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Osaka University
ON RING THEORETIC QUASI-ISOMETRY INVARIANTS

HOKUTO MORISHIMA

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Abstract

We introduce an algebraic version of the translation algebra of a group. We prove that a quasi-isometry of two finitely generated groups induces Morita equivalence of their algebraic translation algebras.

1. Introduction

Ring theoretic approaches for a quasi-isometry of groups were started by Shalom and Sauer [20], [18]. Shalom proved quasi-isometry invariance of the cohomological dimensions of finitely generated amenable groups, and of the R-Betti numbers of finitely generated nilpotent groups. In his proof, it was important that there exists a good topological coupling induced by a quasi-isometry. Sauer refined a part of Shalom’s argument. He showed that a good topological coupling induces a Morita equivalence between Sauer rings of the coupled group actions (see Section 3). He applied this result to quasi-isometry invariance of the (co)homological dimensions of finitely generated groups with finite dimensions, and of the R-cohomology rings of finitely generated nilpotent groups.

Morita theory of Sauer rings is important for classifying groups by quasi-isometry. However, Sauer rings of the same group are not always Morita equivalent. In order to study ring theoretic invariants we should determine a ring for each finitely generated group. We propose considering the rings as follows: Let \( k \) be a ring with the multiplicative identity element 1, and \( G \) a finitely generated group. We consider the skew group ring \( G \ast l^f(G, k) \), where \( l^f(G, k) \) is the ring of functions with finite image. We denote this ring by \( R(G, k) \), and call it an algebraic translation algebra of \( G \) with the coefficient \( k \). In the case where \( k = \mathbb{Q} \) or \( \mathbb{C} \), we see that \( R(G, k) \) is a subring of Roe’s translation algebra [17, p.68]. In fact, \( R(G, k) \) is isomorphic to the Sauer ring of a natural action of \( G \) on \( \beta G \), where \( \beta G \) is the Stone–Čech compactification of \( G \) endowed with the discrete topology (see Lemma 3.2). We have the main theorem:

**Theorem 1.** If finitely generated groups \( G \) and \( G' \) are quasi-isometric, then \( R(G, k) \) and \( R(G', k) \) are Morita equivalent.

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Two groups always have a good topological coupling such that their Stone–Čech compactifications are coupled (see Section 3). Therefore Theorem 1 is the special case of [18]. Without using a topological coupling and [18], we prove this result in Section 4. The cores of a quasi-isometry (see Definition 2.1) play important roles.

Morita invariants of \( \mathcal{R}(G, k) \) are quasi-isometry invariants by Theorem 1. In Section 5, we give a formula to calculate the global dimension and the weak global dimension of \( \mathcal{R}(G, k) \). They are well-known Morita invariants. The global dimension of \( \mathcal{R}(G, k) \) is estimated by the cohomological dimension of \( G \) and the global dimension of \( l^1(G, k) \). The same result is true for the weak global dimension. It should be noted that some of well-known Morita invariants are trivial. For example, the center of \( \mathcal{R}(G, k) \) coincides with that of \( k \) (see Lemma 2.4).

The Morita equivalence in the proof of Theorem 1 preserves some special modules (see Theorem 4.7). For example, \( l^1(G, k) \) and \( l_c(G, k) \) are preserved. The coarse cohomology \( H^n_G(G, G_k) \) is isomorphic to \( \operatorname{Ext}^n_{\mathcal{R}(G, k)}(l^1(G, k), l_c(G, k)) \) (see Section 4.3), and hence the coarse cohomology is a quasi-isometry invariant as already known. The coarse \( l^1 \)-cohomology ([6]) is also obtained in this way.

If \( G \) is not amenable, then the Morita equivalence of Theorem 1 can be replaced by a ring isomorphism. It is proved in Corollary 4.5. In this case, isomorphism invariants of rings are also quasi-isometry invariants.

In Section 6, a geometric description of \( \mathcal{R}(G, k) \) is given by \( \operatorname{Mod}_k(G \rtimes \beta G) \) by using [4]. Indeed, \( \mathcal{R}(G, k) \)-Mod is additively equivalent to \( \operatorname{Mod}_k(G \rtimes \beta G) \) (see Theorem 6.6). From this we can construct \( \mathcal{R}(G, k) \)-modules by the geometry of Stone–Čech compactification. In Appendix 7.1, we give an alternative proof of Theorem 1 using the result in Section 6.

2. Preliminaries

2.1. Geometric group theory. We recall the basic notion of geometric group theory and cores of a quasi-isometry [8, 0.2.C. p. 4, 5].

Let \( G \) be a finitely generated group with a finite generating system \( S \). \( G \) has a metric \( d_{(G, S)} \) defined by

\[
d_{(G, S)}(x, y) = \begin{cases} \min\{n \in \mathbb{N} \mid x = s_1^{i_1} \cdots s_n^{i_n}y, s_k \in S, i_k \in \{-1, 1\}\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}
\]

which is called the word metric with respect to \( S \).

Let \( Z \) be a metric space. For \( W \subseteq Z \) and a real number \( K \geq 0 \), \( \mathcal{N}_K(W) = \{z \in Z \mid \exists w \in W \text{ s.t. } d(z, w) \leq K\} \) is called a \( K \)-neighborhood of \( W \). If \( \mathcal{N}_K(W) = Z \), the subspace \( W \) is said to be \( K \)-coarsely dense in \( Z \).

A quasi-isometry is a map \( f : X \to Y \) between metric spaces such that for some real number \( K \geq 1 \), \( f \) satisfies
Invariants

1. \((1/K) d(x, x') - K \leq d(f(x), f(x')) \leq K d(x, x') + K\) for every \(x, x' \in X\),
2. \(f(X)\) is \(K\)-coarsely dense in \(Y\).

Two metric spaces are quasi-isometric if there exists a quasi-isometry between them. This gives an equivalence relation for metric spaces. If \(S\) and \(S'\) are finite generating systems of \(G\), then \((G, d_{(G,S)})\) and \((G, d_{(G,S')})\) are quasi-isometric.

The definition of cores of a quasi-isometry is as follows:

**Definition 2.1.** Let \(f: X \to Y\) be a quasi-isometry. \(A \subseteq X\) and \(B \subseteq Y\) are called cores of \(f\) if there exists a real number \(K \geq 0\) such that \(N_K(A) = X\), \(N_K(B) = Y\), \(f(A) = B\) and \(f|_A\) is a bijective quasi-isometry.

For every quasi-isometry \(f: X \to Y\) there exist cores of \(f\). Indeed, we can define a core \(B\) to be \(f(X)\), and \(A\) to be \(\{x_b \in X \mid b \in B\}\) by choosing \(x_b \in f^{-1}(b)\) for each \(b\).

**2.2. Algebraic translation algebra.** Let \(G\) be a group, and \(R\) a ring with the multiplicative identity element \(1\) on which \(G\) acts from the right. For \(r \in R\) and \(g \in G\) this action is denoted by \(r^g\). The skew group ring \(G * R\) is a free right \(R\)-module on \(G\) with the multiplication given by

\[(gr_1)(hr_2) = (gh)(r_1^hr_2)\quad \text{for every}\quad g, h \in G, r_1, r_2 \in R.\]

If \(G\) acts on \(R\) trivially, then we especially write \(G * R\) by \(GR\). It is an ordinary group ring (see [16] about skew group rings).

**Definition 2.2.**

1. Let \(G\) be a group and \(k\) a ring with the multiplicative identity element \(1\).

\[l^f(G, k) = \{F: G \to k \mid \sigma(\text{Im } F) < \infty\}\]

is a ring with the following sum and multiplication:

\[(F_1 + F_2)(x) = F_1(x) + F_2(x),\]
\[(F_1 F_2)(x) = F_1(x)F_2(x)\]

for every \(F_1, F_2 \in l^f(G, k)\) and \(x \in G\).

2. \(G\) acts on \(l^f(G, k)\) from the right by

\[F^g(x) = F(gx)\quad \text{for every}\quad g \in G, x \in G.\]

3. We denote \(G * l^f(G, k)\) by \(R(G, k)\). It is called an algebraic translation algebra of \(G\) with the coefficient \(k\).

The multiplicative identity element of \(l^f(G, k)\) is the constant function \(1\). Since \(k \subseteq l^f(G, k)\) as constant functions, a group ring \(Gk\) is a subring of \(R(G, k)\). \(e \cdot 1\) is
the multiplicative identity element of $R(G, k)$, where $e$ is the identity element of $G$. $k$ is regarded as a left $Gk$-module by $gk = k$ ($g \in G$, $k \in k$).

For $S \subseteq G$ the characteristic function

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}$$

is an element of $l^f(G, k)$.

In the case where $k = \mathbb{Z}$, $Q$ or $C$, we see that $R(G, k)$ is a subring of Roe’s translation algebra [17, p. 68].

### 2.3. Morita theory

We give the basic notion of Morita theory [1]. Let $R$ and $S$ be rings. $R$-$\text{Mod}$ ($\text{Mod-}R$) is the category of left (right) modules over $R$. A left $R$- and right $S$-module $M$ is called a left $R$-right $S$-bimodule if $r(ms) = (rm)s$ ($r \in R$, $s \in S$, $m \in M$). $gM$ means $M$ is a left $R$-module, $M_R$ means $M$ is a right $R$-module, and $gMS$ means $M$ is a left $R$-right $S$-bimodule.

Let $F_1, F_2: B \to C$ be functors, a set $\{\tau_B \in \text{Hom}(F_1(B), F_2(B)) \mid B \in \text{Ob}(B)\}$ is called a natural equivalence if $F_2(f) \circ \tau_B = \tau_B \circ F_1(f)$ ($f \in \text{Hom}(B, B')$) and $\tau_B$ is an isomorphism for every $B \in \text{Ob}(B)$. Then $F_1 \simeq F_2$ if there exists a natural equivalence. A functor $F: R$-$\text{Mod} \to S$-$\text{Mod}$ is called an additive functor if $\text{Hom}(A, B) \to \text{Hom}(F_1(A), F_2(B))$ defined by $f \mapsto F(f)$ is a homomorphism. An additive functor $F_1: R$-$\text{Mod} \to S$-$\text{Mod}$ is called an additive equivalence if there exists an additive functor $F_2: S$-$\text{Mod} \to R$-$\text{Mod}$ such that $F_2 \circ F_1 \simeq \text{id}$ and $F_1 \circ F_2 \simeq \text{id}$. A functor $F_2$ is called an inverse equivalence of $F_1$.

$R$ and $S$ are said to be Morita equivalent if there exists an additive equivalence between $R$-$\text{Mod}$ and $S$-$\text{Mod}$. Let $M$ be a left (right) $R$-module. The module $M$ is said to be finitely generated if there exist $n \in N$ and a surjective homomorphism $f: R^n \to M$. $M$ is called a (finite) generator if there exist $n \in N$ and a surjective homomorphism $f: M^n \to R$. $M$ is said to be projective if it is a direct summand of a free left (right) $R$-module. A generator is called a progenerator if it is finitely generated and projective. $\text{End}(M) = \{f: M \to M \mid f\}$ is a left (right) $R$-homomorphism) is called the endomorphism ring. The multiplication is the opposite composition (ordinary composition) of maps.

$e \in R$ is called an idempotent if $e^2 = e$. If $R$ has the multiplicative identity element 1, then $eRe = \{ere \in R \mid r \in R\}$ is a ring with the multiplicative identity element $e$. $Re$ is a left $R$-module, and $\text{End}(Re)$ is isomorphic to $eRe$. $eR$ is a right $R$-module, and $\text{End}(eR)$ is also isomorphic to $eRe$.

**Theorem 2.3.** [1, Corollary 22.5, p. 265] Let $R$ be a ring. If $P_R$ is a progenerator, then $R$ and $S = \text{End}(P_R)$ are Morita equivalent. Indeed, if $P^@ = \text{Hom}_R(P_R, R)$, then $sP_R$ and $R P_S^@$ are bimodules and $(P \otimes R -): R$-$\text{Mod} \to S$-$\text{Mod}$, $(P^@ \otimes S -): S$-$\text{Mod} \to R$-$\text{Mod}$ are inverse equivalences.
If $P_R = eR$ is a pregroupoid, then $S = eRe$, $SP_R = eReR$ and $kP_{eRe} = kReeR$. If $R$ and $S$ are isomorphic, then $R$ and $S$ are Morita equivalent. Indeed, let $\Phi: S \to R$ be a ring isomorphism. Since $S \simeq R \simeq \text{End}(R_R)$, an additive equivalence $(sR_R \otimes_R -): R\text{-Mod} \to S\text{-Mod}$ is obtained. $kM$ is mapped to $sM$ satisfying $sm = \Phi(s)m$ $(s \in S, m \in M)$. We use the notation $\text{Res} \Phi = (sR_R \otimes_R -)$.

2.4. The center of $\mathcal{R}(G, k)$. The center of a ring $R$ is $\text{Cen}(R) = \{r \in R \mid rx = xr \ (\forall x \in R)\}$. If rings $R$ and $S$ are Morita equivalent, then $\text{Cen}(R)$ and $\text{Cen}(S)$ are isomorphic [1, Proposition 21.10, p. 258].

**Lemma 2.4.** $\text{Cen}(\mathcal{R}(G, k)) = \text{Cen}(k)$.

Proof. Let $\alpha \in \text{Cen}(\mathcal{R}(G, k))$. For each $x \neq e \in G$ there exists no $F \neq 0 \in l^1(G, k)$ such that for every $g \in G$, $\chi_xxF = xF\chi_x$ is satisfied, and hence $\alpha \in e \cdot l^1(G, k)$. Since for every $g \in G$ we have $ga = \alpha g$, $\alpha$ is a constant function. $k\alpha = \alpha k$ is satisfied for every $k \in k$, and hence $\alpha \in \text{Cen}(k)$. □

2.5. Transformation groupoids. Let $G_0, G_1$ be topological spaces, and $s: G_1 \to G_0$, $t: G_1 \to G_0$, $m: G_1 \times_{G_0} G_1 \to \{(g_1, g_2) \in G_1 \times G_1 \mid s(g_1) = t(g_2)\}$ $\to G_1$ continuous maps. We consider the following three conditions:

1. $m(m(g_1, g_2), g_3) = m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3)) = m(g_1, (g_2, g_3)) \in G_1 \times G_0 \times G_1$.
2. There exists a continuous map $u: G_0 \to G_1$ such that $s(u(x)) = t(u(x)) = x$ and $m(u(x), g) = g, m(g', u(x)) = g'$ $(x \in G_0, g, g' \in G_1$ with $t(g) = x = s(g'))$.
3. There exists a continuous map $I: G_1 \to G_1$ such that $m(g, I(g)) = u(t(g)), m(I(g), g) = u(s(g)) (g \in G_1)$.

$\mathcal{G} = (G_0, G_1, s, t, m, u, I)$ satisfying the conditions above is called a topological groupoid. We use the notation $g_1 \cdot g_2 = m(g_1, g_2) = (g_1, g_2) \in G_1 \times_{G_0} G_1$, $e_x = u(x) (x \in G_0)$ and $g^{-1} = I(g) (g \in G_1$). A topological groupoid $\mathcal{G} = (G_0, G_1, s, t, m, u, I)$ is called an étale groupoid if $s$ and $t$ are surjective local homeomorphisms (see [14, Section 5] for more on étale groupoids).

Let $G$ be a finitely generated group, and $G$ acts on a topological space $X$ from the left. We define a transformation groupoid $G \ltimes X$ by the following data:

$$(G \ltimes X)_0 = X, \quad (G \ltimes X)_1 = G \times X,$$

where $G$ is regarded as a discrete space.

$s(g, x) = x \ (g \in G, x \in X), \quad t(g, x) = gx \ (g \in G, x \in X),
(g, x) \cdot (g', x') = (gg', x') \ (g, g' \in G, x, x' \in X$ satisfying $x = g'x'),
(u(x) = (e, x) \ (x \in X),

where $e$ is the identity element of $G$.

$$(G \ltimes X)_0 = (g^{-1}, gx) \ (g \in G, x \in X).$$

$G \ltimes X$ is an étale groupoid.
2.6. Stone–Čech compactifications. We recall the Stone–Čech compactifications for discrete spaces [9]. Let $D$ be a set. $\mathcal{U} \subseteq 2^D$ is called a filter on $D$ if the following conditions are satisfied.

(0) $D \in \mathcal{U}$,
(1) $\emptyset \notin \mathcal{U}$,
(2) if $A_1, A_2 \in \mathcal{U}$, then $A_1 \cap A_2 \in \mathcal{U}$,
(3) if $A \in \mathcal{U}$, $B \in 2^D$ and $A \subseteq B$, then $B \in \mathcal{U}$.

In addition, $\mathcal{U}$ is called an ultra filter if $\mathcal{U}$ satisfies

(4) if $D = A_1 \cup \cdots \cup A_n$, then there exists the unique $1 \leq i \leq n$ such that $A_i \in \mathcal{U}$.

The set of ultra filters on $D$ is denoted by $\beta D$. For $A \subseteq D$ we use the notation $\hat{A} = \{ \mathcal{U} \in \beta D \mid A \in \mathcal{U} \}$. $\mathcal{O}$ is a topology on $D$ generated by an open base $\{ \hat{A} \mid A \in 2^D \}$.

$(\beta D, \mathcal{O})$ is called the Stone–Čech compactification of $D$. Let $G$ be a finitely generated group. $\beta G$ has a natural $G$-action from the left. Indeed, for $\mathcal{U} \in \beta G$ and $g \in G$, $g\mathcal{U} = \{ gA \mid A \in \mathcal{U} \} \in \beta G$. This action is a homeomorphic action.

Lemma 2.5. (1) $\beta D$ is compact and Hausdorff. $D$ is identified with a dense subset of $\beta D$ by an injection $e : D \to \beta D$ satisfying $\{ e(d) \} = \{ \hat{d} \}$ for every $d \in D$.

(2) For $A \in 2^D$, $\overline{D - A} = \beta D - \hat{A}$. Therefore the topology of $\beta D$ is generated by clopen (closed and open) sets.

(3) If $O$ is a clopen set of $\beta D$, then there exists $A \in 2^D$ such that $O = \hat{A}$.

Proof. The proof of (1) is in [9, Theorem 3.18 (a) and (c)], and the proof of (2) is in [9, Theorem 3.17 (c)]. If $O$ is a clopen set of $\beta D$, then there exists $A_x \in 2^D$ for each $x \in O$ such that $x \in A_x$ and $O = \bigcup_{x \in O} A_x$. Since $O$ is a closed set of Hausdorff space, $O$ is compact. Therefore there exists $\{ x_i \in O \}_{i=1}^n$ such that $O = \bigcup_{i=1}^n A_{x_i}$. By [9, Theorem 3.17 (b)] we have $O = \bigcup_{i=1}^n A_{x_i}$.

2.7. Definition of $\text{Mod}_k(\mathcal{G})$. Let $\mathcal{G}$ be an étale groupoid. In [4], the abelian category associated to $\mathcal{G}$ was considered to study a homology theory for $\mathcal{G}$ ([21, Appendix A] is a good reference for abelian categories). This category is denoted by $\text{Mod}_k(\mathcal{G})$. In Section 6, we describe $\text{Mod}_k(G \ltimes \beta G)$ and discuss a relation to the algebraic translation algebra.

First, we recall the definition of $\text{Sh}(\mathcal{G})$. A left étale $\mathcal{G}$-space $X = (X, p_0, p_1)$ is a topological space with continuous maps $p_0 : X \to \mathcal{G}_0$ and $p_1 : \mathcal{G}_1 \times_{p_0} X = \{(g, x) \mid s(g) = p_0(x)\} \to X$ such that

(0) $p_0$ is a surjective local homeomorphism,
(1) $p_0(p_1(g, x)) = t(g)$ ($(g, x) \in \mathcal{G}_1 \times_{p_0} X$),
(2) $p_1(h \cdot g, x) = p_1(h, p_1(g, x))$ ($(g, x) \in \mathcal{G}_1 \times_{p_0} X$, $(h, g) \in \mathcal{G}_1 \times \mathcal{G}_0 \mathcal{G}_1$),
(3) $p_1(e_{p_0(x)}, x) = x$ $(x \in X)$.

$p_1$ is usually denoted by $\cdot$. Let $X = (X, p_0, p_1)$ and $Y = (Y, q_0, q_1)$ be left étale $\mathcal{G}$-spaces. A continuous map $\Phi : X \to Y$ is said to be equivariant if
(1) \( q_0 \circ \Phi = p_0 \).
(2) \( \Phi(p_1(g, x)) = q_1(g, \Phi(x)) \) \((g, x) \in \mathcal{G}_1 \times p_0 X \)
are satisfied. \( \text{Sh}(\mathcal{G}) \) is the category of which objects are left étale \( G \)-spaces and morphisms are equivariant maps. \( \text{Sh}(\mathcal{G}) \) is called the category of left étale \( G \)-spaces.

Second, we recall the definition of \( \text{Mod}_k(\mathcal{G}) \). Let \( X = (X, p_0, p_1) \) and \( Y = (Y, q_0, q_1) \) be left étale \( G \)-spaces. We define a finite product of \( \text{Sh}(\mathcal{G}) \): \( X \oplus Y = (X \times_{\mathcal{G}_0} Y, r_0, r_1) \) by \( X \times_{\mathcal{G}_0} Y = \{(x, y) \in X \times Y \mid p_0(x) = q_0(y)\}, \)
\[ r_0 \colon X \times_{\mathcal{G}_0} Y \to \mathcal{G}_0 \text{ with } r_0(x, y) = p_0(x) = q_0(y) \text{ and } r_1 \colon G_1 \times p_0(X \times_{\mathcal{G}_0} Y) \to X \times_{\mathcal{G}_0} Y \text{ with } r_1(g, (x, y)) = (p_1(g, x), q_1(g, y)) \). \( \Theta = \mathcal{G}_0 \) is \( \mathcal{G}_0 \to \mathcal{G}_0, t \circ \text{Pr}_1 \colon G_1 \times \mathcal{G}_0 \to \mathcal{G}_0 \) is a left étale \( G \)-space and \( \text{Hom}(X, \mathcal{G}_0) = \{ p_0 \} \). \( \Theta \) is a terminal object. \( k \) is regarded as a constant left étale \( G \)-space \( k = (k \times \mathcal{G}_0, p_0^0 = \text{Pr}_2, p_1') \) by \( p_1'(g, (k, z)) = (k, t(g)) \). \( k \) has a natural structure of a ring. We consider \( k \)-module objects of \( (\text{Sh}(\mathcal{G}), \oplus, \Theta) \): \( A = (A, M, U, v, \mathcal{M}) \) is called a \( \mathcal{M}^{\mathcal{A}} \)-module object of \( \text{Sh}(\mathcal{G}), \oplus, \Theta \) if morphisms \( M \colon A \oplus A \to A, U \colon \Theta \to A, v \colon A \to A \text{ and } \mathcal{M} : k \oplus A \to A \text{ satisfy} \)

(1) \( (M, U, v) \) is an usual additive group structure on \( A \),
(2) \( \mathcal{M} \) is an usual \( k \)-action giving a \( k \)-module structure on \( (A, M, U, v) \).

\( M \) is usually denoted by \( + \), \( U \) by \( 0 \), \( v \) by \( - \) and \( \mathcal{M} \) by \( \cdot \). Morphisms between \( k \)-module objects \( A \) and \( B \) of \( (\text{Sh}(\mathcal{G}), \oplus, \Theta) \) are morphisms of \( \text{Sh}(\mathcal{G}) \) preserving structures \( M, U, v \) and \( \mathcal{M} \). Therefore \( k \)-module objects form a category. It is denoted by \( \text{Mod}_k(\mathcal{G}) \). \( \text{Mod}_k(\mathcal{G}) \) is an abelian category. \( \Theta = \mathcal{G}_0 \) is the zero object 0.

\( \text{Mod}_k(\mathcal{G}) \) always has an infinite coproduct. Such a category is called an A.B.3 category [21, A.4]. An infinite coproduct exists as follows: For \( \{ A_\lambda \in \text{Ob}(\text{Mod}_k(\mathcal{G})) \mid \lambda \in \Lambda \} \), \( A_\lambda \) gives a presheaf of \( k \)-modules \( \mathcal{F}_{A_\lambda} \) on \( \mathcal{G}_0 \) by \( O \mapsto \Gamma(O, A_\lambda) = \{ f : O \to A_\lambda \mid f \text{ is continuous and } p_0 \lambda \circ f = \text{id}_O \} \) for every open set \( O \subseteq \mathcal{G}_0 \). Therefore for presheaf \( \bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_\lambda} : O \mapsto \bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_\lambda}(O) \) its sheaf space \( E_{\bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_\lambda}} \) has a natural structure of a \( k \)-module object of \( (\text{Sh}(\mathcal{G}), \oplus, \Theta) \) (about the relation of a presheaf and a sheaf space see [2, 2.3]). This \( E_{\bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_\lambda}} \) is an infinite coproduct. Therefore \( \text{Mod}_k(\mathcal{G}) \) is an A.B.3 category.

3. Proof of quasi-isometry invariance of the algebraic translation algebra using a topological coupling

We recall the definition of Sauer rings. Let \( G \) acts on a compact Hausdorff space \( X \) from the left. \( \mathcal{F}(X, k) = \{ F : X \to k \mid F^{-1}(k) \) is clopen for every \( k \in k \} \) has the right action of \( G \) induced by the left action of \( G \) on \( X \). In the paper [18], the skew group ring \( G \ltimes \mathcal{F}(X, k) \) was considered. We call this ring the Sauer ring of \( G \)-space \( X \).

**Theorem 3.1** ([18]). If two finitely generated groups \( G \) and \( G' \) are quasi-isometric, then there exist compact Hausdorff spaces \( Y_1 \) on which \( G \) acts from the left and \( Y_2 \) on which \( G' \) acts from the left such that their Sauer rings \( G \ltimes \mathcal{F}(Y_1, k) \) and \( G' \ltimes \mathcal{F}(Y_2, k) \) are Morita equivalent. A good topological coupling \( \Omega \) always gives such \( Y_1 = \Omega/G' \) and \( Y_2 = \Omega/G \), where a topological coupling \( \Omega \) is said to be good if it has a compact clopen
fundamental domain for each action.

We relate Sauer rings and our algebraic translation algebras.

**Lemma 3.2.** Let $G$ be a finitely generated group.

$$
\mathcal{R}(G, k) \simeq G \star \mathcal{F}(\beta G, k).
$$

Proof. For $F \in \mathcal{F}(\beta G, k)$, $\beta G = \bigsqcup_{k \in k} F^{-1}(k)$. Since $\beta G$ is compact and for each $k$, $F^{-1}(k)$ is open, and hence there exist $k_1, \ldots, k_n \in k$ such that $\beta G = \bigsqcup_{i=1}^n F^{-1}(k_i)$. For each $i$, $F^{-1}(k_i)$ is clopen, and hence by Lemma 2.5 (3) there exist $A_1, \ldots, A_n \subseteq G$ such that $F^{-1}(k_i) = \tilde{A}_i$. We define $\lambda : \mathcal{F}(\beta G, k) \to l^1(G, k) \to l^1(G, k) = \sum_{i=1}^n k_i \chi_{A_i}$. $\lambda$ is a bijective homomorphism and preserves the action of $G$. Every function in $l^1(G, k)$ has an expression $\sum_{i=1}^n k_i \chi_{A_i}$, and hence $\lambda$ is surjective. This $\lambda$ is extended to $\mathcal{R}(G, k) \simeq G \star \mathcal{F}(\beta G, k)$. \hfill $\Box$

By Lemma 3.2 if we have a good topological coupling such that $Y_1 = \beta G$ and $Y_2 = \beta G'$, then Theorem 1 is the special case of [18]. Indeed, we have the following theorem:

**Theorem 3.3.** Two quasi-isometric finitely generated groups $G$ and $G'$ always have a good topological coupling such that their Stone–Čech compactifications are coupled.

Proof. In the proof of Theorem 7.1 in Appendix, we have essential morphisms $G \times \beta G \to \mathcal{G}(G, [G] \sqcup [G']) \leftarrow G \times \beta G'$ (see [19, Section 3.4]). We take the weak pullback $\mathcal{G}$ of this morphism, and hence surjective essential morphisms $G \times \beta G \to \mathcal{G} \to G \times \beta G'$ are obtained (see [14, Exercise 5.22 (1)]). $\mathcal{G}_0$ has a natural $(G \times G')$-action. $\mathcal{G}_0$ is a topological coupling such that $\mathcal{G}_0/G' = \beta G$ and $\mathcal{G}_0/G = \beta G'$. Since surjective essential morphisms above are étale and the topologies of $\beta G$ and $\beta G'$ are generated by clopen sets, we can construct a compact clopen fundamental domain for each action. As a result, $\mathcal{G}_0$ is a good topological coupling. \hfill $\Box$

We have the main theorem by Lemma 3.2, Theorems 3.1 and 3.3:

**Theorem 1.** If finitely generated groups $G$ and $G'$ are quasi-isometric, then $\mathcal{R}(G, k)$ and $\mathcal{R}(G', k)$ are Morita equivalent.

4. **Proof of quasi-isometry invariance of the algebraic translation algebra without using a topological coupling**

The proof is obtained by elementary argument: cores of a quasi-isometry and basic Morita theory (see Sections 2.1 and 2.3).
4.1. Some lemmas. In order to prove quasi-isometry invariance of the algebraic translation algebra, we need Lemmas 4.1 and 4.4.

Lemma 4.1. Let $H$ be a finitely generated group. For $Z \subseteq H$ if there exists a real number $K \geq 0$ such that $Z$ is $K$-coarsely dense in $H$, then a right $\mathcal{R}(H, k)$-module $I_Z = \chi_Z \mathcal{R}(H, k)$ is a progenerator.

Proof. Since $\mathcal{R}(H, k) = I_Z \oplus I_{H-Z}$, $I_Z$ is finitely generated and projective.

We prove that $I_Z$ is a generator: For the identity element $e \in H$, $N_K(e)$ is finite, and hence we have an expression $N_K(e) = \{h_0 = e, h_1, \ldots, h_n\}$. We define $Z_0, Z_1, \ldots, Z_n$ by

\begin{align*}
Z_0 &= Z, \\
Z_1 &= h_1 Z - Z, \\
Z_2 &= h_2 Z - h_1 Z - Z, \\
& \quad \ldots \\
Z_n &= h_n Z - h_{n-1} Z - \cdots - Z.
\end{align*}

$Z_0 \cup \cdots \cup Z_n = H$ and $h_i^{-1}Z_i \subseteq Z$ are satisfied. We define $p_i : I_Z \to \mathcal{R}(H, k)$ by $p_i(\chi_Z \gamma) = \chi_Z h_i \gamma$ for every $\gamma \in \mathcal{R}(H, k)$ and $0 \leq i \leq n$. They are well-defined as follows:

For every $\gamma, \gamma' \in \mathcal{R}(H, k)$ satisfying $\chi_Z \gamma = \chi_Z \gamma'$, by multiplying $\chi_Z h_i$ to this equation from the left, we have $\chi_Z h_i \chi_Z \gamma = \chi_Z h_i \chi_Z \gamma'$. This implies $\chi_Z h_i z h_i \gamma = \chi_Z h_i \chi_z \gamma'$. Thus $h_i^{-1}Z_i \subseteq Z$ shows that $\chi_Z h_i \gamma = \chi_Z h_i \gamma'$, and hence $p_i$ is a well-defined homomorphism. As a result, we have a homomorphism $p = \bigoplus_{i=0}^n p_i : I_Z \to \mathcal{R}(H, k)$. For each $h \in H$ and $F \in l^1(H, k)$ we have $hF = \sum_{i=0}^n \chi_Z h_i hF = \sum_{i=0}^n \chi_Z (h_i h_i^{-1}) hF = \sum_{i=0}^n \chi_Z h_i (h_i^{-1} hF) = \sum_{i=0}^n p_i (\chi_Z h_i^{-1} hF) = p(\chi_Z h_0^{-1} hF, \ldots, \chi_Z h_n^{-1} hF)$. Therefore $p$ is surjective.

Let $H$ be a group and $Z \subseteq H$. $\mathcal{M}_H$ is the endomorphism ring of the right free $k$-module on $\{\delta_h \mid h \in H\}$. $\mathcal{M}_Z$ is the subring of $\mathcal{M}_H$ generated on $\{\delta_z \mid z \in Z\}$. We consider a map $\epsilon = \epsilon_{(H, Z)} : H \times H \to k$ satisfying

$$
\epsilon(h, z) = \chi_{h^{-1}Z \cap Z}(z) = \begin{cases} 
1 & \text{if } z \in h^{-1}Z \cap Z, \\
0 & \text{if } z \notin h^{-1}Z \cap Z
\end{cases}
$$

for every $h, z \in H$. By using $\epsilon$, an injective homomorphism $i_Z : \chi_Z \mathcal{R}(H, k) \chi_Z \to \mathcal{M}_Z$ can be defined by

$$
i_Z(\chi_Z \alpha \chi_Z) \delta_z = \sum_{i=1}^n \delta_{h_i z} \epsilon(h_i, z) F_i(z)
$$

for every $\alpha = \sum_{i=1}^n h_i F_i \in \mathcal{R}(H, k)$, $h_i \in H$, $F_i \in l^1(H, k)$ and $z \in Z$. This is shown in the next lemma.
Lemma 4.2. \( i_Z \) is well-defined, a homomorphism and injective.

Proof. For every \( h \in H \) and \( z \in Z \) if \( hz \notin Z \) is satisfied, then \( \delta_{hz} \epsilon(h, z) = 0 \). Therefore we have \( i_Z(\chi_Z \alpha \chi_Z) \in M_Z \). Since \( i_Z \) preserves the sum, to prove \( i_Z \) is well-defined we will prove that for every \( \alpha = \sum_{i=1}^{n} h_i F_i \in \mathcal{R}(H, k) \) if \( \chi_Z \alpha \chi_Z = 0 \), then \( i_Z(\chi_Z \alpha \chi_Z) = 0 \). \( \chi_Z \alpha \chi_Z = \sum_{i=1}^{n} h_i \chi_{h_i^{-1}Z} \cap Z F_i = 0 \) implies \( \sum_{h \in H, \chi_{h_i^{-1}Z} \cap Z F_i = 0} \) for each \( h \in H \), and hence \( \sum_{h = h_i} \epsilon(h_i, z) F_i(z) = 0 \) for every \( z \in Z \). This shows \( i_Z(\chi_Z \alpha \chi_Z) \delta_z = 0 \).

In order to prove \( i_Z \) is a homomorphism, we only have to check that \( i_Z \) preserves the multiplication for generators about the sum since \( i_Z \) preserves the sum and the identity element. \( \chi_Z \mathcal{R}(H, k) \chi_Z \) is generated by \( \chi_Z H_{l/f}(H, k) \chi_Z \) as an additive group, and hence for \( g, h \in H, \, F_1, \, F_2 \in l/f(H, k) \), and \( z \in Z \)

\[
i_Z(\chi_Z g F_1 \chi_Z) \circ i_Z(\chi_Z h F_2 \chi_Z) \quad \delta_z =
\]

\[
n \sum_{h = h_i} \epsilon(h_i, z) F_i(z) = \sum_{h = h_i} \epsilon(h_i, z) F_i(z) = 0
\]

This implies \( i_Z \) is a homomorphism.

In order to prove \( i_Z \) is injective we will check that for every \( \alpha \in \mathcal{R}(H, k) \), \( i_Z(\chi_Z \alpha \chi_Z) = 0 \) implies \( \chi_Z \alpha \chi_Z = 0 \). We have an expression \( \alpha = \sum_{i=1}^{n} h_i F_i \) for some \( h_i \in H \) and \( F_i \in l/f(H, k) \), where we can assume that \( h_1, \ldots, h_n \) are different from each other. \( i_Z(\chi_Z \alpha \chi_Z) = 0 \) implies

\[
i_Z(\chi_Z \alpha \chi_Z) \delta_z = n \sum_{i=1}^{n} \delta_{h_i \epsilon(h_i, z)} F_i(z) = n \sum_{i=1}^{n} \delta_{h_i \epsilon(h_i, z)} \chi_{h_i^{-1}Z} \cap Z F_i(z) = 0
\]

for every \( z \in Z \). This shows that \( \chi_{h_i^{-1}Z} F_i \chi_Z = 0 \) for every \( i \). Thus

\[
\chi_Z \alpha \chi_Z = n \sum_{i=1}^{n} h_i \chi_{h_i^{-1}Z} F_i \chi_Z = 0.
\]

Let \( G \) and \( G' \) be finitely generated groups, \( X \subseteq G \), \( Y \subseteq G' \) and \( f: X \to Y \) a bijective quasi-isometry. Since \( f \) is bijective, \( f \) induces a natural isomorphism \( \tilde{f}: \mathcal{M}_X \to \mathcal{M}_Y \) as follows. For every \( A \in \mathcal{M}_X \) and \( x \in X \) we have an expression \( A(\delta_x) = \sum_{i=1}^{n} \delta_{x_i(a_i(x))} \) by some \( x_i(x) \in X \) and \( a_i(x) \in k \). Thus by using this expression of \( A(\delta_x) \), \( \tilde{f} \) satisfies

\[
\tilde{f}(A)(\delta_y) = n \sum_{i=1}^{n} \delta_{f(x_i(f^{-1}(y)))} a_i(f^{-1}(y))
\]
for every \( y \in Y \). By Lemma 4.2 we have injective homomorphisms \( i_X: \chi_X \mathcal{R}(G, k) \chi_X \rightarrow \mathcal{M}_X \) and \( i_Y: \chi_Y \mathcal{R}(G', k) \chi_Y \rightarrow \mathcal{M}_Y \).

**Lemma 4.3.**

\[
\tilde{f} \circ i_X(\chi_X \mathcal{R}(G, k) \chi_X) \subseteq i_Y(\chi_Y \mathcal{R}(G', k) \chi_Y).
\]

**Proof.** For every \( g \in G \) and \( y \in Y \) we have

\[
\tilde{f} \circ i_X(\chi_X g \chi_X)(\delta_y) = \begin{cases} 
\delta_{f(gf^{-1}(y))\epsilon(g, f^{-1}(y))} & \text{if } gf^{-1}(y) \in X, \\
0 & \text{otherwise}
\end{cases}
\]

since

\[
i_X(\chi_X g \chi_X)(\delta_y) = \delta_{gf^{-1}(y)\epsilon(g, f^{-1}(y))}.
\]

Since \( f \) is a quasi-isometry, \( L = \{f(gf^{-1}(y))y^{-1} \mid y \in Y \text{ and } gf^{-1}(y) \in X\} \) is a finite set. Therefore we have an expression \( L = \{h_1, \ldots, h_m\} \). We have \( S_i = \{y \in Y \mid gf^{-1}(y) \in X \text{ and } f(gf^{-1}(y))y^{-1} = h_i\} \). \( S_1, \ldots, S_m \) are disjoint for each other. If \( gf^{-1}(y) \in X \), then there exists \( 1 \leq j \leq m \) such that \( f(gf^{-1}(y))y^{-1} = h_j \) and

\[
\delta_{f(gf^{-1}(y))\epsilon(g, f^{-1}(y))} = \delta_{gf^{-1}(y)\epsilon(g, f^{-1}(y))} = \delta_{h_j\epsilon(g, f^{-1}(y))}
\]

\[
= \delta_{h_j, y}(g, f^{-1}(y)) = \delta{(h_j, y)}_{\chi_S(y)}(g, f^{-1}(y))
\]

\[
= \delta_{h_i, y}(g, f^{-1}(y)) = \chi_h(y)_{\chi_S(y)}(g, f^{-1}(y))
\]

\[
= \chi_h(y)_{\chi_S(y)}(g, f^{-1}(\cdot) \chi_{\chi}) (\delta_y).
\]

where \( \epsilon_{\chi}(g, f^{-1}(\cdot) \chi_{\chi}) \in l^f(G', k) \). This shows that \( \tilde{f} \circ i_X(\chi_X g \chi_X) \) is in the image of \( i_Y \).

On the other hand, for every \( F \in l^f(G, k) \) and \( y \in Y \) we have

\[
\tilde{f} \circ i_X(\chi_X F \chi_X)(\delta_y) = \delta_y F(f^{-1}(y))
\]

\[
= i_Y(\chi_Y (F \circ f^{-1}) \chi_Y)(\delta_y),
\]

where \( (F \circ f^{-1}) \chi_Y \in l^f(G', k) \). This shows that \( \tilde{f} \circ i_X(\chi_X F \chi_X) \) is in the image of \( i_Y \).

\( \chi_X \mathcal{R}(G, k) \chi_X \) is generated by \( \chi_X G \chi_X \) and \( \chi_X l^f(G, k) \chi_X \) as a ring. Therefore we have \( \tilde{f} \circ i_X(\chi_X \mathcal{R}(G, k) \chi_X) \subseteq i_Y(\chi_Y \mathcal{R}(G', k) \chi_Y) \).
By Lemma 4.3 we have a homomorphism \( \Phi : \chi_X \mathcal{R}(G, k) \chi_X \to \chi_Y \mathcal{R}(G', k) \chi_Y \) with \( i_Y \circ \Phi = \tilde{f} \circ i_X \). Similarly, we have \( \psi : \chi_Y \mathcal{R}(G', k) \chi_Y \to \chi_X \mathcal{R}(G, k) \chi_X \) with \( i_X \circ \psi = \tilde{f}^{-1} \circ i_Y \). Therefore \( \Phi \circ \psi = \text{id} \) and \( \psi \circ \Phi = \text{id} \).

We summarize the discussion above as follows:

**Lemma 4.4.** Let \( G \) and \( G' \) be finitely generated, \( X \subseteq G \), and \( Y \subseteq G' \). If a bijective quasi-isometry \( f : X \to Y \) exists, then \( \chi_X \mathcal{R}(G) \chi_X \) and \( \chi_Y \mathcal{R}(G') \chi_Y \) are isomorphic. The isomorphism \( \Phi = \Phi_f : \chi_X \mathcal{R}(G, k) \chi_X \to \chi_Y \mathcal{R}(G', k) \chi_Y \) is given by

\[
\Phi(\chi_X g \chi_X) = \chi_Y \left( \sum_{i=1}^{m} h_i \chi_{X, \epsilon(g, x)} (g, f^{-1}(\cdot)) \chi_Y \right) \quad (g \in G),
\]

\[
\Phi(\chi_X f \chi_X) = \chi_Y (F \circ f^{-1}) \chi_Y \quad (F \in \mathcal{R}(G, k)),
\]

where \( \{ f(gf^{-1}(y))y^{-1} \mid y \in Y \text{ and } gf^{-1}(y) \in X \} = \{ h_1, \ldots, h_m \} \) (\( m, h_i \) depend on \( g \)) and \( S_i = \{ y \in Y \mid gf^{-1}(y) \in X \text{ and } f(gf^{-1}(y))y^{-1} = h_i \} \).

Given two quasi-isometric non-amenable finitely generated groups, we can find a bijective quasi-isometry between them [5, Proposition p. 104], and hence by combining this fact and Lemma 4.4, we have

**Corollary 4.5.** If non-amenable finitely generated groups \( G \) and \( G' \) are quasi-isometric, then \( (\mathcal{R}(G), \mathcal{R}(G', k)) \) and \( (\mathcal{R}(G'), \mathcal{R}(G, k)) \) are isomorphic as pairs of rings.

### 4.2. The proof of the main theorem.

Let \( G \) and \( G' \) be finitely generated groups, and \( f : G \to G' \) a quasi-isometry. There exist cores of \( f : X \subseteq G \) and \( Y \subseteq G' \). By Lemma 4.1 \( I_X = \chi_X \mathcal{R}(G, k) \) is a progenerator. By Theorem 2.3 \( \mathcal{R}(G, k) \) and \( \text{End}(I_X) \) are Morita equivalent. Since \( \chi_X \) is an idempotent, \( \text{End}(I_X) \simeq \chi_X \mathcal{R}(G, k) \chi_X \). Therefore \( \mathcal{R}(G, k) \) and \( \chi_X \mathcal{R}(G, k) \chi_X \) are Morita equivalent. Furthermore, by Theorem 2.3

\[
\chi_X \mathcal{R}(G, k) \chi_X \mathcal{R}(G, k), \quad \mathcal{R}(G, k) \chi_X \mathcal{R}(G, k) \chi_X
\]

are bimodules, and

\( (\chi_X \mathcal{R}(G, k) \otimes \mathcal{R}(G, k) \to) : \mathcal{R}(G, k) \text{-Mod} \to \chi_X \mathcal{R}(G, k) \chi_X \text{-Mod} \),

\( (\mathcal{R}(G, k) \chi_X \otimes_{\chi_X \mathcal{R}(G, k) \chi_X} -) : \chi_X \mathcal{R}(G, k) \chi_X \text{-Mod} \to \mathcal{R}(G, k) \text{-Mod} \)

are inverse equivalences. Similarly, \( \mathcal{R}(G', k) \) and \( \chi_Y \mathcal{R}(G', k) \chi_Y \) are Morita equivalent. Furthermore, by Theorem 2.3

\[
\chi_Y \mathcal{R}(G', k) \chi_Y \mathcal{R}(G', k), \quad \mathcal{R}(G', k) \chi_Y \mathcal{R}(G', k) \chi_Y
\]

are bimodules, and

\( (\chi_Y \mathcal{R}(G', k) \otimes \mathcal{R}(G', k) \to) : \mathcal{R}(G', k) \text{-Mod} \to \chi_Y \mathcal{R}(G', k) \chi_Y \text{-Mod} \),

\( (\mathcal{R}(G', k) \chi_Y \otimes_{\chi_Y \mathcal{R}(G', k) \chi_Y} -) : \chi_Y \mathcal{R}(G', k) \chi_Y \text{-Mod} \to \mathcal{R}(G', k) \text{-Mod} \)

are inverse equivalences.

Similarly, \( \mathcal{R}(G', k) \) and \( \chi_Y \mathcal{R}(G', k) \chi_Y \) are Morita equivalent.
are bimodules, and
\[(\chi_Y \mathcal{R}(G', k) \otimes_{\mathcal{R}(G', k)} -) : \mathcal{R}(G', k)\text{-Mod} \to \chi_Y \mathcal{R}(G', k)\chi_Y\text{-Mod},\]
\[(\mathcal{R}(G', k)\chi_Y \otimes_{\chi_Y \mathcal{R}(G', k)\chi_Y} -) : \chi_Y \mathcal{R}(G', k)\chi_Y\text{-Mod} \to \mathcal{R}(G', k)\text{-Mod}\]
are inverse equivalences. Therefore
\[
\text{Res } \Phi : \chi_Y \mathcal{R}(G', k)\chi_Y\text{-Mod} \to \chi_X \mathcal{R}(G, k)\chi_X\text{-Mod},
\]
\[
\text{Res}(\Phi^{-1}) : \chi_X \mathcal{R}(G, k)\chi_X\text{-Mod} \to \chi_Y \mathcal{R}(G', k)\chi_Y\text{-Mod}
\]
are inverse equivalences. Therefore
\[
\mathcal{F}_1 = (\mathcal{R}(G', k)\chi_Y \otimes_{\chi_Y \mathcal{R}(G', k)\chi_Y} -) \circ \text{Res}(\Phi^{-1}) \circ (\chi_X \mathcal{R}(G, k) \otimes_{\mathcal{R}(G, k)} -),
\]
\[
\mathcal{F}_2 = (\mathcal{R}(G, k)\chi_X \otimes_{\chi_X \mathcal{R}(G, k)\chi_X} -) \circ \text{Res } \Phi \circ (\chi_Y \mathcal{R}(G', k) \otimes_{\mathcal{R}(G', k)} -)
\]
are inverse equivalences. As a result, we obtain the main theorem:

**Theorem 1.** If finitely generated groups $G$ and $G'$ are quasi-isometric, then $\mathcal{R}(G, k)$ and $\mathcal{R}(G', k)$ are Morita equivalent.

### 4.3. On some characteristic modules.

Let $H$ be a finitely generated group. There exist characteristic $\mathcal{R}(H, k)$-modules of functions on $H$ preserved by $\mathcal{F}_1$ of the previous subsection. In this subsection, we use the notation of Section 4.2.

We consider a left $\mathcal{R}(H, k)$-module $l(H, k) = \{F : H \to k\}$ with an action
\[
(hF_1)F = F_1^h F^{h^{-1}} \quad (hF_1 \in \mathcal{R}(H, k), F \in l(H, k)).
\]

$l^H(H, k)$ or $l_c(H, k) = \{F : H \to k \mid \#(\text{supp}(F)) < \infty\}$ are submodules of $l(H, k)$. For $Z \subseteq H$ a left $\chi_Z \mathcal{R}(H, k)\chi_Z$-module $\chi_Z \mathcal{R}(H, k) \otimes_{\mathcal{R}(H, k)} l(H, k)$ is isomorphic to the left $\chi_Z \mathcal{R}(H, k)\chi_Z$-module $l(Z, k)$ with an action
\[
(\chi_Z hF_1 \chi_Z)F = \chi_Z F_1^h \chi_Z F^{h^{-1}} \quad (\chi_Z hF_1 \chi_Z \in \chi_Z \mathcal{R}(H, k)\chi_Z, F \in l(Z, k)).
\]

**Lemma 4.6.** Under the notation of Section 4.2
\[
(\text{Res } \Phi) \circ (\chi_Y \mathcal{R}(G', k) \otimes_{\mathcal{R}(G', k)} -)(l(G', k)) \simeq (\chi_X \mathcal{R}(G, k) \otimes_{\mathcal{R}(G, k)} -)(l(G, k)).
\]
Proof. By Lemma 4.4 \((Res \Phi) \circ (\chi_Y R(G', k) \otimes R(G', k) \rightarrow l(G', k))\) is isomorphic to the left \(\chi_X R(G, k) \chi_X\)-module \(l(Y, k)\) with an action

\[
(\chi_X g \chi_X) F = (\Phi_{f|_k})(\chi_X g \chi_X) F
= \left(\chi_Y \sum_{i=1}^{m} h_i \chi_S, \epsilon_{g, X} (g, f|_X^{-1}(\cdot)) \chi_Y \right) F
= \chi_Y \sum_{i=1}^{m} \chi_{h_i S, \epsilon_{g, X}} (g, f|_X^{-1}(\cdot)) h_i^{-1} \chi_{h_i Y} F h_i^{-1}
\]

for every \(g \in G\) and \(F \in l(Y, k)\), and also

\[
(\chi_X F_1 \chi_X) F = (\Phi_{f|_k})(\chi_X F_1 \chi_X) F
= (\chi_Y (F_1 \circ (f|_X)^{-1}) \chi_Y) F
= \chi_Y (F_1 \circ (f|_X)^{-1}) \chi_Y F
\]

for every \(F_1 \in l^f(G, k)\) and \(F \in l(Y, k)\). We define \(\lambda : l(Y, k) \rightarrow l(X, k)\) by \(F \mapsto F \circ f|_X\). We will prove that \(\lambda\) is a left \(\chi_X R(G, k) \chi_X\)-isomorphism. Since \(\lambda\) is a bijective additive group homomorphism, we only have to check that the action is preserved. For every \(g \in G\), \(F \in l(Y, k)\) and \(x \in X\), under the notation of Lemma 4.4, if \(x \in g X\), then there exists the only \(h_j\) such that \(f(x) \in h_j S_j\), and also if \(x \notin g X\), then there exists no \(h_j\) such that \(f(x) \in h_j S_j\). Therefore

\[
\lambda((\chi_X g \chi_X) F)(x) = (\Phi_{f|_k})(\chi_X g \chi_X) F(f(x))
= \left(\chi_Y \sum_{i=1}^{m} \chi_{h_i S, \epsilon_{g, X}} (g, f|_X^{-1}(\cdot)) h_i^{-1} \chi_{h_i Y} F h_i^{-1}\right)(f(x))
= \begin{cases} F h_i^{-1}(f(x)) & \text{if } x \in g X, \\ 0 & \text{if } x \notin g X \end{cases}
= \begin{cases} F \circ f(g^{-1} x) & \text{if } x \in g X, \\ 0 & \text{if } x \notin g X \end{cases}
= ((\chi_X g \chi_X) \lambda(F))(x).
\]

For every \(F_1 \in l^f(G, k)\), \(F \in l(Y, k)\) and \(x \in X\)

\[
\lambda((\chi_X F_1 \chi_X) F)(x) = (\Phi_{f|_k})(\chi_X F_1 \chi_X) F(f(x))
= (\chi_Y (F_1 \circ (f|_X)^{-1}) \chi_Y F)(f(x))
= F_1(F \circ f)(x)
= ((\chi_X F_1 \chi_X) \lambda(F))(x).
\]
**Theorem 4.7.** Under the notation of Section 4.2
(1) $\mathcal{F}_1(l(G, k)) \cong l(G', k)$, $\mathcal{F}_2(l(G', k)) \cong l(G, k)$.
(2) $\mathcal{F}_1(l^f(G, k)) \cong l^f(G', k)$, $\mathcal{F}_2(l^f(G', k)) \cong l^f(G, k)$.
(3) $\mathcal{F}_1(l_e(G, k)) \cong l_e(G', k)$, $\mathcal{F}_2(l_e(G', k)) \cong l_e(G, k)$.

Proof. (1) We send the equation of Lemma 4.6 by $(\mathcal{R}(G, k))_{X_k} \otimes_{X_k} \mathcal{R}(G, k)_{X_k} \to -)$. Therefore we have $\mathcal{F}_2(l(G', k)) \cong l(G, k)$. We also send this equation by $\mathcal{F}_1$, and hence $\mathcal{F}_1(l(G, k)) \cong l(G', k)$. Since $\lambda$ of the proof of Lemma 4.6 is an isomorphism on $l^f$ and $l_e$, (2) and (3) are also proved. \hfill $\square$

We have a left $\mathcal{R}(H, k)$-module $T = T_H = T_{(H, k)} = \mathcal{R}(H, k) \otimes_{H_k} k$. $T_H$ is isomorphic to $l^f(H, k)$. Indeed, $\theta: T_H \to l^f(H, k)$ defined by $\theta(\sum_{i=1}^n g_n F_n \otimes k) = \sum_{i=1}^n F_n^{g_n} k$ gives an isomorphism. Let $M$ be a left $\mathcal{R}(H, k)$-module. Since $\mathcal{R}(H, k)$ is a subring of $\mathcal{R}(H, k)$, $M$ is regarded as a left $\mathcal{R}(H, k)$-module. By the flatness of $\mathcal{R}(H, k)_{H_k}$ (see Lemma 5.1 of the next section), we have $\text{Ext}_{\mathcal{R}(H, k)}^n(l^f(H, k), M) = \text{Ext}_{\mathcal{R}(H, k)}^n(T_H, M) = \text{Ext}_{\mathcal{R}(H, k)}^n(k, M) = H^n(H, M)$. Since $l_e(H, k)$ is isomorphic to $H_k$, $H^n(H, l_e(H, k)) = H^n(H, H_k)$. This cohomology group is the coarse cohomology (see [7]). By Theorem 4.7 $H^n(H, H_k)$ is a quasi-isometry invariant.

In the case of $k = C$ (or $\mathbb{R}$) for $0 < p \leq \infty$ we have a module of $p$-summable functions $l^p(G, C) \subseteq l(G, C)$. We can also prove $\mathcal{F}_1(l^p(G, C)) \cong l^p(G', C)$, $\mathcal{F}_2(l^p(G', C)) \cong l^p(G, C)$. Therefore $\text{Ext}_{\mathcal{R}(H, k)}^n(l^p(H, C), l^p(H, C))$ is a quasi-isometry invariant. This cohomology group is isomorphic to $H^n(H, l^p(H, C))$: the coarse $l^p$-cohomology (see [6]).

5. The global dimension and the weak global dimension of algebraic translation algebras

Let $G$ be a finitely generated group. We see that $\mathcal{R}(G, k)_{G_k}$ is a flat $G_k$-module. Let $\Lambda = \{S = \{S_1, \ldots, S_n\} \mid S_i \subseteq G, \bigcup_{i=1}^n S_i = G\}$ be the set of finite decompositions of $G$. For $\alpha$ and $\beta \in \Lambda$ we denote $\alpha < \beta$ if $\beta$ is a refinement of $\alpha$. Let $L_{\alpha} = \bigoplus_{\alpha \in \Lambda}^n (G_k)$ be a right free $G_k$-module. If $\alpha < \beta$, then for each $1 \leq k \leq n_{\beta}$ there exists $1 \leq i_k \leq n_{\alpha}$ such that $\beta_k \subseteq \alpha_{i_k}$. Let $f_{\beta\alpha}: L_{\alpha} \to L_{\beta}$ be a $G_k$-homomorphism such that $f_{\beta\alpha}(x_1, \ldots, x_{n_{\alpha}}) = (x_{i_1}, \ldots, x_{i_{n_{\beta}}})$ if $x_k \in G_k$. These data define a direct system of right $G_k$-modules, and hence we have a right $G_k$-module $\lim_{\to} L_{\alpha}$. $L_{\alpha} = (\bigoplus_{\alpha \in \Lambda}^n L_{\alpha})/N$, where $N$ is a submodule of $\bigoplus_{\alpha \in \Lambda}^n L_{\alpha}$ generated by $\{l_\beta \circ f_{\beta\alpha}(x) - l_\alpha(x) \mid \alpha < \beta, x \in L_{\alpha}\}$. $l_{\alpha}$ and $l_{\beta}$ are injections to $\bigoplus_{\alpha \in \Lambda}^n L_{\alpha}$.

**Lemma 5.1.** $\mathcal{R}(G, k)_{G_k}$ is a direct limit of flat $G_k$-modules:

$$\lim_{\to} L_{\alpha} \cong \mathcal{R}(G, k)_{G_k}.$$ Therefore $\mathcal{R}(G, k)_{G_k}$ is a flat $G_k$-module, and the functor $\mathcal{R}(G, k) \otimes_{G_k} -: G_k\text{-Mod} \to \mathcal{R}(G, k)\text{-Mod}$ is exact.
We assume that $G$ acts on $l$ from the right trivially, we have $R(l) = \sum_{i=1}^{n_x} \chi_i^x x_i \in Gk$. The direct sum of $\{t_{\alpha} \mid \alpha \in \Lambda\}$ defines an isomorphism.

We recall the definitions of some homological dimensions. Let $R$ be a ring, and $M$ a left $R$-module.

1. $\text{fgd}(M) = \sup\{n \mid \exists a \text{ right } R\text{-module } N \text{ with } \text{Tor}_n^R(N, M) \neq 0\}$. This number is equal to the minimal number $n$ such that there exists an $n$-length flat resolution of $M$.

2. $\text{pd}_g(M) = \sup\{n \mid \exists a \text{ left } R\text{-module } N \text{ with } \text{Ext}_n^g(M, N) \neq 0\}$. This number is also equal to the minimal number $n$ such that there exists an $n$-length projective resolution of $M$.

3. $\text{wd}(R) = \sup\{\text{fgd}(M) \mid M \text{ is a left } R\text{-module}\}$.

4. $\text{l.gl.dim}(R) = \sup\{\text{pd}_g(M) \mid M \text{ is a left } R\text{-module}\}$ ([21, Section 3, 4] is a good reference for Tor or Ext and homological dimensions).

$\text{wd}$ and $\text{l.gl.dim}$ are Morita invariants. We discuss $\text{l.gl.dim}(R(G, k))$ and $\text{wd}(R(G, k))$.

**Lemma 5.2.** Let $G$ be a finitely generated group, and $l$ a ring containing $k$ as a subring. We assume that $G$ acts on $l$ from the right trivially on $k$. Let $R = G \ast l$, and $T = R \otimes_{Gk} k$. For every left $R$-module $A$ and $B$, and an $R$-projective resolution $C$ of $A$ we have a spectral sequence

$$\text{Ext}_R^p(T, H^q(\widetilde{\text{Hom}}(C, B))) \Rightarrow \text{Ext}_{R}^{p+q}(A, B),$$

where for a left $R$-module $C$, $\widetilde{\text{Hom}}(C, B) = \text{Hom}(C, B)$ is a left module with a left action of $R$ defined by

$$(gF)\varphi(x) = Fg^\varphi(g^{-1}x)$$

for every $\varphi \in \widetilde{\text{Hom}}(C, B)$, $g \in G$, $F \in l$ and $x \in C$. The notation of a spectral sequence is that of [3, Chapter XV].

**Proof.** This spectral sequence is obtained by modifying a spectral sequence of Cartan and Leray [3, Proposition 8.2].

First, we prove $\text{Ext}_R^p(T, \widetilde{\text{Hom}}(R, B)) = 0$ (if $p > 0$) by direct calculation. Since $R = \bigoplus_{g \in G} l \cdot g$ and $\varphi \in \widetilde{\text{Hom}}(R, B)$ is decided by $\varphi(g) \in B$, we have $\widetilde{\text{Hom}}(R, B) = \prod_{g \in G} B_g$, where $B_g$ is a copy of $B$. For $(b_g)_{g \in G} \in \prod_{g \in G} B_g$, the $R$-action is given by $(xF)(b_g)_{g \in G} = (F^x^{-1} x b_{g^{-1}x})_{g \in G}$ ($x \in G$, $F \in l$). We consider free right $k$-modules $I_p = \{(\sigma_0, \ldots, \sigma_p) \mid \sigma_i \in G\}k$ and $\varphi_{p-1}(\sigma_0, \ldots, \sigma_p) = \sum_{i=0}^{p} (-1)^i (\sigma_0, \ldots, \tilde{\sigma}_i, \ldots, \sigma_p)$. $I = \{I_p, \varphi_p\}$ is the $Gk$-standard resolution of $k$. Then $\tilde{I} = R \otimes_{Gk} I$ is an $R$-projective resolution of $T$. Since $f \in \text{Hom}(\tilde{I}_p, \widetilde{\text{Hom}}(R, B))$ is decided by $f(1, \sigma_1, \ldots, \sigma_p)(g) \in B_g$, we have

$$\text{Hom}(\tilde{I}_p, \widetilde{\text{Hom}}(R, B)) = \prod_{\sigma_1, \ldots, \sigma_p \in G} B_{\sigma_1, \ldots, \sigma_p, g}.$$
where \( B_{\sigma_1, \ldots, \sigma_p, g} \) is a copy of \( B \). \( \partial_p = \text{Hom}(\tilde{\varphi}_p, \text{Hom}_\mathcal{R}(\mathcal{T}, B)) \) satisfies

\[
\partial_p((b_{\sigma_1, \ldots, \sigma_p, g})_{\sigma_1, \ldots, \sigma_p, g} \in G) = \left( \sigma_1 b_{\sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_p^{-1} \sigma_{p+1}^{-1} g} + \sum_{i=1}^{p+1} (-1)^i b_{\sigma_i \ldots, \sigma_{p+1}, g} \right)_{\sigma_1, \ldots, \sigma_{p+1}, g} \in G.
\]

By the definition of Ext, \( \text{Ext}^p_\mathcal{R}(T, \text{Hom}_\mathcal{R}(\mathcal{T}, B)) = \text{Ker}\partial_p/\text{Im}\partial_{p-1} \). For every \((b_{\sigma_1, \ldots, \sigma_p, g})_{\sigma_1, \ldots, \sigma_p, g} \in G \subset \text{Ker}\partial_p\) and in the case of \( p \geq 1 \), we have \( c_{\sigma_1, \ldots, \sigma_{p-1}, g} = (-1)^p b_{\sigma_1, \ldots, \sigma_{p-1}, g} \). This satisfies

\[
\partial_{p-1}(c_{\sigma_1, \ldots, \sigma_{p-1}, g})_{\sigma_1, \ldots, \sigma_{p-1}, g} \in G = (b_{\sigma_1, \ldots, \sigma_p, g})_{\sigma_1, \ldots, \sigma_p, g} \in G.
\]

In fact,

\[
\begin{align*}
\partial_{p-1}(c_{\sigma_1, \ldots, \sigma_{p-1}, g})_{\sigma_1, \ldots, \sigma_{p-1}, g} &= \left( \sigma_1 c_{\sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_p^{-1} \sigma_{p-1}^{-1} g} + \sum_{i=1}^{p-1} (-1)^i c_{\sigma_i \ldots, \sigma_{p-1}, g} \right)_{\sigma_1, \ldots, \sigma_{p-1}, g} \\
&= \left( (-1)^p \sigma_1 b_{\sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_p^{-1} \sigma_{p-1}^{-1} g} + \sum_{i=1}^{p-1} (-1)^{p+i} b_{\sigma_i \ldots, \sigma_{p-1}, g} \right)_{\sigma_1, \ldots, \sigma_{p-1}, g}.
\end{align*}
\]

By substituting \( g \) for \( \sigma_{p+1} \) in (i), we have

\[
\begin{align*}
\sigma_1 b_{\sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_p^{-1} \sigma_{p-1}^{-1} g} + \sum_{i=1}^{p} (-1)^i b_{\sigma_i \ldots, \sigma_p, g} + (-1)^{p+1} b_{\sigma_1 \ldots, \sigma_p, g} &= 0.
\end{align*}
\]

(ii) is obtained by (iii) and (iv). Therefore \( \text{Ext}^p_\mathcal{R}(T, \text{Hom}_\mathcal{R}(\mathcal{T}, B)) = 0 \) if \( p > 0 \) is proved. This shows \( \text{Ext}^p_\mathcal{R}(T, \text{Hom}_\mathcal{R}(\mathcal{P}, B)) = 0 \) if \( p > 0 \) for every projective left \( \mathcal{R} \)-module \( \mathcal{P} \).

Second, for left \( \mathcal{R} \)-modules \( X \) and \( Y \), \( \rho : \text{Hom}_\mathcal{R}(T, \text{Hom}_\mathcal{R}(X, Y)) \to \text{Hom}_\mathcal{R}(X, Y) \) defined by \( f \mapsto f(1) \) is an isomorphism.

Let \( X \) be a projective resolution of \( T \) and \( Y = \text{Hom}_\mathcal{R}(\mathcal{C}, B) \). \( \text{Hom}_\mathcal{R}(\mathcal{X}, \mathcal{Y}) \) is a double complex, and hence we have two spectral sequences with the same limit:

\[
\begin{align*}
\Pi^p_q &= \text{Hom}(\text{Tot}^p(\text{Hom}_\mathcal{R}(\mathcal{X}, \mathcal{Y}))), \\
\Pi^q_p &= \text{Hom}(\text{Tot}^q(\text{Hom}_\mathcal{R}(\mathcal{X}, \mathcal{Y}))).
\end{align*}
\]

By the first assertion above \( \text{Hom}(\text{Hom}_\mathcal{R}(\mathcal{X}, \mathcal{Y})) = \text{Hom}_\mathcal{R}(T, Y) \) if \( p = 0 \) and \( \text{Hom}(\text{Hom}_\mathcal{R}(\mathcal{X}, \mathcal{Y})) = 0 \) (otherwise). We have \( \Pi^p_q = \text{Hom}(\text{Hom}_\mathcal{R}(\mathcal{T}, \mathcal{Y})) = \text{Hom}(\mathcal{C}, B) = \text{Ext}^q_\mathcal{R}(\mathcal{A}, \mathcal{B}) \) if \( p = 0 \) and \( \Pi^p_q = 0 \) (otherwise) by the second assertion above. Therefore \( \text{Tot}^p(\text{Hom}_\mathcal{R}(\mathcal{X}, \mathcal{Y})) = \text{Ext}^p_\mathcal{R}(\mathcal{A}, \mathcal{B}) \). Since each term \( X_p \) of \( X \) is projective,
Hom$_R(X_p, -)$ is exact.

$$H^q(\text{Hom}_R(X_p, Y)) = H^q(\text{Hom}_R(X_p, \overline{\text{Hom}}_r(C, B)))$$

$$= \text{Hom}_R(X_p, H^q(\text{Hom}_r(C, B)))$$

shows that $l^p_{d_2} = \text{Ext}^p_R(T, H^q(\overline{\text{Hom}}_r(C, B)))$.

\[ \square \]

**Corollary 5.3.**

$$\text{pd}_{R(G, k)}(T) \leq \text{l.gl.dim}(R(G, k)) \leq \text{pd}_{R(G, k)}(T) + \text{l.gl.dim}(l^f(G, k)).$$

We can also prove the Tor version of Lemma 5.2, and hence

$$\text{fd}_{R(G, k)}(T) \leq \text{wd}(R(G, k)) \leq \text{fd}_{R(G, k)}(T) + \text{wd}(l^f(G, k)).$$

We estimate $\text{pd}_{R(G, k)}(T)$ and $\text{fd}_{R(G, k)}(T)$ by the homological dimensions of $G$.

**Lemma 5.4.** (1) $\text{pd}_{R(G, k)}(T) \leq \text{cd}_k(G)$, where $\text{cd}_k(G) = \text{pd}_{G_k}(k)$.

(2) $\text{fd}_{R(G, k)}(T) \leq \text{hd}_k(G)$, where $\text{hd}_k(G) = \text{fd}_{G_k}(k)$.

(3) If $\text{cd}_k(G) < \infty$, then $\text{cd}_k(G) = \text{pd}_{R(G, k)}(T)$.

(4) If $\text{hd}_k(G) < \infty$, then $\text{hd}_k(G) = \text{fd}_{R(G, k)}(T)$.

Proof. Since the functor $R(G, k) \otimes_{G_k} -$ is exact by Lemma 5.1, a projective resolution (flat resolution) of $k$ is mapped to a projective resolution (flat resolution) of $T$ by $R(G, k) \otimes_{G_k} -$ Therefore (1) and (2) are obtained.

(3) and (4) are proved by the same argument as [18, Section 4].

We estimate $\text{l.gl.dim}(l^f(G, k))$ and $\text{wd}(l^f(G, k))$.

**Lemma 5.5.** Let $\Delta$ be a countably infinite set.

(1) If $k$ is a field, then $\text{wd}(l^f(\Delta, k)) = 0$.

(2) If $k$ is $\mathbb{Z}$, then $\text{wd}(l^f(\Delta, k)) = 1$.

If the continuum hypothesis is true, then

(3) if $k$ is a field, then $\text{l.gl.dim}(l^f(\Delta, k)) = 2$.

(4) if $k$ is $\mathbb{Z}$, then $\text{l.gl.dim}(l^f(\Delta, k)) \leq 3$.

Proof. (1) We see that for every $F \in l^f(\Delta, k)$, $F \in F \cdot l^f(\Delta, k) \cdot F$ is satisfied. Therefore $l^f(\Delta, k)$ is von-Neumann regular [11, xviii, the third paragraph]. This implies that $\text{wd}(l^f(\Delta, k)) = 0$ [11, (5.62a) p. 185].

(2) $l^f(\Delta, k)$ is not von-Neumann regular since $2 \notin 2 \cdot l^f(\Delta, k) \cdot 2$. Every ideal of $l^f(\Delta, k)$ is generated by projective modules with the form $l^f(\Delta, k) \cdot \chi \eta$, and hence every ideal of $l^f(\Delta, k)$ is flat. This implies $\text{wd}(l^f(\Delta, k)) = 1$ [11, (5.69) p. 187].
(3) By the theorem of Osofsky [15, Corollary 2.47] for every ring $R$ if every left ideal of $R$ is generated by $\mathfrak{s}_h$ elements, then

$$\text{l.gl.dim}(R) \leq \text{wd}(R) + h + 1.$$ 

Every ideal of $I^f(\Delta, k)$ is generated by characteristic functions on $\Delta$. Therefore if the continuum hypothesis is true, then $\text{l.gl.dim}(I^f(\Delta, k)) \leq 2$. Since $I^f(\Delta, k)$ has a non-projective ideal $(I^f(\Delta, k)$ is not semi-simple and not hereditary), $\text{l.gl.dim}(I^f(\Delta, k)) = 2$ [11, (5.14) p.169].

(4) It is proved by the theorem of Osofsky and (2). $\blacksquare$

We remark that $\text{wd}(I^\infty(\Delta, C)) \geq 1$. Indeed, for $s_1, \ldots, s_n, \ldots \in \Delta$ there exists a function $f \in I^\infty(G, C)$ such that $f(s_n) = \exp(-n)$, and $f \notin f \cdot I^\infty(\Delta, C) \cdot f$.

**Theorem 5.6.** If $k$ is a field, then

1. if $\text{hd}_k(G) < \infty$, then $\text{wd}(R(G, k)) = \text{hd}_k(G)$,
   and if the continuum hypothesis is true, then
2. if $\text{cd}_k(G) < \infty$, then $\text{cd}_k(G) \leq \text{l.gl.dim}(R(G, k)) \leq \text{cd}_k(G) + 2$.

**Proof.** The assertions (1) and (2) follow from Corollary 5.3, Lemmas 5.4 and 5.5. $\blacksquare$

6. Geometric description of $R(G, k)$

In this section, we need some theorems of groupoid theory and category theory. Everything needed in this section is in Sections 2.5, 2.6 and 2.7. Let $G$ be a finitely generated group with the identity element $e$, and $\mathcal{G} = G \ltimes \beta G$ an étale groupoid. We consider $\text{Mod}_k(\mathcal{G})$. We define the characteristic object $U \in \text{Mod}_k(\mathcal{G})$.

**Definition 6.1.** We define $U = \beta G \times G k$, where $G k$ has the discrete topology. An element $(x, \alpha) \in \beta G \times G k$ is denoted by $_x\alpha$. We also define

- $p_0 : U \rightarrow \beta G$ by $p_0(_x\alpha) = x$,
- $p_1 : (G \ltimes \beta G) \times_{p_0} U \rightarrow U$ by $p_1((g, x), _x\alpha) = _x(g\alpha)$,
- $M : U \oplus U \rightarrow U$ by $M(_x\alpha, _y\beta) = _x(\alpha + \beta)$ ($x \in \beta G$),
- $\mathcal{U} : \emptyset \rightarrow U$ by $\mathcal{U}(x) = _x(0)$ ($x \in \beta G$),
- $\nu : U \rightarrow U$ by $\nu(_x\alpha) = _x(-\alpha)$ ($x \in \beta G$),
- $\mathcal{M} : k \oplus U \rightarrow U$ by $\mathcal{M}(k, _x\alpha) = _x(k\alpha)$ ($x \in \beta G$).

$U = ((U, p_0, p_1), M, \mathcal{U}, \nu, \mathcal{M})$ is an object of $\text{Mod}_k(\mathcal{G})$. $p_0, p_1, M, \mathcal{U}, \nu$ and $\mathcal{M}$ are open maps. $\{_x\alpha \mid x \in \beta G\}$ can be identified with $\beta G$ by $p_0$. 
A morphism from the characteristic object $U$ is determined on $\beta G \subseteq U$:

**Lemma 6.2.** For every object $S = ((S, q_0, q_1), M_S, U_S, v_S, \mathcal{M}_S)$ of $\Mod_k(G)$ a continuous map $w: \beta G \to S$ satisfying $q_0 \circ w = \id_{\beta G}$ defines the unique morphism $f: U \to S$ such that $f|_{\beta G} = w$.

Proof. We define $f$ by $f((\sum_{i=1}^{n} g_i k_i)) = \sum_{i=1}^{n} k_i \cdot (g_i, g_i^{-1} x) \cdot w(g_i^{-1} x(e))$ for every $x \in \beta G, g_i \in G$ and $k_i \in k$. Therefore if $f$ is a morphism, then $f$ is uniquely defined. Since $q_0 \circ w = \id_{\beta G}, f$ satisfies $q_0 \circ f = p_0$. In order to prove $f$ is a morphism we may prove that $f$ is continuous, but this is a routine. \hfill $\Box$

Let $A$ be an A.B.3 category. $U \in \text{Ob}(A)$ is called a projective object if for every epi $b \in \text{Hom}(B, C)$ and morphism $a \in \text{Hom}(U, C)$ there exists $c \in \text{Hom}(U, B)$ such that $a = b \circ c$. A projective object $U \in \text{Ob}(A)$ is called a projective generator if every non-zero $A \in \text{Ob}(A)$ satisfies $\text{Hom}(U, A) \neq 0$. $U \in \text{Ob}(A)$ is said to be small if every morphism from $U$ into a coproduct $s: U \to \bigoplus_{\lambda \in \Lambda} A_\lambda$ factors as $U \to \bigoplus_{\lambda \in J} A_\lambda \to \bigoplus_{\lambda \in \Lambda} A_\lambda$, where $J$ is a finite subset of $\Lambda$ and morphism between the coproducts is the one which preserves injections.

**Theorem 6.3** ([12, Theorem 3.1, p. 631]). Let $A$ be an A.B.3 category with a small projective generator $U$ and $\text{End}_A(U)$ denote the endomorphism ring of $U$. Then the functor $T: A \to \text{Mod-End}_A(U)$ defined by $T(A) = \text{Hom}(U, A)$ is an additive equivalence.

To see that Theorem 6.3 is applied to $\Mod_k(G)$ we prove that the characteristic object $U$ is a small projective generator of $\Mod_k(G)$.

**Lemma 6.4.** (1) $U \in \text{Ob}(\Mod_k(G))$ is a projective object.
(2) $U \in \text{Ob}(\Mod_k(G))$ is a projective generator.
(3) $U \in \text{Ob}(\Mod_k(G))$ is small.

Proof. (1) For every $B = (B, p_{0,B}, p_{1,B}), C = (C, p_{0,C}, p_{1,C}) \in \text{Ob}(\Mod_k(G))$, a morphism $a: U \to C$ and an epi $b: B \to C$ there exists an open set $V_x \subseteq C$ such that $a(x(e)) \in V_x$ and $p_{0,C}|_{V_x}$ is a homeomorphism for each $x \in \beta G$. Since $b: B \to C$ is an epi, $b$ is surjective. Therefore there exists $y_x \in B$ such that $b(y_x) = a(x(e))$, $b$ is continuous, and hence $y_x \in b^{-1}(V_x)$ is open. We have an open set $L_x \subseteq b^{-1}(V_x)$ such that $y_x \in L_x$ and $p_{0,B}|_{L_x}$ is a homeomorphism. We also have a clopen set $L''_x \subseteq p_{0,B}(L'_x) \cap p_{0,C}(V_x)$ since the topology of $\beta G$ is generated by clopen sets (Lemma 2.5 (2)). $b$ satisfies $p_{0,C} \circ b = p_{0,B}$, and hence $L_x = (p_{0,B}|_{L'_x})^{-1}(L''_x)$ satisfies $L_x \subseteq b^{-1}(V_x), y_x \in L_x$ and $b|_{L_x}$ is a homeomorphism. Clopen sets $W_x = a^{-1}(b(L_x))$ satisfy $\bigcup_{x \in \beta G} W_x = \beta G$. $\beta G$ is compact, and hence $\bigcup_{j=1}^{m} W_{x_j} = \beta G$. We have a refinement $\{A_i\}_{i=1}^{m}$ of $\{W_{x_j}\}_{j=1}^{m}$ such that $\beta G = \bigcup_{i=1}^{m} A_i$. We can chose $k_i$ for each $i$ such that $A_i \subseteq W_{x_i}$, and hence
we define a continuous map \( w \) by \( w|_{A_i} = (b|_{L_{x_i}})^{-1} \circ a|_{A_i} \). By Lemma 6.2 there exists a morphism \( c \) such that \( c|_{\beta G} = w \). This \( c \) satisfies \( b \circ c = a \).

(2) By (1) \( U \) is a projective object. We will prove that every non-zero object \( A = (A, p_{0,A}, p_{1,A}) \in \text{Ob}(\text{Mod}_k(G)) \) satisfies \( \text{Hom}(U, A) \neq 0 \). Since \( A \) is non-zero, there exists \( a \neq 0 \in A \). For \( x = p_{0,A}(a) \in \beta G \) since \( p_{0,A} \) is a local homeomorphism and by Lemma 2.5 (2) the topology of \( \beta G \) is generated by clopen sets, there exists clopen \( W_x \subseteq A \) such that \( a \in W_x \) and \( p_{0,A}|_{W_x} \) is a homeomorphism. We define a continuous map \( w \) by \( w|_{p_{0,A}(W_x)} = (p_{0,A}|_{W_x})^{-1} \) and \( w|_{\beta G - p_{0,A}(W_x)} = 0 \). By Lemma 6.2 there exists a morphism \( f \) such that \( f|_{\beta G} = w \). \( f(x) = a \neq 0 \) shows that \( \text{Hom}(U, A) \neq 0 \).

(3) For every morphism from \( U \) into a coproduct \( s: U \to \bigoplus_{x \in \Lambda} A_x \) there exists a finite set \( \Lambda_1 \subseteq \Lambda \) such that \( s(x) \in \bigoplus_{x \in \Lambda_1} A_x \subseteq \bigoplus_{x \in \Lambda} A_x \) for each \( x \in \beta G \). \( \bigoplus_{x \in \Lambda_1} A_x \) is open in \( \bigoplus_{x \in \Lambda} A_x \). \( s \) is continuous and the topology of \( \beta G \) is generated by clopen sets (Lemma 2.5 (2)), and hence there exists clopen \( W_x \subseteq \beta G \) such that \( s(W_x) \subseteq \bigoplus_{x \in \Lambda} A_x \) and \( x \in W_x \). \( \beta G \) is compact, and hence there exist \( x_1, \ldots, x_m \) such that \( \bigcup_{i=1}^{m} W_{x_i} = \beta G \). Therefore \( s(\beta G) \subseteq \bigoplus_{x \in \bigcup_{i=1}^{m} \Lambda_{x_i}} A_x \). This shows that \( s: U \to \bigoplus_{x \in \Lambda} A_x \) factors as \( U \to \bigoplus_{x \in J} A_x \to \bigoplus_{x \in \Lambda} A_x \), where \( J = \bigcup_{i=1}^{m} \Lambda_{x_i} \) and morphism between the coproducts is the one which preserves injections.

Since \( U \) is a small projective generator, by Theorem 6.3 the functor \( T: \text{Mod}_k(G) \to \text{Mod-End}_{\text{Mod}_k(G)}(U) \) defined by \( T(A) = \text{Hom}(U, A) \) is an additive equivalence. We describe the ring \( \text{End}_{\text{Mod}_k(G)}(U) \).

**Lemma 6.5.**

(1) A morphism \( f: U \to U \) is determined by a finite decomposition \( G = \bigsqcup_{i=1}^{n} L_i \) and \( \alpha_i \in Gk \) such that \( f(x(e)) = x(\alpha_i) \) for every \( x \in \hat{L}_i \).

(2) \( \text{End}_{\text{Mod}_k(G)}(U) \) is isomorphic to the ring \( \mathcal{R}(G, k)^{op} \).

**Proof.** (1) By Lemma 6.2 a continuous map \( w = f|_{\beta G}: \beta G \to U \) uniquely defines \( f \). Since \( U = \beta G \times Gk \), we have a continuous projection \( \pi_2: \beta G \times Gk \to Gk \). The topology of \( Gk \) is discrete, and hence for \( \alpha \in Gk \), \( \{\alpha\} \) is clopen. For clopen \( W_\alpha = (\pi_2 \circ w)^{-1}(\{\alpha\}) \), \( \beta G = \bigsqcup_{\alpha \in Gk} W_\alpha \). \( \beta G \) is compact, and hence \( \beta G = \bigsqcup_{i=1}^{n} W_{\alpha_i} \). By Lemma 2.5 (3), \( W_{\alpha_i} = \hat{L}_i \) by some \( L_i \subseteq G \), and \( G = \bigsqcup_{i=1}^{n} L_i \). Since \( \pi_2 \circ w|_{L_i} = \alpha_i \), \( w(x) = x(\alpha_i) \) for every \( x \in \hat{L}_i \).

(2) We define a map \( \theta: \text{End}_{\text{Mod}_k(G \times \beta G)}(A) \to \mathcal{R}(G, k) \) by

\[
\theta(f) = \sum_{i=1}^{n} \chi_{L_i} \alpha_i ,
\]

where \( L_i \) and \( \alpha_i \) are determined by (1). Since \( \theta \) is bijective by (1), we will prove that it is a ring homomorphism. For every \( f \) and \( g \in \text{End}_{\text{Mod}_k(G \times \beta G)}(A) \) such that \( \theta(f) = \sum_{i=1}^{n} \chi_{L_i} \alpha_i \) and \( \theta(g) = \sum_{j=1}^{m} \chi_{M_j} \beta_j \) we have an expression \( \alpha_i = \sum_{l=1}^{n} h_{i,l}k_{i,l} \) by \( h_{i,l} \in \{0, 1\} \).
$G$ and $k_{i,l} \in k$. We have the following equations:

$$
\theta(f)\theta(g) = \left( \sum_{i=1}^{n} \chi_{Li, \alpha_i} \right) \left( \sum_{j=1}^{m} \chi_{M_j, \beta_j} \right) = \\
= \left( \sum_{i=1}^{n} \sum_{l=1}^{n_i} \chi_{Li, h_{i,l}M_j} \right) \left( \sum_{j=1}^{m} \chi_{M_j, \beta_j} \right) = \\
= \left( \sum_{i=1}^{n} \sum_{l=1}^{n_i} \sum_{j=1}^{m} \chi_{Li, \cap h_{i,l}M_j} h_{i,l}k_{i,l} \beta_j \right).
$$

On the other hand, we have $G = \bigsqcup_{i,j} (L_i \cap h_{i,l}M_j)$, and for every $x \in (L_i \cap h_{i,l}M_j)$

$$
g \circ f_{(x)}(e) = g_{(x, \alpha_i)} = g \left( x \left( \sum_{l=1}^{n_i} h_{i,l}k_{i,l} \beta_j \right) \right) = \\
= \sum_{l=1}^{n_i} k_{i,l}g((h_{i,l}^{-1}x) \cdot h_{i,l}(e)) = \\
= \sum_{l=1}^{n_i} k_{i,l}(h_{i,l}^{-1}x) \cdot h_{i,l}(\beta_j) = \\
= x \left( \sum_{l=1}^{n_i} h_{i,l}k_{i,l} \beta_j \right).
$$

Therefore $\theta(f)\theta(g) = \theta(g \circ f)$.

By Theorem 6.3, Lemmas 6.4 and 6.5, we have

**Theorem 6.6.** The functor $T': \text{Mod}_k(G) \rightarrow \mathcal{R}(G, k)$-Mod defined by $T'(A) = (\text{Res} \theta^{-1})(\text{Hom}(U, A))$ is an (additive) equivalence, where $\theta$ is an isomorphism defined in Lemma 6.5 (2).

7. Appendix

7.1. An alternative proof of Theorem 1. Let $G$ and $G'$ be quasi-isometric finitely generated groups, we have Diagram 1.

Q.I. means quasi-isometric and W.E. means weak equivalent: let $G$ and $H$ be étale groupoids, $G$ and $H$ are said to be weakly equivalent (Morita equivalent) if there exists an étale groupoid $K$ and there exist essential morphisms $\Phi: K \rightarrow G$, $\Psi: K \rightarrow H$ (about precise definitions see [4, 1.4, 1.5] or [14, Section 5]). The following theorems for a quasi-isometry are known.
Let $G$ and $G'$ be finitely generated groups. If $G$ and $G'$ are quasi-isometric, then $G \cong \beta G$ and $G' \cong \beta G'$ are weakly equivalent.

Theorem 7.1 is proved by the notion of the coarse space. The converse of the theorem is also true:

**Theorem 7.2.** Let $G$ and $G'$ be finitely generated groups. If $G \cong \beta G$ and $G' \cong \beta G'$ are weakly equivalent, then $G$ and $G'$ are quasi-isometric.

Proof. If $G \cong \beta G$ and $G' \cong \beta G'$ are weakly equivalent, then $G$ and $G'$ have a topological coupling $\Omega$. Gromov’s dynamical criterion [8, 0.2. C2] shows that $G$ and $G'$ are quasi-isometric.

(i) in Diagram 1 is obtained by Theorems 7.1 and 7.2. For an étale groupoid $G$ the category of left étale $G$-spaces $\text{Sh}(G)$ is in fact a (Grothendieck) topos (see [13], and about toposes see [10]) and its equivalence class is a weak equivalence invariant of an étale groupoid, and also $\text{Mod}_k(G \ltimes \beta G)$ is a weak equivalence invariant of an étale groupoid:

**Theorem 7.3** ([4, Section 2.3]). Let $G$ and $G'$ be étale groupoids. If $G$ and $G'$ are weakly equivalent, then $\text{Sh}(G)$ and $\text{Sh}(G')$ are equivalent as (Grothendieck) toposes, and hence $\text{Mod}_k(G)$ and $\text{Mod}_k(G')$ are additively equivalent.

We have (iii) in Diagram 1. For $\text{Sh}(G)$ the converse is also true:

**Theorem 7.4** ([13, 7.7 Theorem]). Let $G$ and $G'$ be étale groupoids. $G$ and $G'$ are weakly equivalent if and only if $\text{Sh}(G)$ and $\text{Sh}(G')$ are equivalent as (Grothendieck) toposes.

(ii) in Diagram 1 is obtained by Theorem 7.4. We lose some information about quasi-isometry classes of finitely generated groups by (iii) in Diagram 1. Thus, we have the following problem:
Problem 7.5. Let $G$ and $G'$ be finitely generated groups. Is it true that if $R(G, k)$ and $R(G', k)$ are Morita equivalent, then $G$ and $G'$ are quasi-isometric? If not, give a counter-example.

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