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## On a conformally invariant functional of the space of Riemannian metrics

By Osamu KOBAYASHI

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### Introduction.

Let  $\mathcal{M}(M)$  be the space of all  $C^\infty$  Riemannian metrics on a compact  $n$ -dimensional manifold  $M$ , and  $\nu: \mathcal{M}(M) \rightarrow \mathbf{R}$  be a functional of  $\mathcal{M}$  defined by  $\nu(g) = (2/n) \int_M |W|^{n/2} dv$ , where  $W$  is the Weyl conformal curvature tensor. Our main subject in this paper is to determine  $\inf\{\nu(g); g \in \mathcal{M}\}$ , which will be denoted by  $\nu(M)$ . A little consideration shows that  $\nu(M) > 0$  if some Pontrjagin number of  $M$  is not zero. Thus, in general,  $\nu(M)$  is a nontrivial invariant of a manifold.

In §2, we shall show two general properties of  $\nu(M)$ . One is that  $\nu(M) = 0$  for the total space  $M$  of a principal circle bundle (Theorem 2.1). This provides examples of  $M$  for which  $\nu(M) = 0$  but which has no conformally flat metric. The other is an inequality for connected sum;  $\nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2)$  (Theorem 2.2). This is useful for computing  $\nu(M)$  for certain  $M$ .

However, to determine  $\nu(M)$  for general  $M$  seems to be not so easy. Even for  $S^2 \times S^2$ ,  $\nu(S^2 \times S^2)$  is not known (to the author). We want to show that the standard Einstein metric  $g_0$  of  $S^2 \times S^2$  is a candidate at which  $\nu$  takes a minimum, if  $\nu: \mathcal{M}(S^2 \times S^2) \rightarrow \mathbf{R}$  has a minimum. In fact,  $g_0$  is a minimum point of  $\nu$  restricted to Kähler metrics (Proposition 1.4). Moreover, we shall prove that  $g_0$  is a strictly stable critical point of the functional  $\nu$  (cf. Definition 4.1 and Theorem 4.2).

In the course of proof of stability of  $g_0 \in \mathcal{M}(S^2 \times S^2)$ , we establish the first and the second variational formulas of  $\nu: \mathcal{M}(M) \rightarrow \mathbf{R}$  for 4 dimensional  $M$  (Propositions 3.1 and 3.7; The first variational formula has already appeared in [2]). From these formulas, we can also see that other than conformally flat metrics, Einstein metrics are critical points of the functional  $\nu$ , and Ricci flat metrics are stable critical points of  $\nu$ .

### §1. Preliminary definitions and remarks.

Throughout this paper,  $M$  denotes a compact  $C^\infty$  manifold of dimension  $n$ , and  $\mathcal{M}(M)$  denotes the space of  $C^\infty$  Riemannian metrics on  $M$ . For  $g \in \mathcal{M}(M)$ ,

the curvature tensor  $R^i_{jkl}$ , the Ricci tensor  $R_{ij}=R^k_{ikj}$  and the scalar curvature  $R=g^{ij}R_{ij}$  are defined. Our concern in this paper is the Weyl tensor defined by

$$(1.1) \quad W^i_{jkl} = R^i_{jkl} - \frac{1}{n-2} (L^i_k g_{jl} + g^i_k L_{jl} - L^i_l g_{jk} - g^i_l L_{jk}),$$

where  $L_{ij}=R_{ij}-(R/2(n-1))g_{ij}$  (we put  $W=0$  if  $n \leq 2$ ). From the second Bianchi identity, we have  $R^i_{jkl;i}=R_{jl;k}-R_{jk;l}$  and hence,

$$(1.2) \quad W^i_{jkl;i} = \frac{n-3}{n-2} C_{jkl},$$

where  $C_{jkl}=L_{jl;k}-L_{jk;l}$ .

DEFINITION 1.1. We define a functional  $\nu: \mathcal{M}(M) \rightarrow \mathbf{R}$  by  $\nu(g) = \frac{2}{n} \int_M |W|^{n/2} dv_g$ , where  $|W|^{n/2} = \langle W, W \rangle^{n/4} = (g_{ip} g^{jq} g^{kr} g^{ls} W^i_{jkl} W^p_{qrs})^{n/4}$ . For a subset  $U$  of  $M$  and  $g \in \mathcal{M}(M)$ , we write  $\nu(g; U) = \frac{2}{n} \int_U |W|^{n/2} dv_g$ .

LEMMA 1.2. (i)  $\nu(e^{2f}g) = \nu(g)$  for any  $f \in C^\infty(M)$  and  $g \in \mathcal{M}(M)$ . (ii)  $\nu(\varphi^*g) = \nu(g)$  for any diffeomorphism  $\varphi$  of  $M$ . (iii)  $\nu=0$  if  $\dim M \leq 3$ .

PROOF. Let  $W$  and  $W'$  be the Weyl tensors of the metrics  $g$  and  $g'=e^{2f}g$ , respectively. Since the Weyl tensor is invariant under a conformal change of metric, we have  $\langle W', W' \rangle_{g'} = \langle W, W \rangle_{g'} = e^{-4f} \langle W, W \rangle_g$ . Hence, from  $dv' = e^{nf} dv$  for the volume elements, we get  $|W'|^{n/2} dv' = |W|^{n/2} dv$ , which proves (i). (ii) is trivial, and (iii) is well-known.  $\square$

For the dimensions higher than three, we first remark the following:

PROPOSITION 1.3. If  $\dim M \geq 4$ , then  $\sup \{\nu(g); g \in \mathcal{M}(M)\} = \infty$ .

PROOF. Let  $T^n = \mathbf{R}^n / \mathbf{Z}^n$  be the  $n$ -dimensional torus. If  $n \geq 4$ , there exists a metric  $g \in \mathcal{M}(T^n)$  with  $c := \nu(g) > 0$ . Then,  $\tilde{g} \in \mathcal{M}(\mathbf{R}^n)$  denoting the lift of  $g$  to the universal covering, we have  $\nu(\tilde{g}; [0, l]^n) = l^n c$  for  $l \in \mathbf{N}$ . Now, let  $(U, \phi)$  be a chart of  $M$  such that  $\phi(U) = \mathbf{R}^n$ . Then, for each  $l \in \mathbf{N}$ , we can take a metric  $g_l$  on  $M$  which coincides with  $\tilde{g}$  in  $[0, l]^n \subset \mathbf{R}^n \cong U$ , i.e.,  $(\phi_* g_l)|_{[0, l]^n} = \tilde{g}|_{[0, l]^n}$ . We thus have  $\nu(g_l) \geq \nu(g_l; U) \geq \nu(\phi_* g_l; [0, l]^n) = l^n c$  and hence,  $\lim_{l \rightarrow \infty} \nu(g_l) = \infty$ .  $\square$

On the other hand, there are non-trivial topological lower bounds for  $\nu$ : Any Pontrjagin class is represented by a differential form composed of only the Weyl tensor ([1]). Namely, the  $m$ -th Pontrjagin class  $p_m \in H^{4m}(M)$  (cf. [5]) is given inductively by  $\Pi_m = -(p_1 \Pi_{m-1} + \cdots + p_{m-1} \Pi_1) - 2m p_m$ , where  $\Pi_m \in H^{4m}(M)$  is represented by the following differential form;

$$(2\pi)^{-2m} \hat{\Omega}^{i_1}_{i_2} \wedge \hat{\Omega}^{i_2}_{i_3} \wedge \cdots \wedge \hat{\Omega}^{i_{2m}}_{i_1},$$

where  $\hat{\Omega}^i_j = (1/2) W^i_{jkl} e^k \wedge e^l$ . So we can see that any Pontrjagin number of a  $4k$ -

dimensional  $M$  is dominated by  $\nu(g)$  multiplied by some universal constant. For example, the following is well-known.

**PROPOSITION 1.4.** *If  $\dim M=4$ , then  $|p_1[M]| \leq \nu(g)/8\pi^2$  for all  $g \in \mathcal{M}(M)$ . Hence, if furthermore  $M$  is oriented,  $|\tau| \leq \nu(g)/24\pi^2$  for all  $g \in \mathcal{M}(M)$ , where  $\tau$  is the signature of  $M$ . Equality holds if and only if  $g$  is a half conformally flat metric.*

Thus, unlike  $\sup \nu$ ,  $\inf \nu$  reflects certain global properties of a manifold.

**DEFINITION 1.5.**  $\nu(M) := \inf \{\nu(g); g \in \mathcal{M}(M)\}$ .

Here are some examples:

1.  $\nu(g)=0$  if  $g$  is conformally flat. Hence, if  $M$  carries a conformally flat metric, then  $\nu(M)=0$ . However,  $\nu(M)=0$  does not imply in general that  $M$  admits a conformally flat metric (see § 2).

2. The Fubini Study metric  $g_0$  of  $CP^2$  is half conformally flat. Hence, by Proposition 1.4,  $\nu(CP^2)=\nu(g_0)=24\pi^2$ . For the connected sum  $kCP^2$  of  $k$  copies of  $CP^2$ , Proposition 1.4 gives only  $\nu(kCP^2) \geq 24\pi^2 |\tau(kCP^2)| = 24k\pi^2$ . We shall show in § 2 that  $\nu(kCP^2) \leq k\nu(CP^2) = 24k\pi^2$ . Hence, we have  $\nu(kCP^2) = 24k\pi^2$ .

3. Although the author does not know the value  $\nu(S^2 \times S^2)$  at present, there are partial results which suggest that  $\nu(S^2 \times S^2)$  may be positive. Let  $\bar{g}, \bar{\bar{g}}$  be two Riemannian metrics on  $S^2$  with the Gauss curvatures  $\bar{K}, \bar{\bar{K}}$ , respectively. Consider the product metric  $g = \bar{g} + \bar{\bar{g}}$  on  $S^2 \times S^2$ . Then,  $\nu(g) = (128/3)\pi^2 + (2/3) \int_{S^2 \times S^2} (\bar{K} - \bar{\bar{K}})^2 dv$ .

**PROOF.** The Weyl tensor is computed as  $W_{mijk} = (1/6)(\bar{K} + \bar{\bar{K}})(2\bar{g}_{mj}\bar{g}_{ik} - 2\bar{g}_{ij}\bar{g}_{km} - \bar{g}_{mj}\bar{\bar{g}}_{ik} + \bar{g}_{ij}\bar{\bar{g}}_{km} + 2\bar{\bar{g}}_{mj}\bar{\bar{g}}_{ik} - 2\bar{\bar{g}}_{ij}\bar{\bar{g}}_{km} - \bar{g}_{mj}\bar{\bar{g}}_{ik} + \bar{g}_{ij}\bar{\bar{g}}_{km})$ . Hence, we have  $|W|^2/2 = 2(\bar{K} + \bar{\bar{K}})^2/3 = 8\bar{K}\bar{\bar{K}}/3 + 2(\bar{K} - \bar{\bar{K}})^2/3$ . Then, from the Gauss Bonnet formula,  $\nu(g) = (8/3) \int \bar{K}\bar{\bar{K}} dv + (2/3) \int (\bar{K} - \bar{\bar{K}})^2 dv = 128\pi^2/3 + (2/3) \int (\bar{K} - \bar{\bar{K}})^2 dv$ .  $\square$

So, among the product metrics of  $S^2 \times S^2$ , the standard Einstein metric attains the smallest value  $128\pi^2/3$ . This is generalized as follows:

**PROPOSITION 1.6.** *Suppose that  $\dim M=4$  and  $g \in \mathcal{M}(M)$  is a Kähler metric for some complex structure of  $M$ . Then,*

$$\nu(g) \geq 24\pi^2 |\tau| + \frac{16}{3}\pi^2 \min \{2\chi - 6\tau, 2\chi + 3\tau\},$$

where  $\tau$  and  $\chi$  are the signature and the Euler number of  $M$  respectively. The equality holds if and only if  $g$  is an Einstein Kähler metric.

**PROOF.** By the four dimensional Gauss Bonnet theorem,

$$(1.3) \quad \nu(g) = 16\pi^2 \chi + \int |E|^2 dv - \frac{1}{12} \int R^2 dv,$$

where  $E$  is the traceless part of the Ricci tensor;  $E_{ij} = R_{ij} - (R/4)g_{ij}$ .

Let  $\rho$  be the Ricci form of the Kähler metric  $g$  ([5]). Then, it is easily seen that

$$(1.4) \quad \rho \wedge \rho = \left( \frac{1}{2} R^2 - 2|E|^2 \right) dv.$$

Since the first Chern class is represented by  $\rho/4\pi$ , we have

$$\int \rho \wedge \rho = 16\pi^2 c_1^2 = 16\pi^2 (2\chi + 3\tau).$$

This, together with (1.3) and (1.4), yields

$$\begin{aligned} \nu(g) &= -8\pi^2\tau + \frac{32}{3}\pi^2\chi + \frac{2}{3}\int |E|^2 dv \\ &= \begin{cases} 24\pi^2|\tau| + \frac{16}{3}\pi^2(2\chi - 6\tau) + \frac{2}{3}\int |E|^2 dv, & \text{if } \tau \geq 0; \\ 24\pi^2|\tau| + \frac{16}{3}\pi^2(2\chi + 3\tau) + \frac{2}{3}\int |E|^2 dv, & \text{if } \tau < 0. \end{cases} \end{aligned}$$

Now, we get the desired inequality.  $\square$

Applying this proposition to  $S^2 \times S^2$ , we have  $\nu(g) \geq 128\pi^2/3$  for any Kähler metric  $g$  of  $S^2 \times S^2$ , because  $\tau(S^2 \times S^2) = 0$  and  $\chi(S^2 \times S^2) = 4$ . Thus, the standard Einstein metric  $g_0$  of  $S^2 \times S^2$  attains the smallest value of  $\nu$  also in the class of Kähler metrics. Moreover, we shall show in § 4 that the functional  $\nu: \mathcal{M}(S^2 \times S^2) \rightarrow \mathbf{R}$  has in fact a local minimum at  $g_0$ .

## § 2. Two general formulas for $\nu(M)$ .

THEOREM 2.1. *If  $S^1$  acts freely and differentiably on  $M$ , then  $\nu(M) = 0$ .*

PROOF. Let  $K$  denote the vector field on  $M$  which generates the  $S^1$  action. Since  $S^1$  is compact, there is an  $S^1$  invariant Riemannian metric  $h$  on  $M$ , for which  $K$  is a Killing vector field. Since the action is free,  $h(K, K)$  is nowhere zero, hence,  $g = (h(K, K))^{-1}h$  defines a Riemannian metric. Then we have

$$(2.1) \quad \mathcal{L}_K g = 0 \quad \text{and} \quad g(K, K) = 1.$$

Now, consider a family of Riemannian metrics  $\{g(t); 0 < t \leq 1\}$  defined by

$$(2.2) \quad g_{ij}(t) = g_{ij} - (1-t^2)\alpha_i\alpha_j,$$

where  $\alpha$  is the 1-form associated with  $K$  with respect to  $g = g(1)$ , i.e.,  $\alpha_i = g_{ij}K^j$ . The inverse matrix  $g^{ij}(t)$  is easily seen to be  $g^{ij} - (1-t^2)K^iK^j$ . Then, using (2.1), we get the relation between the Christoffel symbols of  $g(t)$  and  $g$ :

$$(2.3) \quad \Gamma_{ij}^k(t) - \Gamma_{ij}^k = -(1-t^2)(K^k_{;i}\alpha_j + K^k_{;j}\alpha_i),$$

where the covariant derivation in the right hand side is taken as one with respect to  $g$ . From this and (2.1), we have

$$(2.4) \quad R^i_{jkl}(t) - R^i_{jkl} = -(1-t^2) \{ (K^i_{;j} \alpha_l)_{;k} - (K^i_{;j} \alpha_k)_{;l} + K^m R^i_{mkl} \alpha_j \\ + K^i_{;l} \alpha_{j;k} - K^i_{;k} \alpha_{j;l} \} + (1-t^2)^2 K^i_{;m} \alpha_j (\alpha_k K^m_{;l} - \alpha_l K^m_{;k}).$$

Then, again using (2.1), we have

$$(2.5) \quad g^{jh}(t)(R^i_{jkl}(t) - R^i_{jkl}) = -(1-t^2) \{ (K^{i;h} \alpha_l)_{;k} - (K^{i;h} \alpha_k)_{;l} \\ + K^i_{;l} K^h_{;k} - K^i_{;k} K^h_{;l} \} + (1-t^2) K^h K^m R^i_{mkl}.$$

On the other hand,  $g^{jn}(t)R^i_{jkl} = R^{in}_{kl} - (1-t^2)K^j K^n R^i_{jkl}$ . Hence, we get

$$(2.6) \quad g^{jn}(t)R^i_{jkl}(t) = R^{in}_{kl} - (1-t^2) \{ (K^{i;n} \alpha_l)_{;k} - (K^{i;n} \alpha_k)_{;l} \\ + K^i_{;l} K^n_{;k} - K^i_{;k} K^n_{;l} \}.$$

Note that (2.6) does *not* contain terms of  $t^{-2}$  and that both sides of (2.6) are tensors of type (2, 2). So, there is a constant  $c$  such that  $|R^i_{jkl}(t)|^2_{(t)} < c$  for all  $t \in (0, 1]$ . In particular,  $|W(t)|^2_{(t)} < c$ . On the other hand, the volume form  $dv(t)$  relative to the metric  $g(t)$  is easily computed as  $dv(t) = t dv$ . Thus, we get  $\lim_{t \rightarrow 0} \nu(g(t)) = 0$ . Hence,  $\nu(M) = 0$ .  $\square$

REMARK. There is no conformally flat metric on  $S^p \times T^q$ ,  $p, q \geq 2$ , i.e.,  $\nu(g) > 0$  for any  $g \in \mathcal{M}(S^p \times T^q)$ ,  $p, q \geq 2$ , because, by a theorem of Kuiper [6; Theorem III], the universal covering space of a compact conformally flat space with an infinite Abelian fundamental group must be  $\mathbf{R}^n$  or  $\mathbf{R} \times S^{n-1}$ . However, the above theorem asserts that  $\nu(S^p \times T^q) = 0$ . So, in general,  $\nu(M) = 0$  does not imply the existence of a conformally flat metric of  $M$ .

THEOREM 2.2. *For any compact manifolds  $M_1$  and  $M_2$  of the same dimension,  $\nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2)$ .*

For the proof, we prepare the following lemma.

LEMMA 2.3. *Let  $g \in \mathcal{M}(M)$  be given. Then, for each  $\varepsilon > 0$ , there is a  $\tilde{g} \in \mathcal{M}(M)$  such that  $|\nu(g) - \nu(\tilde{g})| < \varepsilon$  and  $\tilde{g}$  is flat in an open subset of  $M$ .*

PROOF. Let  $(U, \phi)$  be a chart of  $M$  such that  $\phi(U) \supset \{x \in \mathbf{R}^n; |x| < 1\}$  and in the coordinate expression of the metric  $g|_U = g_{ij}(x) dx^i dx^j$ ,

$$(2.7) \quad g_{ij}(0) = \delta_{ij}$$

holds. Take a nonnegative smooth function  $\varphi: \mathbf{R}^n \rightarrow \mathbf{R}$  such that  $\varphi(x) \equiv 1$  if  $|x| \leq 1/2$ , and  $\varphi(x) \equiv 0$  if  $|x| \geq 1$ . We set  $\varphi_t(x) = \varphi(x/t)$ . The support of  $\varphi_t$  is contained in  $B_t = \{x \in \mathbf{R}^n; |x| < t\}$ . For  $0 < t < 1$ , we define a metric  $\tilde{g} \in \mathcal{M}(M)$  by  $\tilde{g}|(M \setminus U) = g|(M \setminus U)$  and

$$(2.8) \quad \tilde{g}_{ij} = (1 - \varphi_t)g_{ij} + \varphi_t \delta_{ij}$$

in  $U$ . We shall show that  $\tilde{g}$  has the desired properties for a sufficiently small  $t$ .

It follows from (2.7) and the definition of  $\varphi_t$  that

$$(2.9) \quad |\varphi_t(g_{ij} - \delta_{ij})| < c_1 t, \quad |\partial \varphi_t| < c_1 t^{-1}, \quad |\partial^2 \varphi_t| < c_1 t^{-2},$$

for some constant  $c_1$  where  $\partial$  is the Euclidean gradient. Hence,  $|\partial \tilde{g}_{ij}| < c_2$  and  $|\partial^2 \tilde{g}_{ij}| < c_2(t^{-1} + 1)$  for some  $c_2$ . Then, putting  $f_t = \tilde{g}(\tilde{W}, \tilde{W})^{n/4}(\det(\tilde{g}_{ij}))^{1/2}$ , we can easily see that

$$(2.10) \quad f_t < c_3(t^{-1} + 1)^{n/2}.$$

Thus, we get

$$(2.11) \quad \begin{aligned} \int_{B_t} f_t dx &\leq c_3(t^{-1} + 1)^{n/2} \int_{B_t} dx \\ &= c_4(t^{-1} + 1)^{n/2} t^n = c_4(t^2 + t)^{n/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\nu(g) - \nu(\tilde{g})| &= |\nu(g; \phi^{-1}(B_t)) - \nu(\tilde{g}; \phi^{-1}(B_t))| \\ &\leq \nu(g; \phi^{-1}(B_t)) + \frac{2}{n} \int_{B_t} f_t dx. \end{aligned}$$

Therefore, from (2.11), we conclude that  $|\nu(g) - \nu(\tilde{g})| < \varepsilon$  for a sufficiently small  $t$ . It is obvious from (2.8) that  $\tilde{g}$  is flat in  $\phi^{-1}(B_t)$ .  $\square$

**PROOF OF THEOREM 2.2.** Let  $\varepsilon$  be an arbitrary positive number. Take  $g_i \in \mathcal{M}(M_i)$  so that  $\nu(g_i) \leq \nu(M_i) + \varepsilon$ ,  $i=1, 2$ . By the above lemma, we can choose  $\tilde{g}_i \in \mathcal{M}(M_i)$  such that  $\nu(\tilde{g}_i) \leq \nu(M_i) + 2\varepsilon$  and  $\tilde{g}_i$  is flat in some neighbourhood of  $M_i$ . Suppose that for some  $r > 0$  and  $p_i \in M_i$ ,  $\tilde{g}_i$  is flat in  $U_i(p_i; [0, 2r]) := \{x \in M_i; \tilde{d}_i(x, p_i) \in [0, 2r]\}$ , where  $\tilde{d}_i$  is the distance function of the metric  $\tilde{g}_i$ ,  $i=1, 2$ .

We define a diffeomorphism  $\varphi: U_1(p_1; (r/2, 2r)) \rightarrow U_2(p_2; (r/2, 2r))$  by  $\varphi(\exp_{p_1} X) = \exp_{p_2}(-(r^2/\tilde{g}_1(X, X))\varphi_0 X)$ , where  $\varphi_0: T_{p_1}M_1 \rightarrow T_{p_2}M_2$  is a linear isometry and  $\exp_{p_i}$  denotes the exponential map at  $p_i \in M_i$  with respect to  $\tilde{g}_i$ . Then we can regard  $M_1 \# M_2$  as  $\{M_1 \setminus U_1(p_1; [0, r/2])\} \cup_{\varphi} \{M_2 \setminus U_2(p_2; [0, r/2])\}$ .

Let  $f_i$  be a positive smooth function on  $M_i$  such that  $f_i(x) = (\tilde{d}_i(p_i, x))^{-2}$  if  $r/2 < \tilde{d}_i(p_i, x) < 2r$ . Then,  $f_i \tilde{g}_i$  is a Riemannian metric on  $M_i$  and is conformally flat in  $U_i(p_i; [0, 2r])$ . Moreover,  $\varphi: (U_1(p_1; (r/2, 2r)), f_1 \tilde{g}_1) \rightarrow (U_2(p_2; (r/2, 2r)), f_2 \tilde{g}_2)$  becomes an isometry. Hence, we can define a Riemannian metric  $g$  on  $M_1 \# M_2$  by  $g|(M_i \setminus U_i(p_i; [0, r/2])) := f_i \tilde{g}_i|(M_i \setminus U_i(p_i; [0, r/2]))$ ,  $i=1, 2$ . Then, we have

$$\begin{aligned} \nu(g) &= \nu(f_1 \tilde{g}_1; M_1 \setminus U_1(p_1; [0, r/2])) + \nu(f_2 \tilde{g}_2; M_2 \setminus U_2(p_2; [0, 2r])) \\ &= \nu(f_1 \tilde{g}_1) + \nu(f_2 \tilde{g}_2) \end{aligned}$$

$$\begin{aligned}
&= \nu(\tilde{g}_1) + \nu(\tilde{g}_2) \\
&\leq \nu(M_1) + \nu(M_2) + 4\varepsilon.
\end{aligned}$$

Therefore,  $\nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2) + 4\varepsilon$  for arbitrary  $\varepsilon > 0$ . That is,  $\nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2)$ .  $\square$

### § 3. Variational formulas in dimension four.

In this section, we use the following abbreviation for shortness' sake;  
 1. *Omitting the summation sign*, e. g.,  $C_{ijk;k} + L_{mk}W^m_{ijk}$  stands for  $\sum_{k,l} g^{kl}C_{ijk;l} + \sum_{k,l,m} g^{kl}L_{mk}W^m_{ijl}$ . 2. *Identification through the duality defined by metric*, e. g.,  $S^k_{ij} = (1/2)(h_{jk;i} + h_{ki;j} - h_{ij;k})$  stands for  $S^k_{ij} = (1/2)g^{kl}(h_{jl;i} + h_{li;j} - h_{ij;l})$ .

PROPOSITION 3.1 ([2]). Suppose that  $M$  is a compact manifold of dimension 4. Then for a smooth curve  $g = g(t)$  in  $\mathcal{M}(M)$ , we have

$$\frac{d}{dt}\nu(g) = \int_M \left\langle X, \frac{d}{dt}g \right\rangle dv,$$

where  $X$  is a symmetric 2-tensor defined by  $X_{ij} = C_{ijk;k} + L_{mk}W^m_{ijk}$  (see § 1 for  $C$ ,  $L$  and  $W$ ).

PROOF. We set  $h_{ij} = (d/dt)g_{ij}$  and  $S^k_{ij} = (d/dt)\Gamma^k_{ij}$ . Then,

$$(3.1) \quad S^k_{ij} = \frac{1}{2}(h_{jk;i} + h_{ki;j} - h_{ij;k}).$$

Hence,

$$(3.2) \quad \frac{d}{dt}R^m_{ijk} = S^m_{ik;j} - S^m_{ij;k}.$$

Then, from (1.1) and elementary algebraic properties of curvature tensor, we have

$$W^m_{ijk} \left( \frac{d}{dt} W^m_{ijk} \right) = W^m_{ijk} (2h_{ij;k} + L_{mk}h_{ij}).$$

Therefore, using  $(d/dt)g^{ij} = -h^{ij}$  and  $(d/dt)dv = (1/2)h_{ii}dv$ , we get

$$\begin{aligned}
\frac{d}{dt}\nu(g) &= \int \left\{ W^m_{ijk} (2h_{ij;k} + L_{mk}h_{ij}) - W^m_{ijk} W^m_{ijl} h^{kl} + \frac{1}{4} |W|^2 h_{ii} \right\} dv \\
&= \int \left\{ X_{ij} h_{ij} - (W^m_{ijk} W^m_{ijl} - \frac{1}{4} |W|^2 g_{kl}) h_{kl} \right\} dv,
\end{aligned}$$

where we use Stokes' formula and (1.2). Thus, we have only to prove  $W^m_{ijk} W^m_{ijl} = (1/4) |W|^2 g_{kl}$ .

It is known that a symmetric linear transformation on the space of 2-forms  $\Lambda^2$  commutes with the Hodge star  $*$ :  $\Lambda^2 \rightarrow \Lambda^2$  (since the argument is local, we



need not assume  $M$  is orientable) if and only if its Ricci contraction is proportional to  $g$  (cf. [7; Theorem 1.3]). Viewed as a symmetric transformation on  $\Lambda^2$ , the Weyl tensor commutes with the Hodge star, because the Ricci contraction of  $W$  is zero. Hence, so does  $W \circ W$ , and the Ricci contraction of  $W \circ W$  is proportional to  $g$ . That is,  $W^{ik}_{ab} W^{ab}_{jk} = \lambda g^i_j$  for some scalar  $\lambda$ , which implies  $W^m_{ijk} W^m_{ijl} = (1/4) |W|^2 g_{kl}$ .

That  $X$  is symmetric is not difficult to see.  $\square$

COROLLARY 3.2. *If  $\dim M=4$ , the tensor  $X$  has the following properties;*

- (i)  $X_{ii}=0$ ; (ii)  $X_{ij;j}=0$ ; (iii)  $X \otimes g$  is conformally invariant.

PROOF. Easy consequence of Lemma 1.2.  $\square$

COROLLARY 3.3. *If  $\dim M=4$  and  $g \in \mathcal{M}(M)$  is conformal to an Einstein metric, then  $g$  is a critical point of  $\nu: \mathcal{M}(M) \rightarrow \mathbf{R}$ .*

PROOF. Obviously,  $X=0$  if  $g$  is an Einstein metric. Thus, the assertion follows from Corollary 3.2 (iii).  $\square$

Next, we shall compute the second variational formula. To do this, we review the Lichnerowicz Laplacian and decomposition of the space of symmetric 2-tensor fields (cf. [3]).

DEFINITION 3.4. The tangent space  $T_g \mathcal{M}(M)$  of  $\mathcal{M}(M)$  at  $g$  is naturally identified with the space of  $C^\infty$  symmetric 2-tensor fields on  $M$ . The *Lichnerowicz Laplacian*  $\Delta_L: T_g \mathcal{M} \rightarrow T_g \mathcal{M}$  is defined by

$$(\Delta_L h)_{ij} := (\tilde{\Delta} h)_{ij} + h_{ik;kj} - h_{ik;jk} + h_{jk;ki} - h_{jk;ik},$$

where  $\tilde{\Delta}: T_g \mathcal{M} \rightarrow T_g \mathcal{M}$  is the *rough Laplacian*;  $(\tilde{\Delta} h)_{ij} = h_{ij;kk}$  (our sign convention of Laplacians is opposite to that used in [3]).

LEMMA 3.5. (i)  $(\Delta_L h)_{ij} = (\tilde{\Delta} h)_{ij} - h_{ik} R_{kj} - R_{ik} h_{kj} - 2h_{mk} R^m_{ijk}$ . Hence, if  $\dim M=4$ , then

$$\begin{aligned} (\Delta_L h)_{ij} = & (\tilde{\Delta} h)_{ij} - 2(h_{ik} E_{kj} + E_{ik} h_{kj}) + \langle E, h \rangle g_{ij} \\ & + (\text{tr } h)(E_{ij} + (R/6)g_{ij}) - (2R/3)h_{ij} - 2h_{mk} W^m_{ijk}, \end{aligned}$$

where  $E_{ij} = R_{ij} - (R/4)g_{ij}$ .

$$(ii) \quad \int_M \langle h', \Delta_L h'' \rangle dv = - \int_M \{ \langle h', \tilde{\Delta} h'' \rangle + (h'_{ij;k} - h'_{ik;j})(h''_{ij;k} - h''_{ik;j}) + 2h'_{ij;j} h''_{ik;k} \} dv$$

for  $h', h'' \in T_g \mathcal{M}$ .

$$(iii) \quad \int_M \langle h', \Delta_L h'' \rangle dv = \int_M \{ h'_{ij;k} (h''_{jk;i} + h''_{ki;j} - h''_{ij;k}) - 2h'_{ij;j} h''_{ik;k} \} dv, \text{ for } h', h''$$

$\in T_g \mathcal{M}$ .

PROOF. Easy and omitted.  $\square$

LEMMA 3.6. If  $M$  is a compact manifold,  $T_g\mathcal{M}(M)$  has the following decomposition;  $T_g\mathcal{M} = \mathcal{S}_0(g) \oplus \mathcal{S}_1(g)$ , where  $\mathcal{S}_0(g) = \{h \in T_g\mathcal{M}; \operatorname{tr} h = 0, \operatorname{div} h = 0\}$  and  $\mathcal{S}_1(g) = \{h \in T_g\mathcal{M}; h = \mathcal{L}_u g + fg \text{ for some } u \in \mathfrak{X}(M) \text{ and } f \in C^\infty(M)\}$ . This decomposition is orthogonal with respect to the  $L_2$  inner product defined by  $g$ .

PROOF. Put  $P(u) = \mathcal{L}_u g - (1/n)(\operatorname{tr} \mathcal{L}_u g)g$  for a vector field  $u$ . Then, it is easy to check that the principal symbol of the linear differential operator  $P: \mathfrak{X} \rightarrow T_g\mathcal{M}$  is injective. Hence,  $T_g\mathcal{M} = \operatorname{Ker} P^* \oplus \operatorname{Im} P$  (cf. [3]), where  $P^*$  is the adjoint operator of  $P$ .  $P^*$  is computed as  $P^*(h) = -2(h_{ij} - (1/n)(\operatorname{tr} h)g_{ij})_{,j}$ . From this, we have  $\operatorname{Ker} P^* = \mathcal{S}_0 \oplus C^\infty(M) \cdot g$ . Then, putting  $\mathcal{S}_1 = C^\infty(M) \cdot g \oplus \operatorname{Im} P$ , we get the desired decomposition.  $\square$

REMARKS. 1. Let  $G$  be the semi direct product of the diffeomorphism group  $\mathcal{D}(M)$  and  $C^\infty(M)$  with multiplication;  $(\varphi_1, f_1) \cdot (\varphi_2, f_2) = (\varphi_1 \circ \varphi_2, f_1 \circ \varphi_2 + f_2)$  for  $\varphi_1, \varphi_2 \in \mathcal{D}(M)$  and  $f_1, f_2 \in C^\infty(M)$ . Then,  $G$  acts on  $\mathcal{M}(M)$  on the right as follows;  $(g, (\varphi, f)) \mapsto e^{2f} \varphi^* g$ ,  $g \in \mathcal{M}(M)$ ,  $\varphi \in \mathcal{D}(M)$  and  $f \in C^\infty(M)$ . Lemma 1.2 says that  $\nu$  is constant on every  $G$ -orbit in  $\mathcal{M}(M)$ .  $\mathcal{S}_1(g)$  in the above lemma is regarded as the tangent space at  $g$  of the  $G$ -orbit of  $g$ .

2. Using the orthogonality between  $\mathcal{S}_0$  and  $\mathcal{S}_1$ , we have an isomorphism  $\mathcal{S}_0(g) \cong \mathcal{S}_0(e^{2f}g)$ ;  $h \mapsto e^{(2-n)f}h$ . In particular, we see that  $\mathcal{S}_0(g) = \mathcal{S}_0(e^{2f}g)$  if  $\dim M = 2$ ,  $\mathcal{S}_0(g) \otimes g = \mathcal{S}_0(e^{2f}g) \otimes e^{2f}g$  if  $\dim M = 4$ , and so on.

PROPOSITION 3.7. Suppose that  $M$  is a compact manifold of dimension 4, and  $g \in \mathcal{M}(M)$  is a critical point of  $\nu: \mathcal{M}(M) \rightarrow \mathbf{R}$ . Let  $g_t$  be a smooth variation of  $g$  with  $g_0 = g$ . Then,

$$\begin{aligned} \left( \frac{d}{dt} \right)^2 \nu(g_t) \Big|_{t=0} &= \int_M \left[ \frac{1}{2} \left\langle \Delta_L h + \frac{R}{2} h, \Delta_L h + \frac{R}{3} h \right\rangle \right. \\ &\quad + \left\langle E \circ h, 2\Delta_L h + \frac{3}{2} h \circ E + R h \right\rangle + \frac{1}{2} |E|^2 |h|^2 + \frac{1}{2} |E \circ h|^2 - \frac{2}{3} \langle E, h \rangle^2 \\ &\quad + S_{ij}^m \left\{ (2E_{mk} h_{ij})_{,k} - \frac{1}{3} h_{ij} R_{,m} - \frac{1}{2} (E \circ h - h \circ E)_{mi,j} + 2h_{ik} C_{mjk} - h_{kj} C_{kmi} \right\} \\ &\quad \left. - h_{ij} h_{km} C_{ijk;m} \right] dv, \end{aligned}$$

where  $h$  is the  $\mathcal{S}_0(g)$  component of  $(dg/dt)|_{t=0} \in T_g\mathcal{M}$  (cf. Lemma 3.6),  $(E \circ h)_{ij} = E_{ik} h_{kj}$  and  $S_{ij}^k = (1/2)(h_{jk,i} + h_{ki,j} - h_{ij,k})$ .

PROOF. From the first variational formula (Proposition 3.1),  $\langle d\nu(g_t)/dt \rangle = \int_M \langle X_t, (dg_t/dt) \rangle_t dv_t$ . Since  $X_t \in \mathcal{S}_0(g_t)$  (Corollary 3.2), we have  $\langle d\nu(g_t)/dt \rangle = \int_M \langle X_t, h_t \rangle_t dv_t$ , where  $h_t$  is the  $\mathcal{S}_0(g_t)$  component of  $(dg_t/dt)$ . Thus, since  $X = X_0 = 0$ , we get

$$(3.3) \quad \left(\frac{d}{dt}\right)^2 \nu(g_t) \Big|_{t=0} = \int_M \langle \dot{X}, h \rangle dv,$$

where  $\cdot$  means  $(d/dt)|_{t=0}$ , i.e.,  $\dot{X} = (dX_t/dt)|_{t=0}$ . If we write  $\dot{X} = (DX)(\dot{g})$ , we can see  $\dot{X} = (DX)(h)$ , because that  $X=0$  is a conformally invariant property by Corollary 3.2 (iii). Hence, it suffices to prove the formula under the assumption that  $\dot{g} = h \in \mathcal{S}_0(g)$ . So, we assume in the following that  $h_{ij} = \dot{g}_{ij}$  and  $h_{ii} = 0$ ,  $h_{ij;j} = 0$ .

From the definition of  $X$ ,

$$(3.4) \quad \begin{aligned} \dot{X}_{ij} &= (g^{km} C_{ijk;m}) \cdot + (g^{kl} L_{ml} W_{ijk}^m) \cdot \\ &= (\dot{C}_{ijk};_k - h_{km} C_{ijk;m} - S_{ki}^m C_{mjk} - S_{kj}^m C_{imk} \\ &\quad + \dot{L}_{mk} W_{ijk}^m + L_{mk} \dot{W}_{ijk}^m - h_{kl} L_{ml} W_{ijk}^m). \end{aligned}$$

From the definition of  $C$ ,

$$(3.5) \quad \dot{C}_{ijk} = (\dot{L}_{ik};_j - (\dot{L}_{ij};_k + S_{ki}^m L_{mj} - S_{ji}^m L_{mk}).$$

From the definition of  $L$  and the Lichnerowicz Laplacian, and (3.2),

$$(3.6) \quad \dot{L}_{ij} = -\frac{1}{2}(\Delta_L h)_{ij} - \frac{R}{6}h_{ij} + \frac{1}{6}\langle E, h \rangle g_{ij}.$$

From this and Lemma 3.5 (ii),

$$(3.7) \quad \begin{aligned} (\dot{L}_{ij};_k \langle h_{ij};_k - h_{ik;j} \rangle &\doteq \frac{1}{4}|\Delta_L h|^2 + \frac{R}{12}\langle h, \Delta_L h \rangle + \frac{1}{4}\langle \Delta_L h, \tilde{\Delta} h \rangle \\ &\quad + \frac{R}{12}\langle h, \tilde{\Delta} h \rangle, \end{aligned}$$

where the meaning of the notation  $\doteq$  is as follows; for  $f_1$  and  $f_2 \in C^\infty(M)$ , we write  $f_1 \doteq f_2$  if  $\int_M f_1 dv = \int_M f_2 dv$ .

Then, from (3.5), (3.6), (3.7) and Lemma 3.5 (i),

$$(3.8) \quad \begin{aligned} (\dot{C}_{ijk};_k h_{ij} + \dot{L}_{mk} W_{ijk}^m h_{ij} &\doteq -\dot{C}_{ijk} h_{ij;k} + \dot{L}_{mk} W_{ijk}^m h_{ij} \\ &= ((\dot{L}_{ij};_k + S_{ij}^m L_{mk})(h_{ij;k} - h_{ik;j}) + \dot{L}_{mk} W_{ijk}^m h_{ij} \\ &\doteq \frac{1}{4}|\Delta_L h|^2 + \frac{R}{12}\langle h, \Delta_L h \rangle + \frac{1}{4}\langle \Delta_L h, \tilde{\Delta} h \rangle + \frac{R}{12}\langle h, \tilde{\Delta} h \rangle \\ &\quad + S_{ij}^m L_{mk} (h_{ij;k} - h_{ik;j}) \\ &\quad - \frac{1}{2}(\Delta_L h)_{ij} h_{km} W_{ijk}^m - \frac{R}{6}h_{ij} h_{mk} W_{ijk}^m \\ &= \frac{1}{2}|\Delta_L h|^2 + \frac{R}{3}\langle h, \Delta_L h \rangle + \frac{1}{18}R^2|h|^2 \end{aligned}$$

$$+\left\langle E \circ h, \Delta_L h + \frac{R}{3} h \right\rangle + S_{ij}^m L_{mk} (h_{ij;k} - h_{ik;j}).$$

From (1.1) and (3.6)

$$\begin{aligned} (3.9) \quad h_{ij} L_{mk} \dot{W}_{ijk}^m &= h_{ij} L_{mk} \left\{ \dot{R}_{ijk}^m - \frac{1}{2} (L_{mj} h_{ik} + \dot{L}_{mj} g_{ik} \right. \\ &\quad \left. - h_{ml} L_{lj} g_{ik} + g_{mj} \dot{L}_{ik} - L_{mk} h_{ij} - \dot{L}_{mk} g_{ij} + h_{ml} L_{lk} g_{ij} - g_{mk} \dot{L}_{ij}) \right\} \\ &= h_{ij} L_{mk} \dot{R}_{ijk}^m - \frac{1}{2} \left( |L \circ h|^2 + 2 \langle L \circ h, \dot{L} \rangle - \langle L \circ h, h \circ L \rangle - |L|^2 |h|^2 - \frac{R}{3} \langle \dot{L}, h \rangle \right) \\ &= h_{ij} L_{mk} \dot{R}_{ijk}^m - \frac{R}{24} \langle h, \Delta_L h \rangle + \left\langle E \circ h, \frac{1}{2} \Delta_L h + \frac{1}{2} h \circ E + \frac{R}{6} h \right\rangle \\ &\quad + \frac{1}{2} |E|^2 |h|^2 - \frac{1}{2} |E \circ h|^2 - \frac{1}{6} \langle E \circ h \rangle^2. \end{aligned}$$

From Lemma 3.5 (i),

$$\begin{aligned} (3.10) \quad -h_{ij} h_{kl} L_{ml} W_{ijk}^m &= \frac{R}{24} \langle h, \Delta_L h \rangle - \frac{R}{24} \langle h, \tilde{\Delta} h \rangle + \frac{1}{36} R^2 |h|^2 \\ &\quad - \frac{1}{2} \langle E \circ h, \tilde{\Delta} h \rangle + \left\langle E \circ h, \frac{1}{2} \Delta_L h + h \circ E + \frac{R}{2} h \right\rangle + |E \circ h|^2 - \frac{1}{2} \langle E, h \rangle^2. \end{aligned}$$

From (3.2), Bianchi's identity  $L_{i;j} = R_{;i}/3$  and using Lemma 3.5 (iii), we get

$$\begin{aligned} (3.11) \quad S_{ij}^m L_{mk} (h_{ij;k} - h_{ik;j}) &+ h_{ij} L_{mk} \dot{R}_{ijk}^m \\ &= S_{ij}^m (L_{mk} h_{ij};_k - S_{ij}^m L_{mk;k} h_{ij} - S_{ij}^m (L_{mk} h_{ik});_j + S_{ij}^m L_{mk;j} h_{ik} \\ &\quad + h_{ij} L_{mk} (S_{ik;j}^m - S_{ij;k}^m)) \\ &\doteq S_{ij}^m (L_{mk} h_{ij};_k - \frac{1}{3} S_{ij}^m h_{ij} R_{;m} \\ &\quad - \frac{1}{2} S_{ij}^m (L_{mk} h_{ik} + h_{mk} L_{ik});_j - \frac{1}{2} S_{ij}^m (L_{mk} h_{ik} - h_{mk} L_{ik});_j \\ &\quad + S_{ij}^m L_{mk;j} h_{ik} - S_{ik}^m L_{mk;j} h_{ij} + S_{ij}^m (h_{ij} L_{mk});_k) \\ &= 2 S_{ij}^m (L_{mk} h_{ij};_k - \frac{1}{4} h_{mi;j} (L_{mk} h_{ik} + h_{mk} L_{ik});_j \\ &\quad - S_{ij}^m \left( \frac{1}{3} h_{ij} R_{;m} + \frac{1}{2} (E \circ h - h \circ E)_{mi;j} - h_{ik} C_{mj k} \right)) \\ &\doteq 2 S_{ij}^m (E_{mk} h_{ij};_k + \frac{1}{6} S_{ij}^k (R h_{ij});_k + \frac{1}{2} \langle L \circ h, \tilde{\Delta} h \rangle \\ &\quad - S_{ij}^m \left( \frac{1}{3} h_{ij} R_{;m} + \frac{1}{2} (E \circ h - h \circ E)_{mi;j} - h_{ik} C_{mj k} \right)) \end{aligned}$$

$$\begin{aligned}
&= \frac{R}{12} \langle h, \Delta_L h \rangle + \frac{1}{2} \langle E \circ h, \tilde{\Delta} h \rangle + \frac{R}{24} \langle h, \tilde{\Delta} h \rangle \\
&\quad + S_{ij}^m \left\{ 2(E_{mk} h_{ij})_{;k} - \frac{1}{3} h_{ij} R_{;m} - \frac{1}{2} (E \circ h - h \circ E)_{mi;j} + h_{ik} C_{mj k} \right\}.
\end{aligned}$$

Summing up (3.8), (3.9) and (3.10), then substituting (3.11), we obtain from (3.4) the following:

$$\begin{aligned}
\dot{X}_{ij} h_{ij} &= \frac{1}{2} |\Delta_L h|^2 + \frac{5}{12} R \langle h, \Delta_L h \rangle + \frac{1}{12} R^2 |h|^2 \\
&\quad + \left\langle E \circ h, 2\Delta_L h + \frac{3}{2} h \circ E + Rh \right\rangle + \frac{1}{2} |E|^2 |h|^2 + \frac{1}{2} |E \circ h|^2 - \frac{2}{3} \langle E, h \rangle^2 \\
&\quad + S_{ij}^m \left\{ 2(E_{mk} h_{ij})_{;k} - \frac{1}{3} h_{ij} R_{;m} - \frac{1}{2} (E \circ h - h \circ E)_{mi;j} + 2h_{ik} C_{mj k} - h_{kj} C_{kmi} \right\} \\
&\quad - h_{ij} h_{km} C_{ijk;m}.
\end{aligned}$$

Thus, from (3.3), we have the desired formula.  $\square$

**COROLLARY 3.8.** *Under the same assumptions and notations as in Proposition 3.7, the second variational formula at an Einstein metric  $g$  (cf. Corollary 3.3) is as follows:*

$$\left( \frac{d}{dt} \right)^2 \nu(g_t) \Big|_{t=0} = \frac{1}{2} \int_M \left\langle \Delta_L h + \frac{R}{2} h, \Delta_L h + \frac{R}{3} h \right\rangle dv.$$

#### § 4. Stability of the standard Einstein metric of $S^2 \times S^2$ .

**DEFINITION 4.1.** Let  $g \in \mathcal{M}(M)$  be a critical point of the functional  $\nu: \mathcal{M}(M) \rightarrow \mathbf{R}$ . Then  $g$  is said to be *stable* if

$$(4.1) \quad \left( \frac{d}{dt} \right)^2 \nu(g_t) \Big|_{t=0} \geq 0$$

for all smooth variation  $g_t$  with  $g_0 = g$ . Moreover,  $g$  is said to be *strictly stable* if  $g$  is stable and if equality of (4.1) holds only when  $(dg_t/dt)|_{t=0} \in \mathcal{S}_1(g)$  (cf. Lemma 3.6).

**REMARK.** It follows from Lemma 1.2 that if  $g$  is a (strictly) stable critical point of  $\nu$ , then so is any metric conformal to  $g$ .

**EXAMPLES.** 1. Any conformally flat metric is stable. Any half conformally flat metric of a compact orientable 4-manifold is stable (cf. Proposition 1.4).

2. Setting the scalar curvature  $R=0$  in Corollary 3.8, we see that any Ricci flat metric of a compact 4-manifold is stable.

3. Let  $g \in \mathcal{M}(S^4)$  be the standard metric of constant curvature 1. Then,  $g$  is strictly stable.

PROOF. Since  $g$  is an Einstein metric, we have only to prove that

$$\frac{1}{2} \int_{S^4} \left\langle \Delta_L h + \frac{R}{2} h, \Delta_L h + \frac{R}{3} h \right\rangle dv \geq 0$$

for all  $h \in \mathcal{S}_0(g)$  and that equality holds only when  $h=0$  (cf. Corollary 3.8).

We have  $R=12$  and  $\Delta_L h = \tilde{\Delta} h - 8h$  for  $h \in \mathcal{S}_0(g)$  (cf. Lemma 3.5 (i)). Hence,

$$\begin{aligned} \frac{1}{2} \int \left\langle \Delta_L h + \frac{R}{2} h, \Delta_L h + \frac{R}{3} h \right\rangle dv &= \frac{1}{2} \int \langle \tilde{\Delta} h - 2h, \tilde{\Delta} h - 4h \rangle dv \\ &= \frac{1}{2} \int \{ |\tilde{\Delta} h|^2 - 6 \langle h, \tilde{\Delta} h \rangle + 8 |h|^2 \} dv \\ &= \frac{1}{2} \int (|\tilde{\Delta} h|^2 + 6 |\nabla h|^2 + 8 |h|^2) dv \geq 0. \end{aligned}$$

Obviously, equality implies that  $h=0$ .  $\square$

The purpose of this section is to prove the following:

**THEOREM 4.2.** *Let  $g$  be the standard Einstein metric on  $S^2 \times S^2$ , that is,  $g = \bar{g} + \bar{\bar{g}}$ , where  $\bar{g}$  and  $\bar{\bar{g}}$  are Riemannian metrics on  $S^2$  with constant Gauss curvature 1. Then,  $g$  is a strictly stable critical point of the functional  $\nu: \mathcal{M}(S^2 \times S^2) \rightarrow \mathbf{R}$ .*

**PROOF.** First, we remark that  $\bar{g}$  and  $\bar{\bar{g}}$  are parallel tensor fields, and the curvature tensor is given as

$$(4.2) \quad R_{mijk} = (\bar{g}_{mj} \bar{g}_{ik} - \bar{g}_{ij} \bar{g}_{km}) + (\bar{\bar{g}}_{mj} \bar{\bar{g}}_{ik} - \bar{\bar{g}}_{ij} \bar{\bar{g}}_{km}).$$

For  $h \in \mathcal{S}_0(g)$ , we define  $f \in C^\infty(S^2 \times S^2)$  and  $\bar{h}, \bar{\bar{h}}, \check{h} \in T_g \mathcal{M}$  as follows:

$$(4.3) \quad \begin{cases} f = \frac{1}{2} h_{ij} \bar{g}_{ij} = -\frac{1}{2} h_{ij} \bar{\bar{g}}_{ij}, \\ \bar{h}_{ij} = \bar{g}_{ik} h_{km} \bar{g}_{mj} - f \bar{g}_{ij}, & \bar{\bar{h}}_{ij} = \bar{\bar{g}}_{ik} h_{km} \bar{\bar{g}}_{mj} + f \bar{\bar{g}}_{ij}, \\ \check{h}_{ij} = \bar{g}_{ik} h_{km} \bar{\bar{g}}_{mj} + \bar{\bar{g}}_{ik} h_{km} \bar{g}_{mj}. \end{cases}$$

Then,  $h = \bar{h} + \bar{\bar{h}} + \check{h} + f \bar{g} - f \bar{\bar{g}}$  and this decomposition is orthogonal. By (4.2), we have

$$(4.4) \quad h_{mk} R_{mijk} = \bar{h}_{ij} + \bar{\bar{h}}_{ij} - f \bar{g}_{ij} + f \bar{\bar{g}}_{ij}.$$

Then, a straightforward computation gives

$$(4.5) \quad \begin{cases} |h_{mk} R_{mijk}|^2 = |\bar{h}|^2 + |\bar{\bar{h}}|^2 + 4f^2, \\ h_{ij} h_{mk} R_{mijk} = |\bar{h}|^2 + |\bar{\bar{h}}|^2 - 4f^2, \\ (\tilde{\Delta} h)_{ij} h_{mk} R_{mijk} = \langle \bar{h}, \tilde{\Delta} \bar{h} \rangle + \langle \bar{\bar{h}}, \tilde{\Delta} \bar{\bar{h}} \rangle - 4f \Delta f, \\ |\tilde{\Delta} h|^2 = |\tilde{\Delta} \bar{h}|^2 + |\tilde{\Delta} \bar{\bar{h}}|^2 + 4(\Delta f)^2 + |\tilde{\Delta} \check{h}|^2, \\ |\nabla h|^2 = |\nabla \bar{h}|^2 + |\nabla \bar{\bar{h}}|^2 + 4|\nabla f|^2 + |\nabla \check{h}|^2. \end{cases}$$

Then, using Lemma 3.5 (i), (4.2) and (4.4), we get

$$\begin{aligned}
 (4.6) \quad & \frac{1}{2} \int \left\langle \Delta_L h + 2h, \Delta_L h + \frac{4}{3} h \right\rangle dv \\
 &= \int \left\{ 2 |h_{mk} R_{mijk}|^2 + \frac{2}{3} h_{mk} h_{ij} R_{mijk} - 2 (\tilde{\Delta} h)_{ij} h_{km} R_{mijk} + \frac{1}{2} |\tilde{\Delta} h|^2 + \frac{1}{3} |\nabla h|^2 \right\} dv \\
 &= \int \left\{ \frac{8}{3} |\bar{h}|^2 - 2 \langle \bar{h}, \tilde{\Delta} \bar{h} \rangle + \frac{1}{3} |\nabla \bar{h}|^2 + \frac{1}{2} |\tilde{\Delta} \bar{h}|^2 \right. \\
 &\quad \left. + \frac{8}{3} |\check{h}|^2 - 2 \langle \check{h}, \tilde{\Delta} \check{h} \rangle + \frac{1}{3} |\nabla \check{h}|^2 + \frac{1}{2} |\tilde{\Delta} \check{h}|^2 \right. \\
 &\quad \left. + \frac{1}{3} |\nabla \check{h}|^2 + \frac{1}{2} |\tilde{\Delta} \check{h}|^2 + \frac{16}{3} f^2 + 8f\Delta f + \frac{4}{3} |\nabla f|^2 + 2|\tilde{\Delta} f|^2 \right\} dv \\
 &= \int \left\{ \frac{8}{3} (|\bar{h}|^2 + |\check{h}|^2) + \frac{7}{3} (|\nabla \bar{h}|^2 + |\nabla \check{h}|^2) + \frac{1}{2} (|\tilde{\Delta} \bar{h}|^2 + |\tilde{\Delta} \check{h}|^2) \right. \\
 &\quad \left. + \frac{1}{3} |\nabla \check{h}|^2 + \frac{1}{2} |\tilde{\Delta} \check{h}|^2 + \frac{4}{3} (\Delta f + 2f)^2 + \frac{2}{3} \Delta f (\Delta f + 2f) \right\} dv \\
 &\geq 0,
 \end{aligned}$$

because the first eigenvalue of the Laplacian  $-\Delta$  of  $(S^2 \times S^2, g)$  is 2, and hence

$$\int \Delta f (\Delta f + 2f) dv \geq 0.$$

Next, we consider when the equality of (4.6) holds. Obviously, the equality holds if and only if  $\bar{h} = \check{h} = 0$ ,  $\nabla \check{h} = 0$  and  $\Delta f + 2f = 0$ . Since  $\operatorname{div} h = 0$  (see the definition of  $\mathcal{S}_0(g)$  in Lemma 3.6), the conditions  $\bar{h} = \check{h} = 0$  and  $\nabla \check{h} = 0$  yield  $f = \text{constant}$ . Then from  $\Delta f + 2f = 0$ , we have  $f \equiv 0$ . That is,  $h = \check{h}$  and  $\nabla h = 0$ . In particular,  $\Delta_L h = \tilde{\Delta} h = 0$ . Then from Lemma 3.5 (i) and  $R_{ij} = g_{ij}$ , we get

$$(4.7) \quad h_{ij} + h_{mk} R_{mijk} = 0.$$

On the other hand, from (4.4),  $h_{mk} R_{mijk} = 0$ , since  $h = \check{h}$ . Hence, from (4.7), we have  $h = 0$ . Thus, the equality of (4.6) holds only when  $h = 0$ .

Now the assertion follows from Corollary 3.8.  $\square$

## §5. Additional remarks.

LEMMA 5.1. Suppose that  $\dim M = 4$  and  $g \in \mathcal{M}(M)$  is a metric with nonnegative sectional curvature. Then the following pointwise inequality holds;  $3|W|^2 \leq 2R^2$ .

PROOF. Let  $\{e_1, e_2, e_3, e_4\}$  be an orthonormal frame. Then,  $f_1 = e_1 \wedge e_2 + e_3 \wedge e_4$ ,  $f_2 = e_1 \wedge e_3 + e_4 \wedge e_2$ ,  $f_3 = e_1 \wedge e_4 + e_2 \wedge e_3$ ,  $f_4 = e_1 \wedge e_2 - e_3 \wedge e_4$ ,  $f_5 = e_1 \wedge e_3 - e_4 \wedge e_2$  and  $f_6 = e_1 \wedge e_4 - e_2 \wedge e_3$  form an orthonormal frame of  $\Lambda^2$  (in our convention,  $e_i \wedge e_j = (1/2)(e_i \otimes e_j - e_j \otimes e_i)$ ). We regard the curvature tensor as a linear transformation of  $\Lambda^2$ . Then, with respect to the frame  $\{f_a\}$ , we have the following

matrix representation of the curvature tensor ;

$$\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix},$$

where  $A$  and  $C$  are  $3 \times 3$  symmetric matrices with  $\text{tr} A = \text{tr} C = R/2$ , and

$$B = (B_{\alpha\beta}) = \begin{pmatrix} E_{11} + E_{22} & E_{23} - E_{41} & E_{42} + E_{13} \\ E_{23} + E_{41} & E_{11} + E_{33} & E_{43} - E_{21} \\ E_{42} - E_{13} & E_{43} + E_{21} & E_{11} + E_{44} \end{pmatrix},$$

where  $E_{ij} = R_{ij} - (R/4)g_{ij}$ . It is known that  $A$  and  $C$  can be diagonalized for some orthonormal frame  $\{e_i\}$  ([7; Theorem 2.1]). So, we write

$$A = \begin{pmatrix} (R/6) + \lambda_1 & 0 & 0 \\ 0 & (R/6) + \lambda_2 & 0 \\ 0 & 0 & (R/6) + \lambda_3 \end{pmatrix}, \quad C = \begin{pmatrix} (R/6) + \mu_1 & 0 & 0 \\ 0 & (R/6) + \mu_2 & 0 \\ 0 & 0 & (R/6) + \mu_3 \end{pmatrix}.$$

Then,  $\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3 = 0$  and

$$(5.1) \quad |W|^2 = \sum_{\alpha=1}^3 \lambda_{\alpha}^2 + \sum_{\beta=1}^3 \mu_{\beta}^2.$$

A 2-form corresponding to a plane section is of the form  $\sum_{\alpha=1}^3 \xi_{\alpha} f_{\alpha} + \sum_{\beta=1}^3 \eta_{\beta} f_{\beta+3}$  with  $\sum \xi_{\alpha}^2 = \sum \eta_{\beta}^2 = 1/2$ . Therefore, if the sectional curvature is non-negative then,

$$(5.2) \quad \sum_{\alpha} \xi_{\alpha}^2 \left( \frac{R}{6} + \lambda_{\alpha} \right) + \sum_{\beta} \eta_{\beta}^2 \left( \frac{R}{6} + \mu_{\beta} \right) + 2 \sum_{\alpha, \beta} \xi_{\alpha} \eta_{\beta} B_{\alpha\beta} \geq 0,$$

for all  $\{\xi_{\alpha}\}$  and  $\{\eta_{\beta}\}$  with  $\sum \xi_{\alpha}^2 = \sum \eta_{\beta}^2 = 1/2$ . From this, it is easily seen that

$$\frac{R}{3} + \lambda_{\alpha} + \mu_{\beta} \geq 0 \quad \text{for all } \alpha, \beta.$$

Hence,

$$\begin{aligned} 3R^2 &= \left\{ \left( \frac{R}{3} + \lambda_1 + \mu_1 \right) + \left( \frac{R}{3} + \lambda_2 + \mu_2 \right) + \left( \frac{R}{3} + \lambda_3 + \mu_3 \right) \right\}^2 \\ &\quad + \left\{ \left( \frac{R}{3} + \lambda_1 + \mu_2 \right) + \left( \frac{R}{3} + \lambda_2 + \mu_3 \right) + \left( \frac{R}{3} + \lambda_3 + \mu_1 \right) \right\}^2 \\ &\quad + \left\{ \left( \frac{R}{3} + \lambda_1 + \mu_3 \right) + \left( \frac{R}{3} + \lambda_2 + \mu_1 \right) + \left( \frac{R}{3} + \lambda_3 + \mu_2 \right) \right\}^2 \\ &\geq \sum_{\alpha, \beta} \left( \frac{R}{3} + \lambda_{\alpha} + \mu_{\beta} \right)^2 = R^2 + 3(\sum \lambda_{\alpha}^2 + \sum \mu_{\beta}^2). \end{aligned}$$

Therefore, from (5.1), we have  $2R^2 \geq 3|W|^2$ .  $\square$



PROPOSITION 5.2. *Let  $M$  be a compact 4-dimensional manifold.*

- (i) *If  $M$  admits an Einstein metric, then  $\nu(M) \leq 16\pi^2\chi$ ;*  
(ii) *If  $M$  admits an Einstein metric with nonnegative sectional curvature, then  $\nu(M) \leq (64/5)\pi^2\chi$ , where  $\chi$  is the Euler characteristic of  $M$ .*

PROOF. (i) follows from the Gauss Bonnet formula (1.3).

(ii): If  $g \in \mathcal{M}(M)$  is an Einstein metric with nonnegative sectional curvature, then from (1.3) and Lemma 5.1, we have

$$\begin{aligned}\nu(g) &= 16\pi^2\chi - \frac{1}{12} \int R^2 dv \leq 16\pi^2\chi - \frac{1}{8} \int |W|^2 dv \\ &= 16\pi^2\chi - \frac{1}{4} \nu(g).\end{aligned}$$

Hence,  $\nu(M) \leq \nu(g) \leq 64\pi^2\chi/5$ .  $\square$

COROLLARY 5.3. *Let  $M$  be a compact oriented 4-dimensional manifold. If  $M$  admits an Einstein metric with nonnegative sectional curvature, then  $|\tau| \leq 8\chi/15$ , and equality holds if and only if  $M$  has a flat metric.*

PROOF. Let  $g \in \mathcal{M}(M)$  be the Einstein metric with nonnegative curvature. Then by Propositions 1.4 and 5.2,  $|\tau| \leq \nu(M)/24\pi^2 \leq \nu(g)/24\pi^2 \leq 8\chi/15$ .

If the equality holds, then  $g$  is half conformally flat. So, we assume that  $*(W_{mijk}e^j \wedge e^k) = W_{mijk}e^j \wedge e^k$  (resp.  $*(W_{mijk}e^j \wedge e^k) = -W_{mijk}e^j \wedge e^k$ ). From the Weitzenböck formula, we have for any harmonic 2-form  $\alpha$ ,

$$\alpha_{ij;k k} = \frac{R}{3} \alpha_{ij} + \alpha_{km} W^{km}_{ij}.$$

Hence, if furthermore  $*\alpha = -\alpha$  (resp.  $*\alpha = +\alpha$ ), then

$$(5.3) \quad \alpha_{ij;k k} = \frac{R}{3} \alpha_{ij}.$$

Now, suppose that  $g$  is not flat, i.e.,  $R > 0$ . Then, from (5.3),  $\alpha = 0$  for any harmonic 2-form with  $*\alpha = -\alpha$  (resp.  $*\alpha = \alpha$ ). Therefore,  $\tau = \pm 2$ nd Betti number of  $M$ . The 1st Betti number is zero since the Ricci curvature is positive. So,  $|\tau| = \chi - 2$ . It is easy to see that  $|\tau| = \chi - 2$  with  $|\tau| = 8\chi/15$  does not have integral solutions. This is a contradiction. Hence,  $g$  is flat.  $\square$

REMARK. This proposition slightly improves Theorem 2 of [4], where  $8/15$  is replaced by  $(2/3)^{1.5} (> 8/15)$ .

## References

- [1] A. Avez, Characteristic classes and Weyl tensor, Proc. Nat. Acad. Sci. U.S.A., **66** (1970), 265-268.  
[2] R. Bach, Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des

- Krümmungstensorbegriffs, *Math. Z.*, **9** (1921), 110-135.
- [3] M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, *J. Diff. Geom.*, **3** (1969), 379-392.
  - [4] N. Hitchin, Compact four-dimensional Einstein manifolds, *J. Diff. Geom.*, **9** (1974), 435-441.
  - [5] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 2, Interscience, 1969.
  - [6] N. H. Kuiper, On compact conformally Euclidean spaces of dimension  $>2$ , *Ann. of Math.*, **52** (1950), 478-490.
  - [7] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, *Global Analysis, Papers in Honor of K. Kodaira*, Princeton Univ. Press, 1969, 355-365.

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