



Title	A variational problem for Affine Connections
Author(s)	Kobayashi, Osamu
Citation	Archiv der Mathematik. 2006, 86(5), p. 464-469
Version Type	VoR
URL	<a href="https://hdl.handle.net/11094/26340">https://hdl.handle.net/11094/26340</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

# A Variational Problem for Affine Connections

Osamu Kobayashi

Abstract. We give a variational problem for affine connections which characterizes the Riemannian connection of an Einstein metric of negative scalar curvature.

## 1. Introduction

Let  $\mathcal{A}(M)$  denote the sapce of all torsion free affine connections of a compact connected manifold  $M$ , each of which preserves a volume element of  $M$ . Every affine connections is projectively equivalent to such a connection. For  $\nabla \in \mathcal{A}(M)$  we can define the Ricci curvature tensor  $\text{Ric}$ . Since  $\nabla$  preserves a volume element,  $\text{Ric}$  is a symmetric 2-tensor. Moreover using the volume element we can define the determinant  $\det \text{Ric}$  of the Ricci tensor. We put for  $\nabla \in \mathcal{A}(M)$

$$E(\nabla) = \int_M \det \text{Ric} d\mu \int_M d\mu.$$

It is easy to see that the right hand side is independent of the choice of a volume element  $d\mu$  which is preserved by  $\nabla$ . Moreover  $E(\nabla)$  depends on  $\nabla$  differentiably. This functional may be compared with the normalized Einstein-Hilbert functional of the total scalar curvature of a Riemannian metric ([1]). In this paper we will show the following:

**Theorem.** *Let  $M$  be a compact connected manifold. Suppose  $\nabla \in \mathcal{A}(M)$  is a critical point of  $E$ , and its Ricci curvature is negative semidefinite and negative definite somewhere. Then  $\nabla$  is the Riemannian connection of an Einstein metric of  $M$ .*

**Remark.** A Ricci flat connection  $\nabla \in \mathcal{A}(M)$  is a critical point of the functional  $E$ . It is known [2; p. 211] that there is a Ricci flat affine connection which is not a Riemannian connection.

I would like to thank the referee for useful comments on the paper.

## 2. Projective equivalence of affine connections

Given two affine connections  $\nabla$  and  $\tilde{\nabla}$  on  $M$ , we say that they are *projectively equivalent* if geodesics ignoring their parameters are the same for  $\nabla$  and  $\tilde{\nabla}$ . That is to say, there is a function  $\lambda: TM \rightarrow \mathbf{R}$  such that

$$\tilde{\nabla}_X X = \nabla_X X + 2\lambda(X)X \tag{2.1}$$

for  $X \in TM$ . Then it is immediate to see that  $\lambda$  is linear if we put  $\lambda(0) = 0$ , and thus  $\lambda$  is a smooth 1-form of  $M$ .

We will see that any affine connection of  $M$  is projectively equivalent to some connection in  $\mathcal{A}(M)$ . Let  $\nabla$  be an arbitrary affine connection of  $M$ . Then an affine connection  $\nabla'$  defined as  $\nabla'_X Y = \frac{1}{2}(\nabla_X Y + \nabla_Y X + [X, Y])$  is a torsion free connection which is projectively equivalent to  $\nabla$ . So we assume  $\nabla$  is torsion free. Let  $\tilde{\nabla}$  be another torsion free connection which is projectively equivalent to  $\nabla$ . Then from (2.1),

$$\tilde{\nabla}_X Y = \nabla_X Y + \lambda(X)Y + \lambda(Y)X$$

for some 1-form  $\lambda$ . Take a volume element  $d\mu$  and  $u \in C^\infty(M)$ , we then have

$$\tilde{\nabla}_X(e^{(n+1)u}d\mu) = e^{(n+1)u}\nabla_X d\mu + (n+1)e^{(n+1)u}(du - \lambda)(X)d\mu, \quad (2.2)$$

where  $n = \dim M$ . Hence if we choose  $u = 0$  and  $\lambda$  to be such that  $(n+1)\lambda(X)d\mu = \nabla_X d\mu$ , we have  $\tilde{\nabla}d\mu = 0$ , i.e.,  $\tilde{\nabla} \in \mathcal{A}(M)$ .

The formula (2.2) says more and we have the following.

**Lemma 2.1.** *Suppose that  $\nabla, \tilde{\nabla} \in \mathcal{A}(M)$  are projectively equivalent. Then there is a  $u \in C^\infty(M)$  such that*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xu)Y + (Yu)X.$$

Moreover if  $\nabla d\mu = 0$ , then  $\tilde{\nabla}(e^{(n+1)u}d\mu) = 0$ .

**Corollary 2.2.** *Suppose  $\nabla, \tilde{\nabla} \in \mathcal{A}(M)$ . Then we have*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xu)Y + (Yu)X + S(X, Y), \quad (2.3)$$

where  $u \in C^\infty(M)$  and  $S = (S_{jk}^i)$  is a  $(1, 2)$ -tensor such that  $S_{jk}^i = S_{kj}^i$  and  $S_{ji}^i = 0$ . The tensor  $S$  is uniquely determined by  $\nabla$  and  $\tilde{\nabla}$ , and  $u$  is determined up to a constant.

**Proof.** Let  $T = \tilde{\nabla} - \nabla$  be the difference of the two connections. Put  $S_{jk}^i = T_{jk}^i - \frac{1}{n+1}(T_{jl}^l \delta_k^i + T_{kl}^l \delta_j^i)$ . Then  $\nabla + S \in \mathcal{A}(M)$  is projectively equivalent to  $\tilde{\nabla}$ .  $\square$

**Corollary 2.3.** *Denote by  $\mathcal{V}(M)$  the space of smooth volume elements of  $M$ . Then we have a smooth map  $\varphi: \mathcal{A}(M) \rightarrow \mathcal{V}(M)$  such that  $\nabla \varphi(\nabla) = 0$ .*

**Proof.** Fix  $\nabla \in \mathcal{A}(M)$  and  $d\mu \in \mathcal{V}(M)$  such that  $\nabla d\mu = 0$ . Fix a point  $x \in M$ . Then any  $\tilde{\nabla} \in \mathcal{A}(M)$  is given as in (2.3), and we can determine a unique  $u$  on condition that  $u(x) = 0$ . Then put  $\varphi(\tilde{\nabla}) = e^{(n+1)u}d\mu$ .  $\square$

### 3. The variational formula for the functional $E$

First we note that the Ricci curvature of  $\nabla \in \mathcal{A}(M)$  is a symmetric 2-tensor.

Let  $\omega \in \Lambda^n T_x^* M \setminus \{0\}$  be a volume form at  $x \in M$ . Define  $\omega^* \in \Lambda^n T_x M$  by  $\omega(\omega^*) = 1$ . Then we have  $(a\omega)^* = \frac{1}{a}\omega^*$  for  $a \in \mathbf{R} \setminus \{0\}$ . Thinking of the Ricci tensor as  $\text{Ric}: T_x M \rightarrow T_x^* M$ , we define  $\det_\omega \text{Ric}$  as

$$\text{Ric}^* \omega^* = (\det_\omega \text{Ric})\omega.$$

For  $a \in \mathbf{R} \setminus \{0\}$ , we have  $\det_{a\omega} \text{Ric} = \frac{1}{a^n} \det_\omega \text{Ric}$ . In particular  $\det_{-\omega} \text{Ric} = \det_\omega \text{Ric}$ . Hence we can define  $\det \text{Ric} = \det_{d\mu} \text{Ric}$  with respect to a volume element  $d\mu$ .

Now suppose  $\nabla d\mu = \nabla d\mu'$ . Then  $d\mu' = ad\mu$  for some positive constant  $a$  since  $M$  is connected. So we have  $\int_M \det_{d\mu'} \text{Ric} d\mu' = \frac{1}{a} \int_M \det_{d\mu} \text{Ric} d\mu$ . Thus  $E(\nabla) = \int_M \det \text{Ric} d\mu \int_M d\mu$  depends only on  $\nabla \in \mathcal{A}(M)$ , and  $E: \mathcal{A}(M) \rightarrow \mathbf{R}$  is differentiable by virtue of Corollary 2.3.

We define a contravariant symmetric 2-tensor  $\hat{\text{Ric}} = (\hat{R}^{ij})$  as the cofactor tensor of  $\text{Ric} = (R_{ij})$ . Namely

$$\hat{R}^{ik} R_{kj} = (\det \text{Ric}) \delta_j^i. \quad (3.1)$$

We remark that this tensor field depends on the choice of a volume element  $d\mu$ . We also define  $\rho$  and  $\rho^\circ$  as

$$\rho = \det \text{Ric}$$

and

$$\rho^\circ = \det \text{Ric} - \frac{\int_M \det \text{Ric} d\mu}{\int_M d\mu}.$$

With these notations we can state the first variational formula of the functional  $E$ .

**Proposition 3.1.**  $\nabla \in \mathcal{A}(M)$  is a critical point of  $E: \mathcal{A}(M) \rightarrow \mathbf{R}$  if and only if the following two conditions are satisfied:

$$\hat{R}^{ij}{}_{;ij} = -\frac{n+1}{n-1}\rho^\circ \quad (3.2)$$

and

$$\hat{R}^{ij}{}_{;k} = \frac{1}{n+1}(\hat{R}^{il}{}_{;l}\delta_k^j + \hat{R}^{lj}{}_{;l}\delta_k^i), \quad (3.3)$$

where  $n = \dim M$ .

**Proof.** Fix  $\nabla \in \mathcal{A}(M)$  with  $\nabla d\mu = 0$ . We first consider a projective variation of  $\nabla$ . Let  ${}^t\nabla$  be defined as

$${}^t\nabla_X Y = \nabla_X Y + t(Xu)Y + t(Yu)X,$$

where  $u \in C^\infty(M)$  is an arbitrary function. Put

$$d\mu(t) = e^{(n+1)tu} d\mu,$$

and we have  ${}^t\nabla d\mu(t) = 0$ . The Ricci curvature is calculated as

$$R_{ij}(t) = R_{ij} - (n-1)(tu_{;ij} - t^2 u_{;i} u_{;j}).$$

Since

$$\det_{d\mu(t)} \text{Ric}(t) d\mu(t) = e^{-(n+1)tu} \det_{d\mu} \text{Ric}(t) d\mu,$$

we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \det_{d\mu(t)} \text{Ric}(t) d\mu(t) &= -(n+1)u\rho d\mu + \hat{R}^{ij} \frac{d}{dt} \Big|_{t=0} R_{ij}(t) d\mu \\ &= -(n+1)u\rho d\mu - (n-1)\hat{R}^{ij} u_{;ij} d\mu. \end{aligned}$$

Hence,

$$\frac{d}{dt} \Big|_{t=0} \int_M \det_{d\mu(t)} \text{Ric}(t) d\mu(t) = \int_M u(-(n+1)\rho - (n-1)\hat{R}^{ij}{}_{;ij}) d\mu.$$

On the other hand,

$$\frac{d}{dt} \Big|_{t=0} \int_M d\mu(t) = (n+1) \int_M u d\mu.$$

Therefore we get

$$\frac{d}{dt} \Big|_{t=0} E({}^t\nabla) = \int_M u(-(n+1)\rho^\circ) - (n-1)\hat{R}^{ij}{}_{;ij} d\mu \int_M d\mu.$$

The equation (3.2) follows from this formula.

From Corollary 2.2, we have only to check the variation of  $\nabla$  in the direction of  $S = (S_{jk}^i)$  with  $S_{jk}^i = S_{kj}^i$  and  $S_{ji}^i = 0$ . Now we put

$${}^t\nabla_X Y = \nabla_X Y + tS(X, Y).$$

We have  ${}^t\nabla d\mu = 0$  for any  $t$ . The Ricci curvature is then

$$R_{ij}(t) = R_{ij} + tS_{ij;k}^k - t^2 S_{il}^k S_{jk}^l.$$

Hence we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E({}^t\nabla) &= \int_M \frac{d}{dt} \Big|_{t=0} (\det \text{Ric}(t)) d\mu \int_M d\mu \\ &= \int_M \hat{R}^{ij} S_{ij;k}^k d\mu \int_M d\mu \\ &= - \int_M \hat{R}^{ij}{}_{;k} S_{ij}^k d\mu \int_M d\mu \\ &= - \int_M (\hat{R}^{ij}{}_{;k} - \frac{1}{n+1}(\hat{R}^{il}{}_{;l}\delta_k^j + \hat{R}^{lj}{}_{;l}\delta_k^i)) S_{ij}^k d\mu \int_M d\mu \end{aligned}$$

The equation (3.3) follows from this formula.  $\square$

The equations (3.2) and (3.3) are not independent. In fact we can show the following.

**Proposition 3.2.** *On a connected manifold, the equation (3.3) implies the equation (3.2).*

**Proof.** The derivative of the determinant  $\rho$  is given as  $\rho_{;k} = \hat{R}^{ij} R_{ij;k}$ . This together with (3.1) yields

$$(n-1)\rho_{;k} = \hat{R}^{ij}_{;k} R_{ij}. \quad (3.4)$$

Applying this to (3.3) we have

$$\frac{n^2-1}{2}\rho_{;k} = \hat{R}^{ij}_{;i} R_{jk}. \quad (3.5)$$

Then we get

$$\hat{R}^{ij}_{;ijk} - \hat{R}^{ij}_{;ikj} = \hat{R}^{il}_{;i} R^j_{lkj} = -\hat{R}^{il}_{;i} R_{lk} = -\frac{n^2-1}{2}\rho_{;k}. \quad (3.6)$$

On the other hand we have

$$\begin{aligned} \hat{R}^{ij}_{;ikj} - \hat{R}^{ij}_{;kij} &= (\hat{R}^{lj} R^i_{lki} + \hat{R}^{il} R^j_{lki})_{;j} \\ &= (-\hat{R}^{lj} R_{lk} + \hat{R}^{il} R^j_{lki})_{;j} \\ &= -\rho_{;k} + \hat{R}^{il}_{;j} R^j_{lki} + \hat{R}^{il} R^j_{lki;j} \\ &= -\rho_{;k} + \frac{1}{n+1}(\hat{R}^{im}_{;m} \delta^l_j + \hat{R}^{ml}_{;m} \delta^i_j) R^j_{lki} - \hat{R}^{il} (R^j_{lij;k} + R^j_{ljk;i}) \\ &= -\rho_{;k} - \frac{1}{n+1} \hat{R}^{ml}_{;m} R_{lk} + \hat{R}^{il} R_{li;k} - \hat{R}^{il} R_{lk;i} \\ &= -\rho_{;k} - \frac{n-1}{2} \rho_{;k} + \rho_{;k} - (\hat{R}^{il} R_{lk})_{;i} + \hat{R}^{il}_{;i} R_{lk} \\ &= \frac{(n+1)(n-2)}{2} \rho_{;k}, \end{aligned} \quad (3.7)$$

where in the fourth equality we used the equation (3.3) and the second Bianchi identity. Hence from (3.6) and (3.7) we have

$$\hat{R}^{ij}_{;ijk} - \hat{R}^{ij}_{;kij} = -\frac{n+1}{2} \rho_{;k}. \quad (3.8)$$

From (3.3) and (3.6) we have

$$\hat{R}^{ij}_{;kij} = \frac{1}{n+1}(\hat{R}^{il}_{;lik} + \hat{R}^{lj}_{;lkj}) = \frac{2}{n+1} \hat{R}^{ij}_{;ijk} + \frac{n-1}{2} \rho_{;k}. \quad (3.9)$$

Then it is easy to see from (3.8) and (3.9) that

$$\hat{R}^{ij}_{;ijk} = -\frac{n+1}{n-1} \rho_{;k},$$

which implies the condition (3.2).  $\square$

#### 4. Proof of Theorem

It follows from (3.5) and (3.1) that

$$\rho_{;i} \hat{R}^{ij} = \frac{2}{n^2-1} \rho \hat{R}^{ij}_{;i}. \quad (4.1)$$

Hence using (3.2), we have

$$\rho_{;ij} \hat{R}^{ij} + \frac{n^2 - 3}{n^2 - 1} \rho_{;i} \hat{R}^{ij}{}_{;j} = -\frac{2}{(n-1)^2} \rho \rho^\circ. \quad (4.2)$$

The proof is divided into two cases. One is the case when  $n = \dim M$  is even. Then since  $\text{Ric} \leq 0$ , we have  $\rho \geq 0$  and  $\hat{\text{Ric}} \leq 0$ . At a point where  $\rho^\circ$  and therefore  $\rho$  too take their maximums, the left hand side of (4.2) is nonnegative because  $\hat{\text{Ric}} \leq 0$ , and the right hand side of (4.2) is nonpositive. Hence we conclude that  $\rho^\circ = 0$ , that is,  $\rho$  is a constant, which must be positive.

The other is the case when  $n$  is odd. Then since  $\text{Ric} \leq 0$ , we have  $\rho \leq 0$  and  $\hat{\text{Ric}} \geq 0$ . At a point where  $\rho$  takes its minimum, the left hand side of (4.2) is nonnegative, because  $\hat{\text{Ric}} \geq 0$ , and the right hand side of (4.2) is nonpositive. Hence we have  $\rho^\circ = 0$ , that is,  $\rho$  is constant.

In both cases we have that  $\rho$  is a nonzero constant. Then it follows from (4.1) that  $\hat{R}^{ij}{}_{;j} = 0$ . Hence from (3.3), we have  $\nabla \hat{\text{Ric}} = 0$ . This implies  $\nabla \text{Ric} = 0$  because  $\rho \neq 0$ . Now put  $g = -\text{Ric}$ , which is a Riemannian metric, and is parallel with respect to  $\nabla$ . Therefore  $\nabla$  is the Riemannian connection of  $g$ .

### References

- [1] Besse, A. L., Einstein manifolds, Springer, 1987
- [2] Kobayashi, S. and Nomizu, K., Foundation of differential geometry, Vol. 1, Interscience publishers, 1963

Department of Mathematics  
Kumamoto University  
Kumamoto 860-8555, Japan  
e-mail: ok@math.sci.kumamoto-u.ac.jp