a coalition on a Bird tree allocation are elements of the core of the mst-game. However, the w.d.o. by a coalition is not well-defined, and in fact some allocations obtained through it on Bird tree allocations are not elements of the core of the mst-game. Furthermore, some obtained allocations may have negative elements. This means not only that they do not belong to the core of the mc-game but also that some players obtain payoff; hence this allocation is not acceptable in many real situations.

In this thesis, we add an appropriate restriction to the weak demand operation. Furthermore, we propose more rational operation than the weak demand operation.

1.2 Contributions and Organization of the Thesis

In this thesis, we give definitions of solution concepts in cooperative games and investigate their properties in the following structure.

In Chapter 2, we give preliminaries to discuss solution concepts in cooperative fuzzy games and minimum spanning tree games.

In Chapter 3, we discuss solution concepts in cooperative fuzzy games in general. We define an imputation function, an imputation set-valued function and a fuzzy population monotonic allocation scheme (FPMAS) as an extension of an imputation, an imputation set and a PMAS, respectively.

Natural extensions of a carrier and a null player are given to introduce an axiomatic definition of the Shapley function. We also introduce a core function and a dominance core function. They are functions which derive the core and the dominance core from a given pair of a fuzzy game and a fuzzy coalition, respectively. It is shown that they coincide if a given fuzzy game is superadditive and monotone nondecreasing with respect to rates of players' participation. Balancedness is also defined. It is also shown that the core of a fuzzy game is nonempty if and only if the game is balanced.

In Chapter 4, we introduce a particular class of fuzzy games where any game is both monotone nondecreasing and continuous with regard to rates of players' participation to give some explicit form of a Shapley function. The class can be considered natural in terms of continuity and monotonicity. An explicit form of the Shapley function on the proposed class is given. We show that the Shapley function on the class has rational properties. Furthermore, we show that the center of gravity of the core coincides with the Shapley value for any convex game in the proposed class of fuzzy games.

In Chapter 5, we discuss solution concepts in minimum spanning tree games. We add a certain restriction to the weak demand operation by a coalition so that it is well-defined. It will be called the demand operation. Then, any allocation obtained through it on a Bird tree allocation is an element of the core of the mst-game. However, some of the obtained allocations still do not belong to the core of the mc-game. We propose a revised weak demand operation such that any allocation obtained through it on a Bird tree allocation is an element of both the cores of the mst-game and the mc-game. It follows that every obtained allocation has no negative element, and hence the revised weak demand operation is more applicable than the original one.

Chapter 6 concludes this thesis.

Chapter 2

Preliminaries

2.1 Cooperative Fuzzy Games

In this section, we shall provide preliminaries to introduce solution concepts in cooperative fuzzy games, i.e., cooperative games with fuzzy coalitions.

2.1.1 Cooperative Fuzzy Games

In this thesis, we consider cooperative fuzzy games with the set of players $N = \{1, \dots, n\}$. A cooperative fuzzy game is a cooperative game based on fuzzy coalitions. A fuzzy coalition is a fuzzy subset of N, which is identified with a function from N to $[0, 1]$. Then for a fuzzy coalition S and player i, $S(i)$ indicates the membership grade of i in S, i.e., the rate of the ith player's participation in S. For a fuzzy coalition S, the level set is denoted by $[S]_h = \{i \in N \mid S(i) \geq h\}$ for any $h \in [0, 1]$, and the support is denoted by Supp $S = \{i \in N \mid S(i) > 0\}$. The class of all fuzzy subsets of a fuzzy set $U \subseteq N$ is denoted by $L(U)$. Let $P(W)$ denote the set of all crisp subsets of a crisp set $W \subseteq N$.

A fuzzy game is a function $v : L(N) \to \mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ such that $v(\emptyset) = 0$. $G(N)$ denotes the set of all fuzzy games. A function $v : P(N) \to \mathbb{R}_+$ such that $v(\emptyset) = 0$ is called a crisp game. According to the classical interpretation by von Neumann et al. [29], for any crisp coalition S, $v(S)$ is regarded as the least profit which can be achieved by the members of

S when S is formed. It follows that any crisp game v is superadditive; hence v is monotone nondecreasing with respect to set inclusion. We denote the set of all superadditive crisp game by $G_0(N)$.

Union and intersection of two fuzzy sets are defined as usual, i.e.,

$$\begin{aligned} (S \cup T)(i) &= \max\{S(i), T(i)\}, \qquad \forall\, i \in N, \\ (S \cap T)(i) &= \min\{S(i), T(i)\}, \qquad \forall\, i \in N. \end{aligned}$$

Then superadditivity and convexity in fuzzy games are defined as follows:

Definition 2.1 *A game $v \in G(N)$ is said to be superadditive if*

$$v(S \cup T) \geq v(S) + v(T), \quad \forall\, S, T \in L(N) \text{ s.t. } S \cap T = \emptyset.$$

Definition 2.2 *A game $v \in G(N)$ is said to be convex if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \forall\, S, T \in L(N).$$

As can be seen easily, any convex game is superadditive. Note that the restrictions of the definitions above to crisp games coincide with usual superadditivity and convexity, respectively.

As will be seen later, any fuzzy game dealt with in this thesis is superadditive. It can be considered that the interpretation of coalition value by von Neumann et al. is adopted in fuzzy games as well as in crisp games in this thesis. It is to be noted that not all superadditive fuzzy games are monotone nondecreasing whereas any superadditive crisp games is monotone nondecreasing as far as the range is $\mathbb{R}_+$.

2.1.2 Basic Concepts in Cooperative Crisp Games

In usual cooperative crisp game theory, a rational distribution of revenue of a grand coalition, say a crisp coalition N, is discussed. We call such rational distributions usual solution concepts. In Chapters 3 and 4, we discuss rational distributions of revenue of a fuzzy coalition defined by a fuzzy subset of the grand coalition N, which are regarded as solution concepts in Chapters 3 and 4. It is to be noted that many concepts in Chapters 3 and 4 depend on not only a game but also a coalition while many concepts in usual cooperative game theory on a game. Preparatory to discussions in Chapters 3 and 4, we prepare some basic concepts which depend on both a game and a coalition as follows.

In this thesis, a vector $\boldsymbol{x} \in \mathbb{R}^n$ can be called an imputation of $W \in P(N)$ for $v \in G_0(N)$ if it satisfies

1. $x_i = 0, \quad \forall\, i \notin W,$

2. $\sum\limits_{i \in N} x_i = v(W),$

3. $x_i \geq v(\{i\}), \quad \forall\, i \in W,$

where $\boldsymbol{x} = (x_i)_{i \in N}$. Let us denote the set of all imputations of $W \in P(N)$ for $v \in G_0(N)$ by $I'(v)(W)$. An imputation function and an imputation set-valued function based on an imputation have defined as follows:

Definition 2.3 *A function x from $P(N)$ to $\mathbb{R}^n_+$ is said to be an imputation function of a crisp game v if $x(W) \in I(v)(W)$ for any $W \in P(N)$. A function X from $P(N)$ to $2^{\mathbb{R}^n_+}$ is called an imputation set-valued function of a crisp game v if $x(W) \subseteq I(v)(W)$ for any $W \in P(N)$.*

Note that $x(N)$ is a usual imputation of v if x is an imputation function.

Sprumont [25] proposed a population monotonic allocation scheme (PMAS) as a reasonable solution concept as follows:

Definition 2.4 *[25] A vector $\boldsymbol{x} = (x_i(W))_{i \in W,\, W \in P(N)}$ is said to be a population monotonic allocation scheme (PMAS) if it satisfies the following:*

$$\begin{cases} \sum\limits_{i \in W} x_i(W) = v(W), & \forall\, W \in P(N), \\ x_i(S) \leq x_i(T), & \forall\, i \in S, \quad \forall\, S, T \in P(N) \text{ s.t. } S \subseteq T. \end{cases}$$

PMAS's also meet the need for solution concepts which specify not only how to allocate $v(N)$ but also how to allocate the worth $v(S)$ of every coalition $S \in P(N)$.

2.1.3 Solution Concepts in Cooperative Crisp Games

In this subsection, functions which derive solutions from a given pair of a crisp game and a crisp coalition are introduced. First, a core function on the class of crisp games is defined as follows:

Definition 2.5 *A function C' from $G_0(N)$ to $(2^{\mathbb{R}^n_+})^{P(N)}$ is said to be a core function on $G_0(N)$ if it satisfies*

$$C'(v)(W) = \left\{ \boldsymbol{x} \in I'(v)(W) \;\middle|\; \sum_{i \in W} x_i \geq v(S), \quad \forall\, S \in P(W) \right\}.$$

To introduce a dominance core function on the class of crisp games, we define the concept of dominance in crisp games. In this thesis, it is said that an imputation $\boldsymbol{x}$ dominates an imputation $\boldsymbol{y}$ via a coalition $S \in P(N)$ if

$$\begin{cases} \displaystyle\sum_{i \in S} x_i \leq v(S), \\ x_i > y_i, \quad \forall\, i \in S. \end{cases}$$

Now, a dominance core function on the class of crisp games is defined as follows:

Definition 2.6 *A function DC' from $G_0(N)$ to $(2^{\mathbb{R}^n_+})^{P(N)}$ is said to be a dominance core function on $G_0(N)$ if it satisfies:*

$$\begin{aligned} &DC'(v)(W) \\ &= \{\boldsymbol{x} \in I'(v)(W) \mid \textit{there exists no } \boldsymbol{y} \in I'(v)(W) \\ &\qquad\qquad\qquad\qquad \textit{which dominates } \boldsymbol{x} \textit{ in } W \,\} \end{aligned}$$

In order to define the Shapley function on the class of all superadditive crisp games in $G_0(N)$, we introduce a carrier in a coalition $W \in P(N)$ for a game $v \in G_0(N)$, which can be regarded as an extension of the traditional carrier defined in the grand coalition N.

Definition 2.7 *Let $v \in G_0(N)$ and $W \in P(N)$. $S \in P(W)$ is called a carrier in a coalition W for a game v if*

$$v(S \cap T) = v(T), \quad \forall\, T \in P(W).$$

We will denote the set of all carriers in W for v by $C(W \mid v)$, i.e.,

$$C(W \mid v) = \{S \in P(W) \mid v(S \cap T) = v(T), \quad \forall\, T \in P(W)\}.$$

A carrier is closely related to the concept of a null player defined below.

Definition 2.8 *Let $v \in G_0(N)$ and $W \in P(N)$. Player $i \in W$ is called a null player in W for v if*

$$v(S) = v(S \cup \{i\}), \quad \forall\, S \in P(W \backslash \{i\}).$$

Note that carriers and null players defined above are specified not only by a game but also by a coalition. Carriers and null players in a coalition N coincide with the usual carriers and the usual null players, respectively. As can be seen easily, if $S \in C(W \mid v)$ then any $i \notin S$ is a null player in W for v.

Now we introduce a Shapley function on $G_0(N)$.

Definition 2.9 *A function $f' : G_0(N) \to (\mathbb{R}^n_+)^{P(N)}$ is said to be a Shapley function on $G_0(N)$ if it satisfies the following four axioms.*

Axiom C_1: *If $v \in G_0(N)$ and $W \in P(N)$ then*

$$\begin{cases} \sum_{i \in N} f'_i(v)(W) = v(W), \\ f'_i(v)(W) = 0, \quad \forall\, i \notin W, \end{cases}$$

where $f'_i(v)(W)$ is the i-th element of $f'(v)(W) \in \mathbb{R}^n$.

Axiom C_2: *If $v \in G_0(N)$, $W \in P(N)$ and $T \in C(W \mid v)$ then*

$$f'_i(v)(W) = f'_i(v)(T), \quad \forall\, i \in N.$$

Axiom C_3: *If $v \in G_0(N)$, $W \in P(N)$, $i, j \in W$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ holds for any $S \in P(W \backslash \{i, j\})$ then*

$$f'_i(v)(W) = f'_j(v)(W).$$

Axiom C_4: *For any $v_1, v_2 \in G_0(N)$, define a game $v_1 + v_2 \in G_0(N)$ by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \in P(N)$. If v_1, $v_2 \in G_0(N)$ and $W \in P(N)$ then*

$$f'_i(v_1 + v_2)(W) = f'_i(v_1)(W) + f'_i(v_2)(W), \quad \forall\, i \in N.$$

Butnariu [7] also introduced a carrier and a Shapley function on $G_0(N)$ implicitly. His axioms of the Shapley function are slightly different from ours, as will be explained later in the fuzzy case.

The unique Shapley function on $G_0(N)$ is explicitly obtained by extending the Shapley value for grand coalition N.

Theorem 2.1 *Define a function $f' : G_0(N) \to (\mathbb{R}^n_+)^{P(N)}$ by*

$$f'_i(v)(W) = \begin{cases} \sum\limits_{T \in P_i(W)} \beta(|T|; |W|) \cdot \{v(T) - v(T \backslash \{i\})\}, & \text{if } i \in W, \\ 0, & \text{otherwise,} \end{cases}$$

where $P_i(W) = \{T \in P(W) \mid T \ni i\}$ and $\beta(|T|; |W|) = (|T| - 1)! \cdot (|W| - |T|)!/|W|!$. Then the function f' is the unique Shapley function on $G_0(N)$.

Proof. Let $v \in G_0(N)$ and $W \in P(N)$. Then $f'(v)(W)$ can be obtained in the same manner as the usual Shapley value, namely $f'(v)(N)$. It was proved for $f'(v)(N)$ in [22]. □

Note that $f'(v)$ is an imputation function of a crisp game v. It is also noted that if $v \in G_0(N)$ then the vector $(f'_i(v)(W))_{i \in W, W \in P(N)}$ coincides with the extended Shapley value proposed by Sprumont [25]. Hence, the following proposition holds:

Proposition 2.1 *[25] The vector $(f'_i(v)(W))_{i \in W, W \in P(N)}$ is a PMAS if $v \in G_0(N)$ is convex.*

This proposition implies that if $v \in G_0(N)$ is convex, $i \in N$ and $S \subseteq T$ then $f'_i(v)(S) \leq f'_i(v)(T)$.

2.2 Minimum Spanning Tree Games

In this section, a definition of a minimum spanning tree problem is given. A minimum spanning tree game and a monotone cover game are derived from this problem. A basic solution concept, a Bird tree allocation [6], is given and its properties are clarified.

2.2.1 Minimum Spanning Tree Problem

In this subsection, various basic concepts concerning networks are defined preparatory to define minimum spanning tree problems.

Let $G = (V, E)$ represent a graph with the set of vertices V and the set of edges E. An edge $e \in E$ is denoted by (u, v) if the end points of e are u and v. Edges (u, v) and (v, u) denote the same edge for any $u, v \in V$. In this thesis, we do not take account of edges (u, u) for any $u \in V$. If there exists an edge (u, v) in a graph G for any $u, v \in V$, G is said to be a complete graph. Consider a sequence described as follows:

$$P = (v_0, e_1, v_1, \ldots, e_l, v_l),$$

where $v_0, \ldots, v_l \in V$ and $e_1, \ldots, e_l \in E$. A sequence P is said to be a path of length l from vertex v_0 to vertex v_l if $e_i = (v_{i-1}, v_i)$ for any $i \in \{1, 2, \ldots, l\}$. A path P is said to be a circuit if v_0 and v_l denote the same vertex. A graph G is said to be connected if there exists a path from u to v for any $u, v \in V$. It is clear that a graph G is connected if it is complete. A connected subgraph without any circuits is called a tree. A tree is said to be a spanning tree in a graph $G = (V, E)$ if its vertex set is equal to V.

A graph $G = (V, E)$ is said to be a network if a value $c_{uv} \geq 0$ is given for each edge $(u, v) \in E$. In this thesis, the value c_{uv} is regarded as the cost of constructing a link between u and v and called the edge cost of (u, v). Any network can be considered complete without loss of generality by letting c_{uv} be positively infinite or sufficiently large if a link between u and v cannot be constructed. For subnetwork (V', E') satisfying $V' \subseteq V$ and $E' \subseteq E$, we call the sum of its edge costs, say $\sum_{(u,v)\in E'} c_{uv}$, the cost of the subnetwork. Then a minimum spanning tree is defined as follows:

Definition 2.10 *A spanning tree in a network $G = (V, E)$ is said to be a minimum spanning tree in G and denoted by $\Gamma = (V, E_\Gamma)$ if it is minimum of all spanning trees in G in terms of the cost.*

It is to be noted that there exist some distinct minimum spanning trees in many networks.

Let a vertex 0 be regarded as a source. For a minimum spanning tree Γ in a network G whose vertex set includes the source 0, let $O_\Gamma(i)$ denote the length of the unique path from the source 0 to a vertex i in Γ. If a vertex i is on the unique path from the source 0 to a vertex j in Γ, the notation $i \preceq_\Gamma j$ is used and vertices i and j are called a predecessor of j and a follower of i, respectively. It is apparent that $O_\Gamma(i) \leq O_\Gamma(j)$ if $i \preceq_\Gamma j$. In particular, if $i \preceq_\Gamma j$ and $O_\Gamma(j) - O_\Gamma(i) = 1$, then the vertices i and j are said to be the adjacent predecessor of vertex j and an adjacent follower of vertex i in Γ, respectively. For any $i \in N$, let $p(i)$ and $F(i)$ denote the adjacent predecessor of i and the set of the adjacent followers of i in Γ, respectively.

2.2.2 Minimum Spanning Tree Games

In this subsection, we shall define minimum spanning tree games and their solution concepts. A minimum spanning tree game is a kind of cost-sharing game which is defined as follows. Suppose that each player needs cost to make products or something. In many of such situations, they may be able to reduce the cost by cooperating with someone, which means that it is rational to make the coalition if possible. Game theory on cost-sharing games gives an answer to allocate the cost among members of the formed coalition. Let the formed coalition be denoted by a set of players $N = \{1, 2, \dots, n\}$. The cost which is to be paid by members of a coalition $S \subseteq N$ when S is formed is denoted by $c(S)$. By convention, let $c(\emptyset) = 0$. Then a function $c : 2^N \longrightarrow \mathbb{R}_+$ is said to be a discrete cost function or a cost-sharing game, where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$. A cost-sharing game c is said to be monotone if $c(S) \leq c(T)$ for any $S \subseteq T \subseteq N$. Let x_i denote the amount charged to player i and let $x(S) = \sum_{i \in S} x_i$. An n-vector $\boldsymbol{x} = (x_i)_{i \in N}$ is called an allocation if $x(N) = c(N)$ and x_i shows the cost to be paid by player i. An allocation or a set of allocations selected by reasonable bases are regarded as solution concepts in cost-sharing games. The core of a cost-sharing game c is one of solution concepts and defined as follows:

$$C(c) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x(N) = c(N),\ x(S) \leq c(S), \quad \forall\, S \subset N\}.$$

Next, a minimum spanning tree game (mst-game) and a monotone cover game (mc-game) are introduced. Suppose that players in N would like to make up an intelligence network system with which all players in N and information source 0 are covered. If G denotes a network whose vertex set is $N_0 = N \cup \{0\}$, then such intelligence network systems can be regarded as subnetworks including spanning trees in a network G. In such a case, it is rational in terms of cost to construct a minimum spanning tree in G. If once players find a minimum spanning tree, they may face a problem of how to allocate the cost of the minimum spanning tree. Optimal solution to this problem can be given by mst-games or mc-games, which will be defined later. In order to define those games, we prepare some notations. For $S \subseteq N$, let $S_0 = S \cup \{0\}$. For $Q \subseteq N_0$, let $G_Q = (Q, E_Q)$ and $\Gamma_Q = (Q, E_{\Gamma_Q})$ denote a complete network with a vertex set Q and a minimum spanning tree in G_Q, respectively. We will call a minimum spanning tree Γ_{S_0} in a network G_{S_0} a minimum spanning tree for the coalition S. In particular, let $\Gamma = (N_0, E_\Gamma)$ denote a minimum spanning tree for N. Then an mst-game is defined as follows:

Definition 2.11 *[10, 13] A function $c : 2^N \longrightarrow \mathbb{R}_+$ is said to be the minimum spanning tree game (mst-game for short) for a network G if it satisfies*

$$c(S) = \sum_{(i,j) \in E_{\Gamma_{S_0}}} c_{ij}, \quad \forall\, S \subseteq N,$$

where $c(\emptyset) = 0$.

Table 2.1: The Costs to Construct Power-transmission Line ($\times$ \$ 10,000)

	E P P	City 1	City 2	City 3	City 4
E P P	0	2	6	1	5
City 1	2	0	2	5	4
City 2	6	2	0	5	1
City 3	1	5	5	0	5
City 4	5	4	1	5	0

Table 2.2: The Mst-game c and the Mc-game $\bar{c}$ of Example 1

S	$c(S)$	$\bar{c}(S)$	S	$c(S)$	$\bar{c}(S)$
$\{1\}$	2	2	$\{2,4\}$	6	5
$\{2\}$	6	4	$\{3,4\}$	6	6
$\{3\}$	1	1	$\{1,2,3\}$	5	5
$\{4\}$	5	5	$\{1,2,4\}$	5	5
$\{1,2\}$	4	4	$\{1,3,4\}$	7	6
$\{1,3\}$	3	3	$\{2,3,4\}$	7	6
$\{1,4\}$	6	5	$\{1,2,3,4\}$	6	6
$\{2,3\}$	6	5	$\emptyset$	0	0

Note that the mst-game is uniquely determined for any network. An example of an mst-game is given as follows:

Example 2.1 Suppose that 4 cities which are denoted by 1,2,3,4 are geographically separated and that they have neither an electric power plant nor a power-transmission line. A new electric power plant (EPP) which could provide the electricity to the 4 cities was constructed in the place denoted by 0. The mayors of the cities have decided to cooperate in laying a power-transmission line over the cities. The costs of constructing power-transmission lines are given in Table 2.1. Figure 2.1 illustrates the situations, where 0 denotes the electric power plant and the number assigned to each edge denotes the cost of constructing the link. Then the associated mst-game c is shown in Table 2.2.

At the first glance, one may be considered that any mst-game is monotone

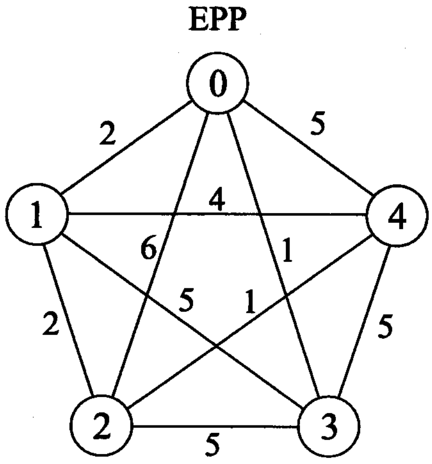

Figure 2.1: The Network corresponding to Table 1

since a minimum spanning tree for a larger coalition consists of more edges than that for a smaller one. However, some mst-games are not monotone. In fact, the mst-game in Example 1 is not monotone since $c(\{1,4\}) = 6 > c(\{1,2,4\}) = 5$ holds. In such a case in a real situation, Cities 1 and 4 will propose to construct a minimum spanning tree not for the coalition $\{1,4\}$ but for the coalition $\{1,2,4\}$ without asking City 2 for a shared cost but for a permission to pass. It can be considered that City 2 will accept the proposal. From this point of view, the mc-game for a network G is defined as follows:

Definition 2.12 *[10, 13] Let c be the mst-game for a network G. A function $\bar{c} : 2^N \longrightarrow \mathbb{R}_+$ is said to be the monotonic cover of a game c or the monotone cover game (mc-game for short) for a network G if it satisfies*

$$\bar{c}(S) = \min_{S \subseteq T \subseteq N} c(T), \quad \forall\, S \subseteq N.$$

It is noted that for a network, i.e., for the mst-game derived from it, the monotonic cover of it is uniquely determined. In this thesis, c and $\bar{c}$ represent the mst-game and the mc-game for a network G, respectively, if there is no fear of confusion. The monotonic cover of the mst-game in Example 1 is also shown in Table 2.2.

From Definition 2.12, it is apparent that $\bar{c}(S) \leq c(S)$ holds for any $S \subseteq N$. Hence, it is apparent that the following remark holds.

Remark 2.1 *If c is the monotonic cover of an mst-game c, then the following holds:*

$$C(\bar{c}) \subseteq C(c).$$

For many networks, some allocations included in the core of the mst-game have negative elements. In fact, the allocation $(2, -1, 1, 4)$ is included in the core of the mst-game in Example 1. On the other hand, if an allocation is an element of the core of an mc-game, then all the elements of the allocation are negative, as is shown in the following lemma.

Lemma 2.1 *[10, 13] If $\bar{c}$ is an mc-game and $\boldsymbol{x} \in C(\bar{c})$, then the following holds:*

$$x_i \geq 0, \quad \forall\, i \in N.$$

Bird [6] proposed a solution concept defined as follows:

Definition 2.13 *[6, 10, 11, 13] Let Γ be a minimum spanning tree in a network G. For $i \in N$, let $p(i)$ denote the adjacent predecessor of i in Γ. A vector $\boldsymbol{l} = (l_i)_{i \in N}$ is said to be the Bird tree allocation for the minimum spanning tree Γ or a Bird tree allocation for the network G if it satisfies*

$$l_i = c_{p(i)i}, \quad \forall\, i \in N.$$

A Bird tree allocation depends on and is uniquely determined by the selected minimum spanning tree. In many networks some distinct minimum spanning trees exist. So, for many of such networks, there exist distinct Bird tree allocations. Any Bird tree allocation is an element of the core of the mst-game, as is shown in the following lemma.

Lemma 2.2 *[10, 13] If $\boldsymbol{l}$ is a Bird tree allocation, then the following holds:*

$$\boldsymbol{l} \in C(c).$$

The following theorem is proved by using Lemma 2.2.

Theorem 2.2 *[10, 13] If* $\boldsymbol{l}$ *is a Bird tree allocation, then the following holds:*

$$\boldsymbol{l} \in C(\bar{c}).$$

Lemma 2.2 and Theorem 2.2 guarantee that both the cores of the mst-game and the mc-game are not empty for any network.

Note that $\bar{c}(\{i\}) = c(\{i\}) = l_i$ for any adjacent follower i of the source in a minimum spanning tree $\Gamma = (N_0, E_\Gamma)$ in a network G, as explained in what follows. If player i is an adjacent follower of the source, $(N, E_\Gamma \backslash \{(0,i)\})$ represents two minimum spanning trees for the disjoint subnetworks of G_{N_0}. The one's edge set is $\{(u,v) \in E_\Gamma \mid u, v \succeq_\Gamma i\}$ and the other's vertex set includes source 0. Since Γ is a minimum spanning tree, it can be confirmed from Kruskal's algorithm [17] that $l_i = c_{0i} = c(\{i\})$ is minimum of all edge costs such that edges connect the two trees. It means that at least $l_i = c(\{i\})$ is necessary to have a connection from the source to vertex i. It follows that $\bar{c}(\{i\}) = c(\{i\}) = l_i$ for any $i \in F(0)$.

Chapter 3

Solution Concepts in Cooperative Fuzzy Games

3.1 Introduction

Cooperative fuzzy games (fuzzy games) are extensions of conventional cooperative games, cooperative crisp games (crisp games). Fuzzy games are based on fuzzy coalitions while crisp games on conventional coalitions, crisp coalitions. Fuzzy coalitions are defined by the rate of each player's participation while crisp coalitions by whether each player participate or not. This means that fuzzy games are more flexible to be applied to the real situations than conventional one.

Although there are many situations which can be described by fuzzy games but which cannot be described by crisp games, there are not so many researches which deal with rational imputations in fuzzy games. Butnariu [7] gave an approach to this problem by defining functions which derived solution concepts from a given pair of a fuzzy game and a fuzzy coalition. However, his definitions are unacceptable on some points, which will be discussed in this chapter.

On the other hand, multi-choice cooperative games [15, 18, 19] and continuously-many-choice cooperative games [16] have been discussed. Those games deal with situations where each player has finite or infinite action levels, which may correspond to rates of his/her participation in coalitions. Several solution concepts for such games are considered [15, 16, 18, 19]. Based on those solution concepts, each player's rational gain in an action level is independent of the other players' action levels, and not all n-vectors of the rational gains satisfy the efficiency of the corresponding (fuzzy) coalition. Hence, it can be considered that these games do not deal with rational distributions of the revenue when a fuzzy coalition is formed.

On the other hand, Sprumont [25] proposed a population monotonic allocation scheme (PMAS) as a reasonable solution concept. It as well as solution concepts by Butnariu meets the need for solution concepts which specify not only how to allocate $v(N)$ but also how to allocate the worth of every coalition $S \subseteq N$. But it was introduced only for crisp games.

In this chapter, we follow Butnariu's approach and discuss rational distributions of the revenue when a fuzzy coalition is formed. To do so, we extend an imputation function, an imputation set-valued function and a PMAS to fuzzy games. Concepts related to the Shapley value are introduced in fuzzy games. We introduce an axiomatic definition of the Shapley function which derives the Shapley value based on those concepts from a pair of a fuzzy game and a fuzzy coalition. Furthermore, we define a core function and a dominance core function as functions which derive the core and the dominance core, respectively. We investigate properties of the core function and the dominance core function and give a necessary and sufficient condition for the core to be nonempty.

3.2 Basic Concepts in Cooperative Fuzzy Games

In this section, we introduce basic concepts closely related to solution concepts in fuzzy games.

First, we extend an imputation to a fuzzy game. Define $e^i \in \mathbb{R}^n$ by

$$e^i(k) = \begin{cases} 1, & \text{if } k = i, \\ 0, & \text{otherwise.} \end{cases}$$

In this thesis, a vector $\boldsymbol{x} \in \mathbb{R}^n$ is called a fuzzy imputation of $U \in L(N)$ for $v \in G(N)$ if it satisfies

1. $x_i = 0, \quad \forall\, i \notin \text{Supp } U$,
2. $\sum_{i \in N} x_i = v(U)$,
3. $x_i \geq v(U(i)e^i), \quad \forall\, i \in \text{Supp } U$,

where $\boldsymbol{x} = (x_i)_{i \in N}$. Let us denote the set of all fuzzy imputations of $U \in L(N)$ for $v \in G(N)$ by $I(v)(U)$.

A fuzzy imputation function and a fuzzy imputation set-valued function based on a fuzzy imputation have defined as follows:

Definition 3.1 *A function x from $L(N)$ to $\mathbb{R}^n_+$ is said to be a fuzzy imputation function of a fuzzy game v if $x(U) \in I(v)(U)$ for any $U \in L(N)$. A function X from $L(N)$ to $2^{\mathbb{R}^n_+}$ is called a fuzzy imputation set-valued function of a fuzzy game v if $x(U) \subseteq I(v)(U)$ for any $U \in L(N)$.*

Butnariu [7] also defined a fuzzy imputation function, which he called a payoff function, but it is different from that of Definition 3.1.

We extend a PMAS in order to treat fuzzy games.

Definition 3.2 *A vector $\boldsymbol{x} = (x_i(U))_{i \in \text{Supp}\, U,\, U \in L(N)}$ is said to be a fuzzy population monotonic allocation scheme (FPMAS) if it satisfies the following:*

$$\begin{cases} \displaystyle\sum_{i \in \text{Supp } U} x_i(U) = v(U), & \forall\, U \in L(N), \\ x_i(S) \leq x_i(T), & \forall\, i \in \text{Supp } S, \quad \forall\, S, T \in L(N) \ s.t. \ S \subseteq T. \end{cases}$$

It is noted that the restriction of an FPMAS to crisp game will be a PMAS, i.e., if the class $L(N)$ is replaced by the class $P(N)$ in the above definition, a PMAS is obtained.

3.3 The Shapley Function

This section is dedicated to give a definition of a Shapley function, which is applicable to any class of fuzzy games. In order to give the definition, we extend a null player and a carrier to fuzzy games. It will be found that they are closely connected with each other and with a Shapley function.

First, we define a γ-null player as an extension of a null player to fuzzy games. There may exist some player who cannot contribute to the coalition value further if the rate of his/her participation exceeds a certain rate γ. We call such a player a γ-null player. Preparatory to its definition, let us define $S_i^U \in L(U)$ by

$$S_i^U(j) = \begin{cases} U(i), & \text{if } j = i, \\ S(j), & \text{otherwise,} \end{cases}$$

for any $U \in L(N)$, $S \in L(U)$ and $i \in N$. Then a γ-null player is defined as follows:

Definition 3.3 *Let γ satisfy $0 \leq \gamma < U(i)$. Player $i \in \mathrm{Supp}\, U$ is said to be a γ-null player in U for $v \in G(N)$ if*

$$v(S) = v(S_i^U), \quad \forall\, S \in L(U), \text{ s.t. } S(i) > \gamma.$$

For monotonous fuzzy games, if player i is a γ-null player in U for v then he/she is a γ'-null player in U for v for any γ' satisfying $\gamma \leq \gamma' < U(i)$.

We also define an f-carrier in a fuzzy coalition for a fuzzy game as an extension of a carrier in a crisp coalition for a crisp game.

Definition 3.4 *Let $v \in G(N)$ and $U \in L(N)$. $S \in L(U)$ is called an f-carrier in U for v if it satisfies*

$$v(S \cap T) = v(T), \quad \forall\, T \in L(U).$$

The set of all f-carriers in U for v is denoted by $FC(U \mid v)$.

From the definition it is clear that U is an f-carrier in U for any $v \in G(N)$. For a monotonous fuzzy game v, it can be found that if $S \in FC(U \mid v)$, then $S' \in FC(U \mid v)$ for any $S' \in L(U)$ such that $S' \supseteq S$.

It is to be noted that an f-carrier is specified not only with a game but also with a coalition whereas a carrier defined by Hsiao [16] as well as a usual carrier in a crisp game only with a game. Especially, f-carriers in a coalition N coincide with carriers defined by Hsiao for any fuzzy game. Thus, f-carriers are extensions of Hsiao's carriers. Butnariu [7] also extended a carrier to fuzzy games. However, an f-carrier can be considered more straightforward than the carrier defined by Butnariu. It is because an f-carrier is based on standard set inclusion whereas the carrier by Butnariu on nonstandard set inclusion.

Preparatory to the definition of a Shapley function, we define the following. Let $U \in L(N)$ and $i, j \in N$. For any $S \in L(U)$, define $S_{ij}^U \in L(U)$ by

$$S_{ij}^U(k) = \begin{cases} \min\{S(i), U(j)\}, & \text{if } \ k = i, \\ \min\{S(j), U(i)\}, & \text{if } \ k = j, \\ S(k), & \text{otherwise.} \end{cases}$$

For any $S \in L(N)$, define $\mathcal{P}_{ij}[S]$ by

$$\mathcal{P}_{ij}[S](k) = \begin{cases} S(j), & \text{if } \ k = i, \\ S(i), & \text{if } \ k = j, \\ S(k), & \text{otherwise.} \end{cases}$$

Clearly, $S_{ij}^U, \mathcal{P}_{ij}[S_{ij}^U] \in L(U)$. Now, we define a Shapley function as follows.

Definition 3.5 *Let $G'(N) \subseteq G(N)$. A function $f : G'(N) \to (\mathbb{R}_+^n)^{L(N)}$ is said to be a Shapley function on $G'(N)$ if it satisfies the following four axioms.*

Axiom F_1: *If $v \in G'(N)$ and $U \in L(N)$ then*

$$\begin{cases} \displaystyle\sum_{i \in N} f_i(v)(U) = v(U), \\ f_i(v)(U) = 0, \quad \forall\, i \notin \operatorname{Supp} U, \end{cases}$$

where $f_i(v)(U)$ is the i-th element of $f(v)(U) \in \mathbb{R}_+^n$.

Axiom F_2: *If $v \in G'(N)$, $U \in L(N)$ and $T \in FC(U \mid v)$ then*

$$f_i(v)(U) = f_i(v)(T), \quad \forall\, i \in N.$$

Axiom F_3: *If $v \in G'(N)$, $U \in L(N)$, $U_{ij}^U \in FC(U \mid v)$ and $v(S) = v(\mathcal{P}_{ij}[S])$ for any $S \in L(U_{ij}^U)$ then*

$$f_i(v)(U) = f_j(v)(U).$$

Axiom F_4: *For any $v_1, v_2 \in G'(N)$, define a game v_1+v_2 by $(v_1+v_2)(S) = v_1(S) + v_2(S)$ for any $S \in L(N)$. If $v_1 + v_2 \in G'(N)$ and $U \in L(N)$ then*

$$f_i(v_1 + v_2)(U) = f_i(v_1)(U) + f_i(v_2)(U), \quad \forall\, i \in N.$$

Note that the definition above can be adopted for any class of fuzzy games.

Butnariu [7] also gave a definition of a Shapley function. One of the axioms of his Shapley function uses an extended carrier defined by him. As is stated above, his definition of an extended carrier is not quite natural, because it is based on a nonstandard inclusion relation between fuzzy sets. This is why we have proposed a new extended carrier. From this point of view, a Shapley function should be defined without using Butnariu's extended carrier. One conceivable approach is to replace Butnariu's extended carrier with an f-carrier. However, by this approach, we encounter an unacceptable result, i.e., a contributory player cannot always receive a profit by forming the coalition U. Hence, we abandoned this approach and gave a new definition of a Shapley function in Definition 3.5.

3.4 The Core and the Dominance Core Function

We introduce a core function as a function which derives the core from a given pair of a fuzzy game and a fuzzy coalition.

Definition 3.6 *Let $G'(N)$ be a subset of $G(N)$. A function C from $G'(N)$*

to $(2^{\mathbb{R}^n_+})^{L(N)}$ *is said to be a core function on* $G'(N)$ *if it satisfies:*

$$C(v)(U) = \left\{ \boldsymbol{x} \in I(v)(U) \;\middle|\; \sum_{i \in \text{Supp } S} x_i \geq v(S), \quad \forall\, S \in L(U) \right\}.$$

Note that $C(v)(U)$ is a convex polyhedron if it is nonempty. It is also noted that $C(v)$ can be regarded as an imputation set-valued function for any $v \in G(N)$.

For $v \in G(N)$ and $U \in L(N)$, define a crisp game $v_*^U \in G_0(N)$ by $v_*^U(T) = \max_{S \in L(U): T = \text{Supp } S, S \neq U} v(S)$.

Remark 3.1 *If* $v \in G(N)$ *and* $U \in L(N)$, *then we have*

$$\begin{aligned} &C(v)(U) \\ &= \left\{ \boldsymbol{x} \in I(v)(U) \;\middle|\; \sum_{i \in \text{Supp } S} x_i \geq v(S), \quad \forall\, S \in L(U) \backslash \{\emptyset, U\} \right\} \\ &= \left\{ \boldsymbol{x} \in I(v)(U) \;\middle|\; \sum_{i \in T} x_i \geq v_*^U(T), \quad \forall\, T \in P(\text{Supp } U) \backslash \{\emptyset\} \right\}. \end{aligned}$$

Let $L_0(U) = \{S \in L(U) \mid S(i) \in \{0, U(i)\}, \quad \forall\, i \in N\}$. For $v \in G(N)$ and $U \in L(N)$, let us define a crisp game $v^U \in G_0(N)$ by $v^U(T) = \max_{S \in L(U): T = \text{Supp } S} v(S)$. If $v \in G(N)$ is monotone nondecreasing, then we have $v^U(\text{Supp } U) = v(U)$ and $v^U(\{i\}) = v(U(i)e^i)$ for any $i \in N$; hence $I'(v^U)(\text{Supp } U) = I(v)(U)$. We obtain the following remark.

Remark 3.2 *If* $v \in G(N)$ *is monotone nondecreasing with respect to rates of players' participation, we have*

$$\begin{aligned} &C(v)(U) \\ &= \left\{ \boldsymbol{x} \in I(v)(U) \;\middle|\; \sum_{i \in \text{Supp } S} x_i \geq v(S), \quad \forall\, S \in L_0(U) \right\} \\ &= \left\{ \boldsymbol{x} \in I'(v^U)(\text{Supp } U) \;\middle|\; \sum_{i \in T} x_i \geq v^U(T), \quad \forall\, T \in P(\text{Supp } U) \right\} \\ &= C'(v^U)(\text{Supp } U). \end{aligned}$$

To introduce a dominance core function, we define the concept of dominance in fuzzy games. We say that a fuzzy imputation $\boldsymbol{x}$ dominates a fuzzy imputation $\boldsymbol{y}$ via a coalition $S \in L(N)$ if

$$\begin{cases} \displaystyle\sum_{i \in \text{Supp } S} x_i \leq v(S), \\ x_i > y_i, \quad \forall\, i \in \text{Supp } S. \end{cases}$$

In such a case, the notation $\boldsymbol{x}$ dom_S $\boldsymbol{y}$ is used.

Note that there exists no $\boldsymbol{x} \in I(v)(U)$ which dominates some fuzzy imputation $\boldsymbol{y} \in I(v)(U)$ via U or $U(i)e^i$ for some $i \in \text{Supp } U$, as in a crisp game. In fact, if $\boldsymbol{x} \in I(v)(U)$, $\boldsymbol{y} \in \mathbb{R}^n$ and there exists $i \in \text{Supp } U$ satisfying $\boldsymbol{x}$ $\text{dom}_{U(i)e^i}$ $\boldsymbol{y}$ then $y_i < x_i \leq v(U(i)e^i)$, which means $\boldsymbol{y} \notin I(v)(U)$. Thus, a dominance via $U(i)e^i$ is not possible for any $i \in N$. If $\boldsymbol{x} \in I(v)(U)$ and $\boldsymbol{x}$ dom_U $\boldsymbol{y}$ then $x_i > y_i$ for any $i \in \text{Supp } U$. Hence, $v(U) = \sum_{i \in \text{Supp } U} x_i > \sum_{i \in \text{Supp } U} y_i$. It means $\boldsymbol{y} \notin I(v)(U)$. A dominance via U is neither possible.

If $\boldsymbol{x}, \boldsymbol{y} \in I(v)(U)$ and there exists $S \in L(U)$ satisfying $\boldsymbol{x}$ dom_S $\boldsymbol{y}$, we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ in U. The dominance core function can be defined.

Definition 3.7 *Let $G'(N)$ be a subset of $G(N)$. A function C from $G'(N)$ to $(2^{\mathbb{R}^n_+})^{L(N)}$ is said to be a dominance core function on $G'(N)$ if it satisfies:*

$$\begin{aligned} &DC(v)(U) \\ &= \{\boldsymbol{x} \in I(v)(U) \mid \textit{there exists} \\ &\qquad\qquad\qquad \textit{no } \boldsymbol{y} \in I(v)(U) \textit{ which dominates } \boldsymbol{x} \textit{ in } U\ \} \\ &= \{\boldsymbol{x} \in I(v)(U) \mid \boldsymbol{y} \text{ dom}_S\ \boldsymbol{x} \textit{ does not hold} \\ &\qquad\qquad\qquad \textit{for any pair } (S, \boldsymbol{y}) \in L(U) \times I(v)(U)\ \}. \end{aligned}$$

Consider a pair $(S, S') \in L_0(U) \times L(U)$ satisfying $\text{Supp } S = \text{Supp } S'$. Then it is easy to confirm that $\boldsymbol{x}$ dom_S $\boldsymbol{y}$ if v is monotone nondecreasing and $\boldsymbol{x}$ $\text{dom}_{S'}$ $\boldsymbol{y}$. Thus, we have the following remark.

Remark 3.3 *If $v \in G(N)$ is monotone nondecreasing with respect to rates of players' participation, we have*

$$\begin{aligned} &DC(v)(U) \\ &= \{\boldsymbol{x} \in I(v)(U) \mid \boldsymbol{y} \text{ dom}_S\ \boldsymbol{x} \textit{ does not hold} \\ &\qquad\qquad\qquad \textit{for any pair } (S, \boldsymbol{y}) \in L_0(U) \times I(v)(U)\ \}. \end{aligned}$$

To show the relationship between the core function and the dominance core function, we prepare the following lemma.

Lemma 3.1 *Let* $U \in L(N)$ *and* $\boldsymbol{x} \in I(v)(U)$. *Let* $S \in L_0(U)$ *satisfy* $S \neq U, U(i)e^i$ *for any* $i \in \text{Supp } U$. *If* $v \in G(N)$ *is superadditive, then there exists* $\boldsymbol{y} \in I(v)(U)$ *which dominates* $\boldsymbol{x}$ *via* S *if and only if* $\sum_{i \in \text{Supp } S} x_i < v(S)$.

Proof. If there exists a fuzzy imputation $\boldsymbol{y} \in I(v)(U)$ satisfying $\boldsymbol{y}$ dom_S $\boldsymbol{x}$, then $v(S) \geq \sum_{i \in \text{Supp } S} y_i > \sum_{i \in \text{Supp } S} x_i$ holds. Hence, it is sufficient to prove that the converse relationship holds. Suppose that $\sum_{i \in \text{Supp } S} x_i < v(S)$. Let $\delta = v(S) - \sum_{i \in \text{Supp } S} x_i$ and $\delta' = v(U) - v(S) - \sum_{i \in \text{Supp } U \backslash \text{Supp } S} v(U(i)e^i)$. From the assumption, $\delta > 0$. For $(U, W) \in L(N) \times P(N)$, define $U \backslash W \in L(U)$ by

$$(U \backslash W)(i) = \begin{cases} U(i), & \forall\, i \notin W, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall\, i \in N.$$

Note that $\cup_{i \in \text{Supp } U \backslash \text{Supp } S} U(i)e^i = U \backslash \text{Supp } S$ and $S \cup (U \backslash \text{Supp } S) = U$. Then, by virtue of superadditivity of v, we have

$$\begin{aligned} \delta' &= v(U) - v(S) - \sum_{i \in \text{Supp } U \backslash \text{Supp } S} v(U(i)e^i) \\ &\geq v(U) - v(S) - v(U \backslash \text{Supp } S) \geq 0. \end{aligned}$$

Define $\boldsymbol{y} \in \mathbb{R}^n$ by

$$y_i = \begin{cases} x_i + \dfrac{\delta}{|\text{Supp } S|}, & \forall\, i \in \text{Supp } S, \\ v(U(i)e^i) + \dfrac{\delta'}{|\text{Supp } U| - |\text{Supp } S|}, & \forall\, i \in \text{Supp } U \backslash \text{Supp } S, \\ 0, & \forall\, i \notin \text{Supp } U, \end{cases}$$

for any $i \in N$. Then it is easy to confirm $\boldsymbol{y} \in I(v)(U)$ and $\boldsymbol{y}$ dom_S $\boldsymbol{x}$. □

Theorem 3.1 *If* $v \in G(N)$ *is superadditive and monotone nondecreasing with respect to rates of players' participation then* $C(v) = DC(v)$.

Proof. Let $U \in L(N)$ and $\boldsymbol{x} \in I(v)(U)$. Let $S \in L_0(U)$ satisfy $S \neq U, U(i)e^i$ for any $i \in \text{Supp } U$. As noted above, no $\boldsymbol{y} \in I(v)(U)$ dominates $\boldsymbol{x} \in I(v)(U)$ via U or $U(i)e^i$ for some $i \in \text{Supp } U$. Thus, by using Remarks 3.2 and 3.3, it is sufficient to prove that $\boldsymbol{y}$ dom$_S$ $\boldsymbol{x}$ does not hold for any $\boldsymbol{y} \in I(v)(U)$ if and only if $\sum_{i \in \text{Supp } S} x_i \geq v(S)$, since v is monotone nondecreasing. From Lemma 3.1, we have the relation above when v is superadditive. The proof is completed. □

Note that $C(v) = DC(v)$ if $v \in G_C(N)$, since any $v \in G_C(N)$ is superadditive and monotone nondecreasing.

We define balancedness in fuzzy games as follows:

Definition 3.8 *A collection $\{\gamma_T\}_{T \in P(\text{Supp } U) \backslash \{\emptyset\}}$ is said to be a balanced collection in U if it satisfies*

$$\begin{cases} \displaystyle\sum_{T \in P(\text{Supp } U) \backslash \{\emptyset\} : T \ni i} \gamma_T = 1, & \forall\, i \in \text{Supp } U, \\ \gamma_T \geq 0, & \forall\, T \in P(\text{Supp } U) \backslash \{\emptyset\}. \end{cases}$$

A fuzzy game $v \in G(N)$ is said to be balanced in U if the following holds:

$$\sum_{T \in P(\text{Supp } U) \backslash \{\emptyset\}} \gamma_T \cdot v_*^U(T) \leq v(U),$$

for any balanced collection $\{\gamma_T\}_{T \in P(\text{Supp } U) \backslash \{\emptyset\}}$ in U.

Then the following theorem holds:

Theorem 3.2 *Let $v \in G(N)$ and $U \in L(N)$. Then the core of U for v, say $C(v)(U)$, is nonempty if and only if v is balanced in U.*

Proof. Consider the following problem:

$$\begin{aligned} (P)\quad &\textit{minimize} \quad \sum_{i \in \text{Supp } U} x_i \\ &\textit{subject to} \quad \sum_{i \in T} x_i \geq v_*^U(T) \quad \forall\, T \in P(\text{Supp } U) \backslash \{\emptyset\} \end{aligned}$$

It is apparent from Remark 3.1 that the core of U for v is nonempty if and only if the optimal value of (P) is not greater than $v(U)$. The dual problem for the problem (P) is described as follows:

$$
\begin{array}{llll}
(D) & \textit{maximize} & \displaystyle\sum_{T \in P(\mathrm{Supp}\ U)\setminus\{\emptyset\}} \gamma_T \cdot v_*^U(T) & \\
 & \textit{subject to} & \displaystyle\sum_{T \in P(\mathrm{Supp}\ U)\setminus\{\emptyset\}:T\ni i} \gamma_T = 1 & \forall\, i \in \mathrm{Supp}\ U \\
 & & \gamma_T \geq 0 & \forall\, T \in P(\mathrm{Supp}\ U)\setminus\{\emptyset\}
\end{array}
$$

It is apparent that the problem (P) has the optimal solutions. From the duality theorem, its dual problem (D) has also the optimal solutions and the optimal values of the two problems coincides with each other. Therefore, the core of U for v is nonempty if and only if $\sum_{T\in P(\mathrm{Supp}\ U)\setminus\{\emptyset\}} \gamma_T \cdot v_*^U(T) \leq v(U)$ for any $\{\gamma_T\}_{T\in P(\mathrm{Supp}\ U)\setminus\{\emptyset\}}$ satisfying the constraints of the problem (D), i.e., for any balanced collection in U. □

The following remark shows a relationship between the cores for two strategically equivalent games in $G_C(N)$.

Remark 3.4 *Let $x'_i = c \cdot x_i + U(i) \cdot a_i$ for any $i \in N$. If $v, v' \in G_C(N)$ satisfy $v'(S) = c \cdot v(S) + \sum_{i\in N} S(i) \cdot a_i$ for any $S \in L(U)$ where $c > 0$, then $\boldsymbol{x} \in C(v)(U)$ if and only if $\boldsymbol{x}' \in C(v')(U)$.*

3.5 Concluding Remarks

In this chapter, we have defined some basic concepts related to solution concepts, i.e., a fuzzy imputation function, a fuzzy imputation set-valued function and an FPMAS, which are applicable to any fuzzy games. We have given definitions of an f-carrier and a γ-null player in fuzzy games to introduce an axiomatic definition of the Shapley function. The core function and the dominance core function have been defined. We have shown that those two functions coincide with each other if a given fuzzy game is superadditive and monotone nondecreasing with respect to rates of players' participation.

Balancedness in fuzzy games has also been introduced. The core and the dominance core are not always nonempty. It has been shown that the core is nonempty if and only if a given fuzzy game is balanced. Strategical equivalence on the core function is discussed.

Chapter 4

Solution Concepts in a Class of Cooperative Fuzzy Games

4.1 Introduction

In Chapter 3, we have given an axiomatic definition of the Shapley function, which is applicable to any class of fuzzy games. However, it is not easy to give an explicit form of the Shapley function on the class of all fuzzy games. Butnariu [7] introduced a limited class of fuzzy games in order to obtain an explicit form of the Shapley function defined by him. However, the class cannot be quite natural since most games in the class are neither continuous nor monotone nondecreasing with respect to rates of players' participation.

In this chapter, in order to give an explicit form of the function, we shall define a new class of fuzzy games, which is considered natural in terms of monotonicity and continuity. We investigate properties of concepts closely related to the Shapley function in the class of fuzzy games. An explicit form of the Shapley function is given and its rational properties are discussed.

4.2 A Class of Cooperative Fuzzy Games

We define the following class of fuzzy games:

Definition 4.1 *Given $S \in L(N)$, let $Q(S) = \{S(i) \mid S(i) > 0,\ i \in N\}$ and let $q(S)$ be the cardinality of $Q(S)$. We write the elements of $Q(S)$ in the* *increasing order as* $h_1 < \cdots < h_{q(S)}$. *Then a game* $v \in G(N)$ *is said to be a fuzzy game 'with Choquet integral form' if and only if the following holds:*

$$v(S) = \sum_{l=1}^{q(S)} v([S]_{h_l}) \cdot (h_l - h_{l-1}), \tag{4.1}$$

for any $S \in L(N)$ *where* $h_0 = 0$. *We denote by* $G_C(N)$ *the set of all fuzzy games with Choquet integral forms.*

It is apparent that (4.1) is a Choquet integral [8, 12, 26] of the function S with regard to v. There is a one-to-one correspondence between a crisp game and a fuzzy game with Choquet integral form. We call the crisp game corresponding to a fuzzy game with Choquet integral form the associated crisp game, and the fuzzy game with Choquet integral form corresponding to a crisp game the associated fuzzy game. For the sake of simplicity, we will denote the crisp game associated with $v \in G_C(N)$ by $v \in G_0(N)$ if there is no fear of confusion.

Remark 4.1 *Let* $v \in G_C(N)$ *and* $S \in L(N)$. *Consider a set* $\{k_1, \ldots, k_m\} \supseteq Q(S)$ *such that* $0 \leq k_1 < \cdots < k_m \leq 1$. *Then the following holds:*

$$v(S) = \sum_{l=1}^{m} v([S]_{k_l}) \cdot (k_l - k_{l-1}), \quad \forall\, S \in L(N),$$

where $k_0 = 0$.

From the definition of $G_C(N)$, we obtain the following relations. It is clear that $v \in G_C(N)$ is superadditive since the associated game $v \in G_0(N)$ is superadditive. Any $v \in G_C(N)$ is monotone nondecreasing with respect to rates of players' participation since the associated game $v \in G_0(N)$ is monotone nondecreasing, which is shown in the next lemma. It is also apparent that $v \in G_C(N)$ is convex if and only if the associated game $v \in G_0(N)$ is convex.

Lemma 4.1 *Let* $v \in G_C(N)$. *Then the following holds:*

$$v(S) \leq v(T), \quad \forall\, S, T \in L(N), \text{ s.t. } S \subseteq T.$$

In other words, any $v \in G_C(N)$ is monotone nondecreasing with respect to each player's grade of membership.

Proof. Note that $S \subseteq T$ if and only if $[S]_h \subseteq [T]_h$ for any $h \in (0,1]$. Hence, the lemma is apparent from the definition of $v \in G_C(N)$ since any game $v \in G_0(N)$ is monotone nondecreasing with respect to set inclusion. □

The following theorem is obtained.

Theorem 4.1 *Define the distance d in $L(N)$ by $d(S,T) = \max_{i \in N} |S(i) - T(i)|$ for any $S, T \in L(N)$. Any $v \in G_C(N)$ is continuous.*

Proof. Let $S \in L(N)$. Let δ be sufficiently small and let $T \in L(N)$ satisfy $\max_{i \in N} |S(i) - T(i)| < \delta$. We shall prove that $v(T) \to v(S)$ if $\delta \to 0$. Let $Q'(S) = \{S(i) \mid i \in N\} = \{k_1, \dots, k_m, \dots, k_{q'(S)}\}$, where $q'(S)$ denotes the cardinality of $Q'(S)$. Let $S^m = \{i \in N \mid S(i) = k_m\}$. We write the elements of $T[m] = \{T(i) \mid i \in S^m\}$ in the increasing order as $k_1^m < \cdots < k_p^m < \cdots < k_{t[m]}^m$, where $t[m]$ denotes the cardinality of $T[m]$. We can assume that $k_{t[m-1]}^{m-1} \leq k_1^m$ holds for any $m \geq 2$ when δ is sufficiently small. Thus,

$$v(T) = \sum_{m=1}^{q'(S)} \sum_{p=1}^{t[m]} v([T]_{k_p^m}) \cdot (k_p^m - k_{p-1}^m),$$

where $k_0^1 = 0$ and $k_0^m = k_{t[m-1]}^{m-1}$ $(m \geq 2)$. Note that $v([T]_{k_1^m}) = v([S]_{k_m})$ for any m since δ is sufficiently small. Hence, for any m, we have

$$\begin{aligned} & \sum_{p=1}^{t[m]} v([T]_{k_p^m}) \cdot (k_p^m - k_{p-1}^m) \\ = \quad & v([S]_{k_m}) \cdot (k_1^m - k_{t[m-1]}^{m-1}) + \sum_{p=2}^{t[m]} v([T]_{k_p^m}) \cdot (k_p^m - k_{p-1}^m), \end{aligned}$$

where $k_{t[0]}^0 = 0$. Note that $k_m - k_{m-1} - 2\delta < k_1^m - k_{t[m-1]}^{m-1} < k_m - k_{m-1} + 2\delta$ and $0 < k_p^m - k_{p-1}^m < 2\delta$ $(p \geq 2)$ hold for any $m \geq 2$ because of the definition of T. Thus, the following holds:

$$\begin{aligned}
& v([S]_{k_m}) \cdot (k_m - k_{m-1} - 2\delta) \\
< & \sum_{p=1}^{t[m]} v([T]_{k_p^m}) \cdot (k_p^m - k_{p-1}^m) \\
< & v([S]_{k_m}) \cdot (k_m - k_{m-1} + 2\delta) + \sum_{p=2}^{t[m]} v([T]_{k_p^m}) \cdot 2\delta.
\end{aligned}$$

Note that $v(W) < \infty$ holds for any $W \in P(N)$. Hence, if $\delta \to 0$ then $\sum_{p=1}^{t[m]} v([T]_{k_p^m}) \cdot (k_p^m - k_{p-1}^m) \to v([S]_{k_m}) \cdot (k_m - k_{m-1})$. It follows that if $\delta \to 0$ then $v(T) \to \sum_{k=1}^{q'(S)} v([S]_{k_m}) \cdot (k_m - k_{m-1}) = v(S)$. The proof is completed.

□

From Lemma 4.1 and Theorem 4.1, $G_C(N)$ can be considered more natural than Butnariu's class in the sense of monotonicity and continuity.

Let us show next lemma preliminary to the following section.

Lemma 4.2 *Let $v \in G_C(N)$ and $S, T \in L(N)$ such that $S \subseteq T$. Then $v(S) = v(T)$ if and only if $v([S]_h) = v([T]_h)$ for any $h \in (0, 1]$.*

Proof. It is apparent that $v(S) = v(T)$ if $v([S]_h) = v([T]_h)$ for any $h \in (0, 1]$. We shall show the reverse relationship. Let $Q(S) = \{S(i) \mid S(i) > 0,\ i \in N\}$, $Q(T) = \{T(i) \mid T(i) > 0,\ i \in N\}$ and $Q(S,T) = Q(S) \cup Q(T) = \{h_1, \ldots, h_{q(S,T)}\}$, where $h_1 < \cdots < h_{q(S,T)}$ and $q(S,T) = |Q(S,T)|$. From Remark 4.1, we have

$$v(T) - v(S) \;=\; \sum_{l=1}^{q(S,T)} \{v([T]_{h_l}) - v([S]_{h_l})\} \cdot (h_l - h_{l-1}).$$

The associated game $v \in G_0(N)$ is nondecreasing with respect to set inclusion, and $S \subseteq T$ if and only if $[S]_h \subseteq [T]_h$ for any $h \in [0, 1]$; hence $v([T]_h) \geq v([S]_h)$ holds for any $h \in [0, 1]$. Thus, if $v(S) = v(T)$, $v([T]_{h_l}) - v([S]_{h_l}) = 0$ holds for any $h_l \in Q(S,T)$. Note that $[S]_h = [S]_{h_l}$ and $[T]_h = [T]_{h_l}$ holds for any h satisfying $h_{l-1} < h \leq h_l$; hence, we have $v([T]_h) = v([S]_h)$ for any $h \in (0, h_{q(S,T)}]$. For any h satisfying $h > h_{q(S,T)}$, $[T]_h = [S]_h = \emptyset$;

Table 4.1: An example of crisp games

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	120	$\{1,2\}$	450	$\{2,4\}$	480	$\{1,3,4\}$	900
$\{2\}$	150	$\{1,3\}$	480	$\{3,4\}$	600	$\{2,3,4\}$	900
$\{3\}$	180	$\{1,4\}$	480	$\{1,2,3\}$	840	$\{1,2,3,4\}$	1500
$\{4\}$	180	$\{2,3\}$	480	$\{1,2,4\}$	840		

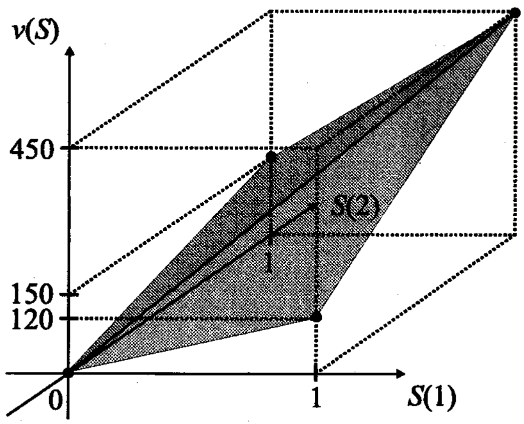

Figure 4.1: $v \in G_C(N)$ in Example 4.1

hence $v([T]_h) = v([S]_h) = v(\emptyset)$ holds. Accordingly, if $v(S) = v(T)$ then $v([S]_h) = v([T]_h)$ for any $h \in (0,1]$. This completes the proof. □

Some properties of $G_C(N)$ are illustrated with the following examples.

Example 4.1 Let $N = \{1,2\}$. For any $S \in L(N)$, if $S(1) \le S(2)$ then $v(S) = v(\{1,2\}) \cdot S(1) + v(\{2\}) \cdot \{S(2) - S(1)\}$; otherwise, $v(S) = v(\{1,2\}) \cdot S(2) + v(\{1\}) \cdot \{S(1) - S(2)\}$. Suppose that $v \in G_0(N)$ is given by Table 4.1. Then we have the associated fuzzy game $v \in G_C(N)$ explicitly as follows. If $S(1) \le S(2)$ then $v(S) = 300 \cdot S(1) + 150 \cdot S(2)$. Otherwise, $v(S) = 120 \cdot S(1) + 330 \cdot S(2)$. It is shown in Figure 4.1.

Example 4.2 Let $N = \{1,2,3,4\}$. Let $S \in L(N)$ satisfy $S(2) = 0.4$ and $S(3) = S(4) = 0.7$. We have the associated fuzzy game $v \in G_C(N)$ explicitly as follows. If $S(1) \le 0.4$ then $v(S) = v(\{1,2,3,4\}) \cdot S(1) + v(\{2,3,4\}) \cdot \{0.4 - S(1)\} + v(\{3,4\}) \cdot (0.7 - 0.4)$. If $0.4 \le S(1) \le 0.7$ then $v(S) = v(\{1,2,3,4\}) \cdot$

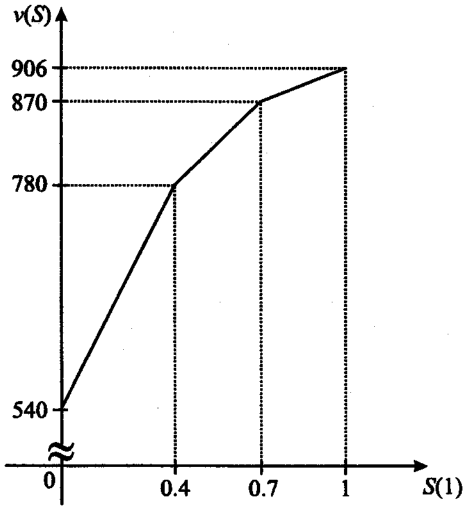

Figure 4.2: $v \in G_C(N)$ in Example 4.2

$0.4+v(\{1,3,4\})\cdot\{S(1)-0.4\}+v(\{3,4\})\cdot\{0.7-S(1)\}$. If $0.7 \leq S(1) \leq 1$ then $v(S) = v(\{1,2,3,4\}) \cdot 0.4 + v(\{1,3,4\}) \cdot (0.7 - 0.4) + v(\{1\}) \cdot \{S(1) - 0.7\}$. Here, suppose that $v \in G_0(N)$ is given by Table 4.1. If $S(1) \leq 0.4$ then $v(S) = 600 \cdot S(1) + 540$. If $0.4 \leq S(1) \leq 0.7$ then $v(S) = 300 \cdot S(1) + 660$. If $0.7 \leq S(1) \leq 1$ then $v(S) = 120 \cdot S(1) + 786$. A gradual variation of the coalition values with that of the rate of the first player's participation is shown in Figure 4.2.

4.3 Concepts Related to the Shapley Function in the Class of Cooperative Fuzzy Games

In this section, we discuss properties of a γ-null player and an f-carrier in the proposed class.

As is shown in Lemma 4.1, any $v \in G_C(N)$ is monotone nondecreasing with respect to rates of players' participation. Hence, if player i is a γ-null player in U for $v \in G_C(N)$ then he is a γ'-null player in U for v for any γ' satisfying $\gamma \leq \gamma' < U(i)$.

The following theorem shows a relationship between a γ-null player in U for $v \in G_C(N)$ and a null player in $[U]_h$ for the associated crisp game $v \in G_0(N)$.

Theorem 4.2 *Let $v \in G_C(N)$ and $U \in L(N)$. Player i is a γ-null player in U for v if and only if he is a null player in $[U]_h$ for the associated $v \in G_0(N)$ for any h satisfying $\gamma < h \leq U(i)$.*

Proof. We define $L[\gamma](U) = \{[S]_h \mid S \in L(U),\ 0 \leq \gamma < S(i) < h \leq U(i)\}$ and $P[\gamma](U) = \{T \in P([U]_h\backslash\{i\}) \mid 0 \leq \gamma < h \leq U(i)\}$. Let us show $L[\gamma](U) = P[\gamma](U)$ holds. If $[S]_h \in L[\gamma](U)$ is given, then the corresponding $T \in P[\gamma](U)$ is immediately obtained by setting $T = [S]_h$. It means that $L[\gamma](U) \subseteq P[\gamma](U)$. We shall show that $L[\gamma](U) \supseteq P[\gamma](U)$. Let $T \in P[\gamma](U)$, i.e., let $T \in P([U]_h\backslash\{i\})$ for some h satisfying $0 \leq \gamma < h \leq U(i)$. Let S be defined by

$$S(j) = \begin{cases} U(j), & \text{if } j \in T \in P([U]_h\backslash\{i\}), \\ t, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

where $\gamma < t < h$. Then we have $S \in L(U)$, $0 \leq \gamma < S(i) < h \leq U(i)$ and

$$[S]_h = \{j \in N \mid j \in T \in P([U]_h\backslash\{i\})\} = T.$$

Hence, $T \in L[\gamma](U)$, which means that $L[\gamma](U) \supseteq P[\gamma](U)$.

Since $S \subseteq S_i^U$, by using Lemma 4.2 we obtain

$$\begin{array}{lll} v(S) = v(S_i^U), & & \\ \iff & v([S]_h) = v([S_i^U]_h), & \forall\, h \in (0,1], \\ \iff & v([S]_h) = v([S]_h \cup \{i\}), & S(i) < \forall h \leq U(i). \end{array}$$

Hence, using $L[\gamma](U) = P[\gamma](U)$, we have

$$\begin{array}{lll} v(S) = v(S_i^U), & \forall\, S \in L(U), \text{ s.t. } S(i) > \gamma \geq 0, & \\ \iff & [\ v([S]_h) = v([S]_h \cup \{i\}),\ \ S(i) < \forall h \leq U(i)\ ], & \\ & & \forall\, S \in L(U), \text{ s.t. } S(i) > \gamma \geq 0, \\ \iff & v([S]_h) = v([S]_h \cup \{i\}), & \forall\, [S]_h \in L[\gamma](U) \\ \iff & v(T) = v(T \cup \{i\}), & \forall\, T \in P[\gamma](U) \\ \iff & [\ v(T) = v(T \cup \{i\}), & \forall T \in P([U]_h\backslash\{i\})\ ], \\ & & 0 \leq \gamma < \forall h \leq U(i). \end{array}$$

Therefore, player i is a γ-null player in U for v if and only if he/she is a null player in $[U]_h$ for the associated $v \in G_0(N)$ for any h satisfying $\gamma < h \leq U(i)$.

□

Next, we shall discuss an f-carrier for $v \in G_C(N)$. Since any $v \in G_C(N)$ is monotone nondecreasing, it can be found that if $v \in G_C(N)$ and $S \in FC(U \mid v)$, then $S' \in FC(U \mid v)$ for any $S' \in L(U)$ such that $S' \supseteq S$.

We have relationships between an f-carrier in U for $v \in G_C(N)$ and a carrier in $[U]_h$ for the associated crisp game $v \in G_0(N)$, as shown in the following theorems.

Theorem 4.3 *Let $v \in G_C(N)$ and $U \in L(N)$. If S is an f-carrier in U for v then $[S]_h$ is a carrier in $[U]_h$ for the associated crisp game $v \in G_0(N)$ for any $h \in (0,1]$.*

Proof. The following always holds:

$$\{[T]_h \mid T \in L(U)\} = P([U]_h), \quad \forall\, h \in (0,1].$$

By using Lemma 4.2 and the above relationship,

$$\begin{array}{llll} & S \in FC(U \mid v), & & \\ \Longleftrightarrow & v(S \cap T) = v(T), & \forall\, T \in L(U), & \\ \Longleftrightarrow & v([S \cap T]_h) = v([T]_h), & \forall\, h \in (0,1], & \forall\, T \in L(U), \\ \Longleftrightarrow & v([S]_h \cap [T]_h) = v([T]_h), & \forall\, T \in L(U), & \forall\, h \in (0,1], \\ \Longleftrightarrow & [\ v([S]_h \cap R) = v(R), & \forall R \in P([U]_h)\ ], & \forall\, h \in (0,1], \\ \Longleftrightarrow & [S]_h \in C([U]_h \mid v), & \forall\, h \in (0,1]. & \end{array}$$

Hence, if $S \in FC(U \mid v)$, then $[S]_h \in C([U]_h \mid v)$ for any $h \in (0,1]$. □

In general, the following lemma holds.

Lemma 4.3 *[21, Lemma 3.1] Let $U \in L(N)$. Let a set-valued function τ : $[0,1] \to P(N)$ satisfy the following properties:*

Property 1. $\begin{cases} \textit{For any } h \in (0,1],\ \tau[h] \subseteq [U]_h, \\ \tau[0] = N. \end{cases}$

Property 2. *For any pair (h_1, h_2) satisfying $0 \leq h_1 < h_2 \leq 1$, $\tau[h_1] \supseteq \tau[h_2]$.*

Property 3. *For any* $h^* \in (0,1]$, $\bigcap_{h<h^*} \tau[h] = \tau[h^*]$.

Then S *defined by* $S(i) = \sup\{h \in [0,1] \mid i \in \tau[h]\}$ *is an element of* $L(U)$ *and satisfies* $[S]_h = \tau[h]$ *for any* $h \in [0,1]$.

Theorem 4.4 *Let* $v \in G_0(N)$ *and* $U \in L(N)$. *Let* τ *be a set-valued function satisfying Properties 1 - 3 in Lemma 4.3. If* $\tau[h]$ *is a carrier in* $[U]_h$ *for* v *for any* $h \in (0,1]$, *then* S *defined by* $S(i) = \sup\{h \in [0,1] \mid i \in \tau[h]\}$ *is an* f*-carrier in* U *for the associated fuzzy game* $v \in G_C(N)$ *and satisfies* $[S]_h = \tau[h]$ *for any* $h \in (0,1]$.

Proof. From Lemma 4.3, S defined by $S(i) = \sup\{h \in [0,1] \mid i \in \tau[h]\}$ is an element of $L(U)$ and satisfies $[S]_h = \tau[h]$ for any $h \in (0,1]$. If $\tau[h]$ is a carrier in $[U]_h$ for v for any $h \in (0,1]$ then we can adopt the relation proved in Theorem 4.3, namely

$$\begin{array}{lll}
\tau[h] \in C([U]_h \mid v), \quad \forall\, h \in (0,1], & & \\
\iff \quad [\, v(\tau[h] \cap R) = v(R), \ \ \forall R \in P([U]_h)\ ], & & \forall\, h \in (0,1], \\
\iff \quad [\, v([S]_h \cap R) = v(R), \ \ \forall R \in P([U]_h)\ ], & & \forall\, h \in (0,1], \\
\iff \quad v(S \cap T) = v(T), & \forall\, T \in L(U), & \\
\iff \quad S \in FC(U \mid v). & &
\end{array}$$

Thus, the proof is completed. □

Preparatory to the proof of the existence of the smallest f-carrier, we show the following remark and lemmas.

Remark 4.2 *Let* $U \in L(N)$ *and* $h^* \in (0,1]$. *Then* $\bigcap_{h<h^*} [U]_h = [U]_{h^*}$.

Next lemma is obtained from the definition.

Lemma 4.4 *Let* $v \in G_0(N)$, $U \in L(N)$ *and* $0 \le h_1 \le h_2 \le 1$. *If* $S \in C([U]_{h_1} \mid v)$ *then* $S \cap [U]_{h_2} \in C([U]_{h_2} \mid v)$.

Proof. Since $h_1 \le h_2$ implies $[U]_{h_1} \supseteq [U]_{h_2}$, we obtain $\{R \mid R = T \cap$

$[U]_{h_2}, T \in P([U]_{h_1})\} = P([U]_{h_2})$. Hence, the following holds:

$$\begin{array}{lll} S \in C([U]_{h_1} \mid v), & & \\ \iff & v(S \cap T) = v(T), & \forall\, T \in P([U]_{h_1}), \\ \implies & v(S \cap (T \cap [U]_{h_2})) = v(T \cap [U]_{h_2}), & \forall\, T \in P([U]_{h_1}), \\ \iff & v(S \cap R) = v(R), & \forall\, R \in P([U]_{h_2}), \\ \iff & v(S \cap ([U]_{h_2} \cap R)) = v(R), & \forall\, R \in P([U]_{h_2}), \\ \iff & v((S \cap [U]_{h_2}) \cap R) = v(R), & \forall\, R \in P([U]_{h_2}). \end{array}$$

Obviously, $S \cap [U]_{h_2} \in P([U]_{h_2})$. This completes the proof. □

Lemma 4.5 *Let $v \in G_0(N)$ and $W \in P(N)$. Then $\bigcap_{S \in C(W|v)} S$ is a carrier in W for v.*

Proof. Suppose that $S, T \in C(W \mid v)$. Then the following relationships hold:

$$\begin{array}{llll} S \in C(W \mid v) & \iff & v(S \cap R) = v(R), & \forall\, R \in P(W), \\ & \implies & v(S \cap (T \cap R)) = v(T \cap R), & \forall\, R \in P(W). \end{array}$$

$$T \in C(W \mid v) \iff v(T \cap R) = v(R), \quad \forall\, R \in P(W).$$

Consequently, if $S, T \in C(W \mid v)$, we obtain

$$v((S \cap T) \cap R) = v(R), \quad \forall\, R \in P(W).$$

It implies $S \cap T \in C(W \mid v)$. Since $|C(W \mid v)|$ is finite for any $v \in G_0(N)$, $\bigcap_{S \in C(W|v)} S$ is the smallest carrier in W for v. □

Lemma 4.5 implies that there exists the smallest (possibly empty) carrier for any crisp game. Now we are ready to prove the existence of the smallest f-carrier.

Theorem 4.5 *Let $v \in G_C(N)$ and $U \in L(N)$. S defined by*

$$S(i) = \sup\left\{ h \in [0,1] \;\middle|\; i \in \bigcap_{R \in C([U]_h|v)} R \right\}$$

is the smallest f-carrier in U for v.

Proof. Let $\tau[h] = \cap_{R \in C([U]_h|v)} R$ for any $h \in (0,1]$ and $\tau[0] = N$. It is apparent that $\tau[h]$ is the smallest carrier in $[U]_h$ for v from Lemma 4.5. We shall show that the set-valued function τ satisfies Properties 1 - 3 in Lemma 4.3 so that we ensure the existence of $T \in L(U)$ such that $[T]_h = \tau[h]$ for any $h \in [0,1]$.

It is apparent that, for any $h \in (0,1]$, $\tau[h] \subseteq [U]_h$ since $\tau[h]$ is a carrier in $[U]_h$ for v. Hence, Property 1 has been proved.

From Lemma 4.4, for any pair (h_1, h_2) satisfying $0 \le h_1 \le h_2 \le 1$, if $R \in C([U]_{h_1} \mid v)$ then $R' = R \cap [U]_{h_2} \in C([U]_{h_2} \mid v)$. Note that $P([U]_{h_2}) = \{R' \mid R' = R \cap [U]_{h_2}, \text{ for } R \in P([U]_{h_1})\}$ since $[U]_{h_1} \supseteq [U]_{h_2}$. We have $\tau[h_1] = \bigcap_{R \in C([U]_{h_1}|v)} R \supseteq \bigcap_{R \in C([U]_{h_1}|v)} (R \cap [U]_{h_2}) \supseteq \bigcap_{R' \in C([U]_{h_2}|v)} R' = \tau[h_2]$. Property 2 has been proved.

Let $Q(U) = \{U(i) \mid U(i) > 0\} = \{h_1, \ldots, h_{q(U)}\}$, $h_0 = 0$ and $h_{q(U)+1} = 1$ where $h_0 < h_1 < \cdots < h_{q(U)} \le h_{q(U)+1}$. For any $h^* \in (0,1]$, there exists $l \in \{1, \ldots, q(U)+1\}$ such that $h_{l-1} < h^* \le h_l$. Then $[U]_{h^*} = [U]_{h_l}$ holds and there exists $h' > 0$ such that $h_{l-1} < h' < h^* \le h_l$. We have $[U]_{h'} = [U]_{h^*} = [U]_{h_l}$. Thus, from the definition, $C([U]_{h'} \mid v) = C([U]_{h^*} \mid v) = C([U]_{h_l} \mid v)$ holds, from which we derive that $\tau[h'] = \tau[h^*] = \tau[h_l]$. From the above discussion, for any $h^* \in (0,1]$, there exists h' such that $0 < h' < h^*$ and $\tau[h'] = \tau[h^*] = \tau[h_l]$. We have $\tau[h] \supseteq \tau[h^*]$ for any $h \in (0, h^*)$ from Property 2 proved above. Consequently we obtain $\cap_{h<h^*} \tau[h] = \tau[h^*]$. Hence, Property 3 has been proved.

Therefore the set-valued function τ satisfies Properties 1 - 3 in Lemma 4.3; hence $[S]_h = \tau[h]$ holds for any $h \in [0,1]$ from Lemma 4.3, and S is an f-carrier in U for v from Theorem 4.4. Moreover, Theorem 4.3 and definition of $\tau[h]$ imply that if T is an f-carrier in U then $[T]_h \supseteq \tau[h] = [S]_h$ for any $h \in (0,1]$, i.e., $T \supseteq S$. Hence, S is the smallest f-carrier in U for v. □

We can show that the smallest f-carrier in U for $v \in G_C(N)$ is nonempty under a natural condition.

Corollary 4.1 *Let $v \in G_C(N)$ and $U \in L(N)$. The smallest f-carrier in U*

for v is nonempty if and only if $v(\mathrm{Supp}\ U) > 0$.

Proof. From Theorem 4.5 there exists the smallest f-carrier in U for v but it is possibly empty. It is sufficient to show that $\emptyset$ is not an f-carrier in U for v if and only if $v(\mathrm{Supp}\ U) > 0$. By using Lemma 4.1, we obtain

$$\begin{aligned} & \emptyset \in FC(U \mid v), \\ \Longleftrightarrow\ & v(T) = v(\emptyset), \qquad \forall\, T \in L(U), \\ \Longleftrightarrow\ & [\ v(R) = v(\emptyset), \qquad \forall\, R \in P([U]_h)\ ], \qquad \forall\, h \in (0,1], \\ \Longleftrightarrow\ & v(\mathrm{Supp}\ U) = v(\emptyset) = 0. \end{aligned}$$

Hence, the emptyset $\emptyset$ is not an f-carrier in U for $v \in G_C(N)$ if and only if $v(\mathrm{Supp}\ U) > 0$. □

We showed a close relationship between an f-carrier and carriers as well as that between a γ-null player and null players. Through those results and the relationship between null players and carriers in a crisp game, we have the following remark.

Remark 4.3 *Let $v \in G_C(N)$ and $U \in L(N)$. Suppose that there exists the smallest f-carrier R in U for v satisfying $R \subsetneq U$ and that player i is a γ-null player in U for v for some γ. Then we have $R(i) \leq \gamma$.*

4.4 Properties of the Shapley Function and its Relationship to the Core Function

In this section, we discuss properties of an explicit Shapley function and a relationship between the explicit Shapley function and the core function in the proposed class of fuzzy games.

Define a function $f : G_C(N) \to (\mathbb{R}^n_+)^{L(N)}$ by

$$f_i(v)(U) = \sum_{l=1}^{q(U)} f'_i(v)([U]_{h_l}) \cdot (h_l - h_{l-1}), \tag{4.2}$$

where f' is the function given in Theorem 2.1. Note that (4.2) is a Choquet integral of the function U with regard to $f'_i(v)$. Now we show that f is a Shapley function on $G_C(N)$.

Theorem 4.6 *The function defined by (4.2) is a Shapley function on $G_C(N)$.*

Proof. We shall prove that the function f defined by (4.2) satisfies the Axioms F_1 - F_4.

Axiom F_1: Let $v \in G_C(N)$ and $U \in L(N)$. Since $\sum_{i \in N} f_i'(v)([U]_{h_l}) = v([U]_{h_l})$ holds for any $l \in \{1, \dots, q(U)\}$ from Axiom C_1 in Definition 2.9, we obtain

$$
\begin{aligned}
\sum_{i \in N} f_i(v)(U) &= \sum_{l=1}^{q(U)} \sum_{i \in N} f_i'(v)([U]_{h_l}) \cdot (h_l - h_{l-1}) \\
&= \sum_{l=1}^{q(U)} v([U]_{h_l}) \cdot (h_l - h_{l-1}) = v(U).
\end{aligned}
$$

Next, consider an arbitrary $i \notin \text{Supp } U$. Then $i \notin [U]_{h_l}$ holds for any $l \in \{1, \dots, q(U)\}$. We have $f_i'(v)([U]_{h_l}) = 0$ from Axiom C_1. Thus, $f_i(v)(U) = \sum_{l=1}^{q(U)} f_i'(v)([U]_{h_l}) \cdot (h_l - h_{l-1}) = 0$.

Axiom F_2: Let $v \in G_C(N)$, $U \in L(N)$ and $T \in FC(U \mid v)$. From Theorem 4.3, $T \in FC(U \mid v)$ implies $[T]_h \in C([U]_h \mid v)$ for any $h \in (0, 1]$. By Axiom C_2 in Definition 2.9, $f_i'(v)([U]_h) = f_i'(v)([T]_h)$ for any $h \in (0, 1]$. Hence, we obtain $f_i(v)(U) = f_i(v)(T)$.

Axiom F_3: Let $v \in G_C(N)$ and $U \in L(N)$. Note that $U_{ij}^U(i) = U_{ij}^U(j)$. If $U_{ij}^U(i) = U_{ij}^U(j) = 0$, then $f_i(v)(U_{ij}^U) = f_j(v)(U_{ij}^U) = 0$ from Axiom F_1 proved above. We shall discuss the case where $U_{ij}^U(i) = U_{ij}^U(j) > 0$, i.e.,

$U(i), U(j) > 0$. In this case, the following is valid.

$$
\begin{aligned}
& v(S) - v(\mathcal{P}_{ij}[S]) = 0, \quad \forall\, S \in L(U_{ij}^U), \\
\Longrightarrow\quad & v(S) - v(\mathcal{P}_{ij}[S]) = 0, \\
& \quad \forall\, S \in L(U_{ij}^U), \text{ s.t. } S(j) = 0 \text{ and } S(k) \in \{S(i), 0\}, \\
& \quad \forall\, k \in \operatorname{Supp} U, \\
\Longleftrightarrow\quad & \Big[\, v(S) - v(\mathcal{P}_{ij}[S]) = 0, \\
& \quad \forall\, S \in L(U_{ij}^U), \\
& \qquad \text{s.t. } \; S(i) = h, S(j) = 0 \\
& \qquad\qquad \text{and } S(k) \in \{h, 0\}, \forall k \in \operatorname{Supp} U \,\Big], \\
& \quad \forall\, h \in (0, U_{ij}^U(i)], \\
\Longleftrightarrow\quad & \Big[\, \{v([S']_h \cup \{i\}) - v([S']_h \cup \{j\})\} \cdot h = 0, \\
& \quad \forall\, S' \in L(U_{ij}^U), \\
& \qquad \text{s.t. } \; S'(i) = S'(j) = 0 \\
& \qquad\qquad \text{and } S'(k) \in \{h, 0\}, \forall k \in \operatorname{Supp} U \,\Big], \\
& \quad \forall\, h \in (0, U_{ij}^U(i)], \\
\Longleftrightarrow\quad & \Big[\, v([S']_h \cup \{i\}) - v([S']_h \cup \{j\}) = 0, \\
& \quad \forall\, S' \in L(U_{ij}^U), \\
& \qquad \text{s.t. } \; S'(i) = S'(j) = 0 \\
& \qquad\qquad \text{and } S'(k) \in \{h, 0\}, \forall k \in \operatorname{Supp} U \,\Big], \\
& \quad \forall\, h \in (0, U_{ij}^U(i)], \\
\Longleftrightarrow\quad & \Big[\, v(T \cup \{i\}) - v(T \cup \{j\}) = 0, \\
& \quad \forall\, T \in P([U_{ij}^U]_h \backslash \{i, j\}) \,\Big], \\
& \quad \forall\, h \in (0, U_{ij}^U(i)].
\end{aligned}
$$

Consequently, if $v(S) = v(\mathcal{P}_{ij}[S])$ for any $S \in L(U_{ij}^U)$ then $v(T \cup \{i\}) = v(T \cup \{j\})$ for any $T \in P([U_{ij}^U]_h \backslash \{i, j\})$ and $h \in (0, U_{ij}^U(i)]$. Hence, we have $f_i'(v)([U_{ij}^U]_h) = f_j'(v)([U_{ij}^U]_h)$ for any $h \in (0, U_{ij}^U(i)]$ from Axiom C_3 in Definition 2.9. $f_i'(v)([U_{ij}^U]_h) = f_j'(v)([U_{ij}^U]_h) = 0$ holds for any $h \in (U_{ij}^U(i), 1]$ from Axiom C_1 in Definition 2.9. Hence, $f_i'(v)([U_{ij}^U]_h) = f_j'(v)([U_{ij}^U]_h)$ for any $h \in (0, 1]$. It follows that

$$
\begin{aligned}
f_i(v)(U_{ij}^U) &= \sum_{l=1}^{q(U_{ij}^U)} f_i'(v)([U_{ij}^U]_{h_l}) \cdot (h_l - h_{l-1}) \\
&= \sum_{l=1}^{q(U_{ij}^U)} f_j'(v)([U_{ij}^U]_{h_l}) \cdot (h_l - h_{l-1}) = f_j(v)(U_{ij}^U).
\end{aligned}
$$

This completes the proof of Axiom F_3.

Axiom F_4: Let $U \in L(N)$ and $v_1, v_2 \in G_C(N)$. It is clear that $v_1 + v_2 \in G_C(N)$ from the definition of $G_C(N)$. Using Axiom C_4 in Definition 2.9, we have

$$\begin{aligned} f_i(v_1 + v_2)(U) &= \sum_{l=1}^{q(U)} f'_i(v_1 + v_2)([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \{f'_i(v_1)([U]_{h_l}) + f'_i(v_2)([U]_{h_l})\} \cdot (h_l - h_{l-1}) \\ &= f_i(v_1)(U) + f_i(v_2)(U). \end{aligned}$$

□

In the reminder of this section, we investigate properties of a Shapley function f on $G_C(N)$.

Theorem 4.7 *The vector $(f_i(v)(U))_{i \in \mathrm{Supp}\, U,\, U \in L(N)}$ is an FPMAS if $v \in G_C(N)$ is convex.*

Proof. From Axiom F_1, it is apparent that $\sum_{i \in \mathrm{Supp}\, U} f_i(v)(U) = v(U)$ for any $U \in L(N)$. Hence we shall show $f_i(v)(S) \leq f_i(v)(T)$ for any $i \in N$ if $S \subseteq T$. Note that $S \subseteq T$ if and only if $[S]_h \subseteq [T]_h$ for any $h \in (0, 1]$. From Proposition 2.1, if $v \in G_0(N)$ is convex and $[S]_h \subseteq [T]_h$ then $f'_i(v)([S]_h) \leq f'_i(v)([T]_h)$ for any $i \in N$; therefore $f_i(v)(S) \leq f_i(v)(T)$ for any $i \in N$. □

Theorem 4.7 suggests that $f_i(v)$ is monotone nondecreasing with respect to rates of players' participation if $v \in G_C(N)$ is convex. $f_i(v)$ is continuous if $v \in G_C(N)$, as shown in next theorem.

Theorem 4.8 *Define the distance d in $L(N)$ by $d(S,T) = \max_{i \in N} |S(i) - T(i)|$ for any $S, T \in L(N)$. If $v \in G_C(N)$ then $f_i(v)$ is continuous for any $i \in N$.*

Proof. The theorem can be proved in the same manner as Theorem 4.1. □

The following theorem shows that $f(v)$ is a fuzzy imputation function of v if $v \in G_C(N)$ is convex.

Theorem 4.9 *Let $v \in G_C(N)$ be convex. Then $f(v)$ is a fuzzy imputation function of v.*

Proof. Let $U \in L(N)$. From Axiom F_1, it is apparent that $f_i(v)(U) = 0$ for any $i \notin \text{Supp } U$ and $\sum_{i \in N} f_i(v)(U) = v(U)$. We shall show $f_i(v)(U) \geq U(i) \cdot v(\{i\})$ for any $i \in N$. Let $i \in N$ be arbitrary. From Axiom C_1 in Definition 2.9, $f_i'(v)([U]_h) = 0$ for any $h > U(i)$. From Proposition 2.1, $f_i'(v)([U]_h) \geq f_i'(v)(\{i\}) = v(\{i\})$ for any $h \leq U(i)$ if v is convex. Then

$$\begin{aligned} f_i(v)(U) &= \sum_{l=1}^{q(U)} f_i'(v)([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ &= \sum_{l\,:\,h_l \leq U(i)} f_i'(v)([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ &\geq \sum_{l\,:\,h_l \leq U(i)} v(\{i\}) \cdot (h_l - h_{l-1}) = v(\{i\}) \cdot U(i). \end{aligned}$$

□

The following theorem provides a relationship between a pair of Shapley values for two strategically equivalent games in $G_C(N)$.

Theorem 4.10 *Let $v \in G_C(N)$. Let v' be defined by*

$$v'(S) = c \cdot v(S) + \sum_{i \in N} S(i) \cdot a_i, \qquad \forall\, S \in L(N),$$

where $c > 0$ and a_i are real constants. Then $v' \in G_C(N)$ and

$$f_i(v')(U) = c \cdot f_i(v)(U) + U(i) \cdot a_i, \qquad \forall\, i \in N.$$

Proof. Consider a game $w \in G(N)$ defined by $w(S) = \sum_{j \in N} S(j) \cdot a_j$ for any $S \in L(N)$. Then $v'(S) = c \cdot v(S) + w(S)$ for any $S \in L(N)$. The following holds:

$$\begin{aligned} w(S) &= \sum_{j \in N} a_j \sum_{l:h_l \leq S(j)} (h_l - h_{l-1}) = \sum_{l=1}^{q(S)} \sum_{j \in [S]_{h_l}} a_j \cdot (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(S)} w([S]_{h_l}) \cdot (h_l - h_{l-1}). \end{aligned}$$

This implies $w \in G_C(N)$. From the definition of $G_C(N)$, it is clear that $v' = c \cdot v + w \in G_C(N)$. Let $U \in L(N)$. The following holds if $h_l \leq U(i)$:

$$\begin{aligned} f_i'(w)([U]_{h_l}) &= \sum_{S \in P_i([U]_{h_l})} \beta(|S|; |[U]_{h_l}|) \cdot \{w(S) - w(S \backslash \{i\})\} \\ &= \sum_{S \in P_i([U]_{h_l})} \beta(|S|; |[U]_{h_l}|) \cdot \left\{ \sum_{j \in S} a_j - \sum_{j \in S, j \neq i} a_j \right\} \\ &= \sum_{S \in P_i([U]_{h_l})} \beta(|S|; |[U]_{h_l}|) \cdot a_i = a_i. \end{aligned}$$

Note that $f_i'(w)([U]_{h_l}) = 0$ if $h_l > U(i)$. We have

$$\begin{aligned} f_i(w)(U) &= \sum_{l=1}^{q(U)} f_i'(w)([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ &= \sum_{h_l \in Q(U) \,:\, h_l \leq U(i)} a_i \cdot (h_l - h_{l-1}) = U(i) \cdot a_i. \end{aligned}$$

The following also holds.

$$\begin{aligned} & f_i(c \cdot v)(U) \\ &= \sum_{l=1}^{q(U)} f_i'(c \cdot v)([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \sum_{S \in P_i([U]_{h_l})} \beta(|S|; |[U]_{h_l}|) \cdot \{c \cdot v(S) - c \cdot v(S \backslash \{i\})\} \cdot (h_l - h_{l-1}) \\ &= c \sum_{l=1}^{q(U)} \sum_{S \in P_i([U]_{h_l})} \beta(|S|; |[U]_{h_l}|) \cdot \{v(S) - v(S \backslash \{i\})\} \cdot (h_l - h_{l-1}) \\ &= c \sum_{l=1}^{q(U)} f_i'(v)([U]_{h_l}) \cdot (h_l - h_{l-1}) = c \cdot f_i(v)(U). \end{aligned}$$

By virtue of Axiom F_4, for any $i \in N$, we have

$$\begin{aligned} f_i(v')(U) &= f_i(c \cdot v + w)(U) = f_i(c \cdot v)(U) + f_i(w)(U) \\ &= c \cdot f_i(v)(U) + U(i) \cdot a_i. \end{aligned}$$

□

A gradual variation of each element of the Shapley value can be illustrated with the following example.

Table 4.2: The first element of the Shapley value for Example 4.3

S	$f_1'(v)(S)$	S	$f_1'(v)(S)$	S	$f_1'(v)(S)$	S	$f_1'(v)(S)$
$\{1\}$	120	$\{1,3\}$	210	$\{1,2,3\}$	260	$\{1,3,4\}$	240
$\{1,2\}$	210	$\{1,4\}$	210	$\{1,2,4\}$	260	$\{1,2,3,4\}$	340

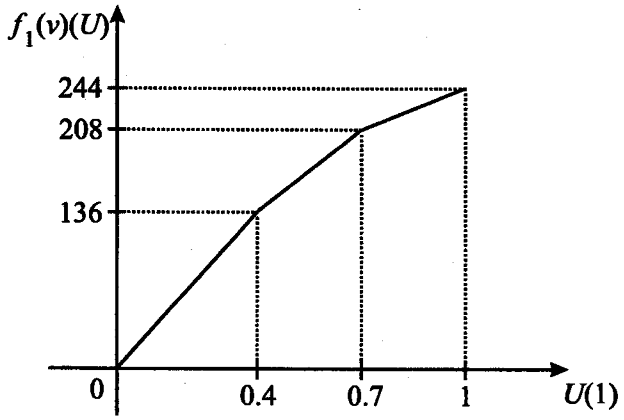

Figure 4.3: $f_i(v)(U)$ in Example 4.3

Example 4.3 Consider the game in Example 4.2. Note that the game is convex. Table 4.2 shows $f_1'(v)(W)$ for any $W \in P(N)$ such that $W \ni 1$. $f_1'(v)(W) = 0$ for any $W \in P(N)$ such that $W \not\ni 1$. The first element of the Shapley value in U for v will be

$$f_1(v)(U) = \begin{cases} 340 \cdot U(1), & \text{if } 0 \leq U(1) \leq 0.4, \\ 240 \cdot U(1) + 40, & \text{if } 0.4 \leq U(1) \leq 0.7, \\ 120 \cdot U(1) + 124, & \text{if } 0.7 \leq U(1) \leq 1. \end{cases}$$

A gradual variation of $f_1(v)(U)$ with respect to the rate of the first player's participation can be confirmed in Figure 4.3.

Preparatory to the following theorem, we define some notations. Consider bijections from W to W. They can be regarded as renumberings of the players in W. Let $\Omega(W)$ denote the set of all bijections from W to W. Let $\omega \in \Omega(\text{Supp } U)$ and $k \in N$. Then let $S_{\omega,k} \in L(U)$ be a fuzzy coalition defined by

$$S_{\omega,k}(i) = \begin{cases} U(i), & \text{if } \omega(i) \leq k, \\ 0, & \text{otherwise,} \end{cases} \quad \forall\, i \in N.$$

For $l \in \{1, \ldots, q(U)\}$, let $\Omega^l(U)$ denote the set of all bijections from $[U]_{h_l}$ to $[U]_{h_l}$. For $\omega^l \in \Omega^l(U)$ and $k \in N$, let us define

$$S^{(l)}_{\omega^l,k} = \{i \in [U]_{h_l} \mid \omega^l(i) \leq k\}.$$

Now, we shall show the relationship between the Shapley function and the core function on the restriction of our proposed class of fuzzy games, $G_C(N)$, to convex games.

Theorem 4.11 *If $v \in G_C(N)$ is convex and $U \in L(N)$, then $f(v)(U)$ coincides with the center of gravity of $C(v)(U)$.*

Proof. Let $v \in G_C(N)$ be convex and $U \in L(N)$. From a well-known property of the Shapley value for a crisp game [24], the following holds:

$$f'_i(v)([U]_{h_l}) = \frac{1}{u_{h_l}!} \sum_{\omega^l \in \Omega^l(U)} \{v(S^{(l)}_{\omega^l,\omega^l(i)}) - v(S^{(l)}_{\omega^l,\omega^l(i)-1})\},$$

where u_{h_l} is the cardinality of $[U]_{h_l}$. Hence, we have

$$\begin{aligned}
&f_i(v)(U)\\
&= \sum_{l=1}^{q(U)} f'_i(v)([U]_{h_l}) \cdot (h_l - h_{l-1})\\
&= \sum_{l=1}^{q(U)} \frac{1}{u_{h_l}!} \sum_{\omega^l \in \Omega^l(U)} \left\{v\left(S^{(l)}_{\omega^l,\omega^l(i)}\right) - v\left(S^{(l)}_{\omega^l,\omega^l(i)-1}\right)\right\} \cdot (h_l - h_{l-1})\\
&= \sum_{l\,:\,h_l \leq U(i)} \frac{1}{u_{h_l}!} \sum_{\omega^l \in \Omega^l(U)} \left\{v\left(S^{(l)}_{\omega^l,\omega^l(i)}\right) - v\left(S^{(l)}_{\omega^l,\omega^l(i)-1}\right)\right\} \cdot (h_l - h_{l-1}).
\end{aligned}$$

The last equality follows from $S^{(l)}_{\omega^l,\omega^l(i)} = S^{(l)}_{\omega^l,\omega^l(i)-1}$ for any l satisfying $h_l > U(i)$.

From Remark 3.2, $C(v)(U) = C'(v^U)(\mathrm{Supp}\ U)$ holds since any $v \in G_C(N)$ is monotone nondecreasing with respect to rates of players' participation. Note that $v^U \in G_0(N)$ is convex since the corresponding $v \in G_C(N)$ is convex, and that the core for a convex crisp game is nonempty [24]. These facts derive that $C(v)(U) = C'(v^U)(\mathrm{Supp}\ U) \neq \emptyset$.

We shall show that $f_i(v)(U)$ and the ith element a_i of the center of gravity of $C'(v^U)(\text{Supp } U)$ coincide with each other. From a well-known property of the core for a convex crisp game [24], the ith element a_i^ω of each extreme point of $C'(v^U)(\text{Supp } U)$ is represented as follows with the corresponding permutation $\omega \in \Omega(\text{Supp } U)$:

$$\begin{aligned}
a_i^\omega &= v^U(S^1_{\omega,\omega(i)}) - v^U(S^1_{\omega,\omega(i)-1}) \\
&= v(S_{\omega,\omega(i)}) - v(S_{\omega,\omega(i)-1}) \\
&= \sum_{l=1}^{q(U)} \{v([S_{\omega,\omega(i)}]_{h_l}) - v([S_{\omega,\omega(i)-1}]_{h_l})\} \cdot (h_l - h_{l-1}) \\
&= \sum_{l\,:\,h_l<U(i)} \{v([S_{\omega,\omega(i)}]_{h_l}) - v([S_{\omega,\omega(i)-1}]_{h_l})\} \cdot (h_l - h_{l-1}). \qquad (4.3)
\end{aligned}$$

The second equality follows from monotonicity of $v \in G_C(N)$. The last equality follows from the fact that $[S_{\omega,\omega(i)}]_{h_l} = [S_{\omega,\omega(i)-1}]_{h_l}$ if $i \notin [U]_{h_l}$.

Note that

$$[S_{\omega,\omega(i)}]_{h_l} = \{j \in [U]_{h_l} \mid \omega(j) \leq \omega(i)\}.$$

Thus, if $i \in [U]_{h_l}$, then $[S_{\omega,\omega(i)}]_{h_l} = S^{(l)}_{\omega^l,\omega^l(i)}$ and $[S_{\omega,\omega(i)-1}]_{h_l} = S^{(l)}_{\omega^l,\omega^l(i)-1}$ for any pair of bijections $(\omega, \omega^l) \in \Omega(\text{Supp } U) \times \Omega^l(U)$ such that $\omega(j) < \omega(k)$ if and only if $\omega^l(j) < \omega^l(k)$ for any pair $(j,k) \in [U]_{h_l} \times [U]_{h_l}$. For l satisfying $h_l \leq U(i)$ and a bijection $\omega^l \in \Omega^l(U)$, the number of bijections $\omega \in \Omega(\text{Supp } U)$ such that $\omega(j) < \omega(k)$ if and only if $\omega^l(j) < \omega^l(k)$ for any pair $(j,k) \in [U]_{h_l} \times [U]_{h_l}$ coincides with that of permutations of Supp U without distinction of elements of $[U]_{h_l}$, i.e., $|\text{Supp } U|!/u_{h_l}!$. Hence, for each l satisfying $h_l \leq U(i)$, we have

$$\begin{aligned}
&\sum_{\omega\in\Omega(\text{Supp } U)} \{v([S_{\omega,\omega(i)}]_{h_l}) - v([S_{\omega,\omega(i)-1}]_{h_l}) \\
&= \sum_{\omega^l\in\Omega^l(U)} \frac{|\text{Supp } U|!}{u_{h_l}!} \cdot \{v(S^{(l)}_{\omega^l,\omega^l(i)}) - v(S^{(l)}_{\omega^l,\omega^l(i)-1})\}.
\end{aligned}$$

Thus, from (4.3), the ith element a_i of the center of gravity of $C'(v^U)(\text{Supp } U)$

will be

$$\begin{aligned}
a_i &= \frac{1}{|\text{Supp } U|!} \sum_{\omega \in \Omega(\text{Supp } U)} a_i^\omega \\
&= \frac{1}{|\text{Supp } U|!} \sum_{l\,:\,h_l \le U(i)} \sum_{\omega \in \Omega(\text{Supp } U)} \{v([S_{\omega,\omega(i)}]_{h_l}) - v([S_{\omega,\omega(i)-1}]_{h_l})\} \\
&\qquad \cdot (h_l - h_{l-1}) \\
&= \frac{1}{|\text{Supp } U|!} \sum_{l\,:\,h_l \le U(i)} \sum_{\omega^l \in \Omega^l(U)} \frac{|\text{Supp } U|!}{u_{h_l}!} \\
&\qquad \cdot \left\{ v\left(S^{(l)}_{\omega^l,\omega^l(i)}\right) - v\left(S^{(l)}_{\omega^l,\omega^l(i)-1}\right) \right\} \cdot (h_l - h_{l-1}) \\
&= \sum_{l\,:\,h_l \le U(i)} \frac{1}{u_{h_l}!} \sum_{\omega^l \in \Omega^l(U)} \left\{ v\left(S^{(l)}_{\omega^l,\omega^l(i)}\right) - v\left(S^{(l)}_{\omega^l,\omega^l(i)-1}\right) \right\} \cdot (h_l - h_{l-1}) \\
&= f_i(v)(U).
\end{aligned}$$

Therefore the center of the gravity of $C(v)(U)$ coincides with $f(v)(U)$. □

4.5 An Illustrative Example

Consider three economic companies, simply named 1, 2 and 3. Company i has 100 units of resource R_i $(i = 1, 2, 3)$. Company i can obtain gains $v(\{i\})$ by producing 100 units of Product P_i from 100 units of Resource R_i. Valuable products can be produced by compounding two or three resources among R_1, R_2 and R_3. Namely, one unit of Product P_{ij} can be produced by compounding one unit of R_i and one unit of R_j $(i < j,\ i, j \in \{1, 2, 3\})$. Moreover, one unit of Product P_{123} can be produced by compounding one unit of R_1, one unit of R_2 and one unit of R_3. However, to produce Product P_{ij} $(i < j)$, Companies i and j have to make up a cooperative relationship, and to produce Product P_{123}, Companies 1, 2 and 3 have to. If Companies i and j make up a full cooperative relationship, i.e., a crisp coalition $\{i, j\}$, then they can obtain gains $v(\{i, j\})$ by producing 100 units of Product P_{ij} $(i < j)$. Similarly, by a crisp coalition $\{1, 2, 3\}$, they can obtain gains $v(\{1, 2, 3\})$ by producing 100 units of Product P_{123}.

As is in the real life, each company does not need to supply all units of resource to the formed cooperation. Thus, we have to consider a fuzzy game. For example, when Company i can supply only 40 units of R_i to the cooperation between i and j, we regard the rate of ith Player's participation (membership degree) as $0.4 = 40/100$. In such a way, a fuzzy coalition is interpreted. On the other hand, in the setting of this example, the value of a fuzzy coalition can be obtained by Choquet integral. Consider a fuzzy coalition U defined by

$$U(1) = 0.2,\ U(2) = 0.4,\ U(3) = 0.5.$$

This fuzzy coalition means that a cooperation among Companies 1, 2 and 3 is formed and Companies 1, 2 and 3 supply 20, 40 and 50 units of R_1, R_2 and R_3 to the cooperation, respectively. Under this cooperation, they can produce 20 units of P_{123}, 20 units of P_{23} and 10 units of P_3. Thus the value of this fuzzy coalition is evaluated by Choquet integral of U with respect to v, i.e.,

$$\begin{aligned}
&v(U)\\
&= (C)\int_N S\ dv\\
&= \sum_{l=1}^{q(S)} v([S]_{h_l}) \cdot (h_l - h_{l-1})\\
&= 0.2 \cdot v(\{1,2,3\}) + (0.4-0.2) \cdot v(\{2,3\}) + (0.5-0.4) \cdot v(\{3\}).
\end{aligned}$$

Now, let us estimate each company's share of $v(U)$ in the fuzzy coalition U. To do this, we can employ the proposed Shapley function or the proposed core function. If v is defined by

$$\begin{aligned}
&v(\{1\}) = 100, \quad v(\{2,3\}) = 800,\\
&v(\{2\}) = 200, \quad v(\{1,3\}) = 600,\\
&v(\{3\}) = 200, \quad v(\{1,2\}) = 600,\\
&\text{and } v(\{1,2,3\}) = 1800,
\end{aligned}$$

we obtain $v(U) = 540$. Note that v is convex. The ordinary Shapley values are obtained as in Table 1. The share of Company i by the Shapley value can be calculated by

$$\begin{aligned}
&f_i(v)(U)\\
&= 0.2 \cdot f_i'(v)([U]_{0.2}) + (0.4-0.2) \cdot f_i'(v)([U]_{0.4})\\
&\qquad + (0.5-0.4) \cdot f_i'(v)([U]_{0.5})\\
&= 0.2 \cdot f_i'(v)(\{1,2,3\}) + 0.2 \cdot f_i'(v)(\{2,3\}) + 0.1 \cdot f_i'(v)(\{3\}).
\end{aligned}$$

Table 1: $f_i'(v)(U)$

$U\backslash$Company i	1	2	3
$\{1\}$	100	0	0
$\{2\}$	0	200	0
$\{3\}$	0	0	200
$\{1,2\}$	250	350	0
$\{1,3\}$	250	0	350
$\{2,3\}$	0	400	400
$\{1,2,3\}$	500	650	650

Table 2: $v(S)$

$S(1)$	$S(2)$	$S(3)$	$v(S)$
0.2	0	0	20
0	0.4	0	80
0	0	0.5	100
0.2	0.4	0	160
0.2	0	0.5	180
0	0.4	0.5	340

Therefore the share of Company 1 can be calculated as follows:

$$\begin{aligned}
& f_1(v)(U) \\
= \; & 0.2 \cdot f_1'(v)(\{1,2,3\}) + 0.2 \cdot f_1'(v)(\{2,3\}) + 0.1 \cdot f_1'(v)(\{3\}) \\
= \; & 100.
\end{aligned}$$

In the same way, Companies 2 and 3's shares can be calculated as $f_2(v)(U) = 210$ and $f_3(v)(U) = 230$.

The values of the fuzzy coalitions needed to calculate the core function is obtained as in Table 2. Hence, we have

$$\begin{aligned}
& C(v)(U) \\
=& \{\boldsymbol{x} \in I(v)(U) \mid x_1 + x_2 \geq 160, x_1 + x_3 \geq 180, x_2 + x_3 \geq 340\} \\
=& \Big\{\boldsymbol{x} \in \mathbb{R}^3 \;\Big|\; x_1 \geq 20, x_2 \geq 80, x_3 \geq 100, \\
& \qquad x_1 + x_2 \geq 160, x_1 + x_3 \geq 180, x_2 + x_3 \geq 340, \\
& \qquad x_1 + x_2 + x_3 = 540\Big\} \\
=& \mathrm{co}\{\ (20,140,380), (20,360,160), (80,80,380), \\
& \qquad (80,360,100), (200,80,260), (200,240,100)\ \}.
\end{aligned}$$

Then the center of gravity of the core is the Shapley value.

4.6 Concluding Remarks

We have given a new class of fuzzy games. It has been shown that any fuzzy game in the class is continuous and monotone nondecreasing with respect to rates of players' participation. The class can be considered natural in these point.

Relations between an f-carrier and a carrier and relations between a γ-null player and a null player have been clarified in this class. In particular, the existence of the smallest f-carrier has been discussed and it has been shown that the smallest f-carrier can be identified with a carrier for the corresponding crisp game.

Furthermore, we have given an explicit form of the Shapley function. We have shown that a collection obtained from this explicit function is an FPMAS and that the function obtained from this explicit function is a fuzzy imputation function, if a given game is convex and belongs to this class. The former implies that the function is monotone nondecreasing with respect to rates of players' participation. Furthermore, it has been shown that the explicit function is continuous with respect to rates of players' participation if a given fuzzy game is in the class. Strategical equivalence on the explicit function has been discussed. Furthermore, we have proved that the Shapley value and the center of gravity of the core coincide with each other. The above discussions means that many of properties which hold in crisp games are valid also in this class.

Finally, an illustrative example has been given.

Chapter 5

Solution Concepts in Minimum Spanning Tree Games

5.1 Introduction

Suppose that players in a set $N = \{1, 2, \dots, n\}$ are geographically separated and need some good that is provided by a common supplier 0. Then a distribution system from the supplier 0 to all members in N has to be built. They are willing to construct a distribution system with the minimum cost. The algorithm to find distribution systems with the minimum cost has been studied for a long time [17, 20]. If it is once found, then the problem of how to allocate the cost to each member will arise. Such a problem was first introduced by Claus and Kleitman [9].

Bird [6] was the first who suggested a game theoretic approach to this problem. He also proposed rational allocations called Bird tree allocations. For each coalition $S \subset N$, let $c(S)$ be determined by the minimum cost of all distribution systems from supplier 0 to players in S. Let $\bar{c}(S)$ be a game defined by the minimum cost all over distribution systems from supplier 0 and to players in $T \supset S$. Then games c and $\bar{c}$ are called the minimum spanning tree game (mst-game) and the monotonic cover game (mc-game), respectively. A Bird tree allocation is an element of both the cores of the mst-game and the mc-game [10, 13]. In a Bird tree allocation, each player should pay the cost of the edge which connects him/her and his/her immediate predecessor. It means that the allocation does not depend on the length of the unique path from the source to him/her. In particular, if the vertex i be an adjacent follower of the source in Γ, i.e., $i \in F(0)$ $(p(i) = 0)$, and $\boldsymbol{l}$ is a Bird tree allocation for Γ, then $l_i = c_{p(i)i} = c(\{i\}) = \bar{c}(\{i\})$ holds as noted in Chapter 2. It means that players directly connected to the source in Γ do not benefit in the allocation $\boldsymbol{l}$ although his/her cooperation is important for connecting the supplier and his/her followers. However, from the definition of a minimum spanning tree, each player may have to pay more cost but for a presence of his/her predecessor. So, it may be considered that each player can demand that his/her followers should pay the cost which his/her absence caused. To give such players the benefit in the grand coalition formation, Granot and Huberman [14] proposed the original weak demand operations (original w.d.o.'s for short). They are defined as operations which map an allocation to an allocation.

However, it can be considered better to give a slight modification of the original w.d.o. by player since the original w.d.o. brings an unfair allocations from some allocations. Furthermore, the original w.d.o. by a coalition is not well-defined. In this chapter, we define a demand operation by a player by giving a slight modification of the original w.d.o. by a player. We also define a demand operation by a coalition by adding a certain restriction to the original w.d.o. by a coalition so that it becomes well-defined and any allocation obtained through it is an element of the core of the mst-game.

The demand operation by player i is based on the idea that players who belong to $F(i)$ should share the cost which the ith player's absence caused. However, if the cost of a minimum spanning tree in G_{N_0} is smaller than that in $G_{N_0 \setminus \{i\}}$, then players in $N \setminus \{i\}$ will propose to construct a minimum spanning tree in G_{N_0} by themselves. It can be considered that player i will accept the proposal. Based on this idea, a revised demand operation is defined.

5.2 Demand Operations

In this section, we define modified demand operations both by a player and by a coalition. They are defined by modifying or restricting the weak demand operations both by a player and by a coalition, which were proposed by Granot and Huberman [14] and which will be called the original weak demand operations (original w.d.o.'s) in what follows.

In order to define the original w.d.o.'s and demand operations, various notations are required, which will be represented as follows. Let Γ be a minimum spanning tree in a network G. Let $F(i)$ be the set of the adjacent followers of vertex i in Γ, namely $F(i) = \{j \in N \mid j \succeq_\Gamma i,\ O_\Gamma(j) - O_\Gamma(i) = 1\}$. For $k \in F(i)$, let V_k denote the set of the followers of vertex k in Γ and k itself, i.e., $V_k = \{j \in N \mid j \succeq_\Gamma k\}$. Let E_k be the edge set consisting of all edges which are included in E_Γ and whose end points belong to V_k, i.e., $E_k = \{(u,v) \in E_\Gamma \mid u, v \in V_k\}$. Define the subnetwork $(V_\Gamma^{-i}, E_\Gamma^{-i})$ by $V_\Gamma^{-i} = N_0 \backslash \{\{i\} \cup (\cup_{k \in F(i)} V_k)\}$ and $E_\Gamma^{-i} = E_\Gamma \backslash \{\{(p(i), i)\} \cup \{\cup_{k \in F(i)} (E_k \cup \{(i,k)\})\}\}$.

Let a spanning tree Γ_i in the network $G_{N_0 \backslash \{i\}}$ be minimum of all spanning trees whose edge sets include $E_\Gamma \backslash \{\{(p(i), i)\} \cup (\cup_{k \in F(i)} \{(i,k)\})\} = E_\Gamma^{-i} \cup (\cup_{k \in F(i)} E_k)$ in terms of the cost. For any $k \in F(i)$, there exists a unique vertex q_k such that $q_k \in V_k$ and $q_k \preceq_{\Gamma_i} j$ for any $j \in V_k$. We denote the unique adjacent predecessor of q_k in Γ_i by vertex p_k. The edge set of Γ_i is represented by $E_\Gamma^{-i} \cup \{\cup_{k \in F(i)} (E_k \cup \{(p_k, q_k)\})\}$. Then Γ_i is a minimum spanning tree in $G_{N_0 \backslash \{i\}}$. In fact, (V_r, E_r) for any $r \in F(i)$ and $(V_\Gamma^{-i}, E_\Gamma^{-i})$ are disjoint subnetworks of the minimum spanning tree Γ in G_{N_0}. Hence, from Kruskal's algorithm [17], spanning trees obtained by adding some appropriate edges to them can be minimum spanning trees in $G_{N_0 \backslash \{i\}}$. From the definition of (p_k, q_k) for $k \in F(i)$, Γ_i is one of minimum spanning trees in $G_{N_0 \backslash \{i\}}$.

Let $c_k = c_{p_k q_k}$ for $k \in F(i)$. Then we define a demand operation by a player as follows.

Definition 5.1 *Let $i \in N$ and let $\boldsymbol{y}$ be an allocation. For $r \in N$ and for a* *minimum spanning tree* Γ *in* G_{N_0}, *let*

$$d_r^i(\boldsymbol{y}) = \begin{cases} c_r, & \text{if } r \in F(i), \\ y_r - \sum_{k \in F(i)} (d_k^i(\boldsymbol{y}) - y_k), & \text{if } r = i, \\ y_r, & \text{otherwise.} \end{cases}$$

The operation $\boldsymbol{d}^i$ *which associates the vector* $\boldsymbol{d}^i(\boldsymbol{y}) = (d_r^i(\boldsymbol{y}))_{r \in N}$ *with each allocation* $\boldsymbol{y}$ *is said to be the demand operation by player* i *in* Γ.

Definition 5.1 is similar to but slightly different from the definition of the original w.d.o. by a player. The difference is that $d_r^i(\boldsymbol{y}) = c_r - (\sum_{(u,v) \in E_r} c_{uv} - y(V_r \backslash \{r\}))$ for $r \in F(i)$ in the original w.d.o. by a player. We consider that the original w.d.o. by a player is not very rational in the following reason. Consider an allocation $\boldsymbol{y}$ satisfying $y(V_r \backslash \{r\}) > \sum_{(u,v) \in E_r} c_{uv}$ for some $r \in F(i)$, which means that the followers of player k, i.e., the members of $V_k \backslash \{k\}$, pay more cost in $\boldsymbol{y}$ than the cost of minimum spanning tree (V_k, E_k). Then player k has to pay more cost than c_k in the allocation obtained through the original w.d.o. on $\boldsymbol{y}$. It follows that player i will gain too much benefit in the obtained allocation. We think it more rational that each player $r \in F(i)$ should pay c_r. Because of this reason, we have slightly modified the definition of a weak demand operation by a player.

If $y(V_r \backslash \{r\}) = l(V_r \backslash \{r\}))$, then $c_r - (\sum_{(u,v) \in E_r} c_{uv} - l(V_r \backslash \{r\})) = c_r$. Thus, our modification does not make difference as far as $y(V_r \backslash \{r\}) = l(V_r \backslash \{r\}))$ for any $r \in F(i)$.

It is apparent that $\boldsymbol{d}^i(\boldsymbol{y})$ is an allocation if $\boldsymbol{y}$ is an allocation. Note that $\boldsymbol{d}^i(\boldsymbol{y}) = \boldsymbol{y}$ if $F(i) = \emptyset$. It is also noted that $\sum_{r \in N \backslash \{i\}} d_r^i(\boldsymbol{l})$ is the cost of the minimum spanning tree Γ_i in $G_{N_0 \backslash \{i\}}$, i.e., $\sum_{r \in N \backslash \{i\}} d_r^i(\boldsymbol{l}) = c(N \backslash \{i\})$.

A subnetwork $(N, E_\Gamma \backslash \{(i,r)\})$ represents two minimum spanning trees for the disjoint subnetworks of G_{N_0} for any $r \in F(i)$; the one includes vertex r and the other includes source 0 and vertex i. For Γ to be a minimum spanning tree means that c_{ir} is minimum among the costs of all edges connecting the two trees from Kruskal's algorithm. It follows that $l_r = c_{ir} \leq c_r$ holds for any $r \in F(i)$. Hence, for any $r \in F(i)$, $d_r^i(\boldsymbol{l}) = c_r \geq l_r$.

The following theorem holds.

Table 5.1: Preparation for obtaining the result of the demand operation in Example 2

i	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
$F(i)$	$\{2\}$	$\{4\}$	$\emptyset$	$\emptyset$
$\sum_{k\in F(i)} c_k$	5	4	–	–
$c_{p(i)i} + \sum_{k\in F(i)} c_{ik}$	4	3	–	–

r	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
c_r	–	5	–	4
$\sum_{(u,v)\in E_{\Gamma_r}} c_{uv}$	–	1	–	0
$V_r\backslash\{r\}$	–	$\{4\}$	–	$\emptyset$

Theorem 5.1 *If $\boldsymbol{l}$ and $\boldsymbol{d}^i$ are the Bird tree allocation and the demand operation both associated with Γ, respectively, then the following holds:*

$$\boldsymbol{d}^i(\boldsymbol{l}) \in C(c), \quad \forall\, i \in N.$$

Proof. It can be proved in a similar manner to Theorem 2 in [14]. □

For a Bird tree allocation $\boldsymbol{l}$ in a network, $\boldsymbol{d}^i(\boldsymbol{l})$ is not always included in the core of the associated mc-game. An example is given as follows:

Example 5.1 Suppose that G is given by Figure 2.1. Then $(N_0, \{(0,1),(1,2),(2,4),(0,3)\})$ is a minimum spanning tree in G. The associated Bird tree allocation is $(2,2,1,1)$. Let $\boldsymbol{l} = (2,2,1,1)$. Then, $F(i)$, $\sum_{k\in F(i)} c_k$ and $c_{p(i)i} + \sum_{k\in F(i)} c_{ik}$ for any $i \in N$, and c_r, $\sum_{(u,v)\in E_{\Gamma_r}} c_{uv}$ and $V_r\backslash\{r\}$ for any $r \in F(i)$ are given in Table 3.

Since $F(1) = \{2\}$ and $c_2 = 5$, $d_2^1(\boldsymbol{l}) = c_2 = 5$ and $d_1^1(\boldsymbol{l}) = l_1 - (d_2^1(\boldsymbol{l}) - l_2) = 2 - (5 - 2) = -1$. We have $\boldsymbol{d}^1(\boldsymbol{l}) = (-1,5,1,1)$. From Lemma 2.1, it is apparent that $\boldsymbol{d}^1(\boldsymbol{l}) \notin C(\bar{c})$.

A weak demand operation by a coalition is also proposed in [14]. It is defined as the operation to apply w.d.o.'s by a player repeatedly for all members of the coalition. In the paper [14], it was considered that the operation by a coalition is independent of the order of applying the w.d.o. by a player. In fact, however, the operation depends on the order, as is shown in Examples 3 and 4 later. Therefore, we revise the weak demand operation by a coalition by taking the order into account.

Let Π be the set of all permutations of N. Let Π_Γ be defined by

$$\Pi_\Gamma = \{\pi \in \Pi \mid \pi(j) < \pi(i), \ \ \forall i, j \in N, \text{ s.t. } j \prec_\Gamma i\}.$$

Then a demand operation by a coalition is defined as follows.

Definition 5.2 *Let $S \subseteq N$ and $\pi \in \Pi_\Gamma$. Let $\boldsymbol{d}^i$ be the demand operation by player i in Γ for any $i \in S$. Consider the following algorithm which generates $\boldsymbol{d}^S(\boldsymbol{y})$ from $\boldsymbol{y}$.*

Step 1: *Set $Q = \emptyset$.*

Step 2: *Choose player $i \in S \backslash Q$ satisfying $\pi(i) \leq \pi(j)$ for any $j \in S \backslash Q$ and update $Q := Q \cup \{i\}$.*

Step 3: *Calculate $\boldsymbol{d}^Q(\boldsymbol{y}) = \boldsymbol{d}^i(\boldsymbol{d}^{Q \backslash \{i\}}(\boldsymbol{y}))$, where we define $\boldsymbol{d}^\emptyset(\boldsymbol{y}) = \boldsymbol{y}$ for convenience.*

Step 4: *If $Q = S$, then it is terminated. Otherwise, return to* Step 2.

Then the operation $\boldsymbol{d}^S$ which associates $\boldsymbol{d}^S(\boldsymbol{y})$ with the vector each allocation $\boldsymbol{y}$ is said to be the demand operation by a coalition S in Γ.

Our definition of a demand operation is different from the original w.d.o. in terms of Step 2. Step 2 of the original w.d.o. can be described as 'select $i \in S \backslash Q$ and set $Q := Q \cup \{i\}$.' The original weak demand operation is not well-defined since the result depends on which player is chosen at Step 2, as is shown in Examples 3, 4 and 5 later.

Note that there exists a unique demand operation by S for any $S \subseteq N$ and for any minimum spanning tree Γ, namely $\boldsymbol{d}^S$ is independent of the choice of $\pi \in \Pi_\Gamma$. This is because the demand operation by player i has an influence only on the r-th element of $\boldsymbol{y}$ for $r \in \{i\} \cup F(i)$. It follows that the order of applying the operations by player j and by player l does not have an influence on the result if they satisfy neither $j \preceq_\Gamma l$ nor $j \succeq_\Gamma l$.

The demand operation by a coalition in Definition 5.2 is based on that by a player defined in Definition 5.1. That demand operation based on the original w.d.o. by a player can be also considered. It is noted that these two demand operations by a coalition bring a Bird tree allocation to the same allocation. This is because an original demand operation by player i depends on $y(V_r \backslash \{r\})$ for $r \in F(i)$ and has an influence only on the r-th element of $\boldsymbol{y}$ for $r \in \{i\} \cup F(i)$.

The following Theorem holds.

Theorem 5.2 *If $\boldsymbol{l}$ and $\boldsymbol{d}^i$ are the Bird tree allocation and the demand operation both associated with Γ respectively, then the following holds:*

$$\boldsymbol{d}^S(\boldsymbol{l}) \in C(c), \quad \forall\, S \subseteq N.$$

Proof. It can be proved in a similar manner to Theorem 3 in [14]. □

It was considered that the theorem corresponding to Theorem 5.2 holds for the original w.d.o. by a coalition. The proof of the theorem is based on the idea that w.d.o. by a coalition is independent of the order. However, the original w.d.o. depends on the order; hence the theorem for original w.d.o. does not hold. In fact, not all allocations obtained through the original w.d.o. are included in the core of the mst-game, as is shown in Example 5 later.

An example of the demand operation is given as follows:

Example 5.2 Consider the same network G and the same minimum spanning tree Γ as Example 2. Then $\boldsymbol{l} = (2, 2, 1, 1)$ is the Bird tree allocation for Γ. Let $\boldsymbol{d}^S$ be the demand operation by $S \subset N$ in Γ. Then $\boldsymbol{d}^N(\boldsymbol{l})$ is obtained as follows. Let π be defined by $\pi(1) = 1$, $\pi(2) = 2$, $\pi(3) = 3$ and $\pi(4) = 4$. In Step 1, set $Q = \emptyset$. Then $\pi(1) = \min\{\pi(j) \mid j \in S\backslash Q\}$, and reset $Q = \{1\}$ in Step 2. $\boldsymbol{d}^{\{1\}}(\boldsymbol{l}) = \boldsymbol{d}^1(\boldsymbol{d}^{\emptyset}(\boldsymbol{l})) = \boldsymbol{d}^1(\boldsymbol{l}) = \boldsymbol{d}^1((2, 2, 1, 1))$. From Example 2, we have $\boldsymbol{d}^1((2, 2, 1, 1)) = (-1, 5, 1, 1)$. Since $Q \neq S$, go to Step 2. Since $\pi(2) = \min\{\pi(j) \mid j \in S\backslash Q\}$, update $Q = \{1\} \cup \{2\} = \{1, 2\}$. $\boldsymbol{d}^{\{1,2\}}(\boldsymbol{l}) = \boldsymbol{d}^2(\boldsymbol{d}^1(\boldsymbol{l})) = \boldsymbol{d}^2((-1, 5, 1, 1))$. Since $F(2) = \{4\}$ and $c_4 = 4$,

$d_4^2((-1,5,1,1)) = 4$. It follows that $d_2^2((-1,5,1,1)) = 5-(4-1) = 2$. Hence, we have $\boldsymbol{d}^{\{1,2\}}(\boldsymbol{l}) = \boldsymbol{d}^2((-1,5,1,1)) = (-1,2,1,4)$. Since $F(3) = F(4) = \emptyset$, we have $\boldsymbol{d}^N(\boldsymbol{l}) = \boldsymbol{d}^4(\boldsymbol{d}^3(\boldsymbol{d}^2(\boldsymbol{d}^1(\boldsymbol{l})))) = \boldsymbol{d}^4(\boldsymbol{d}^3((-1,2,1,4))) = (-1,2,1,4)$. It can be easily seen that $(-1,2,1,4) \in C(c)$.

Not all allocations obtained by the sequential application of the demand operations by a player on a Bird tree allocation in an order $\pi \notin \Pi_\Gamma$ are elements of the core of the mst-game. It is shown in the following example.

Example 5.3 (cf. Example 3) Consider the same network G and the same minimum spanning tree Γ as Example 2. Then the Bird tree allocation for Γ is $\boldsymbol{l} = (2,2,1,1)$. Let π' be defined by $\pi'(1) = 2$, $\pi'(2) = 1$, $\pi'(3) = 3$ and $\pi'(4) = 4$. Then $\pi' \notin \Pi_\Gamma$. For $S \subset N$, let $\boldsymbol{f}^S$ be obtained by the algorithm in Definition 5.2 but based on $\pi' \notin \Pi_\Gamma$. Then $\boldsymbol{f}^N(\boldsymbol{l})$ is obtained as follows. Set $Q = \emptyset$ in Step 1. Then $\pi'(2) = \min\{\pi'(j) \mid j \in S\backslash Q\}$, and reset $Q = \{2\}$ in Step 2. $\boldsymbol{f}^{\{2\}}(\boldsymbol{l}) = \boldsymbol{f}^2(\boldsymbol{f}^{\emptyset}(\boldsymbol{l})) = \boldsymbol{f}^2(\boldsymbol{l}) = \boldsymbol{f}^2((2,2,1,1))$. Since $F(2) = \{4\}$ and $c_4 = 4$, $f_4^2((2,2,1,1)) = 4$. It follows that $f_2^2((2,2,1,1)) = 2-(4-1) = -1$. Hence, we have $\boldsymbol{f}^2(\boldsymbol{l}) = \boldsymbol{f}^2((2,2,1,1)) = (2,-1,1,4)$. Since $Q \neq S$, go to Step 2. $\pi'(1) = \min\{\pi'(j) \mid j \in S\backslash Q\}$. Update $Q = \{2\} \cup \{1\} = \{1,2\}$. Since $F(1) = \{2\}$ and $c_2 = 5$, $f_2^1((2,-1,1,4)) = 5$. It follows that $f_1^1((2,-1,1,4)) = 2-(5-(-1)) = -4$. Hence, we have $\boldsymbol{f}^{\{1,2\}}(\boldsymbol{l}) = \boldsymbol{f}^1(\boldsymbol{f}^2(\boldsymbol{l})) = \boldsymbol{f}^1((2,-1,1,4)) = (-4,5,1,4)$. Since $F(3) = F(4) = \emptyset$, we have $\boldsymbol{f}^N(\boldsymbol{l}) = \boldsymbol{f}^4(\boldsymbol{f}^3(\boldsymbol{f}^1(\boldsymbol{f}^2(\boldsymbol{l})))) = \boldsymbol{f}^4(\boldsymbol{f}^3((-4,5,1,4))) = (-4,5,1,4)$. Since $\sum_{r\in\{2,4\}} f_r^N(\boldsymbol{l}) = 9 > c(\{2,4\}) = 6$, $\boldsymbol{f}^N(\boldsymbol{l}) \notin C(c)$.

Not all allocations obtained by the sequential application of the original w.d.o. by a coalition on a Bird tree allocation in an order $\pi' \notin \Pi_\Gamma$ are elements of the core of the mst-game, as is shown in the following example.

Example 5.4 (cf. Example 3 and Example 4) Consider the same network G and the same minimum spanning tree Γ as Example 2. Then the Bird tree allocation for Γ is $\boldsymbol{l} = (2,2,1,1)$. Let π' be the same permutation as Example 4. For $S \subset N$, let $\boldsymbol{g}^S$ be obtained by the algorithm in Definition

5.2 but based on $\pi' \notin \Pi_\Gamma$ and the original w.d.o. by a player. Then $\boldsymbol{g}^N(\boldsymbol{l}) = \boldsymbol{g}^4(\boldsymbol{g}^3(\boldsymbol{g}^1(\boldsymbol{g}^2(\boldsymbol{g}((2,2,1,1)))))) = \boldsymbol{g}^4(\boldsymbol{g}^3(\boldsymbol{g}^1((2,-1,1,4)))) = \boldsymbol{g}^4(\boldsymbol{g}^3((-7,8,1,4))) = (-7,8,1,4)$. Since $\sum_{r\in\{2,4\}} g_r^N(\boldsymbol{l}) = 12 > c(\{2,4\}) = 6$, $\boldsymbol{g}^N(\boldsymbol{l}) \notin C(c)$.

5.3 Revised Demand Operations

In this section, we define a revised demand operation both by a player and a coalition.

Let $r \in F(i)$. Let Γ be a minimum spanning tree in the network G_{N_0}. Let $p(i)$ be the adjacent predecessor of i in Γ. Then we define β_r with respect to Γ as follows:

$$\beta_r = \begin{cases} c_r, & \text{if } \sum_{k\in F(i)} c_k \le c_{p(i)i} + \sum_{k\in F(i)} c_{ik}, \\ \alpha_r c_{p(i)i} + c_{ir}, & \text{otherwise,} \end{cases}$$

where $0 \le \alpha_r \le (c_r - c_{ir})/c_{p(i)i}$ and $\sum_{r\in F(i)} \alpha_r = 1$.

Note that there exists $(\alpha_r)_{r\in F(i)}$ satisfying the above conditions, i.e., $0 \le \alpha_r \le (c_r - c_{ir})/c_{p(i)i}$ and $\sum_{r\in F(i)} \alpha_r = 1$. In fact, $(c_r - c_{ir})/c_{p(i)i} \ge 0$ holds since $c_r - c_{ir} \ge 0$ and $c_{p(i)i} \ge 0$. Furthermore, $\sum_{r\in F(i)}(c_r - c_{ir})/c_{p(i)i} > 1$ also holds, since $\sum_{k\in F(i)} c_k > c_{p(i)i} + \sum_{k\in F(i)} c_{ik}$ holds when $(\alpha_r)_{r\in F(i)}$ is necessary. Thus, the existence of $(\alpha_r)_{r\in F(i)}$ is apparent.

Note that Γ and Γ_i defined in the previous section are minimum spanning trees in G_{N_0} and $G_{N_0\setminus\{i\}}$, respectively. It follows that the cost of a minimum spanning tree in G_{N_0} is smaller than that in $G_{N_0\setminus\{i\}}$ if and only if $\sum_{k\in F(i)} c_k > c_{p(i)i} + \sum_{k\in F(i)} c_{ik}$. Consider the case where $\sum_{k\in F(i)} c_k > c_{p(i)i} + \sum_{k\in F(i)} c_{ik}$. In such a case, players in $F(i)$ will share the cost $c_{p(i)i} + \sum_{k\in F(i)} c_{ik}$, which means that player i will pay nothing. Then $0 \le \alpha_r \le (c_r - c_{ir})/c_{p(i)i}$ should hold. In fact, $\alpha_r > (c_r - c_{ir})/c_{p(i)i}$ means $c_r < \alpha_r c_{p(i)i} + c_{ir}$; then player r prefer to pay c_r. Hence, $\alpha_r \le (c_r - c_{ir})/c_{p(i)i}$ should hold for any $r \in F(i)$. $\alpha_r \ge 0$ should also hold for any $r \in F(i)$ since at least c_{ir} is needed in order that players in V_r have a link.

It is noted that $\delta_r^i(\boldsymbol{l}) \ge l_r$ for any $r \in F(i)$ if $\boldsymbol{l}$ is the Bird tree allocation for Γ. In fact, it is apparent that $\alpha_r c_{p(i)i} + c_{ir} \ge c_{ir}$, and as noted in the previous section $c_r \geq l_r = c_{ir}$. Hence, $\beta_r \geq c_{ir} = l_r$ holds if $\boldsymbol{l}$ is the Bird tree allocation for Γ.

Then we define a revised demand operation as follows.

Definition 5.3 *Let $i \in N$ and let $\boldsymbol{y}$ be an allocation. For $r \in N$ and for a minimum spanning tree Γ in G_{N_0}, let*

$$\delta_r^i(\boldsymbol{y}) = \begin{cases} \beta_r, & \text{if } r \in F(i), \\ y_r - \sum_{k \in F(i)} (\delta_k^i(\boldsymbol{y}) - y_k), & \text{if } r = i, \\ y_r, & \text{otherwise.} \end{cases}$$

The operation $\boldsymbol{\delta}^i$ which associates the vector $\boldsymbol{\delta}^i(\boldsymbol{y}) = (\delta_r^i(\boldsymbol{y}))_{r \in N}$ is said to be the revised demand operation by player i in Γ.

It is obvious that $\boldsymbol{\delta}^i(\boldsymbol{y})$ is an allocation if $\boldsymbol{y}$ is an allocation. It is noted that $\delta_r^i(\boldsymbol{y}) \leq d_r^i(\boldsymbol{y})$ for any $r \in F(i)$ and that $\delta_i^i(\boldsymbol{y}) \geq d_i^i(\boldsymbol{y})$. In particular, $l_r = c_{ir} \leq \delta_r^i(\boldsymbol{l}) \leq d_r^i(\boldsymbol{l})$ holds for any $r \in F(i)$ and $l_i \geq \delta_i^i(\boldsymbol{l}) \geq d_i^i(\boldsymbol{l})$ holds. Note that $\boldsymbol{\delta}^i(\boldsymbol{y}) = \boldsymbol{y}$ if $F(i) = \emptyset$.

Lemma 5.1 *If $\boldsymbol{l}$ and $\boldsymbol{\delta}^i$ are the Bird tree allocation and the revised demand operation by player i both associated with Γ respectively, then the following holds:*

$$\boldsymbol{\delta}^i(\boldsymbol{l}) \in C(c), \quad \forall\, i \in N.$$

Proof. It is sufficient to show that $\sum_{r \in S} \delta_r^i(\boldsymbol{l}) \leq c(S)$ holds for any $S \subseteq N$. Let $S \subseteq N$ satisfy $S \ni i$. Note that $\delta_r^i(\boldsymbol{l}) \geq l_r$ for any $r \in F(i)$ and $\delta_r^i(\boldsymbol{l}) = l_r$ for any $r \notin F(i) \cup \{i\}$. It follows that $\delta_r^i(\boldsymbol{l}) \geq l_r$ for any $r \in N \backslash S$. Hence, we obtain the following:

$$\sum_{r \in S} \delta_r^i(\boldsymbol{l}) = c(N) - \sum_{r \in N \backslash S} \delta_r^i(\boldsymbol{l}) \leq c(N) - l(N \backslash S) = l(S).$$

Let $T \subseteq N$ satisfy $T \not\ni i$. Note that $\delta_r^i(\boldsymbol{l}) \leq d_r^i(\boldsymbol{l})$ holds for any $r \in F(i)$ and $\delta_r^i(\boldsymbol{l}) = d_r^i(\boldsymbol{l})$ holds for any $r \notin F(i) \cup \{i\}$, which means that $\delta_r^i(\boldsymbol{l}) \leq d_r^i(\boldsymbol{l})$ holds for any $r \in T$. Thus, $\sum_{r \in T} \delta_r^i(\boldsymbol{l}) \leq \sum_{r \in T} d_r^i(\boldsymbol{l}) \leq c(T)$ by Theorem 5.2. The proof is completed. □

Lemma 5.2 *If $\boldsymbol{l}$ and $\boldsymbol{\delta}^i$ are the Bird tree allocation and the revised demand operation by player i both associated with Γ respectively, then the following holds:*

$$\delta_r^i(\boldsymbol{l}) \geq 0, \quad \forall\, r \in N.$$

Proof. If $r \in F(i)$, then the following holds:

$$\delta_r^i(\boldsymbol{l}) \geq l_r \geq 0.$$

If $r = i$, then the following holds:

$$\begin{aligned}
\delta_r^i(\boldsymbol{l}) &= \delta_i^i(\boldsymbol{l}) \\
&= l_i - \sum_{k \in F(i)} (\delta_k^i(\boldsymbol{l}) - l_k) \\
&= l_i + \sum_{k \in F(i)} l_k - \min\left\{ \sum_{k \in F(i)} c_k, \sum_{k \in F(i)} (\alpha_k c_{p(i)i} + c_{ik}) \right\} \\
&\geq l_i + \sum_{k \in F(i)} l_k - \sum_{k \in F(i)} (\alpha_k c_{p(i)i} + c_{ik}) \\
&= l_i + \sum_{k \in F(i)} l_k - \left(c_{p(i)i} + \sum_{k \in F(i)} c_{ik} \right) = 0
\end{aligned}$$

The last equality comes from $l_i = c_{p(i)i}$ and $l_k = c_{ik}$ for any $k \in F(i)$.

If $r \notin F(i) \cup \{i\}$, we have

$$\delta_r^i(\boldsymbol{l}) = l_r \geq 0.$$

Thus, $\delta_r^i(\boldsymbol{l}) \geq 0$ for any $r \in N$. □

Theorem 5.3 *If $\boldsymbol{l}$ and $\boldsymbol{\delta}^i$ are the Bird tree allocation and the revised demand operation by player i both associated with Γ respectively, then the following holds:*

$$\boldsymbol{\delta}^i(\boldsymbol{l}) \in C(\bar{c}), \quad \forall\, i \in N.$$

Proof. It is apparent that $\sum_{r\in N}\delta_r^i(\boldsymbol{l}) = \bar{c}(N)$. It will be shown that $\sum_{r\in S}\delta_r^i(\boldsymbol{l}) \geq \bar{c}(S)$ for any $S \in N$. Suppose that there exists $S \subseteq N$ satisfying $\sum_{r\in S}\delta_r^i(\boldsymbol{l}) > \bar{c}(S)$. From the definition of an mc-game, there exists a coalition $R \subseteq N$ satisfying $R \supseteq S$ and $c(R) = \bar{c}(S)$. From Lemma 5.2, the following holds:

$$\sum_{r\in R}\delta_r^i(\boldsymbol{l}) \geq \sum_{r\in S}\delta_r^i(\boldsymbol{l}) > \bar{c}(S) = c(R).$$

Hence, $\boldsymbol{\delta}^i(\boldsymbol{l}) \notin C(c)$. This contradicts Lemma 5.1. Therefore, $\sum_{r\in S}\delta_r^i(\boldsymbol{l}) \leq \bar{c}(S)$ for any $S \subseteq N$. □

Definition 5.4 *Let $S \subseteq N$ and $\pi \in \Pi_\Gamma$. Let $\boldsymbol{\delta}^i$ be the revised demand operation by player i in Γ. Let $\boldsymbol{\delta}^i$ be the demand operation by player i in Γ for any $i \in S$. Consider the following algorithm which generates $\boldsymbol{\delta}^S(\boldsymbol{y})$ from $\boldsymbol{y}$.*

Step 1: *Set $Q = \emptyset$.*

Step 2: *Choose player $i \in S\backslash Q$ satisfying $\pi(i) \leq \pi(j)$ for any $j \in S\backslash Q$ and update $Q := Q \cup \{i\}$.*

Step 3: *Calculate $\boldsymbol{\delta}^Q(\boldsymbol{y}) = \boldsymbol{\delta}^i(\boldsymbol{\delta}^{Q\backslash\{i\}}(\boldsymbol{y}))$, where we define $\boldsymbol{\delta}^\emptyset(\boldsymbol{y}) = \boldsymbol{y}$ for convenience.*

Step 4: *If $Q = S$, it is terminated. Otherwise, return to* Step 2.

Then the operation $\boldsymbol{\delta}^S$ which associates each allocation $\boldsymbol{y}$ with the vector $\boldsymbol{\delta}^S(\boldsymbol{y})$ is said to be the demand operation by a coalition S in Γ.

Then the following lemma holds.

Lemma 5.3 *If $\boldsymbol{l}$ and $\boldsymbol{\delta}^S$ are the Bird tree allocation and the revised demand operation by a coalition S both associated with Γ respectively, then the following holds:*

$$\boldsymbol{\delta}^S(\boldsymbol{l}) \in C(c), \quad \forall\, S \subseteq N.$$

Proof. From Lemma 5.1, $\boldsymbol{\delta}^S(\boldsymbol{l}) \in C(c)$ holds for any $S \subseteq N$ satisfying $|S| = 1$. Let $T \subseteq N$ satisfying $|T| \geq 2$. Consider $i \in T$ such that $i \prec_\Gamma j$ does not hold for any $j \in T$. It will be shown that $\boldsymbol{\delta}^T(\boldsymbol{l}) \in C(c)$ holds, i.e., $\sum_{r \in Q} \delta_r^T(\boldsymbol{l}) \leq c(Q)$ for any $Q \subseteq N$, if $\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l}) \in C(c)$.

Let $Q \subseteq N$ satisfy $Q \ni i$. Note that $\delta_r^{T\backslash\{i\}}(\boldsymbol{l}) = l_r$ for any $r \in F(i)$ because any $r \in F(i)$ is included in neither $T\backslash\{i\}$ nor $\cup_{j \in T\backslash\{i\}} F(j)$. We have $\delta_r^i(\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l})) = \beta_r \geq l_r = \delta_r^{T\backslash\{i\}}(\boldsymbol{l})$ for any $r \in F(i)$. $\delta_r^i(\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l})) = \delta_r^{T\backslash\{i\}}(\boldsymbol{l})$ for any $r \notin \{i\} \cup F(i)$ also holds. Thus, the following holds:

$$
\begin{aligned}
\sum_{r \in Q} \delta_r^T(\boldsymbol{l}) &= \sum_{r \in Q} \delta_r^i(\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l})) \\
&= c(N) - \sum_{r \in N\backslash Q} \delta_r^i(\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l}) \\
&\leq c(N) - \sum_{r \in N\backslash Q} \delta_r^{T\backslash\{i\}}(\boldsymbol{l}) \\
&= \sum_{r \in Q} \delta_r^{T\backslash\{i\}}(\boldsymbol{l}) \\
&\leq c(Q).
\end{aligned}
$$

Let $Q' \subseteq N$ satisfy $Q \not\ni i$. Note that $\delta_r^i(\boldsymbol{y}) \leq d_r^i(\boldsymbol{y})$ for any pair of $r \neq i$ and $\boldsymbol{y}$. The following holds by using Theorem 5.2:

$$
\sum_{r \in Q} \delta_r^i(\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l})) \leq \sum_{r \in Q} d_r^i(\boldsymbol{\delta}^{T\backslash\{i\}}(\boldsymbol{l})) = \sum_{r \in Q} d_r^T(\boldsymbol{l}) \leq c(Q).
$$

□

Lemma 5.4 *If $\boldsymbol{l}$ and $\boldsymbol{\delta}^i$ are the Bird tree allocation and the revised demand operation by a coalition $S \subseteq N$ both associated with $\boldsymbol{\delta}^i$, then the following holds:*

$$
\delta_r^S(\boldsymbol{l}) \geq 0, \quad \forall\, r \in N.
$$

Proof. From Lemma 5.2, $\delta_r^S(\boldsymbol{l}) \geq 0$ holds for any $r \in N$ when $|S| = 1$.

Let $T \subseteq N$ satisfy $|T| \geq 2$. Consider $i \in T$ such that $i \prec_\Gamma j$ does not hold for any $j \in T$. It is sufficient to show that $\delta_r^T(\boldsymbol{l}) \geq 0$ holds for any $r \in N$ if $\delta_r^{T\backslash\{i\}}(\boldsymbol{l}) \geq 0$ for any $r \in N$.

From the definition, $\delta^i_r(\boldsymbol{y}) \geq 0$ holds for any pair of allocation $\boldsymbol{y}$ and $r \in F(i)$. Hence, we obtain

$$\delta^T_r(\boldsymbol{l}) = \delta^i_r(\boldsymbol{\delta}^{T\setminus\{i\}}(\boldsymbol{l})) \geq 0, \quad \forall\, r \in F(i).$$

Consider the case where $r = i$. Note that $\delta^i_k(\boldsymbol{y}) = \delta^i_k(\boldsymbol{l})$ for any pair of $k \in F(i)$ and an allocation $\boldsymbol{y}$. Note that $\delta^{T\setminus\{i\}}_r(\boldsymbol{l}) = l_r$ for any $r \in F(i)$ because any $r \in F(i)$ is included in neither $T\setminus\{i\}$ nor $\cup_{j\in T\setminus\{i\}}F(j)$. It is also noted that

$$\delta^{T\setminus\{i\}}_i(\boldsymbol{l}) = \begin{cases} \beta_i \geq l_i, & \text{if } p(i) \in T\setminus\{i\}, \\ l_i, & \text{otherwise}. \end{cases}$$

Hence, by using Lemma 5.2, we obtain

$$\begin{aligned} \delta^T_r(\boldsymbol{l}) &= \delta^T_i(\boldsymbol{l}) = \delta^i_i(\boldsymbol{\delta}^{T\setminus\{i\}}(\boldsymbol{l})) \\ &= \delta^{T\setminus\{i\}}_i(\boldsymbol{l}) - \sum_{k\in F(i)} (\delta^i_k(\boldsymbol{\delta}^{T\setminus\{i\}}(\boldsymbol{l})) - \delta^{T\setminus\{i\}}_k(\boldsymbol{l})) \\ &= \delta^{T\setminus\{i\}}_i(\boldsymbol{l}) - \sum_{k\in F(i)} (\delta^i_k(\boldsymbol{l}) - l_k) \\ &\geq l_i - \sum_{k\in F(i)} (\delta^i_k(\boldsymbol{l}) - l_k) = \delta^i_i(\boldsymbol{l}) \geq 0. \end{aligned}$$

For $r \notin F(i)\cup\{i\}$, it is apparent that $\delta^T_r(\boldsymbol{l}) = \delta^i_r(\boldsymbol{\delta}^{T\setminus\{i\}}(\boldsymbol{l})) = \delta^{T\setminus\{i\}}_r(\boldsymbol{l}) \geq 0$. □

Theorem 5.4 *If $\boldsymbol{l}$ and $\boldsymbol{\delta}^S$ are the Bird tree allocation and the revised demand operation by a coalition S both associated with Γ, then the following holds:*

$$\boldsymbol{\delta}^S(\boldsymbol{l}) \in C(\bar{c}), \quad \forall\, S \subseteq N.$$

Proof. Suppose that there exists $T \subseteq N$ satisfying $\sum_{r\in T}\delta^S_r(\boldsymbol{l}) > \bar{c}(T)$. From the definition of an mc-game, there exist $R \subseteq N$ satisfying $R \supseteq T$ and $c(R) = \bar{c}(T)$. From Lemma 5.4, the following holds:

$$\sum_{r\in R}\delta^S_r(\boldsymbol{l}) \geq \sum_{r\in T}\delta^S_r(\boldsymbol{l}) > \bar{c}(T) = c(R).$$

This contradicts Lemma 5.3. □

An example of revised demand operations is given as follows:

Example 5.5 (cf. Example 3) Consider the same network G and the same minimum spanning tree Γ as Example 2. Then the Bird tree allocation for Γ is $\boldsymbol{l} = (2,2,1,1)$. Let π be the same permutation as Example 3. Thus, $\pi \in \Pi_\Gamma$. Let $\boldsymbol{\delta}^S$ be the revised demand operation by $S \subset N$ in Γ. Then $\boldsymbol{\delta}^N(\boldsymbol{l})$ is obtained as follows. In Step 1, set $Q = \emptyset$. Then $\pi(1) = \min\{\pi(j) \mid j \in S \backslash Q\}$, and reset $Q = \{1\}$ in Step 2. $\boldsymbol{\delta}^{\{1\}}(\boldsymbol{l}) = \boldsymbol{\delta}^1(\boldsymbol{\delta}^{\emptyset}(\boldsymbol{l})) = \boldsymbol{\delta}^1(\boldsymbol{l}) = \boldsymbol{\delta}^1((2,2,1,1))$. Since $F(1) = \{2\}$, $\sum_{k \in F(1)} c_k = 5 > c_{p(1)1} + \sum_{k \in F(1)} c_{1k} = 4$ holds. Hence, $\delta_2^1(\boldsymbol{l}) = \beta_2 = 1 \cdot c_{01} + c_{12} = 4$ and $\delta_1^1(\boldsymbol{l}) = l_1 - (\delta_2^1(\boldsymbol{l}) - l_2) = 2 - (4-2) = 0$. We have $\boldsymbol{\delta}^1(\boldsymbol{l}) = (0,4,1,1)$. Since $Q \neq S$, go to Step 2. Since $\pi(2) = \min\{\pi(j) \mid j \in S \backslash Q\}$, set $Q = \{1\} \cup \{2\} = \{1,2\}$. $\boldsymbol{\delta}^{\{1,2\}}(\boldsymbol{l}) = \boldsymbol{\delta}^2(\boldsymbol{\delta}^1(\boldsymbol{l})) = \boldsymbol{\delta}^2((0,4,1,1))$. Since $F(2) = \{4\}$, $\sum_{k \in F(i)} c_k = c_4 = 4 > c_{p(i)i} + \sum_{k \in F(i)} c_{ik} = c_{12} + c_{24} = 3$ holds. Thus, $\delta_4^2((0,4,1,1)) = \beta_4 = 1 \cdot c_{12} + c_{24} = 3$ and $\delta_2^2((0,4,1,1)) = 4 - (\delta_4^2((0,4,1,1)) - 1) = 4 - (3-1) = 2$. Hence, we have $\boldsymbol{\delta}^{\{1,2\}}(\boldsymbol{l}) = \boldsymbol{\delta}^2((0,4,1,1)) = (0,2,1,3)$. Since $F(3) = F(4) = 0$, we have $\boldsymbol{\delta}^N(\boldsymbol{l}) = \boldsymbol{\delta}^4(\boldsymbol{\delta}^3(\boldsymbol{\delta}^2(\boldsymbol{\delta}^1(\boldsymbol{l})))) = \boldsymbol{\delta}^4(\boldsymbol{\delta}^3((0,2,1,3))) = (0,2,1,3)$. As can be easily seen, $(0,2,1,3) \in C(\bar{c})$

5.4 Concluding Remarks

We have defined the demand operation by a coalition by modifying the original w.d.o. by a coalition so that it is well-defined. We have confirmed that any allocation obtained through the demand operation on a Bird tree allocation is always included in the core of the corresponding mst-game. However, it is not always included in the core of the corresponding mc-game.

Furthermore, we have defined a revised demand operation. It has been shown that allocations obtained through the revised demand operations both by a player and by a coalition on a Bird tree allocation are elements of both cores of the corresponding mst-game and mc-game. We have also shown that any elements of allocations obtained through the revised demand operations both by a player and by a coalition on a Bird tree allocation are nonnegative. It means that the revised demand operation is more acceptable than the demand operation.

Chapter 6

Conclusion

In this thesis, we have introduced solution concepts and investigated their rational properties in both cooperative fuzzy games (fuzzy games) and minimum spanning tree games. We have given realistic and appropriate decision making by discussing those solution concepts.

Chapter 3 has dealt with solution concepts in fuzzy games in general. We have defined a fuzzy imputation function, a fuzzy imputation set-valued function and an FPMAS as basic concepts in fuzzy games. We have introduced the Shapley function, the core function and the dominance core function, which are applicable to any class of fuzzy games. Properties of the latter two functions have been discussed. Furthermore, we have introduced balancedness in fuzzy games and shown that the core is nonempty if and only if a given fuzzy game is balanced, since the core is not always nonempty as in crisp games.

In Chapter 4, in order to obtain an explicit form of the Shapley function, we have defined a new class of fuzzy games and discussed solution concepts in the class. This class has some good properties, i.e., continuity and monotonicity with respect to rates of players' participation. Properties of concepts related to the Shapley function in this class has been discussed. We have given an explicit form of the Shapley function on the class. We have also discussed its properties which cannot be obtained in general and which are special for the proposed class. It has been shown that an imputation obtained through this explicit function coincides with the center of gravity of a set of imputations obtained through the core function, if a given fuzzy game in this class is convex. Furthermore, an illustrative example has been given.

Chapter 5 has dealt with solution concepts in minimum spanning tree games. We have defined the demand operation by a coalition by modifying the original weak demand operation by a coalition so that it is well-defined. It has been confirmed that any allocation obtained through the demand operation on a Bird tree allocation is always included in the core of the corresponding minimum spanning tree game. However, it is not always an element of the core of the corresponding monotone cover game. We have defined a revised demand operation. It has been shown that allocations obtained through the revised demand operations on a Bird tree allocation are elements of both cores of the corresponding mst-game and mc-game. We have proved that any elements of allocations obtained through the revised demand operations on a Bird tree allocation are nonnegative. It means that the revised demand operation is more acceptable than the demand operation.

The following topics need further research:

- Several solution concepts which are considered important in crisp games should be introduced.
- The uniqueness of the Shapley function in the proposed class of fuzzy games should be proven.
- The revised demand operation includes parameters. Those parameters effect on the obtained allocation. The rational determination of those parameters should be discussed.

We hope that this thesis makes an impotant contribution to the progress of the field of cooperative fuzzy games and minimum spanning tree games and gives a guide for an appropriate decision making in cooperation.

Bibliography

[1] J.-P. Aubin, "Coeur et valeur des jeux flous à paiements latéraux," *Comptes Rendus Hebdomadaires des Séances de 1'Académie des Sciences*, vol. 279-A, pp.891–894, 1974.

[2] J.-P. Aubin, "Coeur et équilibres des jeux flous sans paiements latéraux," *Comptes Rendus Hebdomadaires des Séances de 1'Académie des Sciences*, vol. 279-A, pp.963-966, 1974.

[3] J.-P. Aubin, *Mathematical Methods of Game and Economic Theory*, Rev. ed., North-Holland Publishing Company, Amsterdam, 1982.

[4] R.J. Aumann and L.S. Shapley, *Values of Non-Atomic Games*, Princeton University Press, Princeton, 1974.

[5] A. Billot, "Fuzzy convexity and peripherial core of an exchange economy represented as a fuzzy game," in: J. Kacprzyk, M. Fedrizzi (Eds.), *Multiperson Decision Making Models using Fuzzy Sets and Possibility Theory*, Kluwer Academic Publishers, Dordrecht, pp. 311-335, 1990.

[6] C.G. Bird, "On cost allocation for a spanning tree: a game theoretic approach," *Networks*, vol. 6, pp. 335-350, 1976.

[7] D. Butnariu, "Stability and Shapley value for an n-persons fuzzy game," *Fuzzy Sets and Systems*, vol. 4, pp. 63-72, 1980.

[8] G. Choquet, "Theory of capacities," *Annales de I'institut Fourier*, vol. 5, pp.131-295, 1953.

[9] A. Claus and D.J. Kleitman, "Cost allocation for a spanning tree," *Networks*, vol. 3, pp. 289-304, 1973.

[10] I. Curiel, *Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems*, Kluwer Academic Publishers, Netherlands, 1997.

[11] V. Feltkamp, S.H. Tijs and S. Muto, "Bird's tree allocations revisited," *CentER Discussion Paper 9435*, Tilburg University, Tilburg, The Netherlands, 1994.

[12] M. Grabisch, T. Murofushi and M. Sugeno, "Fuzzy measure of fuzzy events defined by fuzzy integrals," *Fuzzy Sets and Systems*, vol. 50, pp. 293-313, 1992.

[13] D. Granot and G. Huberman, "Minimum cost spanning tree games," *Mathematical Programming*, vol. 21, pp. 1-18, 1981.

[14] D. Granot and G. Huberman, "On the core and nucleolus of minimum cost spanning tree games," *Mathematical Programming*, vol. 29, pp. 323-347, 1984.

[15] C.-R. Hsiao and T.E.S. Raghavan, "Shapley value for multi-choice cooperative games, I," *Games and Economic Behavior*, vol. 5, no. 2, pp. 240-256, 1993.

[16] C.-R. Hsiao, "A value for continuously-many-choice cooperative games," *International Journal of Game Theory*, vol. 24, pp. 273-292, 1995.

[17] J.B. Kruskal, "On the shortest spanning subtree of a graph and the travelling salesman problem," *Proceedings of the American Mathematical Society*, vol. 7, pp. 48-50, 1956.

[18] A. van den Nouweland, Chapter 9: Multi-choice games, in: "Games and Graphs in Economic Situations," *PhD Dissertation Tilburg University*, The Netherlands, pp. 119-148, 1993.

[19] A. van den Nouweland, S. Tijs, J. Potters and J. Zarzuelo, "Cores and related solution concepts for multi-choice games," *ZOR - Mathematical Methods of Operations Research*, vol. 41, no. 3, pp. 289-311, 1995.

[20] R.C. Prim, "Shortest connection networks and some generalizations," *Bell Systems Technical Journal*, vol. 36, pp. 1389-1401, 1957.

[21] M.L. Puri and D.A. Ralescu, "Fuzzy random variables," *Journal of Mathematical Analysis and Applications*, vol. 114, pp. 409-422, 1986.

[22] L.S. Shapley, "A value for n-person games," *Annals of Mathematics Studies*, vol. 28, pp. 307-318, 1953.

[23] L.S. Shapley, "On balanced sets and cores," *Naval Research Logistics Quarterly*, vol. 14, pp. 453-460, 1967.

[24] L.S. Shapley, "Cores of convex games," *International Journal of Game Theory*, vol. 1, no. 1, pp. 11-26, 1971.

[25] Y. Sprumont, "Population monotonic allocation schemes for cooperative games with transferable utility," *Games and Economic Behavior*, vol. 2, pp. 378-394, 1990.

[26] M. Sugeno and T. Murofushi, "Choquet's integral as an integral form for a general class of fuzzy measures," *Preprints of Second IFSA Congress*, vol. 1, pp. 408-411, 1987.

[27] J. von Neumann, "Zur Theorie der Gesellschaftsspiele," *Mathematische Annalen*, vol. 100, pp. 295-320, 1928.

[28] J. von Neumann, "On the theory of games of strategy," in: A.W. Tucker and R.D. Luce (Eds.), *Contibutions to the Theory of Games* IV, Annals of Mathematics Studies Series, vol. 40, pp. 13-42, Princeton University Press, Princeton, 1957.

[29] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, 3rd ed., Princeton University Press, Princeton, 1953.

A List of the Author's Works

I. Transactions

1. M. Tsurumi, T. Minamiura, T. Tanino and M. Inuiguchi, "Demand operations in minimum spanning tree games," *Game Theory and Applications*, vol. 5, 1999.
2. M. Tsurumi, T. Tanino and M. Inuiguchi, "The Shapley function on a class of cooperative fuzzy games," *European Journal of Operational Research* (to appear).
3. M. Tsurumi, T. Tanino and M. Inuiguchi, "The core and the related solution concepts in cooperative fuzzy games," *Journal of Japan Society for Fuzzy Theory and Systems* (to appear).

II. International Conferences

1. M. Tsurumi, T. Tanino and M. Inuiguchi, "A solution concept in cooperative fuzzy games," *Proceedings of the Third Asian Fuzzy Systems Symposium*, pp. 669-673, 1998.
2. M. Tsurumi, T. Tanino and M. Inuiguchi, "solution concepts in cooperative fuzzy games," *Proceedings of 1999 IEEE, Systems, Man, Cybernetics Conference*, pp. III 22-27, 1999.

III. Others

1. M. Tsurumi, S. Shiode, T. Tanino and H. Ishii, "A bargaining set with partial information in a political situation," VIII *International Symposium on Applied Stochastic Models and Data Analysis*

– The Ins and Outs of Solving Real Problems –, *A Conference of the Quantitative Methods in Business and Industry Society, Contributed Papers,* pp. 425–430, 1997.

2. M. Tsurumi, T. Tanino and M. Inuiguchi, "Solution concepts in bargaining," *Nonlinear Analysis and Convex Analysis — Proceedings of the International Conference*, pp. 360–367, 1999.