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Title	Ricci curvature of affine connections
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Citation	Tohoku Mathematical Journal. 2008, 60(3), p. 357-364
Version Type	VoR
URL	https://hdl.handle.net/11094/26348
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Ricci curvature of affine connections

Dedicated to Professor Seiki Nishikawa on his sixtieth birthday

Osamu Kobayashi

Abstract

We show various examples of torsion-free affine connections which preserve volume elements and have definite Ricci curvature tensors.

1 Introduction

We will show various examples of affine connections with characteristic properties on the Ricci curvature. All connections which we treat in this paper are torsion-free and have parallel volume elements. We will give projectively flat affine connections with negative definite Ricci curvature on the torus T^n (Proposition 3.1). We will also give affine connections with positive Ricci curvature on compact parallelizable manifolds of dimension greater than two (Theorem 3.3). It follows from this result that every compact orientable 3-manifold has an affine connection of positive definite Ricci curvature. In dimension two, we will give affine connections of negative Ricci curvature on the torus T^2 and Klein's bottle $\mathbf{R}\mathrm{P}^2 \# \mathbf{R}\mathrm{P}^2$ (Proposition 4.1). In view of an eminent theorem by J. Lohkamp [2], what remain to be examined about negative Ricci curvature are the sphere S^2 and the real projective plane $\mathbf{R}\mathrm{P}^2$. We will add an observation on this matter (Proposition 4.6).

2 Preliminaries

For a smooth manifold M we denote by $\mathcal{A}(M)$ the space of all torsion-free affine connections of M that preserve some volume elements. Suppose that $\nabla \in \mathcal{A}(M)$, $n = \dim M$

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and $\nabla d\mu = 0$, where $d\mu$ is a volume element of M. Then the curvature tensor is defined as $R^\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The Ricci curvature $\mathrm{Ric}(X,Y)$ is defined as the trace of linear transformation; $Z \in T_pM \mapsto R^\nabla(Z,Y)X \in T_pM$. Note that $\mathrm{Ric}^\nabla \in T_p^*M \otimes T_p^*M = \mathrm{Hom}(T_pM, T_p^*M)$. Since $\nabla d\mu = 0$, the first Bianchi identity implies that $\mathrm{Ric}^\nabla(X,Y) = \mathrm{Ric}^\nabla(Y,X)$.

Let $A = A_{ij}dx^i \otimes dx^j \in T_p^*M \otimes T_p^*M = \mathrm{Hom}(T_pM, T_p^*M)$ be a 2-tensor, and $\omega = a\,dx^1 \wedge \cdots \wedge dx^n = \omega_{i_1\ldots i_n}dx^{i_1} \otimes \cdots \otimes dx^{i_n} = \omega_{i_1\ldots i_n}dx^{i_1} \wedge \cdots \wedge dx^{i_n}$ be (local) volume form representing $d\mu$. Here we adopt the summation convention for repeated indices. Then,

$$\omega_{i_1\ldots i_n} = \begin{cases} a/n! & \text{if } (i_1,\ldots,i_n) \text{ is an even permutation of } (1,\ldots,n), \\ -a/n! & \text{if } (i_1,\ldots,i_n) \text{ is an odd permutation of } (1,\ldots,n), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(dx^1 \wedge \cdots \wedge dx^n)(\partial_1,\ldots,\partial_n) = (dx^1 \wedge \cdots \wedge dx^n)(\partial_1 \wedge \cdots \wedge \partial_n) = 1/n!$, where $\partial_i = \partial/\partial x^i$. Hence, put $\omega^* = (n!/a)\,\partial_1 \wedge \cdots \wedge \partial_n = \omega^{i_1\ldots i_n}\partial_{i_1} \otimes \cdots \otimes \partial_{i_n} = \omega^{i_1\ldots i_n}\partial_{i_1} \wedge \cdots \wedge \partial_{i_n}$, and we have $\omega(\omega^*) = \omega^*(\omega) = 1$, where

$$\omega^{i_1\ldots i_n} = \begin{cases} 1/a & \text{if } (i_1,\ldots.i_n) \text{ is an even permutation of } (1,\ldots,n), \\ -1/a & \text{if } (i_1,\ldots.i_n) \text{ is an odd permutation of } (1,\ldots,n), \\ 0 & \text{otherwise.} \end{cases}$$

The *determinant* of A with respect to $d\mu$ is then defined as[†]

$$A^*\omega^* = n!(\det_{d\mu}A)\omega.$$

We note that $\det_{d\mu} A = \det(A_{ij})$ if $d\mu = dx^1 \cdots dx^n$. The *cofactor* tensor $\hat{A} = \hat{A}^{ij}\partial_i \otimes \partial_j \in T_pM \otimes T_pM$ is defined so that

$$\hat{A}^{ij}A_{ik} = (\det_{d\mu}A)\delta_k^j.$$

Lemma 2.1. *Notation being as above, we have*

(i) $\omega^{i_1\ldots i_n}A_{i_1j_1} \cdots A_{i_nj_n}\omega^{j_1\ldots j_n} = n!\det_{d\mu} A$;

(ii) $\hat{A}^{ij} = \dfrac{1}{(n-1)!}\omega^{i_1\ldots i_{n-1}i}A_{i_1j_1} \cdots A_{i_{n-1}j_{n-1}}\omega^{j_1\ldots j_{n-1}j}$.

Proof. By definition, $\omega^*(A^*\omega^*) = n!\det_{d\mu} A$. This shows (i). If $j \neq k$, it is easy to see that $\omega^{i_1\ldots i_{n-1}i}A_{i_1j_1} \cdots A_{i_{n-1}j_{n-1}}A_{ik}\omega^{j_1\ldots j_{n-1}j} = 0$. Then (ii) follows from (i). □

[†]The definition in [1; p. 466, l. 6] is incorrect.

COROLLARY 2.2. *If $A_{ij;k} = A_{ik;j}$, then $\hat{A}^{ij}{}_{;j} = 0$.*

PROOF. Under the assumption we have $A_{i_1 j_1} \cdots A_{i_k j_k;j} \cdots A_{i_{n-1} j_{n-1}} \omega^{i_1 \ldots j_k \ldots j_{n-1} j} = 0$. Since $\nabla d\mu = 0$, the assertion follows from (ii) of the above lemma. $\qquad\square$

An affine connection $\nabla \in \mathcal{A}(M)$ is said to be *projectively flat* if the following conditions are satisfied:

$$(1) \qquad R^{\nabla}(X,Y)Z = \frac{1}{n-1}(\mathrm{Ric}^{\nabla}(Y,Z)X - \mathrm{Ric}^{\nabla}(Z,X)Y)$$

and

$$(2) \qquad (\nabla_Z \mathrm{Ric}^{\nabla})(X,Y) = (\nabla_Y \mathrm{Ric}^{\nabla})(X,Z).$$

If $n = 2$, the equality (1) always holds because of the first Bianchi identity. If $n \geq 3$, (1) implies (2) from the second Bianchi identity. It follows from Corollary 2.2 that $\mathrm{div}_{\nabla} \hat{\mathrm{Ric}} = 0$ if ∇ is projectively flat.

Now, let $\bar{\nabla} \in \mathcal{A}(M)$ be a reference connection, and suppose $\bar{\nabla} d\mu = 0$. Let (e_i) be a frame of $T_p M$, and (e^i) its dual. We say that a $(1,2)$-tensor $S = S^i_{jk} e_i \otimes e^j \otimes e^k \in \mathrm{Hom}(T_p M \otimes T_p M, T_p M)$ is *symmetric* if $S^i_{jk} = S^i_{kj}$. We use the following notation:

$$\mathrm{tr}\, S := S^j_{ij}\, e^i, \qquad \mathrm{div}_{\bar{\nabla}} S := S^k_{ij|k}\, e^i \otimes e^j, \qquad S \cdot S := S^l_{ik} S^k_{jl}\, e^i \otimes e^j,$$

where $|$ stands for the covariant differentiation with respect to $\bar{\nabla}$.

LEMMA 2.3. *Suppose $\bar{\nabla} \in \mathcal{A}(M)$ and $\bar{\nabla} d\mu = 0$. Let S be a symmetric $(1,2)$-tensor field, and $\nabla = \bar{\nabla} + S$, that is, $\nabla_X Y = \bar{\nabla}_X Y + S(X,Y)$. Then the following hold.*

(i) $\nabla \in \mathcal{A}(M)$ if and only if $\mathrm{tr}\, S$ is an exact 1-form.

(ii) $\nabla d\mu = 0$ if and only if $\mathrm{tr}\, S = 0$.

(iii) The Ricci curvature is given as

$$(3) \qquad \mathrm{Ric}^{\nabla} = \mathrm{Ric}^{\bar{\nabla}} + \mathrm{div}_{\bar{\nabla}} S - S \cdot S - \bar{\nabla} \mathrm{tr}\, S + (\mathrm{tr}\, S) \cdot S.$$

This is written in index notation as

$$R_{ij} = \bar{R}_{ij} + S^k_{ij|k} - S^l_{ik} S^k_{jl} - s_{i|j} + s_k S^k_{ij},$$

where $s_i = S^j_{ij}$.

PROOF. For a function u, we have $\nabla(e^u d\mu) = e^u d\mu \otimes (du - \mathrm{tr}\, S)$. This yields (i) and (ii) immediately. (iii) is shown by a direct calculation. $\qquad\square$

From (i), $\mathcal{A}(M)$ itself is an affine space.

LEMMA 2.4. *Suppose M is compact. If there is a symmetric $(1,2)$-tensor field S with* $\operatorname{tr} S = 0$ *such that* $-S \cdot S$ *is positive (resp. negative) definite, then there is a* $\nabla \in \mathcal{A}(M)$ *such that* $\operatorname{Ric}^{\nabla}$ *is positive (resp. negative) definite.*

PROOF. It follows from (3) that $\nabla = \bar{\nabla} + tS$ has the required property for sufficiently large $t \in \mathbf{R}$. $\qquad \square$

3 Ricci curvature of parallelizable manifolds

PROPOSITION 3.1. *Let T^n be the n-dimensional torus. If $n \geq 2$, there is a $\nabla \in \mathcal{A}(T^n)$ with the following properties*:

(i) $\operatorname{Ric}^{\nabla}$ *is negative definite.*

(ii) ∇ *is projectively flat.*

(iii) $\det \hat{\operatorname{Ric}}^{\nabla}$ *is constant and* $\operatorname{div}_{\nabla} \hat{\operatorname{Ric}}^{\nabla} = 0$.

PROOF. Let g be the Euclidean metric of $\mathbf{R}^n$, and $\bar{\nabla}$ the standard flat connection. Take vectors $X_0, X_1, ..., X_n$ of $\mathbf{R}^n$ such that $\bar{\nabla} X_i = 0$ and

$$g(X_i, X_j) = \begin{cases} 1, & \text{if } i = j, \\ -1/n, & \text{if } i \neq j. \end{cases}$$

Then $\{X_1, ..., X_n\}$ is a basis of $\mathbf{R}^n$. Define covectors $\xi_0, \xi_1, ..., \xi_n$ as

$$\xi_i(X_j) = g(X_i, X_j).$$

Then,

$$\sum_{i=0}^{n} X_i = 0, \qquad \sum_{i=0}^{n} \xi_i = 0.$$

Put

$$S := \sum_{i=0}^{n} X_i \otimes \xi_i \otimes \xi_i.$$

Clearly, this is symmetric and $\operatorname{tr} S = 0$. Furthermore,

$$S \cdot S = \left(\sum_i X_i \otimes \xi_i \otimes \xi_i \right) \cdot \left(\sum_j X_j \otimes \xi_j \otimes \xi_j \right) = \sum_{i,j} \xi_i(X_j) \xi_j(X_i) \xi_i \otimes \xi_j$$

$$= \sum_{i=j} \xi_i \otimes \xi_j + \frac{1}{n^2} \sum_{i \neq j} \xi_i \otimes \xi_j = \left(1 - \frac{1}{n^2} \right) \sum_i \xi_i \otimes \xi_i.$$

It is easy to see that $\sum \xi_i \otimes \xi_i = ((n+1)/n)g$. Thus, by putting $\nabla = \bar{\nabla} + S$, we have from (3)

$$\mathrm{Ric}^\nabla = -\frac{(n-1)(n+1)^2}{n^3}g,$$

which is negative definite.

Direct calculations show that for $i \neq j$,

$$\nabla_{X_i} X_i = \frac{n^2-1}{n^2} X_i, \qquad \nabla_{X_i} X_j = -\frac{n+1}{n^2}(X_i + X_j).$$

By further calculation, we have for mutually distinct i, j, k

$$R^\nabla(X_i, X_j)X_j = -\frac{(n+1)^2}{n^4}(nX_i + X_j),$$

$$R^\nabla(X_i, X_j)X_k = \frac{(n+1)^2}{n^4}(X_i - X_j).$$

From these, (1) follows. A direct calculation also shows (2). Thus ∇ is projectively flat.

For the volume element $d\mu_g$ of the metric g, we have $\nabla d\mu_g = 0$. It is clear that $\det \mathrm{Ric}^\nabla$ is constant with respect to $d\mu_g$. The equality $\mathrm{div}_\nabla \hat{\mathrm{Ric}}^\nabla = 0$ follows from (ii) and Corollary 2.2.

Since the connection ∇ is invariant under translations of $\mathbf{R}^n$ in the usual sense, it defines a connection of $T^n = \mathbf{R}^n/\mathbf{Z}^n$. $\qquad\square$

REMARKS 3.2.

(i) The connection obtained in this proof is *not* geodesically complete.

(ii) For any $t \in \mathbf{R} \setminus \{0\}$, $\nabla := \bar{\nabla} + tS$ also has the same properties.

(iii) If $n = 3$, $S := \partial_x \otimes (dy \otimes dz + dz \otimes dy) + \partial_y \otimes (dz \otimes dx + dx \otimes dz) + \partial_z \otimes (dx \otimes dy + dy \otimes dx)$ also defines a connection with the same properties.

THEOREM 3.3. *Suppose M is compact, $n = \dim M \geq 3$ and M is parallelizable. Then there is a $\nabla \in \mathcal{A}(M)$ such that Ric^∇ is positive definite.*

PROOF. Let $(e_1, e_2, ..., e_n)$ be a global frame field of the tangent bundle TM, and $(e^1, e^2, ..., e^n)$ its dual so that $e^i(e_j) = \delta^i_j$. For a, $b \in \mathbf{R}$, which will be determined later, define $(1,2)$-tensor field S as

$$S = \sum_{i=1}^n e_i \otimes (ae^{i-1} \otimes e^{i-1} - e^{i-1} \otimes e^i - e^i \otimes e^{i-1} + e^i \otimes e^i + be^{i+1} \otimes e^{i+1}),$$

where we put $e^0 = e^n$ and $e^{n+1} = e^1$. Note that e^{i-1}, e^i, e^{i+1} are mutually distinct as $n \geq 3$. Obviously, S is symmetric, and $\mathrm{tr}\, S = 0$. A calculation shows that

$$-S \cdot S = 2(b-1)\sum_{i=1}^n e^i \otimes e^i + (1-ab)\sum_{i=1}^n (e^{i-1} \otimes e^i + e^i \otimes e^{i-1}).$$

Therefore, putting $a = 2/3$ and $b = 3/2$, we have

$$-S \cdot S = \sum_{i=1}^{n} e^i \otimes e^i,$$

which is positive definite everywhere. Thus the assertion follows from Lemma 2.4. $\square$

From Stiefel's theorem that every compact orientable 3-manifold is parallelizable, we have the following.

COROLLARY 3.4. *If M is a compact orientable 3-manifold, there is a $\nabla \in \mathcal{A}(M)$ such that Ric^∇ is positive definite.*

REMARK 3.5. In the way as in the proof of Proposition 3.1, we get $\nabla \in \mathcal{A}(T^n)$, using S of the above proof. Then this connection is *not* projectively flat. Neither (1) nor (2) is satisfied. In addition, $\mathrm{div}_\nabla \hat{\mathrm{Ric}}^\nabla \neq 0$, but $\mathrm{div}_\nabla \mathrm{div}_\nabla \hat{\mathrm{Ric}}^\nabla = 0$ and $\det \mathrm{Ric}^\nabla$ is constant (cf. [1; Proposition 3.1]).

4   The case of dimension two

PROPOSITION 4.1. *Let M be the torus T^2 or Klein's bottle $\mathbf{R}\mathrm{P}^2 \# \mathbf{R}\mathrm{P}^2$. Then there is a $\nabla \in \mathcal{A}(M)$ with the following properties:*

(i) Ric^∇ *is negative definite.*

(ii) ∇ *is projectively flat.*

(iii) $\det \hat{\mathrm{Ric}}^\nabla$ *is constant, and $\mathrm{div}_\nabla \hat{\mathrm{Ric}}^\nabla = 0$.*

PROOF. Though we have already given a proof for $M = T^2$, we repeat it. Let x, y be the standard coordinates of $\mathbf{R}^2$, $d\mu = dxdy$ and $\bar\nabla \in \mathcal{A}(\mathbf{R}^2)$ the standard flat connection. Put

$$S := \partial_x \otimes (dx \otimes dx - dy \otimes dy) - \partial_y \otimes (dx \otimes dy + dy \otimes dx),$$

and we have $\mathrm{tr}\, S = 0$ and $-S \cdot S = -2(dx \otimes dx + dy \otimes dy)$. We note that this S is $4/3$ times the S in the proof of Proposition 3.1. Thus from (3) we have for $\nabla = \bar\nabla + S$

$$\mathrm{Ric}^\nabla = -2(dx \otimes dx + dy \otimes dy),$$

which is negative definite. It is easy to check (ii) and (iii).

Since ∇ is invariant under translations of $\mathbf{R}^2$ in the usual sense, it descends to $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. Also it is invariant under transformations: $\varphi \colon \mathbf{R}^2 \to \mathbf{R}^2; (x,y) \mapsto (x, y+1)$ and $\psi \colon \mathbf{R}^2 \to \mathbf{R}^2; (x,y) \mapsto (x+1, 1-y)$. Hence it induces a connection of $\mathbf{R}\mathrm{P}^2 \# \mathbf{R}\mathrm{P}^2 = \mathbf{R}^2/\langle \varphi, \psi \rangle$. $\square$

Now, suppose $\dim M = 2$, and let x, y be local coordinates of $U \subset M$. Define $(1,2)$-tensor fields A, B, C, D on U as follows:

(4)
$$\begin{cases} A = \partial_x \otimes (dx \otimes dx - dy \otimes dy) - \partial_y \otimes (dx \otimes dy + dy \otimes dx), \\ B = \partial_y \otimes (dy \otimes dy - dx \otimes dx) - \partial_x \otimes (dx \otimes dy + dy \otimes dx), \\ C = \partial_x \otimes dy \otimes dy, \\ D = \partial_y \otimes dx \otimes dx. \end{cases}$$

Any symmetric $(1,2)$-tensor field S on U with $\operatorname{tr} S = 0$ can be decomposed as

$$S = aA + bB + cC + dD,$$

where a, b, c, $d \in C^\infty(U)$. The matrix form of $-S \cdot S$ is given as

$$-S \cdot S = \begin{pmatrix} -2(a^2 + b^2 - bd) & ad + bc - cd \\ ad + bc - cd & -2(a^2 + b^2 - ac) \end{pmatrix}.$$

LEMMA 4.2. *If $-S \cdot S$ is negative definite at $p \in U$, then $a(p)^2 + b(p)^2 > 0$.*

The following lemma shows that the method of the proof of Theorem 3.3 does not work in dimension two.

LEMMA 4.3. *If $-S \cdot S$ is positive semi-definite at $p \in U$, then $S \cdot S = 0$ at p.*

PROOF. Transform the coordinates x and y by $SO(2)$, if necessary, and we may assume that $ad + bc - cd = 0$ at p. Then the proof is easy. $\square$

This lemma does not hold in dimension greater than two. A counterexample is the S in the proof of Theorem 3.3. A simpler counterexample is given by $S = \partial_y \otimes (dx \otimes dz + dz \otimes dx) - \partial_z \otimes (dx \otimes dy + dy \otimes dx)$, for which $-S \cdot S = 2\, dx \otimes dx$ is positive semi-definite.

QUESTION 4.4. Let M be a closed surface with $\chi(M) \leq 0$. Does there exist a $\nabla \in \mathcal{A}(M)$ such that $\operatorname{Ric}^\nabla$ is positive definite?

It follows from Proposition 4.1 that for a closed surface M with $\chi(M) \leq 0$ there is a $\nabla \in \mathcal{A}(M)$ such that $\operatorname{Ric}^\nabla$ is negative definite. Thus the remainning cases to be considered are those of the sphere and the real projective plane.

QUESTION 4.5. Let M be S^2 or $\mathbf{R}\mathrm{P}^2$. Does there exist a $\nabla \in \mathcal{A}(M)$ such that $\operatorname{Ric}^\nabla$ is negative definite?

The following result makes clear the problem involved.

PROPOSITION 4.6. *There is no symmetric $(1,2)$-tensor field S on the sphere S^2 with* $\operatorname{tr} S = 0$ *such that* $-S \cdot S$ *is negative definite everywhere.*

PROOF. To the contrary, we assume that there exists such an S on the sphere. We regard S^2 as $\mathbf{R}^2 \cup \{\infty\}$. Let x, y be the coordinates of $\mathbf{R}^2$, and

$$\tilde{x} = \frac{x}{x^2 + y^2}, \qquad \tilde{y} = \frac{y}{x^2 + y^2}$$

be the coordinates of $S^2 \setminus \{0\}$. Let A, B, C and D be the tensor fields on $\mathbf{R}^2$ defined in (4). By $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ we denote those tensor fields on $S^2 \setminus \{0\}$ obtained by replacing x, y with $\tilde{x}$, $\tilde{y}$ in (4). Functions a, b and so on are defined such that

$$S_{|S^2\setminus\{\infty\}} = aA + bB + cC + dD, \qquad S_{|S^2\setminus\{0\}} = \tilde{a}\tilde{A} + \tilde{b}\tilde{B} + \tilde{c}\tilde{C} + \tilde{d}\tilde{D}.$$

From Lemma 4.2, we have $a^2 + b^2 > 0$ and $\tilde{a}^2 + \tilde{b}^2 > 0$. Since being negative definite is an open condition, we may assume that there is $\rho_0 \in \mathbf{R}_+$ such that a, b, c and d are constant where $x^2 + y^2 \le e^{-2\rho_0}$, and that $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are constant where $\tilde{x}^2 + \tilde{y}^2 \le e^{-2\rho_0}$.

Define coordinates ρ, θ of $\mathbf{R} \times \mathbf{R}/2\pi\mathbf{Z} \cong S^2 \setminus \{0, \infty\}$ as

$$x = e^\rho \cos\theta, \qquad y = e^\rho \sin\theta.$$

By $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$, we denote those tensor fields obtained by replacing x, y with ρ, θ in (4). $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are functions such that $S_{|S^2\setminus\{0,\infty\}} = \hat{a}\hat{A} + \hat{b}\hat{B} + \hat{c}\hat{C} + \hat{d}\hat{D}$.

For $\rho \in \mathbf{R}$, define a closed curve γ_ρ as

$$\gamma_\rho \colon S^1 \to \mathbf{R}^2; \ \theta \bmod 2\pi \mapsto (\hat{a}(\rho, \theta), \hat{b}(\rho, \theta)).$$

From Lemma 4.2, the closed curves γ_ρ never pass through $(0,0) \in \mathbf{R}^2$. We put

$$\hat{a}_t = e^\rho(a\cos 3\theta - b\sin 3\theta + \frac{t}{2}\sin 2\theta(c\sin\theta + d\cos\theta)),$$

$$\hat{b}_t = e^\rho(a\sin 3\theta + b\cos 3\theta + \frac{t}{2}\sin 2\theta(-c\cos\theta + d\sin\theta)).$$

Then by calculations, we have $(\hat{a}, \hat{b}) = (\hat{a}_1, \hat{b}_1)$. Hence, by putting

$$\gamma_{\rho,t} \colon S^1 \to \mathbf{R}^2; \ \theta \bmod 2\pi \mapsto (\hat{a}_t(\rho, \theta), \hat{b}_t(\rho, \theta)),$$

we have $\gamma_\rho = \gamma_{\rho,1}$. Moreover, since a, b, c, d are constant where $\rho \le -\rho_0$, we get

$$\gamma_{\rho,t}(\theta + \frac{\pi}{2}) = J(\gamma_{\rho,t}(\theta)) \quad \text{for } \rho \le -\rho_0,$$

where $J \colon \mathbf{R}^2 \to \mathbf{R}^2; \ (\xi, \eta) \mapsto (-\eta, \xi)$. Thus if $\rho \le -\rho_0$, the closed curves $\gamma_{\rho,t}$ have a cyclic symmetry of order 4 around $(0,0) \in \mathbf{R}^2$. Therefore,

$$w(\gamma_0, o) = w(\gamma_{-\rho_0}, o) = w(\gamma_{-\rho_0,1}, o) \equiv w(\gamma_{-\rho_0,0}, o) = 3 \pmod 4,$$

where, $w(\cdot, o)$ stands for the winding number around $o = (0, 0) \in \mathbf{R}^2$.

On the other hand, direct calculations show that

$$\hat{a} = -e^{-\rho}(\tilde{a}\cos 3\theta - \tilde{b}\sin 3\theta + \frac{1}{2}\sin 2\theta(\tilde{c}\sin\theta + \tilde{d}\cos\theta)),$$

$$\hat{b} = e^{-\rho}(\tilde{a}\sin 3\theta + \tilde{b}\cos 3\theta + \frac{1}{2}\sin 2\theta(-\tilde{c}\cos\theta + \tilde{d}\sin\theta)).$$

Hence a similar argument leads to the congruence

$$w(\gamma_0, o) \equiv -3 \pmod 4,$$

which is a contradiction. $\square$

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