J. Math. Soc. Japan
Vol. 37, No. 3, 1985

On a conformally invariant functional of the space of Riemannian metrics

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(Received April 26, 1984)

Introduction.

Let $\mathcal{M}(M)$ be the space of all C^∞ Riemannian metrics on a compact n-dimensional manifold M, and $\nu: \mathcal{M}(M)\to \boldsymbol{R}$ be a functional of $\mathcal{M}$ defined by $\nu(g)=(2/n)\int_M |W|^{n/2}dv$, where W is the Weyl conformal curvature tensor. Our main subject in this paper is to determine $\inf\{\nu(g)\,;\, g\in\mathcal{M}\}$, which will be denoted by $\nu(M)$. A little consideration shows that $\nu(M)>0$ if some Pontrjagin number of M is not zero. Thus, in general, $\nu(M)$ is a nontrivial invariant of a manifold.

In § 2, we shall show two general properties of $\nu(M)$. One is that $\nu(M)=0$ for the total space M of a principal circle bundle (Theorem 2.1). This provides examples of M for which $\nu(M)=0$ but which has no conformally flat metric. The other is an inequality for connected sum; $\nu(M_1 \# M_2)\leqq \nu(M_1)+\nu(M_2)$ (Theorem 2.2). This is useful for computing $\nu(M)$ for certain M.

However, to determine $\nu(M)$ for general M seems to be not so easy. Even for $S^2\times S^2$, $\nu(S^2\times S^2)$ is not known (to the author). We want to show that the standard Einstein metric g_0 of $S^2\times S^2$ is a candidate at which ν takes a minimum, if $\nu: \mathcal{M}(S^2\times S^2)\to \boldsymbol{R}$ has a minimum. In fact, g_0 is a minimum point of ν restricted to Kähler metrics (Proposition 1.4). Moreover, we shall prove that g_0 is a strictly stable critical point of the functional ν (cf. Definition 4.1 and Theorem4 .2).

In the course of proof of stability of $g_0\in\mathcal{M}(S^2\times S^2)$, we establish the first and the second variational formulas of $\nu: \mathcal{M}(M)\to\boldsymbol{R}$ for 4 dimensional M (Propositions 3.1 and 3.7; The first variational formula has already appeared in [2]). From these formulas, we can also see that other than conformally flat metrics, Einstein metrics are critical points of the functional ν, and Ricci flat metrics are stable critical points of ν.

§ 1. Preliminary definitions and remarks.

Throughout this paper, M denotes a compact C^∞ manifold of dimension n, and $\mathcal{M}(M)$ denotes the space of C^∞ Riemannian metrics on M. For $g\in\mathcal{M}(M)$, the curvature tensor $R^i{}_{jkl}$, the Ricci tensor $R_{ij}=R^k{}_{ikj}$ and the scalar curvature $R=g^{ij}R_{ij}$ are defined. Our concern in this paper is the Weyl tensor defined by

$$W^i{}_{jkl}=R^i{}_{jkl}-\frac{1}{n-2}(L^i{}_k g_{jl}+g^i{}_k L_{jl}-L^i{}_l g_{jk}-g^i{}_l L_{jk}), \tag{1.1}$$

where $L_{ij}=R_{ij}-(R/2(n-1))g_{ij}$ (we put $W=0$ if $n\leqq 2$). From the second Bianchi identity, we have $R^i{}_{jkl;i}=R_{jl;k}-R_{jk;l}$ and hence,

$$W^i{}_{jkl;i}=\frac{n-3}{n-2}C_{jkl}, \tag{1.2}$$

where $C_{jkl}=L_{jl;k}-L_{jk;l}$.

DEFINITION 1.1. We define a functional $\nu: \mathcal{M}(M)\to \boldsymbol{R}$ by $\nu(g)=\frac{2}{n}\int_M |W|^{n/2}dv_g$, where $|W|^{n/2}=\langle W, W\rangle^{n/4}=(g_{ip}g^{jq}g^{kr}g^{ls}W^i{}_{jkl}W^p{}_{qrs})^{n/4}$. For a subset U of M and $g\in\mathcal{M}(M)$, we write $\nu(g\,;U)=\frac{2}{n}\int_U |W|^{n/2}dv_g$.

LEMMA 1.2. (i) *$\nu(e^{2f}g)=\nu(g)$ for any $f\in C^\infty(M)$ and $g\in\mathcal{M}(M)$.* (ii) *$\nu(\varphi^*g)=\nu(g)$ for any diffeomorphism φ of M.* (iii) *$\nu=0$ if $\dim M\leqq 3$.*

PROOF. Let W and W' be the Weyl tensors of the metrics g and $g'=e^{2f}g$, respectively. Since the Weyl tensor is invariant under a conformal change of metric, we have $\langle W', W'\rangle_{g'}=\langle W, W\rangle_{g'}=e^{-4f}\langle W, W\rangle_g$. Hence, from $dv'=e^{nf}dv$ for the volume elements, we get $|W'|^{n/2}dv'=|W|^{n/2}dv$, which proves (i). (ii) is trivial, and (iii) is well-known. □

For the dimensions higher than three, we first remark the following:

PROPOSITION 1.3. *If $\dim M\geqq 4$, then $\sup\{\nu(g)\,;\,g\in\mathcal{M}(M)\}=\infty$.*

PROOF. Let $T^n=\boldsymbol{R}^n/\boldsymbol{Z}^n$ be the n-dimensional torus. If $n\geqq 4$, there exists a metric $g\in\mathcal{M}(T^n)$ with $c:=\nu(g)>0$. Then, $\tilde{g}\in\mathcal{M}(\boldsymbol{R}^n)$ denoting the lift of g to the universal covering, we have $\nu(\tilde{g}\,;[0, l]^n)=l^n c$ for $l\in\boldsymbol{N}$. Now, let (U, ϕ) be a chart of M such that $\phi(U)=\boldsymbol{R}^n$. Then, for each $l\in\boldsymbol{N}$, we can take a metric g_l on M which coincides with $\tilde{g}$ in $[0, l]^n\subset\boldsymbol{R}^n\cong U$, i.e., $(\phi_* g_l)|[0, l]^n=\tilde{g}|[0, l]^n$. We thus have $\nu(g_l)\geqq\nu(g_l\,;U)\geqq\nu(\phi_* g_l\,;[0, l]^n)=l^n c$ and hence, $\lim_{l\to\infty}\nu(g_l)=\infty$. □

On the other hand, there are non-trivial topological lower bounds for ν: Any Pontrjagin class is represented by a differential form composed of only the Weyl tensor ([**1**]). Namely, the m-th Pontrjagin class $p_m\in H^{4m}(M)$ (cf. [**5**]) is given inductively by $\Pi_m=-(p_1\Pi_{m-1}+\cdots+p_{m-1}\Pi_1)-2mp_m$, where $\Pi_m\in H^{4m}(M)$ is represented by the following differential form;

$$(2\pi)^{-2m}\mathring{\Omega}^{i_1}{}_{i_2}\wedge\mathring{\Omega}^{i_2}{}_{i_3}\wedge\cdots\wedge\mathring{\Omega}^{i_{2m}}{}_{i_1},$$

where $\mathring{\Omega}^i{}_j=(1/2)W^i{}_{jkl}e^k\wedge e^l$. So we can see that any Pontrjagin number of a $4k$-

dimensional M is dominated by $\nu(g)$ multiplied by some universal constant. For example, the following is well-known.

PROPOSITION 1.4. *If* $\dim M=4$, *then* $|p_1[M]|\leqq\nu(g)/8\pi^2$ *for all* $g\in\mathcal{M}(M)$. *Hence, if furthermore* M *is oriented,* $|\tau|\leqq\nu(g)/24\pi^2$ *for all* $g\in\mathcal{M}(M)$, *where* τ *is the signature of* M. *Equality holds if and only if* g *is a half conformally flat metric.*

Thus, unlike $\sup\nu$, $\inf\nu$ reflects certain global properties of a manifold.

DEFINITION 1.5. $\nu(M):=\inf\{\nu(g)\,;\ g\in\mathcal{M}(M)\}$.

Here are some examples:

1. $\nu(g)=0$ if g is conformally flat. Hence, if M carries a conformally flat metric, then $\nu(M)=0$. However, $\nu(M)=0$ does not imply in general that M admits a conformally flat metric (see § 2).

2. The Fubini Study metric g_0 of $\boldsymbol{CP}^2$ is half conformally flat. Hence, by Proposition 1.4, $\nu(\boldsymbol{CP}^2)=\nu(g_0)=24\pi^2$. For the connected sum $k\boldsymbol{CP}^2$ of k copies of $\boldsymbol{CP}^2$, Proposition 1.4 gives only $\nu(k\boldsymbol{CP}^2)\geqq 24\pi^2|\tau(k\boldsymbol{CP}^2)|=24k\pi^2$. We shall show in § 2 that $\nu(k\boldsymbol{CP}^2)\leqq k\nu(\boldsymbol{CP}^2)=24k\pi^2$. Hence, we have $\nu(k\boldsymbol{CP}^2)=24k\pi^2$.

3. Although the author does not know the value $\nu(S^2\times S^2)$ at present, there are partial results which suggest that $\nu(S^2\times S^2)$ may be positive. Let $\bar{g}$, $\bar{\bar{g}}$ be two Riemannian metrics on S^2 with the Gauss curvatures $\bar{K}$, $\bar{\bar{K}}$, respectively. Consider the product metric $g=\bar{g}+\bar{\bar{g}}$ on $S^2\times S^2$. Then, $\nu(g)=(128/3)\pi^2+(2/3)\int_{S^2\times S^2}(\bar{K}-\bar{\bar{K}})^2dv$.

PROOF. The Weyl tensor is computed as $W_{mijk}=(1/6)(\bar{K}+\bar{\bar{K}})(2\bar{g}_{mj}\bar{g}_{ik}-2\bar{g}_{ij}\bar{g}_{km}-\bar{\bar{g}}_{mj}\bar{g}_{ik}+\bar{\bar{g}}_{ij}\bar{g}_{km}+2\bar{\bar{g}}_{mj}\bar{\bar{g}}_{ik}-2\bar{\bar{g}}_{ij}\bar{\bar{g}}_{km}-\bar{g}_{mj}\bar{\bar{g}}_{ik}+\bar{g}_{ij}\bar{\bar{g}}_{km})$. Hence, we have $|W|^2/2=2(\bar{K}+\bar{\bar{K}})^2/3=8\bar{K}\bar{\bar{K}}/3+2(\bar{K}-\bar{\bar{K}})^2/3$. Then, from the Gauss Bonnet formula, $\nu(g)=(8/3)\int\bar{K}\bar{\bar{K}}dv+(2/3)\int(\bar{K}-\bar{\bar{K}})^2dv=128\pi^2/3+(2/3)\int(\bar{K}-\bar{\bar{K}})^2dv$. □

So, among the product metrics of $S^2\times S^2$, the standard Einstein metric attains the smallest value $128\pi^2/3$. This is generalized as follows:

PROPOSITION 1.6. *Suppose that* $\dim M=4$ *and* $g\in\mathcal{M}(M)$ *is a Kähler metric for some complex structure of* M. *Then,*

$$\nu(g)\geqq 24\pi^2|\tau|+\frac{16}{3}\pi^2\min\{2\chi-6\tau,\ 2\chi+3\tau\},$$

where τ *and* χ *are the signature and the Euler number of* M *respectively. The equality holds if and only if* g *is an Einstein Kähler metric.*

PROOF. By the four dimensional Gauss Bonnet theorem,

$$\nu(g)=16\pi^2\chi+\int|E|^2dv-\frac{1}{12}\int R^2dv, \tag{1.3}$$

where E is the traceless part of the Ricci tensor; $E_{ij}=R_{ij}-(R/4)g_{ij}$.

Let ρ be the Ricci form of the Kähler metric g ([5]). Then, it is easily seen that

$$\rho\wedge\rho=\left(\frac{1}{2}R^2-2|E|^2\right)dv. \tag{1.4}$$

Since the first Chern class is represented by $\rho/4\pi$, we have

$$\int\rho\wedge\rho=16\pi^2c_1^2=16\pi^2(2\chi+3\tau).$$

This, together with (1.3) and (1.4), yields

$$\begin{aligned}\nu(g)&=-8\pi^2\tau+\frac{32}{3}\pi^2\chi+\frac{2}{3}\int|E|^2dv\\&=\begin{cases}24\pi^2|\tau|+\dfrac{16}{3}\pi^2(2\chi-6\tau)+\dfrac{2}{3}\displaystyle\int|E|^2dv, & \text{if } \tau\geqq 0;\\[2ex] 24\pi^2|\tau|+\dfrac{16}{3}\pi^2(2\chi+3\tau)+\dfrac{2}{3}\displaystyle\int|E|^2dv, & \text{if } \tau<0.\end{cases}\end{aligned}$$

Now, we get the desired inequality. □

Applying this proposition to $S^2\times S^2$, we have $\nu(g)\geqq 128\pi^2/3$ for any *Kähler* metric g of $S^2\times S^2$, because $\tau(S^2\times S^2)=0$ and $\chi(S^2\times S^2)=4$. Thus, the standard Einstein metric g_0 of $S^2\times S^2$ attains the smallest value of ν also in the class of Kähler metrics. Moreover, we shall show in § 4 that the functional $\nu:\mathcal{M}(S^2\times S^2)\to\boldsymbol{R}$ has in fact a local minimum at g_0.

§ 2. Two general formulas for $\nu(M)$.

THEOREM 2.1. *If S^1 acts freely and differentiably on M, then* $\nu(M)=0$.

PROOF. Let K denote the vector field on M which generates the S^1 action. Since S^1 is compact, there is an S^1 invariant Riemannian metric h on M, for which K is a Killing vector field. Since the action is free, $h(K, K)$ is nowhere zero, hence, $g=(h(K, K))^{-1}h$ defines a Riemannian metric. Then we have

$$\mathcal{L}_Kg=0 \quad\text{and}\quad g(K, K)=1. \tag{2.1}$$

Now, consider a family of Riemannian metrics $\{g(t);\ 0<t\leqq 1\}$ defined by

$$g_{ij}(t)=g_{ij}-(1-t^2)\alpha_i\alpha_j, \tag{2.2}$$

where α is the 1-form associated with K with respect to $g=g(1)$, i.e., $\alpha_i=g_{ij}K^j$. The inverse matrix $g^{ij}(t)$ is easily seen to be $g^{ij}-(1-t^{-2})K^iK^j$. Then, using (2.1), we get the relation between the Christoffel symbols of $g(t)$ and g:

$$\Gamma^k_{ij}(t)-\Gamma^k_{ij}=-(1-t^2)(K^k{}_{;i}\alpha_j+K^k{}_{;j}\alpha_i), \tag{2.3}$$

where the covariant derivation in the right hand side is taken as one with respect to g. From this and (2.1), we have

$$R^i{}_{jkl}(t)-R^i{}_{jkl}=-(1-t^2)\{(K^i{}_{;j}\alpha_l)_{;k}-(K^i{}_{;j}\alpha_k)_{;l}+K^mR^i{}_{mkl}\alpha_j+K^i{}_{;l}\alpha_{j;k}-K^i{}_{;k}\alpha_{j;l}\}+(1-t^2)^2K^i{}_{;m}\alpha_j(\alpha_kK^m{}_{;l}-\alpha_lK^m{}_{;k}). \tag{2.4}$$

Then, again using (2.1), we have

$$g^{jh}(t)(R^i{}_{jkl}(t)-R^i{}_{jkl})=-(1-t^2)\{(K^{i;h}\alpha_l)_{;k}-(K^{i;h}\alpha_k)_{;l}+K^i{}_{;l}K^h{}_{;k}-K^i{}_{;k}K^h{}_{;l}\}+(1-t^{-2})K^hK^mR^i{}_{mkl}. \tag{2.5}$$

On the other hand, $g^{jn}(t)R^i{}_{jkl}=R^{in}{}_{kl}-(1-t^{-2})K^jK^nR^i{}_{jkl}$. Hence, we get

$$g^{jn}(t)R^i{}_{jkl}(t)=R^{in}{}_{kl}-(1-t^2)\{(K^{i;n}\alpha_l)_{;k}-(K^{i;n}\alpha_k)_{;l}+K^i{}_{;l}K^n{}_{;k}-K^i{}_{;k}K^n{}_{,}\}. \tag{2.6}$$

Note that (2.6) does *not* contain terms of t^{-2} and that both sides of (2.6) are tensors of type (2, 2). So, there is a constant c such that $|R^{\cdot}\ldots(t)|^2_{(t)}<c$ for all $t\in(0, 1]$. In particular, $|W(t)|^2_{(t)}<c$. On the other hand, the volume form $dv(t)$ relative to the metric $g(t)$ is easily computed as $dv(t)=t\,dv$. Thus, we get $\lim_{t\to 0}\nu(g(t))=0$. Hence, $\nu(M)=0$. □

REMARK. There is no conformally flat metric on $S^p\times T^q$, $p, q\geqq 2$, i.e., $\nu(g)>0$ for any $g\in\mathcal{M}(S^p\times T^q)$, $p, q\geqq 2$, because, by a theorem of Kuiper [**6**; Theorem III], the universal covering space of a compact conformally flat space with an infinite Abelian fundamental group must be $\boldsymbol{R}^n$ or $\boldsymbol{R}\times S^{n-1}$. However, the above theorem asserts that $\nu(S^p\times T^q)=0$. So, in general, $\nu(M)=0$ does not imply the existence of a conformally flat metric of M.

THEOREM 2.2. *For any compact manifolds M_1 and M_2 of the same dimension, $\nu(M_1\# M_2)\leqq\nu(M_1)+\nu(M_2)$.*

For the proof, we prepare the following lemma.

LEMMA 2.3. *Let $g\in\mathcal{M}(M)$ be given. Then, for each $\varepsilon>0$, there is a $\tilde{g}\in\mathcal{M}(M)$ such that $|\nu(g)-\nu(\tilde{g})|<\varepsilon$ and $\tilde{g}$ is flat in an open subset of M.*

PROOF. Let (U, ϕ) be a chart of M such that $\phi(U)\supset\{x\in\boldsymbol{R}^n;\ |x|<1\}$ and in the coordinate expression of the metric $g|U=g_{ij}(x)dx^idx^j$,

$$g_{ij}(0)=\delta_{ij} \tag{2.7}$$

holds. Take a nonnegative smooth function $\varphi:\boldsymbol{R}^n\to\boldsymbol{R}$ such that $\varphi(x)\equiv 1$ if $|x|\leqq 1/2$, and $\varphi(x)\equiv 0$ if $|x|\geqq 1$. We set $\varphi_t(x)=\varphi(x/t)$. The support of φ_t is contained in $B_t=\{x\in\boldsymbol{R}^n;\ |x|<t\}$. For $0<t<1$, we define a metric $\tilde{g}\in\mathcal{M}(M)$ by $\tilde{g}|(M\setminus U)=g|(M\setminus U)$ and

$$\tilde{g}_{ij}=(1-\varphi_t)g_{ij}+\varphi_t\delta_{ij} \tag{2.8}$$

in U. We shall show that $\tilde{g}$ has the desired properties for a sufficiently small t.

It follows from (2.7) and the definition of φ_t that

$$|\varphi_t(g_{ij}-\delta_{ij})|<c_1 t, \quad |\partial\varphi_t|<c_1 t^{-1}, \quad |\partial^2\varphi_t|<c_1 t^{-2}, \tag{2.9}$$

for some constant c_1 where ∂ is the Euclidean gradient. Hence, $|\partial\tilde{g}_{ij}|<c_2$ and $|\partial^2\tilde{g}_{ij}|<c_2(t^{-1}+1)$ for some c_2. Then, putting $f_t=\tilde{g}(\widetilde{W}, \widetilde{W})^{n/4}(\det(\tilde{g}_{ij}))^{1/2}$, we can easily see that

$$f_t<c_3(t^{-1}+1)^{n/2}. \tag{2.10}$$

Thus, we get

$$\int_{B_t} f_t dx \leqq c_3(t^{-1}+1)^{n/2}\int_{B_t} dx = c_4(t^{-1}+1)^{n/2}t^n=c_4(t^2+t)^{n/2}. \tag{2.11}$$

On the other hand,

$$|\nu(g)-\nu(\tilde{g})|=|\nu(g\,;\,\phi^{-1}(B_t))-\nu(\tilde{g}\,;\,\phi^{-1}(B_t))| \leqq \nu(g\,;\,\phi^{-1}(B_t))+\frac{2}{n}\int_{B_t} f_t dx.$$

Therefore, from (2.11), we conclude that $|\nu(g)-\nu(\tilde{g})|<\varepsilon$ for a sufficiently small t. It is obvious from (2.8) that $\tilde{g}$ is flat in $\phi^{-1}(B_t)$. □

PROOF OF THEOREM 2.2. Let ε be an arbitrary positive number. Take $g_i \in \mathcal{M}(M_i)$ so that $\nu(g_i)\leqq\nu(M_i)+\varepsilon$, $i=1, 2$. By the above lemma, we can choose $\tilde{g}_i\in\mathcal{M}(M_i)$ such that $\nu(\tilde{g}_i)\leqq\nu(M_i)+2\varepsilon$ and $\tilde{g}_i$ is flat in some neighbourhood of M_i. Suppose that for some $r>0$ and $p_i\in M_i$, $\tilde{g}_i$ is flat in $U_i(p_i\,;\,[0, 2r)):=\{x\in M_i\,;\,\tilde{d}_i(x, p_i)\in[0, 2r)\}$, where $\tilde{d}_i$ is the distance function of the metric $\tilde{g}_i$, $i=1, 2$.

We define a diffeomorphism $\varphi\colon U_1(p_1\,;\,(r/2, 2r))\to U_2(p_2\,;\,(r/2, 2r))$ by $\varphi(\exp_{p_1}X)=\exp_{p_2}(-(r^2/\tilde{g}_1(X, X))\varphi_0 X)$, where $\varphi_0\colon T_{p_1}M_1\to T_{p_2}M_2$ is a linear isometry and $\exp_{p_i}$ denotes the exponential map at $p_i\in M_i$ with respect to $\tilde{g}_i$. Then we can regard $M_1\# M_2$ as $\{M_1\setminus U_1(p_1\,;\,[0, r/2])\}\cup_\varphi\{M_2\setminus U_2(p_2\,;\,[0, r/2])\}$.

Let f_i be a positive smooth function on M_i such that $f_i(x)=(\tilde{d}_i(p_i, x))^{-2}$ if $r/2<\tilde{d}_i(p_i, x)<2r$. Then, $f_i\tilde{g}_i$ is a Riemannian metric on M_i and is conformally flat in $U_i(p_i\,;\,[0, 2r))$. Moreover, $\varphi\colon(U_1(p_1\,;\,(r/2, 2r)), f_1\tilde{g}_1)\to(U_2(p_2\,;\,(r/2, 2r)), f_2\tilde{g}_2)$ becomes an isometry. Hence, we can define a Riemannian metric g on $M_1\# M_2$ by $g|(M_i\setminus U_i(p_i\,;\,[0, r/2])):=f_i\tilde{g}_i|(M_i\setminus U_i(p_i\,;\,[0, r/2]))$, $i=1, 2$. Then, we have

$$\nu(g)=\nu(f_1\tilde{g}_1\,;\,M_1\setminus U_1(p_1\,;\,[0, r/2]))+\nu(f_2\tilde{g}_2\,;\,M_2\setminus U_2(p_2\,;\,[0, 2r]))$$
$$=\nu(f_1\tilde{g}_1)+\nu(f_2\tilde{g}_2)$$

$$=\nu(\tilde{g}_1)+\nu(\tilde{g}_2)$$
$$\leqq\nu(M_1)+\nu(M_2)+4\varepsilon.$$

Therefore, $\nu(M_1\#M_2)\leqq\nu(M_1)+\nu(M_2)+4\varepsilon$ for arbitrary $\varepsilon>0$. That is, $\nu(M_1\#M_2)\leqq\nu(M_1)+\nu(M_2)$. □

§3. Variational formulas in dimension four.

In this section, we use the following abbreviation for shortness' sake; 1. *Omitting the summation sign*, e.g., $C_{ijk;k}+L_{mk}W^m{}_{ijk}$ stands for $\sum_{k,l}g^{kl}C_{ijk;l}+\sum_{k,l,m}g^{kl}L_{mk}W^m{}_{ijl}$. 2. *Identification through the duality defined by metric*, e.g., $S^k_{ij}=(1/2)(h_{jk;i}+h_{ki;j}-h_{ij;k})$ stands for $S^k_{ij}=(1/2)g^{kl}(h_{jl;i}+h_{li;j}-h_{ij;l})$.

PROPOSITION 3.1 ([2]). *Suppose that M is a compact manifold of dimension 4. Then for a smooth curve $g=g(t)$ in $\mathcal{M}(M)$, we have*

$$\frac{d}{dt}\nu(g)=\int_M\left\langle X,\frac{d}{dt}g\right\rangle dv,$$

where X is a symmetric 2-tensor defined by $X_{ij}=C_{ijk;k}+L_{mk}W^m{}_{ijk}$ (see §1 *for C, L and W).*

PROOF. We set $h_{ij}=(d/dt)g_{ij}$ and $S^k_{ij}=(d/dt)\Gamma^k_{ij}$. Then,

$$S^k_{ij}=\frac{1}{2}(h_{jk;i}+h_{ki;j}-h_{ij;k}). \tag{3.1}$$

Hence,

$$\frac{d}{dt}R^m{}_{ijk}=S^m_{ik;j}-S^m_{ij;k}. \tag{3.2}$$

Then, from (1.1) and elementary algebraic properties of curvature tensor, we have

$$W^m{}_{ijk}\left(\frac{d}{dt}W^m{}_{ijk}\right)=W^m{}_{ijk}(2h_{ij;km}+L_{mk}h_{ij}).$$

Therefore, using $(d/dt)g^{ij}=-h^{ij}$ and $(d/dt)dv=(1/2)h_{ii}dv$, we get

$$\frac{d}{dt}\nu(g)=\int\left\{W^m{}_{ijk}(2h_{ij;km}+L_{mk}h_{ij})-W^m{}_{ijk}W^m{}_{ijl}h^{kl}+\frac{1}{4}|W|^2h_{ii}\right\}dv$$
$$=\int\left\{X_{ij}h_{ij}-(W^m{}_{ijk}W^m{}_{ijl}-\frac{1}{4}|W|^2g_{kl})h_{kl}\right\}dv,$$

where we use Stokes' formula and (1.2). Thus, we have only to prove $W^m{}_{ijk}W^m{}_{ijl}=(1/4)|W|^2g_{kl}$.

It is known that a symmetric linear transformation on the space of 2-forms Λ^2 commutes with the Hodge star $*:\Lambda^2\to\Lambda^2$ (since the argument is local, we need not assume M is orientable) if and only if its Ricci contraction is proportional to g (cf. [**7**; Theorem 1.3]). Viewed as a symmetric transformation on Λ^2, the Weyl tensor commutes with the Hodge star, because the Ricci contraction of W is zero. Hence, so does $W \circ W$, and the Ricci contraction of $W \circ W$ is proportional to g. That is, $W^{ik}{}_{ab}W^{ab}{}_{jk} = \lambda g^i{}_j$ for some scalar λ, which implies $W^m{}_{ijk}W^m{}_{ijl} = (1/4)|W|^2 g_{kl}$.

That X is symmetric is not difficult to see. □

COROLLARY 3.2. *If* $\dim M=4$, *the tensor* X *has the following properties;* (i) $X_{ii}=0$; (ii) $X_{ij;j}=0$; (iii) $X \otimes g$ *is conformally invariant.*

PROOF. Easy consequence of Lemma 1.2. □

COROLLARY 3.3. *If* $\dim M=4$ *and* $g \in \mathcal{M}(M)$ *is conformal to an Einstein metric, then* g *is a critical point of* $\nu : \mathcal{M}(M) \to \boldsymbol{R}$.

PROOF. Obviously, $X=0$ if g is an Einstein metric. Thus, the assertion follows from Corollary 3.2 (iii). □

Next, we shall compute the second variational formula. To do this, we review the Lichnerowicz Laplacian and decomposition of the space of symmetric 2-tensor fields (cf. [**3**]).

DEFINITION 3.4. The tangent space $T_g\mathcal{M}(M)$ of $\mathcal{M}(M)$ at g is naturally identified with the space of C^∞ symmetric 2-tensor fields on M. The *Lichnerowicz Laplacian* $\Delta_L : T_g\mathcal{M} \to T_g\mathcal{M}$ is defined by

$$(\Delta_L h)_{ij} := (\tilde{\Delta} h)_{ij} + h_{ik;kj} - h_{ik;jk} + h_{jk;ki} - h_{jk;ik},$$

where $\tilde{\Delta} : T_g\mathcal{M} \to T_g\mathcal{M}$ is the *rough Laplacian*; $(\tilde{\Delta} h)_{ij} = h_{ij;kk}$ (our sign convention of Laplacians is opposite to that used in [**3**]).

LEMMA 3.5. (i) $(\Delta_L h)_{ij} = (\tilde{\Delta} h)_{ij} - h_{ik}R_{kj} - R_{ik}h_{kj} - 2h_{mk}R^m{}_{ijk}$. *Hence, if* $\dim M=4$, *then*

$$(\Delta_L h)_{ij} = (\tilde{\Delta} h)_{ij} - 2(h_{ik}E_{kj} + E_{ik}h_{kj}) + \langle E, h\rangle g_{ij} + (\mathrm{tr}\, h)(E_{ij} + (R/6)g_{ij}) - (2R/3)h_{ij} - 2h_{mk}W^m{}_{ijk},$$

where $E_{ij} = R_{ij} - (R/4)g_{ij}$.

(ii) $\int_M \langle h', \Delta_L h''\rangle dv = -\int_M \{\langle h', \tilde{\Delta} h''\rangle + (h'_{ij;k} - h'_{ik;j})(h''_{ij;k} - h''_{ik;j}) + 2h'_{ij;j}h''_{ik;k}\}\, dv$ *for* $h', h'' \in T_g\mathcal{M}$.

(iii) $\int_M \langle h', \Delta_L h''\rangle dv = \int_M \{h'_{ij;k}(h''_{jk;i} + h''_{ki;j} - h''_{ij;k}) - 2h'_{ij;j}h''_{ik;k}\}\, dv$, *for* $h', h'' \in T_g\mathcal{M}$.

PROOF. Easy and omitted. □

LEMMA 3.6. *If M is a compact manifold, $T_g\mathcal{M}(M)$ has the following decomposition; $T_g\mathcal{M}=\mathcal{S}_0(g)\oplus\mathcal{S}_1(g)$, where $\mathcal{S}_0(g)=\{h\in T_g\mathcal{M};\ \mathrm{tr}\,h=0,\ \mathrm{div}\,h=0\}$ and $\mathcal{S}_1(g)=\{h\in T_g\mathcal{M};\ h=\mathcal{L}_u g+fg$ for some $u\in\mathfrak{X}(M)$ and $f\in C^\infty(M)\}$. This decomposition is orthogonal with respect to the L_2 inner product defined by g.*

PROOF. Put $P(u)=\mathcal{L}_u g-(1/n)(\mathrm{tr}\,\mathcal{L}_u g)g$ for a vector field u. Then, it is easy to check that the principal symbol of the linear differential operator $P:\mathfrak{X}\to T_g\mathcal{M}$ is injective. Hence, $T_g\mathcal{M}=\mathrm{Ker}\,P^*\oplus\mathrm{Im}\,P$ (cf. [**3**]), where P^* is the adjoint operator of P. P^* is computed as $P^*(h)=-2(h_{ij}-(1/n)(\mathrm{tr}\,h)g_{ij})_{;j}$. From this, we have $\mathrm{Ker}\,P^*=\mathcal{S}_0\oplus C^\infty(M)\cdot g$. Then, putting $\mathcal{S}_1=C^\infty(M)\cdot g\oplus\mathrm{Im}\,P$, we get the desired decomposition. □

REMARKS. 1. Let G be the semi direct product of the diffeomorphism group $\mathcal{D}(M)$ and $C^\infty(M)$ with multiplication; $(\varphi_1, f_1)\cdot(\varphi_2, f_2)=(\varphi_1\circ\varphi_2, f_1\circ\varphi_2+f_2)$ for $\varphi_1, \varphi_2\in\mathcal{D}(M)$ and $f_1, f_2\in C^\infty(M)$. Then, G acts on $\mathcal{M}(M)$ on the right as follows; $(g, (\varphi, f))\mapsto e^{2f}\varphi^*g$, $g\in\mathcal{M}(M)$, $\varphi\in\mathcal{D}(M)$ and $f\in C^\infty(M)$. Lemma 1.2 says that ν is constant on every G-orbit in $\mathcal{M}(M)$. $\mathcal{S}_1(g)$ in the above lemma is regarded as the tangent space at g of the G-orbit of g.

2. Using the orthogonality between $\mathcal{S}_0$ and $\mathcal{S}_1$, we have an isomorphism $\mathcal{S}_0(g)\cong\mathcal{S}_0(e^{2f}g)$; $h\mapsto e^{(2-n)f}h$. In particular, we see that $\mathcal{S}_0(g)=\mathcal{S}_0(e^{2f}g)$ if $\dim M=2$, $\mathcal{S}_0(g)\otimes g=\mathcal{S}_0(e^{2f}g)\otimes e^{2f}g$ if $\dim M=4$, and so on.

PROPOSITION 3.7. *Suppose that M is a compact manifold of dimension 4, and $g\in\mathcal{M}(M)$ is a critical point of $\nu:\mathcal{M}(M)\to\boldsymbol{R}$. Let g_t be a smooth variation of g with $g_0=g$. Then,*

$$\begin{aligned}\left(\frac{d}{dt}\right)^2\nu(g_t)\Big|_{t=0}=\int_M\Big[&\frac{1}{2}\Big\langle\Delta_L h+\frac{R}{2}h,\ \Delta_L h+\frac{R}{3}h\Big\rangle\\&+\Big\langle E\circ h,\ 2\Delta_L h+\frac{3}{2}h\circ E+Rh\Big\rangle+\frac{1}{2}|E|^2|h|^2+\frac{1}{2}|E\circ h|^2-\frac{2}{3}\langle E, h\rangle^2\\&+S^m_{ij}\Big\{(2E_{mk}h_{ij})_{;k}-\frac{1}{3}h_{ij}R_{;m}-\frac{1}{2}(E\circ h-h\circ E)_{mi;j}+2h_{ik}C_{mjk}-h_{kj}C_{kmi}\Big\}\\&-h_{ij}h_{km}C_{ijk;m}\Big]dv,\end{aligned}$$

where h is the $\mathcal{S}_0(g)$ component of $(dg/dt)|_{t=0}\in T_g\mathcal{M}$ (cf. Lemma 3.6), $(E\circ h)_{ij}=E_{ik}h_{kj}$ and $S^k_{ij}=(1/2)(h_{jk;i}+h_{ki;j}-h_{ij;k})$.

PROOF. From the first variational formula (Proposition 3.1), $(d\nu(g_t)/dt)=\int_M\langle X_t, (dg_t/dt)\rangle_t dv_t$. Since $X_t\in\mathcal{S}_0(g_t)$ (Corollary 3.2), we have $(d\nu(g_t)/dt)=\int_M\langle X_t, h_t\rangle_t dv_t$, where h_t is the $\mathcal{S}_0(g_t)$ component of (dg_t/dt). Thus, since $\boldsymbol{X}=X_0=0$, we get

$$\left(\frac{d}{dt}\right)^2 \nu(g_t)\Big|_{t=0} = \int_M \langle \dot{X}, h\rangle dv, \tag{3.3}$$

where $\dot{}$ means $(d/dt)|_{t=0}$, i.e., $\dot{X}=(dX_t/dt)|_{t=0}$. If we write $\dot{X}=(DX)(\dot{g})$, we can see $\dot{X}=(DX)(h)$, because that $X=0$ is a conformally invariant property by Corollary 3.2 (iii). Hence, it suffices to prove the formula under the assumption that $\dot{g}=h\in\mathcal{S}_0(g)$. So, we assume in the following that $h_{ij}=\dot{g}_{ij}$ and $h_{ii}=0$, $h_{ij;j}=0$.

From the definition of X,

$$\begin{aligned}\dot{X}_{ij} &= (g^{km}C_{ijk;m})^{\cdot}+(g^{kl}L_{ml}W^m{}_{ijk})^{\cdot} \\ &= (\dot{C}_{ijk})_{;k}-h_{km}C_{ijk;m}-S^m_{ki}C_{mjk}-S^m_{kj}C_{imk} \\ &\quad +\dot{L}_{mk}W^m{}_{ijk}+L_{mk}\dot{W}^m{}_{ijk}-h_{kl}L_{ml}W^m{}_{ijk}.\end{aligned} \tag{3.4}$$

From the definition of C,

$$\dot{C}_{ijk}=(\dot{L}_{ik})_{;j}-(\dot{L}_{ij})_{;k}+S^m_{ki}L_{mj}-S^m_{ji}L_{mk}. \tag{3.5}$$

From the definition of L and the Lichnerowicz Laplacian, and (3.2),

$$\dot{L}_{ij}=-\frac{1}{2}(\Delta_L h)_{ij}-\frac{R}{6}h_{ij}+\frac{1}{6}\langle E, h\rangle g_{ij}. \tag{3.6}$$

From this and Lemma 3.5 (ii),

$$\begin{aligned}(\dot{L}_{ij})_{;k}(h_{ij;k}-h_{ik;j}) &\doteq \frac{1}{4}|\Delta_L h|^2+\frac{R}{12}\langle h, \Delta_L h\rangle+\frac{1}{4}\langle\Delta_L h, \tilde{\Delta}h\rangle \\ &\quad +\frac{R}{12}\langle h, \tilde{\Delta}h\rangle,\end{aligned} \tag{3.7}$$

where the meaning of the notation $\doteq$ is as follows; for f_1 and $f_2\in C^\infty(M)$, we write $f_1\doteq f_2$ if $\int_M f_1 dv=\int_M f_2 dv$.

Then, from (3.5), (3.6), (3.7) and Lemma 3.5 (i),

$$\begin{aligned}(\dot{C}_{ijk})_{;k}h_{ij}+\dot{L}_{mk}W^m{}_{ijk}h_{ij} &\doteq -\dot{C}_{ijk}h_{ij;k}+\dot{L}_{mk}W^m{}_{ijk}h_{ij} \\ &= ((\dot{L}_{ij})_{;k}+S^m_{ij}L_{mk})(h_{ij;k}-h_{ik;j})+\dot{L}_{mk}W^m{}_{ijk}h_{ij} \\ &\doteq \frac{1}{4}|\Delta_L h|^2+\frac{R}{12}\langle h, \Delta_L h\rangle+\frac{1}{4}\langle\Delta_L h, \tilde{\Delta}h\rangle+\frac{R}{12}\langle h, \tilde{\Delta}h\rangle \\ &\quad +S^m_{ij}L_{mk}(h_{ij;k}-h_{ik;j}) \\ &\quad -\frac{1}{2}(\Delta_L h)_{ij}h_{km}W^m{}_{ijk}-\frac{R}{6}h_{ij}h_{mk}W^m{}_{ijk} \\ &= \frac{1}{2}|\Delta_L h|^2+\frac{R}{3}\langle h, \Delta_L h\rangle+\frac{1}{18}R^2|h|^2\end{aligned} \tag{3.8}$$

$$+\left\langle E\circ h,\ \Delta_L h+\frac{R}{3}h\right\rangle+S^m_{ij}L_{mk}(h_{ij;k}-h_{ik;j}).$$

From (1.1) and (3.6)

$$\text{(3.9)}\qquad h_{ij}L_{mk}\dot{W}^m{}_{ijk}=h_{ij}L_{mk}\Big\{\dot{R}^m{}_{ijk}-\frac{1}{2}(L_{mj}h_{ik}+\dot{L}_{mj}g_{ik}$$

$$-h_{ml}L_{lj}g_{ik}+g_{mj}\dot{L}_{ik}-L_{mk}h_{ij}-\dot{L}_{mk}g_{ij}+h_{ml}L_{lk}g_{ij}-g_{mk}\dot{L}_{ij})\Big\}$$

$$=h_{ij}L_{mk}\dot{R}^m{}_{ijk}-\frac{1}{2}\left(|L\circ h|^2+2\langle L\circ h,\ \dot{L}\rangle-\langle L\circ h,\ h\circ L\rangle-|L|^2|h|^2-\frac{R}{3}\langle \dot{L},\ h\rangle\right)$$

$$=h_{ij}L_{mk}\dot{R}^m{}_{ijk}-\frac{R}{24}\langle h,\ \Delta_L h\rangle+\left\langle E\circ h,\ \frac{1}{2}\Delta_L h+\frac{1}{2}h\circ E+\frac{R}{6}h\right\rangle$$

$$+\frac{1}{2}|E|^2|h|^2-\frac{1}{2}|E\circ h|^2-\frac{1}{6}\langle E\circ h\rangle^2.$$

From Lemma 3.5 (i),

$$\text{(3.10)}\qquad -h_{ij}h_{kl}L_{ml}W^m_{ijk}=\frac{R}{24}\langle h,\ \Delta_L h\rangle-\frac{R}{24}\langle h,\ \tilde{\Delta}h\rangle+\frac{1}{36}R^2|h|^2$$

$$-\frac{1}{2}\langle E\circ h,\ \tilde{\Delta}h\rangle+\left\langle E\circ h,\ \frac{1}{2}\Delta_L h+h\circ E+\frac{R}{2}h\right\rangle+|E\circ h|^2-\frac{1}{2}\langle E,\ h\rangle^2.$$

From (3.2), Bianchi's identity $L_{ij;j}=R_{;i}/3$ and using Lemma 3.5 (iii), we get

$$\text{(3.11)}\qquad S^m_{ij}L_{mk}(h_{ij;k}-h_{ik;j})+h_{ij}L_{mk}\dot{R}^m{}_{ijk}$$

$$=S^m_{ij}(L_{mk}h_{ij})_{;k}-S^m_{ij}L_{mk;k}h_{ij}-S^m_{ij}(L_{mk}h_{ik})_{;j}+S^m_{ij}L_{mk;j}h_{ik}$$

$$+h_{ij}L_{mk}(S^m_{ik;j}-S^m_{ij;k})$$

$$\doteq S^m_{ij}(L_{mk}h_{ij})_{;k}-\frac{1}{3}S^m_{ij}h_{ij}R_{;m}$$

$$-\frac{1}{2}S^m_{ij}(L_{mk}h_{ik}+h_{mk}L_{ik})_{;j}-\frac{1}{2}S^m_{ij}(L_{mk}h_{ik}-h_{mk}L_{ik})_{;j}$$

$$+S^m_{ij}L_{mk;j}h_{ik}-S^m_{ik}L_{mk;j}h_{ij}+S^m_{ij}(h_{ij}L_{mk})_{;k}$$

$$=2S^m_{ij}(L_{mk}h_{ij})_{;k}-\frac{1}{4}h_{mi;j}(L_{mk}h_{ik}+h_{mk}L_{ik})_{;j}$$

$$-S^m_{ij}\left(\frac{1}{3}h_{ij}R_{;m}+\frac{1}{2}(E\circ h-h\circ E)_{mi;j}-h_{ik}C_{mjk}\right)$$

$$\doteq 2S^m_{ij}(E_{mk}h_{ij})_{;k}+\frac{1}{6}S^k_{ij}(Rh_{ij})_{;k}+\frac{1}{2}\langle L\circ h,\ \tilde{\Delta}h\rangle$$

$$-S^m_{ij}\left(\frac{1}{3}h_{ij}R_{;m}+\frac{1}{2}(E\circ h-h\circ E)_{mi;j}-h_{ik}C_{mjk}\right)$$

$$\doteq \frac{R}{12}\langle h, \Delta_L h\rangle + \frac{1}{2}\langle E\circ h, \tilde{\Delta} h\rangle + \frac{R}{24}\langle h, \tilde{\Delta} h\rangle$$

$$+S^m_{ij}\left\{2(E_{mk}h_{ij})_{;k} - \frac{1}{3}h_{ij}R_{;m} - \frac{1}{2}(E\circ h - h\circ E)_{mi;j} + h_{ik}C_{mjk}\right\}.$$

Summing up (3.8), (3.9) and (3.10), then substituting (3.11), we obtain from (3.4) the following:

$$\dot{X}_{ij}h_{ij} \doteq \frac{1}{2}|\Delta_L h|^2 + \frac{5}{12}R\langle h, \Delta_L h\rangle + \frac{1}{12}R^2|h|^2$$

$$+\left\langle E\circ h, 2\Delta_L h + \frac{3}{2}h\circ E + Rh\right\rangle + \frac{1}{2}|E|^2|h|^2 + \frac{1}{2}|E\circ h|^2 - \frac{2}{3}\langle E, h\rangle^2$$

$$+S^m_{ij}\left\{2(E_{mk}h_{ij})_{;k} - \frac{1}{3}h_{ij}R_{;m} - \frac{1}{2}(E\circ h - h\circ E)_{mi;j} + 2h_{ik}C_{mjk} - h_{kj}C_{kmi}\right\}$$

$$-h_{ij}h_{km}C_{ijk;m}.$$

Thus, from (3.3), we have the desired formula. □

COROLLARY 3.8. *Under the same assumptions and notations as in Proposition 3.7, the second variational formula at an Einstein metric g (cf. Corollary 3.3) is as follows:*

$$\left(\frac{d}{dt}\right)^2 \nu(g_t)\Big|_{t=0} = \frac{1}{2}\int_M \left\langle \Delta_L h + \frac{R}{2}h, \Delta_L h + \frac{R}{3}h\right\rangle dv.$$

§ 4. Stability of the standard Einstein metric of $S^2\times S^2$.

DEFINITION 4.1. Let $g\in\mathcal{M}(M)$ be a critical point of the functional $\nu: \mathcal{M}(M)\to \boldsymbol{R}$. Then g is said to be *stable* if

$$\left(\frac{d}{dt}\right)^2 \nu(g_t)\Big|_{t=0} \geqq 0 \tag{4.1}$$

for all smooth variation g_t with $g_0=g$. Moreover, g is said to be *strictly stable* if g is stable and if equality of (4.1) holds only when $(dg_t/dt)|_{t=0}\in\mathcal{S}_1(g)$ (cf. Lemma 3.6).

REMARK. It follows from Lemma 1.2 that if g is a (strictly) stable critical point of ν, then so is any metric conformal to g.

EXAMPLES. 1. Any conformally flat metric is stable. Any half conformally flat metric of a compact orientable 4-manifold is stable (cf. Proposition 1.4).

2. Setting the scalar curvature $R=0$ in Corollary 3.8, we see that any Ricci flat metric of a compact 4-manifold is stable.

3. Let $g\in\mathcal{M}(S^4)$ be the standard metric of constant curvature 1. Then, g is strictly stable.

PROOF. Since g is an Einstein metric, we have only to prove that

$$\frac{1}{2}\int_{S^4}\left\langle \Delta_L h+\frac{R}{2}h,\ \Delta_L h+\frac{R}{3}h\right\rangle dv \geqq 0$$

for all $h\in\mathcal{S}_0(g)$ and that equality holds only when $h=0$ (cf. Corollary 3.8).

We have $R=12$ and $\Delta_L h=\tilde{\Delta}h-8h$ for $h\in\mathcal{S}_0(g)$ (cf. Lemma 3.5 (i)). Hence,

$$\begin{aligned}\frac{1}{2}\int\left\langle \Delta_L h+\frac{R}{2}h,\ \Delta_L h+\frac{R}{3}h\right\rangle dv &= \frac{1}{2}\int\langle\tilde{\Delta}h-2h,\ \tilde{\Delta}h-4h\rangle dv\\ &=\frac{1}{2}\int\{|\tilde{\Delta}h|^2-6\langle h,\ \tilde{\Delta}h\rangle+8|h|^2\}\,dv\\ &=\frac{1}{2}\int(|\tilde{\Delta}h|^2+6|\nabla h|^2+8|h|^2)dv\geqq 0.\end{aligned}$$

Obviously, equality implies that $h=0$. □

The purpose of this section is to prove the following:

THEOREM 4.2. *Let g be the standard Einstein metric on $S^2\times S^2$, that is, $g=\bar{g}+\bar{\bar{g}}$, where $\bar{g}$ and $\bar{\bar{g}}$ are Riemannian metrics on S^2 with constant Gauss curvature 1. Then, g is a strictly stable critical point of the functional ν: $\mathcal{M}(S^2\times S^2)\to\boldsymbol{R}$.*

PROOF. First, we remark that $\bar{g}$ and $\bar{\bar{g}}$ are parallel tensor fields, and the curvature tensor is given as

$$R_{mijk}=(\bar{g}_{mj}\bar{g}_{ik}-\bar{g}_{ij}\bar{g}_{km})+(\bar{\bar{g}}_{mj}\bar{\bar{g}}_{ik}-\bar{\bar{g}}_{ij}\bar{\bar{g}}_{km}). \tag{4.2}$$

For $h\in\mathcal{S}_0(g)$, we define $f\in C^\infty(S^2\times S^2)$ and $\bar{h},\ \bar{\bar{h}},\ \check{h}\in T_g\mathcal{M}$ as follows:

$$\begin{cases} f=\dfrac{1}{2}h_{ij}\bar{g}_{ij}=-\dfrac{1}{2}h_{ij}\bar{\bar{g}}_{ij}, \\ \bar{h}_{ij}=\bar{g}_{ik}h_{km}\bar{g}_{mj}-f\bar{g}_{ij}, \qquad \bar{\bar{h}}_{ij}=\bar{\bar{g}}_{ik}h_{km}\bar{\bar{g}}_{mj}+f\bar{\bar{g}}_{ij}, \\ \check{h}_{ij}=\bar{g}_{ik}h_{km}\bar{\bar{g}}_{mj}+\bar{\bar{g}}_{ij}h_{km}\bar{g}_{mj}. \end{cases} \tag{4.3}$$

Then, $h=\bar{h}+\bar{\bar{h}}+\check{h}+f\bar{g}-f\bar{\bar{g}}$ and this decomposition is orthogonal. By (4.2), we have

$$h_{mk}R_{mijk}=\bar{h}_{ij}+\bar{\bar{h}}_{ij}-f\bar{g}_{ij}+f\bar{\bar{g}}_{ij}. \tag{4.4}$$

Then, a straightforward computation gives

$$\begin{cases} |h_{mk}R_{mijk}|^2=|\bar{h}|^2+|\bar{\bar{h}}|^2+4f^2, \\ h_{ij}h_{mk}R_{mijk}=|\bar{h}|^2+|\bar{\bar{h}}|^2-4f^2, \\ (\tilde{\Delta}h)_{ij}h_{mk}R_{mijk}=\langle\bar{h},\ \tilde{\Delta}\bar{h}\rangle+\langle\bar{\bar{h}},\ \tilde{\Delta}\bar{\bar{h}}\rangle-4f\Delta f, \\ |\tilde{\Delta}h|^2=|\tilde{\Delta}\bar{h}|^2+|\tilde{\Delta}\bar{\bar{h}}|^2+4(\Delta f)^2+|\tilde{\Delta}\check{h}|^2, \\ |\nabla h|^2=|\nabla\bar{h}|^2+|\nabla\bar{\bar{h}}|^2+4|\nabla f|^2+|\nabla\check{h}|^2. \end{cases} \tag{4.5}$$

Then, using Lemma 3.5 (i), (4.2) and (4.4), we get

$$
\begin{aligned}
(4.6)\qquad & \frac{1}{2}\int\left\langle \Delta_L h+2h,\ \Delta_L h+\frac{4}{3}h\right\rangle dv \\
&=\int\left\{2|h_{mk}R_{mijk}|^2+\frac{2}{3}h_{mk}h_{ij}R_{mijk}-2(\tilde{\Delta}h)_{ij}h_{km}R_{mijk}+\frac{1}{2}|\tilde{\Delta}h|^2+\frac{1}{3}|\nabla h|^2\right\}dv \\
&=\int\left\{\frac{8}{3}|\bar{h}|^2-2\langle\bar{h},\ \tilde{\Delta}\bar{h}\rangle+\frac{1}{3}|\nabla\bar{h}|^2+\frac{1}{2}|\tilde{\Delta}\bar{h}|^2\right. \\
&\quad+\frac{8}{3}|\bar{\bar{h}}|^2-2\langle\bar{\bar{h}},\ \tilde{\Delta}\bar{\bar{h}}\rangle+\frac{1}{3}|\nabla\bar{\bar{h}}|^2+\frac{1}{2}|\tilde{\Delta}\bar{\bar{h}}|^2 \\
&\quad\left.+\frac{1}{3}|\nabla\check{h}|^2+\frac{1}{2}|\tilde{\Delta}\check{h}|^2+\frac{16}{3}f^2+8f\Delta f+\frac{4}{3}|\nabla f|^2+2|\tilde{\Delta}f|^2\right\}dv \\
&=\int\left\{\frac{8}{3}(|\bar{h}|^2+|\bar{\bar{h}}|^2)+\frac{7}{3}(|\nabla\bar{h}|^2+|\nabla\bar{\bar{h}}|^2)+\frac{1}{2}(|\tilde{\Delta}\bar{h}|^2+|\tilde{\Delta}\bar{\bar{h}}|^2)\right. \\
&\quad\left.+\frac{1}{3}|\nabla\check{h}|^2+\frac{1}{2}|\tilde{\Delta}\check{h}|^2+\frac{4}{3}(\Delta f+2f)^2+\frac{2}{3}\Delta f(\Delta f+2f)\right\}dv \\
&\geqq 0,
\end{aligned}
$$

because the first eigenvalue of the Laplacian $-\Delta$ of $(S^2\times S^2,\ g)$ is 2, and hence $\int\Delta f(\Delta f+2f)dv\geqq 0$.

Next, we consider when the equality of (4.6) holds. Obviously, the equality holds if and only if $\bar{h}=\bar{\bar{h}}=0$, $\nabla\check{h}=0$ and $\Delta f+2f=0$. Since $\operatorname{div}h=0$ (see the definition of $\mathcal{S}_0(g)$ in Lemma 3.6), the conditions $\bar{h}=\bar{\bar{h}}=0$ and $\nabla\check{h}=0$ yield $f=$ constant. Then from $\Delta f+2f=0$, we have $f\equiv 0$. That is, $h=\check{h}$ and $\nabla h=0$. In particular, $\Delta_L h=\tilde{\Delta}h=0$. Then from Lemma 3.5 (i) and $R_{ij}=g_{ij}$, we get

$$
(4.7)\qquad h_{ij}+h_{mk}R_{mijk}=0.
$$

On the other hand, from (4.4), $h_{mk}R_{mijk}=0$, since $h=\check{h}$. Hence, from (4.7), we have $h=0$. Thus, the equality of (4.6) holds only when $h=0$.

Now the assertion follows from Corollary 3.8. □

§5. Additional remarks.

LEMMA 5.1. *Suppose that* $\dim M=4$ *and* $g\in\mathcal{M}(M)$ *is a metric with nonnegative sectional curvature. Then the following pointwise inequality holds;* $3|W|^2\leqq 2R^2$.

PROOF. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal frame. Then, $f_1=e_1\wedge e_2+e_3\wedge e_4$, $f_2=e_1\wedge e_3+e_4\wedge e_2$, $f_3=e_1\wedge e_4+e_2\wedge e_3$, $f_4=e_1\wedge e_2-e_3\wedge e_4$, $f_5=e_1\wedge e_3-e_4\wedge e_2$ and $f_6=e_1\wedge e_4-e_2\wedge e_3$ form an orthonormal frame of Λ^2 (in our convention, $e_i\wedge e_j=(1/2)(e_i\otimes e_j-e_j\otimes e_i)$). We regard the curvature tensor as a linear transformation of Λ^2. Then, with respect to the frame $\{f_\alpha\}$, we have the following matrix representation of the curvature tensor;

$$\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix},$$

where A and C are 3×3 symmetric matrices with $\mathrm{tr}A=\mathrm{tr}\,C=R/2$, and

$$B=(B_{\alpha\beta})=\begin{pmatrix} E_{11}+E_{22} & E_{23}-E_{41} & E_{42}+E_{13} \\ E_{23}+E_{41} & E_{11}+E_{33} & E_{43}-E_{21} \\ E_{42}-E_{13} & E_{43}+E_{21} & E_{11}+E_{44} \end{pmatrix},$$

where $E_{ij}=R_{ij}-(R/4)g_{ij}$. It is known that A and C can be diagonalized for some orthonormal frame $\{e_i\}$ ([7; Theorem 2.1]). So, we write

$$A=\begin{pmatrix} (R/6)+\lambda_1 & 0 & 0 \\ 0 & (R/6)+\lambda_2 & 0 \\ 0 & 0 & (R/6)+\lambda_3 \end{pmatrix},\quad C=\begin{pmatrix} (R/6)+\mu_1 & 0 & 0 \\ 0 & (R/6)+\mu_2 & 0 \\ 0 & 0 & (R/6)+\mu_3 \end{pmatrix}.$$

Then, $\lambda_1+\lambda_2+\lambda_3=\mu_1+\mu_2+\mu_3=0$ and

$$(5.1)\qquad |W|^2=\sum_{\alpha=1}^{3}\lambda_\alpha^2+\sum_{\beta=1}^{3}\mu_\beta^2.$$

A 2-form corresponding to a plane section is of the form $\sum_{\alpha=1}^{3}\xi_\alpha f_\alpha+\sum_{\beta=1}^{3}\eta_\beta f_{\beta+3}$ with $\sum\xi_\alpha^2=\sum\eta_\beta^2=1/2$. Therefore, if the sectional curvature is non-negative then,

$$(5.2)\qquad \sum_\alpha\xi_\alpha^2\left(\frac{R}{6}+\lambda_\alpha\right)+\sum_\beta\eta_\beta^2\left(\frac{R}{6}+\mu_\beta\right)+2\sum_{\alpha,\beta}\xi_\alpha\eta_\beta B_{\alpha\beta}\geqq 0,$$

for all $\{\xi_\alpha\}$ and $\{\eta_\beta\}$ with $\sum\xi_\alpha^2=\sum\eta_\beta^2=1/2$. From this, it is easily seen that

$$\frac{R}{3}+\lambda_\alpha+\mu_\beta\geqq 0\qquad\text{for all}\quad \alpha,\ \beta.$$

Hence,

$$\begin{aligned}3R^2=&\left\{\left(\frac{R}{3}+\lambda_1+\mu_1\right)+\left(\frac{R}{3}+\lambda_2+\mu_2\right)+\left(\frac{R}{3}+\lambda_3+\mu_3\right)\right\}^2\\ &+\left\{\left(\frac{R}{3}+\lambda_1+\mu_2\right)+\left(\frac{R}{3}+\lambda_2+\mu_3\right)+\left(\frac{R}{3}+\lambda_3+\mu_1\right)\right\}^2\\ &+\left\{\left(\frac{R}{3}+\lambda_1+\mu_3\right)+\left(\frac{R}{3}+\lambda_2+\mu_1\right)+\left(\frac{R}{3}+\lambda_3+\mu_2\right)\right\}^2\\ \geqq&\sum_{\alpha,\beta}\left(\frac{R}{3}+\lambda_\alpha+\mu_\beta\right)^2=R^2+3\left(\sum\lambda_\alpha^2+\sum\mu_\beta^2\right).\end{aligned}$$

Therefore, from (5.1), we have $2R^2\geqq 3|W|^2$. $\square$

PROPOSITION 5.2. *Let M be a compact 4-dimensional manifold.*

(i) *If M admits an Einstein metric, then $\nu(M) \leqq 16\pi^2\chi$;*

(ii) *If M admits an Einstein metric with nonnegative sectional curvature, then $\nu(M) \leqq (64/5)\pi^2\chi$, where χ is the Euler characteristic of M.*

PROOF. (i) follows from the Gauss Bonnet formula (1.3).

(ii): If $g \in \mathcal{M}(M)$ is an Einstein metric with nonnegative sectional curvature, then from (1.3) and Lemma 5.1, we have

$$\begin{aligned}\nu(g) &= 16\pi^2\chi - \frac{1}{12}\int R^2 dv \leqq 16\pi^2\chi - \frac{1}{8}\int |W|^2 dv \\ &= 16\pi^2\chi - \frac{1}{4}\nu(g).\end{aligned}$$

Hence, $\nu(M) \leqq \nu(g) \leqq 64\pi^2\chi/5$. □

COROLLARY 5.3. *Let M be a compact oriented 4-dimensional manifold. If M admits an Einstein metric with nonnegative sectional curvature, then $|\tau| \leqq 8\chi/15$, and equality holds if and only if M has a flat metric.*

PROOF. Let $g \in \mathcal{M}(M)$ be the Einstein metric with nonnegative curvature. Then by Propositions 1.4 and 5.2, $|\tau| \leqq \nu(M)/24\pi^2 \leqq \nu(g)/24\pi^2 \leqq 8\chi/15$.

If the equality holds, then g is half conformally flat. So, we assume that $*(W_{mijk}e^j \wedge e^k) = W_{mijk}e^j \wedge e^k$ (resp. $*(W_{mijk}e^j \wedge e^k) = -W_{mijk}e^j \wedge e^k$). From the Weitzenböck formula, we have for any harmonic 2-form α,

$$\alpha_{ij;kk} = \frac{R}{3}\alpha_{ij} + \alpha_{km}W^{km}{}_{ij}.$$

Hence, if furthermore $*\alpha = -\alpha$ (resp. $*\alpha = +\alpha$), then

$$\alpha_{ij;kk} = \frac{R}{3}\alpha_{ij}. \tag{5.3}$$

Now, suppose that g is not flat, i.e., $R>0$. Then, from (5.3), $\alpha=0$ for any harmonic 2-form with $*\alpha=-\alpha$ (resp. $*\alpha=\alpha$). Therefore, $\tau=\pm$2nd Betti number of M. The 1st Betti number is zero since the Ricci curvature is positive. So, $|\tau| = \chi - 2$. It is easy to see that $|\tau| = \chi - 2$ with $|\tau| = 8\chi/15$ does not have integral solutions. This is a contradiction. Hence, g is flat. □

REMARK. This proposition slightly improves Theorem 2 of [**4**], where 8/15 is replaced by $(2/3)^{1.5}$ ($>8/15$).

References

[1] A. Avez, Characteristic classes and Weyl tensor, Proc. Nat. Acad. Sci. U.S.A., **66** (1970), 265–268.

[2] R. Bach, Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des

Krümmungstensorbegriffs, Math. Z., **9** (1921), 110-135.
[3] M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, J. Diff. Geom., **3** (1969), 379-392.
[4] N. Hitchin, Compact four-dimensional Einstein manifolds, J. Diff. Geom., **9** (1974), 435-441.
[5] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 2, Interscience, 1969.
[6] N. H. Kuiper, On compact conformally Euclidean spaces of dimension >2, Ann. of Math., **52** (1950), 478-490.
[7] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, Global Analysis, Papers in Honor of K. Kodaira, Princeton Univ. Press, 1969, 355-365.

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