Title	ALMOST COMPLEX STRUCTURE, BLOWDOWNS AND MCKAY CORRESPONDENCE IN QUASITORIC ORBIFOLDS
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Citation	Osaka Journal of Mathematics. 2013, 50(4), p. 977-1005
Version Type	VoR
URL	https://doi.org/10.18910/26818
rights	
Note	

Ganguli, S. and Poddar, M.
Osaka J. Math.
50 (2013), 977–1005

ALMOST COMPLEX STRUCTURE, BLOWDOWNS AND McKAY CORRESPONDENCE IN QUASITORIC ORBIFOLDS

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(Received March 12, 2012)

Abstract

We prove the existence of invariant almost complex structure on any positively omnioriented quasitoric orbifold. We construct blowdowns. We define Chen–Ruan cohomology ring for any omnioriented quasitoric orbifold. We prove that the Euler characteristic of this cohomology is preserved by a crepant blowdown. We prove that the Betti numbers are also preserved if dimension is less or equal to six. In particular, our work reveals a new form of McKay correspondence for orbifold toric varieties that are not Gorenstein. We illustrate with an example.

1. Introduction

McKay correspondence [16] has been studied widely for complex algebraic varieties with only Gorenstein or *SL* orbifold singularities. A cohomological version of this correspondence says that the Hodge numbers (and Betti numbers) of Chen–Ruan cohomology (with compact support) [5] are preserved under crepant blowup. This was proved in [12] and [17] for complete algebraic varieties with *SL* quotient singularities following fundamental work of [3] and [8] in the local case. It makes sense to ask if such a correspondence holds for Betti numbers when the orbifold has almost complex structure only. However the main ingredients in the algebraic proof, namely motivic integration and Hodge structure, may no longer be available.

From a different perspective, the topological properties of quasitoric spaces introduced by Davis and Januskiewicz [6], have been studied extensively. However not much attention has been given to the study of equivariant maps between them. In this article, which is a sequel to [9], we construct equivariant blowdown maps between primitive omnioriented quasitoric orbifolds and prove certain McKay type correspondence for them. These spaces do not have complex or almost complex structure in general.

Quasitoric orbifolds [15] are topological generalizations of projective simplicial toric varieties or symplectic toric orbifolds [11]. They are even dimensional spaces with action of the compact torus of half dimension such that the orbit space has the structure of a simple polytope. We only work with primitive quasitoric orbifolds. The orbifold

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2000 Mathematics Subject Classification. Primary 55N32, 57R18; Secondary 53C15, 14M25.

singularities of these spaces correspond to analytic singularities. An omniorientation is a choice of orientation for the quasitoric orbifold as well as for each invariant suborbifold of codimension two. When these orientations are compatible the quasitoric orbifold is called positively omnioriented, see Section 2.9 for details. We prove the existence of invariant almost complex structure on positively omnioriented quasitoric orbifolds (Theorem 3.1) by adapting the technique of Kustarev [10] for quasitoric manifolds. We also build a stronger version of Kustarev's result: Theorem 3.2 and Corollary 3.3. These may be of use to even those who are mainly interested in quasitoric manifolds.

Chen–Ruan cohomology was originally defined for almost complex orbifolds in [5]. There the almost complex structure on normal bundles of singular strata is used to determine the grading of the cohomology. An omniorientation, together with the torus action, determines a complex structure on the normal bundle of every invariant suborbifold of a quasitoric orbifold. Moreover the singular locus is a subset of the union of invariant suborbifolds. Thus we can define Chen–Ruan cohomology groups for any omnioriented quasitoric orbifold, see Section 7. We also define a ring structure for this cohomology in Section 9 following the approach of [4]. The Chen–Ruan cohomology of the same quasitoric orbifold is in general different for different omniorientations. For a positively omnioriented quasitoric orbifold with the almost complex structure of Theorem 3.1, our definition of Chen–Ruan cohomology ring agrees with that of [5].

The blowdown maps are continuous, and they are diffeomorphism of orbifolds away from the exceptional set. They are not morphisms of orbifolds (see [1] for definition). In some cases they are analytic near the exceptional set, see Lemma 5.1. (In these cases they are pseudoholomorphic in a natural sense, see Definition 5.1.) For these we can compute the pull-back of the canonical sheaf and test if the blowdown is crepant in the sense of complex geometry: The pull back of the canonical sheaf of the blowdown is the canonical sheaf of the blowup. However the combinatorial condition this corresponds to, makes sense in general and may be applied to an arbitrary blowdown. We work with this generalized notion of crepant blowdown, see Section 6.

We prove the conservation of Betti numbers of Chen–Ruan cohomology under crepant blowdowns when the quasitoric orbifold has dimension less than or equal to six (Theorem 8.4). We also prove the conservation of Euler characteristic of this cohomology under crepant blowdowns in arbitrary dimension (Theorem 8.3). This implies that the rational orbifold K-groups [2] are also preserved, see Section 8.2. These statements hold under the condition that the omnioriented quasitoric orbifolds are quasi-SL, a generalization of SL; see Definition 8.1.

The validity of McKay correspondence for Betti numbers remains an interesting open problem in higher dimensions. One might try to make use of the local results from motivic integration, namely correspondence of Betti numbers of Chen–Ruan cohomology with compact support for crepant blowup of a Gorentstein quotient singularity $\mathbb{C}^n/G$ [3, 8]. However such efforts are impeded by the fact that the correspondence obtained from motivic integration is not natural. However, we prove a very basic inequality about the behavior of the second Betti number under crepant blowup in Lemma 8.5. We also give an example of McKay correspondence for Betti numbers when dimension is eight in Section 8.4. This example is particularly interesting as it corresponds to the weighted projective space $\mathbb{P}(1, 1, 3, 3, 3)$ which is not a Gorenstein or *SL* orbifold. Hence McKay correspondence as studied in complex algebraic geometry does not apply to it. However under suitable choice of omniorientation it is quasi-*SL* and McKay correspondence holds. Note that the blowup is not a toric blowup in the sense of algebraic geometry.

In [9], we constructed examples of four dimensional quasitoric orbifolds that are not toric varieties. We also constructed pseudoholomorphic blowdowns between them. Our brief study of pseudo-holomorphicity of blowdowns in Section 5 shows that every primitive positively omnioriented quasitoric orbifold of dimension four has a pseudo-holomorphic resolution of singularities, see Theorem 5.4. The result may hold in dimension six as well, but developing pseudoholomorphic blowdowns in dimension six and higher would need further work.

2. Quasitoric orbifolds

In this section we review the combinatorial construction of quasitoric orbifolds. We also construct an explicit orbifold atlas for them and list a few important properties. The notations established here will be important for the rest of the article.

2.1. Construction. Fix a copy N of $\mathbb{Z}^n$ and let $T_N := (N \otimes_{\mathbb{Z}} \mathbb{R})/N \cong \mathbb{R}^n/N$ be the corresponding n-dimensional torus. A primitive vector in N, modulo sign, corresponds to a circle subgroup. of T_N. More generally, suppose M is a submodule of N of rank m. Then

$$T_M := (M \otimes_{\mathbb{Z}} \mathbb{R})/M \tag{2.1}$$

is a torus of dimension m. Moreover there is a natural homomorphism of Lie groups $\xi_M \colon T_M \to T_N$ induced by the inclusion $M \hookrightarrow N$.

DEFINITION 2.1. Define T(M) to be the image of T_M under ξ_M. If M is generated by a vector $\lambda \in N$, denote T_M and $T(M)$ by T_λ and $T(\lambda)$ respectively.

Usually a polytope is defined to be the convex hull of a finite set of points in $\mathbb{R}^n$. To keep our notation manageable, we will take a more liberal interpretation of the term polytope.

DEFINITION 2.2. A polytope P will denote a subset of $\mathbb{R}^n$ which is diffeomorphic, as manifold with corners, to the convex hull Q of a finite number of points in $\mathbb{R}^n$. Faces of P are the images of the faces of Q under the diffeomorphism.

Let P be a simple polytope in $\mathbb{R}^n$, i.e. every vertex of P is the intersection of exactly n codimension one faces (facets). Consequently every k-dimensional face F of P is the intersection of a unique collection of $n-k$ facets. Let $\mathcal{F} := \{F_1, \dots, F_m\}$ be the set of facets of P.

DEFINITION 2.3. A function $\Lambda\colon \mathcal{F} \to N$ is called a characteristic function for P if $\Lambda(F_{i_1}), \dots, \Lambda(F_{i_k})$ are linearly independent whenever $F_{i_1}, \dots, F_{i_k}$ intersect at a face in P. We write λ_i for $\Lambda(F_i)$ and call it a characteristic vector.

REMARK 2.1. In this article we assume that all characteristic vectors are primitive. Corresponding quasitoric orbifolds have been termed primitive quasitoric orbifold in [15]. They are characterized by the codimension of singular locus being greater than or equal to four.

DEFINITION 2.4. For any face F of P, let $\mathcal{I}(F) = \{i : F \subset F_i\}$. Let Λ be a characteristic function for P. Let $N(F)$ be the submodule of N generated by $\{\lambda_i : i \in \mathcal{I}(F)\}$. Note that $\mathcal{I}(P)$ is empty and $N(P) = \{0\}$.

For any point $p \in P$, denote by $F(p)$ the face of P whose relative interior contains p. Define an equivalence relation $\sim$ on the space $P \times T_N$ by

$$(p,t) \sim (q,s) \quad \text{if and only if} \quad p = q \quad \text{and} \quad s^{-1}t \in T(N(F(p))). \tag{2.2}$$

Then the quotient space $X := P \times T_N/\sim$ can be given the structure of a $2n$-dimensional quasitoric orbifold. Moreover any $2n$-dimensional primitive quasitoric orbifold may be obtained in this way, see [15]. We refer to the pair (P, Λ) as a model for the quasitoric orbifold. The space X inherits an action of T_N with orbit space P from the natural action on $P \times T_N$. Let $\pi : X \to P$ be the associated quotient map.

The space X is a manifold if the characteristic vectors $\lambda_{i_1}, \dots, \lambda_{i_k}$ generate a unimodular subspace of N whenever the facets $F_{i_1}, \dots, F_{i_k}$ intersect. The points $\pi^{-1}(v) \in X$, where v is any vertex of P, are fixed by the action of T_N. For simplicity we will denote the point $\pi^{-1}(v)$ by v when there is no confusion.

2.2. Orbifold charts. Consider open neighborhoods $U_v \subset P$ of the vertices v such that U_v is the complement in P of all edges that do not contain v. Let

$$X_v := \pi^{-1}(U_v) = U_v \times T_N/\sim. \tag{2.3}$$

For a face F of P containing v there is a natural inclusion of $N(F)$ in $N(v)$. It induces an injective homomorphism $T_{N(F)} \to T_{N(v)}$ since a basis of $N(F)$ extends to a basis of $N(v)$. We will regard $T_{N(F)}$ as a subgroup of $T_{N(v)}$ without confusion. Define an equivalence relation $\sim_v$ on $U_v \times T_{N(v)}$ by $(p,t) \sim_v (q,s)$ if $p = q$ and $s^{-1}t \in T_{N(F)}$

where F is the face whose relative interior contains p. Then the space

$$\tilde{X}_v := U_v \times T_{N(v)}/{\sim_v} \tag{2.4}$$

is θ-equivariantly diffeomorphic to an open set in $\mathbb{C}^n$, where $\theta\colon T_{N(v)} \to U(1)^n$ is an isomorphism, see [6]. This means that there exists a diffeomorphism $f\colon \tilde{X}_v \to B \subset \mathbb{C}^n$ such that $f(t \cdot x) = \theta(t) \cdot f(x)$ for all $x \in \tilde{X}_v$. This will be evident from the subsequent discussion.

The map $\xi_{N(v)}\colon T_{N(v)} \to T_N$ induces a map $\xi_v\colon \tilde{X}_v \to X_v$ defined by $\xi_v([(p,t)]^{\sim_v}) = [(p, \xi_{N(v)}(t))]^{\sim}$ on equivalence classes. The kernel of $\xi_{N(v)}$, $G_v = N/N(v)$, is a finite subgroup of $T_{N(v)}$ and therefore has a natural smooth, free action on $T_{N(v)}$ induced by the group operation. This induces smooth action of G_v on $\tilde{X}_v$. This action is not free in general. Since $T_N \cong T_{N(v)}/G_v$, X_v is homeomorphic to the quotient space $\tilde{X}_v/G_v$. An orbifold chart (or uniformizing system) on X_v is given by $(\tilde{X}_v, G_v, \xi_v)$.

Let $(p_1, \dots, p_n)$ denote the standard coordinates on $\mathbb{R}^n \supset P$. Let $(q_1, \dots, q_n)$ be the coordinates on $N \otimes \mathbb{R}$ that correspond to the standard basis of N. Let $\{u_1, \dots, u_n\}$ be the standard basis of N. Suppose the characteristic vectors u_i are assigned to the facets $p_i = 0$ of the cone $\mathbb{R}^n_{\geq}$. In this case there is a homeomorphism $\phi\colon (\mathbb{R}^n_{\geq} \times T_N/{\sim}) \to \mathbb{R}^{2n}$ given by

$$x_i = \sqrt{p_i}\cos(2\pi q_i), \quad y_i = \sqrt{p_i}\sin(2\pi q_i) \quad \text{where} \quad i = 1, \dots, n. \tag{2.5}$$

REMARK 2.2. The square root over p_i is necessary to ensure that the orbit map $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n_{\geq}$ is smooth.

We define a homeomorphism $\phi_v\colon \tilde{X}_v \to \mathbb{R}^{2n}$ as follows. Assume without loss of generality that $F_1, \dots, F_n$ are the facets of U_v. Let the equation of F_i be $p_{i,v} = 0$. Assume that $p_{i,v} > 0$ in the interior of U_v for every i. Let Λ_v be the corresponding matrix of characteristic vectors

$$\Lambda_v = [\lambda_1 \dots \lambda_n]. \tag{2.6}$$

If $\mathbf{q}_v = (q_{1,v}, \dots, q_{n,v})^t$ are angular coordinates of an element of T_N with respect to the basis $\{\lambda_1, \dots, \lambda_n\}$ of $N \otimes \mathbb{R}$, then the standard coordinates $\mathbf{q} = (q_1, \dots, q_n)^t$ may be expressed as

$$\mathbf{q} = \Lambda_v \mathbf{q}_v. \tag{2.7}$$

Then define the homeomorphism $\phi_v\colon \tilde{X}_v \to \mathbb{R}^{2n}$ by

$$x_i = x_{i,v} := \sqrt{p_{i,v}}\cos(2\pi q_{i,v}), \quad y_i = y_{i,v} := \sqrt{p_{i,v}}\sin(2\pi q_{i,v}) \quad \text{for} \quad i = 1, \dots, n. \tag{2.8}$$

We write

$$z_i = x_i + \sqrt{-1}y_i \quad \text{and} \quad z_{i,v} = x_{i,v} + \sqrt{-1}y_{i,v}. \tag{2.9}$$

Now consider the action of $G_v = N/N(v)$ on $\tilde{X}_v$. An element g of G_v is represented by a vector $\sum_{i=1}^{n} a_i\lambda_i$ in N where each $a_i \in \mathbb{Q}$. The action of g transforms the coordinates $q_{i,v}$ to $q_{i,v} + a_i$. Therefore

$$g \cdot (z_{1,v}, \dots, z_{n,v}) = (e^{2\pi\sqrt{-1}a_1}z_{1,v}, \dots, e^{2\pi\sqrt{-1}a_n}z_{n,v}). \tag{2.10}$$

We may identify G_v with the cokernel of the linear map $\Lambda_v \colon N \to N$. Then standard arguments using the Smith normal form of the matrix Λ_v imply that

$$o(G_v) = |\det \Lambda_v|. \tag{2.11}$$

2.3. Compatibility of charts. We show the compatibility of the charts $(\tilde{X}_v, G_v, \xi_v)$. Let v_1 and v_2 be two vertices so that the minimal face S of P containing both has dimension $s \geq 1$. Then $X_{v_1} \cap X_{v_2}$ is nonempty. Assume facets $(F_1, \dots, F_s, F_{s+1}, \dots, F_n)$ meet at vertex v_1 and facets $(F_{n+1}, \dots, F_{n+s}, F_{s+1}, \dots, F_n)$ meet at v_2. We take

$$\begin{aligned} \Lambda_{v_1} &= [\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_n] \quad \text{and} \\ \Lambda_{v_2} &= [\lambda_{n+1}, \dots, \lambda_{n+s}, \lambda_{s+1}, \dots, \lambda_n]. \end{aligned} \tag{2.12}$$

Then

$$\mathbf{q}_{v_2} = \Lambda_{v_2}^{-1}\Lambda_{v_1}\mathbf{q}_{v_1}. \tag{2.13}$$

Suppose

$$\lambda_k = \sum_{j=s+1}^{n+s} c_{j,k}\lambda_j, \quad 1 \leq k \leq s. \tag{2.14}$$

Then by (2.13),

$$\begin{aligned} q_{j,v_2} &= \sum_{k=1}^{s} c_{n+j,k}q_{k,v_1} \quad \text{if} \quad 1 \leq j \leq s, \\ q_{j,v_2} &= \sum_{k=1}^{s} c_{j,k}q_{k,v_1} + q_{j,v_1} \quad \text{if} \quad s+1 \leq j \leq n. \end{aligned} \tag{2.15}$$

Let the facets F_j, $j = 1, \dots, n+s$, be defined by $\hat{p}_j = 0$ such that $\hat{p}_j > 0$ in the interior of the polytope P. Then the coordinates (2.8) on $\tilde{X}_{v_2}$ and $\tilde{X}_{v_1}$ are related as follows.

$$
\begin{aligned}
z_{j,v_2} &= \prod_{k=1}^{s} z_{k,v_1}^{c_{n+j,k}} \sqrt{\hat{p}_{n+j} \prod_{k=1}^{s} \hat{p}_k^{-c_{n+j,k}}} \quad \text{if} \quad 1 \le j \le s, \\
z_{j,v_2} &= z_{j,v_1} \prod_{k=1}^{s} z_{k,v_1}^{c_{j,k}} \sqrt{\prod_{k=1}^{s} \hat{p}_k^{-c_{j,k}}} \quad \text{if} \quad s+1 \le j \le n.
\end{aligned}
\tag{2.16}
$$

Take any point $x \in X_{v_1} \cap X_{v_2}$. Let $\tilde{x}$ be a preimage of x with respect to ξ_{v_1}. Suppose $\pi(x)$ belongs to the relative interior of the face $F \subset S$. Suppose F is the intersection of facets $F_{i_1}, \dots, F_{i_t}$ where $s+1 \le i_1 < \cdots < i_t \le n$. Then the coordinate $z_{j,v_1}(\tilde{x})$ is zero if and only if $j \in \mathcal{I}(F) = \{i_1, \dots, i_t\}$. Consider the isotropy subgroup G_x of $\tilde{x}$ in G_{v_1}. It consists of all elements that do not affect the nonzero coordinates of $\tilde{x}$,

$$
G_x = \{g \in G_{v_1} \colon g \cdot z_{j,v_1} = z_{j,v_1} \text{ if } j \notin \mathcal{I}(F)\}. \tag{2.17}
$$

It is clear that G_x is independent of the choice of $\tilde{x}$ and

$$
G_x = \left\{ [\eta] \in N/N(v_1) \colon \eta = \sum_{j \in \mathcal{I}(F)} a_j \lambda_j \right\}. \tag{2.18}
$$

Note that $j \in \mathcal{I}(F)$ if and only if $\lambda_j \in N(F)$. It follows from the linear independence of $\lambda_1, \dots, \lambda_n$ that

$$
G_x \cong G_F := ((N(F) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N)/N(F). \tag{2.19}
$$

Note that G_P is the trivial group.

Choose a small ball $B(\tilde{x}, r)$ around $\tilde{x}$ such that $(g \cdot B(\tilde{x}, r)) \cap B(\tilde{x}, r)$ is empty for all $g \in G_{v_1} - G_x$. Then $B(\tilde{x}, r)$ is stable under the action of G_x and $(B(\tilde{x}, r), G_x, \xi_{v_1})$ is an orbifold chart around x induced by $(\tilde{X}_{v_1}, G_{v_1}, \xi_{v_1})$. We show that for sufficiently small value of r, this chart embeds into $(\tilde{X}_{v_2}, G_{v_2}, \xi_{v_2})$ as well.

Note that the rational numbers $c_{j,k}$ in (2.14) are integer multiples of $1/\Delta$ where $\Delta = \det(\Lambda_{v_2})$. Choose a branch of $z_{k,v_1}^{1/\Delta}$ for each $1 \le k \le s$, so that the branch cut does not intersect $B(\tilde{x}, r)$. Assume r to be small enough so that the functions $z_{k,v_1}^{c_{j,k}}$ are one-to-one on $B(\tilde{x}, r)$ for each $s+1 \le j \le n+s$ and $1 \le k \le s$. Then equation (2.16) defines a smooth embedding ψ of $B(\tilde{x}, r)$ into $\tilde{X}_{v_2}$. Note that $\hat{p}_k$, $1 \le k \le s$, and $\hat{p}_{n+j}$, $1 \le j \le s$ are smooth non-vanishing functions on $\xi_{v_1}^{-1}(X_{v_1} \cap X_{v_2})$. Let $i_{v_2} \colon G_x \to G_{v_2}$ be the natural inclusion obtained using equation (2.19). Then $(\psi, i_{v_2}) \colon (B(\tilde{x}, r), G_x, \xi_{v_1}) \to (\tilde{X}_{v_2}, G_{v_2}, \xi_{v_2})$ is an embedding of orbifold charts.

We denote the space X with the above orbifold structure by **X**. In general we will use a boldface letter to denote an orbifold and the same letter in normal font to denote the underlying topological space.

2.4. Independence of shape of polytope.

Lemma 2.3. *Suppose* **X** *and* **Y** *are quasitoric orbifolds whose orbit spaces* P *and* Q *are diffeomorphic and the characteristic vector of any edge of* P *matches with the characteristic vector of the corresponding edge of* Q. *Then* **X** *and* **Y** *are equivariantly diffeomorphic.*

Proof. Pick any vertex v of P. For simplicity we will write p_i for $p_{i,v}$, and q_i for $q_{i,v}$. Suppose the diffeomorphism $f\colon P_1 \to P_2$ is given near v by $f(p_1, p_2, \dots, p_n) = (f_1, f_2, \dots, f_n)$. It induces a map of local charts $\tilde{X}_v \to \tilde{Y}_{f(v)}$ by

$$(\sqrt{p_i}\cos(2\pi q_i), \sqrt{p_i}\sin(2\pi q_i)) \mapsto (\sqrt{f_i}\cos(2\pi q_i), \sqrt{f_i}\sin(2\pi q_i)) \quad \text{for} \quad i = 1, \dots, n. \tag{2.20}$$

This is a smooth map if the functions $\sqrt{f_i/p_i}$ are smooth functions of $p_1, \dots, p_n$. Without loss of generality let us consider the case of $\sqrt{f_1/p_1}$. We may write

$$f_1(p_1, p_2, \dots p_n) = f_1(0, p_2, \dots p_n) + p_1\frac{\partial f_1}{\partial p_1}(0, p_2, \dots p_n) + p_1^2 g(p_1, p_2, \dots p_n) \tag{2.21}$$

where g is smooth, see Section 8.14 of [7]. Note that $f_1(0, p_2, \dots, p_n) = 0$ as f maps the facet $p_1 = 0$ to the facet $f_1 = 0$. Then it follows from equation (2.21) that f_1/p_1 is smooth. We have

$$f_1/p_1 = \frac{\partial f_1}{\partial p_1}(0, p_2, \dots, p_n) + p_1 g(p_1, p_2, \dots, p_n). \tag{2.22}$$

Note that f_1/p_1 is nonvanishing away from $p_1 = 0$. Moreover we have

$$\frac{f_1}{p_1} = \frac{\partial f_1}{\partial p_1}(0, p_2, \dots, p_n) \quad \text{when} \quad p_1 = 0. \tag{2.23}$$

Since $f_1(0, p_2, \dots, p_n)$ is identically zero, $(\partial f_1/\partial p_j)(0, p_2, \dots, p_n) = 0$ for each $2 \leq j \leq n$. As the Jacobian of f is nonsingular we must have

$$\frac{\partial f_1}{\partial p_1}(0, p_2, \dots, p_n) \neq 0. \tag{2.24}$$

Thus f_1/p_1 is nonvanishing even when $p_1 = 0$. Consequently $\sqrt{f_1/p_1}$ is smooth. Therefore the map (2.20) is smooth and induces an isomorphism of orbifold charts. □

2.5. Torus action. An action of a group H on an orbifold $\mathbf{Y}$ is an action of H on the underlying space Y with some extra conditions. In particular for every sufficiently small H-stable neighborhood U in Y with uniformizing system (W, G, π), the action should lift to an action of H on W that commutes with the action of G. The T_N-action on the underlying topological space of a quasitoric orbifold does not lift to an action on the orbifold in general.

2.6. Metric. By a torus invariant metric on $\mathbf{X}$ we will mean a metric on $\mathbf{X}$ which is $T_{N(F)}$-invariant in some uniformizing neighborhood of x for any point $x \in \pi^{-1}(F^\circ)$.

Any cover of X by T_N-stable open sets induces an open cover of P. Choose a smooth partition of unity on the polytope P subordinate to this induced cover. Composing with the projection map $\pi\colon X \to P$ we obtain a partition of unity on X subordinate to the given cover, which is T_N-invariant. Such a partition of unity is smooth as the map π is smooth, being locally given by maps $p_j = x_j^2 + y_j^2$. For instance, choose a $T_{N(v)}$-invariant metric on each $\tilde{X}_v$. Then using a partition of unity as above we can define an invariant metric on $\mathbf{X}$.

2.7. Invariant suborbifolds. The T_N-invariant subset $X(F) = \pi^{-1}(F)$, where F is a face of P, has a natural structure of a quasitoric orbifold [15]. This structure is obtained by taking F as the polytope for $\mathbf{X}(F)$ and projecting the characteristic vectors to $N/N^*(F)$ where $N^*(F) = (N(F) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N$. With this structure $\mathbf{X}(F)$ is a suborbifold of $\mathbf{X}$. It is called a characteristic suborbifold if F is a facet. Suppose λ is the characteristic vector attached to the facet F. Then $\pi^{-1}(F)$ is fixed by the circle subgroup $T(\lambda)$ of T_N. We denote the relative interior of a face F by F° and the corresponding invariant space $\pi^{-1}(F^\circ)$ by $X(F^\circ)$. Note that $v^\circ = v$ if v is a vertex.

2.8. Orientation. Note that for any vertex v, $dp_{i,v} \wedge dq_{i,v} = dx_{i,v} \wedge dy_{i,v}$. Therefore $\omega_v := dp_{1,v} \wedge \cdots \wedge dp_{n,v} \wedge dq_{1,v} \wedge \cdots \wedge dq_{n,v}$ equals $dx_{1,v} \wedge \cdots \wedge dx_{n,v} \wedge dy_{1,v} \wedge \cdots \wedge dy_{n,v}$. The standard coordinates $(p_1, \ldots, p_n)$ are related to $(p_{1,v}, \ldots, p_{n,v})$ by a diffeomorphism. The same holds for $\mathbf{q}$ and $\mathbf{q}_v$. Therefore $\omega := dp_1 \wedge \cdots \wedge dp_n \wedge dq_1 \wedge \cdots \wedge dq_n$ is a nonzero multiple of each ω_v. The action of G_v on $\tilde{X}_v$, see equation (2.10), preserves ω_v for each vertex v as $dx_{i,v} \wedge dy_{i,v} = (\sqrt{-1}/2) dz_{i,v} \wedge d\bar{z}_{i,v}$. The action of G_v affects only the angular coordinates. Since $dq_1 \wedge \cdots \wedge dq_n = \det(\Lambda_v)\, dq_{1,v} \wedge \cdots \wedge dq_{n,v}$ and the right hand side is G_v-invariant, we conclude that ω is G_v-invariant. Therefore ω defines a nonvanishing $2n$-form on $\mathbf{X}$. Consequently a choice of orientations for $P \subset \mathbb{R}^n$ and T_N induces an orientation for $\mathbf{X}$.

2.9. Omniorientation. An omniorientation is a choice of orientation for the orbifold as well as an orientation for each characteristic suborbifold. For any vertex v, there is a representation of G_v on the tangent space $\mathcal{T}_0 \tilde{X}_v$. This representation splits into the direct sum of n representations corresponding to the normal spaces of $z_{i,v} = 0$. Thus we have a decomposition of the orbifold tangent space $\mathcal{T}_v \mathbf{X}$ as a direct sum of the normal spaces of the characteristic suborbifolds that meet at v. Given an omniorientation, we say that the sign of a vertex v is positive if the orientations of $\mathcal{T}_v(\mathbf{X})$ determined by the orientation of $\mathbf{X}$ and orientations of characteristic suborbifolds coincide. Otherwise we say that sign of v is negative. An omniorientation is then said to be positive if each vertex has positive sign.

It is easy to verify that reversing the sign of any number of characteristic vectors does not affect the topology or differentiable structure of the quasitoric orbifold. There is a circle action of T_{λ_i} on the normal bundle of $\mathbf{X}(F_i)$ producing a complex structure and orientation on it. This action and orientation varies with the sign of λ_i. Therefore, given an orientation on $\mathbf{X}$, omniorientations correspond bijectively to choices of signs for the characteristic vectors. We will assume the standard orientations on P and T^n so that omniorientations will be solely determined by signs of characteristic vectors.

At any vertex v, we may order the incident facets in such a way that their inward normal vectors form a positively oriented basis of $\mathbb{R}^n \supset P$. Facets at a vertex ordered in this way will be called positively ordered. We denote the matrix of characteristic vectors ordered accordingly by $\Lambda_{(v)}$. Then the sign of v equals the sign of $\det(\Lambda_{(v)})$.

3. Almost complex structure

Let $\mathbf{X}$ be a positively omnioriented primitive quasitoric orbifold.

DEFINITION 3.1. We say that an almost complex structure on $\mathbf{X}$ torus invariant if it is $T_{N(F)}$-invariant in some uniformizing neighborhood of each point $x \in X(F^\circ)$.

Theorem 3.1. *Let* $\mathbf{X}$ *be a positively omnioriented quasitoric orbifold and* μ *an invariant metric on it. Then there exists an orthogonal invariant almost complex structure on* $\mathbf{X}$ *that respects the omniorientation.*

Proof. Consider the subset $R_v \subset \tilde{X}_v$ consisting of points whose coordinates (2.9) are real and nonnegative,

$$R_v = \{x \in \tilde{X}_v : z_{j,v}(x) \in \mathbb{R}_{\geq},\ \forall 1 \leq j \leq n\}. \tag{3.1}$$

In other words,

$$R_v = \{x \in \tilde{X}_v : z_{j,v}(x) = \sqrt{p_{j,v}(x)},\ j = 1, \ldots, n\}. \tag{3.2}$$

We glue the spaces R_v according to the transition maps (2.16), choosing the branches uniformly as $-\pi < q_{k,v} < \pi$. We obtain a manifold with boundary R.

Let x be any point in R_{v_1} such that $\xi_{v_1}(x) \in X_{v_1} \cap X_{v_2}$. Then the transition maps (2.16), with above choice of cuts, define a local diffeomorphism ϕ_{12} from a neighborhood of x in $\tilde{X}_{v_1}$ to a neighborhood of the image of x in $\tilde{X}_{v_2}$.

Let $\mathcal{E}_v$ denote the restriction of $\mathcal{T}\tilde{X}_v$ to R_v. The last paragraph shows that these bundles glue to form a smooth rank $2n$ real vector bundle $\mathcal{E}$ on R. The metric μ on $\mathcal{T}\mathbf{X}$ induces a metric on the bundle $\mathcal{E}$.

The restriction of the quotient map $\xi_v|_{R_v} \colon R_v \to X_v$ is a homeomorphism onto its image. As a result the space R is homeomorphic to the subspace $\iota(P)$ of X used by Kustarev [10]. The map $\iota \colon P \to X$ is a homeomorphism given by the composition $P \xrightarrow{i} P \times T_N \xrightarrow{j} X$ where i is the inclusion given by $i(p_1, \dots, p_n) = (p_1, \dots, p_n, 1, \dots, 1)$ and j is the quotient map that defines X. For any face F of P we denote its image in R under the composition of above homeomorphisms as $R(F)$. The restriction of this homeomorphism to the relative interior of F is smooth, and we denote the image by $R(F^\circ)$.

Let $\tilde{X}_v(F)$ be the preimage of $X(F)$ in $\tilde{X}_v$. If F is the intersection of facets $F_{i_1}, \dots, F_{i_t}$, then $\tilde{X}_v(F)$ is the submanifold of $\tilde{X}_v$ defined by the equations $z_{i_j,v} = 0$, $1 \leq j \leq t$. Then arguments similar to the case of $\mathcal{E}$ show that the restrictions $\mathcal{T}\tilde{X}_v(F)|_{R_v \cap R(F)}$ glue together to produce a subbundle $\mathcal{E}_F$ of $\mathcal{E}|_{R(F)}$.

It is easy to check from (2.16) that

$$\left.\frac{\partial}{\partial z_{i_j,v_1}}\right|_x = \left.\frac{\partial}{\partial z_{i_j,v_2}}\right|_x \tag{3.3}$$

at any point x in $R_{v_1} \cap R_{v_2} \cap R(F)$. Therefore we obtain a subbundle $\mathcal{N}_F$ of $\mathcal{E}|_{R(F)}$ corresponding to the normal bundles of $\tilde{X}_{F,v}$ in $\tilde{X}_v$. The bundle $\mathcal{N}_F$ obviously splits into the direct sum of the rank 2 bundles $\mathcal{N}_{F_k}$ where $k \in \mathcal{I}(F) := \{i_1, \dots, i_t\}$.

Recall the torus $T_{N(F)}$ corresponding to the face F of P from equation (2.1) and Definition 2.1. For any vertex v of F, the module $N(F)$ is a direct summand of the module $N(v)$. Consequently, $T_{N(F)}$ injects into $T_{N(v)}$. Suppose x is a point in $R(F^\circ)$. Then $T_{N(F)}$ is the stabilizer of any preimage of x in $\tilde{X}_v$.

$T_{N(F)}$ is the product of the circles T_{λ_k}, $k \in \mathcal{I}(F)$. The circle T_{λ_k} acts nontrivially on $\mathcal{N}_{F_k}$ and induces an almost complex structure on it corresponding to rotation by $\pi/2$. Note that this structure depends on the sign of λ_k or, in other words, the specific omniorientation. Thus the $T_{N(F)}$ action induces an almost complex structure on $\mathcal{N}_F$.

Using the method of Kustarev [10] it is possible to construct an orthogonal almost complex structure J on $\mathcal{E}$ that satisfies the following condition:

($\star$) For any face F of P of dimension less than n, the restriction of J to $\mathcal{N}_F|_{R(F^\circ)}$ agrees with the complex structure induced by the $T_{N(F)}$ action and the omniorientation.

For future use, we give a brief outline of the proof of existence of such a structure. The details may be found in [10]. In our case, the bundles $\mathcal{E}_F$ and $\mathcal{N}_{F_k}$ play the roles of the bundles $\tau(M_F)$ and ξ_k in [10].

An orthogonal almost complex structure on $\mathcal{E}$ may be regarded as a map $J \colon R \to SO(2n)/U(n)$. We proceed by induction. Let $sk_i(R)$ denote the union of all i-dimensional faces of R. For $i = 0$, existence of J is trivial. Extension to $sk_1(R)$ is possible due to positivity of omniorientation. For $i \geq 2$, suppose J is a structure on $sk_{i-1}(R)$ satisfying the condition ($\star$). Then J may be regarded as a map from $sk_{i-1}(R)$ to $SO(2i-2)/U(i-1)$ as it is fixed in the normal directions by the torus action. Construct a cellular cochain $\sigma_J^i \in C^i(R, \pi_{i-1}(SO(2i)/U(i))$ by defining the value of σ_J^i on an i-dimensional face of R to be the homotopy class of the value of J on the boundary of the face, composed with a canonical isomorphism between $\pi_{i-1}(SO(2i-2)/U(i-1))$ and $\pi_{i-1}(SO(2i)/U(i))$. J extends to $sk_i(R)$ if and only if $\sigma_J^i = 0$. Following [10], one proves that σ_J^i is a cocycle. Therefore, by contractibility of R it is a coboundary. Suppose $\sigma_J^i = \delta\beta$, where $\beta \in C^{i-1}(R, \pi_{i-1}(SO(2i)/U(i))$. Note that $\delta\beta(Q) = \pm \sum_{G \subset \partial Q} \beta(G)$. For each $H \in sk_{i-1}(R)$, one perturbs J in the interior of H by a factor of $-\beta(H)$. This makes $\sigma_J^i = 0$. (Note that if $\beta(H) = 0$, no change is required for face H. This will be used crucially in Lemma 3.2.)

By ($\star$) the structure J on $\mathcal{E}_v$ is invariant under the action of isotropy groups. We can therefore use the action of $T_{N(v)}$ to produce an invariant almost complex structure on $\mathcal{T}\tilde{X}_v$ as follows,

$$J(t \cdot x) = dt \circ J(x) \circ dt^{-1}, \quad \forall x \in R_v, \text{ and } \forall t \in T_{N(v)}. \tag{3.4}$$

The local group G_v of orbifold chart $(\tilde{X}_v, G_v, \xi_v)$ is a subgroup of $T_{N(v)}$. Thus J is G_v-invariant on $\tilde{X}_v$.

The compatibility of J across charts may be verified as follows. Take any point $x \in X_{v_1} \cap X_{v_2}$. Let $\tilde{x} \in \tilde{X}_{v_1}$ be a preimage of x under ξ_{v_1}. Suppose $\tilde{x} = t_1 \cdot x_0$ where $x_0 \in R$ and $t_1 \in T_{N(v_1)}$. Choose an embedding $\tilde{\phi}_{12}$ of a small G_x-stable neighborhood of $\tilde{x}$ into $\tilde{X}_{v_2}$ as outlined in Section 2.3. Suppose $\tilde{\phi}_{12}(\tilde{x}) = t_2 \cdot x_0$ where $t_2 \in T_{N(v_2)}$. Then

$$\tilde{\phi}_{12} = t_2 \circ \phi_{12} \circ {t_1}^{-1}. \tag{3.5}$$

By construction of J on $\mathcal{E}$, J commutes with $d\phi_{12}|_R$. J commutes with dt_i and dt_i^{-1} by its construction on $\tilde{X}_{v_i}$. Therefore J commutes with $d\tilde{\phi}_{12}$, as desired. □

Theorem 3.2. *Suppose an orthogonal invariant almost complex structure is given on a characteristic suborbifold* $\mathbf{X}(F)$*. Then it can be extended to* $\mathbf{X}$.

Proof. We follow the notation of the previous theorem. J has been already specified on $\mathbf{X}(F)$ where $\dim(F) = n-1$. This determines J on the subbundle $\mathcal{E}_F$ of $\mathcal{E}$ over $R(F)$. We use the torus action and omniorientation to extend J to $\mathcal{E}|_{R(F)}$.

We construct an extension of J to R skeleton-wise. Extension up to $sk_1(R) \cup F$ is achieved using positivity of omniorientation. For extension to higher skeletons we need to use obstruction theory. We need to take care so that J is preserved on sub-faces of F. We use induction. Suppose J has been extended to $sk_{d-1}(R) \cup F$, where $d < n$. (We will deal with the $d = n$ case separately.)

Let $\sigma^d \in C^d(R, \pi_{d-1}(SO(2d)/U(d)))$ be the obstruction cocycle. Let $i\colon R(F) \hookrightarrow R$ be inclusion map. Restriction to F produces a cochain

$$i^*(\sigma^d) \in C^d(R(F), \pi_{d-1}(SO(2d)/U(d))).$$

Then $i^*(\sigma^d) = 0$ since we know that J extends to $R(F)$. Since $\sigma^d = \delta\beta$, $i^*(\beta)$ is a cocycle. As $R(F)$ is contractible $i^*(\beta)$ is a coboundary. Let $i^*(\beta) = \delta\beta_1$ where $\beta_1 \in C^{d-2}(R(F))$. Define a chain $\beta_2 \in C^{d-2}(R)$ such that

$$\beta_2(H) = \begin{cases} \beta_1(H) & \text{for any } (d-2) \text{ face } H \subset R(F), \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Then define $\beta_3 = \beta - \delta(\beta_2)$. This new cochain has the property that $\delta(\beta_3) = \sigma^d$ and its action $(d-1)$-dimensional faces of $R(F)$ is zero. So we can now extend the structure to $sk_d \cup R(F)$ without affecting the sub-faces of $R(F)$.

By induction, we may assume that J has been extended to $sk_{n-1}(R) \cup R(F)$. Let $\sigma^n \in C^n(R, \pi_{n-1}(SO(2n)/U(n))$ be the corresponding obstruction cochain for extension to sk_n. Since R is contractible we have $\sigma^n = \delta\beta$. We modify β as follows. Suppose K is a facet adjacent to F. Define $\beta' \in C^{n-1}$ as follows.

$$\beta'(H) = \begin{cases} 0 & \text{if} \quad H = R(F), \\ \beta(R(F)) + \beta(R(K)) & \text{if} \quad H = R(K), \\ \beta(H) & \text{otherwise.} \end{cases} \tag{3.7}$$

Then $\delta\beta' = \delta\beta = \sigma^n$ and $\beta'(R(F)) = 0$. So we may extend J to R without changing it on $R(F)$. □

Corollary 3.3. *Suppose an orthogonal invariant almost complex structure is given on a suborbifold* $\mathbf{X}(F)$ *where* F *is any face of* P*. Then it can be extended to* $\mathbf{X}$*.*

Proof. Consider a nested sequence of faces $F = H_0 \subset H_1 \cdots \subset H_k = P$ where $\dim(H_i) = \dim(F) + i$. Extend the structure inductively from $\mathbf{X}(H_i)$ to $\mathbf{X}(H_{i+1})$ using Theorem 3.2. □

4. Blowdowns

Topologically the blowup will correspond to replacing an invariant suborbifold by the projectivization of its normal bundle. Combinatorially we replace a face by a facet with a new characteristic vector. Suppose F is a face of P. We choose a hyperplane $H = \{\hat{p}_0 = 0\}$ such that $\hat{p}_0$ is negative on F and $\hat{P} := \{\hat{p}_0 > 0\} \cap P$ is a simple polytope having one more facet than P. Suppose $F_1, \ldots, F_m$ are the facets of P. Denote the facets $F_i \cap \hat{P}$ by F_i without confusion. Denote the extra facet $H \cap P$ by F_0.

Without loss of generality let $F = \bigcap_{j=1}^{k} F_j$. Suppose there exists a primitive vector $\lambda_0 \in N$ such that

$$\lambda_0 = \sum_{j=1}^{k} b_j \lambda_j, \quad b_j > 0, \ \forall j. \tag{4.1}$$

Then the assignment $F_0 \mapsto \lambda_0$ extends the characteristic function of P to a characteristic function $\hat{\Lambda}$ on $\hat{P}$. Denote the omnioriented quasitoric orbifold derived from the model $(\hat{P}, \hat{\Lambda})$ by $\mathbf{Y}$.

Consider a small open neighborhood $U := \{x \in P \colon \hat{p}_0(x) < \epsilon\}$ of the face F, where $0 < \epsilon < 1$. Denote $U \cap \hat{P}$ by $\hat{U}$. By Lemma 2.3 we may assume that

$$f \colon U = F \times [0, 1)^k. \tag{4.2}$$

We also assume without loss of generality that the defining function $\hat{p}_j$ of the facet F_j equals the j-th coordinate p_j of $\mathbb{R}^n$ on U, for each $1 \leq j \leq k$.

Choose small positive numbers $\epsilon_1 < \epsilon_2 < \epsilon$ and a smooth non-decreasing function $\delta \colon [0, \infty) \to \mathbb{R}$ such that

$$\delta(t) = \begin{cases} t & \text{if} \quad t < \epsilon_1, \\ 1 & \text{if} \quad t > \epsilon_2. \end{cases} \tag{4.3}$$

Then define $\tau \colon \hat{P} \to P$ to be the map given by

$$\tau(p_1, \ldots, p_k, p_{k+1}, \ldots, p_n) = (\delta(\hat{p}_0)^{b_1} p_1, \ldots, \delta(\hat{p}_0)^{b_k} p_k, p_{k+1}, \ldots, p_n). \tag{4.4}$$

The blow down map $\rho \colon (\hat{P} \times T_N/\sim) \to (P \times T_N/\sim)$ is defined by

$$\rho(\mathbf{p}, \mathbf{q}) = (\tau(\mathbf{p}), \mathbf{q}). \tag{4.5}$$

Since $\delta = 1$ if $\hat{p}_0 > \epsilon_2$, ρ is a diffeomorphism of orbifolds away from a tubular neighborhood of $X(F)$. We study the map ρ near $X(F)$.

Let $w = \bigcap_{j=1}^{n} F_j$ be a vertex of F. Suppose v be a vertex of F_0 such that $\tau(v) = w$. Then the edge joining v and w is the intersection of $n-1$ facets common to both which must include $F_{k+1}, \ldots, F_n$. Therefore there are k choices for v, namely $v_i = \bigcap_{0 \leq j \neq i \leq n} F_j$ with $1 \leq i \leq k$.

Let $\hat{p}_j = 0$ be the defining equation of the facet F_j for $k+1 \leq j \leq n$. Order the facets at w as $F_1, \ldots, F_n$, and those at v_i as $F_1, \ldots, F_{i-1}, F_0, F_{i+1}, \ldots, F_n$. Let $z_{j,w}$ and z_{j,v_i} be the coordinates on $\tilde{X}_w$ and $\tilde{Y}_{v_i}$ defined according to (2.8) and (2.9). Then by using a process similar to the one used for (2.16), we obtain the following description of ρ near Y_{v_i},

$$\begin{aligned} z_{i,w} \circ \rho &= z_{i,v_i}^{b_i} \sqrt{p_i \delta(\hat{p}_0)^{b_i} (\hat{p}_0)^{-b_i}}, \\ z_{j,w} \circ \rho &= z_{i,v_i}^{b_j} z_{j,v_i} \sqrt{\delta(\hat{p}_0)^{b_j} (\hat{p}_0)^{-b_j}} \quad \text{if} \quad 1 \leq j \neq i \leq k, \\ z_{j,w} \circ \rho &= z_{j,v_i} \quad \text{if} \quad k+1 \leq j \leq n. \end{aligned} \tag{4.6}$$

We define a new coordinate system on $\tilde{Y}_{v_i}$, for each $1 \leq i \leq k$, as follows.

$$\begin{aligned} z'_{i,v_i} &= z_{i,v_i} (\sqrt{p_i})^{1/b_i} \sqrt{\delta(\hat{p}_0)(\hat{p}_0)^{-1}}, \\ z'_{j,v_i} &= z_{j,v_i} (\sqrt{p_i})^{-b_j/b_i} \quad \text{if} \quad 1 \leq j \neq i \leq k, \\ z'_{j,v_i} &= z_{j,v_i} \quad \text{if} \quad k+1 \leq j \leq n. \end{aligned} \tag{4.7}$$

This is a valid change of coordinates as p_i is positive on $\tilde{Y}_{v_i}$ and $\delta(\hat{p}_0)(\hat{p}_0)^{-1}$ is identically one near $\hat{p}_0 = 0$.

In these new coordinates, ρ can be expressed as

$$\begin{aligned} z_{i,w} \circ \rho &= (z'_{i,v_i})^{b_i}, \\ z_{j,w} \circ \rho &= (z'_{i,v_i})^{b_j} z'_{j,v_i} \quad \text{if} \quad 1 \leq j \neq i \leq k, \\ z_{j,w} \circ \rho &= z'_{j,v_i} \quad \text{if} \quad k+1 \leq j \leq n. \end{aligned} \tag{4.8}$$

Lemma 4.1. *The restriction* $\rho\colon \mathbf{Y} - \mathbf{Y}(F_0) \to \mathbf{X} - \mathbf{X}(F)$ *is a diffeomorphism of orbifolds.*

Proof. This is obvious outside $\pi^{-1}(U)$. On $\pi^{-1}(U) - X(F)$, by formula (4.8), ρ is locally equivalent to a blowup in complex geometry. Therefore ρ is an analytic isomorphism on $\pi^{-1}(U) - X(F)$. However since our quasitoric orbifolds are primitive, there is no complex reflection in our orbifold groups. Hence using the results of [13], analytic isomorphism yields diffeomorphism of orbifolds. □

Lemma 4.2. *If* $\mathbf{X}$ *is positively omnioriented, then so is a blowup* $\mathbf{Y}$.

Proof. Recall the positive ordering of facets at a vertex v in Section 2.9 to define the matrix $\Lambda_{(v)}$ whose determinant has the same sign as sign of v.

Let w be any vertex of F and v_i be any vertex in $\rho^{-1}(w)$. Let $F_1, \dots, F_n$ be positively ordered facets at w. An inward normal vector to F_0 is a positive linear combination of the inward normal vectors to $F_1, \dots, F_k$. Therefore $F_1, \dots, F_{i-1}, F_0, F_{i+1}, \dots, F_n$ are positively ordered for each $i = 1, \dots, k$. So the matrix $\Lambda_{(v_i)}$ is obtained by replacing the i-th column of $\Lambda_{(w)}$, namely λ_i, by $\lambda_0 = \sum_{j=1}^{k} b_j \lambda_j$. Therefore $\det \Lambda_{(v_i)} = b_i \det \Lambda_{(w)}$. The lemma follows. □

DEFINITION 4.1. A blowdown ρ is said to be a resolution if for any vertex w of the exceptional face F and any vertex $v_i \in \rho^{-1}(F)$ we have $o(G_{v_i}) < o(G_w)$.

Lemma 4.3. *A blowdown ρ is a resolution if $b_i < 1$ for each i.*

Proof. The lemma holds since by (2.11) we have $o(G_{v_i}) = |\det \Lambda_{v_i}| = b_i |\det \Lambda_w| = b_i o(G_w)$. □

5. Pseudoholomorphic blowdowns

Lemma 5.1. *Let $\rho \colon Y \to X$ be a blowdown along a subset $X(F)$. Suppose there exist holomorphic coordinate systems $z^*_{1,w}, \dots, z^*_{n,w}$ on the uniformizing chart $\tilde{X}_w$ for every vertex w of F, which produce an analytic structure on a neighborhood $\pi^{-1}(U)$ of $X(F)$. Assume further that this analytic structure extends to an almost complex structure on* **X***. Then the blowup induces an almost complex structure on* **Y** *which is analytic near the exceptional set $Y(F_0)$. Moreover, with respect to these structures ρ is analytic near $Y(F_0)$ and an almost complex diffeomorphism of orbifolds away from $Y(F_0)$.*

Proof. Note that for two vertices w_1, w_2 of F, the coordinates must be related as

$$z^*_{j,w_2} = \prod_{i=1}^{n} (z^*_{i,w_1})^{d_{ij}} \tag{5.1}$$

where the d_{ij}s are rational numbers determined from the matrix $\Lambda^{-1}_{w_2}\Lambda_{w_1}$, see (2.13) and (2.16).

Also the coordinates $z^*_{j,w}$ have to relate to the coordinates defined in (2.8) and (2.9) as follows,

$$z^*_{j,w} = z_{j,w} f_j, \quad 1 \le j \le n \tag{5.2}$$

where each f_j is smooth and non-vanishing on $\tilde{X}_w$. For each $v_i \in \rho^{-1}(w)$ we define coordinates in its neighborhood, by modifying the coordinates of (4.7) as follows,

$$\begin{aligned} z^*_{i,v_i} &= z'_{i,v_i}(f_i \circ \tau)^{1/b_i}, \\ z^*_{j,v_i} &= z'_{j,v_i}(f_j \circ \tau)(f_i \circ \tau)^{-b_j/b_i} \quad \text{if} \quad 1 \le j \ne i \le k, \\ z^*_{j,v_i} &= z'_{j,v_i} \quad \text{if} \quad k+1 \le j \le n. \end{aligned} \tag{5.3}$$

In these coordinates ρ takes the following form near v_i,

$$\begin{aligned} z^*_{i,w} \circ \rho &= (z^*_{i,v_i})^{b_i}, \\ z^*_{j,w} \circ \rho &= (z^*_{i,v_i})^{b_j} z^*_{j,v_i} \quad \text{if} \quad 1 \le j \ne i \le k, \\ z^*_{j,w} \circ \rho &= z^*_{j,v_i} \quad \text{if} \quad k+1 \le j \le n. \end{aligned} \tag{5.4}$$

We define an almost complex structure $\hat{J}$ on $\mathbf{Y}$ by defining the coordinates z^*_{j,v_i} to be holomorphic near $Y(F)$ and by $\hat{J} = d\rho^{-1} \circ J \circ d\rho$ away from it. This is consistent as ρ is a diffeomorphism of orbifolds on the complement of Y_F.

By (5.1) and (5.4), for any two vertices u_1 and u_2 of F_0, we have

$$z^*_{j,u_2} = \prod_{i=1}^{n} (z^*_{i,u_1})^{e_{ij}} \tag{5.5}$$

for some rational numbers e_{ij}. But these numbers are determined by the matrix $\Lambda^{-1}_{u_2}\Lambda_{u_1}$. It is then obvious from the arguments about compatibility of charts in Section 2.2 that the patching of the charts Y_{u_1} and Y_{u_2} is holomorphic. □

Examples of blowdowns that satisfy the hypothesis of Lemma 5.1 include blowdowns of four dimensional positively omnioriented quasitoric orbifolds constructed in [9] and toric blow-ups of simplicial toric varieties.

DEFINITION 5.1 ([9]). A function f on X is said to be smooth if $f \circ \xi$ is smooth for every uniformizing system $(\tilde{U}, G, \xi)$. A complex valued smooth function f on an almost complex orbifold $(\mathbf{X}, J)$ is said to be J-holomorphic if the differential $d(f \circ \xi)$ commutes with J for every chart $(\tilde{U}, G, \xi)$. We denote the sheaf of J-holomorphic functions on $\mathbf{X}$ by $\Omega^0_{J,X}$. A continuous map $\rho \colon Y \to X$ between almost complex orbifolds $(\mathbf{Y}, J_2)$ and $(\mathbf{X}, J_1)$ is said to be pseudo-holomorphic if $f \circ \rho \in \Omega^0_{J_2,Y}(\rho^{-1}(U))$ for every $f \in \Omega^0_{J_1,X}(U)$ for any open set $U \subset X$; that is, ρ pulls back pseudo-holomorphic functions to pseudo-holomorphic functions.

Lemma 5.2. *Blowdowns that satisfy the hypothesis of* Lemma 5.1 *are pseudo-holomorphic.*

Proof. Suppose $\rho \colon Y \to X$ is such a blowdown. Since ρ is an almost complex diffeomorphism of orbifolds away from the exceptional set $Y(F_0)$, it suffices to check the statement near $Y(F_0)$. Pick any vertex w of F. Define $W = X_w \cap \pi^{-1}(U)$. For any vertex $v_i \in \rho^{-1}(w)$, let $V_i = Y_{v_i} \cap \rho^{-1}(\pi^{-1}(U))$. We will denote the characteristic vectors at v_i by $\hat{\lambda}_j$, $j = 1, \ldots, n$. Note that

$$\hat{\lambda}_j = \begin{cases} \lambda_j & \text{if} \quad j \neq i, \\ \lambda_0 & \text{if} \quad j = i. \end{cases} \tag{5.6}$$

The ring $\Omega^0_{J_1,X}(W)$ is the G_w-invariant subring of convergent power series in variables $z^*_{j,w}$. It is generated by monomials of the form

$$f = \prod_{j=1}^{n} (z^*_{j,w})^{d_j} \tag{5.7}$$

where the d_js are integers such that $\sum a_j d_j$ is an integer whenever the vector $\sum a_j \lambda_j \in N$. This last condition follows from invariance under action of the element $g \in G_w$ corresponding to $\sum a_j \lambda_j$.

Using (5.4) and $\lambda_0 = \sum_{j=1}^{n} b_j \lambda_j$ with $b_j = 0$ for $j \geq k+1$, we get

$$f \circ \rho = (z^*_{i,v_i})^{\sum b_j d_j} \prod_{j \neq i} (z^*_{j,v_i})^{d_j}. \tag{5.8}$$

Take any element h in G_{v_i}. Suppose h is represented by $\sum c_j \hat{\lambda}_j \in N$. The action of h on $f \circ \rho$ is multiplication by $e^{2\pi\sqrt{-1}\alpha}$, where

$$\alpha = c_i \sum_j b_j d_j + \sum_{j \neq i} c_j d_j = c_i b_i d_i + \sum_{j \neq i} (c_j + c_i b_j) d_j. \tag{5.9}$$

Note that $\eta := c_i b_i \lambda_i + \sum_{j \neq i} (c_j + c_i b_j) \lambda_j = c_i \sum_j b_j \lambda_j + \sum_{j \neq i} c_j \lambda_j = \sum c_j \hat{\lambda}_j$. Hence this is an element of N.

Suppose f is a generator of $\Omega^0_{J_1,X}(W)$ as in (5.7). Consider the action of the element of G_w corresponding to η on f. It is multiplication by $e^{2\pi\sqrt{-1}\alpha}$. Since f is G_w-invariant, α is an integer. Hence $f \circ \rho$ is G_{v_i} invariant. The ring $\Omega^0_{J_1,Y}(V_i)$ is the G_{v_i}-invariant subring of convergent power series in variables z^*_{j,v_i}. Therefore $f \circ \rho \in \Omega^0_{J_1,Y}(V_i)$. □

The proof of the following corollary of Lemma 5.1 is straightforward.

Corollary 5.3. *Consider a sequence of blowups $\rho_i \colon Y_i \to Y_{i-1}$ where $1 \leq i \leq r$ and ρ_1 satisfies the hypothesis of* Lemma 5.1. *Assume that the locus of the i-th blowup is contained in the exceptional set of the $(i-1)$-st blowup for every i. Then we can inductively choose almost complex structures so that each blowdown map in the sequence is pseudoholomorphic.*

Theorem 5.4. *There exists a pseudoholomorphic resolution of singularity for any primitive positively omnioriented four dimensional quasitoric orbifold.*

Proof. For any primitive positively omnioriented four dimensional quasitoric orbifold, Theorem 3.1 of [9] produces an almost complex structure that satisfies the hypothesis of Lemma 5.1 for every vertex. The singularities are all cyclic. We can resolve them by applying a sequence of blow-ups as in Corollary 5.3. □

6. Crepant blowdowns

DEFINITION 6.1. A blowdown is called crepant if $\sum b_j = 1$.

This has the following geometric interpretation.

DEFINITION 6.2. Given an almost complex $2n$-dimensional orbifold $(\mathbf{X}, J)$, we define the canonical sheaf K_X to be the sheaf of continuous $(n, 0)$-forms on X; that is, for any orbifold chart $(\tilde{U}, G, \xi)$ over an open set $U \subset X$, $K_X(U) = \Gamma\left(\bigwedge^n \mathcal{T}^{1,0}(\tilde{U})^*\right)^G$ where Γ is the functor that takes continuous sections.

An almost complex orbifold is called Gorenstein or *SL* orbifold if the linearization of every local group element g belongs to $SL(n,\mathbb{C})$. For an *SL*-orbifold $\mathbf{X}$, the canonical sheaf is a complex line bundle over X.

Lemma 6.1. *Suppose $\rho\colon Y \to X$ is a pseudoholomorphic blowdown of SL quasitoric orbifolds along a face F satisfying the hypothesis of* Lemma 5.1. *Then ρ is crepant if and only if $\rho^* K_X = K_Y$.*

Proof. We consider the canonical sheaf K_X as a sheaf of modules over the sheaf of continuous functions $\mathcal{C}^0_X$. Since ρ is an almost complex diffeomorphism away from the exceptional set it suffices to check the equality of the $\rho^* K_Y$ and K_X on the neighborhood $\rho^{-1}(\pi^{-1}(U)) \subset Y$ of the exceptional set. Choose any vertex w of F. On $X_w \cap \pi^{-1}(U)$, the sheaf K_X is generated over the sheaf $\mathcal{C}^0_X$ by the form $dz^*_{1,w} \wedge \cdots \wedge dz^*_{n,w}$, see (5.2). Let v_i be any preimage of w under ρ. Similarly on $Y_{v_i} \cap \rho^{-1}(\pi^{-1}(U))$, K_Y is generated over the sheaf $\mathcal{C}^0_Y$ by the form $dz^*_{1,v_i} \wedge \cdots \wedge dz^*_{n,v_i}$.

Using (5.4) we have

$$\begin{aligned} &\rho^* dz^*_{i,w} = b_i (z^*_{i,v_i})^{b_i - 1} dz^*_{i,v_i}, \\ &\rho^* dz^*_{j,w} = (z^*_{i,v_i})^{b_j} dz^*_{j,v_i} + b_j (z^*_{i,v_i})^{b_j - 1} z^*_{j,v_i} dz^*_{i,v_i} \quad \text{if} \quad 1 \le j \ne i \le k, \\ &\rho^* dz^*_{j,w} = dz^*_{j,v_i} \quad \text{if} \quad k+1 \le j \le n. \end{aligned} \tag{6.1}$$

Therefore we have

$$\rho^*(dz^*_{1,w} \wedge \cdots \wedge dz^*_{n,w}) = b_i (z^*_{i,v_i})^{b_1 + \cdots + b_k - 1} dz^*_{1,v_i} \wedge \cdots \wedge dz^*_{n,v_i}. \tag{6.2}$$

The lemma follows. □

7. Chen–Ruan Cohomology

The Chen–Ruan cohomology group is built out of the ordinary cohomology of certain copies of singular strata of an orbifold called twisted sectors. The twisted sectors of orbifold toric varieties was computed in [14]. The determination of such sectors for quasitoric orbifolds is similar in essence. Another important feature of Chen–Ruan cohomology is the grading which is rational in general. In our case the grading will depend on the omniorientation.

Let $\mathbf{X}$ be an omnioriented quasitoric orbifold. Consider any element g of the group G_F (2.19). Then g may be represented by a vector $\sum_{j \in \mathcal{I}(F)} a_j \lambda_j$. We may restrict a_j

to $[0,1)\cap\mathbb{Q}$. Then the above representation is unique. Then define the degree shifting number or age of g to be

$$\iota(g)=\sum a_j. \tag{7.1}$$

For faces F and H of P we write $F\le H$ if F is a sub-face of H, and $F<H$ if it is a proper sub-face. If $F\le H$ we have a natural inclusion of G_H into G_F induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard G_H as a subgroup of G_F. Define the set

$$G_F^{\circ}=G_F-\bigcup_{F<H}G_H. \tag{7.2}$$

Note that $G_F^{\circ}=\left\{\sum_{j\in\mathcal{I}(F)}a_j\lambda_j\colon 0<a_j<1\right\}\cap N$, and $G_P^{\circ}=G_P=\{0\}$.

DEFINITION 7.1. We define the Chen–Ruan orbifold cohomology of an omni-oriented quasitoric orbifold $\mathbf{X}$ to be

$$H^*_{\mathrm{CR}}(\mathbf{X},\mathbb{R})=\bigoplus_{F\le P}\bigoplus_{g\in G_F^{\circ}}H^{*-2\iota(g)}(X(F),\mathbb{R}).$$

Here H^* refers to singular cohomology or equivalently to de Rham cohomology of invariant forms when $X(F)$ is considered as the orbifold $\mathbf{X}(F)$. The pairs $(X(F),g)$ where $F<P$ and $g\in G_F^{\circ}$ are called twisted sectors of $\mathbf{X}$. The pair $(X(P),1)$, i.e. the underlying space X, is called the untwisted sector. We denote the Betti number $\mathrm{rank}(H^d_{\mathrm{CR}}(\mathbf{X}))$ by h^d_{CR}.

Note that if $\mathbf{X}$ is a manifold then its Chen–Ruan cohomology is same as its singular cohomology.

7.1. Poincaré duality. Poincaré duality is established in a similar fashion as for compact almost complex orbifolds. We need to distinguish the copies of $X(F)$ corresponding to different twisted sectors. Therefore for $g\in G_F^{\circ}$, we define the space

$$S(F,g)=\{(x,g)\colon x\in X(F)\}. \tag{7.3}$$

Of course $S(F,g)$ is homeomorphic to $X(F)$. It is denoted by $\mathbf{S}(F,g)$ when endowed with an orbifold structure which is the structure of $\mathbf{X}(F)$ with an additional trivial action of G_F at each point. With this structure, it is a suborbifold of $\mathbf{X}$ in a natural way. The untwisted sector is denoted by $S(P,1)$. In this notation the Chen–Ruan groups may be written as

$$H^*_{\mathrm{CR}}(\mathbf{X},\mathbb{R})=\bigoplus_{F\le P}\bigoplus_{g\in G_F^{\circ}}H^{*-2\iota(g)}(S(F,g),\mathbb{R}). \tag{7.4}$$

Lemma 7.1. *Suppose* $g \in G_F^{\circ}$. *Then* $2\iota(g) + 2\iota(g^{-1}) = 2n - \dim(X(F))$.

Proof. When $F = P$, $G_P^{\circ} = \{0\}$ and the result is obvious. Suppose $F = \bigcap_{i=1}^k F_i$. Then $g = \sum_{i=1}^k a_i\lambda_i$ where each $0 < a_i < 1$. Then g^{-1} is represented by the vector $\sum_{i=1}^k -a_i\lambda_i$ in N modulo $N(F)$. Therefore g^{-1} may be identified with the vector $\sum_{i=1}^k (1-a_i)\lambda_i$. Note that $0 < 1 - a_i < 1$ for each i. Therefore the age of g^{-1}, $\iota(g^{-1}) = \sum_{i=1}^k (1-a_i)$. Hence $2\iota(g) + 2\iota(g^{-1}) = 2\sum_{i=1}^k a_i + 2\sum_{i=1}^k (1-a_i) = 2k = 2n - \dim(X(F))$. □

For any compact orientable orbifold, there exists a notion of orbifold integration $\int^{\text{orb}}$ for invariant top dimensional forms which gives Poincaré duality for the de Rham cohomology of the orbifold, see [5]. For a chart $\mathbf{U} = (\tilde{U}, G, \xi)$ orbifold integration for an invariant form ω on $\tilde{U}$ is defined by

$$\int_{\mathbf{U}}^{\text{orb}} \omega = \frac{1}{o(G)} \int_{\tilde{U}} \omega. \tag{7.5}$$

Let $I \colon \mathbf{S}(F,g) \to \mathbf{S}(F,g^{-1})$ be the diffeomorphism of orbifolds defined by $I(x,g) = (x, g^{-1})$. We define a bilinear pairing

$$\langle\ ,\ \rangle_{(F,g)}^{\text{orb}} \colon H^{d-2\iota(g)}(S(F,g)) \times H^{2n-d-2\iota(g^{-1})}(S(F,g^{-1})) \to \mathbb{R} \tag{7.6}$$

for every $0 \le d \le 2n$ by

$$\langle \alpha, \beta\rangle_{(F,g)}^{\text{orb}} = \int_{\mathbf{S}(F,g)}^{\text{orb}} \alpha \wedge I^*(\beta). \tag{7.7}$$

This pairing is nondegenerate because of Lemma 7.1. By taking a direct sum of the pairing (7.6) over all pairs of sectors $((F,g),(F,g^{-1}))$ for $F \le P$, we get a nonsingular pairing for each $0 \le d \le 2n$

$$\langle\ ,\ \rangle^{\text{orb}} \colon H_{\text{CR}}^d(\mathbf{X}) \times H_{\text{CR}}^{2n-d}(\mathbf{X}) \to \mathbb{R}. \tag{7.8}$$

8. McKay correspondence

First we introduce some notation. Consider a codimension k face $F = F_1 \cap \cdots \cap F_k$ of P where $k \ge 1$. Define a k-dimensional cone C_F in $N \otimes \mathbb{R}$ as follows,

$$C_F = \left\{ \sum_{j=1}^k a_j\lambda_j : a_j \ge 0 \right\}. \tag{8.1}$$

The group G_F can be identified with the subset Box_F of C_F, where

$$Box_F := \left\{ \sum_{j=1}^{k} a_j \lambda_j : 0 \leq a_j < 1 \right\} \cap N. \tag{8.2}$$

Consequently the set G_F° is identified with the subset

$$Box_F^\circ := \left\{ \sum_{j=1}^{k} a_j \lambda_j : 0 < a_j < 1 \right\} \cap N \tag{8.3}$$

of the interior of C_F. We define $Box_P = Box_P^\circ = \{0\}$.

Suppose $v = F_1 \cap \cdots \cap F_n$ is a vertex of P. Then $Box_v = \bigsqcup_{v \leq F} Box_F^\circ$. This implies

$$G_v = \bigsqcup_{v \leq F} G_F^\circ. \tag{8.4}$$

8.1. Euler characteristic. An almost complex orbifold is *SL* if the linearization of each g is in $SL(n, \mathbb{C})$. This is equivalent to $\iota(g)$ being integral for every twisted sector. Therefore, to suit our purposes, we make the following definition.

DEFINITION 8.1. An omnioriented quasitoric orbifold is said to be quasi-*SL* if the age of every twisted sector is an integer.

Lemma 8.1. *Suppose* $\mathbf{X}$ *is a quasi-SL quasitoric orbifold. Then the Chen–Ruan Euler characteristic of* $\mathbf{X}$ *is given by*

$$\chi_{\mathrm{CR}}(\mathbf{X}) = \sum_{v} o(G_v)$$

where v *varies over all vertices of* P.

Proof. Note that each $X(F)$ is a quasitoric orbifold. So its cohomology is concentrated in even degrees, see [15]. Since $\mathbf{X}$ is quasi-*SL*, the shifts $2\iota(g)$ in grading are also even integers. Therefore the Euler characteristic of Chen–Ruan cohomology is given by

$$\chi_{\mathrm{CR}}(\mathbf{X}) = \sum_{F \leq P} \chi(X(F)) \cdot o(G_F^\circ). \tag{8.5}$$

Each $X(F)$ admits a decomposition into even dimensional strata as follows

$$X(F) = \bigsqcup_{H \leq F} X(H^\circ) \tag{8.6}$$

where H° is the relative interior of H and $X(H^\circ) = \pi^{-1}(H^\circ)$. We have

$$\chi(X(F)) = \sum_{H \leq F} \chi(X(H^\circ)). \tag{8.7}$$

However $X(H^\circ)$ is homeomorphic to the product of H° with $(S^1)^{\dim(H)}$. Therefore $\chi(X(H^\circ)) = 0$unless H is a vertex. Hence

$$\chi(X(F)) = \text{number of vertices of } F. \tag{8.8}$$

This formula also follows from the description of the homology groups of a quasitoric orbifold in [15].

Using (8.4), (8.5) and (8.8), we have the desired formula for $\chi_{\mathrm{CR}}(\mathbf{X})$. □

Lemma 8.2. *The crepant blowup of a quasi-SL quasitoric orbifold is quasi-SL.*

Proof. Suppose the blowup is along a face $F = F_1 \cap \cdots \cap F_k$. The new sectors that appear correspond to G_H° where $H < F_0$. Take any vertex v in H. Suppose v projects to the vertex w of F under the blowdown. Without loss of generality assume $w = \bigcap_{j=1}^n F_j$. Then $v = \bigcap_{0 \leq j \neq i \leq n} F_j$ for some $1 \leq i \leq k$. Without loss of generality assume $i = 1$. Since $v \leq H$, $\mathcal{I}(H) \subset \{0, 2, \ldots, n\}$. Therefore any $g \in G_H^\circ$ may be represented by an element $\eta = c_0\lambda_0 + \sum_{j=2}^n c_j\lambda_j$ of N where each $c_j \in [0,1) \cap \mathbb{Q}$. We need to show that the age of g, namely $c_0 + \sum_{j=2}^n c_j$, is an integer.

But using $\lambda_0 = \sum_{j=1}^k b_j\lambda_j$ we get that $\eta \in C_w$. In fact

$$\eta = c_0 b_1 \lambda_1 + \sum_{j=2}^{k} (c_0 b_j + c_j)\lambda_j + \sum_{j=k+1}^{n} c_j \lambda_j. \tag{8.9}$$

We may write $\eta = \sum_{j=1}^n (m_j + a_j)\lambda_j$ where each m_j is an integer and each $a_j \in [0,1) \cap \mathbb{Q}$. Then $\sum_{j=1}^n a_j\lambda_j$ corresponds to an element of G_w. Since $\mathbf{X}$ is quasi-*SL*, $\sum_{j=1}^n a_j$ must be an integer. Therefore $\sum_{j=1}^n (m_j + a_j)$ is an integer. Hence $c_0 b_1 + \sum_{j=2}^k (c_0 b_j + c_j) + \sum_{j=k+1}^n c_j$ is an integer. Using $\sum_{j=1}^k b_j = 1$, this yields that $c_0 + \sum_{j=2}^n c_j$ is an integer. □

Theorem 8.3. *The Euler characteristic of Chen–Ruan cohomology is preserved under a crepant blowup of a quasi-SL quasitoric orbifold.*

Proof. Let $\rho\colon Y \to X$ be a crepant blowdown along a face $F = \bigcap_{j=1}^k F_j$ of P. Let w be any vertex of P and let $v_1, \ldots, v_k$ be the vertices of $\hat{P}$ such that $\rho(v_i) = w$. Suppose $w = \bigcap_{1 \leq j \leq n} F_j$. Then $v_i = F_0 \cap \bigcap_{1 \leq j \neq i \leq n} F_j$.

The contribution of w to $\chi_{\mathrm{CR}}(\mathbf{X})$ is $o(G_w) = |\det \Lambda_w|$, see (2.11). The contribution of each v_i to $\chi_{\mathrm{CR}}(\mathbf{Y})$ is $o(G_{v_i}) = |\det \Lambda_{v_i}| = b_i |\det \Lambda_w| = b_i o(G_w)$. As the blowdown is crepant, we have $o(G_w) = \sum_{i=1}^{k} o(G_{v_i})$. The theorem follows. ☐

8.2. Orbifold *K*-groups. Orbifold K-theory is the K-theory of orbifold vector bundles. Adem and Ruan [2] proved that there is an isomorphism of groups between orbifold K-theory and $\mathbb{Z}_2$-graded orbifold cohomology theory of any reduced differentiable orbifold, with field coefficients. Almost complex structure is not necessary for this result as the grading for orbifold cohomology is the ordinary grading. For a quasi-*SL* quasitoric orbifold, since the degrees of cohomology classes as well degree shifting numbers are even integers, K^0_{orb} has rank same as the Euler characteristic of Chen–Ruan cohomology and K^1_{orb} is trivial. Hence by Theorem 8.3, the orbifold K-groups are preserved under crepant blowup of quasi-*SL* quasitoric orbifolds.

8.3. Betti numbers. We prove a stronger version of McKay correspondence, namely the invariance of Betti numbers of Chen–Ruan cohomology under crepant blowdown, when dimension of $\mathbf{X}$ is less or equal to six. A more restrictive result was proved for dimension four in [9].

Theorem 8.4. *Suppose $\rho\colon Y \to X$ is a crepant blowdown of quasi-SL quasitoric orbifolds of dimension ≤ 6. Then the Betti numbers of Chen–Ruan cohomology of $\mathbf{X}$ and $\mathbf{Y}$ are equal.*

Proof. Assume that $\dim(\mathbf{X}) = 6$. Note that there are no facet sectors as every characteristic vector is primitive. Therefore the twisted sectors correspond to either vertices or edges. The age of a vertex sector is either 1 or 2 and such a sector contributes a generator to H^2_{CR} or H^4_{CR} respectively. An edge sector always has age 1. Since such a sector is a sphere it contributes a generator to H^2_{CR} as well as H^4_{CR}. There is only one generator in H^0_{CR} and H^6_{CR} coming from the untwisted sector. Therefore h^0_{CR} and h^6_{CR} are unchanged under blowup. If h^2_{CR} changes under blowup then by Poincaré duality, h^4_{CR} must change by the same amount. That would contradict the conservation of Euler characteristic. Therefore all Betti numbers are unchanged.

The proof for dimension four is similar. ☐

Lemma 8.5. *Suppose $\rho\colon Y \to X$ is a crepant blowdown of quasi-SL quasitoric orbifolds of dimension ≥ 8. Then $h^2_{\mathrm{CR}}(\mathbf{Y}) \geq h^2_{\mathrm{CR}}(\mathbf{X})$.*

Proof. The sectors that contribute to h^2_{CR} are the untwisted sector and twisted sectors of age one. Each age one sector contributes one to h^2_{CR}. The untwisted sector contributes h^2. It is proved in [15] that $h^2 = m - n$ where m is the number of facets and n is the dimension of the polytope.

Suppose the blowup is along a face F. The twisted sectors that may get affected by the blowup are the ones that intersect $X(F)$. These must be of the form (S, g) where g belongs to $\bigcup_w G_w$ where w varies over vertices of F. Consider any such w. Suppose $\lambda_1, \dots, \lambda_n$ are the corresponding characteristic vectors. Note that the age one sectors of X coming from G_w belong to the set

$$A_w = \left\{ \sum_{j=1}^{n} a_j \lambda_j \colon \sum_{j=1}^{n} a_j = 1 \right\}. \tag{8.10}$$

Since $\lambda_1, \dots, \lambda_n$ are linearly independent, there exists a unique vector v such that the dot product $\langle \lambda_i, v \rangle = 1$ for each i. Hence A_w is a hyperplane given by

$$A_w = \{ x \in N \otimes \mathbb{R} \colon \langle x, v \rangle = 1 \}. \tag{8.11}$$

Note that since the blowup is crepant, $\lambda_0 \in A_w \cap C_F \cap N$. The sector corresponding to λ_0 is lost under the blowup. However the loss in h^2_{CR} because of it is compensated by the contribution from the untwisted sector on account of the new facet F_0.

Consider any other age one sector g of $\mathbf{X}$ in G_w. C_w is partitioned into n subcones by the introduction of λ_0. Accordingly g may be represented by $\sum_{0 \le j \ne i \le n} c_j \lambda_j$ with each $c_j \ge 0$, for some $1 \le i \le n$. This means that g becomes a sector of Y coming from G_{v_i} where $v_i = \bigcap_{0 \le j \ne i \le n} F_j$. Now $g \in A_w$ as it is an age one sector of $\mathbf{X}$. Also each $\lambda_j \in A_w$. Therefore by (8.11), $\sum_{0 \le j \ne i \le n} c_j = 1$. This implies that each $0 \le c_j < 1$ and age of g as a sector of $\mathbf{Y}$ is one as well. The lemma follows. $\square$

8.4. Example. We will consider the weighted projective space $\mathbf{X} = \mathbb{P}(1,3,3,3,1)$ which is a toric variety. The generators of the one dimensional cones of the fan of X are $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$ and $e_5 = (-1, -3, -3, -3)$. $\mathbf{X}$ may be realized as a quasitoric orbifold with the 4-dimensional simplex as the polytope and the e_is as characteristic vectors. However $\mathbb{P}(1, 3, 3, 3, 1)$ is not an *SL* orbifold and this choice of characteristic vectors coming from the fan does not make it an omnioriented quasi-*SL* quasitoric orbifold. So we choose a different omniorientation.

To be precise, by the correspondence established in [11], we can consider $\mathbf{X}$ as a symplectic toric orbifold with a simple rational moment polytope P whose facets have inward normal vectors $e_1, \dots, e_5$. The moment polytope may be identified with the orbit space of the torus action. The denominations of the polytope are related to the choice of the symplectic form and is not important for us. Denote the facet of P with normal vector e_i by F_i. We assign the characteristic vectors as follows

$$\lambda_i = \begin{cases} e_i & \text{if} \quad 1 \le i \le 4, \\ -e_5 & \text{if} \quad i = 5. \end{cases} \tag{8.12}$$

The singular locus of $\mathbf{X}$ is the subset $X(F)$ where $F = F_1 \cap F_5$. The group G_F is isomorphic to $\mathbb{Z}_3$ and

$$G_F^\circ = \left\{ g = \frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_5,\ g^2 = \frac{1}{3}\lambda_1 + \frac{2}{3}\lambda_5 \right\} = \{(1, 1, 1, 1), (1, 2, 2, 2)\}. \tag{8.13}$$

Thus there are only two twisted sectors $S(F, g)$ and $S(F, g^2)$, each of age one. Since F is a triangle, the 4-dimensional quasitoric orbifold $\mathbf{X}(F)$ has $h^0 = h^2 = h^4 = 1$. Therefore each twisted sector contributes one to $h^k_{\mathrm{CR}}(\mathbf{X})$ for $k = 2, 4, 6$.

We consider a crepant blowup $\mathbf{Y}$ of $\mathbf{X}$ along $X(F)$ with $\lambda_0 = (1, 1, 1, 1)$. The singular locus of $\mathbf{Y}$ equals $Y(H)$ where $H = F_0 \cap F_5$. $G_H \cong \mathbb{Z}_2$ and $G_H^\circ = \{h = (1/2)\lambda_0 + (1/2)\lambda_5\} = \{(1, 2, 2, 2)\}$. The age one twisted sector $S(H, h)$ contributes one to $h^k_{\mathrm{CR}}(\mathbf{Y})$ for $k = 2, 4, 6$. But $h^2_{\mathrm{CR}}(\mathbf{Y})$ also has an additional contribution from the new facet. Therefore $h^2_{\mathrm{CR}}(\mathbf{Y}) = h^2_{\mathrm{CR}}(\mathbf{X})$. Then by Poincaré duality, h^6_{CR} are also equal. Finally by conservation of Euler characteristic we get equality of h^4_{CR}.

It is also possible to directly ascertain the change in the ordinary Betti numbers due to blowup. The new facet F_0 is diffeomorphic to $F \times [0, 1]$. So the new polytope has three extra vertices. We can arrange them to have indices 1, 2, 3 and keep indices of other vertices unchanged, see [15] for definition of index. This means that ordinary homology, and therefore cohomology, of Y is richer than that of X by a generator in degrees 2, 4, 6.

If we perform a further blowup of $\mathbf{Y}$ along H with $(1,2,2,2)$ as the new characteristic vector, we obtain a quasitoric manifold Z. It is easy to observe that Betti numbers of Chen–Ruan cohomologies of $\mathbf{Y}$ and Z are equal. If we switched the choice of characteristic vectors for the two blowups, McKay correspondence for Betti numbers would still hold.

Finally consider other choices of omniorientation that could make $\mathbf{X}$ quasi-*SL*. Switching the sign(s) of λ_2, λ_3 or λ_4 does not affect quasi-*SL*ness or the calculations of Betti numbers. Another option is to take $\lambda_1 = -e_1$ and $\lambda_5 = e_5$. The calculations for this choice are analogous to the ones above.

9. Ring structure of Chen–Ruan cohomology

We will follow [4] and define the structure of an associative ring on Chen–Ruan cohomology of an omnioriented quasitoric orbifold.

The normal bundle of a characteristic suborbifold has an almost complex structure determined by the omniorientation. More generally suppose $F = \bigcap_{i=1}^k F_i$ is an arbitrary face of P. The normal bundle of the suborbifold $\mathbf{S}(F, g)$, see Section 7.1, decomposes into the direct sum of complex orbifold line bundles L_i which are restrictions of the normal bundles corresponding to facets F_i that contain F. Each of these line bundles L_i have a Thom form θ_i. (Note that the Thom forms of $\mathbf{X}(F)$ and $\mathbf{S}(F, g)$ in $\mathbf{X}$ may differ at most by a constant factor.) For any $g = \sum_{0\le i\le k} a_i\lambda_i \in Box_F^\circ$ define the formal form (twist factor)

$$t(g) = \prod_{1\le i\le k} \theta_i^{a_i}. \tag{9.1}$$

The order of the θ_is in the above product is not important. The degree of $t(g)$ is defined to be $2\iota(g)$. For any invariant form ω on $\mathbf{S}(F, g)$ define a corresponding twisted form $\omega t(g)$. Define the degree of $\omega t(g)$ to be the sum of the degrees of ω and $t(g)$. Define

$$\Omega^p_{\mathrm{CR}}(F, g) = \{\omega t(g)\colon \omega \in \Omega^*(\mathbf{S}(F, g)),\ \deg(\omega t(g)) = p\}. \tag{9.2}$$

Define the de Rham complex of twisted forms by

$$\Omega^p_{\mathrm{CR}} = \bigoplus_{F\le P, g\in Box_F^\circ} \Omega^p_{\mathrm{CR}}(F, g) \tag{9.3}$$

with differential

$$d\Big(\sum \omega_i t(g_i)\Big) = \sum d(\omega_i)t(g_i). \tag{9.4}$$

It is easy to see that the cohomology of this complex coincides with the Chen–Ruan cohomology defined in Section 7.

Now we define a product $\star\colon \Omega^{p_1}_{\mathrm{CR}}(K_1, g_1) \times \Omega^{p_2}_{\mathrm{CR}}(K_2, g_2) \to \Omega^{p_1+p_2}_{\mathrm{CR}}(K, g_1g_2)$ of twisted forms as follows,

$$\omega_1 t(g_1) \star \omega_2 t(g_2) = i_1^*\omega_1 \wedge i_2^*\omega_2 \wedge \Theta(g_1, g_2)t(g_1g_2). \tag{9.5}$$

Here K is the unique face such that $(K_1 \cap K_2) \le K$ and $g_1g_2 \in G_K^\circ$. The map i_j is the inclusion of $\mathbf{X}(K_1 \cap K_2)$ in $\mathbf{X}(K_j)$. The form $\Theta(g_1, g_2)$ is obtained as follows.

Consider the product $t(g_1)t(g_2)$. We can think of the g_js as elements of Box_v where v is a vertex of $K_1 \cap K_2$. Write $g_j = \sum_{i=1}^n a_{ij}\lambda_i$. Write the twist factor $t(g_j)$ as $\prod_{1\le i\le n} \theta_i^{a_{ij}}$. A term in the product $t(g_1)t(g_2)$ looks $\theta_i^{a_{i1}+a_{i2}}$. We may ignore the i's for which both a_{i1} and a_{i2} are zero. Then there can be three cases:

(1) $a_{i1} + a_{i2} < 1$. Then $\theta_i^{a_{i1}+a_{i2}}$ contributes to $t(g_1g_2)$.

(2) $a_{i1} + a_{i2} > 1$. Then fractional part $\theta_i^{a_{i1}+a_{i2}-1}$ contributes to $t(g_1g_2)$ and the integral part is the Thom form θ_i which contributes as an invariant 2-form to $\Theta(g_1, g_2)$.

(3) $a_{i1} + a_{i2} = 1$. When this happens $g_1g_2 \in Box_K^\circ$ where $(K_1 \cap K_2) < K$ and θ_i contributes to $\Theta(g_1, g_2)$.

If case (3) does not occur for any i, then $K = K_1 \cap K_2$ and $i_1^*\omega_1 \wedge i_2^*\omega_2 \wedge \Theta(g_1, g_2)$ restricts to $\mathbf{S}(K, g_1g_2)$ without problem. If case (3) occurs for some i's then the product of the restrictions of corresponding θ_is to $\mathbf{X}(K)$ is, up to a constant factor, the Thom form of the normal bundle of $\mathbf{X}(K_1 \cap K_2)$ in $\mathbf{X}(K)$. The wedge of this Thom form with $i_1^*\omega_1 \wedge i_2^*\omega_2$ and the restriction of the contributions from case (2) to $\mathbf{X}(K)$ defines a form on $\mathbf{X}(K)$. Thus the star product is well-defined.

We extend the star product to a product on $\Omega^*{}_{\mathrm{CR}}$ by bilinearity. The differential acts on the star product as follows,

$$d(\omega_1 t(g_1) \star \omega_2 t(g_2)) = d(\omega_1 t(g_1)) \star \omega_2 t(g_2) + (-1)^{\deg(\omega_1)+\deg(\omega_2)} \omega_1 t(g_1) \star d(\omega_2 t(g_2)). \tag{9.6}$$

Hence the star product induces a product on the Chen–Ruan cohomology.

Observe that the form $i_1^*\omega_1 \wedge i_2^*\omega_2 \wedge \Theta(g_1, g_2)$ is supported in a small neighborhood of $X(K_1 \cap K_2)$. Therefore the star product of three forms $\omega_i t(g_i) \in \Omega_{\mathrm{CR}}^{p_i}(K_i, g_i)$, $1 \leq i \leq 3$, is nonzero only if $K_1 \cap K_2 \cap K_3$ is nonempty. Now it is fairly straightforward to check that the star product is associative.

ACKNOWLEDGEMENT. We thank Yongbin Ruan for an useful suggestion. The second author thanks Cheol-Hyun Cho, Shintaro Kuroki and Bernardo Uribe for helpful and interesting discussions. We both thank la Universidad de los Andes for providing financial support for our research related activities.

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