Title	ON BLOCKS OF NORMAL SUBGROUPS OF FINITE GROUPS
Author(s)	Murai, Masafumi
Citation	Osaka Journal of Mathematics. 50(4) P.1007-P.1020
Issue Date	2013-12
Text Version	publisher
URL	https://doi.org/10.18910/26820
DOI	10.18910/26820
rights	
Note	

Murai, M.
Osaka J. Math.
50 (2013), 1007–1020

ON BLOCKS OF NORMAL SUBGROUPS OF FINITE GROUPS

MASAFUMI MURAI

(Received November 16, 2011, revised March 19, 2012)

Abstract

For a block b of a normal subgroup of a finite group G, E.C. Dade has defined a subgroup $G[b]$ of G. We give a character-theoretical interpretation of his result on $G[b]$. In the course of proofs we determine a defect group of a block of $G[b]$ covering b. We also consider character-theoretical characterizations of isomorphic blocks with respect to normal subgroups.

Introduction

Let G be a finite group and p a prime. Let $(\mathcal{K}, R, k)$ be a p-modular system. We assume that $\mathcal{K}$ is sufficiently large for G. In this paper a block of G means a block ideal of RG. For a normal subgroup K of G and a block b of K, Dade [3] has defined a normal subgroup $G[b]$ of the inertial group of b in G such that $G[b] \geq K$. More precisely put $C = C_{RG}(K)$. We have $C = \bigoplus_{\bar{x} \in \bar{G}} C_{\bar{x}}$, where $\bar{G} = G/K$ and $C_{\bar{x}} = C \cap RKx$. Let e_b be the block idempotent of b. The subgroup $G[b]$ is defined by

$$G[b] = \{x \in G \mid (e_b C_{\bar{x}})(e_b C_{\bar{x}^{-1}}) = e_b C_{\bar{1}}\}.$$

(Strictly speaking, Dade defines a subgroup $(G/K)[b]$ of G/K. The subgroup $G[b]$ is the preimage of $(G/K)[b]$ in G.) In [3, Corollary 12.6] Dade has determined $G[b]$ in terms of $C_G(Q)$ and a root of b in $C_K(Q)$, where Q is a defect group of b. In Section 3 we shall give a character-theoretical characterization of elements of $G[b]$ and give a character-theoretical interpretation of Dade's result above. In the course of proofs we determine a defect group of a block of $G[b]$ covering b, which is a refinement of a result in [9]. In Section 1, we shall consider weakly regular and regular blocks with respect to normal subgroups. In Section 4, character-theoretical characterizations of isomorphic blocks with respect to normal subgroups which involve $G[b]$ will be obtained. More applications of $G[b]$ will be given in a separate paper [11].

Notation

Let B be a block of G. The block idempotent of B will be denoted by e_B. For an irreducible character χ in B, put $\omega_B(z) = \omega_\chi(z)$, $z \in Z(RG)$. For a subset S of

2010 Mathematics Subject Classification. 20C20.

G, let $\hat{S} = \sum_{s \in S} s \in RG$. For $x \in G$, let K_x be the conjugacy class of G containing x, and so $\hat{K}_x$ is the class sum of K_x. Let $D(K_x)$ be a defect group of K_x. Let $e_B = \sum_y a_B(K_y)\hat{K}_y$, where y runs through a set of representatives of conjugacy classes of G.

Let $B_0(G)$ be the principal block of G. Let $\mathrm{Irr}(B)$ be the set of irreducible characters in B. Let $\mathrm{Irr}_0(B)$ be the set of irreducible characters of height 0 in B. $\mathrm{d}(B)$ is the defect of B. For a block b of a normal subgroup K of G, let G_b be the inertial group of b in G and let $\mathrm{BL}(G \mid b)$ be the set of blocks of G covering b. For an irreducible character ξ of K and a block B of G, let $\mathrm{Irr}(B \mid \xi)$ be the set of irreducible characters in B lying over ξ. Put

$$\mathrm{Irr}_0(B \mid \xi) = \{\chi \in \mathrm{Irr}(B \mid \xi) \mid \mathrm{ht}(\chi) = \mathrm{ht}(\xi)\},$$

where $\mathrm{ht}(\chi)$ is the height of χ. Let ${}^*\colon R \to k$ be the natural map. For a function $\varphi\colon S \to R$ defined on a set S, the function $\varphi^*\colon S \to k$ is defined by $\varphi^*(s) = \varphi(s)^*$, $s \in S$. Let ν be the valuation of $\mathcal{K}$ normalized so that $\nu(p) = 1$.

1. Weakly regular and regular blocks with respect to normal subgroups

In this section we strengthen Theorem 2.1 of [9].

Proposition 1.1. *Let N be a normal subgroup of G. Let b be a block of N covered by a block B of G. Let D be a defect group of B. The following conditions are equivalent.*

(i) *B is a unique weakly regular block of G covering b.*

(ii) *For a block $\hat{b}$ of DN, we have $\hat{b}^G = B$.*

(iii) *For any p'-element x of G satisfying $\omega_B^*(\hat{K}_x) \neq 0$, we have $x \in N$.*

(iv) *For any p'-element x of G satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in N$.*

(v) *For any $x \in G$ satisfying $a_B(K_x)^* \neq 0$ and $D(K_x) =_G D$, we have $x \in N$.*

Proof. (i) $\Rightarrow$ (ii). By replacing D by a conjugate, we may assume D is a defect group of the Fong–Reynolds correspondent of B over b in the inertial group of b in G. Let $\hat{b}$ be a unique block of DN covering b. Then $\hat{b}^G = B$, see the proof of Theorem 2.1 of [9].

(ii) $\Rightarrow$ (iii). This is easy to see.

(iii) $\Rightarrow$ (iv). This is trivial.

(iv) $\Rightarrow$ (v). Since $a_B(K_x)^* \neq 0$, x is a p'-element. Let $\{\varphi_i\}$ be the set of irreducible Brauer characters in B. Let Φ_i be the principal indecomposable character corresponding to φ_i. Let $\{\chi_j\}$ be the set of irreducible characters in B. Put

$$\varphi_i = \sum_j n_{ij}\chi_j \quad \text{(on the set of } p'\text{-elements of } G),$$

where n_{ij} are integers. Then

$$\begin{aligned} a_B(K_x) &= \frac{1}{|G|}\sum_i \Phi_i(1)\varphi_i(x^{-1}) \\ &= \sum_i \frac{\Phi_i(1)}{|G|}\sum_j n_{ij}\omega_{\chi_j}(\hat{K}_{x^{-1}})\frac{\chi_j(1)}{|K_x|}. \end{aligned}$$

Since $\Phi_i(1)/|G|$ and $\chi_j(1)/|K_x|$ lie in R for any i and j, we obtain

$$a_B(K_x) \equiv \omega_B(\hat{K}_{x^{-1}})\sum_i \frac{\Phi_i(1)\varphi_i(1)}{|G|\,|K_x|} \quad \mathrm{mod}\ \mathrm{J}(R).$$

Since $\Phi_i(1)\varphi_i(1)/(|G|\,|K_x|)$ lies in R for any i, $a_B(K_x)^* \neq 0$ implies $\omega_B^*(\hat{K}_{x^{-1}}) \neq 0$. Hence $x \in N$ by (iv).

(v) $\Rightarrow$ (i). Let K_s be a defect class for B ([12, p. 311]). Then $K_s \subset N$ by (v). Since $\omega_B^*(\hat{K}_s) \neq 0$ and $D(K_s) =_G D$, B is weakly regular with respect to N by definition ([12, p. 344]). Let B_1 be any weakly regular block of G covering b. Put $e_B = s_N(e_B)+a$, where $s_N(e_B) = \sum_{K_y\subset N} a_B(K_y)\hat{K}_y$. We claim $\omega_{B_1}^*(a) = 0$. Assume this were false. Then there would be an element $x \notin N$ such that $a_B(K_x)^*\omega_{B_1}^*(\hat{K}_x) \neq 0$. Since $a_B(K_x)^* \neq 0$, $D(K_x) \leq_G D$. Since $\omega_{B_1}^*(\hat{K}_x) \neq 0$, $D(K_x) \geq_G D_1$, where D_1 is a defect group of B_1. By Fong's theorem $D =_G D_1$. Thus $D(K_x) =_G D$. So $x \in N$ by (v), a contradiction, and the claim follows. Now $\omega_{B_1}^*(e_B) = \omega_{B_1}^*(s_N(e_B)) = \omega_b^*(s_N(e_B))$ by [12, Theorem 5.5.5]. Since B is weakly regular, $\omega_b^*(s_N(e_B)) = \omega_b^*(s_N(e_B)e_b) \neq 0$ by [9, Theorem 1.10]. Thus $\omega_{B_1}^*(e_B) \neq 0$. Hence $B_1 = B$ and (i) follows. The proof is complete. $\square$

REMARK 1.2. The equivalence of (i) and (ii) is proved in [4, Theorem 2.4].

Theorem 1.3. *Let N be a normal subgroup of G. Let b be a block of N covered by a block B of G. Let D be a defect group of B. The following conditions are equivalent.*

(i) *B is a unique weakly regular block of G covering b and $Z(D) \leq N$.*

(ii) *$B = b^G$.*

(iii) *For any $x \in G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in N$.*

(iv) (iv a) *For any p'-element x of G satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in N$, and*

(iv b) *$Z(D) \leq N$.*

(v) (v a) *For any $x \in G$ satisfying $a_B(K_x)^* \neq 0$ and $D(K_x) =_G D$, we have $x \in N$, and*

(v b) *$Z(D) \leq N$.*

Proof. (i) $\Leftrightarrow$ (ii). This is Theorem 2.1 of [9].

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (iv). (iv a) is trivial. Let K_s be a defect class for B ([12, p. 311]). So s is a p'-element. We may assume D is a Sylow p-subgroup of $C_G(s)$. Let $u \in Z(D)$. Then, as in [13, Lemma 5.15], $D(K_{us}) =_G D$ and $\omega_B^*(\hat{K}_{us}) \neq 0$. Then $us \in N$ by (iii). So $u \in N$, and $Z(D) \leq N$.

(iv) $\Rightarrow$ (v). This follows from (iv) $\Rightarrow$ (v) of Proposition 1.1.

(v) $\Rightarrow$ (i). This follows from (v) $\Rightarrow$ (i) of Proposition 1.1. □

REMARK 1.4. The equivalences of (i), (ii), (v) have been proved in Fan [4, Theorem 2.3] in a different way.

2. A lemma on $G[b]$

In the rest of this paper, K is a normal subgroup of a group G, and b is a block of K with a defect group Q. The following lemma is certainly well-known. We give a proof for completeness sake. We shall use this lemma without explicit reference.

Lemma 2.1. *Let x be an element of G. The following are equivalent.*
(i) $x \in G[b]$; *that is,* $(e_bC_{\bar{x}})(e_bC_{\bar{x}^{-1}}) = e_bC_{\bar{1}}$.
(ii) $e_bC_{\bar{x}}$ *contains a unit of* e_bC.
(iii) ([6, *p.* 210]) $x \in G_b$ *and* x *induces an inner automorphism of* b.

Proof. (i) $\Rightarrow$ (ii). This follows from [15, p. 551, ll. 5–7][1].

(ii) $\Rightarrow$ (iii). This follows from [3, Proposition 2.17] and [15, p. 551, ll. 7–9][2].

(iii) $\Rightarrow$ (i). Let u be a unit of b such that $v^x = v^u$ for all $v \in b$. We claim $ux^{-1} \in e_bC_{\bar{x}^{-1}}$. Indeed, $(ux^{-1})v = v(ux^{-1})$ for all $v \in b$. Let b' be any block of K with $b' \neq b$. Let $v' \in b'$. Then $(ux^{-1})v' = uv'^x x^{-1} = 0 = v'(ux^{-1})$. So $ux^{-1} \in C$. Then the claim follows. Let u' be an element of b such that $uu' = u'u = e_b$. Then we obtain similarly that $xu' \in e_bC_{\bar{x}}$. We have $(xu')(ux^{-1}) = e_b$. So $(e_bC_{\bar{x}})(e_bC_{\bar{x}^{-1}}) \ni e_b$, which implies $(e_bC_{\bar{x}})(e_bC_{\bar{x}^{-1}}) = e_bC_{\bar{1}}$. The proof is complete. □

REMARK 2.2. See Hida–Koshitani [5, Lemma 3.2] for a module-theoretical reformulation of the definition of $G[b]$.

3. The subgroup $G[b]$

Navarro [14] has obtained a relative version of a well-known theorem of Burnside as follows (letting $K = 1$, we recover the original theorem of Burnside):

Lemma 3.1 (Navarro [14, Theorem A]). *Let χ be an irreducible character of G. The following are equivalent.*

[1]Note that $e_bC_{\bar{1}} = Z(b)$ is a local R-algebra.
[2]In l. 9 $\mathfrak{O}G$ should be $e\mathfrak{O}G$.

(i) χ_K *is irreducible.*
(ii) *For any* $x \in G$, *there is an element* y *in* xK *such that* $\chi(y) \neq 0$.

Proposition 3.2. *Assume that* G/K *is abelian. Let* B *be a block of* G *covering* b. *The following are equivalent.*
(i) $G = G[b]$ *and for any irreducible character* χ *in* B, χ_K *is irreducible.*
(ii) *For any* $x \in G$, *there is an element* y *in* xK *such that* $\omega_B^*(\hat{K}_y) \neq 0$.

Proof. In both cases, the following holds:
($*$) For any irreducible character χ in B, χ_K is irreducible.
Indeed, if (i) holds, trivially ($*$) holds. Assume (ii) holds. Let χ be an irreducible character in B. Since $\omega_B^*(\hat{K}_y) \neq 0$, we have $\chi(y) \neq 0$. Then, by Lemma 3.1, χ_K is irreducible.

Let $\{B_i\}$ be the set of blocks of G covering b. We show that ($*$) implies the following:
($**$) For any irreducible character χ in B_i for any i, χ_K is an irreducible character in b.
Indeed, let $\xi \in \mathrm{Irr}(b)$ be an irreducible constituent of χ_K. Let ζ be an irreducible character in B lying over ξ. By ($*$), $\zeta_K = \xi$. Hence $\chi = \zeta \otimes \theta$ for some $\theta \in \mathrm{Irr}(G/K)$. Since G/K is abelian, we have $\chi_K = \xi$. Hence ($**$) holds. Thus for the proof of proposition we may assume ($**$) holds.

Recall that $C = C_{RG}(K)$. We claim the following:
($***$) $e_b C = Z(Gb) = \bigoplus_i Z(B_i)$,
where $Gb = RGe_b$. By ($**$), b is G-invariant. This yields the second equality. We prove the first equality. Clearly $Z(Gb) \subseteq e_b C$. To prove the reverse containment, let $a \in e_b C$ and $v \in \mathcal{K}Gb$, where $\mathcal{K}Gb = \mathcal{K}Ge_b$. Let T be any irreducible matrix representation of $\mathcal{K}Gb$. By ($**$), restriction of T to $\mathcal{K}b$ is irreducible, where $\mathcal{K}b = \mathcal{K}Ke_b$. Since $e_b C \subseteq \mathcal{K}Gb \cap C(\mathcal{K}b)$, $T(a)$ is a scalar matrix by Schur's lemma. So $T(av - va) = 0$. It follows that $av - va = 0$, since $\mathcal{K}Gb$ is semi-simple. Therefore, $e_b C \subseteq Z(\mathcal{K}Gb) \cap RG = Z(Gb)$. ($***$) is proved.

(i) $\Rightarrow$ (ii). Let $x \in G$. By (i), there exists a unit u of $e_b C$ in $e_b C_{\bar{x}}$. Then, by ($***$), $\omega_B^*(u) \neq 0$. Since $u \in Z(RG)$ by ($***$) and $u \in RKx$, u is an R-linear combination of $\hat{K}_z$ for $z \in xK$. Thus there is some $y \in xK$ such that $\omega_B^*(\hat{K}_y) \neq 0$. Thus (ii) follows.

(ii) $\Rightarrow$ (i). The latter part follows from ($**$). Let ξ be an irreducible character in b. Then, by ($**$), any irreducible character of G lying over ξ is an extension of ξ. Therefore for any i, there is a linear character $\lambda_i\colon G/K \to k^*$, where k^* is the multiplicative group of k, such that $\omega_{B_i}^*(\hat{K}_g) = \omega_B^*(\hat{K}_g)\lambda_i(gK)$ for any $g \in G$. Let $x \in G$ and let y be as in (ii). Then $\omega_{B_i}^*(e_b\hat{K}_y) = \omega_{B_i}^*(\hat{K}_y) = \omega_B^*(\hat{K}_y)\lambda_i(yK) \neq 0$. Therefore, by ($***$), $e_b\hat{K}_y$ is a unit of $e_b C$. Since G/K is abelian, $e_b\hat{K}_y$ lies in $e_b C_{\bar{x}}$. Thus we obtain $G = G[b]$. The proof is complete. □

The following corollary will be used repeatedly.

Corollary 3.3. *Assume that G/K is cyclic, and let $G = \langle x, K\rangle$ for an element $x \in G$. Let B be a block of G covering b. The following are equivalent.*
(i) $x \in G[b]$; *that is*, $G = G[b]$.
(ii) *There exists an element y in xK such that $\omega_B^*(\hat{K}_y) \neq 0$.*

Proof. (i) $\Rightarrow$ (ii). G induces inner automorphisms of b, so any irreducible character in b is G-invariant. Then, since G/K is cyclic, any irreducible character in B restricts irreducibly to K. Thus (ii) holds by Proposition 3.2.

(ii) $\Rightarrow$ (i). For any positive integer i, $\omega_B^*((\hat{K}_y)^i) \neq 0$. Since $y \in xK$, $(\hat{K}_y)^i$ is an integral combination of $\hat{K}_z$ with $z \in x^i K$. So $\omega_B^*(\hat{K}_z) \neq 0$ for some $z \in x^i K$. Thus (i) holds by Proposition 3.2. The proof is complete. □

Proposition 3.4. *Assume that G/K is a cyclic p-group. Let b be G-invariant. Let B be a unique block of G covering b. The following are equivalent.*
(i) $G = G[b]$.
(ii) *For any defect group S of B with $S \geq Q$, $S = Z(S)Q$.*
(ii)′ *For some defect group S of B, $S = Z(S)Q$.*
(iii) *For any defect group S of B with $S \geq Q$, $S = C_S(Q)Q$; that is, S induces inner automorphisms of Q.*
(iii)′ *For some defect group S of B, $S = C_S(Q)Q$.*

Proof. The assertion is trivial if $G = K$. So we assume $G \neq K$. Put $G = \langle x, K\rangle$. Let β be a block of $\langle x^p, K\rangle$ covered by B.

(i) $\Rightarrow$ (ii). Assume $S \neq Z(S)Q$. Since b is G-invariant, $G = SK$. So $S/Q \simeq G/K$ is cyclic. Therefore $Z(S) \leq \langle x^p, K\rangle$. Then $B = \beta^G$ by Theorem 1.3. Thus $\omega_B^*(K_y) = 0$ for all $y \in xK$. Then $x \notin G[b]$ by Corollary 3.3, a contradiction.

(ii) $\Rightarrow$ (i). Assume $x \notin G[b]$. Then $x^i \notin G[b]$ for any p'-integer i. Thus $\omega_B^*(\hat{K}_y) = 0$ for any $y \in G - \langle x^p, K\rangle$ by Corollary 3.3. Hence $B = \beta^G$. Then $Z(S) \leq \langle x^p, K\rangle$ by Theorem 1.3. Since b is G-invariant, $G = SK$. Therefore $G = SK = Z(S)QK \leq \langle x^p, K\rangle < G$, a contradiction. Thus $x \in G[b]$, and $G = G[b]$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (ii). Since b is G-invariant, $G = SK$. So $G/K \simeq S/Q \simeq C_S(Q)/Z(Q)$ is cyclic. Hence $C_S(Q)$ is abelian, and $C_S(Q) \leq Z(S)$. Thus $S = Z(S)Q$.

(iii) $\Rightarrow$ (iii)′. Trivial.

(iii)′ $\Rightarrow$ (iii). Let U be any defect group of B with $U \geq Q$. We have $U = S^g$ for some $g \in G$. Then $Q = U \cap K = S^g \cap K = (S \cap K)^g = Q^g$. So $Q = Q^g$. Then $C_U(Q)Q = C_{S^g}(Q^g)Q^g = S^g = U$.

(ii) $\Leftrightarrow$ (ii)′. This is proved similarly.

This completes the proof. □

Theorem 3.5. *Let b be G-invariant. Let B be a block of G covering b. We choose a block B' of $G[b]$ so that B covers B' (and B' covers b). Let D, S be defect groups of B, B', respectively, such that $Q \leq S \leq D$. The following holds.*
(i) $B = B'^G$. *In particular, B is a unique block of G that covers B'.*
(ii) $S = QC_D(Q)$.

Proof. We first note that $G[b] \lhd G$, so the statement makes sense.

(i) We show $B = B'^G$. By Theorem 1.3, it suffices to show the following:
($*$) For any $x \in G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in G[b]$.
We may assume D is a Sylow p-subgroup of $C_G(x)$. Let χ be an irreducible character of height 0 in B. Put $\chi_{\langle x,K\rangle} = \sum_i n_i \zeta_i$, where ζ_i are distinct irreducible characters of $\langle x, K\rangle$ and n_i are positive integers. Then

$$\omega_\chi(\hat{K}_x) = \sum_i n_i \omega_{\zeta_i}(\hat{L}_x) \frac{\zeta_i(1)|G|\,|C_K(x)|}{\chi(1)|K|\,|C_G(x)|},$$

where L_x is the conjugacy class of $\langle x, K\rangle$ containing x. For any i, let b_i be the block of $\langle x, K\rangle$ containing ζ_i. Then b_i covers b. We claim $\mathrm{d}(b_i) - \mathrm{d}(b) = \nu(|\langle x, K\rangle|) - \nu(|K|)$. Indeed, let H/K be a (normal) Sylow p-subgroup of $\langle x, K\rangle/K$. Let $\hat{b}$ be a unique block of H covering b. Then, since b_i covers $\hat{b}$, $\mathrm{d}(b_i) = \mathrm{d}(\hat{b})$. Furthermore, $\mathrm{d}(\hat{b}) - \mathrm{d}(b) = \nu(|H|) - \nu(|K|)$. Thus the claim follows. On the other hand, since D is a Sylow p-subgroup of $C_G(x)$, $D \cap K$ is a Sylow p-subgroup of $C_K(x)$. Furthermore $D \cap K$ is a defect group of b. Thus

$$\begin{aligned}\nu\left(\frac{\zeta_i(1)|G|\,|C_K(x)|}{\chi(1)|K|\,|C_G(x)|}\right) &= \nu(|\langle x, K\rangle|) - \mathrm{d}(b_i) + \mathrm{ht}(\zeta_i) + \nu(|G|) + \nu(|C_K(x)|)\\ &\quad - \{\nu(|G|) - \mathrm{d}(B) + \nu(|K|) + \nu(|C_G(x)|)\}\\ &= \nu(|\langle x, K\rangle|) - \nu(|K|) - \mathrm{d}(b_i) + \mathrm{d}(b) + \mathrm{ht}(\zeta_i)\\ &= \mathrm{ht}(\zeta_i) \geq 0.\end{aligned}$$

Since $\omega_\chi^*(\hat{K}_x) \neq 0$, there exists i such that $\omega_{\zeta_i}^*(\hat{L}_x) \neq 0$. Then $x \in \langle x, K\rangle[b]$ by Corollary 3.3, and $x \in G[b]$. Thus ($*$) follows and $B = B'^G$.

If B_1 is another block of G covering B', then similarly $B_1 = B'^G$. So $B_1 = B$.

(ii) Since $Q = D \cap K$, Q is a normal subgroup of D. Put

$$I = \{u \in D \mid u \text{ induces an inner automorphism of } Q\}.$$

Clearly $I = QC_D(Q)$, so it suffices to show $I = S$. For any $u \in D$, put $Q_u = \langle u, Q\rangle$. If b_u is a unique block of Q_uK covering b, then Q_u is a defect group of b_u, cf. Lemma 4.13 of [9].

Let $u \in I$. Then Q_u induces inner automorphisms of Q. Since $Q_uK = \langle u, K\rangle$, $Q_uK = (Q_uK)[b] \leq G[b]$ by Proposition 3.4. So $u \in G[b]$, and $I \leq G[b] \cap D = S$.

Conversely let $u \in S$. Then, since $u \in G[b]$ and $Q_u K = \langle u, K\rangle$, we have $Q_u K = (Q_u K)[b]$. Thus Q_u induces inner automorphisms of Q by Proposition 3.4. So $u \in I$, and $S \leq I$. Thus $I = S$. The proof is complete. □

REMARK 3.6. (1) Theorem 3.5 sharpens Lemma 4.14 of [9].
(2) Theorem 3.5 (i) is implicit in [3]. It follows from Lemma 3.3 and Proposition 1.9 of [3].
(3) Proposition 3.1 of [1] follows immediately from Theorem 3.5 (ii). (The assumption made there that c is nilpotent is unnecessary.)

The following extends Proposition 3.4.

Corollary 3.7. *Assume that G/K is a p-group. Let B be a unique block of G covering b. Let D be a defect group of B such that $D \geq Q$. Then the following are equivalent.*
(i) $G = G[b]$.
(ii) *b is G-invariant and $D = QC_D(Q)$.*
In particular, if D is abelian and b is G-invariant, then $G = G[b]$.

Proof. (i) ⇒ (ii). This follows from Theorem 3.5.

(ii) ⇒ (i). Let B' be a block of $G[b]$ such that B covers B' and that $S := D \cap G[b]$ is a defect group of B'. Then B' covers b. Since b is G-invariant, $G = DK$ and $G[b] = SK$. By Theorem 3.5, $S = QC_D(Q) = D$. Therefore $G = G[b]$. □

REMARK 3.8. The last statement of Corollary 3.7 is implicit in the proof of Theorem of [7].

Proposition 3.9. *Assume that G/K is a cyclic p'-group. The following are equivalent.*
(i) $G = G[b]$.
(ii) $|\mathrm{BL}(G \mid b)| = |G/K|$.

Proof. (i) ⇒ (ii). Put $G = \langle x, K\rangle$. Let B be a block of G covering b. By Corollary 3.3, there exists some y in xK such that $\omega_B^*(K_y) \neq 0$. Let χ be an irreducible character in B. Let λ be any linear character of G/K. Assume that $\chi \otimes \lambda$ lies in B. Then $\omega^*_{\chi\otimes\lambda}(\hat{K}_y) = \omega^*_\chi(\hat{K}_y)$, which implies $\lambda^*(y) = 1$. Since G/K is a p'-group, we see that λ is a trivial character. Therefore we obtain $|\mathrm{BL}(G \mid b)| \geq |G/K|$. To prove the reverse inequality, let $\xi \in \mathrm{Irr}(b)$. Let m be the number of irreducible characters of G lying over ξ. Any block of G covering b contains an irreducible character lying over ξ, so $|\mathrm{BL}(G \mid b)| \leq m$. On the other hand, $m \leq (\xi^G, \xi^G)_G = ((\xi^G)_K, \xi)_K \leq |G/K|$. Thus $|\mathrm{BL}(G \mid b)| \leq |G/K|$, and (ii) follows.

(ii) $\Rightarrow$ (i). We claim that any block B in $\mathrm{BL}(G \mid b)$ is induced from a block in $\mathrm{BL}(G[b] \mid b)$. To see this, let $\tilde{B}$ be the Fong–Reynolds correspondent of B in G_b. Choose a block B' of $G[b]$ such that $\tilde{B}$ covers B' and B' covers b. Then $\tilde{B} = B'^{G_b}$ by Theorem 3.5. So $B = \tilde{B}^G = (B'^{G_b})^G = B'^G$. Thus the claim is proved. Then $|\mathrm{BL}(G[b] \mid b)| \geq |\mathrm{BL}(G \mid b)|$. Since $|\mathrm{BL}(G[b] \mid b)| \leq |G[b]/K|$ (as above), it follows that $|G/K| \leq |G[b]/K|$. Thus $G = G[b]$. The proof is complete. □

REMARK 3.10. Application of Theorem 3.7 of [3] would shorten the proof of Proposition 3.9.

The following gives a necessary and sufficient condition for G to coincide with $G[b]$ when G/K is an arbitrary group.

Theorem 3.11. *Let B_w be a weakly regular block of G covering b. Let D_w be a defect group of B_w such that $D_w \geq Q$. The following are equivalent.*

(i) $G = G[b]$.

(ii) (ii a) *b is G-invariant*;

(ii b) *For any subgroup L of G such that $L \geq K$ and that L/K is a cyclic p'-group, it holds that* $|\mathrm{BL}(L \mid b)| = |L/K|$; *and*

(ii c) $D_w = QC_{D_w}(Q)$.

Proof. (i) $\Rightarrow$ (ii). This follows from Proposition 3.9 and Theorem 3.5.

(ii) $\Rightarrow$ (i). Let x be a p'-element of G and put $H = \langle x, K\rangle$. By (ii b) and Proposition 3.9, $x \in H = H[b]$. So $x \in G[b]$. Let x be a p-element of G. By (ii a) and Fong's theorem D_wK/K is a Sylow p-subgroup of G/K. So $x^g \in D_wK$ for some $g \in G$. By (ii a) and [9, Lemma 2.2], D_w is a defect group of a unique block of D_wK covering b. So by (ii c) and Corollary 3.7, $(D_wK)[b] = D_wK$. Thus $x^g \in G[b]$. Since $G[b] \lhd G$ by (ii a), $x \in G[b]$. Hence $G = G[b]$. □

We introduce some notation. Let $\tilde{b}$ be the Brauer correspondent of b in $N_K(Q)$ and let β be a block of $QC_K(Q)$ covered by $\tilde{b}$. Put $L_0 = QC_K(Q)$. Let β_0 be a block of $C_K(Q)$ covered by β. Let θ be the canonical character of β and let φ be the restriction of θ to $C_K(Q)$. So φ is the canonical character of β_0. Let $S = N_G(Q)_\beta$ and $T = N_K(Q)_\beta$. So T is the inertial group of β_0 in $N_K(Q)$. Put $L = QC_G(Q)$ and $C = C_G(Q)$.

Noting that T and L_β are normal subgroups of S, we have $[T, L_\beta] \leq L_\beta \cap T = L_0$. So we can define (after Isaacs [6, Section 2]) $\langle\langle t, x\rangle\rangle_\theta \in \mathcal{K}^*$ for $(t, x) \in T \times L_\beta$, where $\mathcal{K}^*$ is the multiplicative group of $\mathcal{K}$. The definition is as follows: let $x \in L_\beta$ and let $\hat{\theta}$ be an extension of θ to $\langle x, L_0\rangle$. Let $t \in T$. Then, since $\hat{\theta}^t$ is also an extension of θ to $\langle x, L_0\rangle$, there exists a unique linear character λ_t of $\langle x, L_0\rangle/L_0$ such that $\hat{\theta}^t = \hat{\theta} \otimes \lambda_t$. Then put $\langle\langle t, x\rangle\rangle_\theta = \lambda_t(x)$. This definition is independent of the choice of $\hat{\theta}$. It is bilinear in the sense that $\langle\langle ts, x\rangle\rangle_\theta = \langle\langle t, x\rangle\rangle_\theta \langle\langle s, x\rangle\rangle_\theta$ for $t, s \in T$ and $x \in L_\beta$

and $\langle\langle t, xy\rangle\rangle_\theta = \langle\langle t, x\rangle\rangle_\theta\langle\langle t, y\rangle\rangle_\theta$ for $t \in T$ and $x, y \in L_\beta$, see [6, Lemma 2.1 and Theorem 2.3]. Similarly we can define $\langle\langle t, x\rangle\rangle_\varphi \in \mathcal{K}^*$ for $(t, x) \in T \times C_{\beta_0}$. It is also bilinear. Define

$$L_\omega = \{x \in L_\beta \mid \langle\langle t, x\rangle\rangle_\theta = 1 \text{ for all } t \in T\},$$
$$C_\omega = \{x \in C_{\beta_0} \mid \langle\langle t, x\rangle\rangle_\varphi = 1 \text{ for all } t \in T\}.$$

By definition, we see that for $x \in L_\beta$, the condition that $x \in L_\omega$ is equivalent to the condition that any (equivalently, some) extension of θ to $\langle x, L_0\rangle$ is T-invariant.

Lemma 3.12. (i) *L_ω is a normal subgroup of L_β such that L_β/L_ω is a p'-group.* (ii) *$L_\omega K = C_\omega K$.*

Proof. (i) Put $\alpha_x(t) = \langle\langle t, x\rangle\rangle_\theta$ for $(t, x) \in T \times L_\beta$. Since $\alpha_x(t) = 1$ for $t \in L_0$, α_x may be regarded as an element of $\mathrm{Hom}(T/L_0, \mathcal{K}^*)$. Then the map α sending x to α_x is a group homomorphism from L_β to $\mathrm{Hom}(T/L_0, \mathcal{K}^*)$. Since $\mathrm{Ker}\,\alpha = L_\omega$ and T/L_0 is a p'-group, the result follows.

(ii) We have $L_\beta = C_{\beta_0}L_0$. So $L_\omega = (L_\omega \cap C_{\beta_0})L_0$. It is easy to see $\langle\langle t, x\rangle\rangle_\varphi = \langle\langle t, x\rangle\rangle_\theta$ for $t \in T$ and $x \in C_{\beta_0}$. So $L_\omega \cap C_{\beta_0} = C_\omega$. Thus $L_\omega = C_\omega L_0$, and hence $L_\omega K = C_\omega K$. $\square$

Theorem 3.13. *We have $G[b] = C_\omega K$.*

Proof. By Lemma 3.12 it suffices to show $G[b] = L_\omega K$. We fix a block B of G covering b. Let $\tilde{B}$ be the Harris–Knörr correspondent of B over b in $N_G(Q)$.

We first claim $G[b] \leq L_\beta K$. Let $x \in G[b]$. Put $G_x = \langle x, K\rangle$ and $L_x = L \cap G_x$. Then $L_x = QC_{G_x}(Q)$. Since the condition that $x \in G[b]$ is equivalent to the condition that b is $\langle x\rangle$-invariant and $\langle x\rangle$ acts on b as inner automorphisms, $x \in G[b]$ if and only if $x \in G_x[b]$. Thus it suffices to show $G_x[b] \leq (L_x)_\beta K$, where $(L_x)_\beta$ is the inertial group of β in L_x. Thus we may assume $G = G_x = \langle x, K\rangle$. By Corollary 3.3, there is some $y \in xK$ such that $\omega_B^*(\hat{K}_y) \neq 0$. Since $\tilde{B}$ covers $\tilde{b}$, $\tilde{B}$ covers β. So there is a block B' of L such that $\tilde{B}$ covers B' and B' covers β. Let β' be the Fong–Reynolds correspondent of B' over β in L_β. Since a defect group of B' contains Q, we have $B'^H = \tilde{B}$. This implies $B = \beta'^G$. So $\omega_B^*(\hat{K}_y) = \omega_{\beta'}^*(\widehat{K_y \cap L_\beta})$. Thus there is $g \in G$ such that $y^g \in L_\beta \leq L_\beta K$. Then $y \in L_\beta K$, since G/K is abelian. Thus $x \in L_\beta K$, and the claim is proved.

Then $G[b] = (L_\beta \cap G[b])K$. Therefore it suffices to prove $L_\beta \cap G[b] = L_\omega$. We shall show both sides contain the same p-elements and p'-elements. It suffices to show that under the assumption that x is either a p-element or a p'-element, it holds that $x \in L_\beta \cap G[b]$ if and only if $x \in L_\omega$. Since $x \in L_\beta \cap G[b]$ if and only if $x \in (L_x)_\beta \cap G_x[b]$ and $x \in L_\omega$ if and only if $x \in (L_x)_\omega$ (here $(L_x)_\omega$ is defined in a manner similar to L_ω), we may assume $G = G_x$.

Let x be a p-element. If $x \in L_\beta \cap G[b]$, then $x \in L_\omega$, since L_β/L_ω is a p'-group by Lemma 3.12. Conversely let $x \in L_\omega$. Then $L = \langle x, L_0\rangle$. So $L = L_\beta \leq S$. Then $S = \langle x, T\rangle = LT$. Thus $S/L \simeq T/L_0$, and S/L is a p'-group. Let B_1 be the Fong–Reynolds correspondent of $\tilde{B}$ over β in S. Let D be a defect group of B_1. Then $D \geq Q$. Since S/L is a p'-group, $D \leq L$. So $D = QC_D(Q)$. By the Fong–Reynolds theorem, D is a defect group of $\tilde{B}$. So D is a defect group of B. Since β is $\langle x\rangle$-invariant, $b = \beta^K$ is G-invariant. Therefore, $G = G[b]$ by Proposition 3.4, and $x \in L_\beta \cap G[b]$. The proof is complete in this case.

Let x be a p'-element. It suffices to show that under the assumption that $x \in L_\beta$, $x \in G[b]$ if and only if $x \in L_\omega$. Assume $x \in L_\beta$. Then $L = \langle x, L_0\rangle = L_\beta$. We have

$$\begin{aligned}|\mathrm{BL}(G \mid b)| &= |\mathrm{BL}(N_G(Q) \mid \tilde{b})| \quad \text{(by the Harris–Knörr theorem)}\\ &= |\mathrm{BL}(N_G(Q) \mid \beta)| \quad (\text{since } \tilde{b} \text{ is a unique block of } N_K(Q) \text{ covering } \beta)\\ &= |\mathrm{BL}(S \mid \beta)| \quad \text{(by the Fong–Reynolds theorem).}\end{aligned}$$

Since β is S-invariant, if $B_1 \in \mathrm{BL}(S \mid \beta)$ covers a block B' of L, then $B' \in \mathrm{BL}(L \mid \beta)$. If $B' \in \mathrm{BL}(L \mid \beta)$ and a block B_1 of S covers B', then $B_1 \in \mathrm{BL}(S \mid \beta)$. Further in this case B' is determined up to S-conjugacy by B_1 and $B_1 = B'^S$, since $L = QC_G(Q)$. Thus $|\mathrm{BL}(S \mid \beta)| = |\mathrm{BL}(L \mid \beta)/S|$, where $\mathrm{BL}(L \mid \beta)/S$ is a set of representatives of S-conjugacy classes of $\mathrm{BL}(L \mid \beta)$. Since $G = \langle x, K\rangle$, we have $S = \langle x, T\rangle$. So $|\mathrm{BL}(L \mid \beta)/S| = |\mathrm{BL}(L \mid \beta)/T| \leq |\mathrm{BL}(L \mid \beta)|$.

Since L/L_0 is cyclic and θ is L-invariant, there is an extension of θ to L. Let $\mathcal{E}$ be the set of such extensions. We show there is a bijection of $\mathrm{BL}(L \mid \beta)$ onto $\mathcal{E}$. For any $B' \in \mathrm{BL}(L \mid \beta)$, B' contains an irreducible character $\hat{\theta}$ lying over θ. Then $\hat{\theta} \in \mathcal{E}$. Since L/L_0 is a p'-group, B' has defect group Q. Therefore $\hat{\theta}$ is the canonical character of B' and $\hat{\theta}$ is uniquely determined. Of course any $\hat{\theta} \in \mathcal{E}$ is contained in some $B' \in \mathrm{BL}(L \mid \beta)$. Therefore the map $B' \mapsto \hat{\theta}$ is the required bijection. So $|\mathrm{BL}(L \mid \beta)| = |\mathcal{E}| = |L/L_0|$.

Since $|L/L_0| = |G/K|$, we obtain $|\mathrm{BL}(G \mid b)| \leq |G/K|$. By Proposition 3.9, $x \in G[b]$ if and only if the equality holds here. The last condition is equivalent to the condition that any extension of θ to L is T-invariant. Thus it is equivalent to the condition that $x \in L_\omega$, since $L = \langle x, L_0\rangle$. Thus $x \in G[b]$ if and only if $x \in L_\omega$. This completes the proof. □

Corollary 3.14. *Our C_ω in* Theorem 3.13 *is the same as C_ω* ($= C(D$ *in* $H)_\omega$ *in Dade's notation*) *appearing in* Corollary 12.6 *of* [3].

Proof. If we denote by C'_ω the group C_ω defined above, then Theorem 3.13 becomes $G[b] = C'_\omega K$. Then $C'_\omega = C \cap G[b]$. From Dade's theorem that $G[b] = C_\omega K$ [3, Corollary 12.6], we also obtain $C_\omega = C \cap G[b]$. Thus (our) $C_\omega = C'_\omega =$ (Dade's) C_ω. □

Corollary 3.15 (Külshammer [8, Proposition 9]). $G[b] = N_G(Q)[\tilde{b}]K$.

Proof. Use Theorem 3.13 to $G[b]$ and $N_G(Q)[\tilde{b}]$. □

4. Isomorphic blocks

The following theorem gives characterizations of isomorphic blocks with respect to normal subgroups. For isomorphic blocks, see [5, Section 4] and references therein.

Theorem 4.1. *Let B be a block of G covering b. The following are equivalent.*
(i) $G = G[b]$, $\mathrm{d}(B) = \mathrm{d}(b)$ *and for some irreducible character χ in B, χ_K is irreducible.*
(ii) *G/K is a p'-group and for any $x \in G$, there is an element y in xK such that* $\omega^*_B(\hat{K}_y) \neq 0$.
(iii) *The restriction $\chi \mapsto \chi_K$ is a bijection of* $\mathrm{Irr}(B)$ *onto* $\mathrm{Irr}(b)$.
(iv) *The restriction $\chi \mapsto \chi_K$ is a bijection of* $\mathrm{Irr}_0(B)$ *onto* $\mathrm{Irr}_0(b)$.
(v) *For some character $\xi \in \mathrm{Irr}(b)$, we have* $\mathrm{Irr}(B \mid \xi) = \{\chi\}$ *with $\chi_K = \xi$.*
(vi) *For some character $\xi \in \mathrm{Irr}(b)$, we have* $\mathrm{Irr}_0(B \mid \xi) = \{\chi\}$ *with $\chi_K = \xi$.*

Proof. (i) $\Rightarrow$ (ii). Since $\chi = \chi \otimes 1_{G/K}$, we see $B_0(G/K)$ is χ-dominated by B (for χ-domination see [10, p. 35]). So a defect group of $B_0(G/K)$ is contained in $QK/K = 1$ by [10, Corollary 1.5]. Thus G/K is a p'-group.

Let $x \in G$ and put $H = \langle x, K\rangle$. Since $H = H[b]$, by Corollary 3.3, there is some $y \in xK$ such that $\omega^*_\chi(\hat{L}_y) \neq 0$, where L_y is the conjugacy class of H containing y. Now $C_G(y)$ normalizes H. So $C_G(y)H$ is a subgroup of G containing K. Thus $|G : C_G(y)H|$ is a p'-integer. On the other hand, we have $\omega_\chi(\hat{K}_y) = \omega_\chi(\hat{L}_y)|G : C_G(y)H|$. Therefore $\omega^*_\chi(\hat{K}_y) \neq 0$.

(ii) $\Rightarrow$ (iii). Let $\chi \in \mathrm{Irr}(B)$. For any $x \in G$, there is an element $y \in xK$ such that $\chi(y) \neq 0$ by (ii). Then, by Lemma 3.1, χ_K is irreducible and $\chi_K \in \mathrm{Irr}(b)$. Of course, then the restriction is surjective. Let $\chi' \in \mathrm{Irr}(B)$ such that $\chi_K = \chi'_K$. Then $\chi' = \chi \otimes \theta$ for a linear character θ of G/K. For any $x \in G$, let $y \in xK$ be such that $\omega^*_\chi(\hat{K}_y) \neq 0$. We have

$$\omega^*_\chi(\hat{K}_y) = \omega^*_{\chi'}(\hat{K}_y) = \omega^*_\chi(\hat{K}_y)\theta(x)^*.$$

So $\theta(x)^* = 1$. Since G/K is a p'-group, we see that θ is the trivial character. Thus $\chi' = \chi$.

(iii) $\Rightarrow$ (iv). Put $a = \nu(|G|)$ and $a' = \nu(|K|)$. We have $a - \mathrm{d}(B) + \mathrm{ht}(\chi) = a' - \mathrm{d}(b) + \mathrm{ht}(\chi_K)$ for all $\chi \in \mathrm{Irr}(B)$. If $\mathrm{ht}(\chi) = 0$, we obtain $a - \mathrm{d}(B) \geq a' - \mathrm{d}(b)$. If $\mathrm{ht}(\chi_K) = 0$, we obtain $a' - \mathrm{d}(b) \geq a - \mathrm{d}(B)$. Thus $a - \mathrm{d}(B) = a' - \mathrm{d}(b)$. Hence $\mathrm{ht}(\chi) = \mathrm{ht}(\chi_K)$ for all $\chi \in \mathrm{Irr}(B)$. Thus (iv) follows.

(iii) $\Rightarrow$ (v). This is trivial.

(iv) $\Rightarrow$ (vi). This is trivial.

(v) $\Rightarrow$ (vi). Let a and a' be as above. We have $a - \mathrm{d}(B) + \mathrm{ht}(\chi) = a' - \mathrm{d}(b) + \mathrm{ht}(\xi)$. Let B_w be a weakly regular block of G covering b. Since b is G-invariant, we have $a - \mathrm{d}(B_w) = a' - \mathrm{d}(b)$. Thus $a - \mathrm{d}(B) \geq a - \mathrm{d}(B_w) = a' - \mathrm{d}(b)$. On the other hand, we have $\mathrm{ht}(\chi) \geq \mathrm{ht}(\xi)$ by [10, Lemma 2.2]. Thus equality holds throughout and $\mathrm{ht}(\chi) = \mathrm{ht}(\xi)$. So $\mathrm{Irr}_0(B \mid \xi) = \{\chi\}$.

(vi) $\Rightarrow$ (i). Let θ be an irreducible character of p'-degree in $B_0(G/K)$. Then $\chi \otimes \theta \in \mathrm{Irr}(B \mid \xi)$. We have $\mathrm{ht}(\chi \otimes \theta) = \mathrm{ht}(\chi) = \mathrm{ht}(\xi)$. Thus $\chi \otimes \theta = \chi$, and θ is the trivial character. So $B_0(G/K)$ has defect 0 by the Cliff–Plesken–Weiss theorem [2, Proposition 3.3] ([13, Problem 3.11]), and G/K is a p'-group. So $\mathrm{d}(B) = \mathrm{d}(b)$. Put $\zeta = \chi_{G[b]}$. We claim $\mathrm{Irr}(G \mid \zeta) = \{\chi\}$. Let $\chi' \in \mathrm{Irr}(G \mid \zeta)$. Then $\nu(\chi'(1)) = \nu(\zeta(1)) = \nu(\chi(1))$. Since χ' lies in B by Theorem 3.5, $\mathrm{ht}(\chi') = \mathrm{ht}(\chi)$. Therefore $\chi' = \chi$ by assumption, and the claim follows. Then, by Frobenius reciprocity, $\zeta^G = \chi$. Since $\zeta(1) = \chi(1)$, we obtain $G = G[b]$.

The proof is complete. $\square$

REMARK 4.2. The equivalence of (i) and (iii) in Theorem 4.1 follows from [5, Proposition 2.6, Theorem 3.5, and Theorem 4.1].

ACKNOWLEDGEMENT. The author would like to express his thanks to the referee for pointing out an error in an earlier version of the manuscript and for valuable suggestions.

References

[1] D.A. Craven, C.W. Eaton, R. Kessar and M. Linckelmann: *The structure of blocks with a Klein four defect group*, Math. Z. **268** (2011), 441–476.

[2] G.H. Cliff, W. Plesken and A. Weiss: *Order-theoretic properties of the center of a block*; in The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math. **47**, Part 1, Amer. Math. Soc., Providence, RI, 1987, 413–420.

[3] E.C. Dade: *Block extensions*, Illinois J. Math. **17** (1973), 198–272.

[4] Y. Fan: *Remarks on Brauer correspondences*, J. Algebra **157** (1993), 213–223.

[5] A. Hida and S. Koshitani: *Morita equivalent blocks in non-normal subgroups and p-radical blocks in finite groups*, J. London Math. Soc. (2) **59** (1999), 541–556.

[6] I.M. Isaacs: *Characters of solvable and symplectic groups*, Amer. J. Math. **95** (1973), 594–635.

[7] S. Koshitani and B. Külshammer: *A splitting theorem for blocks*, Osaka J. Math. **33** (1996), 343–346.

[8] B. Külshammer: *Morita equivalent blocks in Clifford theory of finite groups*, Astérisque **181–182** (1990), 209–215.

[9] M. Murai: *Block induction, normal subgroups and characters of height zero*, Osaka J. Math. **31** (1994), 9–25.

[10] M. Murai: *Normal subgroups and heights of characters*, J. Math. Kyoto Univ. **36** (1996), 31–43.

[11] M. Murai: *Blocks of normal subgroups, automorphisms of groups and the Alperin–McKay conjecture*, preprint.

[12] H. Nagao and Y. Tsushima: Representations of Finite Groups, Academic Press, Boston, MA, 1989.

[13] G. Navarro: Characters and Blocks of Finite Groups, Cambridge Univ. Press, Cambridge, 1998.

[14] G. Navarro: *Irreducible restriction and zeros of characters*, Proc. Amer. Math. Soc. **129** (2001), 1643–1645.

[15] A. Watanabe: *The Glauberman character correspondence and perfect isometries for blocks of finite groups*, J. Algebra **216** (1999), 548–565.

Meiji-machi 2-27
Izumi Toki-shi
Gifu 509-5146
Japan
e-mail: m.murai@train.ocn.ne.jp
Passed away on July 2012