What Tomoko *Does* Know: A *Normal* and *Simple* Solution to the Problem of Logical Omniscience

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Introduction

Epistemic logic is the logic of knowing. Statements such as,

(1) Tomoko knows if she was born in Nara, then she was born in Japan.

(2) Tomoko knows she was born in Nara,

and their relations are rendered accessible to logical analysis through the careful construction of such logic. Thus an epistemic logic allows us to infer the following from 1 and 2:

(3) Tomoko knows she was born in Japan.

A classical logic would not allow us to make such an inference.[1] Epistemic logic allows us to make such an inference by appropriating the constructive tools of modal logic and interpreting the modal operator as the knowledge operator K. However, so far as it does, it creates a serious problem for itself, for if it follows the standard modal form, then we can infer,

(4) Tomoko knows $\pi = 355/113$

from,

[1] Knowledge creates the need for an intensional analysis of meaning, which classical logic cannot deal with.

(5) Tomoko knows $\pi = 3.1415$

This is problematic because she does assent to (5), but not (4). In this paper, I will appropriate the tools of modal logic to construct a logic of knowledge and show how the logic constructed avoids the aforementioned problem. The key to avoiding the problem is to interpret contexts as inferential contexts or states indexed to a single knower and driven by inferential processes, rather than worlds, and interpret the modal operator as "It is known that...", where the operator is not indexed to any knower but is bound to the set of inferential contexts delimited.[2]

K and NEML

We show how a normal epistemic modal logic (NEML) is constructed. To a formal proposition logic (PL) the following is added: a modal operator, $\Box$ (pronounced "box")[3], a syntactic rule governing its syntactic relations: "if A is a sentence, so too is $\Box$A."[4] Further, a set of axioms and rules:

(Dist) If $\Box(A \rightarrow B)$ is a theorem, then $\Box A \rightarrow \Box B$[5]

(Nec) If A is a theorem, then $\Box$A.[6]

And the rule of modus ponens.

This system is called *K*, after Saul Kripke, and is the basis for different systems of normal modal logic.[7]

Epistemic logic, like all normal modal logics, is an extension of *K*.[8] In such a logic, the modal operator is replaced by a knowledge operator, K.[9] The operator is usually tagged to a single knower using subscripts, so, for example, K_t abbreviates, "Tomoko knows that…" Validity in epistemic logic is defined by providing a model <W, R, V>, which is a frame <W, R> and an evaluation V. Then, W is a non-empty set of possible worlds (w, v, r…) ; R is an accessibility relation that connects possible worlds compatible with w; and V, a function that evaluates A at w (denoted V(A, w) = 1 or 0) for each complex formula of NEML:

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2 It is also possible to tag a community of knowers or rational agents.

3 Box should not be confused with the modal operator interpreted for an alethic logic.

4 "A" is understood to be a meta-variable.

5 This is an axiom of the system in question.

6 This is a derivable rule of the system in question.

7 See, for example, Chellas, B., F. (1980), *Modal Logic: An Introduction*, Cambridge: Cambridge University Press; Cresswell, M. J., and Hughes, G. E., (1996), A New Introduction to Modal Logic, Routledge; Meyer, J. J., "Epistemic Logic" in Goble, L. (Ed.) (2001) *The Blackwell Guide to Philosophical Logic*, Oxford: Blackwell.

8 For example, ibid., Meyer (2001)

9 The knowledge operator, K, should not be confused with *K*, which stands for the system *K*.

(~) V(~A, w) = 1 iff V(A, w) = 0

(→) V(A → B) = 1 iff V(A, w) = 0 or V(B, w) = 1

(K_i) V(KA, w) = 1 iff for each v such that wRv, V(A, v) = 1[10]

An argument is valid just in case any model whose valuation assigns the premises 1 at a world also assigns the conclusion 1 at the same world. A sentence is valid where an argument with no premises is valid. All such sentences are theorems. All theorems are valid. NEML based on *K* is valid.[11] This defines a normal epistemic modal logic (NEML). Below we present a slightly modified interpretation, but a normal logic nonetheless.

tNEML

tNEML is exactly the same as NEML except that a new model is interpreted for the logic. For tNEML we interpret the model <T, R, V>, which is a frame <T, R> and a evaluation V. A model is T which is a non-empty set of inferential contexts (t, s, r,…), which all belong to a single knower, in this case Tomoko; R is an accessibility relation; and V, which is a function, evaluates A at t (denoted V(A, t) = 1 or 0) for each complex formula in the language as in NEML and validity is defined in the same manner, except for K. The operator is not tagged to any single knower, and is evaluated in the following manner:

(K) V(KA, t) = 1 iff for each s such that tRs, V(A, s) = 1

This construction is used in order to do away with the problem of logical omniscience. The problem for NEML is stated below, the solution situated on tNEML, is provided thereafter.

The Problem for NEML

Logical omniscience can be stated for NEML, but it cannot be stated for tNEML. It is stated for NEML:

$_w[$ A

:

[10] It is important to understand the import of this evaluation for epistemic logic. The evaluation *splits* the set of possible worlds into two, those that are compatible with a knowers actual knowledge state and those that are not. See Hendriks, V. F., (2004), *Forcing Epistemology*, New York: Cambridge University Press; Hintikka, J., (1962) *Knowledge and Belief*, Cornell: Cornell University; Op cit., Meyer (2001) puts it like this: “the epistemic alternatives for the agent are given by the set $\{t \in S \mid R(s, t)\}$, i.e. all possible worlds *t* that are accessible from *s* by means of the relation *R*.” (p.185)

[11] Ibid., Meyer (2001)

:

:

———

B

———

$A \to B$ (CP)[12]

———

$K_t(A \to B)$ (Nec)

———

$K_tA \to K_tB]_w$ (Dist)[13]

K_t abbreviates "Tomoko knows that…" The conclusion, then, states that if Tomoko knows that A, she knows that B——that is, she knows the consequents of every antecedent she accepts. This can be stated in the following manner:

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12 CP abbreviates conditional proof. A conditional proof says that if B can be inferred directly from A, then $A \to B$. The CP rule is valid based on the acceptance of the "deduction theorem".

13 Worlds are represented by square brackets, [...] Worlds are indexed by subscripts attached to the world in question, w, v, u... A proof occurs within a world, represented thus:

$_w[A$

:

:

:

———

$B]_w$

A world accessible from another world is represented simply by the accessible world sitting below the world that sees it.

$_w[...]$
$_v[...]$

An inference is denoted by —— and a rule is placed on the same line as the inferred conclusion. Assumptions aren't inferred and labeled clearly by "Assume".

$_w[A$ (Assume)
$A \to B$ (Assume)

———

B (MPP)

———

$A \to B]_w$ (CP)

We also assume the CP rule.

These explanations suffice for this paper.

$_w[K_tA$ (Assume)
$K_t(A \rightarrow B)$ (Reiteration)

$K_tB]_w$ (Dist. and MP)

Let us put some flesh on these bones. Let A be replaced by the proposition that *Water is tasteless*, and B be replaced by the proposition *H_2O is tasteless*. Then:

$_w$[Water is tasteless
:
:
:

H_2O is tasteless

Water is tasteless $\rightarrow$ H_2O is tasteless (CP)

K_t(Water is tasteless $\rightarrow$ H_2O is tasteless) (Nec)

K_t(Water is tasteless) $\rightarrow$ K_t(H_2O is tasteless)$]_w$ (Dist)

The conclusion, whilst being presumably true, should not be true as a matter of logic. The argument is just as, if not more, persuasive when we replace A and B with mathematical truths. For example, let A be replaced by the proposition, $\pi = 3.1415$, and B the proposition, $\pi = 355/113$. In this case, while Tomoko readily assents to knowing that $\pi = 3.1415$, she says she is not certain that $\pi = 355/113$. So the following is off the mark:

$_w[\pi = 3.1415$
:
:
:

$\pi = 355/113)$

$(\pi = 3.1415 \rightarrow \pi = 355/113)$	(CP)
$K_t(\pi = 3.1415 \rightarrow \pi = 355/113)$	(Nec)
$K_t(\pi = 3.1415 \rightarrow K_t(\pi = 355/113)$	(Dist)
$K_t(\pi = 3.1415)$	(Assume)
$K_t(\pi = 355/113)]_w$	(MP)

While the latter *assumption* is true, so far as Tomoko is concerned, the conclusion that follows is false, because she is just not certain that $\pi = 103993/33102$. This is the problem of logical omniscience laid bare for both a *posteriori* and *a priori* truths.

The Solution, tNEML

Logical omniscience does not infect tNEML because in that logic the inferential process is tagged to a rational agent who is *able* to *infer* the conditional, *only whereby* the knowledge operator is introduced and *thereof* knowledge ascribed to the rational agent in question. Thus:

$_t[A$
:
:
:

B

$A \rightarrow B$	(CP)
$K(A \rightarrow B)$	(Nec)
$KA \rightarrow KB]_t$	(Dist)[14]

[14] Above we talked about worlds. Here we talk about inferential contexts or states. These are represented in the same way as worlds. For example, inferential states are represented by square brackets, [...]. Inferential states are indexed by subscripts attached to the context in question, for example, t, s, r... all index states of one knower in tNEM. See previous footnote for more details.

Here each inferential step from premise to conclusion is placed within an inferential context that is indexed to a rational agent, Tomoko, who is the knower in question. On this basis, there is no gratuity in attributing knowledge of the kind attributed to her. *All the inferences are attributed to her, so, too, then the knowledge*. To make this more concrete consider the following set of inferences:

$_t$[Water is tasteless
:
:
:

H_2O is tasteless

Water is tasteless → H_2O is tasteless (CP)

K(Water is tasteless → H_2O is tasteless) (Nec)

K(Water is tasteless) → K(H_2O is tasteless)]$_t$ (Dist)

Here each concrete inferential step from the premise to the conclusion is placed within the inferential context that is indexed to our rational heroine, Tomoko, who is the knower in question. *Every one of the inferences is attributed to her, so, too, then the knowledge*. Given this happy state of affairs, the following is clearly of no issue:

$_t$[K(Water is tasteless) (Assume)
K(Water is tasteless → H_2O is tasteless) (Reiteration)

K(H_2O is tasteless)]$_t$ (Dist. MP)

Further, if we let A be replaced by the proposition, π =3.1417, and B the proposition, π =103993/33102, then we can represent the lack of a logical relation, for Tomoko, between the two propositions in the following way:

$_t$[π = 3.1415
X
X
X

$\pi = 355/113)]_t$[15]

Here there are no inferential steps from the premise to the conclusion. This lack of *process* is placed within the inferential context we are tagging to our rational agent, Tomoko, since she is the knower as defined by the interpretation of tNEML. Sadly, in this case no inferences are attributed to Tomoko, so, too, then no (perfect) knowledge. But this is not to say Tomoko is not rational, this is to say, simply enough, that Tomoko lacks knowledge, as we all do, for none of us are *actually* omniscient knowers!

Conclusion

In this paper we looked at a problem for normal epistemic modal logic. We interpreted a basic modal model based on the system *K* in order to construct a normal epistemic modal logic, which we called NEML. We found this logic suffered the burden of logical omniscience. We reinterpreted the basic modal model based on the system *K* in order to reconstruct a normal epistemic modal logic, which we called tNEML, and showed how the problem for NEML could be avoided by way of this simple reinterpretation. Our reinterpretation involved tagging not only the knower but the process to a rational agent. (NEML does not do this.) We found the problem dissipated under our reinterpretation. There are other solutions to the aforementioned problem. They, like our solution, must be accepted or rejected on their own merits and as needs be. However, we suggest our solution is normal, the simplest and the closest to the actual world. We further suggest there is ample room to take our suggestive logic forward. We think the reinterpretation of epistemic logic offered here may help in advancing a number of philosophical positions, for example, two-dimensional semantics to mention one.

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15 "X" in the diagram represents the lack of any inferential relations between the premise and the conclusion.

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Abstract: Epistemic logic is the logic of knowledge. A normal epistemic logic is based on system *K*, which is taken as the foundation of a normal modal logic. Based on K, epistemic logic is normal. But formulated in this way it is prone to the problem of logical omniscience. This ascribes perfect knowledge to agents tagged to the knowledge operator used in the system. While many authors have presented ways to avoid this problem in a non-normal way, there is no standard way out of this conundrum. The author suggests a simple way of avoiding this problem in a normal way.

「キーワード」
Normal, Epistemic, Logic, Omniscience, Knowledge