A Variational Problem for Affine Connections

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Abstract. We give a variational problem for affine connections which characterizes the Riemannian connection of an Einstein metric of negative scalar curvature.

1. Introduction

Let $\mathcal{A}(M)$ denote the sapce of all torsion free affine connections of a compact connected manifold M, each of which preserves a volume element of M. Every affine connections is projectively equivalent to such a connection. For $\nabla \in \mathcal{A}(M)$ we can define the Ricci curvature tensor Ric. Since ∇ preserves a volume element, Ric is a symmetric 2-tensor. Moreover using the volume element we can define the determinant $\det \mathrm{Ric}$ of the Ricci tensor. We put for $\nabla \in \mathcal{A}(M)$

$$E(\nabla) = \int_M \det \mathrm{Ric}\, d\mu \int_M d\mu.$$

It is easy to see that the right hand side is independent of the choice of a volume element $d\mu$ which is preserved by ∇. Moreover $E(\nabla)$ depends on ∇ differentiably. This functional may be compared with the normalized Einstein-Hilbert functional of the total scalar curvature of a Riemannian metric ([1]). In this paper we will show the following:

Theorem. *Let M be a compact connected manifold. Suppose $\nabla \in \mathcal{A}(M)$ is a critical point of E, and its Ricci curvature is negative semidefinite and negative definite somewhere. Then ∇ is the Riemannian connection of an Einstein metric of M.*

Remark. A Ricci flat connection $\nabla \in \mathcal{A}(M)$ is a critical point of the functional E. It is known [2; p. 211] that there is a Ricci flat affine connection which is not a Riemannian connection.

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2. Projective equivalence of affine connections

Given two affine connections ∇ and $\tilde{\nabla}$ on M, we say that they are *projectively equivalent* if geodesics ignoring their parameters are the same for ∇ and $\tilde{\nabla}$. That is to say, there is a function $\lambda: TM \to \mathbf{R}$ such that

$$\tilde{\nabla}_X X = \nabla_X X + 2\lambda(X)X \tag{2.1}$$

for $X \in TM$. Then it is immediate to see that λ is linear if we put $\lambda(0) = 0$, and thus λ is a smooth 1-form of M.

We will see that any affine connection of M is projectively equivalent to some connection in $\mathcal{A}(M)$. Let ∇ be an arbitrary affine connection of M. Then an affine connection ∇' defined as $\nabla'_X Y = \frac{1}{2}(\nabla_X Y + \nabla_Y X + [X, Y])$ is a torsion free connection which is projectively equivalent to ∇. So we assume ∇ is torsion free. Let $\tilde{\nabla}$ be another torsion free connection which is projectively equivalent to ∇. Then from (2.1),

$$\tilde{\nabla}_X Y = \nabla_X Y + \lambda(X)Y + \lambda(Y)X$$

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for some 1-form λ. Take a volume element $d\mu$ and $u \in C^\infty(M)$, we then have

$$\tilde{\nabla}_X(e^{(n+1)u}d\mu) = e^{(n+1)u}\nabla_X d\mu + (n+1)e^{(n+1)u}(du - \lambda)(X)d\mu, \tag{2.2}$$

where $n = \dim M$. Hence if we choose $u = 0$ and λ to be such that $(n+1)\lambda(X)d\mu = \nabla_X d\mu$, we have $\tilde{\nabla} d\mu = 0$, i.e., $\tilde{\nabla} \in \mathcal{A}(M)$.

The formula (2.2) says more and we have the following.

Lemma 2.1. *Suppose that ∇, $\tilde{\nabla} \in \mathcal{A}(M)$ are projectively equivalent. Then there is a $u \in C^\infty(M)$ such that*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xu)Y + (Yu)X.$$

Moreover if $\nabla d\mu = 0$, then $\tilde{\nabla}(e^{(n+1)u}d\mu) = 0$.

Corollary 2.2. *Suppose ∇, $\tilde{\nabla} \in \mathcal{A}(M)$. Then we have*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xu)Y + (Yu)X + S(X,Y), \tag{2.3}$$

where $u \in C^\infty(M)$ and $S = (S^i_{jk})$ is a $(1,2)$-tensor such that $S^i_{jk} = S^i_{kj}$ and $S^i_{ji} = 0$. The tensor S is uniquely determined by ∇ and $\tilde{\nabla}$, and u is determined up to a constant.

Proof. Let $T = \tilde{\nabla} - \nabla$ be the difference of the two connections. Put $S^i_{jk} = T^i_{jk} - \frac{1}{n+1}(T^l_{jl}\delta^i_k + T^l_{kl}\delta^i_j)$. Then $\nabla + S \in \mathcal{A}(M)$ is projectively equivalent to $\tilde{\nabla}$. ▯

Corollary 2.3. *Denote by $\mathcal{V}(M)$ the space of smooth volume elements of M. Then we have a smooth map $\varphi\colon \mathcal{A}(M) \to \mathcal{V}(M)$ such that $\nabla\varphi(\nabla) = 0$.*

Proof. Fix $\nabla \in \mathcal{A}(M)$ and $d\mu \in \mathcal{V}(M)$ such that $\nabla d\mu = 0$. Fix a point $x \in M$. Then any $\tilde{\nabla} \in \mathcal{A}(M)$ is given as in (2.3), and we can determine a unique u on condition that $u(x) = 0$. Then put $\varphi(\tilde{\nabla}) = e^{(n+1)u}d\mu$. ▯

3. The variational formula for the functional E

First we note that the Ricci curvature of $\nabla \in \mathcal{A}(M)$ is a symmetric 2-tensor.

Let $\omega \in \Lambda^n T^*_x M \setminus \{0\}$ be a volume form at $x \in M$. Define $\omega^* \in \Lambda^n T_x M$ by $\omega(\omega^*) = 1$. Then we have $(a\omega)^* = \frac{1}{a}\omega^*$ for $a \in \mathbf{R} \setminus \{0\}$. Thinking of the Ricci tensor as $\mathrm{Ric}\colon T_xM \to T^*_xM$, we define $\det_\omega \mathrm{Ric}$ as

$$\mathrm{Ric}^*\omega^* = (\det_\omega \mathrm{Ric})\omega.$$

For $a \in \mathbf{R} \setminus \{0\}$, we have $\det_{a\omega} \mathrm{Ric} = \frac{1}{a^2} \det_\omega \mathrm{Ric}$. In particular $\det_{-\omega} \mathrm{Ric} = \det_\omega \mathrm{Ric}$. Hence we can define $\det \mathrm{Ric} = \det_{d\mu} \mathrm{Ric}$ with respect to a volume element $d\mu$.

Now suppose $\nabla d\mu = \nabla d\mu'$. Then $d\mu' = a d\mu$ for some positive constant a since M is connected. So we have $\int_M \det_{d\mu'} \mathrm{Ric}\, d\mu' = \frac{1}{a}\int_M \det_{d\mu} \mathrm{Ric}\, d\mu$. Thus $E(\nabla) = \int_M \det \mathrm{Ric}\, d\mu \int_M d\mu$ depends only on $\nabla \in \mathcal{A}(M)$, and $E\colon \mathcal{A}(M) \to \mathbf{R}$ is differentiable by virture of Corollary 2.3.

We define a contravariant symetric 2-tensor $\hat{\mathrm{Ric}} = (\hat{R}^{ij})$ as the cofactor tensor of $\mathrm{Ric} = (R_{ij})$. Namely

$$\hat{R}^{ik}R_{kj} = (\det \mathrm{Ric})\delta^i_j. \tag{3.1}$$

We remark that this tensor field depends on the choice of a volume element $d\mu$. We also define ρ and ρ° as

$$\rho = \det \mathrm{Ric}$$

and

$$\rho^\circ = \det \mathrm{Ric} - \frac{\int_M \det \mathrm{Ric}\, d\mu}{\int_M d\mu}.$$

With these notations we can state the first variational formula of the functional E.

Proposition 3.1. *$\nabla \in \mathcal{A}(M)$ is a critical point of $E\colon \mathcal{A}(M) \to \mathbf{R}$ if and only if the following two conditions are satisfied:*

$$\hat{R}^{ij}{}_{;ij} = -\frac{n+1}{n-1}\rho^\circ \tag{3.2}$$

and

$$\hat{R}^{ij}{}_{;k} = \frac{1}{n+1}(\hat{R}^{il}{}_{;l}\delta^j_k + \hat{R}^{lj}{}_{;l}\delta^i_k), \tag{3.3}$$

where $n = \dim M$.

Proof. Fix $\nabla \in \mathcal{A}(M)$ with $\nabla d\mu = 0$. We first consider a projective variation of ∇. Let ${}^t\nabla$ be defined as

$${}^t\nabla_X Y = \nabla_X Y + t(Xu)Y + t(Yu)X,$$

where $u \in C^\infty(M)$ is an arbitrary function. Put

$$d\mu(t) = e^{(n+1)tu} d\mu,$$

and we have ${}^t\nabla d\mu(t) = 0$. The Ricci curvature is calculated as

$$R_{ij}(t) = R_{ij} - (n-1)(tu_{;ij} - t^2 u_{;i}u_{;j}).$$

Since

$$\det{}_{d\mu(t)}\mathrm{Ric}(t)d\mu(t) = e^{-(n+1)tu}\det{}_{d\mu}\mathrm{Ric}(t)d\mu,$$

we have

$$\begin{aligned}\frac{d}{dt}_{|t=0}\det{}_{d\mu(t)}\mathrm{Ric}(t)d\mu(t) &= -(n+1)u\rho d\mu + \hat{R}^{ij}\frac{d}{dt}_{|t=0}R_{ij}(t)d\mu \\ &= -(n+1)u\rho d\mu - (n-1)\hat{R}^{ij}u_{;ij}d\mu.\end{aligned}$$

Hence,

$$\frac{d}{dt}_{|t=0}\int_M \det{}_{d\mu(t)}\mathrm{Ric}(t)d\mu(t) = \int_M u(-(n+1)\rho - (n-1)\hat{R}^{ij}{}_{;ij})d\mu.$$

On the other hand,

$$\frac{d}{dt}_{|t=0}\int_M d\mu(t) = (n+1)\int_M u\, d\mu.$$

Therefore we get

$$\frac{d}{dt}_{|t=0}E({}^t\nabla) = \int_M u(-(n+1)\rho^\circ) - (n-1)\hat{R}^{ij}{}_{;ij})d\mu \int_M d\mu.$$

The equation (3.2) follows from this formula.

From Corollary 2.2, we have only to check the variation of ∇ in the direction of $S = (S^i_{jk})$ with $S^i_{jk} = S^i_{kj}$ and $S^i_{ji} = 0$. Now we put

$${}^t\nabla_X Y = \nabla_X Y + tS(X,Y).$$

We have ${}^t\nabla d\mu = 0$ for any t. The Ricci curvature is then

$$R_{ij}(t) = R_{ij} + tS^k_{ij;k} - t^2 S^k_{il}S^l_{jk}.$$

Hence we have

$$\begin{aligned}\frac{d}{dt}_{|t=0}E({}^t\nabla) &= \int_M \frac{d}{dt}_{|t=0}(\det\mathrm{Ric}(t))d\mu \int_M d\mu \\ &= \int_M \hat{R}^{ij}S^k_{ij;k}d\mu \int_M d\mu \\ &= -\int_M \hat{R}^{ij}{}_{;k}S^k_{ij}d\mu \int_M d\mu \\ &= -\int_M (\hat{R}^{ij}{}_{;k} - \frac{1}{n+1}(\hat{R}^{il}{}_{;l}\delta^j_k + \hat{R}^{lj}{}_{;l}\delta^i_k))S^k_{ij}d\mu \int_M d\mu\end{aligned}$$

The equation (3.3) follows from this formula. □

The equations (3.2) and (3.3) are not independent. In fact we can show the following.

Proposition 3.2. *On a connected manifold, the equation* (3.3) *implies the equation* (3.2).

Proof. The derivative of the determinant ρ is given as $\rho_{;k} = \hat{R}^{ij} R_{ij;k}$. This together with (3.1) yields

$$(n-1)\rho_{;k} = \hat{R}^{ij}{}_{;k} R_{ij}. \tag{3.4}$$

Applying this to (3.3) we have

$$\frac{n^2-1}{2}\rho_{;k} = \hat{R}^{ij}{}_{;i} R_{jk}. \tag{3.5}$$

Then we get

$$\hat{R}^{ij}{}_{;ijk} - \hat{R}^{ij}{}_{;ikj} = \hat{R}^{il}{}_{;i} R^{j}{}_{lkj} = -\hat{R}^{il}{}_{;i} R_{lk} = -\frac{n^2-1}{2}\rho_{;k}. \tag{3.6}$$

On the other hand we have

$$\begin{aligned}
\hat{R}^{ij}{}_{;ikj} - \hat{R}^{ij}{}_{;kij} &= (\hat{R}^{lj} R^{i}{}_{lki} + \hat{R}^{il} R^{j}{}_{lki})_{;j} \\
&= (-\hat{R}^{lj} R_{lk} + \hat{R}^{il} R^{j}{}_{lki})_{;j} \\
&= -\rho_{;k} + \hat{R}^{il}{}_{;j} R^{j}{}_{lki} + \hat{R}^{il} R^{j}{}_{lki;j} \\
&= -\rho_{;k} + \frac{1}{n+1}(\hat{R}^{im}{}_{;m}\delta^{l}_{j} + \hat{R}^{ml}{}_{;m}\delta^{i}_{j}) R^{j}{}_{lki} - \hat{R}^{il}(R^{j}{}_{lij;k} + R^{j}{}_{ljk;i}) \\
&= -\rho_{;k} - \frac{1}{n+1}\hat{R}^{ml}{}_{;m} R_{lk} + \hat{R}^{il} R_{li;k} - \hat{R}^{il} R_{lk;i} \\
&= -\rho_{;k} - \frac{n-1}{2}\rho_{;k} + \rho_{;k} - (\hat{R}^{il} R_{lk})_{;i} + \hat{R}^{il}{}_{;i} R_{lk} \\
&= \frac{(n+1)(n-2)}{2}\rho_{;k},
\end{aligned} \tag{3.7}$$

where in the fourth equality we used the equation (3.3) and the second Bianchi identity. Hence from (3.6) and (3.7) we have

$$\hat{R}^{ij}{}_{;ijk} - \hat{R}^{ij}{}_{;kij} = -\frac{n+1}{2}\rho_{;k}. \tag{3.8}$$

From (3.3) and (3.6) we have

$$\hat{R}^{ij}{}_{;kij} = \frac{1}{n+1}(\hat{R}^{il}{}_{;lik} + \hat{R}^{lj}{}_{;lkj}) = \frac{2}{n+1}\hat{R}^{ij}{}_{;ijk} + \frac{n-1}{2}\rho_{;k}. \tag{3.9}$$

Then it is easy to see from (3.8) and (3.9) that

$$\hat{R}^{ij}{}_{;ijk} = -\frac{n+1}{n-1}\rho_{;k},$$

which implies the condition (3.2). □

4. Proof of Theorem

It follows from (3.5) and (3.1) that

$$\rho_{;i}\hat{R}^{ij} = \frac{2}{n^2-1}\rho\hat{R}^{ij}{}_{;i}. \tag{4.1}$$

Hence using (3.2), we have

$$\rho_{;ij}\hat{R}^{ij} + \frac{n^2-3}{n^2-1}\rho_{;i}\hat{R}^{ij}{}_{;j} = -\frac{2}{(n-1)^2}\rho\rho^{\circ}. \tag{4.2}$$

The proof is divided into two cases. One is the case when $n = \dim M$ is even. Then since $\mathrm{Ric} \leq 0$, we have $\rho \geq 0$ and $\hat{\mathrm{Ric}} \leq 0$. At a point where ρ° and therefore ρ too take their maximums, the left hand side of (4.2) is nonnegative because $\hat{\mathrm{Ric}} \leq 0$, and the right hand side of (4.2) is nonpositive. Hence we conclude that $\rho^{\circ} = 0$, that is, ρ is a constant, which must be positive.

The other is the case when n is odd. Then since $\mathrm{Ric} \leq 0$, we have $\rho \leq 0$ and $\hat{\mathrm{Ric}} \geq 0$. At a point where ρ takes its minimum, the left hand side of (4.2) is nonnegative, because $\hat{\mathrm{Ric}} \geq 0$, and the right hand side of (4.2) is nonpostive. Hence we have $\rho^{\circ} = 0$, that is, ρ is constant.

In both cases we have that ρ is a nonzero constant. Then it follows from (4.1) that $\hat{R}^{ij}{}_{;j} = 0$. Hence from (3.3), we have $\nabla\hat{\mathrm{Ric}} = 0$. This implies $\nabla\mathrm{Ric} = 0$ because $\rho \neq 0$. Now put $g = -\mathrm{Ric}$, which is a Riemannian metric, and is parallel with respect to ∇. Therefore ∇ is the Riemannian connection of g.

References

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