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A differential equation arising from scalar curvature function

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0. Introduction.

In this paper, we consider on a Riemannian manifold (M, g) a differential equation

$$-g\Delta f+\mathrm{Hess}\, f-f\,\mathrm{Ric}=0\,, \tag{0.1}$$

where $\mathrm{Hess}\, f$ is the second covariant derivative of f, $\Delta f=\mathrm{trace}\,\mathrm{Hess}\, f$, and Ric denotes the Ricci curvature tensor. Since (0.1) is linear with respect to f, $f\equiv 0$ is a solution to (0.1) for any (M, g). Conversely, from overdetermined character of the equation, $f\equiv 0$ is often the unique solution for generic (M, g) (see Lemma A below). We ask what the exceptional cases are. The purpose of this paper is to determine Riemannian manifolds (M, g) admitting a non-trivial solution to (0.1) when they are complete and conformally flat.

The equation (0.1) was originally derived from the linearization of the scalar curvature equation. Suppose for a while that M is compact. Let $\mathfrak{M}^s$ (resp. $\mathfrak{F}^s$) denote the space of Riemannian metrics (resp. functions) on M whose derivatives up to order s are L_2-integrable and let $\mathfrak{M}^\infty=\bigcap_s\mathfrak{M}^s$ (resp. $\mathfrak{F}^\infty=\bigcap_s\mathfrak{F}^s$) be the space of C^∞ Riemannian metrics (resp. C^∞ functions) on M. Then scalar curvature is considered as a non-linear mapping R of $\mathfrak{M}^s$ to $\mathfrak{F}^{s-2}$. It is known that the mapping $R:\mathfrak{M}^s\to\mathfrak{F}^{s-2}$ is differentiable if $s>\frac{n}{2}+1$, $n=\dim M$, and that the derivative dR_g at $g\in\mathfrak{M}^s$ is given by

$$dR_g(h)=\frac{d}{dt}\Big|_{t=0}R(g+th)=-\Delta\,\mathrm{tr}\,h+\mathrm{div}\,\mathrm{div}\,h-h\cdot\mathrm{Ric}$$

$$(=-h_i{}^i{}_{;j}{}^j+h_{ij;}{}^{ij}-h_{ij}R^{ij} \text{ in classical tensor notation}).$$

Using Stokes' theorem, the L_2-adjoint operator $dR_g{}^*$ of dR_g with respect to the canonical L_2-inner product defined by g is computed to be

$$dR_g{}^*(f)=-g\Delta f+\mathrm{Hess}\, f-f\,\mathrm{Ric}.$$

Since dR_g is not surjective when $dR_g{}^*$ has a non-trivial kernel, it can be said that a Riemannian metric g such that (0.1) has a non-trivial solution f, which implies that $dR_g{}^*$ has a non-trivial kernel, is a singular point of the mapping $R: \mathfrak{M} \to \mathfrak{F}$. In their papers, Bourguignon [1] and Fischer-Marsden [3] proved the following.

LEMMA A. *Let (M, g) be a complete Riemannian manifold admitting a non-trivial solution f (i.e., $f \not\equiv 0$) to* (0.1). *Then (M, g) has constant scalar curvature, which is necessarily non-negative if we assume further that M is compact.*

Using this lemma, they showed that *for a compact M and for $\frac{n}{2}+1<s\leqq\infty$, dR_g is surjective and $R: \mathfrak{M}^s \to \mathfrak{F}^{s-2}$ maps any neighborhood of $g \in \mathfrak{M}^s$ onto a neighborhood of $R(g) \in \mathfrak{F}^{s-2}$ unless $R(g)$ is a non-negative constant.* On the other hand, this lemma says that the class of Riemannian manifolds admitting a non-trivial solution to (0.1) is, in a sense, very small. So they conjectured in [3] that *(M, g) in the above lemma might be an Einstein space when it is compact.* If so, by a theorem of Obata [5], such a space must be a standard sphere or a Ricci flat space. But it will be proved that the conjecture is not true. There are many counter examples even if we restrict our attention to conformally flat spaces. The simplest one is given by a Riemannian product $S^1(r/\sqrt{n-2}) \times S^{n-1}(r)$, $n \geqq 3$, where $S^N(\rho)$ denotes the Euclidean N-sphere in $\boldsymbol{R}^{N+1}$ with radius ρ. Namely it is compact conformally flat and Ricci parallel but not Einstein, and admits a non-trivial solution to (0.1). In the present paper, we shall determine the other compact conformally flat examples admitting a non-trivial solution to (0.1) and also give a refinement of the result of Fischer-Marsden [3] on local surjectivity of $R: \mathfrak{M} \to \mathfrak{F}$.

1. Preliminaries.

It is easy to see that, if (0.1) holds, then so does

$$f\,\mathrm{Ric} - \mathrm{Hess}\, f = \frac{1}{n}(fR - \Delta f)g\,, \tag{1.1}$$

where R denotes the scalar curvature, i.e., $R = \mathrm{trace}\,\mathrm{Ric}$. In fact, (0.1) is equivalent to (1.1) with an additional condition

$$\Delta f = -\frac{R}{n-1} f\,. \tag{1.2}$$

The equation (1.1) is known as the *static perfect fluid equation*. In this respect the following is known.

LEMMA B ([4]). *Let (M, g) be a complete conformally flat Riemannian n-manifold with $n \geqq 3$, f a solution to* (1.1) *and let M_0 be a connected component of the open submanifold $\{df \neq 0\} \subset M$. Then,*

(i) *M_0 is isometric to a warped product $I\times_r N$ of (I, ds^2) and $(N, \bar{g})$, i.e., $g|_{M_0}=ds^2+r(s)^2\bar{g}$, where I is an open interval in $\boldsymbol{R}$, r is a positive function on I, and $(N, \bar{g})$ is an $(n-1)$-dimensional complete space of constant sectional curvature.*

(ii) *$f|_{M_0}$ can be regarded as a function on I, i.e., f depends only on the parameter $s\in I$.*

(iii) *$\operatorname{grad} f|_{M_0}$ is a complete vector field on M_0.*

REMARK. The above statement is slightly different from that in [4], but the proof in [4] is valid for the above situation.

By the lemma, the Riemannian structure of M_0 and the solution f restricted to M_0 can be described by some ordinary differential equations. The aim of this section is to give the ordinary differential equations. Before stating the result, we quote a lemma which was used to prove Lemma A.

LEMMA A′ ([1], [3]). *Under the same assumption as in Lemma* A, *$\{x\in M: f(x)\neq 0\}$ is a dense subset in M.*

Keeping Lemmas A, A′ and B in mind, we give the following.

LEMMA 1.1. *Let (M_0, g) be a warped product $I\times_r N$ of an interval I and a complete $(n-1)$-space N, $n\geqq 3$, of constant sectional curvature k. Let $f=f(s)$ be a smooth function on M_0 depending only on the parameter $s\in I$ such that $f\not\equiv 0$ and $\{f\neq 0\}$ is dense in M_0. Assume (M_0, g) has constant scalar curvature R. Then the metric g and the function f satisfy*

$$-g\Delta f+\operatorname{Hess} f-f\operatorname{Ric}=0\,, \tag{1.3}$$

if and only if f and r satisfy the following:

$$\ddot{r}+\frac{R}{n(n-1)}r=ar^{1-n}\qquad \textit{for some constant } a; \tag{1.4}$$

$$\begin{cases} \ddot{f}+\dfrac{R}{n-1}f=0, & \textit{if } r \textit{ is constant;} \\ f=c\dot{r} & \textit{for some non-zero constant } c, \textit{ if } r \textit{ is not constant;} \end{cases} \tag{1.5}$$

where $\dot{}$ (resp. $\ddot{}$) means d/ds (resp. d^2/ds^2).

Moreover, the sectional curvature k of N coincides with an integral constant of (1.4), *that is,*

$$\dot{r}^2+\frac{2a}{n-2}r^{2-n}+\frac{R}{n(n-1)}r^2=k\,. \tag{1.6}$$

PROOF. Since $M_0=I\times_r N$ and $f=f(s)$, a direct calculation yields

$$\operatorname{Ric}=-\Big(n\frac{\ddot{r}}{r}+\frac{R}{n-1}\Big)ds^2+\Big(\frac{\ddot{r}}{r}+\frac{R}{n-1}\Big)g\,, \tag{1.7}$$

$$R=-2(n-1)\frac{\ddot{r}}{r}-(n-1)(n-2)\left(\frac{\dot{r}}{r}\right)^2+(n-1)(n-2)\frac{k}{r^2}, \tag{1.8}$$

$$\mathrm{Hess}\, f=\left(\ddot{f}-\dot{f}\frac{\dot{r}}{r}\right)ds^2+\dot{f}\frac{\dot{r}}{r}g,$$

$$\Delta f=\ddot{f}+(n-1)\dot{f}\frac{\dot{r}}{r}.$$

Hence, we have

$$-g\Delta f+\mathrm{Hess}\, f-f\,\mathrm{Ric}$$
$$=\left(\ddot{f}-\dot{f}\frac{\dot{r}}{r}+nf\frac{\ddot{r}}{r}+\frac{R}{n-1}f\right)ds^2-\left(\ddot{f}+(n-2)\dot{f}\frac{\dot{r}}{r}+f\frac{\ddot{r}}{r}+\frac{R}{n-1}f\right)g.$$

Therefore (1.3) holds if and only if

$$\begin{cases} \ddot{f}-\dot{f}\dfrac{\dot{r}}{r}+nf\dfrac{\ddot{r}}{r}+\dfrac{R}{n-1}f=0, \\ \ddot{f}+(n-2)\dot{f}\dfrac{\dot{r}}{r}+f\dfrac{\ddot{r}}{r}+\dfrac{R}{n-1}f=0. \end{cases}$$

Namely,

$$\begin{cases} \ddot{f}=-\left((n-1)\dfrac{\ddot{r}}{r}+\dfrac{R}{n-1}\right)f, \\ \dot{f}\dot{r}=f\ddot{r}. \end{cases} \tag{1.9}$$

If $r=$const., then we have the condition (1.5) immediately and (1.4) is trivially satisfied. So we assume $\dot{r}\not\equiv 0$. Then, from the assumption that $\{f\neq 0\}$ is dense in M_0, the latter part of (1.9) implies

$$f=c\dot{r} \quad \text{for some non-zero constant } c. \tag{1.10}$$

Thus (1.9) is equivalent to (1.10) together with

$$\dddot{r}+(n-1)\frac{\dot{r}\ddot{r}}{r}+\frac{R}{n-1}\dot{r}=0. \tag{1.11}$$

Since $R=$const., we have

$$\left(r^{n-1}\ddot{r}+\frac{R}{n(n-1)}r^n\right)^{\cdot}=r^{n-1}\left(\dddot{r}+(n-1)\frac{\dot{r}\ddot{r}}{r}+\frac{R}{n-1}\dot{r}\right).$$

Therefore (1.11) implies

$$r^{n-1}\ddot{r}+\frac{R}{n(n-1)}r^n=a=\text{constant}. \tag{1.12}$$

Hence, we obtain (1.4).

Using the equality

$$\left(\dot{r}^2+\frac{2a}{n-2}r^{2-n}+\frac{R}{n(n-1)}r^2\right)^{\cdot}=2\dot{r}\left(\ddot{r}-ar^{1-n}+\frac{R}{n(n-1)}r\right),$$

we see $\dot{r}^2+(2a/(n-2))r^{2-n}+(R/n(n-1))r^2$ is an integral constant for (1.12). It is easy to see from (1.8) and (1.12) that the constant coincides with k. ■

REMARK. Putting $I=\boldsymbol{R}$ and $N=S^{n-1}(1)$, $n\geqq 3$, in the above lemma, we can write the condition (1.5) as

$$\ddot{f}+(n-2)r^{-2}f=0\,, \qquad r=\text{constant}.$$

Then f is a periodic function with a period $2\pi r/\sqrt{n-2}$. Hence, we can construct a non-trivial solution to (1.3) on

$$\boldsymbol{R}/(2\pi r/\sqrt{n-2})\boldsymbol{Z}\times_r S^{n-1}(1)=S^1(r/\sqrt{n-2})\times S^{n-1}(r)\,,$$

which is a usual Riemannian product because the warping function r is constant, and is evidently compact but not an Einstein space. This is the simplest counter example for the Fischer-Marsden conjecture.

REMARK. A geometric meaning of the constant a defined by (1.12) is as follows: By (1.7), the Ricci tensor has, at each point of M_0, two eigenvalues μ and ν with multiplicities 1 and $n-1$ respectively. Then, from (1.7) and (1.12),

$$\mu-\nu=-\left(n\frac{\ddot{r}}{r}+\frac{R}{n-1}\right)=-nr^{-n}a\,. \tag{1.13}$$

Thus, $a=0$ implies g is an Einstein metric.

REMARK. If $R=0$ and $n=3$, the space M_0 satisfying (1.5) is isometric to the spatial section of Schwarzshild solution in general relativity. Then the constant a is nothing but the total mass of the space-time.

REMARK. By considering (1.6) as something like Hamiltonian, we can see some resemblance between (1.6) and a Hamiltonian system of a mixture of harmonic oscillator and Kepler's problem.

2. Analysis on (1.4).

In this section we study the solution to the ordinary differential equation

$$\ddot{r}+\frac{R}{n(n-1)}r=ar^{1-n}, \qquad n>2,\ r>0, \tag{2.1}$$

with initial values $r(0)>0$ and $\dot{r}(0)$, where a and R are constants and r is a positive function in one variable s.

As mentioned in Lemma 1.1,

$$k:=\dot{r}^2+\frac{2a}{n-2}r^{2-n}+\frac{R}{n(n-1)}r^2 \tag{2.2}$$

is a constant determined by the initial values $r(0)$ and $\dot{r}(0)$. We define some other constants depending on a, R, $r(0)$ and $\dot{r}(0)$ as follows:

$$\rho_0=\left(\frac{n(n-1)a}{R}\right)^{1/n} \qquad \text{when } aR>0;$$

$$\rho_1=\begin{cases}\left(\dfrac{a}{(n-2)k}\right)^{1/(n-2)} & \text{when } a>0,\ R\leqq 0,\ k>0,\\ \left(-\dfrac{2n(n-1)a}{(n-2)R}\right)^{1/n} & \text{when } a>0,\ R<0,\ k=0,\\ \left(\dfrac{n(n-1)k}{R}\right)^{1/2} & \text{when } a\geqq 0,\ R<0,\ k<0,\\ \rho_0 & \text{when } a<0,\ R<0;\end{cases}$$

$$\kappa_0=\frac{R}{(n-1)(n-2)}\left(\frac{n(n-1)a}{R}\right)^{2/n} \qquad \text{when } aR>0.$$

Next, we give conditions on a, R and initial values:

I. 1. $a>0,\ R>0,\ k\leqq\kappa_0$.
2. $a<0,\ R<0,\ k\leqq\kappa_0,\ r(s)=\rho_0$ for some s.
3. $a=k=R=0$.

II. $a=R=0,\ k\neq 0$.

III. 1. $a<0,\ R\geqq 0$.
2. $a<0,\ R<0,\ k\leqq\kappa_0,\ r(0)<\rho_0$.
3. $a<0,\ R<0,\ k>\kappa_0$.

IV. 1. $a>0,\ R\leqq 0$.
2. $a=0,\ R<0,\ k<0$.
3. $a<0,\ R<0,\ k\leqq\kappa_0,\ r(0)>\rho_0$.
4. $a=0,\ R<0,\ k=0$.

V. $a>0,\ R>0,\ k>\kappa_0$.

VI. 1. $a=0,\ R>0$.
2. $a=0,\ R<0,\ k>0$.

It should be remarked that these conditions exhaust all of cases.

The following lemma is easy to prove by using (2.2).

LEMMA 2.1. (i) *The condition* (III. 3) *implies* $\dot{r}^2\geqq k-\kappa_0>0$. (ii) *Each one of the condition* (IV. 1)~(IV. 3) *implies* $r\geqq\rho_1>0$.

PROPOSITION 2.2. *Each one of the conditions* (I. 1)~(I. 3) *implies* $r\equiv\rho_0$. *Conversely, if* $r=$*const., then one of the conditions* (I) *is satisfied.*

PROOF. By (2.2), it is easy to see that $k=\kappa_0$ and $\dot{r}\equiv 0$ under the condition (I. 1) or (I. 3). So $r=\text{const.}=\rho_0$ by (2.1).

Similarly, if (I. 2) holds, then $k=\kappa_0$ and $\dot{r}(s)=0$ by (2.2). Hence, by the uniqueness of solution, we have $r\equiv\rho_0$.

The latter part of the assertion is trivial. ■

LEMMA 2.3. *Each one of the conditions* (III. 1), (III. 2) *and* (VI. 1) (*resp.* (IV. 1) $\sim$(IV. 4) *and* (VI. 2)) *implies* $\ddot{r}<0$ (*resp.* $\ddot{r}>0$).

PROOF. We consider only the cases (III. 2) and (IV. 3). The other cases are trivial. So, suppose (III. 2) or (IV. 3) holds. Then r never attains ρ_0 because, if $r(s)=\rho_0$ for some s, then $k=\kappa_0$ by (2.2), so $r\equiv\rho_0$ by Proposition 2.2, which is a contradiction. Hence, in the case (III. 2) (resp. (IV. 3)), we have $r<\rho_0$ (resp. $r>\rho_0$). Thus, by (2.1), we have $\ddot{r}<0$ (resp. $\ddot{r}>0$). ■

PROPOSITION 2.4. (i) *If* (II) *holds, then* $\dot{r}\not\equiv 0$ *and* $\ddot{r}\equiv 0$. (ii) *If one of the conditions* (III. 1)$\sim$(III. 3) *is satisfied, then* $\lim_{s\to s_0}(\dot{r}(s))^2=\infty$ *for some* s_0.

PROOF. (i): Trivial from (2.1) and (2.2).

(ii): In the case of (III. 1) or (III. 2), we have $\ddot{r}<0$ by Lemma 2.3. So there is an s_0 such that $\lim_{s\to s_0} r(s)=0$. Since $a<0$, we have, from (2.2), $\lim_{s\to s_0}(\dot{r}(s))^2=\infty$.

In the case of (III. 3), we have $\dot{r}\geqq\sqrt{k-\kappa_0}>0$ or $\dot{r}\leqq-\sqrt{k-\kappa_0}<0$. Hence, $\lim_{s\to s_0} r(s)=0$ for some s_0, so $\lim_{s\to s_0}(\dot{r}(s))^2=\infty$. ■

PROPOSITION 2.5. *If one of the conditions* (IV. 1)$\sim$(IV. 4) *is satisfied, there exists a positive solution* r *to* (2.1) *defined on* $\boldsymbol{R}$ *with* $\ddot{r}>0$.

PROOF. In the case of (IV. 4), the solution to (2.1) can be given explicitly and we get the assertion. So we consider only the cases (IV. 1)$\sim$(IV. 3). Then, for a, R satisfying one of (IV. 1)$\sim$(IV. 3), we have, by an elementary calculation, a positive function h defined entirely on $\boldsymbol{R}$ such that $\ddot{h}+(Rh/(n(n-1)))=a\rho_0^{1-n}$, $h(0)=r(0)$, $\dot{h}(0)=\dot{r}(0)$. Then, by Lemma 2.1 (ii), we have $\ddot{r}-\ddot{h}\leqq 0$. Since $h(0)=r(0)$ and $\dot{h}(0)=\dot{r}(0)$, we get $r\leqq h$. On the other hand, again by Lemma 2.1 (ii), $r\geqq\rho_1>0$. Thus, the solution r is extendible on $\boldsymbol{R}$. Moreover, by Lemma 2.3, we have $\ddot{r}>0$. ■

PROPOSITION 2.6. *In the case of the condition* (V), *the solution* r *to* (2.1) *is a periodic function.*

PROOF. We consider (2.1) as a problem on a vector field $\dot{r}(\partial/\partial r)+(ar^{1-n}-Rr/(n(n-1)))(\partial/\partial\dot{r})$ on the half plane $P=\{(r,\dot{r})\in\boldsymbol{R}^2 : r>0\}$. Then the flow of the vector field is described by curves defined by (2.2). Under the assumption, it is easy to see that (2.2) defines a regular submanifold in P and that the submanifold is contained in a compact set in P. Hence, the curves defined by (2.2) are diffeomorphic to S^1 and there is no fixed point of the flow on the curves because $k>\kappa_0$. So the solution to (2.1) is a periodic function. ■

PROPOSITION 2.7. *In the case of* (VI), *the solution to* (2.1) *is given by*

$$r(s)=\begin{cases}\sqrt{\dfrac{n(n-1)k}{R}}\sin\sqrt{\dfrac{R}{n(n-1)}}(s+s_0), & -s_0<s<\pi\sqrt{\dfrac{n(n-1)}{R}}-s_0, \\ & \textit{in the case of (VI. 1)}; \\ \sqrt{-\dfrac{n(n-1)k}{R}}\sinh\sqrt{-\dfrac{R}{n(n-1)}}(\pm s+s_0), & \mp s_0\lesseqgtr s\lesseqgtr\pm\infty, \\ & \textit{in the case of (VI. 2)};\end{cases}$$

where s_0 *is a constant.*

PROOF. Elementary. ■

3. Main results.

By using Lemma 1.1 and results in the previous section, we can construct various examples of complete conformally flat Riemannian manifold with a non-trivial solution f to (0.1):

EXAMPLE 1. Let $N(k)$ be an $(n-1)$-dimensional connected complete space of constant sectional curvature k. A Riemannian product $\boldsymbol{R}\times N(k)$, $k\neq 0$ is not an Einstein space, on which

$$f=f(s, x)=\begin{cases} c_1\sin\sqrt{(n-2)k}\cdot s+c_2\cos\sqrt{(n-2)k}\cdot s, & \text{if } k>0, \\ c_1\sinh\sqrt{(2-n)k}\cdot s+c_2\cosh\sqrt{(2-n)k}\cdot s, & \text{if } k<0,\end{cases}$$

is a solution to (0.1), where c_1 and c_2 are constants.

EXAMPLE 2. For $l\in\boldsymbol{N}$ and an isometry $\varphi\in\mathrm{Isom}\,N(k)$, let $\Gamma_{l,\varphi}$ be a transformation group of isometries of $\boldsymbol{R}\times N(k)$, $k>0$, generated by $\Phi:\boldsymbol{R}\times N(k)\to\boldsymbol{R}\times N(k)$; $(s, x)\mapsto(s+2\pi l/\sqrt{(n-2)k}, \varphi(x))$. Since f in Example 1 is a periodic function with period $2\pi l/\sqrt{(n-2)k}$ for $k>0$, f can be considered as a function on $\boldsymbol{R}\times N(k)/\Gamma_{l,\varphi}$, which is compact but not an Einstein space. In particular, $\boldsymbol{R}\times N(k)/\Gamma_{1,\mathrm{Id.}}$ has already been shown in Section 1.

EXAMPLE 3. A warped product $\boldsymbol{R}\times_r N(1)$, where r is a periodic function as in Proposition 2.6, is also conformally flat but not Einstein, and admits a non-trivial f satisfying (0.1): $f(s, x)=c\dfrac{d}{ds}r(s)$ for some constant c.

EXAMPLE 4. Since f and r in Example 3 have a common period, we obtain compact spaces with non-trivial solutions to (0.1) in a similar way as Example 2. These spaces were first found by Ejiri [2] as counter examples to a conjecture of conformal transformations.

EXAMPLE 5. A warped product $\boldsymbol{R}\times_r N(k)$, where r and k are as in Proposition 2.5, is also conformally flat, and admits a non-trivial solution to (0.1): $f(s, x)=c\dfrac{d}{ds}r(s)$ for some constant c. In particular, when $a>0$, $R=0$ (for notations, see Lemma 1.1 and Section 2), the obtained space contains the space section of the well-known Schwarzshild space-time as an open submanifold.

Our theorem is stated as follows:

THEOREM 3.1. *Let (M, g) be a connected complete conformally-flat Riemannian manifold with $n=\dim M \geqq 3$. Then M admits a non-trivial solution f (i.e., $f \not\equiv 0$) to*

$$-g\Delta f + \mathrm{Hess}\, f - f\,\mathrm{Ric} = 0\,, \tag{3.1}$$

if and only if M is isometric to one of the following:

(i) *Euclidean sphere S^n;*

(ii) *a flat space;*

(iii) *Hyperbolic space H^n;*

(iv) *one of the spaces in Examples* 1~5.

PROOF. As was mentioned, there are non-trivial solutions to (3.1) if M is one of the spaces in Examples 1~5 (remark that H^n is a special case in Example 5: put $N(k)=H^{n-1}$ and consider the condition (IV. 2)). The first eigenfunctions for the Laplace-Beltrami operator Δ on S^n satisfy (3.1). Any non-zero constant satisfies (3.1) if M is flat. Thus "if part" of the statement is proved.

Now, suppose f is a non-trivial solution to (3.1) on M. If $f=\text{const.}\neq 0$, then by (3.1) M is Ricci flat, hence it is flat because M is assumed to be conformally flat.

So, in the following, we assume that $\{df \neq 0\} \subset M$ is not empty, and denote by M_0 a connected component of $\{df \neq 0\}$. Then, by Lemma B, M_0 is isometric to a warped product $I \times_r N(k)$. Hence, at each point of M, the Ricci tensor has eigenvalues μ and ν with multiplicities 1 and $n-1$, respectively. Then, by Lemma 1.1, $a := -\dfrac{1}{n} r^n(\mu-\nu) = r^{n-1}\ddot{r} + Rr^n/n(n-1)$ is constant on M_0 (see also (1.13)), where R is the scalar curvature of M, which is constant by Lemma A.

LEMMA 3.2. *a, k, r and R does not satisfy conditions* (II) *and* (III).

PROOF. Suppose a, k, r and R satisfy (II). Then $\dot{r} \not\equiv 0$, hence by Lemma 1.1 $f=c\dot{r}$. But by Proposition 2.4 (i), $\dot{f}=c\ddot{r}\equiv 0$, which contradicts that $\dot{f}\neq 0$ on M_0.

In the case of (III), by Proposition 2.4 (ii), $f=c\dot{r}$ diverges at a point of finite distance from an arbitrarily fixed point of M_0. Hence the conditions in (III) cannot be satisfied. ■

LEMMA 3.3. *Suppose r is not constant on M_0. If $R \leqq 0$ or $a \leqq 0$, then $\overline{M}_0 = M$ and M is isometric to either S^n or H^n or a space in Example* 5.

PROOF. By Proposition 2.2 and Lemma 3.2, we have only to consider the conditions (IV) and (VI).

(IV): By Proposition 2.5, $\ddot{r} = \dfrac{1}{c}\dot{f} > 0$ on $\boldsymbol{R} \times_r N(k)$ where r and k are as in Proposition 2.5. Hence by Lemma B (iii), $\boldsymbol{R} \times_r N(k) \subset M_0$. On the other hand, $\boldsymbol{R} \times_r N(k)$ itself is a complete space, hence $\boldsymbol{R} \times_r N(k) = M_0 = M$.

(VI. 1): By Proposition 2.7 and Lemma B (iii), $M_0=I\times_r N(k)$, where $I=(-s_0, \pi\sqrt{n(n-1)/R}-s_0)$, r and k are as in Proposition 2.7. Since $r(s)\to 0$ as $s\to -s_0$ or $\pi\sqrt{n(n-1)/R}-s_0$, the closure $\overline{M}_0$ in M is $(I\times_r N(k))\cup\{p\}\cup\{q\}$, where p and q are points in M. By considering the exponential mapping at p or q, we see that $N(k)$ is an $(n-1)$-sphere. Hence, again by Proposition 2.5, $\overline{M}_0$ is isometric to a standard n-sphere.

(VI. 2): By a similar argument, we see $\overline{M}_0=M=H^n$. ∎

LEMMA 3.4. *If* (I. 3) *is satisfied on* M_0, *then* M *is flat.*

PROOF. It is easy to see that M_0 is Ricci flat, hence it is flat. Let M_1 be another component of $\{df\neq 0\}$. Then, by Lemma 3.2 and Lemma 3.3, M_1 cannot satisfy the conditions (II), (III), (IV) and (VI). Since $R=0$, M_1 must satisfy the condition (I. 3). Hence, M_1 also is flat. Namely, $\{df\neq 0\}$ is a flat space. On the other hand, the interior $\mathrm{Int}\{df=0\}\subset M$ is a flat space. Hence, $M=\overline{\{df\neq 0\}\cup\mathrm{Int}\{df=0\}}$ is flat. ∎

The remaining cases are (I. 1), (I. 2) and (V). By the above argument, if M_0 satisfies one of (I. 1), (I. 2) and (V), the other components of $\{df\neq 0\}$ must have the same property, and $\overline{\{df\neq 0\}}=M$. Then it is easy to see that each possible combination of these components forms one of the spaces in Examples 1~4. ∎

Finally, we pick up compact spaces in Theorem 3.1.

THEOREM 3.5. *Let* (M, g) *be a compact connected conformally-flat Riemannian manifold with* $n=\dim M\geqq 3$. *Then,*

(i) *M admits non-zero constant as a solution to* (3.1) *if and only if M is flat.*

(ii) *M admits a non-constant solution to* (3.1) *if and only if M is isometric to either* S^n *or one of the spaces in Examples* 2 *and* 4.

Thus, by the argument in [3], we have

THEOREM 3.6. *Let M be a closed* C^∞ *manifold with* $n=\dim M\geqq 3$, *and* $g\in\mathfrak{M}^s$ *(for the notation, see Introduction). Suppose that*

(i) *g is conformally flat but not flat, and*

(ii) (M, g) *is neither* S^n *nor any one of the spaces in Examples* 2 *and* 4. *Then* dR_g *is surjective and* $R:\mathfrak{M}^s\to\mathfrak{F}^{s-2}$ *is locally surjective at g for* $\frac{n}{2}+1<s\leqq\infty$.

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Added in proof: After this paper was submitted, it was pointed out by Prof. M. Obata that a similar result was obtained independently by J. Lafontaine; "*Sur la géométrie d'une généralisation de l'équation différentielle d'Obata*" (to appear).