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On a conformally invariant functional of the space of Riemannian metrics

By Osamu KOBAYASHI

(Received April 26, 1984)

Introduction.

Let $\mathcal{H}(M)$ be the space of all $C^\infty$ Riemannian metrics on a compact $n$-dimensional manifold $M$, and $\nu: \mathcal{H}(M) \to \mathbb{R}$ be a functional of $\mathcal{H}$ defined by $\nu(g) = (2/n) \int_M |W|^n/2\,dv$, where $W$ is the Weyl conformal curvature tensor. Our main subject in this paper is to determine $\inf\{\nu(g) ; g \in \mathcal{H}\}$, which will be denoted by $\nu(M)$. A little consideration shows that $\nu(M) > 0$ if some Pontrjagin number of $M$ is not zero. Thus, in general, $\nu(M)$ is a nontrivial invariant of a manifold.

In § 2, we shall show two general properties of $\nu(M)$. One is that $\nu(M) = 0$ for the total space $M$ of a principal circle bundle (Theorem 2.1). This provides examples of $M$ for which $\nu(M) = 0$ but which has no conformally flat metric. The other is an inequality for connected sum; $\nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2)$ (Theorem 2.2). This is useful for computing $\nu(M)$ for certain $M$.

However, to determine $\nu(M)$ for general $M$ seems to be not so easy. Even for $S^3 \times S^3$, $\nu(S^3 \times S^3)$ is not known (to the author). We want to show that the standard Einstein metric $g_0$ of $S^3 \times S^3$ is a candidate at which $\nu$ takes a minimum, if $\nu: \mathcal{H}(S^3 \times S^3) \to \mathbb{R}$ has a minimum. In fact, $g_0$ is a minimum point of $\nu$ restricted to Kähler metrics (Proposition 1.4). Moreover, we shall prove that $g_0$ is a strictly stable critical point of the functional $\nu$ (cf. Definition 4.1 and Theorem 4.2).

In the course of proof of stability of $g_0 \in \mathcal{H}(S^3 \times S^3)$, we establish the first and the second variational formulas of $\nu: \mathcal{H}(M) \to \mathbb{R}$ for 4 dimensional $M$ (Propositions 3.1 and 3.7; The first variational formula has already appeared in [2]). From these formulas, we can also see that other than conformally flat metrics, Einstein metrics are critical points of the functional $\nu$, and Ricci flat metrics are stable critical points of $\nu$.

§ 1. Preliminary definitions and remarks.

Throughout this paper, $M$ denotes a compact $C^\infty$ manifold of dimension $n$, and $\mathcal{H}(M)$ denotes the space of $C^\infty$ Riemannian metrics on $M$. For $g \in \mathcal{H}(M)$,
the curvature tensor $R^i_{jkl}$, the Ricci tensor $R_{ij} = R^k_{ijkj}$ and the scalar curvature $R = g^{ij} R_{ij}$ are defined. Our concern in this paper is the Weyl tensor defined by

$$W^i_{jkl} = R^i_{jkl} - \frac{1}{n-2} \left( L^i_{jkl} + g^i_{jk} L_{jl} - L^i_{jl} g_{jk} - g^i_{jl} L_{jk} \right),$$

where $L_{ij} = R_{ij} - \frac{1}{n-1} R g_{ij}$ (we put $W = 0$ if $n \leq 2$). From the second Bianchi identity, we have $R^i_{jkl;i} = R_{jkl;i} - R_{jkl;i}$ and hence,

$$W^i_{jkl;i} = \frac{n-3}{n-2} C_{jkl},$$

where $C_{jkl} = L_{jkl;i} - L_{jkl;i}$.

DEFINITION 1.1. We define a functional $\nu : \mathfrak{M}(M) \rightarrow \mathbb{R}$ by

$$\nu(g) = \frac{2}{n} \int_M |W|^\frac{n}{2} dV_g,$$

where $|W|^\frac{n}{2} = \langle W, W \rangle^{\frac{n}{4}} = (g_{ip} g_{jq} g_{kr} g_{ls} W_{ijkl} W^{iprs})^{\frac{n}{4}}$. For a subset $U$ of $M$ and $g \in \mathfrak{M}(M)$, we write $\nu(g; U) = \frac{2}{n} \int_U |W|^\frac{n}{2} dV_g$.

LEMMA 1.2. (i) $\nu(g^2 f g) = \nu(g)$ for any $f \in C^\infty(M)$ and $g \in \mathfrak{M}(M)$. (ii) $\nu(\varphi^* g) = \nu(g)$ for any diffeomorphism $\varphi$ of $M$. (iii) $\nu = 0$ if $\dim M \leq 3$.

PROOF. Let $W$ and $W'$ be the Weyl tensors of the metrics $g$ and $g' = e^{2f} g$, respectively. Since the Weyl tensor is invariant under a conformal change of metric, we have $\langle W', W' \rangle_{g'} = \langle W, W \rangle_{g'} = e^{-4f} \langle W, W \rangle_g$. Hence, from $dv' = e^{2f} dv$ for the volume elements, we get $|W'|^{\frac{n}{2}} dv = |W|^{\frac{n}{2}} dv$, which proves (i). (ii) is trivial, and (iii) is well-known. $\square$

For the dimensions higher than three, we first remark the following:

PROPOSITION 1.3. If $\dim M \geq 4$, then $\operatorname{sup} \{\nu(g); g \in \mathfrak{M}(M)\} = \infty$.

PROOF. Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ be the $n$-dimensional torus. If $n \geq 4$, there exists a metric $g \in \mathfrak{M}(T^n)$ with $c := \nu(g) > 0$. Then, $\tilde{g} \in \mathfrak{M}(\mathbb{R}^n)$ denoting the lift of $g$ to the universal covering, we have $\nu(\tilde{g}) \geq \nu(g)$. If $\tilde{g}$ is a chart of $M$ such that $\varphi(U) = \mathbb{R}^n$. Then, for each $\ell \in \mathbb{N}$, we can take a metric $g_\ell$ on $M$ which coincides with $\tilde{g}$ in $[0, \ell] \subset T^n \cong U$, i.e., $(\varphi_* g_\ell) [0, \ell]^n = \tilde{g} [0, \ell]^n$. We thus have $\nu(g_\ell) \geq \nu(g_\ell; U) \geq \nu(\varphi_* g_\ell; [0, \ell]^n) = \ell^n c$ and hence, $\lim_{\ell \to \infty} \nu(g_\ell) = \infty$. $\square$

On the other hand, there are non-trivial topological lower bounds for $\nu$: Any Pontrjagin class is represented by a differential form composed of only the Weyl tensor ([1]). Namely, the $m$-th Pontrjagin class $p_m \in H^{4m}(M)$ (cf. [5]) is given inductively by $\Pi_m = -(p_1 \Pi_{m-1} + \cdots + p_{m-1} \Pi_1) - 2m p_m$, where $\Pi_m \in H^{4m}(M)$ is represented by the following differential form:

$$(2\pi)^{-2m} \hat{\Omega}^{i_1} \wedge \hat{\Omega}^{i_2} \wedge \cdots \wedge \hat{\Omega}^{i_{2m}},$$

where $\hat{\Omega}^{ij} = (1/2) W^i_{jkl} e^k \wedge e^l$. So we can see that any Pontrjagin number of a $4k$-
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dimensional \( M \) is dominated by \( \nu(g) \) multiplied by some universal constant. For example, the following is well-known.

**Proposition 1.4.** If \( \dim M=4 \), then \( |p_1[M]| \leq \nu(g)/8\pi^2 \) for all \( g \in \mathcal{M}(M) \). Hence, if furthermore \( M \) is oriented, \( |\tau| \leq \nu(g)/24\pi^2 \) for all \( g \in \mathcal{M}(M) \), where \( \tau \) is the signature of \( M \). Equality holds if and only if \( g \) is a half conformally flat metric.

Thus, unlike \( \sup \nu \), \( \inf \nu \) reflects certain global properties of a manifold.

**Definition 1.5.** \( \nu(M) := \inf \{ \nu(g); g \in \mathcal{M}(M) \} \).

Here are some examples:

1. \( \nu(g)=0 \) if \( g \) is conformally flat. Hence, if \( M \) carries a conformally flat metric, then \( \nu(M)=0 \). However, \( \nu(M)=0 \) does not imply in general that \( M \) admits a conformally flat metric (see § 2).

2. The Fubini Study metric \( g_0 \) of \( CP^2 \) is half conformally flat. Hence, by Proposition 1.4, \( \nu(CP^2)=\nu(g_0)=24\pi^2 \). For the connected sum \( kCP^2 \) of \( k \) copies of \( CP^2 \), Proposition 1.4 gives only \( \nu(kCP^2) \geq 24\pi^2|\tau(kCP^2)| = 24k\pi^2 \). We shall show in § 2 that \( \nu(kCP^2) \leq k\nu(CP^2) = 24k\pi^2 \). Hence, we have \( \nu(kCP^2) = 24k\pi^2 \).

3. Although the author does not know the value \( \nu(S^3 \times S^3) \) at present, there are partial results which suggest that \( \nu(S^3 \times S^3) \) may be positive. Let \( \bar{g}, \tilde{g} \) be two Riemannian metrics on \( S^3 \) with the Gauss curvatures \( K, \tilde{K} \), respectively. Consider the product metric \( \bar{g} \times \tilde{g} \) on \( S^3 \times S^3 \). Then, \( \nu(g) = (128/3)\pi^2 + (2/3)\int_{S^3 \times S^3} (\bar{K} - \tilde{K})^2 dv \).

**Proof.** The Weyl tensor is computed as \( W_{ij,kl} = (1/6)(\bar{K} + \tilde{K})(2\bar{g}_{ij}\tilde{g}_{kl} - 2\bar{g}_{ij}\tilde{g}_{kl}) \). Hence, we have \( |W|^2 = 2(\bar{K} + \tilde{K})^2/3 = 8\bar{K}\tilde{K}/3 + 2(\bar{K} - \tilde{K})^2/3 \). Then, from the Gauss Bonnet formula, \( \nu(g) = (8/3)\int_{S^3 \times S^3} (\bar{K} + \tilde{K})^2 dv = (2/3)(\bar{K} - \tilde{K})^2 dv = 128\pi^2/3 + (2/3)\int_{S^3 \times S^3} (\bar{K} - \tilde{K})^2 dv \).}

So, among the product metrics of \( S^3 \times S^3 \), the standard Einstein metric attains the smallest value \( 128\pi^2/3 \). This is generalized as follows:

**Proposition 1.6.** Suppose that \( \dim M=4 \) and \( g \in \mathcal{M}(M) \) is a Kähler metric for some complex structure of \( M \). Then,

\[
\nu(g) \geq 24\pi^2|\tau| + \frac{16}{3} \pi^2 \min \{2X - 6r, 2X + 3r\},
\]

where \( \tau \) and \( X \) are the signature and the Euler number of \( M \) respectively. The equality holds if and only if \( g \) is an Einstein Kähler metric.

**Proof.** By the four dimensional Gauss Bonnet theorem,

\[
(1.3) \quad \nu(g) = 16\pi^2 X + \int |E|^2 dv - \frac{1}{12} \int R\^2 dv,
\]

where \( E \) and \( R \) are the Einstein and scalar curvature of \( g \) respectively. The equality holds if and only if \( g \) is an Einstein metric.

Expanding \( |E|^2 \) and integrating by parts, we obtain

\[
\nu(g) = \int_{S^3 \times S^3} (\bar{K} - \tilde{K})^2 dv \quad \text{and} \quad \nu(g) \leq 24\pi^2|\tau| + \frac{16}{3} \pi^2 \min \{2X - 6r, 2X + 3r\}.
\]
where $E$ is the traceless part of the Ricci tensor; $E_{ij} = R_{ij} - (R/4)g_{ij}$.

Let $\rho$ be the Ricci form of the Kähler metric $g$ ([5]). Then, it is easily seen that

$$\rho \wedge \rho = \left( \frac{1}{2} R^2 - 2 |E|^2 \right) dv.$$

Since the first Chern class is represented by $\rho/4\pi$, we have

$$\int \rho \wedge \rho = 16\pi \gamma \epsilon = 16\pi \gamma (2\gamma + 3\epsilon).$$

This, together with (1.3) and (1.4), yields

$$i(g) = -8\pi^2 \gamma + 32\pi \gamma + \frac{2}{3} \int |E|^2 dv.$$  

Now, we get the desired inequality.

§ 2. Two general formulas for $\nu(M)$.

**Theorem 2.1.** If $S^1$ acts freely and differentiably on $M$, then $\nu(M) = 0$.

**Proof.** Let $K$ denote the vector field on $M$ which generates the $S^1$ action.

Since $S^1$ is compact, there is an $S^1$ invariant Riemannian metric $h$ on $M$, for which $K$ is a Killing vector field. Since the action is free, $h(K, K)$ is nowhere zero, hence, $g = (h(K, K))^{-1} h$ defines a Riemannian metric. Then we have

$$\mathcal{L}_K g = 0 \quad \text{and} \quad g(K, K) = 1.$$  

Now, consider a family of Riemannian metrics $\{g(t); 0 < t \leq 1\}$ defined by

$$g_{ij}(t) = g_{ij} - (1 - t^2) \alpha_i \alpha_j,$$

where $\alpha$ is the 1-form associated with $K$ with respect to $g = g(1)$, i.e., $\alpha_i = g_{ij} K^j$.

The inverse matrix $g^{ij}(t)$ is easily seen to be $g^{ij} - (1 - t^2) K^i K^j$. Then, using (2.1), we get the relation between the Christoffel symbols of $g(t)$ and $g$:

$$\Gamma^t_{ij}(t) - \Gamma^t_{ij} = -(1 - t^2)(K^k \alpha_j + K^k \alpha_i).$$
where the covariant derivation in the right hand side is taken as one with respect to \( g \). From this and (2.1), we have

\[
R^i_{jkh}(t) - R^i_{jkh} = -(1-t^2) \left\{ (K^i_{j;kh})_{;i} - (K^i_{j;kh})_{;i} + K^m R^i_{m;kh} \alpha_j \right\} \\
+ K^i_{j;kh} - K^i_{j;kh} (1-t^2) \left\{ K^i_{j;akh} - (K^i_{j;akh})_{;i} + K^m R^i_{m;akh} \right\}.
\]

Then, again using (2.1), we have

\[
g^{jk}(t)(R^i_{jkh}(t) - R^i_{jkh}) = -(1-t^2) \left\{ (K^i_{i;kh})_{;i} - (K^i_{i;kh})_{;i} + K^m R^i_{m;kh} \right\} \\
+ K^i_{i;kh} - K^i_{i;kh} (1-t^2) K^m R^i_{m;kh}.
\]

On the other hand, \( g^{jn}(t)R^i_{jkh} = R^i_{nkh} - (1-t^2) K^m R^i_{nkzh} \). Hence, we get

\[
g^{jn}(t)R^i_{jkh}(t) = R^i_{nkh} - (1-t^2) \left\{ (K^i_{i;nh})_{;i} - (K^i_{i;nh})_{;i} + K^m R^i_{m;nh} \right\} \\
+ K^i_{i;nh} - K^i_{i;nh} K^m R^i_{m;nh}.
\]

Note that (2.6) does not contain terms of \( t^{-2} \) and that both sides of (2.6) are tensors of type \((2, 2)\). So, there is a constant \( c \) such that \( |R^i_{jkh}(t)| < c \) for all \( t \in (0, 1] \). In particular, \( |W(t)| < c \). On the other hand, the volume form \( dv(t) \) relative to the metric \( g(t) \) is easily computed as \( dv(t) = t dv \). Thus, we get

\[
\lim_{t \to 0^+} \nu(g(t)) = 0.
\]

Hence, \( \nu(M) = 0 \). \( \square \)

REMARK. There is no conformally flat metric on \( S^p \times T^q \), \( p, q \geq 2 \), i.e., \( \nu(g) > 0 \) for any \( g \in \mathcal{M}(S^p \times T^q) \), \( p, q \geq 2 \), because, by a theorem of Kuiper [6; Theorem III], the universal covering space of a compact conformally flat space with an infinite Abelian fundamental group must be \( R^n \) or \( R \times S^{n-1} \). However, the above theorem asserts that \( \nu(S^p \times T^q) = 0 \). So, in general, \( \nu(M) = 0 \) does not imply the existence of a conformally flat metric of \( M \).

**THEOREM 2.2.** For any compact manifolds \( M_1 \) and \( M_2 \) of the same dimension, \( \nu(M_1 \times M_2) \leq \nu(M_1) + \nu(M_2) \).

For the proof, we prepare the following lemma.

**LEMMA 2.3.** Let \( g \in \mathcal{M}(M) \) be given. Then, for each \( \varepsilon > 0 \), there is a \( \tilde{g} \in \mathcal{M}(M) \) such that \( \nu(g) - \nu(\tilde{g}) < \varepsilon \) and \( \tilde{g} \) is flat in an open subset of \( M \).

Proof. Let \( (U, \phi) \) be a chart of \( M \) such that \( \phi(U) \supset \{ x \in R^n ; |x| < 1 \} \) and in the coordinate expression of the metric \( g|U = g_{ij}(x) dx^i dx^j \),

\[
g_{ij}(x) = \delta_{ij}
\]

holds. Take a nonnegative smooth function \( \varphi : R^n \to R \) such that \( \varphi(x) = 1 \) if \( |x| \leq 1/2 \), and \( \varphi(x) = 0 \) if \( |x| \geq 1 \). We set \( \varphi_t(x) = \varphi(x/t) \). The support of \( \varphi_t \) is contained in \( B_t = \{ x \in R^n ; |x| < t \} \). For \( 0 < t < 1 \), we define a metric \( \tilde{g} \in \mathcal{M}(M) \) by \( \tilde{g}|(M \setminus U) = g|(M \setminus U) \) and...
(2.8) \[ \bar{g}_{ij} = (1 - \varphi_i)g_{ij} + \varphi_i \partial_i \]
in \( U \). We shall show that \( \bar{g} \) has the desired properties for a sufficiently small \( t \).

It follows from (2.7) and the definition of \( \varphi_i \) that

(2.9) \[ |\varphi_i(g_{ij} - \partial_i)| < c_1 t, \quad |\partial_i g_{ij}| < c_1 t^{-1}, \quad |\partial_i \varphi_i| < c_1 t^{-2}, \]

for some constant \( c_1 \) where \( \partial \) is the Euclidean gradient. Hence, \( |\partial_i \bar{g}_{ij}| < c_2 \) and \( |\partial_i \bar{g}_{ij}| < c_2(t^{-1} + 1) \) for some \( c_2 \). Then, putting \( f_i = \bar{g}(\bar{W}, \bar{W})^{n/4}(\det(\bar{g}_{ij}))^{1/2} \), we can easily see that

(2.10) \[ f_i < c_3(t^{-1} + 1)^{n/2}. \]

Thus, we get

(2.11) \[ \int_{B_t} f_i dx \leq c_3(t^{-1} + 1)^{n/2} \int_{B_t} dx = c_3(t^{-1} + 1)^{n/2} t^n = c_3(t^2 + t)^{n/2}. \]

On the other hand,

\[ |\nu(g) - \nu(\bar{g})| = |\nu(g; \phi^{-1}(B_i)) - \nu(\bar{g}; \phi^{-1}(B_i))| \]

\[ \leq \nu(g; \phi^{-1}(B_i)) + \frac{2}{n} \int_{B_t} f_i dx. \]

Therefore, from (2.11), we conclude that \( |\nu(g) - \nu(\bar{g})| < \varepsilon \) for a sufficiently small \( t \). It is obvious from (2.8) that \( \bar{g} \) is flat in \( \phi^{-1}(B_i) \). \( \square \)

**Proof of Theorem 2.2.** Let \( \varepsilon \) be an arbitrary positive number. Take \( g_i \in \mathfrak{M}(M_i) \) so that \( \nu(g_i) \leq \nu(M_i) + \varepsilon, i = 1, 2 \). By the above lemma, we can choose \( \bar{g}_i \in \mathfrak{M}(M_i) \) such that \( \nu(\bar{g}_i) \leq \nu(M_i) + 2\varepsilon \) and \( \bar{g}_i \) is flat in some neighbourhood of \( M_i \). Suppose that for some \( r > 0 \) and \( p_i \in M_i, \bar{g}_i \) is flat in \( U_i(p_i; [0, 2r]) := \{ x \in M_i ; d(x, p_i) \in [0, 2r] \} \), where \( d \) is the distance function of the metric \( g_i, i = 1, 2 \).

We define a diffeomorphism \( \varphi : U_i(p_i; (r/2, 2r)) \rightarrow U_i(p_i; (r/2, 2r)) \) by \( \varphi(\exp_{p_i}X) = \exp_{p_i}(-r^2/g_i(X, X)\varphi_oX), \) where \( \varphi_o : T_{p_i}M_i \rightarrow T_{p_i} \bar{M}_i \) is a linear isometry and \( \exp_{p_i} \) denotes the exponential map at \( p_i \in M_i \) with respect to \( g_i \). Then we can regard \( M_i \# M_2 \) as \( \{ M_i \setminus U_i(p_i; [0, r/2]) \cup \{ M_2 \setminus U_i(p_2; [0, r/2]) \} \}. \)

Let \( f_i \) be a positive smooth function on \( M_i \) such that \( f_i(x) = (d_i(x, p_i))^2 \) if \( r/2 < d_i(x, p_i) < 2r \). Then, \( f_i \bar{g}_i \) is a Riemannian metric on \( M_i \) and is conformally flat in \( U_i(p_i; [0, 2r]) \). Moreover, \( \varphi : (U_i(p_i; (r/2, 2r)), f_i \bar{g}_i) \rightarrow (U_i(p_2; (r/2, 2r)), f_2 g_2) \) becomes an isometry. Hence, we can define a Riemannian metric \( g \) on \( M_i \# M_2 \) by \( g(M_i \setminus U_i(p_i; [0, r/2])) := f_i \bar{g}_i(M_i \setminus U_i(p_i; [0, r/2])), i = 1, 2 \). Then, we have

\[ \nu(g) = \nu(f_i \bar{g}_i; M_i \setminus U_i(p_i; [0, r/2])) + \nu(f_2 \bar{g}_2; M_2 \setminus U_i(p_2; [0, r/2])) \]

\[ = \nu(f_i \bar{g}_i) + \nu(f_2 \bar{g}_2) \]
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\[ \nu(M_1) + \nu(M_2) + 4\varepsilon. \]

Therefore, \( \nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2) + 4\varepsilon \) for arbitrary \( \varepsilon > 0 \). That is, \( \nu(M_1 \# M_2) \leq \nu(M_1) + \nu(M_2) \). \( \square \)

§ 3. Variational formulas in dimension four.

In this section, we use the following abbreviation for shortness' sake:
1. Omitting the summation sign, e.g., \( C_{ijk;k} + L_{mk} W_{mjk} \) stands for \( \sum_k C_{ijk;k} + \sum_k L_{mk} W_{mjk} \).
2. Identification through the duality defined by metric, e.g., \( S_{ij} = (1/2)(h_{jk;i} + h_{ki;j} - h_{ij;k}) \) stands for \( S_{ij} = (1/2) g^{kl}(h_{jk;i} + h_{ki;j} - h_{ij;k}) \).

**Proposition 3.1 ([2]).** Suppose that \( M \) is a compact manifold of dimension 4. Then for a smooth curve \( g = g(t) \) in \( \mathcal{A}(M) \), we have

\[
\frac{d}{dt} \nu(g) = \int_M \left( X, \frac{d}{dt} g \right) dv,
\]

where \( X \) is a symmetric 2-tensor defined by \( X_{ij} = C_{ijk;k} + L_{mk} W_{mjk} \) (see § 1 for \( C, L \) and \( W \)).

**Proof.** We set \( h_{ij} = (d/dt)g_{ij} \) and \( S_{ij} = (d/dt)g_{ij} \). Then,

\[
S_{ij} = \frac{1}{2}(h_{jk;i} + h_{ki;j} - h_{ij;k}).
\]

Hence,

\[
\frac{d}{dt} R_{mjk} = S_{njk} - S_{mj};k.
\]

Then, from (1.1) and elementary algebraic properties of curvature tensor, we have

\[
W^m_{ijk}(\frac{d}{dt} W^m_{ijk}) = W^m_{ijk}(2h_{ijk;k} + L_{mk} h_{jk;i}).
\]

Therefore, using \( (d/dt)g^ij = -h^ij \) and \( (d/dt)dv = \gamma h_{ij} dv \), we get

\[
\frac{d}{dt} \nu(g) = \int \left\{ W^m_{ijk}(2h_{ijk;k} + L_{mk} h_{jk;i}) - W^m_{ijk} W_{mji} h^kl + \frac{1}{4} |W|^2 h_{kl} \right\} dv.
\]

where we use Stokes' formula and (1.2). Thus, we have only to prove

\[
W^m_{ijk} W_{mji} = (1/4) |W|^2 g_{kl}.
\]

It is known that a symmetric linear transformation on the space of 2-forms \( \Lambda^2 \) commutes with the Hodge star \( \ast : \Lambda^2 \to \Lambda^2 \) (since the argument is local, we
need not assume $M$ is orientable) if and only if its Ricci contraction is proportional to $g$ (cf. [7; Theorem 1.3]). Viewed as a symmetric transformation on $\mathcal{M}$, the Weyl tensor commutes with the Hodge star, because the Ricci contraction of $W$ is zero. Hence, so does $W \cdot W$, and the Ricci contraction of $W \cdot W$ is proportional to $g$. That is, $W^{\alpha \beta} W^{\alpha \beta}_{jk} = \lambda g^{i j}$ for some scalar $\lambda$, which implies $W^{\mu \nu}_{ij} W^{\mu \nu}_{ijkl} = (1/4) |W|^2 g_{ki}$.

That $X$ is symmetric is not difficult to see. □

**Corollary 3.2.** If $\dim M=4$, the tensor $X$ has the following properties;
(i) $X_i = 0$; (ii) $X_{ij;j} = 0$; (iii) $X \otimes g$ is conformally invariant.

**Proof.** Easy consequence of Lemma 1.2. □

**Corollary 3.3.** If $\dim M=4$ and $g \in \mathcal{M}(M)$ is conformal to an Einstein metric, then $g$ is a critical point of $\nu : \mathcal{M}(M) \to \mathbb{R}$.

**Proof.** Obviously, $X=0$ if $g$ is an Einstein metric. Thus, the assertion follows from Corollary 3.2 (iii). □

Next, we shall compute the second variational formula. To do this, we review the Lichnerowicz Laplacian and decomposition of the space of symmetric 2-tensor fields (cf. [3]).

**Definition 3.4.** The tangent space $T_g \mathcal{M}(M)$ of $\mathcal{M}(M)$ at $g$ is naturally identified with the space of $C^\infty$ symmetric 2-tensor fields on $M$. The Lichnerowicz Laplacian $\Delta_L : T_g \mathcal{M} \to T_g \mathcal{M}$ is defined by

$$ (\Delta_L h)_{ij} := (\tilde{\Delta} h)_{ij} + h_{ik;js} - h_{ks;ij} + h_{js;ki} - h_{js;ik}, $$

where $\tilde{\Delta} : T_g \mathcal{M} \to T_g \mathcal{M}$ is the rough Laplacian; $(\tilde{\Delta} h)_{ij} = h_{ij;ks}$ (our sign convention of Laplacians is opposite to that used in [3]).

**Lemma 3.5.** (i) $(\Delta_L h)_{ij} = (\tilde{\Delta} h)_{ij} - 2(h_{ik} R_{kj} - R_{ik} h_{kj} - 2h_{mk} R_{ijk}^m)$. Hence, if $\dim M=4$, then

$$ (\Delta_L h)_{ij} = (\tilde{\Delta} h)_{ij} - 2(h_{ik} E_{kj} + E_{ik} h_{kj}) + \langle E, h \rangle g_{ij} $$

$$ + (\text{tr} h)(E_{ij} + (R/6) g_{ij}) - (2R/3) h_{ij} - 2h_{mk} W^{\alpha \beta}_{ij j k}, $$

where $E_{ij} = R_{ij} - (R/4) g_{ij}$.

(ii) $\int_M \langle h', \Delta_L h'' \rangle dv = - \int_M \{ \langle h', \tilde{\Delta} h'' \rangle + (h'_{i;jk} - h'_{ij;ks})(h''_{ik} - h''_{ki}) + 2h''_{ij} h'_{ik} \} dv$ for $h', h'' \in T_g \mathcal{M}$.

(iii) $\int_M \langle h', \Delta_L h'' \rangle dv = \int_M \{ h'_{i;jk} (h''_{ik} + h''_{ki}) - h''_{ij} h'_{ik} \} dv$, for $h', h'' \in T_g \mathcal{M}$.

**Proof.** Easy and omitted. □
Lemma 3.6. If M is a compact manifold, \( T_\text{g} \mathcal{M}(M) \) has the following decomposition; 
\( T_\text{g} \mathcal{M} = S_0(g) \oplus S_1(g) \), where \( S_0(g) = \{ h \in T_\text{g} \mathcal{M} ; \text{tr} \ h = 0, \text{div} \ h = 0 \} \) and 
\( S_1(g) = \{ h \in T_\text{g} \mathcal{M} ; h = \mathcal{L}_u \text{g} + f \text{g} \} \) for some \( u \in \mathfrak{X}(M) \) and \( f \in C^\infty(M) \). This decomposition is orthogonal with respect to the \( L^2 \) inner product defined by \( g \).

Proof. Put \( P(u) = \mathcal{L}_u \text{g} - (1/n)(\text{tr} \mathcal{L}_u \text{g}) \text{g} \) for a vector field \( u \). Then, it is easy to check that the principal symbol of the linear differential operator \( P : \mathfrak{X} \rightarrow T_\text{g} \mathcal{M} \) is injective. Hence, \( T_\text{g} \mathcal{M} = \text{Ker} P^* \oplus \text{Im} P \) (cf. [3]), where \( P^* \) is the adjoint operator of \( P \). \( P^* \) is computed as \( P^*(h) = -2(h_1 - (1/n)(\text{tr} h)g_{ij}j) \). From this, we have \( \text{Ker} P^* = S_0 \oplus C^\infty(M) \cdot g \). Then, putting \( S_i = C^\infty(M) \cdot g \oplus \text{Im} P \), we get the desired decomposition. \( \square \)

Remarks. 1. Let \( G \) be the semi direct product of the diffeomorphism group \( \mathfrak{D}(M) \) and \( C^\infty(M) \) with multiplication; \( \langle \varphi_1, f_1 \rangle \cdot \langle \varphi_2, f_2 \rangle = \langle \varphi_1 \varphi_2, f_1 f_2 + f_1 \rangle \) for \( \varphi_1, \varphi_2 \in \mathfrak{D}(M) \) and \( f_1, f_2 \in C^\infty(M) \). Then, \( G \) acts on \( \mathcal{M}(M) \) on the right as follows; \( (g, \langle \varphi, f \rangle) \mapsto e^{i\varphi} \varphi f \text{g}, g \in \mathcal{M}(M), \varphi \in \mathfrak{D}(M) \) and \( f \in C^\infty(M) \). Lemma 1.2 says that \( \nu \) is constant on every \( G \)-orbit in \( \mathcal{M}(M) \). \( S_i(g) \) in the above lemma is regarded as the tangent space at \( g \) of the \( G \)-orbit of \( g \).

2. Using the orthogonality between \( S_0 \) and \( S_1 \), we have an isomorphism 
\( S_0(g) \cong S_0(e^{\varphi} g) ; h \mapsto e^{(n-1)/2} h \). In particular, we see that \( S_0(g) = S_0(e^{\varphi} g) \) if \( \dim M = 2 \), \( S_0(g) \otimes g = S_0(e^{\varphi} g) \otimes e^{\varphi} g \) if \( \dim M = 4 \), and so on.

Proposition 3.7. Suppose that \( M \) is a compact manifold of dimension 4, and \( g \in \mathcal{M}(M) \) is a critical point of \( \nu : \mathcal{M}(M) \rightarrow \mathbb{R} \). Let \( g_t \) be a smooth variation of \( g \) with \( g_0 = g \). Then,
\[
\left( \frac{d}{dt} \nu(g_t) \right)_{t=0} = \int_M \left\{ \frac{1}{2} \left( \Delta_t h + \frac{R}{2} h, \Delta_t h + \frac{R}{3} h \right) + \left( E \cdot h, 2\Delta_t h + \frac{3}{2} h \cdot h + 3R h \right) + \frac{1}{2} |E|^2 |h|^2 + \frac{1}{2} |E \cdot h|^2 - \frac{1}{3} \langle E, h \rangle^2 \right. \\
+ S_0 \left( 2E_{ij} h_{ij}; h - \frac{1}{3} h_{ij} R_{km} - \frac{1}{2} (E \cdot h - h \cdot E)_{mij} + 2h_{ik} C_{mj} - h_{ik} C_{km} \right) \\
- h_{ij} h_{km} C_{ijk;m} \right\} dv ,
\]
where \( h \) is the \( S_0(g) \) component of \( (dg/dt) \) at \( t=0 \in T_\text{g} \mathcal{M} \) (cf. Lemma 3.6), \( (E \cdot h)_{ij} = E_{ij} h_{kl} \) and \( S_0 = (1/2)(h_{ij} k + h_{ik} j - h_{ij} l) \).

Proof. From the first variational formula (Proposition 3.1), \( (d\nu(g_t))/dt = \int_M \langle X_t, (dg_t)/dt \rangle dv_t \). Since \( X_t \in S_0(g_t) \) (Corollary 3.2), we have \( (d\nu(g_t))/dt = \int_M \langle X_t, h_t \rangle dv_t \), where \( h_t \) is the \( S_0(g_t) \) component of \( (dg_t)/dt \). Thus, since \( X = X_0 = 0 \), we get
\begin{equation}
\left( \frac{d}{dt} \right)^2 \nu(g) \bigg|_{t=0} = \int_M \langle \dot{X}, h \rangle dv,
\end{equation}
where \( \cdot \) means \((d/dt)|_{t=0}\), i.e., \( \dot{X} = (dX/dt)|_{t=0} \). If we write \( \dot{X} = (DX)(\dot{g}) \), we can see \( \dot{X} = (DX)(h) \), because that \( X=0 \) is a conformally invariant property by Corollary 3.2 (iii). Hence, it suffices to prove the formula under the assumption that \( \dot{g} = h \in \mathcal{S}_0(g) \). So, we assume in the following that \( h_{ij} = g_{ij} \) and \( h_{ij} = 0 \).

From the definition of \( X \),
\begin{equation}
X_{ij} = (g^{km}C_{ijk;m}) + (g^{kt}L_{mt}W_{tjk}^{m}) = (C_{ijk;k} - S_{kl}C_{mjk} - S_{kl}C_{mik}) + L_{mk}W_{tjk}^{m} = h_{ij}W_{tjk}^{m} - h_{ik}L_{mt}W_{tjk}^{m}.
\end{equation}

From the definition of \( C \),
\begin{equation}
C_{ijk} = (C_{ijk})_{;k} = (C_{ijk})_{;k} + S_{kl}L_{mj} - S_{kl}L_{mk}.
\end{equation}

From the definition of \( L \) and the Lichnerowicz Laplacian, and (3.2),
\begin{equation}
L_{ij} = -\frac{1}{2}(\Delta_L h)_{ij} - \frac{R}{6} h_{ij} + \frac{1}{6} \langle E, h \rangle g_{ij}.
\end{equation}

From this and Lemma 3.5 (ii),
\begin{equation}
(L_{ij})_{;k}(h_{ij;k} - h_{ik;j}) = \frac{1}{4} |\Delta_L h|^2 + \frac{R}{12} \langle h, \Delta_L h \rangle + \frac{1}{4} \langle \Delta_L h, \Delta h \rangle + \frac{R}{12} \langle h, \Delta h \rangle,
\end{equation}
where the meaning of the notation \( \dot{=} \) is as follows: for \( f_1 \) and \( f_2 \in \mathcal{C}^0(M) \), we write \( f_1 = f_2 \) if \( \int_M f_1 dv = \int_M f_2 dv \).

Then, from (3.5), (3.6), (3.7) and Lemma 3.5 (i),
\begin{equation}
(C_{ijk})_{;k} h_{ij} + L_{mk}W_{tjk}^{m} h_{ij} \dot{=} - (C_{ijk})_{;k} h_{ij} + L_{mk}W_{tjk}^{m} h_{ij}
\end{equation}
Conformally invariant functional

\[ + \langle E \ast h, \Delta_L h + \frac{R}{3} h \rangle + S_{ij} L_{mk} (h_{ij;k} - h_{ik;j}) . \]

From (1.1) and (3.6)

\[ (3.9) \quad h_{ij} L_{mk} W_{mjk} = h_{ij} L_{mk} \left\{ \frac{\tilde{R}_{mjk}}{2} \left( L_{mj} h_{ik} + \tilde{L}_{mj} g_{ik} \right) - h_{ml} L_{ijk} g_{ik} - h_{mk} L_{ijk} g_{ij} - h_{mk} \tilde{L}_{ijk} \right\} \]

\[ = h_{ij} L_{mk} \tilde{R}_{mjk} - \frac{1}{2} \left( \langle L \ast h \rangle^2 + 2 \langle L \ast h, \tilde{L} \rangle - \langle L \ast h, h \ast L \rangle - |L|^2 |h|^2 - \frac{R}{3} \langle L, h \rangle \right) \]

\[ = h_{ij} L_{mk} \tilde{R}_{mjk} - \frac{R}{24} \langle h, \Delta_L h \rangle + \langle E \ast h, \frac{1}{2} \Delta_L h + \frac{1}{2} h \ast E + \frac{R}{6} h \rangle \]

\[ + \frac{1}{2} |E|^2 |h|^2 - \frac{1}{2} |E \ast h|^2 - \frac{1}{6} \langle E \ast h \rangle^2 . \]

From Lemma 3.5 (i),

\[ (3.10) \quad -h_{ij} h_{kl} L_{m} W_{mjk} = \frac{R}{24} \langle h, \Delta_L h \rangle - \frac{R}{24} \langle h, \Delta_L h \rangle + \frac{1}{36} R^2 |h|^2 \]

\[ = -\frac{1}{2} \langle E \ast h, \Delta_L h \rangle + \langle E \ast h, \frac{1}{2} \Delta_L h + h \ast E + \frac{R}{2} h \rangle + |E \ast h|^2 - \frac{1}{2} \langle E, h \rangle^2 . \]

From (3.2), Bianchi’s identity \( L_{ij;k} = R_{ijk} / 3 \) and using Lemma 3.5 (iii), we get

\[ (3.11) \quad S_{ij} L_{mk} (h_{ij;k} - h_{ik;j}) + h_{ij} L_{mk} \tilde{R}_{mjk} \]

\[ = S_{ij} (L_{mk} h_{ij})_{;k} - S_{ij} L_{mk ; k} h_{ij} - S_{ij} (L_{mk} h_{ik})_{;j} - S_{ij} L_{mk ; j} h_{ik} \]

\[ + h_{ij} L_{mk} (S_{ik} - S_{ij}) \]

\[ = S_{ij} (L_{mk} h_{ij})_{;k} - \frac{1}{3} S_{ij} h_{ij} R_{;m} \]

\[ - \frac{1}{2} S_{ij} (L_{mk} h_{ik} + h_{mk} L_{ik})_{;j} - \frac{1}{2} S_{ij} (L_{mk} h_{ik} - h_{mk} L_{ik})_{;j} \]

\[ + S_{ij} L_{mk ; j} h_{ik} - S_{ij} L_{mk ; i} h_{jk} + S_{ij} (h_{ij} L_{mk}) ; k \]

\[ = 2 S_{ij} (L_{mk} h_{ij})_{;k} - \frac{1}{4} h_{mj ; j} (L_{mk} h_{ik} + h_{mk} L_{ik})_{;j} \]

\[ - S_{ij} \left( \frac{1}{3} h_{ij} R_{;m} + \frac{1}{2} (E \ast h - h \ast E)_{mj ; j} - h_{ik} C_{mk} \right) \]

\[ = 2 S_{ij} (E_{mk} h_{ij})_{;k} + \frac{1}{6} S_{ij} (R h_{ij})_{;k} + \frac{1}{2} \langle L \ast h, \Delta h \rangle \]

\[ - S_{ij} \left( \frac{1}{3} h_{ij} R_{;m} + \frac{1}{2} (E \ast h - h \ast E)_{mj ; j} - h_{ik} C_{mk} \right) \]
\[ \frac{d}{dt} \langle h, \Delta_E h \rangle + \frac{1}{2} \langle E^* h, \Delta h \rangle + \frac{R}{24} \langle h, \Delta h \rangle \\
+ \sum_{j=0}^{n} \left\{ 2(E_{m,h} R_{;i;j}) - \frac{1}{3} R_{i,j} R_{;m} - \frac{1}{2} \langle E^* h - h^* E \rangle_{m;i,j} + h_{ik} C_{mjk} \right\}. \]

Summing up (3.8), (3.9) and (3.10), then substituting (3.11), we obtain from (3.4) the following:

\[ \sum_{i=1}^{n} \sum_{j=0}^{n} \left[ \frac{1}{2} \Delta_E h - \frac{1}{2} R_{i,j} h_{;m} \right] + \frac{5}{12} R \left\langle h, \Delta_E h \right\rangle + \frac{1}{12} R_{i,j} \left\langle h, \Delta_E h \right\rangle \]
\[ + \left\{ E^* h, 2 \Delta_E h + \frac{1}{2} h^* R + R h \right\} + \frac{1}{2} |E|^2 \langle h, h \rangle + \frac{1}{2} |E* h|^2 - \frac{2}{3} \langle E, h \rangle^2 \\
+ \sum_{j=0}^{n} \left\{ 2(E_{m,h} R_{;i;j}) - \frac{1}{3} R_{i,j} R_{;m} - \frac{1}{2} \langle E^* h - h^* E \rangle_{m;i,j} + 2 h_{ik} C_{mjk} - h_{ik} C_{mjk} \right\} \\
- h_{ik} R_{m;ik}. \]

Thus, from (3.3), we have the desired formula. \( \square \)

**Corollary 3.8.** Under the same assumptions and notations as in Proposition 3.7, the second variational formula at an Einstein metric \( g \) (cf. Corollary 3.3) is as follows:

\[ \left( \frac{d}{dt} \right)^2 \langle g, \Delta_E h \rangle \mid_{t=0} = \frac{1}{2} \left[ \langle \Delta_E h + R_{i,j} h_{;m} + R_{;m} \rangle \right] dv. \]

**§ 4. Stability of the standard Einstein metric of \( S^2 \times S^2. \)**

**Definition 4.1.** Let \( g \in \mathcal{M}(M) \) be a critical point of the functional \( \nu : \mathcal{M}(M) \to \mathbb{R} \). Then \( g \) is said to be stable if

\[ \left( \frac{d}{dt} \right)^t \nu(g_t) \mid_{t=0} \geq 0 \]

for all smooth variation \( g_t \) with \( g_0 = g \). Moreover, \( g \) is said to be strictly stable if \( g \) is stable and if equality of (4.1) holds only when \( (dg_t/dt) \mid_{t=0} = S_t(g) \) (cf. Lemma 3.6).

**Remark.** It follows from Lemma 1.2 that if \( g \) is a (strictly) stable critical point of \( \nu \), then so is any metric conformal to \( g \).

**Examples.** 1. Any conformally flat metric is stable. Any half conformally flat metric of a compact orientable 4-manifold is stable (cf. Proposition 1.4).

2. Setting the scalar curvature \( R = 0 \) in Corollary 3.8, we see that any Ricci flat metric of a compact 4-manifold is stable.

3. Let \( g \in \mathcal{M}(S^4) \) be the standard metric of constant curvature \( 1 \). Then, \( g \) is strictly stable.
PROOF. Since $g$ is an Einstein metric, we have only to prove that
\[
\frac{1}{2} \int_{S^2} \left( \Delta_L h + \frac{R}{2} h, \Delta_L h + \frac{R}{3} h \right) dv \geq 0
\]
for all $h \in \mathcal{S}(g)$ and that equality holds only when $h = 0$ (cf. Corollary 3.8).
We have $R = 12$ and $\Delta_L h = \Delta h - 8h$ for $h \in \mathcal{S}(g)$ (cf. Lemma 3.5 (i)). Hence,
\[
\frac{1}{2} \int_{S^2} \left( \Delta_L h + \frac{R}{2} h, \Delta_L h + \frac{R}{3} h \right) dv = \frac{1}{2} \int_{S^2} \left( \Delta h - 2h, \Delta h - 4h \right) dv
\]
\[
= \frac{1}{2} \int \left( |\Delta h|^2 - 6\langle h, \Delta h \rangle + 8|h|^2 \right) dv
\]
\[
= \frac{1}{2} \int \left( |\Delta h|^2 + 6|\nabla h|^2 + 8|h|^2 \right) dv \geq 0.
\]
Obviously, equality implies that $h = 0$. □

The purpose of this section is to prove the following:

**Theorem 4.2.** Let $g$ be the standard Einstein metric on $S^2 \times S^2$, that is, $g = \tilde{g} + \bar{g}$, where $\tilde{g}$ and $\bar{g}$ are Riemannian metrics on $S^2$ with constant Gauss curvature 1. Then, $g$ is a strictly stable critical point of the functional $\nu : \mathcal{M}(S^2 \times S^2) \rightarrow \mathbb{R}$.

**Proof.** First, we remark that $\tilde{g}$ and $\bar{g}$ are parallel tensor fields, and the curvature tensor is given as
\[
(4.2) \quad R_{ijkl} = (\tilde{g}_{mj} \tilde{g}_{ik} - \tilde{g}_{ij} \tilde{g}_{km}) + (\bar{g}_{mj} \bar{g}_{ik} - \bar{g}_{ij} \bar{g}_{km}).
\]
For $h \in \mathcal{S}(g)$, we define $f \in C^\infty(S^2 \times S^2)$ and $\tilde{h}$, $\bar{h}$, $\tilde{h}$, $\bar{h} \in T_g \mathcal{M}$ as follows:
\[
(4.3) \quad \left\{ \begin{array}{l}
 f = \frac{1}{2} \tilde{h}_{ij} \tilde{g}_{ij} = - \frac{1}{2} \bar{h}_{ij} \bar{g}_{ij}, \\
 \tilde{h}_{ij} = \tilde{g}_{ij} h_{km} \tilde{g}_{mj} - f \tilde{g}_{ij}, \\
 \bar{h}_{ij} = \bar{g}_{ij} h_{km} \bar{g}_{mj} + f \bar{g}_{ij},
\end{array} \right.
\]
Then, $h = \tilde{h} + \bar{h} + \tilde{h} + \bar{h}$ and this decomposition is orthogonal. By (4.2), we have
\[
(4.4) \quad h_{mk} R_{mijk} = \tilde{h}_{ij} + \bar{h}_{ij} - f \tilde{g}_{ij} + f \bar{g}_{ij}.
\]
Then, a straightforward computation gives
\[
(4.5) \quad \left\{ \begin{array}{l}
 |h_{mk} R_{mijk}|^2 = |\tilde{h}|^2 + |\bar{h}|^2 + 4f^2, \\
 h_{ij} h_{mk} R_{mijk} = |\tilde{h}|^2 + |\bar{h}|^2 - 4f^2, \\
 (\Delta h)_{ij} h_{mk} R_{mijk} = \langle \tilde{h}, \Delta \tilde{h} \rangle + \langle \bar{h}, \Delta \bar{h} \rangle - 4f \Delta f, \\
 |\Delta h|^2 = |\Delta \tilde{h}|^2 + |\Delta \bar{h}|^2 + 4(\Delta f)^2 + |\Delta f|^2, \\
 |\nabla h|^2 = |\nabla \tilde{h}|^2 + |\nabla \bar{h}|^2 + 4|\nabla f|^2 + |\nabla f|^2.
\end{array} \right.
\]
Then, using Lemma 3.5 (i), (4.2) and (4.4), we get

\[ \frac{1}{2} \left( \Delta_{\mathcal{L}} h + 2 h, \Delta_{\mathcal{L}} h + \frac{4}{3} \nabla h \right) dv \]

\[ = \int \left[ \frac{8}{3} |h|^2 - 2 \langle \nabla h, \Delta h \rangle + \frac{1}{3} |\nabla h|^2 + \frac{1}{2} |\Delta h|^2 \right] dv \]

\[ + \frac{8}{3} |h|^2 - 2 \langle \nabla h, \Delta h \rangle + \frac{1}{3} |\nabla h|^2 + \frac{1}{2} |\Delta h|^2 \]

\[ + \frac{1}{3} |\nabla h|^2 + \frac{1}{2} |\Delta h|^2 + \frac{16}{3} f^2 + 8 f \Delta f + \frac{4}{3} |\nabla f|^2 + 2 |\Delta f|^2 \right] dv \]

\[ = \left[ \frac{8}{3} \left( |h|^2 + |\Delta h|^2 \right) + \frac{7}{3} \left( |\nabla h|^2 + |\Delta h|^2 \right) + \frac{1}{2} \left( |\Delta h|^2 + |\Delta h|^2 \right) \]

\[ + \frac{1}{3} |\nabla h|^2 + \frac{1}{2} |\Delta h|^2 + \frac{4}{3} (\Delta f + 2f)^2 + \frac{2}{3} \Delta f (\Delta f + 2f) \right] dv \]

\[ \geq 0, \]

because the first eigenvalue of the Laplacian \(-\Delta\) of \((S^2 \times S^2, g)\) is 2, and hence

\[ \int \Delta f (\Delta f + 2f) dv \geq 0. \]

Next, we consider when the equality of (4.6) holds. Obviously, the equality holds if and only if \(h = \h = 0\), \(\nabla h = 0\) and \(\Delta f + 2f = 0\). Since \(\text{div} h = 0\) (see the definition of \(S_0(g)\) in Lemma 3.6), the conditions \(h = \h = 0\) and \(\nabla h = 0\) yield \(f = \text{constant}\). Then from \(\Delta f + 2f = 0\), we have \(f = 0\). That is, \(h = \h\) and \(\nabla h = 0\). In particular, \(\Delta_{\mathcal{L}} h = \Delta h = 0\). Then from Lemma 3.5 (i) and \(R_{1j} = g_{1j}\), we get

\[ h_{1j} + h_{mk} R_{m1j} = 0. \]

On the other hand, from (4.4), \(h_{mk} R_{m1j} = 0\), since \(h = \h\). Hence, from (4.7), we have \(h = 0\). Thus, the equality of (4.6) holds only when \(h = 0\).

Now the assertion follows from Corollary 3.8. □

§ 5. Additional remarks.

**Lemma 5.1.** Suppose that \(\dim M = 4\) and \(g \in \mathcal{M}(M)\) is a metric with nonnegative sectional curvature. Then the following pointwise inequality holds; \(3|W|^2 \leq 2R^2\).

**Proof.** Let \(\{e_1, e_2, e_3, e_4\}\) be an orthonormal frame. Then, \(f_1 = e_1 \wedge e_2 + e_3 \wedge e_4, f_2 = e_1 \wedge e_3 + e_4 \wedge e_2, f_3 = e_1 \wedge e_4 + e_2 \wedge e_3, f_4 = e_1 \wedge e_2 - e_3 \wedge e_4, f_5 = e_1 \wedge e_3 - e_2 \wedge e_4, f_6 = e_1 \wedge e_4 - e_2 \wedge e_3, f_7 = (1/2)(e_1 \wedge e_2 - e_3 \wedge e_4, f_8 = (1/2)(e_1 \wedge e_3 + e_2 \wedge e_4)\). We regard the curvature tensor as a linear transformation of \(A^2\). Then, with respect to the frame \(\{f_i\}\), we have the following...
matrix representation of the curvature tensor;

\[
\begin{pmatrix}
A & B \\
B & C
\end{pmatrix},
\]

where \(A\) and \(C\) are \(3 \times 3\) symmetric matrices with \(\text{tr} A = \text{tr} C = R/2\), and

\[
B = \langle B_{\alpha \beta} \rangle = \begin{pmatrix}
E_{11} + E_{22} & E_{23} - E_{41} & E_{42} + E_{43} \\
E_{23} + E_{41} & E_{11} + E_{33} & E_{43} - E_{21} \\
E_{42} - E_{13} & E_{43} + E_{21} & E_{11} + E_{44}
\end{pmatrix},
\]

where \(E_{ij} = R_{ij} - (R/4)g_{ij}\). It is known that \(A\) and \(C\) can be diagonalized for some orthonormal frame \(\{e_i\}\) ([7; Theorem 2.1]). So, we write

\[
A = \begin{pmatrix}
(R/6) + \lambda_1 & 0 & 0 \\
0 & (R/6) + \lambda_2 & 0 \\
0 & 0 & (R/6) + \lambda_3
\end{pmatrix}, \quad C = \begin{pmatrix}
(R/6) + \mu_1 & 0 & 0 \\
0 & (R/6) + \mu_2 & 0 \\
0 & 0 & (R/6) + \mu_3
\end{pmatrix}.
\]

Then, \(\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3 = 0\) and

\[
|W|^2 = \sum_{\alpha=1}^3 \lambda_\alpha^2 + \sum_{\beta=1}^3 \mu_\beta^2.
\]

A 2-form corresponding to a plane section is of the form \(\sum_{\alpha=1}^3 \xi_\alpha f_\alpha + \sum_{\beta=1}^3 \eta_\beta f_\beta + g_{\alpha \beta} = \sum_{\alpha=1}^3 \eta_\beta f_\beta = 1/2\). Therefore, if the sectional curvature is non-negative then,

\[
\sum_{\alpha} \xi_\alpha^2 \left( \frac{R}{6} + \lambda_\alpha \right) + \sum_{\beta} \eta_\beta^2 \left( \frac{R}{6} + \mu_\beta \right) + 2 \sum_{\alpha, \beta} \xi_\alpha \eta_\beta B_{\alpha \beta} \geq 0,
\]

for all \(\{\xi_\alpha\}\) and \(\{\eta_\beta\}\) with \(\sum_{\alpha} \xi_\alpha^2 = \sum_{\beta} \eta_\beta^2 = 1/2\). From this, it is easily seen that

\[
\frac{R}{3} + \lambda_\alpha + \mu_\beta \geq 0 \quad \text{for all } \alpha, \beta.
\]

Hence,

\[
3R^2 = \left( \frac{R}{3} + \lambda_\alpha + \mu_\beta \right) + \left( \frac{R}{3} + \lambda_\alpha + \mu_\beta \right) + \left( \frac{R}{3} + \lambda_\alpha + \mu_\beta \right) \geq \sum_{\alpha, \beta} \left( \frac{R}{3} + \lambda_\alpha + \mu_\beta \right)^2 = \sum_{\alpha, \beta} \left( \frac{R}{3} + \lambda_\alpha + \mu_\beta \right)^2.
\]

Therefore, from (5.1), we have \(2R^2 \geq 3|W|^2\). □
PROPOSITION 5.2. Let \( M \) be a compact 4-dimensional manifold.

(i) If \( M \) admits an Einstein metric, then \( \nu(M) \leq 16\pi^2 \chi \);

(ii) If \( M \) admits an Einstein metric with nonnegative sectional curvature, then \( \nu(M) \leq (64/5)\pi^2 \chi \), where \( \chi \) is the Euler characteristic of \( M \).

PROOF. (i) follows from the Gauss Bonnet formula (1.3).

(ii): If \( g \in \mathcal{M}(M) \) is an Einstein metric with nonnegative sectional curvature, then from (1.3) and Lemma 5.1, we have

\[
\nu(g) = 16\pi^2 \chi - \frac{1}{12} \int R^2 \, dv \leq 16\pi^2 \chi - \frac{1}{8} \int |W|^2 \, dv \\
= 16\pi^2 \chi - \frac{1}{4} \nu(g).
\]

Hence, \( \nu(M) \leq \nu(g) \leq 64\pi^2 \chi/5. \)

COROLLARY 5.3. Let \( M \) be a compact oriented 4-dimensional manifold. If \( M \) admits an Einstein metric with nonnegative sectional curvature, then \( |\tau| \leq 8\chi/15 \), and equality holds if and only if \( M \) has a flat metric.

PROOF. Let \( g \in \mathcal{M}(M) \) be the Einstein metric with nonnegative curvature. Then by Propositions 1.4 and 5.2, \( |\tau| \leq \nu(M)/24\pi^2 \leq \nu(g)/24\pi^2 \leq 8\chi/15 \).

If the equality holds, then \( g \) is half conformally flat. So, we assume that \( *(W_{mijk}e^j \wedge e^k) = W_{mijk}e^j \wedge e^k \) (resp. \( *(W_{mijk}e^j \wedge e^k) = -W_{mijk}e^j \wedge e^k \)). From the Weitzenböck formula, we have for any harmonic 2-form \( \alpha \),

\[
\alpha_{ij;kk} = \frac{R}{3} \alpha_{ij} + \alpha_{km} W^{km}{}_{ij}.
\]

Hence, if furthermore \( *\alpha = -\alpha \) (resp. \( *\alpha = +\alpha \)), then

(5.3) \[
\alpha_{ij;kk} = \frac{R}{3} \alpha_{ij}.
\]

Now, suppose that \( g \) is not flat, i.e., \( R > 0 \). Then, from (5.3), \( \alpha = 0 \) for any harmonic 2-form with \( *\alpha = -\alpha \) (resp. \( *\alpha = +\alpha \)). Therefore, \( \tau = \pm 2\text{nd Betti number of } M \). The 1st Betti number is zero since the Ricci curvature is positive. So, \( |\tau| = \chi - 2 \). It is easy to see that \( |\tau| = \chi - 2 \) with \( |\tau| = 8\chi/15 \) does not have integral solutions. This is a contradiction. Hence, \( g \) is flat. \( \square \)

REMARK. This proposition slightly improves Theorem 2 of [4], where \( 8/15 \) is replaced by \( (2/3)^{1.5} \) (> 8/15).

References


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Osamu Kobayashi
Department of Mathematics
Faculty of Science and Technology
Keio University
Hiyoshi, Yokohama 223
Japan