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ON RING THEORETIC QUASI-ISOMETRY INVARIANTS

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Abstract

We introduce an algebraic version of the translation algebra of a group. We prove that a quasi-isometry of two finitely generated groups induces Morita equivalence of their algebraic translation algebras.

1. Introduction

Ring theoretic approaches for a quasi-isometry of groups were started by Shalom and Sauer [20], [18]. Shalom proved quasi-isometry invariance of the cohomological dimensions of finitely generated amenable groups, and of the **R**-Betti numbers of finitely generated nilpotent groups. In his proof, it was important that there exists a good topological coupling induced by a quasi-isometry. Sauer refined a part of Shalom's argument. He showed that a good topological coupling induces a Morita equivalence between Sauer rings of the coupled group actions (see Section 3). He applied this result to quasi-isometry invariance of the (co)homological dimensions of finitely generated groups with finite dimensions, and of the **R**-cohomology rings of finitely generated nilpotent groups.

Morita theory of Sauer rings is important for classifying groups by quasi-isometry. However, Sauer rings of the same group are not always Morita equivalent. In order to study ring theoretic invariants we should determine a ring for each finitely generated group. We propose considering the rings as follows: Let $\mathbf{k}$ be a ring with the multiplicative identity element 1, and G a finitely generated group. We consider the skew group ring $G * l^f(G, \mathbf{k})$, where $l^f(G, \mathbf{k})$ is the ring of functions with finite image. We denote this ring by $\mathcal{R}(G, \mathbf{k})$, and call it an algebraic translation algebra of G with the coefficient $\mathbf{k}$. In the case where $\mathbf{k} = \mathbf{Q}$ or $\mathbf{C}$, we see that $\mathcal{R}(G, \mathbf{k})$ is a subring of Roe's translation algebra [17, p. 68]. In fact, $\mathcal{R}(G, \mathbf{k})$ is isomorphic to the Sauer ring of a natural action of G on βG, where βG is the Stone–Čech compactification of G endowed with the discrete topology (see Lemma 3.2). We have the main theorem:

Theorem 1. *If finitely generated groups G and G' are quasi-isometric, then $\mathcal{R}(G, \mathbf{k})$ and $\mathcal{R}(G', \mathbf{k})$ are Morita equivalent.*

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Two groups always have a good topological coupling such that their Stone–Čech compactifications are coupled (see Section 3). Therefore Theorem 1 is the special case of [18]. Without using a topological coupling and [18], we prove this result in Section 4. The cores of a quasi-isometry (see Definition 2.1) play important roles.

Morita invariants of $\mathcal{R}(G, \mathbf{k})$ are quasi-isometry invariants by Theorem 1. In Section 5, we give a formula to calculate the global dimension and the weak global dimension of $\mathcal{R}(G, \mathbf{k})$. They are well-known Morita invariants. The global dimension of $\mathcal{R}(G, \mathbf{k})$ is estimated by the cohomological dimension of G and the global dimension of $l^f(G, \mathbf{k})$. The same result is true for the weak global dimension. It should be noted that some of well-known Morita invariants are trivial. For example, the center of $\mathcal{R}(G, \mathbf{k})$ coincides with that of $\mathbf{k}$ (see Lemma 2.4).

The Morita equivalence in the proof of Theorem 1 preserves some special modules (see Theorem 4.7). For example, $l^f(G, \mathbf{k})$ and $l_c(G, \mathbf{k})$ are preserved. The coarse cohomology $\mathrm{H}^n(G, G\mathbf{k})$ is isomorphic to $\mathrm{Ext}^n_{\mathcal{R}(G,\mathbf{k})}(l^f(G, \mathbf{k}), l_c(G, \mathbf{k}))$ (see Section 4.3), and hence the coarse cohomology is a quasi-isometry invariant as already known. The coarse l^p-cohomology ([6]) is also obtained in this way.

If G is not amenable, then the Morita equivalence of Theorem 1 can be replaced by a ring isomorphism. It is proved in Corollary 4.5. In this case, isomorphism invariants of rings are also quasi-isometry invariants.

In Section 6, a geometric description of $\mathcal{R}(G, \mathbf{k})$ is given by $\underline{\mathrm{Mod}}_{\mathbf{k}}(G \ltimes \beta G)$ by using [4]. Indeed, $\mathcal{R}(G, \mathbf{k})$-Mod is additively equivalent to $\underline{\mathrm{Mod}}_{\mathbf{k}}(G \ltimes \beta G)$ (see Theorem 6.6). From this we can construct $\mathcal{R}(G, \mathbf{k})$-modules by the geometry of Stone–Čech compactification. In Appendix 7.1, we give an alternative proof of Theorem 1 using the result in Section 6.

2. Preliminaries

2.1. Geometric group theory. We recall the basic notion of geometric group theory and cores of a quasi-isometry [8, 0.2.C. p. 4, 5].

Let G be a finitely generated group with a finite generating system S. G has a metric $d_{(G,S)}$ defined by

$$d_{(G,S)}(x, y) = \begin{cases} \min\{n \in \mathbf{N} \mid x = s_1^{i_1} \cdots s_n^{i_n} y,\ s_k \in S,\ i_k \in \{-1, 1\}\} & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y, \end{cases}$$

which is called the word metric with respect to S.

Let Z be a metric space. For $W \subseteq Z$ and a real number $K \geq 0$, $\mathcal{N}_K(W) = \{z \in Z \mid \exists w \in W \text{ s.t. } d(z, w) \leq K\}$ is called a K-neighborhood of W. If $\mathcal{N}_K(W) = Z$, the subspace W is said to be K-coarsely dense in Z.

A quasi-isometry is a map $f \colon X \to Y$ between metric spaces such that for some real number $K \geq 1$, f satisfies

(1) $(1/K)\, d(x, x') - K \leq d(f(x), f(x')) \leq K\, d(x, x') + K$ for every $x, x' \in X$,
(2) $f(X)$ is K-coarsely dense in Y.

Two metric spaces are quasi-isometric if there exists a quasi-isometry between them. This gives an equivalence relation for metric spaces. If S and S' are finite generating systems of G, then $(G, d_{(G,S)})$ and $(G, d_{(G,S')})$ are quasi-isometric.

The definition of cores of a quasi-isometry is as follows:

DEFINITION 2.1. Let $f\colon X \to Y$ be a quasi-isometry. $A \subseteq X$ and $B \subseteq Y$ are called cores of f if there exists a real number $K \geq 0$ such that $\mathcal{N}_K(A) = X$, $\mathcal{N}_K(B) = Y$, $f(A) = B$ and $f|_A$ is a bijective quasi-isometry.

For every quasi-isometry $f\colon X \to Y$ there exist cores of f. Indeed, we can define a core B to be $f(X)$, and A to be $\{x_b \in X \mid b \in B\}$ by choosing $x_b \in f^{-1}(b)$ for each b.

2.2. Algebraic translation algebra. Let G be a group, and R a ring with the multiplicative identity element 1 on which G acts from the right. For $r \in R$ and $g \in G$ this action is denoted by r^g. The skew group ring $G * R$ is a free right R-module on G with the multiplication given by

$$(gr_1)(hr_2) = (gh)(r_1^h r_2) \quad \text{for every} \quad g, h \in G,\ r_1, r_2 \in R.$$

If G acts on R trivially, then we especially write $G * R$ by GR. It is an ordinary group ring (see [16] about skew group rings).

DEFINITION 2.2. (1) Let G be a group and $\mathbf{k}$ a ring with the multiplicative identity element 1.

$$l^f(G, \mathbf{k}) = \{F\colon G \to \mathbf{k} \mid \sharp(\operatorname{Im} F) < \infty\}$$

is a ring with the following sum and multiplication:

$$(F_1 + F_2)(x) = F_1(x) + F_2(x),$$
$$(F_1 F_2)(x) = F_1(x) F_2(x)$$

for every $F_1, F_2 \in l^f(G, \mathbf{k})$ and $x \in G$.
(2) G acts on $l^f(G, \mathbf{k})$ from the right by

$$F^g(x) = F(gx) \quad \text{for every} \quad g \in G, x \in G.$$

(3) We denote $G * l^f(G, \mathbf{k})$ by $\mathcal{R}(G, \mathbf{k})$. It is called an algebraic translation algebra of G with the coefficient $\mathbf{k}$.

The multiplicative identity element of $l^f(G, \mathbf{k})$ is the constant function 1. Since $\mathbf{k} \subseteq l^f(G, \mathbf{k})$ as constant functions, a group ring $G\mathbf{k}$ is a subring of $\mathcal{R}(G, \mathbf{k})$. $e \cdot 1$ is the multiplicative identity element of $\mathcal{R}(G, \mathbf{k})$, where e is the identity element of G. $\mathbf{k}$ is regarded as a left $G\mathbf{k}$-module by $gk = k$ ($g \in G$, $k \in \mathbf{k}$).

For $S \subseteq G$ the characteristic function

$$\chi_S(x) = \begin{cases} 1 & \text{if} \quad x \in S, \\ 0 & \text{if} \quad x \notin S \end{cases}$$

is an element of $l^f(G, \mathbf{k})$.

In the case where $\mathbf{k} = \mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{C}$, we see that $\mathcal{R}(G, \mathbf{k})$ is a subring of Roe's translation algebra [17, p. 68].

2.3. Morita theory. We give the basic notion of Morita theory [1]. Let R and S be rings. R-Mod (Mod-R) is the category of left (right) modules over R. A left R and right S-module M is called a left R-right S-bimodule if $r(ms) = (rm)s$ ($r \in R$, $s \in S$, $m \in M$). ${}_RM$ means M is a left R-module, M_R means M is a right R-module, and ${}_RM_S$ means M is a left R-right S-bimodule.

Let $\mathcal{F}_1$, $\mathcal{F}_2\colon \mathbf{B} \to \mathbf{C}$ be functors, a set $\{\tau_B \in \mathrm{Hom}(\mathcal{F}_1(B), \mathcal{F}_2(B)) \mid B \in \mathrm{Ob}(\mathbf{B})\}$ is called a natural equivalence if $\mathcal{F}_2(f) \circ \tau_B = \tau_{B'} \circ \mathcal{F}_1(f)$ ($f \in \mathrm{Hom}(B, B')$) and τ_B is an isomorphism for every $B \in \mathrm{Ob}(\mathbf{B})$. Then $\mathcal{F}_1 \simeq \mathcal{F}_2$ if there exists a natural equivalence. A functor $\mathcal{F}\colon R\text{-Mod} \to S\text{-Mod}$ is called an additive functor if $\mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ defined by $f \mapsto \mathcal{F}(f)$ is a homomorphism. An additive functor $\mathcal{F}_1\colon R\text{-Mod} \to S\text{-Mod}$ is called an additive equivalence if there exists an additive functor $\mathcal{F}_2\colon S\text{-Mod} \to R\text{-Mod}$ such that $\mathcal{F}_2 \circ \mathcal{F}_1 \simeq \mathrm{id}$ and $\mathcal{F}_1 \circ \mathcal{F}_2 \simeq \mathrm{id}$. A functor $\mathcal{F}_2$ is called an inverse equivalence of $\mathcal{F}_1$.

R and S are said to be *Morita equivalent* if there exists an additive equivalence between R-Mod and S-Mod. Let M be a left (right) R-module. The module M is said to be *finitely generated* if there exist $n \in \mathbf{N}$ and a surjective homomorphism $f\colon R^n \to M$. M is called a (*finite*) *generator* if there exist $n \in \mathbf{N}$ and a surjective homomorphism $f\colon M^n \to R$. M is said to be *projective* if it is a direct summand of a free left (right) R-module. A generator is called a *progenerator* if it is finitely generated and projective. $\mathrm{End}(M) = \{f\colon M \to M \mid f$ is a left (right) R-homomorphism$\}$ is called the endomorphism ring. The multiplication is the opposite composition (ordinary composition) of maps.

$e \in R$ is called an idempotent if $e^2 = e$. If R has the multiplicative identity element 1, then $eRe = \{ere \in R \mid r \in R\}$ is a ring with the multiplicative identity element e. Re is a left R-module, and $\mathrm{End}(Re)$ is isomorphic to eRe. eR is a right R-module, and $\mathrm{End}(eR)$ is also isomorphic to eRe.

Theorem 2.3. [1, Corollary 22.5, p. 265] *Let R be a ring. If P_R is a progenerator, then R and $S = \mathrm{End}(P_R)$ are Morita equivalent. Indeed, if $P^{\circledast} = \mathrm{Hom}_R(P_R, R)$, then ${}_SP_R$ and ${}_RP^{\circledast}_S$ are bimodules and $(P \otimes_R -)\colon R\text{-Mod} \to S\text{-Mod}$, $(P^{\circledast} \otimes_S -)\colon S\text{-Mod} \to R\text{-Mod}$ are inverse equivalences.*

If $P_R = eR$ is a progenerator, then $S = eRe$, ${}_SP_R = {}_{eRe}eR_R$ and ${}_RP^{\circledast}_{eRe} = {}_RRe_{eRe}$.

If R and S are isomorphic, then R and S are Morita equivalent. Indeed, let $\Phi\colon S \to R$ be a ring isomorphism. Since $S \simeq R \simeq \mathrm{End}(R_R)$, an additive equivalence $({}_SR_R \otimes_R -)\colon R\text{-Mod} \to S\text{-Mod}$ is obtained. ${}_RM$ is mapped to ${}_SM$ satisfying $sm = \Phi(s)m$ $(s \in S,\ m \in M)$. We use the notation $\mathrm{Res}\,\Phi = ({}_SR_R \otimes_R -)$.

2.4. The center of $\mathcal{R}(G,\mathbf{k})$. The center of a ring R is $\mathrm{Cen}(R) = \{r \in R \mid rx = xr\ (\forall x \in R)\}$. If rings R and S are Morita equivalent, then $\mathrm{Cen}(R)$ and $\mathrm{Cen}(S)$ are isomorphic [1, Proposition 21.10, p. 258].

Lemma 2.4. $\mathrm{Cen}(\mathcal{R}(G,\mathbf{k})) = \mathrm{Cen}(\mathbf{k})$.

Proof. Let $\alpha \in \mathrm{Cen}(\mathcal{R}(G,\mathbf{k}))$. For each $x \neq e \in G$ there exists no $F \neq 0 \in l^f(G,\mathbf{k})$ such that for every $g \in G$, $\chi_g xF = xF\chi_g$ is satisfied, and hence $\alpha \in e\cdot l^f(G,\mathbf{k})$. Since for every $g \in G$ we have $g\alpha = \alpha g$, α is a constant function. $k\alpha = \alpha k$ is satisfied for every $k \in \mathbf{k}$, and hence $\alpha \in \mathrm{Cen}(\mathbf{k})$. □

2.5. Transformation groupoids. Let $\mathcal{G}_0$, $\mathcal{G}_1$ be topological spaces, and $s\colon \mathcal{G}_1 \to \mathcal{G}_0$, $t\colon \mathcal{G}_1 \to \mathcal{G}_0$, $m\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(g_1, g_2) \in \mathcal{G}_1 \times \mathcal{G}_1 \mid s(g_1) = t(g_2)\} \to \mathcal{G}_1$ continuous maps. We consider the following three conditions:

(1) $m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$ $((g_1, g_2), (g_2, g_3) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1)$,

(2) there exists a continuous map $u\colon \mathcal{G}_0 \to \mathcal{G}_1$ such that $s(u(x)) = t(u(x)) = x$ and $m(u(x), g) = g$, $m(g', u(x)) = g'$ $(x \in \mathcal{G}_0,\ g, g' \in \mathcal{G}_1$ with $t(g) = x = s(g'))$,

(3) there exists a continuous map $I\colon \mathcal{G}_1 \to \mathcal{G}_1$ such that $m(g, I(g)) = u(t(g))$, $m(I(g), g) = u(s(g))$ $(g \in \mathcal{G}_1)$.

$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, s, t, m, u, I)$ satisfying the conditions above is called a *topological groupoid*. We use the notation $g_1 \cdot g_2 = m(g_1, g_2)$ $((g_1, g_2) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1)$, $e_x = u(x)$ $(x \in \mathcal{G}_0)$ and $g^{-1} = I(g)$ $(g \in \mathcal{G}_1)$. A topological groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, s, t, m, u, I)$ is called an *étale groupoid* if s and t are surjective local homeomorphisms (see [14, Section 5] for more on étale groupoids).

Let G be a finitely generated group, and G acts on a topological space X from the left. We define a *transformation groupoid* $G \ltimes X$ by the following data:

$$(G \ltimes X)_0 = X, \quad (G \ltimes X)_1 = G \times X,$$

where G is regarded as a discrete space.

$$s(g, x) = x \quad (g \in G, x \in X), \quad t(g, x) = gx \quad (g \in G, x \in X),$$
$$(g, x) \cdot (g', x') = (gg', x') \quad (g, g' \in G,\ x, x' \in X \text{ satisfying } x = g'x'),$$
$$u(x) = (e, x) \quad (x \in X),$$

where e is the identity element of G.

$$I(g, x) = (g^{-1}, gx) \quad (g \in G, x \in X).$$

$G \ltimes X$ is an étale groupoid.

2.6. Stone–Čech compactifications. We recall the Stone–Čech compactifications for discrete spaces [9]. Let D be a set. $\mathcal{U} \subseteq 2^D$ is called a filter on D if the following conditions are satisfied.
(0) $D \in \mathcal{U}$,
(1) $\emptyset \notin \mathcal{U}$,
(2) if $A_1, A_2 \in \mathcal{U}$, then $A_1 \cap A_2 \in \mathcal{U}$,
(3) if $A \in \mathcal{U}$, $B \in 2^D$ and $A \subseteq B$, then $B \in \mathcal{U}$.
In addition, $\mathcal{U}$ is called an ultra filter if $\mathcal{U}$ satisfies
(4) if $D = A_1 \sqcup \cdots \sqcup A_n$, then there exists the unique $1 \le i \le n$ such that $A_i \in \mathcal{U}$.
The set of ultra filters on D is denoted by βD. For $A \subseteq D$ we use the notation $\hat{A} = \{\mathcal{U} \in \beta D \mid A \in \mathcal{U}\}$. $\mathcal{O}$ is a topology on D generated by an open base $\{\hat{A} \mid A \in 2^D\}$. $(\beta D, \mathcal{O})$ is called the *Stone–Čech compactification* of D. Let G be a finitely generated group. βG has a natural G-action from the left. Indeed, for $\mathcal{U} \in \beta G$ and $g \in G$, $g\mathcal{U} = \{gA \mid A \in \mathcal{U}\} \in \beta G$. This action is a homeomorphic action.

Lemma 2.5. (1) *βD is compact and Hausdorff. D is identified with a dense subset of βD by an injection $e\colon D \to \beta D$ satisfying $\{e(d)\} = \widehat{\{d\}}$ for every $d \in D$.*
(2) *For $A \in 2^D$, $\widehat{D - A} = \beta D - \hat{A}$. Therefore the topology of βD is generated by clopen (closed and open) sets.*
(3) *If O is a clopen set of βD, then there exists $A \in 2^D$ such that $O = \hat{A}$.*

Proof. The proof of (1) is in [9, Theorem 3.18 (a) and (c)], and the proof of (2) is in [9, Theorem 3.17 (c)]. If O is a clopen set of βD, then there exists $A_x \in 2^D$ for each $x \in O$ such that $x \in A_x$ and $O = \bigcup_{x \in O} \hat{A}_x$. Since O is a closed set of Hausdorff space, O is compact. Therefore there exists $\{x_i \in O\}_{i=1}^n$ such that $O = \bigcup_{i=1}^n \hat{A}_{x_i}$. By [9, Theorem 3.17 (b)] we have $O = \widehat{\bigcup_{i=1}^n A_{x_i}}$. □

2.7. Definition of $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$. Let $\mathcal{G}$ be an étale groupoid. In [4], the abelian category associated to $\mathcal{G}$ was considered to study a homology theory for $\mathcal{G}$ ([21, Appendix A] is a good reference for abelian categories). This category is denoted by $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$. In Section 6, we describe $\underline{\mathrm{Mod}}_{\mathbf{k}}(G \ltimes \beta G)$ and discuss a relation to the algebraic translation algebra.

First, we recall the definition of $\underline{\mathrm{Sh}}(\mathcal{G})$. A *left étale $\mathcal{G}$-space* $X = (X, p_0, p_1)$ is a topological space with continuous maps $p_0\colon X \to \mathcal{G}_0$ and $p_1\colon \mathcal{G}_1 \times_{p_0} X = \{(g, x) \mid s(g) = p_0(x)\} \to X$ such that
(0) p_0 is a surjective local homeomorphism,
(1) $p_0(p_1(g, x)) = t(g)$ $((g, x) \in \mathcal{G}_1 \times_{p_0} X)$,
(2) $p_1(h \cdot g, x) = p_1(h, p_1(g, x))$ $((g, x) \in \mathcal{G}_1 \times_{p_0} X,\ (h, g) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1)$,
(3) $p_1(e_{p_0(x)}, x) = x$ $(x \in X)$.
p_1 is usually denoted by $\cdot$. Let $X = (X, p_0, p_1)$ and $Y = (Y, q_0, q_1)$ be left étale $\mathcal{G}$-spaces. A continuous map $\Phi\colon X \to Y$ is said to be *equivariant* if

(1) $q_0 \circ \Phi = p_0$,
(2) $\Phi(p_1(g, x)) = q_1(g, \Phi(x))$ $((g, x) \in \mathcal{G}_1 \times_{p_0} X)$
are satisfied. $\underline{\mathrm{Sh}}(\mathcal{G})$ is the category of which objects are left étale $\mathcal{G}$-spaces and morphisms are equivariant maps. $\underline{\mathrm{Sh}}(\mathcal{G})$ is called *the category of left étale $\mathcal{G}$-spaces*.

Second, we recall the definition of $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$. Let $X = (X, p_0, p_1)$ and $Y = (Y, q_0, q_1)$ be left étale $\mathcal{G}$-spaces. We define a finite product of $\underline{\mathrm{Sh}}(\mathcal{G})$: $X \oplus Y = (X \times_{\mathcal{G}_0} Y, r_0, r_1)$ by $X \times_{\mathcal{G}_0} Y = \{(x, y) \in X \times Y \mid p_0(x) = q_0(y)\}$, $r_0\colon X \times_{\mathcal{G}_0} Y \to \mathcal{G}_0$ with $r_0(x, y) = p_0(x) = q_0(y)$ and $r_1\colon \mathcal{G}_1 \times_{r_0} (X \times_{\mathcal{G}_0} Y) \to X \times_{\mathcal{G}_0} Y$ with $r_1(g,(x,y)) = (p_1(g,x), q_1(g,y))$. $\Theta = \mathcal{G}_0 = (\mathcal{G}_0, \mathrm{id}\colon \mathcal{G}_0 \to \mathcal{G}_0, t \circ \mathrm{Pr}_1\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_0 \to \mathcal{G}_0)$ is a left étale $\mathcal{G}$-space and $\mathrm{Hom}(X, \mathcal{G}_0) = \{p_0\}$. Θ is a terminal object. $\mathbf{k}$ is regarded as a constant left étale $\mathcal{G}$-space $\mathbf{k} = (\mathbf{k} \times \mathcal{G}_0,\ p'_0 = \mathrm{Pr}_2,\ p'_1)$ by $p'_1(g, (k, z)) = (k, t(g))$. $\mathbf{k}$ has a natural structure of a ring. We consider $\mathbf{k}$-module objects of $(\underline{\mathrm{Sh}}(\mathcal{G}), \oplus, \Theta)$: $A = (A, M, \mathcal{U}, v, \mathcal{M})$ is called a *$\mathbf{k}$-module object* of $(\underline{\mathrm{Sh}}(\mathcal{G}), \oplus, \Theta)$ if morphisms $M\colon A \oplus A \to A$, $\mathcal{U}\colon \Theta \to A$, $v\colon A \to A$ and $\mathcal{M}\colon \mathbf{k} \oplus A \to A$ satisfy
(1) $(M, \mathcal{U}, v)$ is an usual additive group structure on A,
(2) $\mathcal{M}$ is an usual $\mathbf{k}$-action giving a $\mathbf{k}$-module structure on $(A, M, \mathcal{U}, v)$.
M is usually denoted by $+$, $\mathcal{U}$ by 0, v by $-$ and $\mathcal{M}$ by $\cdot$. Morphisms between $\mathbf{k}$-module objects A and B of $(\underline{\mathrm{Sh}}(\mathcal{G}), \oplus, \Theta)$ are morphisms of $\underline{\mathrm{Sh}}(\mathcal{G})$ preserving structures M, $\mathcal{U}$, v and $\mathcal{M}$. Therefore $\mathbf{k}$-module objects form a category. It is denoted by $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$. $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ is an abelian category. $\Theta = \mathcal{G}_0$ is the zero object 0.

$\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ always has an infinite coproduct. Such a category is called an A.B.3 category [21, A.4]. An infinite coproduct exists as follows: For $\{A_\lambda \in \mathrm{Ob}(\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})) \mid \lambda \in \Lambda\}$, A_λ gives a presheaf of $\mathbf{k}$-modules $\mathcal{F}_{A_\lambda}$ on $\mathcal{G}_0$ by $O \mapsto \Gamma(O, A_\lambda) = \{f\colon O \to A_\lambda \mid f$ is continuous and $p_{0,\lambda} \circ f = \mathrm{id}_O\}$ for every open set $O \subseteq \mathcal{G}_0$. Therefore for presheaf $\bigoplus_{\lambda\in\Lambda} \mathcal{F}_{A_\lambda}\colon O \mapsto \bigoplus_{\lambda\in\Lambda} \mathcal{F}_{A_\lambda}(O)$ its sheaf space $E_{\bigoplus_{\lambda\in\Lambda} \mathcal{F}_{A_\lambda}}$ has a natural structure of a $\mathbf{k}$-module object of $(\underline{\mathrm{Sh}}(\mathcal{G}), \oplus, \Theta)$ (about the relation of a presheaf and a sheaf space see [2, 2.3]). This $E_{\bigoplus_{\lambda\in\Lambda} \mathcal{F}_{A_\lambda}}$ is an infinite coproduct. Therefore $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ is an A.B.3 category.

3. Proof of quasi-isometry invariance of the algebraic translation algebra using a topological coupling

We recall the definition of Sauer rings. Let G acts on a compact Hausdorff space X from the left. $\mathcal{F}(X, \mathbf{k}) = \{F\colon X \to \mathbf{k} \mid F^{-1}(k)$ is clopen for every $k \in \mathbf{k}\}$ has the right action of G induced by the left action of G on X. In the paper [18], the skew group ring $G * \mathcal{F}(X, \mathbf{k})$ was considered. We call this ring *the Sauer ring of G-space X*.

Theorem 3.1 ([18])**.** *If two finitely generated groups G and G' are quasi-isometric, then there exist compact Hausdorff spaces Y_1 on which G acts from the left and Y_2 on which G' acts from the left such that their Sauer rings $G * \mathcal{F}(Y_1, \mathbf{k})$ and $G' * \mathcal{F}(Y_2, \mathbf{k})$ are Morita equivalent. A good topological coupling Ω always gives such $Y_1 = \Omega/G'$ and $Y_2 = \Omega/G$, where a topological coupling Ω is said to be good if it has a compact clopen* *fundamental domain for each action.*

We relate Sauer rings and our algebraic translation algebras.

Lemma 3.2. *Let G be a finitely generated group.*

$$\mathcal{R}(G, \mathbf{k}) \simeq G * \mathcal{F}(\beta G, \mathbf{k}).$$

Proof. For $F \in \mathcal{F}(\beta G, \mathbf{k})$, $\beta G = \bigsqcup_{k \in \mathbf{k}} F^{-1}(k)$. Since βG is compact and for each k, $F^{-1}(k)$ is open, and hence there exist $k_1, \ldots, k_n \in \mathbf{k}$ such that $\beta G = \bigsqcup_{i=1}^{n} F^{-1}(k_i)$. For each i, $F^{-1}(k_i)$ is clopen, and hence by Lemma 2.5 (3) there exist $A_1, \ldots, A_n \subseteq G$ such that $F^{-1}(k_i) = \hat{A}_i$. We define $\lambda \colon \mathcal{F}(\beta G, \mathbf{k}) \to l^f(G, \mathbf{k})$ by $\lambda(F) = F|_G = \sum_{i=1}^{n} k_i \chi_{A_i}$. λ is a bijective homomorphism and preserves the action of G. Every function in $l^f(G, \mathbf{k})$ has an expression $\sum_{i=1}^{n} k_i \chi_{A_i}$, and hence λ is surjective. This λ is extended to $\mathcal{R}(G, \mathbf{k}) \simeq G * \mathcal{F}(\beta G, \mathbf{k})$. □

By Lemma 3.2 if we have a good topological coupling such that $Y_1 = \beta G$ and $Y_2 = \beta G'$, then Theorem 1 is the special case of [18]. Indeed, we have the following theorem:

Theorem 3.3. *Two quasi-isometric finitely generated groups G and G' always have a good topological coupling such that their Stone–Čech compactifications are coupled.*

Proof. In the proof of Theorem 7.1 in Appendix, we have essential morphisms $G \ltimes \beta G \to \mathbf{G}(|G| \sqcup |G'|) \leftarrow G' \ltimes \beta G'$ (see [19, Section 3.4]). We take the weak pullback $\mathcal{G}$ of this morphisms, and hence surjective essential morphisms $G \ltimes \beta G \leftarrow \mathcal{G} \to G' \ltimes \beta G'$ are obtained (see [14, Exercise 5.22 (1)]). $\mathcal{G}_0$ has a natural $(G \times G')$-action. $\mathcal{G}_0$ is a topological coupling such that $\mathcal{G}_0/G' = \beta G$ and $\mathcal{G}_0/G = \beta G'$. Since surjective essential morphisms above are étale and the topologies of βG and $\beta G'$ are generated by clopen sets, we can construct a compact clopen fundamental domain for each action. As a result, $\mathcal{G}_0$ is a good topological coupling. □

We have the main theorem by Lemma 3.2, Theorems 3.1 and 3.3:

Theorem 1. *If finitely generated groups G and G' are quasi-isometric, then $\mathcal{R}(G, \mathbf{k})$ and $\mathcal{R}(G', \mathbf{k})$ are Morita equivalent.*

4. Proof of quasi-isometry invariance of the algebraic translation algebra without using a topological coupling

The proof is obtained by elementary argument: cores of a quasi-isometry and basic Morita theory (see Sections 2.1 and 2.3).

4.1. Some lemmas. In order to prove quasi-isometry invariance of the algebraic translation algebra, we need Lemmas 4.1 and 4.4.

Lemma 4.1. *Let H be a finitely generated group. For $Z \subseteq H$ if there exists a real number $K \geq 0$ such that Z is K-coarsely dense in H, then a right $\mathcal{R}(H,\mathbf{k})$-module $I_Z = \chi_Z \mathcal{R}(H, \mathbf{k})$ is a progenerator.*

Proof. Since $\mathcal{R}(H, \mathbf{k}) = I_Z \oplus I_{H-Z}$, I_Z is finitely generated and projective.

We prove that I_Z is a generator: For the identity element $e \in H$, $\mathcal{N}_K(e)$ is finite, and hence we have an expression $\mathcal{N}_K(e) = \{h_0 = e, h_1, \dots, h_n\}$. We define $Z_0, \dots, Z_n$ by

$$\begin{aligned} Z_0 &= Z, \\ Z_1 &= h_1 Z - Z, \\ Z_2 &= h_2 Z - h_1 Z - Z, \\ &\cdots \\ Z_n &= h_n Z - h_{n-1} Z - \cdots - Z. \end{aligned}$$

$Z_0 \sqcup \cdots \sqcup Z_n = H$ and $h_i^{-1} Z_i \subseteq Z$ are satisfied. We define $p_i \colon I_Z \to \mathcal{R}(H, \mathbf{k})$ by $p_i(\chi_Z \gamma) = \chi_{Z_i} h_i \gamma$ for every $\gamma \in \mathcal{R}(H,\mathbf{k})$ and $0 \leq i \leq n$. They are well-defined as follows: For every $\gamma, \gamma' \in \mathcal{R}(H, \mathbf{k})$ satisfying $\chi_Z \gamma = \chi_Z \gamma'$, by multiplying $\chi_{Z_i} h_i$ to this equation from the left, we have $\chi_{Z_i} h_i \chi_Z \gamma = \chi_{Z_i} h_i \chi_Z \gamma'$. This implies $\chi_{Z_i} \chi_{h_i Z} h_i \gamma = \chi_{Z_i} \chi_{h_i Z} h_i \gamma'$. Thus $h_i^{-1} Z_i \subseteq Z$ shows that $\chi_{Z_i} h_i \gamma = \chi_{Z_i} h_i \gamma'$, and hence p_i is a well-defined homomorphism. As a result, we have a homomorphism $p = \bigoplus_{i=1}^n p_i \colon I_Z^n \to \mathcal{R}(H, \mathbf{k})$. For each $h \in H$ and $F \in l^f(H, \mathbf{k})$ we have $hF = \sum_{i=0}^n \chi_{Z_i} hF = \sum_{i=0}^n \chi_{Z_i} (h_i h_i^{-1}) hF = \sum_{i=0}^n \chi_{Z_i} h_i (h_i^{-1} hF) = \sum_{i=0}^n p_i(\chi_Z h_i^{-1} hF) = p(\chi_Z h_0^{-1} hF, \dots, \chi_Z h_n^{-1} hF)$. Therefore p is surjective. □

Let H be a group and $Z \subseteq H$. $\mathcal{M}_H$ is the endomorphism ring of the right free $\mathbf{k}$-module on $\{\delta_h \mid h \in H\}$. $\mathcal{M}_Z$ is the subring of $\mathcal{M}_H$ generated on $\{\delta_z \mid z \in Z\}$. We consider a map $\epsilon = \epsilon_{(H,Z)} \colon H \times H \to \mathbf{k}$ satisfying

$$\epsilon(h, z) = \chi_{h^{-1}Z \cap Z}(z) = \begin{cases} 1 & \text{if} \quad z \in h^{-1} Z \cap Z, \\ 0 & \text{if} \quad z \notin h^{-1} Z \cap Z \end{cases}$$

for every $h, z \in H$. By using ϵ, an injective homomorphism $i_Z \colon \chi_Z \mathcal{R}(H, \mathbf{k}) \chi_Z \to \mathcal{M}_Z$ can be defined by

$$i_Z(\chi_Z \alpha \chi_Z) \delta_z = \sum_{i=1}^n \delta_{h_i z} \epsilon(h_i, z) F_i(z)$$

for every $\alpha = \sum_{i=1}^n h_i F_i \in \mathcal{R}(H, \mathbf{k})$, $h_i \in H$, $F_i \in l^f(H, \mathbf{k})$ and $z \in Z$. This is shown in the next lemma.

Lemma 4.2. *i_Z is well-defined, a homomorphism and injective.*

Proof. For every $h \in H$ and $z \in Z$ if $hz \notin Z$ is satisfied, then $\delta_{hz}\epsilon(h, z) = 0$. Therefore we have $i_Z(\chi_Z\alpha\chi_Z) \in \mathcal{M}_Z$. Since i_Z preserves the sum, to prove i_Z is well-defined we will prove that for every $\alpha = \sum_{i=1}^n h_i F_i \in \mathcal{R}(H, \mathbf{k})$ if $\chi_Z\alpha\chi_Z = 0$, then $i_Z(\chi_Z\alpha\chi_Z) = 0$. $\chi_Z\alpha\chi_Z = \sum_{i=1}^n h_i \chi_{h_i^{-1}Z\cap Z} F_i = 0$ implies $\sum_{h=h_i} \chi_{h_i^{-1}Z\cap Z} F_i = 0$ for each $h \in H$, and hence $\sum_{h=h_i} \epsilon(h_i, z)F_i(z) = 0$ for every $z \in Z$. This shows $i_Z(\chi_Z\alpha\chi_Z)\delta_z = 0$.

In order to prove i_Z is a homomorphism, we only have to check that i_Z preserves the multiplication for generators about the sum since i_Z preserves the sum and the identity element. $\chi_Z\mathcal{R}(H, \mathbf{k})\chi_Z$ is generated by $\chi_Z H l^f(H, \mathbf{k})\chi_Z$ as an additive group, and hence for $g, h \in H$, $F_1, F_2 \in l^f(H, \mathbf{k})$, and $z \in Z$

$$\begin{aligned}
&i_Z(\chi_Z g F_1 \chi_Z) \circ i_Z(\chi_Z h F_2 \chi_Z)\delta_z \\
&= i_Z(\chi_Z g F_1 \chi_Z)(\delta_{hz}\epsilon(h, z)F_2(z)) \\
&= \delta_{ghz}\epsilon(g, hz)F_1(hz)\epsilon(h, z)F_2(z) \\
&= \delta_{ghz}\epsilon(g, hz)\epsilon(h, z)F_1(hz)F_2(z) \\
&= \delta_{ghz}\chi_{h^{-1}g^{-1}Z\cap h^{-1}Z}(z)\chi_{h^{-1}Z\cap Z}(z)F_1^h(z)F_2(z) \\
&= \delta_{ghz}\chi_{(gh)^{-1}Z\cap Z}(z)(\chi_{h^{-1}Z}F_1^h F_2)(z) \\
&= i_Z(\chi_Z gh \chi_{h^{-1}Z} F_1^h F_2 \chi_Z)\delta_z \\
&= i_Z(\chi_Z g F_1 \chi_Z \chi_Z h F_2 \chi_Z)\delta_z.
\end{aligned}$$

This implies i_Z is a homomorphism.

In order to prove i_Z is injective we will check that for every $\alpha \in \mathcal{R}(H, \mathbf{k})$, $i_Z(\chi_Z\alpha\chi_Z) = 0$ implies $\chi_Z\alpha\chi_Z = 0$. We have an expression $\alpha = \sum_{i=1}^n h_i F_i$ for some $h_i \in H$ and $F_i \in l^f(H, \mathbf{k})$, where we can assume that $h_1, \dots, h_n$ are different from each other. $i_Z(\chi_Z\alpha\chi_Z) = 0$ implies

$$i_Z(\chi_Z\alpha\chi_Z)\delta_z = \sum_{i=1}^n \delta_{h_i z}\epsilon(h_i, z)F_i(z) = \sum_{i=1}^n \delta_{h_i z}\chi_{h_i^{-1}Z\cap Z}(z)F_i(z) = 0$$

for every $z \in Z$. This shows that $\chi_{h_i^{-1}Z}F_i\chi_Z = 0$ for every i. Thus

$$\chi_Z\alpha\chi_Z = \sum_{i=1}^n h_i \chi_{h_i^{-1}Z} F_i \chi_Z = 0. \qquad \square$$

Let G and G' be finitely generated groups, $X \subseteq G$, $Y \subseteq G'$ and $f\colon X \to Y$ a bijective quasi-isometry. Since f is bijective, f induces a natural isomorphism $\tilde{f}\colon \mathcal{M}_X \to \mathcal{M}_Y$ as follows. For every $A \in \mathcal{M}_X$ and $x \in X$ we have an expression $A(\delta_x) = \sum_{i=1}^n \delta_{x_i(x)}a_i(x)$ by some $x_i(x) \in X$ and $a_i(x) \in \mathbf{k}$. Thus by using this expression of $A(\delta_x)$, $\tilde{f}$ satisfies

$$\tilde{f}(A)(\delta_y) = \sum_{i=1}^n \delta_{f(x_i((f^{-1}(y))))}a_i(f^{-1}(y))$$

for every $y \in Y$. By Lemma 4.2 we have injective homomorphisms $i_X\colon \chi_X\mathcal{R}(G,\mathbf{k})\chi_X \to \mathcal{M}_X$ and $i_Y\colon \chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y \to \mathcal{M}_Y$.

Lemma 4.3.

$$\tilde{f} \circ i_X(\chi_X\mathcal{R}(G,\mathbf{k})\chi_X) \subseteq i_Y(\chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y).$$

Proof. For every $g \in G$ and $y \in Y$ we have

$$\tilde{f} \circ i_X(\chi_X g\chi_X)(\delta_y) = \begin{cases} \delta_{f(gf^{-1}(y))}\epsilon_{(G,X)}(g,\ f^{-1}(y)) & \text{if} \quad gf^{-1}(y) \in X, \\ 0 & \text{otherwise} \end{cases}$$

since

$$i_X(\chi_X g\chi_X)(\delta_{f^{-1}(y)}) = \delta_{gf^{-1}(y)}\epsilon_{(G,X)}(g,\ f^{-1}(y)).$$

Since f is a quasi-isometry, $L = \{f(gf^{-1}(y))y^{-1} \mid y \in Y \text{ and } gf^{-1}(y) \in X\}$ is a finite set. Therefore we have an expression $L = \{h_1, \ldots, h_m\}$. We have $S_i = \{y \in Y \mid gf^{-1}(y) \in X \text{ and } f(gf^{-1}(y))y^{-1} = h_i\}$. $S_1, \ldots, S_m$ are disjoint for each other. If $gf^{-1}(y) \in X$, then there exists $1 \le j \le m$ such that $f(gf^{-1}(y))y^{-1} = h_j$ and

$$\begin{aligned}
\delta_{f(gf^{-1}(y))}\epsilon_{(G,X)}(g,\ f^{-1}(y)) &= \delta_{f(gf^{-1}(y))y^{-1}y}\epsilon_{(G,X)}(g,\ f^{-1}(y)) \\
&= \delta_{h_j y}\epsilon_{(G,X)}(g,\ f^{-1}(y)) \\
&= \delta_{h_j y}\left(\sum_{i=1}^{m} \epsilon_{(G',Y)}(h_i,\ y)\chi_{S_i}(y)\right)\epsilon_{(G,X)}(g,\ f^{-1}(y)) \\
&= \sum_{i=1}^{m} \delta_{h_i y}\epsilon_{(G',Y)}(h_i,\ y)\chi_{S_i}(y)\epsilon_{(G,X)}(g,\ f^{-1}(y)) \\
&= i_Y\left(\chi_Y \sum_{i=1}^{m} h_i\chi_{S_i}\epsilon_{(G,X)}(g,\ f^{-1}(\,\cdot\,))\chi_Y\right)(\delta_y),
\end{aligned}$$

where $\epsilon_{(G,X)}(g,\ f^{-1}(\,\cdot\,))\chi_Y \in l^f(G',\mathbf{k})$. This shows that $\tilde{f} \circ i_X(\chi_X g\chi_X)$ is in the image of i_Y.

On the other hand, for every $F \in l^f(G,\mathbf{k})$ and $y \in Y$ we have

$$\begin{aligned}
\tilde{f} \circ i_X(\chi_X F\chi_X)(\delta_y) &= \delta_y F(f^{-1}(y)) \\
&= i_Y(\chi_Y(F \circ f^{-1})\chi_Y)(\delta_y),
\end{aligned}$$

where $(F \circ f^{-1})\chi_Y \in l^f(G',\mathbf{k})$. This shows that $\tilde{f} \circ i_X(\chi_X F\chi_X)$ is in the image of i_Y.

$\chi_X\mathcal{R}(G,\mathbf{k})\chi_X$ is generated by $\chi_X G\chi_X$ and $\chi_X l^f(G,\mathbf{k})\chi_X$ as a ring. Therefore we have $\tilde{f} \circ i_X(\chi_X\mathcal{R}(G,\mathbf{k})\chi_X) \subseteq i_Y(\chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y)$. □

By Lemma 4.3 we have a homomorphism $\Phi\colon \chi_X\mathcal{R}(G,\mathbf{k})\chi_X \to \chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y$ with $i_Y \circ \Phi = \tilde{f} \circ i_X$. Similarly, we have $\Psi\colon \chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y \to \chi_X\mathcal{R}(G,\mathbf{k})\chi_X$ with $i_X \circ \Psi = \tilde{f}^{-1} \circ i_Y$. Therefore $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$.

We summarize the discussion above as follows:

Lemma 4.4. *Let G and G' be finitely generated groups, $X \subseteq G$, and $Y \subseteq G'$. If a bijective quasi-isometry $f\colon X \to Y$ exists, then $\chi_X\mathcal{R}(G)\chi_X$ and $\chi_Y\mathcal{R}(G')\chi_Y$ are isomorphic. The isomorphism $\Phi = \Phi_f\colon \chi_X\mathcal{R}(G,\mathbf{k})\chi_X \to \chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y$ is given by*

$$\Phi(\chi_X g\chi_X) = \chi_Y \sum_{i=1}^{m} h_i \chi_{S_i}\epsilon_{(G,X)}(g,\ f^{-1}(\,\cdot\,))\chi_Y \quad (g \in G),$$
$$\Phi(\chi_X F\chi_X) = \chi_Y(F \circ f^{-1})\chi_Y \quad (F \in l^f(G,\mathbf{k})),$$

where $\{f(gf^{-1}(y))y^{-1} \mid y \in Y \text{ and } gf^{-1}(y) \in X\} = \{h_1,\dots,h_m\}$ (m, h_i depend on g) and $S_i = \{y \in Y \mid gf^{-1}(y) \in X \text{ and } f(gf^{-1}(y))y^{-1} = h_i\}$.

Given two quasi-isometric non-amenable finitely generated groups, we can find a bijective quasi-isometry between them [5, Proposition p. 104], and hence by combining this fact and Lemma 4.4, we have

Corollary 4.5. *If non-amenable finitely generated groups G and G' are quasi-isometric, then $(\mathcal{R}(G), l^f(G,\mathbf{k}))$ and $(\mathcal{R}(G'), l^f(G',\mathbf{k}))$ are isomorphic as pairs of rings.*

4.2. The proof of the main theorem. Let G and G' be finitely generated groups, and $f\colon G \to G'$ a quasi-isometry. There exist cores of $f\colon X \subseteq G$ and $Y \subseteq G'$. By Lemma 4.1 $I_X = \chi_X\mathcal{R}(G,\mathbf{k})$ is a progenerator. By Theorem 2.3 $\mathcal{R}(G,\mathbf{k})$ and $\mathrm{End}(I_X)$ are Morita equivalent. Since χ_X is an idempotent, $\mathrm{End}(I_X) \simeq \chi_X\mathcal{R}(G,\mathbf{k})\chi_X$. Therefore $\mathcal{R}(G,\mathbf{k})$ and $\chi_X\mathcal{R}(G,\mathbf{k})\chi_X$ are Morita equivalent. Furthermore, by Theorem 2.3

$${}_{\chi_X\mathcal{R}(G,\mathbf{k})\chi_X}\chi_X\mathcal{R}(G,\mathbf{k})_{\mathcal{R}(G,\mathbf{k})},$$
$${}_{\mathcal{R}(G,\mathbf{k})}\mathcal{R}(G,\mathbf{k})\chi_X{}_{\chi_X\mathcal{R}(G,\mathbf{k})\chi_X}$$

are bimodules, and

$$(\chi_X\mathcal{R}(G,\mathbf{k}) \otimes_{\mathcal{R}(G,\mathbf{k})} -)\colon \mathcal{R}(G,\mathbf{k})\text{-Mod} \to \chi_X\mathcal{R}(G,\mathbf{k})\chi_X\text{-Mod},$$
$$(\mathcal{R}(G,\mathbf{k})\chi_X \otimes_{\chi_X\mathcal{R}(G,\mathbf{k})\chi_X} -)\colon \chi_X\mathcal{R}(G,\mathbf{k})\chi_X\text{-Mod} \to \mathcal{R}(G,\mathbf{k})\text{-Mod}$$

are inverse equivalences. Similarly, $\mathcal{R}(G',\mathbf{k})$ and $\chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y$ are Morita equivalent. Furthermore, by Theorem 2.3

$${}_{\chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y}\chi_Y\mathcal{R}(G',\mathbf{k})_{\mathcal{R}(G',\mathbf{k})},$$
$${}_{\mathcal{R}(G',\mathbf{k})}\mathcal{R}(G',\mathbf{k})\chi_Y{}_{\chi_Y\mathcal{R}(G',\mathbf{k})\chi_Y}$$

are bimodules, and

$$(\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G',\mathbf{k})} -)\colon \mathcal{R}(G', \mathbf{k})\text{-Mod} \to \chi_Y \mathcal{R}(G', \mathbf{k})\chi_Y\text{-Mod},$$
$$(\mathcal{R}(G', \mathbf{k})\chi_Y \otimes_{\chi_Y \mathcal{R}(G',\mathbf{k})\chi_Y} -)\colon \chi_Y \mathcal{R}(G', \mathbf{k})\chi_Y\text{-Mod} \to \mathcal{R}(G', \mathbf{k})\text{-Mod}$$

are inverse equivalences.

Since $f|_X$ is a bijective quasi-isometry, by Lemma 4.4 $\chi_X \mathcal{R}(G, \mathbf{k})\chi_X$ and $\chi_Y \mathcal{R}(G', \mathbf{k})\chi_Y$ are isomorphic. $\Phi = \Phi_{f|_X}\colon \chi_X \mathcal{R}(G, \mathbf{k})\chi_X \to \chi_Y \mathcal{R}(G', \mathbf{k})\chi_Y$ is the isomorphism, and hence

$$\operatorname{Res}\Phi\colon \chi_Y \mathcal{R}(G', \mathbf{k})\chi_Y\text{-Mod} \to \chi_X \mathcal{R}(G, \mathbf{k})\chi_X\text{-Mod},$$
$$\operatorname{Res}(\Phi^{-1})\colon \chi_X \mathcal{R}(G, \mathbf{k})\chi_X\text{-Mod} \to \chi_Y \mathcal{R}(G', \mathbf{k})\chi_Y\text{-Mod}$$

are inverse equivalences. Therefore

$$\mathcal{F}_1 = (\mathcal{R}(G', \mathbf{k})\chi_Y \otimes_{\chi_Y \mathcal{R}(G',\mathbf{k})\chi_Y} -) \circ \operatorname{Res}(\Phi^{-1}) \circ (\chi_X \mathcal{R}(G, \mathbf{k}) \otimes_{\mathcal{R}(G,\mathbf{k})} -),$$
$$\mathcal{F}_2 = (\mathcal{R}(G, \mathbf{k})\chi_X \otimes_{\chi_X \mathcal{R}(G,\mathbf{k})\chi_X} -) \circ \operatorname{Res}\Phi \circ (\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G',\mathbf{k})} -)$$

are inverse equivalences. As a result, we obtain the main theorem:

Theorem 1. *If finitely generated groups G and G' are quasi-isometric, then $\mathcal{R}(G, \mathbf{k})$ and $\mathcal{R}(G', \mathbf{k})$ are Morita equivalent.*

4.3. On some characteristic modules. Let H be a finitely generated group. There exist characteristic $\mathcal{R}(H, \mathbf{k})$-modules of functions on H preserved by $\mathcal{F}_1$ of the previous subsection. In this subsection, we use the notation of Section 4.2.

We consider a left $\mathcal{R}(H, \mathbf{k})$-module $l(H, \mathbf{k}) = \{F\colon H \to \mathbf{k}\}$ with an action

$$(hF_1)F = F_1^{h^{-1}} F^{h^{-1}} \quad (hF_1 \in \mathcal{R}(H, \mathbf{k}),\ F \in l(H, \mathbf{k})).$$

$l^f(H, \mathbf{k})$ or $l_c(H, \mathbf{k}) = \{F\colon H \to \mathbf{k} \mid \#(\operatorname{supp}(F)) < \infty\}$ are submodules of $l(H, \mathbf{k})$. For $Z \subseteq H$ a left $\chi_Z \mathcal{R}(H, \mathbf{k})\chi_Z$-module $\chi_Z \mathcal{R}(H, \mathbf{k}) \otimes_{\mathcal{R}(H,\mathbf{k})} l(H, \mathbf{k})$ is isomorphic to the left $\chi_Z \mathcal{R}(H, \mathbf{k})\chi_Z$-module $l(Z, \mathbf{k})$ with an action

$$(\chi_Z hF_1\chi_Z)F = \chi_Z F_1^{h^{-1}} \chi_{hZ} F^{h^{-1}} \quad (\chi_Z hF_1\chi_Z \in \chi_Z \mathcal{R}(H, \mathbf{k})\chi_Z,\ F \in l(Z, \mathbf{k})).$$

Lemma 4.6. *Under the notation of* Section 4.2

$$(\operatorname{Res}\Phi) \circ (\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G',\mathbf{k})} -)(l(G', \mathbf{k})) \simeq (\chi_X \mathcal{R}(G, \mathbf{k}) \otimes_{\mathcal{R}(G,\mathbf{k})} -)(l(G, \mathbf{k})).$$

Proof. By Lemma 4.4 $(\operatorname{Res}\Phi)\circ(\chi_Y\mathcal{R}(G',\mathbf{k})\otimes_{\mathcal{R}(G',\mathbf{k})}-)(l(G',\mathbf{k}))$ is isomorphic to the left $\chi_X\mathcal{R}(G,\mathbf{k})\chi_X$-module $l(Y,\mathbf{k})$ with an action

$$\begin{aligned}(\chi_X g\chi_X)F &= (\Phi_{f|_X})(\chi_X g\chi_X)F\\ &= \left(\chi_Y\sum_{i=1}^m h_i\chi_{S_i}\epsilon_{(G,X)}(g,\ f|_X{}^{-1}(\,\cdot\,))\chi_Y\right)F\\ &= \chi_Y\sum_{i=1}^m \chi_{h_iS_i}\epsilon_{(G,X)}(g,\ f|_X{}^{-1}(\,\cdot\,))^{h_i^{-1}}\chi_{h_iY}F^{h_i^{-1}}\end{aligned}$$

for every $g\in G$ and $F\in l(Y,\mathbf{k})$, and also

$$\begin{aligned}(\chi_X F_1\chi_X)F &= (\Phi_{f|_X})(\chi_X F_1\chi_X)F\\ &= (\chi_Y(F_1\circ(f|_X)^{-1})\chi_Y)F\\ &= \chi_Y(F_1\circ(f|_X)^{-1})\chi_Y F\end{aligned}$$

for every $F_1\in l^f(G,\mathbf{k})$ and $F\in l(Y,\mathbf{k})$. We define $\lambda\colon l(Y,\mathbf{k})\to l(X,\mathbf{k})$ by $F\mapsto F\circ f|_X$. We will prove that λ is a left $\chi_X\mathcal{R}(G,\mathbf{k})\chi_X$-isomorphism. Since λ is a bijective additive group homomorphism, we only have to check that the action is preserved. For every $g\in G$, $F\in l(Y,\mathbf{k})$ and $x\in X$, under the notation of Lemma 4.4, if $x\in gX$, then there exists the only h_j such that $f(x)\in h_jS_j$, and also if $x\notin gX$, then there exists no h_j such that $f(x)\in h_jS_j$. Therefore

$$\begin{aligned}\lambda((\chi_X g\chi_X)F)(x) &= (\Phi_{f|_X}(\chi_X g\chi_X)F)(f(x))\\ &= \left(\chi_Y\sum_{i=1}^m \chi_{h_iS_i}\epsilon_{(G,X)}(g,\ f|_X{}^{-1}(\,\cdot\,))^{h_i^{-1}}\chi_{h_iY}F^{h_i^{-1}}\right)(f(x))\\ &= \begin{cases}F^{h_j{}^{-1}}(f(x)) & \text{if}\quad x\in gX,\\ 0 & \text{if}\quad x\notin gX\end{cases}\\ &= \begin{cases}F\circ f(g^{-1}x) & \text{if}\quad x\in gX,\\ 0 & \text{if}\quad x\notin gX\end{cases}\\ &= ((\chi_X g\chi_X)\lambda(F))(x).\end{aligned}$$

For every $F_1\in l^f(G,\mathbf{k})$, $F\in l(Y,\mathbf{k})$ and $x\in X$

$$\begin{aligned}\lambda((\chi_X F_1\chi_X)F)(x) &= (\Phi_{f|_X}(\chi_X F_1\chi_X)F)(f(x))\\ &= (\chi_Y(F_1\circ(f|_X)^{-1})\chi_Y F)(f(x))\\ &= F_1(F\circ f)(x)\\ &= ((\chi_X F_1\chi_X)\lambda(F))(x).\end{aligned}$$

□

Theorem 4.7. *Under the notation of* Section 4.2
(1) $\mathcal{F}_1(l(G,\mathbf{k})) \simeq l(G',\mathbf{k})$, $\mathcal{F}_2(l(G',\mathbf{k})) \simeq l(G,\mathbf{k})$,
(2) $\mathcal{F}_1(l^f(G,\mathbf{k})) \simeq l^f(G',\mathbf{k})$, $\mathcal{F}_2(l^f(G',\mathbf{k})) \simeq l^f(G,\mathbf{k})$,
(3) $\mathcal{F}_1(l_c(G,\mathbf{k})) \simeq l_c(G',\mathbf{k})$, $\mathcal{F}_2(l_c(G',\mathbf{k})) \simeq l_c(G,\mathbf{k})$.

Proof. (1) We send the equation of Lemma 4.6 by $(\mathcal{R}(G,\mathbf{k})\chi_X \otimes_{\chi_X \mathcal{R}(G,\mathbf{k})\chi_X} -)$. Therefore we have $\mathcal{F}_2(l(G',\mathbf{k})) \simeq l(G,\mathbf{k})$. We also send this equation by $\mathcal{F}_1$, and hence $\mathcal{F}_1(l(G,\mathbf{k})) \simeq l(G',\mathbf{k})$. Since λ of the proof of Lemma 4.6 is an isomorphism on l^f and l_c, (2) and (3) are also proved. □

We have a left $\mathcal{R}(H,\mathbf{k})$-module $T = T_H = T_{(H,\mathbf{k})} = \mathcal{R}(H,\mathbf{k}) \otimes_{H\mathbf{k}} \mathbf{k}$. T_H is isomorphic to $l^f(H,\mathbf{k})$. Indeed, $\theta\colon T_H \to l^f(H,\mathbf{k})$ defined by $\theta\left(\sum_{i=1}^n g_n F_n \otimes k\right) = \sum_{i=1}^n F_n^{g_n^{-1}} k$ ($\sum_{i=1}^n g_n F_n \in \mathcal{R}(H,\mathbf{k})$, $k \in \mathbf{k}$) gives an isomorphism. Let M be a left $\mathcal{R}(H,\mathbf{k})$-module. Since $H\mathbf{k}$ is a subring of $\mathcal{R}(H,\mathbf{k})$, M is regarded as a left $H\mathbf{k}$-module. By the flatness of $\mathcal{R}(H,\mathbf{k})_{H\mathbf{k}}$ (see Lemma 5.1 of the next section), we have $\mathrm{Ext}^n_{\mathcal{R}(H,\mathbf{k})}(l^f(H,\mathbf{k}), M) = \mathrm{Ext}^n_{\mathcal{R}(H,\mathbf{k})}(T_H, M) = \mathrm{Ext}^n_{H\mathbf{k}}(\mathbf{k}, M) = \mathrm{H}^n(H, M)$. Since $l_c(H,\mathbf{k})$ is isomorphic to $H\mathbf{k}$, $\mathrm{H}^n(H, l_c(H,\mathbf{k})) = \mathrm{H}^n(H, H\mathbf{k})$. This cohomology group is the coarse cohomology (see [7]). By Theorem 4.7 $\mathrm{H}^n(H, H\mathbf{k})$ is a quasi-isometry invariant.

In the case of $\mathbf{k} = \mathbf{C}$ (or $\mathbf{R}$) for $0 < p \le \infty$ we have a module of p-summable functions $l^p(G,\mathbf{C}) \subseteq l(G,\mathbf{C})$. We can also prove $\mathcal{F}_1(l^p(G,\mathbf{C})) \simeq l^p(G',\mathbf{C})$, $\mathcal{F}_2(l^p(G',\mathbf{C})) \simeq l^p(G,\mathbf{C})$. Therefore $\mathrm{Ext}^n_{\mathcal{R}(H,\mathbf{k})}(l^f(H,\mathbf{C}), l^p(H,\mathbf{C}))$ is a quasi-isometry invariant. This cohomology group is isomorphic to $\mathrm{H}^n(H, l^p(H,\mathbf{C}))$: the coarse l^p-cohomology (see [6]).

5. The global dimension and the weak global dimension of algebraic translation algebras

Let G be a finitely generated group. We see that $\mathcal{R}(G,\mathbf{k})_{G\mathbf{k}}$ is a flat $G\mathbf{k}$-module. Let $\Lambda = \left\{S = \{S_1,\ldots,S_{n_S}\} \;\middle|\; S_i \subseteq G,\ \bigsqcup_{i=1}^{n_S} S_i = G\right\}$ be the set of finite decompositions of G. For α and $\beta \in \Lambda$ we denote $\alpha < \beta$ if β is a refinement of α. Let $L_\alpha = \bigoplus_{i=1}^{n_\alpha}(G\mathbf{k})$ be a right free $G\mathbf{k}$-module. If $\alpha < \beta$, then for each $1 \le k \le n_\beta$ there exists $1 \le i_k \le n_\alpha$ such that $\beta_k \subseteq \alpha_{i_k}$. Let $f_{\beta\alpha}\colon L_\alpha \to L_\beta$ be a $G\mathbf{k}$-homomorphism such that $f_{\beta\alpha}(x_1,\ldots,x_{n_\alpha}) = \left(x_{i_1},\ldots,x_{i_{n_\beta}}\right)$ ($x_k \in G\mathbf{k}$). These data define a direct system of right $G\mathbf{k}$-modules, and hence we have a right $G\mathbf{k}$-module $\varinjlim L_\alpha$. $\varinjlim L_\alpha = \left(\bigoplus_{\alpha\in\Lambda} L_\alpha\right)/N$, where N is a submodule of $\bigoplus_{\alpha\in\Lambda} L_\alpha$ generated by $\{\iota_\beta \circ f_{\beta\alpha}(x) - \iota_\alpha(x) \mid \alpha < \beta,\ x \in L_\alpha\}$. ι_α and ι_β are injections to $\bigoplus_{\alpha\in\Lambda} L_\alpha$.

Lemma 5.1. *$\mathcal{R}(G,\mathbf{k})_{G\mathbf{k}}$ is a direct limit of flat $G\mathbf{k}$-modules*:

$$\varinjlim L_\alpha \simeq \mathcal{R}(G,\mathbf{k})_{G\mathbf{k}}.$$

Therefore $\mathcal{R}(G,\mathbf{k})_{G\mathbf{k}}$ is a flat $G\mathbf{k}$-module, and the functor $\mathcal{R}(G,\mathbf{k}) \otimes_{G\mathbf{k}} -\colon G\mathbf{k}\text{-Mod} \to \mathcal{R}(G,\mathbf{k})\text{-Mod}$ is exact.

Proof. We define $t_\alpha\colon L_\alpha \to \mathcal{R}(G, \mathbf{k})$ by $t_\alpha(x_1, \dots, x_{n_\alpha}) = \sum_{i=1}^{n_\alpha} \chi_{\alpha_i} x_i$ $(x_i \in G\mathbf{k})$. The direct sum of $\{t_\alpha \mid \alpha \in \Lambda\}$ defines an isomorphism. □

We recall the definitions of some homological dimensions. Let R be a ring, and M a left R-module.
(1) $\mathrm{fd}_R(M) = \sup\{n \mid \exists \text{a right } R\text{-module } N \text{ with } \mathrm{Tor}_n^R(N, M) \neq 0\}$. This number is equal to the minimal number n such that there exists an n-length flat resolution of M.
(2) $\mathrm{pd}_R(M) = \sup\{n \mid \exists \text{a left } R\text{-module } N \text{ with } \mathrm{Ext}_R^n(M, N) \neq 0\}$. This number is also equal to the minimal number n such that there exists an n-length projective resolution of M.
(3) $\mathrm{wd}(R) = \sup\{\mathrm{fd}_R(M) \mid M \text{ is a left } R\text{-module}\}$.
(4) $\mathrm{l.gl.dim}(R) = \sup\{\mathrm{pd}_R(M) \mid M \text{ is a left } R\text{-module}\}$ ([21, Section 3, 4] is a good reference for Tor or Ext and homological dimensions).
wd and l.gl.dim are Morita invariants. We discuss $\mathrm{l.gl.dim}(\mathcal{R}(G, \mathbf{k}))$ and $\mathrm{wd}(\mathcal{R}(G, \mathbf{k}))$.

Lemma 5.2. *Let G be a finitely generated group, and l a ring containing $\mathbf{k}$ as a subring. We assume that G acts on l from the right trivially on $\mathbf{k}$. Let $\mathcal{R} = G * l$, and $T = \mathcal{R} \otimes_{G\mathbf{k}} \mathbf{k}$. For every left $\mathcal{R}$-module A and B, and an $\mathcal{R}$-projective resolution $\mathcal{C}$ of A we have a spectral sequence*

$$\mathrm{Ext}_{\mathcal{R}}^p(T, \mathrm{H}^q(\widetilde{\mathrm{Hom}}_l(\mathcal{C}, B))) \Rightarrow_p \mathrm{Ext}_{\mathcal{R}}^n(A, B),$$

where for a left $\mathcal{R}$-module C, $\widetilde{\mathrm{Hom}}_l(C, B) = \mathrm{Hom}_l(C, B)$ is a left module with a left action of $\mathcal{R}$ defined by

$$(gF)\varphi(x) = F^{g^{-1}} g\varphi(g^{-1}x)$$

for every $\varphi \in \widetilde{\mathrm{Hom}}_l(C, B)$, $g \in G$, $F \in l$ and $x \in C$. The notation of a spectral sequence is that of [3, Chapter XV].

Proof. This spectral sequence is obtained by modifying a spectral sequence of Cartan and Leray [3, Proposition 8.2].

First, we prove $\mathrm{Ext}_{\mathcal{R}}^p(T, \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B)) = 0$ (if $p > 0$) by direct calculation. Since $\mathcal{R} = \bigoplus_{g \in G} l \cdot g$ and $\varphi \in \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B)$ is decided by $\varphi(g) \in B$, we have $\widetilde{\mathrm{Hom}}_l(\mathcal{R}, B) = \prod_{g \in G} B_g$, where B_g is a copy of B. For $(b_g)_{g \in G} \in \prod_{g \in G} B_g$, the $\mathcal{R}$-action is given by $(xF)(b_g)_{g \in G} = (F^{x^{-1}} x b_{x^{-1}g})_{g \in G}$ $(x \in G,\ F \in l)$. We consider free right $\mathbf{k}$-modules $I_p = \{(\sigma_0, \dots, \sigma_p) \mid \sigma_i \in G\}\mathbf{k}$ and $\varphi_{p-1}(\sigma_0, \dots, \sigma_p) = \sum_{i=0}^p (-1)^i (\sigma_0, \dots, \check{\sigma}_i, \dots, \sigma_p)$. $\mathbf{I} = \{I_p, \varphi_p\}$ is the $G\mathbf{k}$-standard resolution of $\mathbf{k}$. Then $\tilde{\mathbf{I}} = \mathcal{R} \otimes_{G\mathbf{k}} \mathbf{I}$ is an $\mathcal{R}$-projective resolution of T. Since $f \in \mathrm{Hom}(\tilde{I}_p, \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B))$ is decided by $f(1, \sigma_1, \dots, \sigma_p)(g) \in B_g$, we have

$$\mathrm{Hom}(\tilde{I}_p, \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B)) = \prod_{\sigma_1, \dots, \sigma_p, g \in G} B_{\sigma_1, \dots, \sigma_p, g},$$

where $B_{\sigma_1,\dots,\sigma_p,g}$ is a copy of B. $\partial_p = \mathrm{Hom}(\tilde{\varphi}_p, \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B))$ satisfies

$$\text{(i)}\qquad \begin{aligned}&\partial_p((b_{\sigma_1,\dots,\sigma_p,g})_{\sigma_1,\dots,\sigma_p,g\in G})\\ &= \left(\sigma_1 b_{\sigma_1^{-1}\sigma_2,\dots,\sigma_1^{-1}\sigma_{p+1},\sigma_1^{-1}g} + \sum_{i=1}^{p+1}(-1)^i b_{\sigma_1,\dots,\check{\sigma}_i,\dots,\sigma_{p+1},g}\right)_{\sigma_1,\dots,\sigma_{p+1},g\in G}.\end{aligned}$$

By the definition of Ext, $\mathrm{Ext}^p_{\mathcal{R}}(T, \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B)) = \mathrm{Ker}\partial_p/\mathrm{Im}\,\partial_{p-1}$. For every $(b_{\sigma_1,\dots,\sigma_p,g})_{\sigma_1,\dots,\sigma_p,g\in G} \in \mathrm{Ker}\partial_p$ and in the case of $p \geq 1$, we have $c_{\sigma_1,\dots,\sigma_{p-1},g} = (-1)^p b_{\sigma_1,\dots,\sigma_{p-1},g,g}$. This satisfies

$$\text{(ii)}\qquad \partial_{p-1}(c_{\sigma_1,\dots,\sigma_{p-1},g})_{\sigma_1,\dots,\sigma_{p-1},g\in G} = (b_{\sigma_1,\dots,\sigma_p,g})_{\sigma_1,\dots,\sigma_p,g\in G}.$$

In fact,

$$\text{(iii)}\qquad \begin{aligned}&\partial_{p-1}(c_{\sigma_1,\dots,\sigma_{p-1},g})_{\sigma_1,\dots,\sigma_{p-1},g\in G}\\ &= \left(\sigma_1 c_{\sigma_1^{-1}\sigma_2,\dots,\sigma_1^{-1}\sigma_p,\sigma_1^{-1}g} + \sum_{i=1}^{p}(-1)^i c_{\sigma_1,\dots,\check{\sigma}_i,\dots,\sigma_p,g}\right)_{\sigma_1,\dots,\sigma_p,g\in G}\\ &= \left((-1)^p\sigma_1 b_{\sigma_1^{-1}\sigma_2,\dots,\sigma_1^{-1}\sigma_p,\sigma_1^{-1}g,\sigma_1^{-1}g} + \sum_{i=1}^{p}(-1)^{p+i} b_{\sigma_1,\dots,\check{\sigma}_i,\dots,\sigma_p,g,g}\right)_{\sigma_1,\dots,\sigma_p,g\in G}.\end{aligned}$$

By substituting g for σ_{p+1} in (i), we have

$$\text{(iv)}\qquad \sigma_1 b_{\sigma_1^{-1}\sigma_2,\dots,\sigma_1^{-1}\sigma_p,\sigma_1^{-1}g,\sigma_1^{-1}g} + \sum_{i=1}^{p}(-1)^i b_{\sigma_1,\dots,\check{\sigma}_i,\dots,\sigma_p,g,g} + (-1)^{p+1} b_{\sigma_1,\dots,\sigma_p,g} = 0.$$

(ii) is obtained by (iii) and (iv). Therefore $\mathrm{Ext}^p_{\mathcal{R}}(T, \widetilde{\mathrm{Hom}}_l(\mathcal{R}, B)) = 0$ (if $p > 0$) is proved. This shows $\mathrm{Ext}^p_{\mathcal{R}}(T, \widetilde{\mathrm{Hom}}_l(P, B)) = 0$ (if $p > 0$) for every projective left $\mathcal{R}$-module P.

Second, for left $\mathcal{R}$-modules X and Y, $\rho\colon \mathrm{Hom}_{\mathcal{R}}(T, \widetilde{\mathrm{Hom}}_l(X, Y)) \to \mathrm{Hom}_{\mathcal{R}}(X, Y)$ defined by $f \mapsto f(1)$ is an isomorphism.

Let $\mathbf{X}$ be a projective resolution of T and $\mathbf{Y} = \widetilde{\mathrm{Hom}}_l(\mathcal{C}, B)$. $\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})$ is a double complex, and hence we have two spectral sequences with the same limit:

$$\begin{aligned}\mathrm{I}_2^{p,q} &= \mathrm{H}^p(\mathrm{H}^q(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y}))) \Rightarrow_p \mathrm{Tot}^n(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})),\\ \mathrm{II}_2^{p,q} &= \mathrm{H}^q(\mathrm{H}^p(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y}))) \Rightarrow_q \mathrm{Tot}^n(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})).\end{aligned}$$

By the first assertion above $\mathrm{H}^p(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})) = \mathrm{Hom}_{\mathcal{R}}(T, \mathbf{Y})$ (if $p = 0$) and $\mathrm{H}^p(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})) = 0$ (otherwise). We have $\mathrm{II}_2^{p,q} = \mathrm{H}^q(\mathrm{Hom}_{\mathcal{R}}(T, \mathbf{Y})) = \mathrm{H}^q(\mathcal{C}, B) = \mathrm{Ext}^q_{\mathcal{R}}(A, B)$ (if $p = 0$) and $\mathrm{II}_2^{p,q} = 0$ (otherwise) by the second assertion above. Therefore $\mathrm{Tot}^n(\mathrm{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})) = \mathrm{Ext}^n_{\mathcal{R}}(A, B)$. Since each term X_p of $\mathbf{X}$ is projective,

$\mathrm{Hom}_{\mathcal{R}}(X_p, -)$ is exact.

$$\begin{aligned}\mathrm{H}^q(\mathrm{Hom}_{\mathcal{R}}(X_p, \mathbf{Y})) &= \mathrm{H}^q(\mathrm{Hom}_{\mathcal{R}}(X_p, \widetilde{\mathrm{Hom}}_l(\mathcal{C}, B))) \\ &= \mathrm{Hom}_{\mathcal{R}}(X_p, \mathrm{H}^q(\widetilde{\mathrm{Hom}}_l(\mathcal{C}, B)))\end{aligned}$$

shows that $\mathrm{I}_2^{p,q} = \mathrm{Ext}^p_{\mathcal{R}}(T, \mathrm{H}^q(\widetilde{\mathrm{Hom}}_l(\mathcal{C}, B)))$. □

Corollary 5.3.

$$\mathrm{pd}_{\mathcal{R}(G,\mathbf{k})}(T) \leq \mathrm{l.gl.dim}(\mathcal{R}(G, \mathbf{k})) \leq \mathrm{pd}_{\mathcal{R}(G,\mathbf{k})}(T) + \mathrm{l.gl.dim}(l^f(G, \mathbf{k})).$$

We can also prove the Tor *version of* Lemma 5.2*, and hence*

$$\mathrm{fd}_{\mathcal{R}(G,\mathbf{k})}(T) \leq \mathrm{wd}(\mathcal{R}(G, \mathbf{k})) \leq \mathrm{fd}_{\mathcal{R}(G,\mathbf{k})}(T) + \mathrm{wd}(l^f(G, \mathbf{k})).$$

We estimate $\mathrm{pd}_{\mathcal{R}(G,\mathbf{k})}(T)$ and $\mathrm{fd}_{\mathcal{R}(G,\mathbf{k})}(T)$ by the homological dimensions of G.

Lemma 5.4. (1) $\mathrm{pd}_{\mathcal{R}(G,\mathbf{k})}(T) \leq \mathrm{cd}_{\mathbf{k}}(G)$, *where* $\mathrm{cd}_{\mathbf{k}}(G) = \mathrm{pd}_{G\mathbf{k}}(\mathbf{k})$.
(2) $\mathrm{fd}_{\mathcal{R}(G,\mathbf{k})}(T) \leq \mathrm{hd}_{\mathbf{k}}(G)$, *where* $\mathrm{hd}_{\mathbf{k}}(G) = \mathrm{fd}_{G\mathbf{k}}(\mathbf{k})$.
(3) *If* $\mathrm{cd}_{\mathbf{k}}(G) < \infty$, *then* $\mathrm{cd}_{\mathbf{k}}(G) = \mathrm{pd}_{\mathcal{R}(G,\mathbf{k})}(T)$.
(4) *If* $\mathrm{hd}_{\mathbf{k}}(G) < \infty$, *then* $\mathrm{hd}_{\mathbf{k}}(G) = \mathrm{fd}_{\mathcal{R}(G,\mathbf{k})}(T)$.

Proof. Since the functor $\mathcal{R}(G,\mathbf{k}) \otimes_{G\mathbf{k}} -$ is exact by Lemma 5.1, a projective resolution (flat resolution) of $\mathbf{k}$ is mapped to a projective resolution (flat resolution) of T by $\mathcal{R}(G, \mathbf{k}) \otimes_{G\mathbf{k}} -$. Therefore (1) and (2) are obtained.

(3) and (4) are proved by the same argument as [18, Section 4]. □

We estimate $\mathrm{l.gl.dim}(l^f(G, \mathbf{k}))$ and $\mathrm{wd}(l^f(G, \mathbf{k}))$.

Lemma 5.5. *Let* Δ *be a countably infinite set.*
(1) *If* $\mathbf{k}$ *is a field, then* $\mathrm{wd}(l^f(\Delta, \mathbf{k})) = 0$.
(2) *If* $\mathbf{k}$ *is* $\mathbf{Z}$*, then* $\mathrm{wd}(l^f(\Delta, \mathbf{k})) = 1$.
If the continuum hypothesis is true, then
(3) *if* $\mathbf{k}$ *is a field, then* $\mathrm{l.gl.dim}(l^f(\Delta, \mathbf{k})) = 2$,
(4) *if* $\mathbf{k}$ *is* $\mathbf{Z}$*, then* $\mathrm{l.gl.dim}(l^f(\Delta, \mathbf{k})) \leq 3$.

Proof. (1) We see that for every $F \in l^f(\Delta, \mathbf{k})$, $F \in F \cdot l^f(\Delta, \mathbf{k}) \cdot F$ is satisfied. Therefore $l^f(\Delta,\mathbf{k})$ is von-Neumann regular [11, xviii, the third paragraph]. This implies that $\mathrm{wd}(l^f(\Delta, \mathbf{k})) = 0$ [11, (5.62a) p. 185].

(2) $l^f(\Delta, \mathbf{k})$ is not von-Neumann regular since $2 \notin 2 \cdot l^f(\Delta, \mathbf{k}) \cdot 2$. Every ideal of $l^f(\Delta, \mathbf{k})$ is generated by projective modules with the form $l^f(\Delta, \mathbf{k}) \cdot \chi_X n$, and hence every ideal of $l^f(\Delta, \mathbf{k})$ is flat. This implies $\mathrm{wd}(l^f(\Delta, \mathbf{k})) = 1$ [11, (5.69) p. 187].

(3) By the theorem of Osofsky [15, Corollary 2.47] for every ring R if every left ideal of R is generated by $\aleph_h$ elements, then

$$\text{l.gl.dim}(R) \le \text{wd}(R) + h + 1.$$

Every ideal of $l^f(\Delta, \mathbf{k})$ is generated by characteristic functions on Δ. Therefore if the continuum hypothesis is true, then $\text{l.gl.dim}(l^f(\Delta, \mathbf{k})) \le 2$. Since $l^f(\Delta, \mathbf{k})$ has a non-projective ideal ($l^f(\Delta, \mathbf{k})$ is not semi-simple and not hereditary), $\text{l.gl.dim}(l^f(\Delta, \mathbf{k})) = 2$ [11, (5.14) p. 169].

(4) It is proved by the theorem of Osofsky and (2). □

We remark that $\text{wd}(l^\infty(\Delta, \mathbf{C})) \ge 1$. Indeed, for $s_1, \dots, s_n, \dots \in \Delta$ there exists a function $f \in l^\infty(G, \mathbf{C})$ such that $f(s_n) = \exp(-n)$, and $f \notin f \cdot l^\infty(\Delta, \mathbf{C}) \cdot f$.

Theorem 5.6. *If* $\mathbf{k}$ *is a field, then*
(1) *if* $\text{hd}_{\mathbf{k}}(G) < \infty$, *then* $\text{wd}(\mathcal{R}(G, \mathbf{k})) = \text{hd}_{\mathbf{k}}(G)$,
and if the continuum hypothesis is true, then
(2) *if* $\text{cd}_{\mathbf{k}}(G) < \infty$, *then* $\text{cd}_{\mathbf{k}}(G) \le \text{l.gl.dim}(\mathcal{R}(G, \mathbf{k})) \le \text{cd}_{\mathbf{k}}(G) + 2$.

Proof. The assertions (1) and (2) follow from Corollary 5.3, Lemmas 5.4 and 5.5. □

6. Geometric description of $\mathcal{R}(G, \mathbf{k})$

In this section, we need some theorems of groupoid theory and category theory. Everything needed in this section is in Sections 2.5, 2.6 and 2.7. Let G be a finitely generated group with the identity element e, and $\mathcal{G} = G \ltimes \beta G$ an étale groupoid. We consider $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$. We define the characteristic object $U \in \underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$.

DEFINITION 6.1. We define $U = \beta G \times G\mathbf{k}$, where $G\mathbf{k}$ has the discrete topology. An element $(x, \alpha) \in \beta G \times G\mathbf{k}$ is denoted by ${}_x(\alpha)$. We also define

$$p_0 \colon U \to \beta G \quad \text{by} \quad p_0({}_x(\alpha)) = x,$$
$$p_1 \colon (G \ltimes \beta G) \times_{p_0} U \to U \quad \text{by} \quad p_1((g, x), {}_x(\alpha)) = {}_{gx}(g\alpha),$$
$$M \colon U \oplus U \to U \quad \text{by} \quad M({}_x(\alpha), {}_x(\beta)) = {}_x(\alpha + \beta) \quad (x \in \beta G),$$
$$\mathcal{U} \colon \Theta \to U \quad \text{by} \quad \mathcal{U}(x) = {}_x(0) \quad (x \in \beta G),$$
$$v \colon U \to U \quad \text{by} \quad v({}_x(\alpha)) = {}_x(-\alpha) \quad (x \in \beta G),$$
$$\mathcal{M} \colon \mathbf{k} \oplus U \to U \quad \text{by} \quad \mathcal{M}(k, {}_x(\alpha)) = {}_x(k\alpha) \quad (x \in \beta G).$$

$U = ((U, p_0, p_1), M, \mathcal{U}, v, \mathcal{M})$ is an object of $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$. p_0, p_1, M, $\mathcal{U}$, v and $\mathcal{M}$ are open maps. $\{{}_x(e) \mid x \in \beta G\}$ can be identified with βG by p_0.

A morphism from the characteristic object U is determined on $\beta G \subseteq U$:

Lemma 6.2. *For every object* $S = ((S, q_0, q_1), M_S, \mathcal{U}_S, v_S, \mathcal{M}_S)$ *of* $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ *a continuous map* $w\colon \beta G \to S$ *satisfying* $q_0 \circ w = \mathrm{id}_{\beta G}$ *defines the unique morphism* $f\colon U \to S$ *such that* $f|_{\beta G} = w$.

Proof. We define f by $f\left({}_x\left(\sum_{i=1}^n g_i k_i\right)\right) = \sum_{i=1}^n k_i \cdot (g_i, g_i{}^{-1}x) \cdot w({}_{g_i{}^{-1}x}(e))$ for every $x \in \beta G$, $g_i \in G$ and $k_i \in \mathbf{k}$. Therefore if f is a morphism, then f is uniquely defined. Since $q_0 \circ w = \mathrm{id}_{\beta G}$, f satisfies $q_0 \circ f = p_0$. In order to prove f is a morphism we may prove that f is continuous, but this is a routine. □

Let $\mathcal{A}$ be an A.B.3 category. $U \in \mathrm{Ob}(\mathcal{A})$ is called a *projective object* if for every epi $b \in \mathrm{Hom}(B, C)$ and morphism $a \in \mathrm{Hom}(U, C)$ there exists $c \in \mathrm{Hom}(U, B)$ such that $a = b \circ c$. A projective object $U \in \mathrm{Ob}(\mathcal{A})$ is called a *projective generator* if every non-zero $A \in \mathrm{Ob}(\mathcal{A})$ satisfies $\mathrm{Hom}(U, A) \neq 0$. $U \in \mathrm{Ob}(\mathcal{A})$ is said to be *small* if every morphism from U into a coproduct $s\colon U \to \bigoplus_{\lambda\in\Lambda} A_\lambda$ factors as $U \to \bigoplus_{\lambda\in J} A_\lambda \to \bigoplus_{\lambda\in\Lambda} A_\lambda$ where J is a finite subset of Λ and morphism between the coproducts is the one which preserves injections.

Theorem 6.3 ([12, Theorem 3.1, p. 631])**.** *Let* $\mathcal{A}$ *be an* A.B.3 *category with a small projective generator* U *and* $\mathrm{End}_{\mathcal{A}}(U)$ *denote the endomorphism ring of* U. *Then the functor* $T\colon \mathcal{A} \to \text{Mod-}\mathrm{End}_{\mathcal{A}}(U)$ *defined by* $T(A) = \mathrm{Hom}(U, A)$ *is an additive equivalence.*

To see that Theorem 6.3 is applied to $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ we prove that the characteristic object U is a small projective generator of $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$.

Lemma 6.4. (1) $U \in \mathrm{Ob}(\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}))$ *is a projective object.*
(2) $U \in \mathrm{Ob}(\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}))$ *is a projective generator.*
(3) $U \in \mathrm{Ob}(\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}))$ *is small.*

Proof. (1) For every $B = (B, p_{0,B}, p_{1,B})$, $C = (C, p_{0,C}, p_{1,C}) \in \mathrm{Ob}(\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}))$, a morphism $a\colon U \to C$ and an epi $b\colon B \to C$ there exists an open set $V_x \subseteq C$ such that $a({}_x(e)) \in V_x$ and $p_{0,C}|_{V_x}$ is a homeomorphism for each $x \in \beta G$. Since $b\colon B \to C$ is an epi, b is surjective. Therefore there exists $y_x \in B$ such that $b(y_x) = a({}_x(e))$. b is continuous, and hence $y_x \in b^{-1}(V_x)$ is open. We have an open set $L'_x \subseteq b^{-1}(V_x)$ such that $y_x \in L'_x$ and $p_{0,B}|_{L'_x}$ is a homeomorphism. We also have a clopen set $L''_x \subseteq p_{0,B}(L'_x) \cap p_{0,C}(V_x)$ since the topology of βG is generated by clopen sets (Lemma 2.5 (2)). b satisfies $p_{0,C} \circ b = p_{0,B}$, and hence $L_x = (p_{0,B}|_{L'_x})^{-1}(L''_x)$ satisfies $L_x \subseteq b^{-1}(V_x)$, $y_x \in L_x$ and $b|_{L_x}$ is a homeomorphism. Clopen sets $W_x = a^{-1}(b(L_x))$ satisfy $\bigcup_{x\in\beta G} W_x = \beta G$. βG is compact, and hence $\bigcup_{j=1}^m W_{x_j} = \beta G$. We have a refinement $\{A_i\}_{i=1}^n$ of $\{W_{x_j}\}_{j=1}^m$ such that $\beta G = \bigsqcup_{i=1}^n A_i$. We can chose k_i for each i such that $A_i \subseteq W_{x_{k_i}}$, and hence we define a continuous map w by $w|_{A_i} = \left(b|_{L_{x_{k_i}}}\right)^{-1} \circ a|_{A_i}$. By Lemma 6.2 there exists a morphism c such that $c|_{\beta G} = w$. This c satisfies $b \circ c = a$.

(2) By (1) U is a projective object. We will prove that every non-zero object $A = (A, p_{0,A}, p_{1,A}) \in \mathrm{Ob}(\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}))$ satisfies $\mathrm{Hom}(U, A) \neq 0$. Since A is non-zero, there exists $a \neq 0 \in A$. For $x = p_{0,A}(a) \in \beta G$ since $p_{0,A}$ is a local homeomorphism and by Lemma 2.5 (2) the topology of βG is generated by clopen sets, there exists clopen $W_x \subseteq A$ such that $a \in W_x$ and $p_{0,A}|_{W_x}$ is a homeomorphism. We define a continuous map w by $w|_{p_{0,A}(W_x)} = (p_{0,A}|_{W_x})^{-1}$ and $w|_{\beta G - p_{0,A}(W_x)} = 0$. By Lemma 6.2 there exists a morphism f such that $f|_{\beta G} = w$. $f(x) = a \neq 0$ shows that $\mathrm{Hom}(U, A) \neq 0$.

(3) For every morphism from U into a coproduct $s\colon U \to \bigoplus_{\lambda\in\Lambda} A_\lambda$ there exists a finite set $\Lambda_x \subseteq \Lambda$ such that $s(x) \in \bigoplus_{\lambda\in\Lambda_x} A_\lambda \subseteq \bigoplus_{\lambda\in\Lambda} A_\lambda$ for each $x \in \beta G$. $\bigoplus_{\lambda\in\Lambda_x} A_\lambda$ is open in $\bigoplus_{\lambda\in\Lambda} A_\lambda$. s is continuous and the topology of βG is generated by clopen sets (Lemma 2.5 (2)), and hence there exists clopen $W_x \subseteq \beta G$ such that $s(W_x) \subseteq \bigoplus_{\lambda\in\Lambda_x} A_\lambda$ and $x \in W_x$. βG is compact, and hence there exist $x_1, \ldots, x_m$ such that $\bigcup_{j=1}^m W_{x_j} = \beta G$. Therefore $s(\beta G) \subseteq \bigoplus_{\lambda\in\bigcup_{j=1}^m \Lambda_{x_j}} A_\lambda$. This shows that $s\colon U \to \bigoplus_{\lambda\in\Lambda} A_\lambda$ factors as $U \to \bigoplus_{\lambda\in J} A_\lambda \to \bigoplus_{\lambda\in\Lambda} A_\lambda$, where $J = \bigcup_{j=1}^m \Lambda_{x_j}$ and morphism between the coproducts is the one which preserves injections. □

Since U is a small projective generator, by Theorem 6.3 the functor $T\colon \underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}) \to \text{Mod-End}_{\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})}(U)$ defined by $T(A) = \mathrm{Hom}(U, A)$ is an additive equivalence. We describe the ring $\mathrm{End}_{\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})}(U)$.

Lemma 6.5. (1) *A morphism $f\colon U \to U$ is determined by a finite decomposition $G = \bigsqcup_{i=1}^n L_i$ and $\alpha_i \in G\mathbf{k}$ such that $f({}_x(e)) = {}_x(\alpha_i)$ for every $x \in \hat{L}_i$.*
(2) *$\mathrm{End}_{\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})}(U)$ is isomorphic to the ring $\mathcal{R}(G, \mathbf{k})^{op}$.*

Proof. (1) By Lemma 6.2 a continuous map $w = f|_{\beta G}\colon \beta G \to U$ uniquely defines f. Since $U = \beta G \times G\mathbf{k}$, we have a continuous projection $\pi_2\colon \beta G \times G\mathbf{k} \to G\mathbf{k}$. The topology of $G\mathbf{k}$ is discrete, and hence for $\alpha \in G\mathbf{k}$, $\{\alpha\}$ is clopen. For clopen $W_\alpha = (\pi_2 \circ w)^{-1}(\{\alpha\})$, $\beta G = \bigsqcup_{\alpha\in G\mathbf{k}} W_\alpha$. βG is compact, and hence $\beta G = \bigsqcup_{i=1}^m W_{\alpha_i}$. By Lemma 2.5 (3), $W_{\alpha_i} = \hat{L}_i$ by some $L_i \subseteq G$, and $G = \bigsqcup_{i=1}^m L_i$. Since $\pi_2 \circ w|_{\hat{L}_i} = \alpha_i$, $w(x) = {}_x(\alpha_i)$ for every $x \in \hat{L}_i$.

(2) We define a map $\theta\colon \mathrm{End}_{\underline{\mathrm{Mod}}_{\mathbf{k}}(G\ltimes\beta G)}(A) \to \mathcal{R}(G, \mathbf{k})$ by

$$\theta(f) = \sum_{i=1}^n \chi_{L_i}\alpha_i,$$

where L_i and α_i are determined by (1). Since θ is bijective by (1), we will prove that it is a ring homomorphism. For every f and $g \in \mathrm{End}_{\underline{\mathrm{Mod}}_{\mathbf{k}}(G\ltimes\beta G)}(A)$ such that $\theta(f) = \sum_{i=1}^n \chi_{L_i}\alpha_i$ and $\theta(g) = \sum_{j=1}^m \chi_{M_j}\beta_j$ we have an expression $\alpha_i = \sum_{l=1}^{n_i} h_{i,l}k_{i,l}$ by $h_{i,l} \in$

G and $k_{i,l} \in \mathbf{k}$. We have the following equations:

$$\begin{aligned}
\theta(f)\theta(g) &= \left(\sum_{i=1}^{n} \chi_{L_i}\alpha_i\right)\left(\sum_{j=1}^{m} \chi_{M_j}\beta_j\right)\\
&= \left(\sum_{i=1}^{n} \chi_{L_i} \sum_{l=1}^{n_i} h_{i,l}k_{i,l}\right)\left(\sum_{j=1}^{m} \chi_{M_j}\beta_j\right)\\
&= \left(\sum_{i=1}^{n}\sum_{l=1}^{n_i}\sum_{j=1}^{m} \chi_{L_i\cap h_{i,l}M_j} h_{i,l}k_{i,l}\beta_j\right).
\end{aligned}$$

On the other hand, we have $G = \bigsqcup_{i,j}(L_i \cap h_{i,l}M_j)$, and for every $x \in \widehat{(L_i \cap h_{i,l}M_j)}$

$$\begin{aligned}
g \circ f({}_x(e)) = g({}_x(\alpha_i)) &= g\left({}_x\left(\sum_{l=1}^{n_i} h_{i,l}k_{i,l}\right)\right)\\
&= \sum_{l=1}^{n_i} k_{i,l}g((h_{i,l}, h_{i,l}^{-1}x)\cdot {}_{h_{i,l}^{-1}x}(e))\\
&= \sum_{l=1}^{n_i} k_{i,l}(h_{i,l}, h_{i,l}^{-1}x)\cdot {}_{h_{i,l}^{-1}x}(\beta_j)\\
&= {}_x\left(\sum_{l=1}^{n_i} h_{i,l}k_{i,l}\beta_j\right).
\end{aligned}$$

Therefore $\theta(f)\theta(g) = \theta(g \circ f)$. □

By Theorem 6.3, Lemmas 6.4 and 6.5, we have

Theorem 6.6. *The functor* $T'\colon \underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}) \to \mathcal{R}(G, \mathbf{k})$*-Mod defined by* $T'(A) = (\mathrm{Res}\,\theta^{-1})(\mathrm{Hom}(U, A))$ *is an* (*additive*) *equivalence, where* θ *is an isomorphism defined in* Lemma 6.5 (2).

7. Appendix

7.1. An alternative proof of Theorem 1. Let G and G' be quasi-isometric finitely generated groups, we have Diagram 1.

Q.I. means quasi-isometric and W.E. means weak equivalent: let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids, $\mathcal{G}$ and $\mathcal{H}$ are said to be weakly equivalent (Morita equivalent) if there exists an étale groupoid $\mathcal{K}$ and there exist essential morphisms $\Phi\colon \mathcal{K} \to \mathcal{G}$, $\Psi\colon \mathcal{K} \to \mathcal{H}$ (about precise definitions see [4, 1.4, 1.5] or [14, Section 5]). The following theorems for a quasi-isometry are known.

Diagram 1.

$$\begin{array}{ccc}
G & \simeq_{\text{Q.I.}} & G' \\
 & \Updownarrow \text{ (i)} & \\
G \ltimes \beta G & \simeq_{\text{W.E.}} & G' \ltimes \beta G' \\
 & \Updownarrow \text{ (ii)} & \\
\underline{\mathrm{Sh}}(G \ltimes \beta G) & \simeq_{\text{as topos}} & \underline{\mathrm{Sh}}(G' \ltimes \beta G') \\
 & \Downarrow \text{ (iii)} & \\
\underline{\mathrm{Mod}}_{\mathbf{k}}(G \ltimes \beta G) & \simeq_{\text{additive e.q.}} & \underline{\mathrm{Mod}}_{\mathbf{k}}(G' \ltimes \beta G') \\
\wr| \text{ Theorem 6.6} & & \wr| \text{ Theorem 6.6} \\
\mathcal{R}(G, \mathbf{k})\text{-Mod} & & \mathcal{R}(G', \mathbf{k})\text{-Mod}
\end{array}$$

Theorem 7.1 ([19, Corollary 3.6, p. 820])**.** *Let G and G' be finitely generated groups. If G and G' are quasi-isometric, then $G \ltimes \beta G$ and $G' \ltimes \beta G'$ are weakly equivalent.*

Theorem 7.1 is proved by the notion of the coarse space. The converse of the theorem is also true:

Theorem 7.2. *Let G and G' be finitely generated groups. If $G \ltimes \beta G$ and $G' \ltimes \beta G'$ are weakly equivalent, then G and G' are quasi-isometric.*

Proof. If $G \ltimes \beta G$ and $G' \ltimes \beta G'$ are weakly equivalent, then G and G' have a topological coupling Ω. Gromov's dynamical criterion [8, $0.2.C_2'$] shows that G and G' are quasi-isometric. □

(i) in Diagram 1 is obtained by Theorems 7.1 and 7.2. For an étale groupoid $\mathcal{G}$ the category of left étale $\mathcal{G}$-spaces $\underline{\mathrm{Sh}}(\mathcal{G})$ is in fact a (Grothendieck) topos (see [13], and about toposes see [10]) and its equivalence class is a weak equivalence invariant of an étale groupoid, and also $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ is a weak equivalence invariant of an étale groupoid:

Theorem 7.3 ([4, Section 2.3])**.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be étale groupoids. If $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent, then $\underline{\mathrm{Sh}}(\mathcal{G})$ and $\underline{\mathrm{Sh}}(\mathcal{G}')$ are equivalent as (Grothendieck) toposes, and hence $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G})$ and $\underline{\mathrm{Mod}}_{\mathbf{k}}(\mathcal{G}')$ are additively equivalent.*

We have (iii) in Diagram 1. For $\underline{\mathrm{Sh}}(\mathcal{G})$ the converse is also true:

Theorem 7.4 ([13, 7.7 Theorem])**.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be étale groupoids. $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent if and only if $\underline{\mathrm{Sh}}(\mathcal{G})$ and $\underline{\mathrm{Sh}}(\mathcal{G}')$ are equivalent as (Grothendieck) toposes.*

(ii) in Diagram 1 is obtained by Theorem 7.4. We lose some information about quasi-isometry classes of finitely generated groups by (iii) in Diagram 1. Thus, we have the following problem:

PROBLEM 7.5. Let G and G' be finitely generated groups. Is it true that if $\mathcal{R}(G, \mathbf{k})$ and $\mathcal{R}(G', \mathbf{k})$ are Morita equivalent, then G and G' are quasi-isometric? If not, give a counter-example.

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