Title	A variational problem for Affine Connections
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Citation	Archiv der Mathematik. 86(5) p.464-p.469
Issue Date	2006-05
oaire:version	VoR
URL	https://hdl.handle.net/11094/26340
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A Variational Problem for Affine Connections

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Abstract. We give a variational problem for affine connections which characterizes the Riemannian connection of an Einstein metric of negative scalar curvature.

1. Introduction

Let $\mathcal{A}(M)$ denote the sapce of all torsion free affine connections of a compact connected manifold M , each of which preserves a volume element of M . Every affine connections is projectively equivalent to such a connection. For $\nabla \in \mathcal{A}(M)$ we can define the Ricci curvature tensor Ric . Since ∇ preserves a volume element, Ric is a symmetric 2-tensor. Moreover using the volume element we can define the determinant $\det \text{Ric}$ of the Ricci tensor. We put for $\nabla \in \mathcal{A}(M)$

$$E(\nabla) = \int_M \det \text{Ric} d\mu \int_M d\mu.$$

It is easy to see that the right hand side is independent of the choice of a volume element $d\mu$ which is preserved by ∇ . Moreover $E(\nabla)$ depends on ∇ differentiably. This functional may be compared with the normalized Einstein-Hilbert functional of the total scalar curvature of a Riemannian metric ([1]). In this paper we will show the following:

Theorem. *Let M be a compact connected manifold. Suppose $\nabla \in \mathcal{A}(M)$ is a critical point of E , and its Ricci curvature is negative semidefinite and negative definite somewhere. Then ∇ is the Riemannian connection of an Einstein metric of M .*

Remark. A Ricci flat connection $\nabla \in \mathcal{A}(M)$ is a critical point of the functional E . It is known [2; p. 211] that there is a Ricci flat affine connection which is not a Riemannian connection.

I would like to thank the referee for useful comments on the paper.

2. Projective equivalence of affine connections

Given two affine connections ∇ and $\tilde{\nabla}$ on M , we say that they are *projectively equivalent* if geodesics ignoring their parameters are the same for ∇ and $\tilde{\nabla}$. That is to say, there is a function $\lambda: TM \rightarrow \mathbf{R}$ such that

$$\tilde{\nabla}_X X = \nabla_X X + 2\lambda(X)X \tag{2.1}$$

for $X \in TM$. Then it is immediate to see that λ is linear if we put $\lambda(0) = 0$, and thus λ is a smooth 1-form of M .

We will see that any affine connection of M is projectively equivalent to some connection in $\mathcal{A}(M)$. Let ∇ be an arbitrary affine connection of M . Then an affine connection ∇' defined as $\nabla'_X Y = \frac{1}{2}(\nabla_X Y + \nabla_Y X + [X, Y])$ is a torsion free connection which is projectively equivalent to ∇ . So we assume ∇ is torsion free. Let $\tilde{\nabla}$ be another torsion free connection which is projectively equivalent to ∇ . Then from (2.1),

$$\tilde{\nabla}_X Y = \nabla_X Y + \lambda(X)Y + \lambda(Y)X$$

for some 1-form λ . Take a volume element $d\mu$ and $u \in C^\infty(M)$, we then have

$$\tilde{\nabla}_X(e^{(n+1)u}d\mu) = e^{(n+1)u}\nabla_X d\mu + (n+1)e^{(n+1)u}(du - \lambda)(X)d\mu, \quad (2.2)$$

where $n = \dim M$. Hence if we choose $u = 0$ and λ to be such that $(n+1)\lambda(X)d\mu = \nabla_X d\mu$, we have $\tilde{\nabla}d\mu = 0$, i.e., $\tilde{\nabla} \in \mathcal{A}(M)$.

The formula (2.2) says more and we have the following.

Lemma 2.1. *Suppose that $\nabla, \tilde{\nabla} \in \mathcal{A}(M)$ are projectively equivalent. Then there is a $u \in C^\infty(M)$ such that*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xu)Y + (Yu)X.$$

Moreover if $\nabla d\mu = 0$, then $\tilde{\nabla}(e^{(n+1)u}d\mu) = 0$.

Corollary 2.2. *Suppose $\nabla, \tilde{\nabla} \in \mathcal{A}(M)$. Then we have*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xu)Y + (Yu)X + S(X, Y), \quad (2.3)$$

where $u \in C^\infty(M)$ and $S = (S_{jk}^i)$ is a $(1, 2)$ -tensor such that $S_{jk}^i = S_{kj}^i$ and $S_{ji}^i = 0$. The tensor S is uniquely determined by ∇ and $\tilde{\nabla}$, and u is determined up to a constant.

Proof. Let $T = \tilde{\nabla} - \nabla$ be the difference of the two connections. Put $S_{jk}^i = T_{jk}^i - \frac{1}{n+1}(T_{jl}^l \delta_k^i + T_{kl}^l \delta_j^i)$. Then $\nabla + S \in \mathcal{A}(M)$ is projectively equivalent to $\tilde{\nabla}$. $\square$

Corollary 2.3. *Denote by $\mathcal{V}(M)$ the space of smooth volume elements of M . Then we have a smooth map $\varphi: \mathcal{A}(M) \rightarrow \mathcal{V}(M)$ such that $\nabla \varphi(\nabla) = 0$.*

Proof. Fix $\nabla \in \mathcal{A}(M)$ and $d\mu \in \mathcal{V}(M)$ such that $\nabla d\mu = 0$. Fix a point $x \in M$. Then any $\tilde{\nabla} \in \mathcal{A}(M)$ is given as in (2.3), and we can determine a unique u on condition that $u(x) = 0$. Then put $\varphi(\tilde{\nabla}) = e^{(n+1)u}d\mu$. $\square$

3. The variational formula for the functional E

First we note that the Ricci curvature of $\nabla \in \mathcal{A}(M)$ is a symmetric 2-tensor.

Let $\omega \in \Lambda^n T_x^* M \setminus \{0\}$ be a volume form at $x \in M$. Define $\omega^* \in \Lambda^n T_x M$ by $\omega(\omega^*) = 1$. Then we have $(a\omega)^* = \frac{1}{a}\omega^*$ for $a \in \mathbf{R} \setminus \{0\}$. Thinking of the Ricci tensor as $\text{Ric}: T_x M \rightarrow T_x^* M$, we define $\det_\omega \text{Ric}$ as

$$\text{Ric}^* \omega^* = (\det_\omega \text{Ric})\omega.$$

For $a \in \mathbf{R} \setminus \{0\}$, we have $\det_{a\omega} \text{Ric} = \frac{1}{a^2} \det_\omega \text{Ric}$. In particular $\det_{-\omega} \text{Ric} = \det_\omega \text{Ric}$. Hence we can define $\det \text{Ric} = \det_{d\mu} \text{Ric}$ with respect to a volume element $d\mu$.

Now suppose $\nabla d\mu = \nabla d\mu'$. Then $d\mu' = ad\mu$ for some positive constant a since M is connected. So we have $\int_M \det_{d\mu'} \text{Ric} d\mu' = \frac{1}{a} \int_M \det_{d\mu} \text{Ric} d\mu$. Thus $E(\nabla) = \int_M \det \text{Ric} d\mu \int_M d\mu$ depends only on $\nabla \in \mathcal{A}(M)$, and $E: \mathcal{A}(M) \rightarrow \mathbf{R}$ is differentiable by virtue of Corollary 2.3.

We define a contravariant symmetric 2-tensor $\hat{\text{Ric}} = (\hat{R}^{ij})$ as the cofactor tensor of $\text{Ric} = (R_{ij})$. Namely

$$\hat{R}^{ik} R_{kj} = (\det \text{Ric}) \delta_j^i. \quad (3.1)$$

We remark that this tensor field depends on the choice of a volume element $d\mu$. We also define ρ and ρ° as

$$\rho = \det \text{Ric}$$

and

$$\rho^\circ = \det \text{Ric} - \frac{\int_M \det \text{Ric} d\mu}{\int_M d\mu}.$$

With these notations we can state the first variational formula of the functional E .

Proposition 3.1. $\nabla \in \mathcal{A}(M)$ is a critical point of $E: \mathcal{A}(M) \rightarrow \mathbf{R}$ if and only if the following two conditions are satisfied:

$$\hat{R}^{ij}{}_{;ij} = -\frac{n+1}{n-1}\rho^\circ \quad (3.2)$$

and

$$\hat{R}^{ij}{}_{;k} = \frac{1}{n+1}(\hat{R}^{il}{}_{;l}\delta_k^j + \hat{R}^{lj}{}_{;l}\delta_k^i), \quad (3.3)$$

where $n = \dim M$.

Proof. Fix $\nabla \in \mathcal{A}(M)$ with $\nabla d\mu = 0$. We first consider a projective variation of ∇ . Let ${}^t\nabla$ be defined as

$${}^t\nabla_X Y = \nabla_X Y + t(Xu)Y + t(Yu)X,$$

where $u \in C^\infty(M)$ is an arbitrary function. Put

$$d\mu(t) = e^{(n+1)tu} d\mu,$$

and we have ${}^t\nabla d\mu(t) = 0$. The Ricci curvature is calculated as

$$R_{ij}(t) = R_{ij} - (n-1)(tu_{;ij} - t^2 u_{;i} u_{;j}).$$

Since

$$\det_{d\mu(t)} \text{Ric}(t) d\mu(t) = e^{-(n+1)tu} \det_{d\mu} \text{Ric}(t) d\mu,$$

we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \det_{d\mu(t)} \text{Ric}(t) d\mu(t) &= -(n+1)u\rho d\mu + \hat{R}^{ij} \frac{d}{dt} \Big|_{t=0} R_{ij}(t) d\mu \\ &= -(n+1)u\rho d\mu - (n-1)\hat{R}^{ij} u_{;ij} d\mu. \end{aligned}$$

Hence,

$$\frac{d}{dt} \Big|_{t=0} \int_M \det_{d\mu(t)} \text{Ric}(t) d\mu(t) = \int_M u(-(n+1)\rho - (n-1)\hat{R}^{ij}{}_{;ij}) d\mu.$$

On the other hand,

$$\frac{d}{dt} \Big|_{t=0} \int_M d\mu(t) = (n+1) \int_M u d\mu.$$

Therefore we get

$$\frac{d}{dt} \Big|_{t=0} E({}^t\nabla) = \int_M u(-(n+1)\rho^\circ - (n-1)\hat{R}^{ij}{}_{;ij}) d\mu \int_M d\mu.$$

The equation (3.2) follows from this formula.

From Corollary 2.2, we have only to check the variation of ∇ in the direction of $S = (S_{jk}^i)$ with $S_{jk}^i = S_{kj}^i$ and $S_{ji}^i = 0$. Now we put

$${}^t\nabla_X Y = \nabla_X Y + tS(X, Y).$$

We have ${}^t\nabla d\mu = 0$ for any t . The Ricci curvature is then

$$R_{ij}(t) = R_{ij} + tS_{ij;k}^k - t^2 S_{il}^k S_{jk}^l.$$

Hence we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E({}^t\nabla) &= \int_M \frac{d}{dt} \Big|_{t=0} (\det \text{Ric}(t)) d\mu \int_M d\mu \\ &= \int_M \hat{R}^{ij} S_{ij;k}^k d\mu \int_M d\mu \\ &= - \int_M \hat{R}^{ij}{}_{;k} S_{ij}^k d\mu \int_M d\mu \\ &= - \int_M (\hat{R}^{ij}{}_{;k} - \frac{1}{n+1}(\hat{R}^{il}{}_{;l}\delta_k^j + \hat{R}^{lj}{}_{;l}\delta_k^i)) S_{ij}^k d\mu \int_M d\mu \end{aligned}$$

The equation (3.3) follows from this formula. $\square$

The equations (3.2) and (3.3) are not independent. In fact we can show the following.

Proposition 3.2. *On a connected manifold, the equation (3.3) implies the equation (3.2).*

Proof. The derivative of the determinant ρ is given as $\rho_{;k} = \hat{R}^{ij} R_{ij;k}$. This together with (3.1) yields

$$(n-1)\rho_{;k} = \hat{R}^{ij}_{;k} R_{ij}. \quad (3.4)$$

Applying this to (3.3) we have

$$\frac{n^2-1}{2}\rho_{;k} = \hat{R}^{ij}_{;i} R_{jk}. \quad (3.5)$$

Then we get

$$\hat{R}^{ij}_{;ijk} - \hat{R}^{ij}_{;ikj} = \hat{R}^{il}_{;i} R^j_{lkj} = -\hat{R}^{il}_{;i} R_{lk} = -\frac{n^2-1}{2}\rho_{;k}. \quad (3.6)$$

On the other hand we have

$$\begin{aligned} \hat{R}^{ij}_{;ikj} - \hat{R}^{ij}_{;kij} &= (\hat{R}^{lj} R^i_{lki} + \hat{R}^{il} R^j_{lki})_{;j} \\ &= (-\hat{R}^{lj} R_{lk} + \hat{R}^{il} R^j_{lki})_{;j} \\ &= -\rho_{;k} + \hat{R}^{il}_{;j} R^j_{lki} + \hat{R}^{il} R^j_{lki;j} \\ &= -\rho_{;k} + \frac{1}{n+1}(\hat{R}^{im}_{;m} \delta^l_j + \hat{R}^{ml}_{;m} \delta^i_j) R^j_{lki} - \hat{R}^{il} (R^j_{lij;k} + R^j_{ljk;i}) \\ &= -\rho_{;k} - \frac{1}{n+1} \hat{R}^{ml}_{;m} R_{lk} + \hat{R}^{il} R_{li;k} - \hat{R}^{il} R_{lk;i} \\ &= -\rho_{;k} - \frac{n-1}{2} \rho_{;k} + \rho_{;k} - (\hat{R}^{il} R_{lk})_{;i} + \hat{R}^{il}_{;i} R_{lk} \\ &= \frac{(n+1)(n-2)}{2} \rho_{;k}, \end{aligned} \quad (3.7)$$

where in the fourth equality we used the equation (3.3) and the second Bianchi identity. Hence from (3.6) and (3.7) we have

$$\hat{R}^{ij}_{;ijk} - \hat{R}^{ij}_{;kij} = -\frac{n+1}{2} \rho_{;k}. \quad (3.8)$$

From (3.3) and (3.6) we have

$$\hat{R}^{ij}_{;kij} = \frac{1}{n+1}(\hat{R}^{il}_{;lik} + \hat{R}^{lj}_{;lkj}) = \frac{2}{n+1} \hat{R}^{ij}_{;ijk} + \frac{n-1}{2} \rho_{;k}. \quad (3.9)$$

Then it is easy to see from (3.8) and (3.9) that

$$\hat{R}^{ij}_{;ijk} = -\frac{n+1}{n-1} \rho_{;k},$$

which implies the condition (3.2). $\square$

4. Proof of Theorem

It follows from (3.5) and (3.1) that

$$\rho_{;i} \hat{R}^{ij} = \frac{2}{n^2-1} \rho \hat{R}^{ij}_{;i}. \quad (4.1)$$

Hence using (3.2), we have

$$\rho_{;ij} \hat{R}^{ij} + \frac{n^2 - 3}{n^2 - 1} \rho_{;i} \hat{R}^{ij}{}_{;j} = -\frac{2}{(n-1)^2} \rho \rho^\circ. \quad (4.2)$$

The proof is divided into two cases. One is the case when $n = \dim M$ is even. Then since $\text{Ric} \leq 0$, we have $\rho \geq 0$ and $\hat{\text{Ric}} \leq 0$. At a point where ρ° and therefore ρ too take their maximums, the left hand side of (4.2) is nonnegative because $\hat{\text{Ric}} \leq 0$, and the right hand side of (4.2) is nonpositive. Hence we conclude that $\rho^\circ = 0$, that is, ρ is a constant, which must be positive.

The other is the case when n is odd. Then since $\text{Ric} \leq 0$, we have $\rho \leq 0$ and $\hat{\text{Ric}} \geq 0$. At a point where ρ takes its minimum, the left hand side of (4.2) is nonnegative, because $\hat{\text{Ric}} \geq 0$, and the right hand side of (4.2) is nonpositive. Hence we have $\rho^\circ = 0$, that is, ρ is constant.

In both cases we have that ρ is a nonzero constant. Then it follows from (4.1) that $\hat{R}^{ij}{}_{;j} = 0$. Hence from (3.3), we have $\nabla \hat{\text{Ric}} = 0$. This implies $\nabla \text{Ric} = 0$ because $\rho \neq 0$. Now put $g = -\text{Ric}$, which is a Riemannian metric, and is parallel with respect to ∇ . Therefore ∇ is the Riemannian connection of g .

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