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# ON RING THEORETIC QUASI-ISOMETRY INVARIANTS

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## Abstract

We introduce an algebraic version of the translation algebra of a group. We prove that a quasi-isometry of two finitely generated groups induces Morita equivalence of their algebraic translation algebras.

## 1. Introduction

Ring theoretic approaches for a quasi-isometry of groups were started by Shalom and Sauer [20], [18]. Shalom proved quasi-isometry invariance of the cohomological dimensions of finitely generated amenable groups, and of the  $\mathbf{R}$ -Betti numbers of finitely generated nilpotent groups. In his proof, it was important that there exists a good topological coupling induced by a quasi-isometry. Sauer refined a part of Shalom’s argument. He showed that a good topological coupling induces a Morita equivalence between Sauer rings of the coupled group actions (see Section 3). He applied this result to quasi-isometry invariance of the (co)homological dimensions of finitely generated groups with finite dimensions, and of the  $\mathbf{R}$ -cohomology rings of finitely generated nilpotent groups.

Morita theory of Sauer rings is important for classifying groups by quasi-isometry. However, Sauer rings of the same group are not always Morita equivalent. In order to study ring theoretic invariants we should determine a ring for each finitely generated group. We propose considering the rings as follows: Let  $\mathbf{k}$  be a ring with the multiplicative identity element 1, and  $G$  a finitely generated group. We consider the skew group ring  $G * l^f(G, \mathbf{k})$ , where  $l^f(G, \mathbf{k})$  is the ring of functions with finite image. We denote this ring by  $\mathcal{R}(G, \mathbf{k})$ , and call it an algebraic translation algebra of  $G$  with the coefficient  $\mathbf{k}$ . In the case where  $\mathbf{k} = \mathbf{Q}$  or  $\mathbf{C}$ , we see that  $\mathcal{R}(G, \mathbf{k})$  is a subring of Roe’s translation algebra [17, p.68]. In fact,  $\mathcal{R}(G, \mathbf{k})$  is isomorphic to the Sauer ring of a natural action of  $G$  on  $\beta G$ , where  $\beta G$  is the Stone–Čech compactification of  $G$  endowed with the discrete topology (see Lemma 3.2). We have the main theorem:

**Theorem 1.** *If finitely generated groups  $G$  and  $G'$  are quasi-isometric, then  $\mathcal{R}(G, \mathbf{k})$  and  $\mathcal{R}(G', \mathbf{k})$  are Morita equivalent.*

Two groups always have a good topological coupling such that their Stone–Čech compactifications are coupled (see Section 3). Therefore Theorem 1 is the special case of [18]. Without using a topological coupling and [18], we prove this result in Section 4. The cores of a quasi-isometry (see Definition 2.1) play important roles.

Morita invariants of  $\mathcal{R}(G, \mathbf{k})$  are quasi-isometry invariants by Theorem 1. In Section 5, we give a formula to calculate the global dimension and the weak global dimension of  $\mathcal{R}(G, \mathbf{k})$ . They are well-known Morita invariants. The global dimension of  $\mathcal{R}(G, \mathbf{k})$  is estimated by the cohomological dimension of  $G$  and the global dimension of  $l^f(G, \mathbf{k})$ . The same result is true for the weak global dimension. It should be noted that some of well-known Morita invariants are trivial. For example, the center of  $\mathcal{R}(G, \mathbf{k})$  coincides with that of  $\mathbf{k}$  (see Lemma 2.4).

The Morita equivalence in the proof of Theorem 1 preserves some special modules (see Theorem 4.7). For example,  $l^f(G, \mathbf{k})$  and  $l_c(G, \mathbf{k})$  are preserved. The coarse cohomology  $H^n(G, G\mathbf{k})$  is isomorphic to  $\text{Ext}_{\mathcal{R}(G, \mathbf{k})}^n(l^f(G, \mathbf{k}), l_c(G, \mathbf{k}))$  (see Section 4.3), and hence the coarse cohomology is a quasi-isometry invariant as already known. The coarse  $l^p$ -cohomology ([6]) is also obtained in this way.

If  $G$  is not amenable, then the Morita equivalence of Theorem 1 can be replaced by a ring isomorphism. It is proved in Corollary 4.5. In this case, isomorphism invariants of rings are also quasi-isometry invariants.

In Section 6, a geometric description of  $\mathcal{R}(G, \mathbf{k})$  is given by  $\underline{\text{Mod}}_{\mathbf{k}}(G \ltimes \beta G)$  by using [4]. Indeed,  $\mathcal{R}(G, \mathbf{k})\text{-Mod}$  is additively equivalent to  $\underline{\text{Mod}}_{\mathbf{k}}(G \ltimes \beta G)$  (see Theorem 6.6). From this we can construct  $\mathcal{R}(G, \mathbf{k})$ -modules by the geometry of Stone–Čech compactification. In Appendix 7.1, we give an alternative proof of Theorem 1 using the result in Section 6.

## 2. Preliminaries

**2.1. Geometric group theory.** We recall the basic notion of geometric group theory and cores of a quasi-isometry [8, 0.2.C. p. 4, 5].

Let  $G$  be a finitely generated group with a finite generating system  $S$ .  $G$  has a metric  $d_{(G, S)}$  defined by

$$d_{(G, S)}(x, y) = \begin{cases} \min\{n \in \mathbf{N} \mid x = s_1^{i_1} \cdots s_n^{i_n} y, s_k \in S, i_k \in \{-1, 1\}\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

which is called the word metric with respect to  $S$ .

Let  $Z$  be a metric space. For  $W \subseteq Z$  and a real number  $K \geq 0$ ,  $\mathcal{N}_K(W) = \{z \in Z \mid \exists w \in W \text{ s.t. } d(z, w) \leq K\}$  is called a  $K$ -neighborhood of  $W$ . If  $\mathcal{N}_K(W) = Z$ , the subspace  $W$  is said to be  $K$ -coarsely dense in  $Z$ .

A quasi-isometry is a map  $f: X \rightarrow Y$  between metric spaces such that for some real number  $K \geq 1$ ,  $f$  satisfies

(1)  $(1/K) d(x, x') - K \leq d(f(x), f(x')) \leq K d(x, x') + K$  for every  $x, x' \in X$ ,  
 (2)  $f(X)$  is  $K$ -coarsely dense in  $Y$ .

Two metric spaces are quasi-isometric if there exists a quasi-isometry between them. This gives an equivalence relation for metric spaces. If  $S$  and  $S'$  are finite generating systems of  $G$ , then  $(G, d_{(G,S)})$  and  $(G, d_{(G,S')})$  are quasi-isometric.

The definition of cores of a quasi-isometry is as follows:

**DEFINITION 2.1.** Let  $f: X \rightarrow Y$  be a quasi-isometry.  $A \subseteq X$  and  $B \subseteq Y$  are called cores of  $f$  if there exists a real number  $K \geq 0$  such that  $\mathcal{N}_K(A) = X$ ,  $\mathcal{N}_K(B) = Y$ ,  $f(A) = B$  and  $f|_A$  is a bijective quasi-isometry.

For every quasi-isometry  $f: X \rightarrow Y$  there exist cores of  $f$ . Indeed, we can define a core  $B$  to be  $f(X)$ , and  $A$  to be  $\{x_b \in X \mid b \in B\}$  by choosing  $x_b \in f^{-1}(b)$  for each  $b$ .

**2.2. Algebraic translation algebra.** Let  $G$  be a group, and  $R$  a ring with the multiplicative identity element 1 on which  $G$  acts from the right. For  $r \in R$  and  $g \in G$  this action is denoted by  $r^g$ . The skew group ring  $G * R$  is a free right  $R$ -module on  $G$  with the multiplication given by

$$(gr_1)(hr_2) = (gh)(r_1^h r_2) \quad \text{for every } g, h \in G, r_1, r_2 \in R.$$

If  $G$  acts on  $R$  trivially, then we especially write  $G * R$  by  $GR$ . It is an ordinary group ring (see [16] about skew group rings).

**DEFINITION 2.2.** (1) Let  $G$  be a group and  $\mathbf{k}$  a ring with the multiplicative identity element 1.

$$l^f(G, \mathbf{k}) = \{F: G \rightarrow \mathbf{k} \mid \#(\text{Im } F) < \infty\}$$

is a ring with the following sum and multiplication:

$$(F_1 + F_2)(x) = F_1(x) + F_2(x),$$

$$(F_1 F_2)(x) = F_1(x) F_2(x)$$

for every  $F_1, F_2 \in l^f(G, \mathbf{k})$  and  $x \in G$ .

(2)  $G$  acts on  $l^f(G, \mathbf{k})$  from the right by

$$F^g(x) = F(gx) \quad \text{for every } g \in G, x \in G.$$

(3) We denote  $G * l^f(G, \mathbf{k})$  by  $\mathcal{R}(G, \mathbf{k})$ . It is called an algebraic translation algebra of  $G$  with the coefficient  $\mathbf{k}$ .

The multiplicative identity element of  $l^f(G, \mathbf{k})$  is the constant function 1. Since  $\mathbf{k} \subseteq l^f(G, \mathbf{k})$  as constant functions, a group ring  $G\mathbf{k}$  is a subring of  $\mathcal{R}(G, \mathbf{k})$ .  $e \cdot 1$  is

the multiplicative identity element of  $\mathcal{R}(G, \mathbf{k})$ , where  $e$  is the identity element of  $G$ .  $\mathbf{k}$  is regarded as a left  $G\mathbf{k}$ -module by  $gk = k$  ( $g \in G$ ,  $k \in \mathbf{k}$ ).

For  $S \subseteq G$  the characteristic function

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}$$

is an element of  $I^f(G, \mathbf{k})$ .

In the case where  $\mathbf{k} = \mathbf{Z}$ ,  $\mathbf{Q}$  or  $\mathbf{C}$ , we see that  $\mathcal{R}(G, \mathbf{k})$  is a subring of Roe's translation algebra [17, p. 68].

**2.3. Morita theory.** We give the basic notion of Morita theory [1]. Let  $R$  and  $S$  be rings.  $R\text{-Mod}$  ( $\text{Mod-}R$ ) is the category of left (right) modules over  $R$ . A left  $R$  and right  $S$ -module  $M$  is called a left  $R$ -right  $S$ -bimodule if  $r(ms) = (rm)s$  ( $r \in R$ ,  $s \in S$ ,  $m \in M$ ).  ${}_R M$  means  $M$  is a left  $R$ -module,  $M_S$  means  $M$  is a right  $R$ -module, and  ${}_R M_S$  means  $M$  is a left  $R$ -right  $S$ -bimodule.

Let  $\mathcal{F}_1, \mathcal{F}_2: \mathbf{B} \rightarrow \mathbf{C}$  be functors, a set  $\{\tau_B \in \text{Hom}(\mathcal{F}_1(B), \mathcal{F}_2(B)) \mid B \in \text{Ob}(\mathbf{B})\}$  is called a natural equivalence if  $\mathcal{F}_2(f) \circ \tau_B = \tau_{B'} \circ \mathcal{F}_1(f)$  ( $f \in \text{Hom}(B, B')$ ) and  $\tau_B$  is an isomorphism for every  $B \in \text{Ob}(\mathbf{B})$ . Then  $\mathcal{F}_1 \simeq \mathcal{F}_2$  if there exists a natural equivalence. A functor  $\mathcal{F}: R\text{-Mod} \rightarrow S\text{-Mod}$  is called an additive functor if  $\text{Hom}(A, B) \rightarrow \text{Hom}(\mathcal{F}(A), \mathcal{F}(B))$  defined by  $f \mapsto \mathcal{F}(f)$  is a homomorphism. An additive functor  $\mathcal{F}_1: R\text{-Mod} \rightarrow S\text{-Mod}$  is called an additive equivalence if there exists an additive functor  $\mathcal{F}_2: S\text{-Mod} \rightarrow R\text{-Mod}$  such that  $\mathcal{F}_2 \circ \mathcal{F}_1 \simeq \text{id}$  and  $\mathcal{F}_1 \circ \mathcal{F}_2 \simeq \text{id}$ . A functor  $\mathcal{F}_2$  is called an inverse equivalence of  $\mathcal{F}_1$ .

$R$  and  $S$  are said to be *Morita equivalent* if there exists an additive equivalence between  $R\text{-Mod}$  and  $S\text{-Mod}$ . Let  $M$  be a left (right)  $R$ -module. The module  $M$  is said to be *finitely generated* if there exist  $n \in \mathbf{N}$  and a surjective homomorphism  $f: R^n \rightarrow M$ .  $M$  is called a *(finite) generator* if there exist  $n \in \mathbf{N}$  and a surjective homomorphism  $f: M^n \rightarrow R$ .  $M$  is said to be *projective* if it is a direct summand of a free left (right)  $R$ -module. A generator is called a *progenerator* if it is finitely generated and projective.  $\text{End}(M) = \{f: M \rightarrow M \mid f \text{ is a left (right) } R\text{-homomorphism}\}$  is called the endomorphism ring. The multiplication is the opposite composition (ordinary composition) of maps.

$e \in R$  is called an idempotent if  $e^2 = e$ . If  $R$  has the multiplicative identity element 1, then  $eRe = \{ere \in R \mid r \in R\}$  is a ring with the multiplicative identity element  $e$ .  $Re$  is a left  $R$ -module, and  $\text{End}(Re)$  is isomorphic to  $eRe$ .  $eR$  is a right  $R$ -module, and  $\text{End}(eR)$  is also isomorphic to  $eRe$ .

**Theorem 2.3.** [1, Corollary 22.5, p. 265] *Let  $R$  be a ring. If  $P_R$  is a progenerator, then  $R$  and  $S = \text{End}(P_R)$  are Morita equivalent. Indeed, if  $P^\otimes = \text{Hom}_R(P_R, R)$ , then  ${}_S P_R$  and  ${}_R P_S^\otimes$  are bimodules and  $(P \otimes_R -): R\text{-Mod} \rightarrow S\text{-Mod}$ ,  $(P^\otimes \otimes_S -): S\text{-Mod} \rightarrow R\text{-Mod}$  are inverse equivalences.*

If  $P_R = eR$  is a progenerator, then  $S = eRe$ ,  ${}_S P_R = {}_{eRe} eR_R$  and  ${}_R P_{eRe}^\otimes = {}_R R e_{eRe}$ .

If  $R$  and  $S$  are isomorphic, then  $R$  and  $S$  are Morita equivalent. Indeed, let  $\Phi: S \rightarrow R$  be a ring isomorphism. Since  $S \simeq R \simeq \text{End}(R_R)$ , an additive equivalence  $({}_S R_R \otimes_R -): R\text{-Mod} \rightarrow S\text{-Mod}$  is obtained.  ${}_R M$  is mapped to  ${}_S M$  satisfying  $sm = \Phi(s)m$  ( $s \in S$ ,  $m \in M$ ). We use the notation  $\text{Res } \Phi = ({}_S R_R \otimes_R -)$ .

**2.4. The center of  $\mathcal{R}(G, \mathbf{k})$ .** The center of a ring  $R$  is  $\text{Cen}(R) = \{r \in R \mid rx = xr \ (\forall x \in R)\}$ . If rings  $R$  and  $S$  are Morita equivalent, then  $\text{Cen}(R)$  and  $\text{Cen}(S)$  are isomorphic [1, Proposition 21.10, p. 258].

**Lemma 2.4.**  $\text{Cen}(\mathcal{R}(G, \mathbf{k})) = \text{Cen}(\mathbf{k})$ .

Proof. Let  $\alpha \in \text{Cen}(\mathcal{R}(G, \mathbf{k}))$ . For each  $x \neq e \in G$  there exists no  $F \neq 0 \in l^f(G, \mathbf{k})$  such that for every  $g \in G$ ,  $\chi_g x F = x F \chi_g$  is satisfied, and hence  $\alpha \in e \cdot l^f(G, \mathbf{k})$ . Since for every  $g \in G$  we have  $g\alpha = \alpha g$ ,  $\alpha$  is a constant function.  $k\alpha = \alpha k$  is satisfied for every  $k \in \mathbf{k}$ , and hence  $\alpha \in \text{Cen}(\mathbf{k})$ .  $\square$

**2.5. Transformation groupoids.** Let  $\mathcal{G}_0$ ,  $\mathcal{G}_1$  be topological spaces, and  $s: \mathcal{G}_1 \rightarrow \mathcal{G}_0$ ,  $t: \mathcal{G}_1 \rightarrow \mathcal{G}_0$ ,  $m: \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(g_1, g_2) \in \mathcal{G}_1 \times \mathcal{G}_1 \mid s(g_1) = t(g_2)\} \rightarrow \mathcal{G}_1$  continuous maps. We consider the following three conditions:

- (1)  $m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$  ( $(g_1, g_2), (g_2, g_3) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ ),
- (2) there exists a continuous map  $u: \mathcal{G}_0 \rightarrow \mathcal{G}_1$  such that  $s(u(x)) = t(u(x)) = x$  and  $m(u(x), g) = g$ ,  $m(g', u(x)) = g'$  ( $x \in \mathcal{G}_0$ ,  $g, g' \in \mathcal{G}_1$  with  $t(g) = x = s(g')$ ),
- (3) there exists a continuous map  $I: \mathcal{G}_1 \rightarrow \mathcal{G}_1$  such that  $m(g, I(g)) = u(t(g))$ ,  $m(I(g), g) = u(s(g))$  ( $g \in \mathcal{G}_1$ ).

$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, s, t, m, u, I)$  satisfying the conditions above is called a *topological groupoid*. We use the notation  $g_1 \cdot g_2 = m(g_1, g_2)$  ( $(g_1, g_2) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ ),  $e_x = u(x)$  ( $x \in \mathcal{G}_0$ ) and  $g^{-1} = I(g)$  ( $g \in \mathcal{G}_1$ ). A topological groupoid  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, s, t, m, u, I)$  is called an *étale groupoid* if  $s$  and  $t$  are surjective local homeomorphisms (see [14, Section 5] for more on étale groupoids).

Let  $G$  be a finitely generated group, and  $G$  acts on a topological space  $X$  from the left. We define a *transformation groupoid*  $G \ltimes X$  by the following data:

$$(G \ltimes X)_0 = X, \quad (G \ltimes X)_1 = G \times X,$$

where  $G$  is regarded as a discrete space.

$$\begin{aligned} s(g, x) &= x \quad (g \in G, x \in X), \quad t(g, x) = gx \quad (g \in G, x \in X), \\ (g, x) \cdot (g', x') &= (gg', x') \quad (g, g' \in G, x, x' \in X \text{ satisfying } x = g'x'), \\ u(x) &= (e, x) \quad (x \in X), \end{aligned}$$

where  $e$  is the identity element of  $G$ .

$$I(g, x) = (g^{-1}, gx) \quad (g \in G, x \in X).$$

$G \ltimes X$  is an étale groupoid.

**2.6. Stone–Čech compactifications.** We recall the Stone–Čech compactifications for discrete spaces [9]. Let  $D$  be a set.  $\mathcal{U} \subseteq 2^D$  is called a filter on  $D$  if the following conditions are satisfied.

- (0)  $D \in \mathcal{U}$ ,
- (1)  $\emptyset \notin \mathcal{U}$ ,
- (2) if  $A_1, A_2 \in \mathcal{U}$ , then  $A_1 \cap A_2 \in \mathcal{U}$ ,
- (3) if  $A \in \mathcal{U}$ ,  $B \in 2^D$  and  $A \subseteq B$ , then  $B \in \mathcal{U}$ .

In addition,  $\mathcal{U}$  is called an ultra filter if  $\mathcal{U}$  satisfies

- (4) if  $D = A_1 \sqcup \cdots \sqcup A_n$ , then there exists the unique  $1 \leq i \leq n$  such that  $A_i \in \mathcal{U}$ .

The set of ultra filters on  $D$  is denoted by  $\beta D$ . For  $A \subseteq D$  we use the notation  $\hat{A} = \{\mathcal{U} \in \beta D \mid A \in \mathcal{U}\}$ .  $\mathcal{O}$  is a topology on  $D$  generated by an open base  $\{\hat{A} \mid A \in 2^D\}$ .  $(\beta D, \mathcal{O})$  is called the *Stone–Čech compactification* of  $D$ . Let  $G$  be a finitely generated group.  $\beta G$  has a natural  $G$ -action from the left. Indeed, for  $\mathcal{U} \in \beta G$  and  $g \in G$ ,  $g\mathcal{U} = \{gA \mid A \in \mathcal{U}\} \in \beta G$ . This action is a homeomorphic action.

**Lemma 2.5.** (1)  $\beta D$  is compact and Hausdorff.  $D$  is identified with a dense subset of  $\beta D$  by an injection  $e: D \rightarrow \beta D$  satisfying  $\{e(d)\} = \widehat{\{d\}}$  for every  $d \in D$ .

(2) For  $A \in 2^D$ ,  $\widehat{D - A} = \beta D - \hat{A}$ . Therefore the topology of  $\beta D$  is generated by clopen (closed and open) sets.

(3) If  $O$  is a clopen set of  $\beta D$ , then there exists  $A \in 2^D$  such that  $O = \hat{A}$ .

Proof. The proof of (1) is in [9, Theorem 3.18 (a) and (c)], and the proof of (2) is in [9, Theorem 3.17 (c)]. If  $O$  is a clopen set of  $\beta D$ , then there exists  $A_x \in 2^D$  for each  $x \in O$  such that  $x \in A_x$  and  $O = \bigcup_{x \in O} \hat{A}_x$ . Since  $O$  is a closed set of Hausdorff space,  $O$  is compact. Therefore there exists  $\{x_i \in O\}_{i=1}^n$  such that  $O = \bigcup_{i=1}^n \hat{A}_{x_i}$ . By [9, Theorem 3.17 (b)] we have  $O = \widehat{\bigcup_{i=1}^n A_{x_i}}$ .  $\square$

**2.7. Definition of  $\underline{\text{Mod}}_k(\mathcal{G})$ .** Let  $\mathcal{G}$  be an étale groupoid. In [4], the abelian category associated to  $\mathcal{G}$  was considered to study a homology theory for  $\mathcal{G}$  ([21, Appendix A] is a good reference for abelian categories). This category is denoted by  $\underline{\text{Mod}}_k(\mathcal{G})$ . In Section 6, we describe  $\underline{\text{Mod}}_k(G \ltimes \beta G)$  and discuss a relation to the algebraic translation algebra.

First, we recall the definition of  $\underline{\text{Sh}}(\mathcal{G})$ . A left étale  $\mathcal{G}$ -space  $X = (X, p_0, p_1)$  is a topological space with continuous maps  $p_0: X \rightarrow \mathcal{G}_0$  and  $p_1: \mathcal{G}_1 \times_{p_0} X = \{(g, x) \mid s(g) = p_0(x)\} \rightarrow X$  such that

- (0)  $p_0$  is a surjective local homeomorphism,
- (1)  $p_0(p_1(g, x)) = t(g)$  ( $(g, x) \in \mathcal{G}_1 \times_{p_0} X$ ),
- (2)  $p_1(h \cdot g, x) = p_1(h, p_1(g, x))$  ( $(g, x) \in \mathcal{G}_1 \times_{p_0} X$ ,  $(h, g) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ ),
- (3)  $p_1(e_{p_0(x)}, x) = x$  ( $x \in X$ ).

$p_1$  is usually denoted by  $\cdot$ . Let  $X = (X, p_0, p_1)$  and  $Y = (Y, q_0, q_1)$  be left étale  $\mathcal{G}$ -spaces. A continuous map  $\Phi: X \rightarrow Y$  is said to be *equivariant* if

- (1)  $q_0 \circ \Phi = p_0$ ,
- (2)  $\Phi(p_1(g, x)) = q_1(g, \Phi(x))$  ( $(g, x) \in \mathcal{G}_1 \times_{p_0} X$ )

are satisfied.  $\underline{\text{Sh}}(\mathcal{G})$  is the category of which objects are left étale  $\mathcal{G}$ -spaces and morphisms are equivariant maps.  $\underline{\text{Sh}}(\mathcal{G})$  is called *the category of left étale  $\mathcal{G}$ -spaces*.

Second, we recall the definition of  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$ . Let  $X = (X, p_0, p_1)$  and  $Y = (Y, q_0, q_1)$  be left étale  $\mathcal{G}$ -spaces. We define a finite product of  $\underline{\text{Sh}}(\mathcal{G})$ :  $X \oplus Y = (X \times_{\mathcal{G}_0} Y, r_0, r_1)$  by  $X \times_{\mathcal{G}_0} Y = \{(x, y) \in X \times Y \mid p_0(x) = q_0(y)\}$ ,  $r_0: X \times_{\mathcal{G}_0} Y \rightarrow \mathcal{G}_0$  with  $r_0(x, y) = p_0(x) = q_0(y)$  and  $r_1: \mathcal{G}_1 \times_{r_0} (X \times_{\mathcal{G}_0} Y) \rightarrow X \times_{\mathcal{G}_0} Y$  with  $r_1(g, (x, y)) = (p_1(g, x), q_1(g, y))$ .  $\Theta = \mathcal{G}_0 = (\mathcal{G}_0, \text{id}: \mathcal{G}_0 \rightarrow \mathcal{G}_0, t \circ \text{Pr}_1: \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_0 \rightarrow \mathcal{G}_0)$  is a left étale  $\mathcal{G}$ -space and  $\text{Hom}(X, \mathcal{G}_0) = \{p_0\}$ .  $\Theta$  is a terminal object.  $\mathbf{k}$  is regarded as a constant left étale  $\mathcal{G}$ -space  $\mathbf{k} = (\mathbf{k} \times \mathcal{G}_0, p'_0 = \text{Pr}_2, p'_1)$  by  $p'_1(g, (k, z)) = (k, t(g))$ .  $\mathbf{k}$  has a natural structure of a ring. We consider  $\mathbf{k}$ -module objects of  $(\underline{\text{Sh}}(\mathcal{G}), \oplus, \Theta)$ :  $A = (A, M, \mathcal{U}, v, \mathcal{M})$  is called a  **$\mathbf{k}$ -module object** of  $(\underline{\text{Sh}}(\mathcal{G}), \oplus, \Theta)$  if morphisms  $M: A \oplus A \rightarrow A$ ,  $\mathcal{U}: \Theta \rightarrow A$ ,  $v: A \rightarrow A$  and  $\mathcal{M}: \mathbf{k} \oplus A \rightarrow A$  satisfy

- (1)  $(M, \mathcal{U}, v)$  is an usual additive group structure on  $A$ ,
- (2)  $\mathcal{M}$  is an usual  $\mathbf{k}$ -action giving a  $\mathbf{k}$ -module structure on  $(A, M, \mathcal{U}, v)$ .

$M$  is usually denoted by  $+$ ,  $\mathcal{U}$  by  $0$ ,  $v$  by  $-$  and  $\mathcal{M}$  by  $\cdot$ . Morphisms between  $\mathbf{k}$ -module objects  $A$  and  $B$  of  $(\underline{\text{Sh}}(\mathcal{G}), \oplus, \Theta)$  are morphisms of  $\underline{\text{Sh}}(\mathcal{G})$  preserving structures  $M$ ,  $\mathcal{U}$ ,  $v$  and  $\mathcal{M}$ . Therefore  $\mathbf{k}$ -module objects form a category. It is denoted by  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$ .  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  is an abelian category.  $\Theta = \mathcal{G}_0$  is the zero object  $0$ .

$\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  always has an infinite coproduct. Such a category is called an A.B.3 category [21, A.4]. An infinite coproduct exists as follows: For  $\{A_{\lambda} \in \text{Ob}(\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})) \mid \lambda \in \Lambda\}$ ,  $A_{\lambda}$  gives a presheaf of  $\mathbf{k}$ -modules  $\mathcal{F}_{A_{\lambda}}$  on  $\mathcal{G}_0$  by  $O \mapsto \Gamma(O, A_{\lambda}) = \{f: O \rightarrow A_{\lambda} \mid f \text{ is continuous and } p_{0, \lambda} \circ f = \text{id}_O\}$  for every open set  $O \subseteq \mathcal{G}_0$ . Therefore for presheaf  $\bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_{\lambda}}: O \mapsto \bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_{\lambda}}(O)$  its sheaf space  $E_{\bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_{\lambda}}}$  has a natural structure of a  $\mathbf{k}$ -module object of  $(\underline{\text{Sh}}(\mathcal{G}), \oplus, \Theta)$  (about the relation of a presheaf and a sheaf space see [2, 2.3]). This  $E_{\bigoplus_{\lambda \in \Lambda} \mathcal{F}_{A_{\lambda}}}$  is an infinite coproduct. Therefore  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  is an A.B.3 category.

### 3. Proof of quasi-isometry invariance of the algebraic translation algebra using a topological coupling

We recall the definition of Sauer rings. Let  $G$  acts on a compact Hausdorff space  $X$  from the left.  $\mathcal{F}(X, \mathbf{k}) = \{F: X \rightarrow \mathbf{k} \mid F^{-1}(k) \text{ is clopen for every } k \in \mathbf{k}\}$  has the right action of  $G$  induced by the left action of  $G$  on  $X$ . In the paper [18], the skew group ring  $G * \mathcal{F}(X, \mathbf{k})$  was considered. We call this ring *the Sauer ring of  $G$ -space  $X$* .

**Theorem 3.1** ([18]). *If two finitely generated groups  $G$  and  $G'$  are quasi-isometric, then there exist compact Hausdorff spaces  $Y_1$  on which  $G$  acts from the left and  $Y_2$  on which  $G'$  acts from the left such that their Sauer rings  $G * \mathcal{F}(Y_1, \mathbf{k})$  and  $G' * \mathcal{F}(Y_2, \mathbf{k})$  are Morita equivalent. A good topological coupling  $\Omega$  always gives such  $Y_1 = \Omega/G'$  and  $Y_2 = \Omega/G$ , where a topological coupling  $\Omega$  is said to be good if it has a compact clopen*

fundamental domain for each action.

We relate Sauer rings and our algebraic translation algebras.

**Lemma 3.2.** *Let  $G$  be a finitely generated group.*

$$\mathcal{R}(G, \mathbf{k}) \simeq G * \mathcal{F}(\beta G, \mathbf{k}).$$

Proof. For  $F \in \mathcal{F}(\beta G, \mathbf{k})$ ,  $\beta G = \bigsqcup_{k \in \mathbf{k}} F^{-1}(k)$ . Since  $\beta G$  is compact and for each  $k$ ,  $F^{-1}(k)$  is open, and hence there exist  $k_1, \dots, k_n \in \mathbf{k}$  such that  $\beta G = \bigsqcup_{i=1}^n F^{-1}(k_i)$ . For each  $i$ ,  $F^{-1}(k_i)$  is clopen, and hence by Lemma 2.5 (3) there exist  $A_1, \dots, A_n \subseteq G$  such that  $F^{-1}(k_i) = \hat{A}_i$ . We define  $\lambda: \mathcal{F}(\beta G, \mathbf{k}) \rightarrow l^f(G, \mathbf{k})$  by  $\lambda(F) = F|_G = \sum_{i=1}^n k_i \chi_{A_i}$ .  $\lambda$  is a bijective homomorphism and preserves the action of  $G$ . Every function in  $l^f(G, \mathbf{k})$  has an expression  $\sum_{i=1}^n k_i \chi_{A_i}$ , and hence  $\lambda$  is surjective. This  $\lambda$  is extended to  $\mathcal{R}(G, \mathbf{k}) \simeq G * \mathcal{F}(\beta G, \mathbf{k})$ .  $\square$

By Lemma 3.2 if we have a good topological coupling such that  $Y_1 = \beta G$  and  $Y_2 = \beta G'$ , then Theorem 1 is the special case of [18]. Indeed, we have the following theorem:

**Theorem 3.3.** *Two quasi-isometric finitely generated groups  $G$  and  $G'$  always have a good topological coupling such that their Stone–Čech compactifications are coupled.*

Proof. In the proof of Theorem 7.1 in Appendix, we have essential morphisms  $G \times \beta G \rightarrow \mathbf{G}(|G| \sqcup |G'|) \leftarrow G' \times \beta G'$  (see [19, Section 3.4]). We take the weak pullback  $\mathcal{G}$  of this morphisms, and hence surjective essential morphisms  $G \times \beta G \leftarrow \mathcal{G} \rightarrow G' \times \beta G'$  are obtained (see [14, Exercise 5.22 (1)]).  $\mathcal{G}_0$  has a natural  $(G \times G')$ -action.  $\mathcal{G}_0$  is a topological coupling such that  $\mathcal{G}_0/G' = \beta G$  and  $\mathcal{G}_0/G = \beta G'$ . Since surjective essential morphisms above are étale and the topologies of  $\beta G$  and  $\beta G'$  are generated by clopen sets, we can construct a compact clopen fundamental domain for each action. As a result,  $\mathcal{G}_0$  is a good topological coupling.  $\square$

We have the main theorem by Lemma 3.2, Theorems 3.1 and 3.3:

**Theorem 1.** *If finitely generated groups  $G$  and  $G'$  are quasi-isometric, then  $\mathcal{R}(G, \mathbf{k})$  and  $\mathcal{R}(G', \mathbf{k})$  are Morita equivalent.*

#### 4. Proof of quasi-isometry invariance of the algebraic translation algebra without using a topological coupling

The proof is obtained by elementary argument: cores of a quasi-isometry and basic Morita theory (see Sections 2.1 and 2.3).

**4.1. Some lemmas.** In order to prove quasi-isometry invariance of the algebraic translation algebra, we need Lemmas 4.1 and 4.4.

**Lemma 4.1.** *Let  $H$  be a finitely generated group. For  $Z \subseteq H$  if there exists a real number  $K \geq 0$  such that  $Z$  is  $K$ -coarsely dense in  $H$ , then a right  $\mathcal{R}(H, \mathbf{k})$ -module  $I_Z = \chi_Z \mathcal{R}(H, \mathbf{k})$  is a progenerator.*

Proof. Since  $\mathcal{R}(H, \mathbf{k}) = I_Z \oplus I_{H-Z}$ ,  $I_Z$  is finitely generated and projective.

We prove that  $I_Z$  is a generator: For the identity element  $e \in H$ ,  $\mathcal{N}_K(e)$  is finite, and hence we have an expression  $\mathcal{N}_K(e) = \{h_0 = e, h_1, \dots, h_n\}$ . We define  $Z_0, \dots, Z_n$  by

$$\begin{aligned} Z_0 &= Z, \\ Z_1 &= h_1 Z - Z, \\ Z_2 &= h_2 Z - h_1 Z - Z, \\ &\dots \\ Z_n &= h_n Z - h_{n-1} Z - \dots - Z. \end{aligned}$$

$Z_0 \sqcup \dots \sqcup Z_n = H$  and  $h_i^{-1} Z_i \subseteq Z$  are satisfied. We define  $p_i: I_Z \rightarrow \mathcal{R}(H, \mathbf{k})$  by  $p_i(\chi_Z \gamma) = \chi_{Z_i} h_i \gamma$  for every  $\gamma \in \mathcal{R}(H, \mathbf{k})$  and  $0 \leq i \leq n$ . They are well-defined as follows: For every  $\gamma, \gamma' \in \mathcal{R}(H, \mathbf{k})$  satisfying  $\chi_Z \gamma = \chi_Z \gamma'$ , by multiplying  $\chi_{Z_i} h_i$  to this equation from the left, we have  $\chi_{Z_i} h_i \chi_Z \gamma = \chi_{Z_i} h_i \chi_Z \gamma'$ . This implies  $\chi_{Z_i} \chi_{h_i Z} h_i \gamma = \chi_{Z_i} \chi_{h_i Z} h_i \gamma'$ . Thus  $h_i^{-1} Z_i \subseteq Z$  shows that  $\chi_{Z_i} h_i \gamma = \chi_{Z_i} h_i \gamma'$ , and hence  $p_i$  is a well-defined homomorphism. As a result, we have a homomorphism  $p = \bigoplus_{i=1}^n p_i: I_Z^n \rightarrow \mathcal{R}(H, \mathbf{k})$ . For each  $h \in H$  and  $F \in l^f(H, \mathbf{k})$  we have  $hF = \sum_{i=0}^n \chi_{Z_i} hF = \sum_{i=0}^n \chi_{Z_i} (h_i h_i^{-1}) hF = \sum_{i=0}^n \chi_{Z_i} h_i (h_i^{-1} hF) = \sum_{i=0}^n p_i(\chi_Z h_i^{-1} hF) = p(\chi_Z h_0^{-1} hF, \dots, \chi_Z h_n^{-1} hF)$ . Therefore  $p$  is surjective.  $\square$

Let  $H$  be a group and  $Z \subseteq H$ .  $\mathcal{M}_H$  is the endomorphism ring of the right free  $\mathbf{k}$ -module on  $\{\delta_h \mid h \in H\}$ .  $\mathcal{M}_Z$  is the subring of  $\mathcal{M}_H$  generated on  $\{\delta_z \mid z \in Z\}$ . We consider a map  $\epsilon = \epsilon_{(H, Z)}: H \times H \rightarrow \mathbf{k}$  satisfying

$$\epsilon(h, z) = \chi_{h^{-1} Z \cap Z}(z) = \begin{cases} 1 & \text{if } z \in h^{-1} Z \cap Z, \\ 0 & \text{if } z \notin h^{-1} Z \cap Z \end{cases}$$

for every  $h, z \in H$ . By using  $\epsilon$ , an injective homomorphism  $i_Z: \chi_Z \mathcal{R}(H, \mathbf{k}) \chi_Z \rightarrow \mathcal{M}_Z$  can be defined by

$$i_Z(\chi_Z \alpha \chi_Z) \delta_z = \sum_{i=1}^n \delta_{h_i z} \epsilon(h_i, z) F_i(z)$$

for every  $\alpha = \sum_{i=1}^n h_i F_i \in \mathcal{R}(H, \mathbf{k})$ ,  $h_i \in H$ ,  $F_i \in l^f(H, \mathbf{k})$  and  $z \in Z$ . This is shown in the next lemma.

**Lemma 4.2.**  $i_Z$  is well-defined, a homomorphism and injective.

Proof. For every  $h \in H$  and  $z \in Z$  if  $hz \notin Z$  is satisfied, then  $\delta_{hz}\epsilon(h, z) = 0$ . Therefore we have  $i_Z(\chi_Z\alpha\chi_Z) \in \mathcal{M}_Z$ . Since  $i_Z$  preserves the sum, to prove  $i_Z$  is well-defined we will prove that for every  $\alpha = \sum_{i=1}^n h_i F_i \in \mathcal{R}(H, \mathbf{k})$  if  $\chi_Z\alpha\chi_Z = 0$ , then  $i_Z(\chi_Z\alpha\chi_Z) = 0$ .  $\chi_Z\alpha\chi_Z = \sum_{i=1}^n h_i \chi_{h_i^{-1}Z \cap Z} F_i = 0$  implies  $\sum_{h=h_i} \chi_{h_i^{-1}Z \cap Z} F_i = 0$  for each  $h \in H$ , and hence  $\sum_{h=h_i} \epsilon(h_i, z) F_i(z) = 0$  for every  $z \in Z$ . This shows  $i_Z(\chi_Z\alpha\chi_Z)\delta_z = 0$ .

In order to prove  $i_Z$  is a homomorphism, we only have to check that  $i_Z$  preserves the multiplication for generators about the sum since  $i_Z$  preserves the sum and the identity element.  $\chi_Z\mathcal{R}(H, \mathbf{k})\chi_Z$  is generated by  $\chi_Z H l^f(H, \mathbf{k})\chi_Z$  as an additive group, and hence for  $g, h \in H$ ,  $F_1, F_2 \in l^f(H, \mathbf{k})$ , and  $z \in Z$

$$\begin{aligned} & i_Z(\chi_Z g F_1 \chi_Z) \circ i_Z(\chi_Z h F_2 \chi_Z) \delta_z \\ &= i_Z(\chi_Z g F_1 \chi_Z)(\delta_{hz}\epsilon(h, z) F_2(z)) \\ &= \delta_{ghz}\epsilon(g, hz)F_1(hz)\epsilon(h, z)F_2(z) \\ &= \delta_{ghz}\epsilon(g, hz)\epsilon(h, z)F_1(hz)F_2(z) \\ &= \delta_{ghz}\chi_{h^{-1}g^{-1}Z \cap h^{-1}Z}(z)\chi_{h^{-1}Z \cap Z}(z)F_1^h(z)F_2(z) \\ &= \delta_{ghz}\chi_{(gh)^{-1}Z \cap Z}(z)(\chi_{h^{-1}Z}F_1^h F_2)(z) \\ &= i_Z(\chi_Z gh\chi_{h^{-1}Z}F_1^h F_2 \chi_Z)\delta_z \\ &= i_Z(\chi_Z g F_1 \chi_Z \chi_Z h F_2 \chi_Z)\delta_z. \end{aligned}$$

This implies  $i_Z$  is a homomorphism.

In order to prove  $i_Z$  is injective we will check that for every  $\alpha \in \mathcal{R}(H, \mathbf{k})$ ,  $i_Z(\chi_Z\alpha\chi_Z) = 0$  implies  $\chi_Z\alpha\chi_Z = 0$ . We have an expression  $\alpha = \sum_{i=1}^n h_i F_i$  for some  $h_i \in H$  and  $F_i \in l^f(H, \mathbf{k})$ , where we can assume that  $h_1, \dots, h_n$  are different from each other.  $i_Z(\chi_Z\alpha\chi_Z) = 0$  implies

$$i_Z(\chi_Z\alpha\chi_Z)\delta_z = \sum_{i=1}^n \delta_{h_i z}\epsilon(h_i, z)F_i(z) = \sum_{i=1}^n \delta_{h_i z}\chi_{h_i^{-1}Z \cap Z}(z)F_i(z) = 0$$

for every  $z \in Z$ . This shows that  $\chi_{h_i^{-1}Z}F_i\chi_Z = 0$  for every  $i$ . Thus

$$\chi_Z\alpha\chi_Z = \sum_{i=1}^n h_i \chi_{h_i^{-1}Z} F_i \chi_Z = 0. \quad \square$$

Let  $G$  and  $G'$  be finitely generated groups,  $X \subseteq G$ ,  $Y \subseteq G'$  and  $f: X \rightarrow Y$  a bijective quasi-isometry. Since  $f$  is bijective,  $f$  induces a natural isomorphism  $\tilde{f}: \mathcal{M}_X \rightarrow \mathcal{M}_Y$  as follows. For every  $A \in \mathcal{M}_X$  and  $x \in X$  we have an expression  $A(\delta_x) = \sum_{i=1}^n \delta_{x_i(x)}a_i(x)$  by some  $x_i(x) \in X$  and  $a_i(x) \in \mathbf{k}$ . Thus by using this expression of  $A(\delta_x)$ ,  $\tilde{f}$  satisfies

$$\tilde{f}(A)(\delta_y) = \sum_{i=1}^n \delta_{f(x_i((f^{-1}(y))))}a_i(f^{-1}(y))$$

for every  $y \in Y$ . By Lemma 4.2 we have injective homomorphisms  $i_X: \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \rightarrow \mathcal{M}_X$  and  $i_Y: \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \rightarrow \mathcal{M}_Y$ .

**Lemma 4.3.**

$$\tilde{f} \circ i_X(\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X) \subseteq i_Y(\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y).$$

Proof. For every  $g \in G$  and  $y \in Y$  we have

$$\tilde{f} \circ i_X(\chi_X g \chi_X)(\delta_y) = \begin{cases} \delta_{f(gf^{-1}(y))} \epsilon_{(G, X)}(g, f^{-1}(y)) & \text{if } gf^{-1}(y) \in X, \\ 0 & \text{otherwise} \end{cases}$$

since

$$i_X(\chi_X g \chi_X)(\delta_{f^{-1}(y)}) = \delta_{gf^{-1}(y)} \epsilon_{(G, X)}(g, f^{-1}(y)).$$

Since  $f$  is a quasi-isometry,  $L = \{f(gf^{-1}(y))y^{-1} \mid y \in Y \text{ and } gf^{-1}(y) \in X\}$  is a finite set. Therefore we have an expression  $L = \{h_1, \dots, h_m\}$ . We have  $S_i = \{y \in Y \mid gf^{-1}(y) \in X \text{ and } f(gf^{-1}(y))y^{-1} = h_i\}$ .  $S_1, \dots, S_m$  are disjoint for each other. If  $gf^{-1}(y) \in X$ , then there exists  $1 \leq j \leq m$  such that  $f(gf^{-1}(y))y^{-1} = h_j$  and

$$\begin{aligned} \delta_{f(gf^{-1}(y))} \epsilon_{(G, X)}(g, f^{-1}(y)) &= \delta_{f(gf^{-1}(y))y^{-1}} \epsilon_{(G, X)}(g, f^{-1}(y)) \\ &= \delta_{h_j y} \epsilon_{(G, X)}(g, f^{-1}(y)) \\ &= \delta_{h_j y} \left( \sum_{i=1}^m \epsilon_{(G', Y)}(h_i, y) \chi_{S_i}(y) \right) \epsilon_{(G, X)}(g, f^{-1}(y)) \\ &= \sum_{i=1}^m \delta_{h_i y} \epsilon_{(G', Y)}(h_i, y) \chi_{S_i}(y) \epsilon_{(G, X)}(g, f^{-1}(y)) \\ &= i_Y \left( \chi_Y \sum_{i=1}^m h_i \chi_{S_i} \epsilon_{(G, X)}(g, f^{-1}(\cdot)) \chi_Y \right) (\delta_y), \end{aligned}$$

where  $\epsilon_{(G, X)}(g, f^{-1}(\cdot)) \chi_Y \in l^f(G', \mathbf{k})$ . This shows that  $\tilde{f} \circ i_X(\chi_X g \chi_X)$  is in the image of  $i_Y$ .

On the other hand, for every  $F \in l^f(G, \mathbf{k})$  and  $y \in Y$  we have

$$\begin{aligned} \tilde{f} \circ i_X(\chi_X F \chi_X)(\delta_y) &= \delta_y F(f^{-1}(y)) \\ &= i_Y(\chi_Y (F \circ f^{-1}) \chi_Y)(\delta_y), \end{aligned}$$

where  $(F \circ f^{-1}) \chi_Y \in l^f(G', \mathbf{k})$ . This shows that  $\tilde{f} \circ i_X(\chi_X F \chi_X)$  is in the image of  $i_Y$ .

$\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$  is generated by  $\chi_X G \chi_X$  and  $\chi_X l^f(G, \mathbf{k}) \chi_X$  as a ring. Therefore we have  $\tilde{f} \circ i_X(\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X) \subseteq i_Y(\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y)$ .  $\square$

By Lemma 4.3 we have a homomorphism  $\Phi: \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \rightarrow \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y$  with  $i_Y \circ \Phi = \tilde{f} \circ i_X$ . Similarly, we have  $\Psi: \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \rightarrow \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$  with  $i_X \circ \Psi = \tilde{f}^{-1} \circ i_Y$ . Therefore  $\Phi \circ \Psi = \text{id}$  and  $\Psi \circ \Phi = \text{id}$ .

We summarize the discussion above as follows:

**Lemma 4.4.** *Let  $G$  and  $G'$  be finitely generated groups,  $X \subseteq G$ , and  $Y \subseteq G'$ . If a bijective quasi-isometry  $f: X \rightarrow Y$  exists, then  $\chi_X \mathcal{R}(G) \chi_X$  and  $\chi_Y \mathcal{R}(G') \chi_Y$  are isomorphic. The isomorphism  $\Phi = \Phi_f: \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \rightarrow \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y$  is given by*

$$\begin{aligned}\Phi(\chi_X g \chi_X) &= \chi_Y \sum_{i=1}^m h_i \chi_{S_i \in (G, X)}(g, f^{-1}(\cdot)) \chi_Y \quad (g \in G), \\ \Phi(\chi_X F \chi_X) &= \chi_Y (F \circ f^{-1}) \chi_Y \quad (F \in l^f(G, \mathbf{k})),\end{aligned}$$

where  $\{f(gf^{-1}(y))y^{-1} \mid y \in Y \text{ and } gf^{-1}(y) \in X\} = \{h_1, \dots, h_m\}$  ( $m, h_i$  depend on  $g$ ) and  $S_i = \{y \in Y \mid gf^{-1}(y) \in X \text{ and } f(gf^{-1}(y))y^{-1} = h_i\}$ .

Given two quasi-isometric non-amenable finitely generated groups, we can find a bijective quasi-isometry between them [5, Proposition p. 104], and hence by combining this fact and Lemma 4.4, we have

**Corollary 4.5.** *If non-amenable finitely generated groups  $G$  and  $G'$  are quasi-isometric, then  $(\mathcal{R}(G), l^f(G, \mathbf{k}))$  and  $(\mathcal{R}(G'), l^f(G', \mathbf{k}))$  are isomorphic as pairs of rings.*

**4.2. The proof of the main theorem.** Let  $G$  and  $G'$  be finitely generated groups, and  $f: G \rightarrow G'$  a quasi-isometry. There exist cores of  $f: X \subseteq G$  and  $Y \subseteq G'$ . By Lemma 4.1  $I_X = \chi_X \mathcal{R}(G, \mathbf{k})$  is a progenerator. By Theorem 2.3  $\mathcal{R}(G, \mathbf{k})$  and  $\text{End}(I_X)$  are Morita equivalent. Since  $\chi_X$  is an idempotent,  $\text{End}(I_X) \simeq \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$ . Therefore  $\mathcal{R}(G, \mathbf{k})$  and  $\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$  are Morita equivalent. Furthermore, by Theorem 2.3

$$\begin{aligned}&\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \chi_X \mathcal{R}(G, \mathbf{k})_{\mathcal{R}(G, \mathbf{k})}, \\ &\mathcal{R}(G, \mathbf{k}) \mathcal{R}(G, \mathbf{k}) \chi_X \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X\end{aligned}$$

are bimodules, and

$$\begin{aligned}(\chi_X \mathcal{R}(G, \mathbf{k}) \otimes_{\mathcal{R}(G, \mathbf{k})} -): \mathcal{R}(G, \mathbf{k})\text{-Mod} &\rightarrow \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X\text{-Mod}, \\ (\mathcal{R}(G, \mathbf{k}) \chi_X \otimes_{\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X} -): \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X\text{-Mod} &\rightarrow \mathcal{R}(G, \mathbf{k})\text{-Mod}\end{aligned}$$

are inverse equivalences. Similarly,  $\mathcal{R}(G', \mathbf{k})$  and  $\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y$  are Morita equivalent. Furthermore, by Theorem 2.3

$$\begin{aligned}&\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \chi_Y \mathcal{R}(G', \mathbf{k})_{\mathcal{R}(G', \mathbf{k})}, \\ &\mathcal{R}(G', \mathbf{k}) \mathcal{R}(G', \mathbf{k}) \chi_Y \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y\end{aligned}$$

are bimodules, and

$$\begin{aligned} (\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G', \mathbf{k})} -) : \mathcal{R}(G', \mathbf{k})\text{-Mod} &\rightarrow \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \text{-Mod}, \\ (\mathcal{R}(G', \mathbf{k}) \chi_Y \otimes_{\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y} -) : \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \text{-Mod} &\rightarrow \mathcal{R}(G', \mathbf{k})\text{-Mod} \end{aligned}$$

are inverse equivalences.

Since  $f|_X$  is a bijective quasi-isometry, by Lemma 4.4  $\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$  and  $\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y$  are isomorphic.  $\Phi = \Phi_{f|_X} : \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \rightarrow \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y$  is the isomorphism, and hence

$$\begin{aligned} \text{Res } \Phi : \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \text{-Mod} &\rightarrow \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \text{-Mod}, \\ \text{Res}(\Phi^{-1}) : \chi_X \mathcal{R}(G, \mathbf{k}) \chi_X \text{-Mod} &\rightarrow \chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y \text{-Mod} \end{aligned}$$

are inverse equivalences. Therefore

$$\begin{aligned} \mathcal{F}_1 &= (\mathcal{R}(G', \mathbf{k}) \chi_Y \otimes_{\chi_Y \mathcal{R}(G', \mathbf{k}) \chi_Y} -) \circ \text{Res}(\Phi^{-1}) \circ (\chi_X \mathcal{R}(G, \mathbf{k}) \otimes_{\mathcal{R}(G, \mathbf{k})} -), \\ \mathcal{F}_2 &= (\mathcal{R}(G, \mathbf{k}) \chi_X \otimes_{\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X} -) \circ \text{Res } \Phi \circ (\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G', \mathbf{k})} -) \end{aligned}$$

are inverse equivalences. As a result, we obtain the main theorem:

**Theorem 1.** *If finitely generated groups  $G$  and  $G'$  are quasi-isometric, then  $\mathcal{R}(G, \mathbf{k})$  and  $\mathcal{R}(G', \mathbf{k})$  are Morita equivalent.*

**4.3. On some characteristic modules.** Let  $H$  be a finitely generated group. There exist characteristic  $\mathcal{R}(H, \mathbf{k})$ -modules of functions on  $H$  preserved by  $\mathcal{F}_1$  of the previous subsection. In this subsection, we use the notation of Section 4.2.

We consider a left  $\mathcal{R}(H, \mathbf{k})$ -module  $l(H, \mathbf{k}) = \{F : H \rightarrow \mathbf{k}\}$  with an action

$$(hF_1)F = F_1^{h^{-1}} F^{h^{-1}} \quad (hF_1 \in \mathcal{R}(H, \mathbf{k}), F \in l(H, \mathbf{k})).$$

$l^f(H, \mathbf{k})$  or  $l_c(H, \mathbf{k}) = \{F : H \rightarrow \mathbf{k} \mid \#(\text{supp}(F)) < \infty\}$  are submodules of  $l(H, \mathbf{k})$ . For  $Z \subseteq H$  a left  $\chi_Z \mathcal{R}(H, \mathbf{k}) \chi_Z$ -module  $\chi_Z \mathcal{R}(H, \mathbf{k}) \otimes_{\mathcal{R}(H, \mathbf{k})} l(H, \mathbf{k})$  is isomorphic to the left  $\chi_Z \mathcal{R}(H, \mathbf{k}) \chi_Z$ -module  $l(Z, \mathbf{k})$  with an action

$$(\chi_Z h F_1 \chi_Z)F = \chi_Z F_1^{h^{-1}} \chi_Z F^{h^{-1}} \quad (\chi_Z h F_1 \chi_Z \in \chi_Z \mathcal{R}(H, \mathbf{k}) \chi_Z, F \in l(Z, \mathbf{k})).$$

**Lemma 4.6.** *Under the notation of Section 4.2*

$$(\text{Res } \Phi) \circ (\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G', \mathbf{k})} -)(l(G', \mathbf{k})) \simeq (\chi_X \mathcal{R}(G, \mathbf{k}) \otimes_{\mathcal{R}(G, \mathbf{k})} -)(l(G, \mathbf{k})).$$

Proof. By Lemma 4.4  $(\text{Res } \Phi) \circ (\chi_Y \mathcal{R}(G', \mathbf{k}) \otimes_{\mathcal{R}(G', \mathbf{k})} -)(l(G', \mathbf{k}))$  is isomorphic to the left  $\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$ -module  $l(Y, \mathbf{k})$  with an action

$$\begin{aligned} (\chi_X g \chi_X) F &= (\Phi_{f|_X})(\chi_X g \chi_X) F \\ &= \left( \chi_Y \sum_{i=1}^m h_i \chi_{S_i} \epsilon_{(G, X)}(g, f|_X^{-1}(\cdot)) \chi_Y \right) F \\ &= \chi_Y \sum_{i=1}^m \chi_{h_i S_i} \epsilon_{(G, X)}(g, f|_X^{-1}(\cdot))^{h_i^{-1}} \chi_{h_i Y} F^{h_i^{-1}} \end{aligned}$$

for every  $g \in G$  and  $F \in l(Y, \mathbf{k})$ , and also

$$\begin{aligned} (\chi_X F_1 \chi_X) F &= (\Phi_{f|_X})(\chi_X F_1 \chi_X) F \\ &= (\chi_Y (F_1 \circ (f|_X)^{-1}) \chi_Y) F \\ &= \chi_Y (F_1 \circ (f|_X)^{-1}) \chi_Y F \end{aligned}$$

for every  $F_1 \in l^f(G, \mathbf{k})$  and  $F \in l(Y, \mathbf{k})$ . We define  $\lambda: l(Y, \mathbf{k}) \rightarrow l(X, \mathbf{k})$  by  $F \mapsto F \circ f|_X$ . We will prove that  $\lambda$  is a left  $\chi_X \mathcal{R}(G, \mathbf{k}) \chi_X$ -isomorphism. Since  $\lambda$  is a bijective additive group homomorphism, we only have to check that the action is preserved. For every  $g \in G$ ,  $F \in l(Y, \mathbf{k})$  and  $x \in X$ , under the notation of Lemma 4.4, if  $x \in gX$ , then there exists the only  $h_j$  such that  $f(x) \in h_j S_j$ , and also if  $x \notin gX$ , then there exists no  $h_j$  such that  $f(x) \in h_j S_j$ . Therefore

$$\begin{aligned} \lambda((\chi_X g \chi_X) F)(x) &= (\Phi_{f|_X})(\chi_X g \chi_X) F(f(x)) \\ &= \left( \chi_Y \sum_{i=1}^m \chi_{h_i S_i} \epsilon_{(G, X)}(g, f|_X^{-1}(\cdot))^{h_i^{-1}} \chi_{h_i Y} F^{h_i^{-1}} \right) (f(x)) \\ &= \begin{cases} F^{h_j^{-1}}(f(x)) & \text{if } x \in gX, \\ 0 & \text{if } x \notin gX \end{cases} \\ &= \begin{cases} F \circ f(g^{-1}x) & \text{if } x \in gX, \\ 0 & \text{if } x \notin gX \end{cases} \\ &= ((\chi_X g \chi_X) \lambda(F))(x). \end{aligned}$$

For every  $F_1 \in l^f(G, \mathbf{k})$ ,  $F \in l(Y, \mathbf{k})$  and  $x \in X$

$$\begin{aligned} \lambda((\chi_X F_1 \chi_X) F)(x) &= (\Phi_{f|_X})(\chi_X F_1 \chi_X) F(f(x)) \\ &= (\chi_Y (F_1 \circ (f|_X)^{-1}) \chi_Y) F(f(x)) \\ &= F_1(F \circ f)(x) \\ &= ((\chi_X F_1 \chi_X) \lambda(F))(x). \end{aligned} \quad \square$$

**Theorem 4.7.** *Under the notation of Section 4.2*

- (1)  $\mathcal{F}_1(l(G, \mathbf{k})) \simeq l(G', \mathbf{k}), \mathcal{F}_2(l(G', \mathbf{k})) \simeq l(G, \mathbf{k}),$
- (2)  $\mathcal{F}_1(l^f(G, \mathbf{k})) \simeq l^f(G', \mathbf{k}), \mathcal{F}_2(l^f(G', \mathbf{k})) \simeq l^f(G, \mathbf{k}),$
- (3)  $\mathcal{F}_1(l_c(G, \mathbf{k})) \simeq l_c(G', \mathbf{k}), \mathcal{F}_2(l_c(G', \mathbf{k})) \simeq l_c(G, \mathbf{k}).$

Proof. (1) We send the equation of Lemma 4.6 by  $(\mathcal{R}(G, \mathbf{k})\chi_X \otimes_{\chi_X \mathcal{R}(G, \mathbf{k})\chi_X} -)$ . Therefore we have  $\mathcal{F}_2(l(G', \mathbf{k})) \simeq l(G, \mathbf{k})$ . We also send this equation by  $\mathcal{F}_1$ , and hence  $\mathcal{F}_1(l(G, \mathbf{k})) \simeq l(G', \mathbf{k})$ . Since  $\lambda$  of the proof of Lemma 4.6 is an isomorphism on  $l^f$  and  $l_c$ , (2) and (3) are also proved.  $\square$

We have a left  $\mathcal{R}(H, \mathbf{k})$ -module  $T = T_H = T_{(H, \mathbf{k})} = \mathcal{R}(H, \mathbf{k}) \otimes_{H\mathbf{k}} \mathbf{k}$ .  $T_H$  is isomorphic to  $l^f(H, \mathbf{k})$ . Indeed,  $\theta: T_H \rightarrow l^f(H, \mathbf{k})$  defined by  $\theta\left(\sum_{i=1}^n g_n F_n \otimes k\right) = \sum_{i=1}^n F_n^{g_n^{-1}} k$  ( $\sum_{i=1}^n g_n F_n \in \mathcal{R}(H, \mathbf{k}), k \in \mathbf{k}$ ) gives an isomorphism. Let  $M$  be a left  $\mathcal{R}(H, \mathbf{k})$ -module. Since  $H\mathbf{k}$  is a subring of  $\mathcal{R}(H, \mathbf{k})$ ,  $M$  is regarded as a left  $H\mathbf{k}$ -module. By the flatness of  $\mathcal{R}(H, \mathbf{k})_{H\mathbf{k}}$  (see Lemma 5.1 of the next section), we have  $\text{Ext}_{\mathcal{R}(H, \mathbf{k})}^n(l^f(H, \mathbf{k}), M) = \text{Ext}_{\mathcal{R}(H, \mathbf{k})}^n(T_H, M) = \text{Ext}_{H\mathbf{k}}^n(\mathbf{k}, M) = H^n(H, M)$ . Since  $l_c(H, \mathbf{k})$  is isomorphic to  $H\mathbf{k}$ ,  $H^n(H, l_c(H, \mathbf{k})) = H^n(H, H\mathbf{k})$ . This cohomology group is the coarse cohomology (see [7]). By Theorem 4.7  $H^n(H, H\mathbf{k})$  is a quasi-isometry invariant.

In the case of  $\mathbf{k} = \mathbf{C}$  (or  $\mathbf{R}$ ) for  $0 < p \leq \infty$  we have a module of  $p$ -summable functions  $l^p(G, \mathbf{C}) \subseteq l(G, \mathbf{C})$ . We can also prove  $\mathcal{F}_1(l^p(G, \mathbf{C})) \simeq l^p(G', \mathbf{C}), \mathcal{F}_2(l^p(G', \mathbf{C})) \simeq l^p(G, \mathbf{C})$ . Therefore  $\text{Ext}_{\mathcal{R}(H, \mathbf{k})}^n(l^f(H, \mathbf{C}), l^p(H, \mathbf{C}))$  is a quasi-isometry invariant. This cohomology group is isomorphic to  $H^n(H, l^p(H, \mathbf{C}))$ : the coarse  $l^p$ -cohomology (see [6]).

## 5. The global dimension and the weak global dimension of algebraic translation algebras

Let  $G$  be a finitely generated group. We see that  $\mathcal{R}(G, \mathbf{k})_{G\mathbf{k}}$  is a flat  $G\mathbf{k}$ -module. Let  $\Lambda = \{S = \{S_1, \dots, S_{n_S}\} \mid S_i \subseteq G, \bigsqcup_{i=1}^{n_S} S_i = G\}$  be the set of finite decompositions of  $G$ . For  $\alpha$  and  $\beta \in \Lambda$  we denote  $\alpha < \beta$  if  $\beta$  is a refinement of  $\alpha$ . Let  $L_\alpha = \bigoplus_{i=1}^{n_\alpha} (G\mathbf{k})$  be a right free  $G\mathbf{k}$ -module. If  $\alpha < \beta$ , then for each  $1 \leq k \leq n_\beta$  there exists  $1 \leq i_k \leq n_\alpha$  such that  $\beta_k \subseteq \alpha_{i_k}$ . Let  $f_{\beta\alpha}: L_\alpha \rightarrow L_\beta$  be a  $G\mathbf{k}$ -homomorphism such that  $f_{\beta\alpha}(x_1, \dots, x_{n_\alpha}) = (x_{i_1}, \dots, x_{i_{n_\beta}})$  ( $x_k \in G\mathbf{k}$ ). These data define a direct system of right  $G\mathbf{k}$ -modules, and hence we have a right  $G\mathbf{k}$ -module  $\varinjlim L_\alpha$ .  $\varinjlim L_\alpha = (\bigoplus_{\alpha \in \Lambda} L_\alpha)/N$ , where  $N$  is a submodule of  $\bigoplus_{\alpha \in \Lambda} L_\alpha$  generated by  $\{\iota_\beta \circ f_{\beta\alpha}(x) - \iota_\alpha(x) \mid \alpha < \beta, x \in L_\alpha\}$ .  $\iota_\alpha$  and  $\iota_\beta$  are injections to  $\bigoplus_{\alpha \in \Lambda} L_\alpha$ .

**Lemma 5.1.**  *$\mathcal{R}(G, \mathbf{k})_{G\mathbf{k}}$  is a direct limit of flat  $G\mathbf{k}$ -modules:*

$$\varinjlim L_\alpha \simeq \mathcal{R}(G, \mathbf{k})_{G\mathbf{k}}.$$

Therefore  $\mathcal{R}(G, \mathbf{k})_{G\mathbf{k}}$  is a flat  $G\mathbf{k}$ -module, and the functor  $\mathcal{R}(G, \mathbf{k}) \otimes_{G\mathbf{k}} -: G\mathbf{k}\text{-Mod} \rightarrow \mathcal{R}(G, \mathbf{k})\text{-Mod}$  is exact.

Proof. We define  $t_\alpha: L_\alpha \rightarrow \mathcal{R}(G, \mathbf{k})$  by  $t_\alpha(x_1, \dots, x_{n_\alpha}) = \sum_{i=1}^{n_\alpha} \chi_{\alpha_i} x_i$  ( $x_i \in G\mathbf{k}$ ). The direct sum of  $\{t_\alpha \mid \alpha \in \Lambda\}$  defines an isomorphism.  $\square$

We recall the definitions of some homological dimensions. Let  $R$  be a ring, and  $M$  a left  $R$ -module.

- (1)  $\text{fd}_R(M) = \sup\{n \mid \exists \text{a right } R\text{-module } N \text{ with } \text{Tor}_n^R(N, M) \neq 0\}$ . This number is equal to the minimal number  $n$  such that there exists an  $n$ -length flat resolution of  $M$ .
- (2)  $\text{pd}_R(M) = \sup\{n \mid \exists \text{a left } R\text{-module } N \text{ with } \text{Ext}_R^n(M, N) \neq 0\}$ . This number is also equal to the minimal number  $n$  such that there exists an  $n$ -length projective resolution of  $M$ .
- (3)  $\text{wd}(R) = \sup\{\text{fd}_R(M) \mid M \text{ is a left } R\text{-module}\}$ .
- (4)  $1.\text{gl.dim}(R) = \sup\{\text{pd}_R(M) \mid M \text{ is a left } R\text{-module}\}$  ([21, Section 3, 4] is a good reference for Tor or Ext and homological dimensions).

$\text{wd}$  and  $1.\text{gl.dim}$  are Morita invariants. We discuss  $1.\text{gl.dim}(\mathcal{R}(G, \mathbf{k}))$  and  $\text{wd}(\mathcal{R}(G, \mathbf{k}))$ .

**Lemma 5.2.** *Let  $G$  be a finitely generated group, and  $l$  a ring containing  $\mathbf{k}$  as a subring. We assume that  $G$  acts on  $l$  from the right trivially on  $\mathbf{k}$ . Let  $\mathcal{R} = G * l$ , and  $T = \mathcal{R} \otimes_{G\mathbf{k}} \mathbf{k}$ . For every left  $\mathcal{R}$ -module  $A$  and  $B$ , and an  $\mathcal{R}$ -projective resolution  $\mathcal{C}$  of  $A$  we have a spectral sequence*

$$\text{Ext}_{\mathcal{R}}^p(T, \text{H}^q(\widetilde{\text{Hom}}_l(\mathcal{C}, B))) \Rightarrow_p \text{Ext}_{\mathcal{R}}^n(A, B),$$

where for a left  $\mathcal{R}$ -module  $C$ ,  $\widetilde{\text{Hom}}_l(C, B) = \text{Hom}_l(C, B)$  is a left module with a left action of  $\mathcal{R}$  defined by

$$(gF)\varphi(x) = F^{g^{-1}} g\varphi(g^{-1}x)$$

for every  $\varphi \in \widetilde{\text{Hom}}_l(C, B)$ ,  $g \in G$ ,  $F \in l$  and  $x \in C$ . The notation of a spectral sequence is that of [3, Chapter XV].

Proof. This spectral sequence is obtained by modifying a spectral sequence of Cartan and Leray [3, Proposition 8.2].

First, we prove  $\text{Ext}_{\mathcal{R}}^p(T, \widetilde{\text{Hom}}_l(\mathcal{R}, B)) = 0$  (if  $p > 0$ ) by direct calculation. Since  $\mathcal{R} = \bigoplus_{g \in G} l \cdot g$  and  $\varphi \in \widetilde{\text{Hom}}_l(\mathcal{R}, B)$  is decided by  $\varphi(g) \in B$ , we have  $\widetilde{\text{Hom}}_l(\mathcal{R}, B) = \prod_{g \in G} B_g$ , where  $B_g$  is a copy of  $B$ . For  $(b_g)_{g \in G} \in \prod_{g \in G} B_g$ , the  $\mathcal{R}$ -action is given by  $(xF)(b_g)_{g \in G} = (F^{x^{-1}} x b_{x^{-1}g})_{g \in G}$  ( $x \in G$ ,  $F \in l$ ). We consider free right  $\mathbf{k}$ -modules  $I_p = \{(\sigma_0, \dots, \sigma_p) \mid \sigma_i \in G\}\mathbf{k}$  and  $\varphi_{p-1}(\sigma_0, \dots, \sigma_p) = \sum_{i=0}^p (-1)^i (\sigma_0, \dots, \check{\sigma}_i, \dots, \sigma_p)$ .  $\mathbf{I} = \{I_p, \varphi_p\}$  is the  $G\mathbf{k}$ -standard resolution of  $\mathbf{k}$ . Then  $\tilde{\mathbf{I}} = \mathcal{R} \otimes_{G\mathbf{k}} \mathbf{I}$  is an  $\mathcal{R}$ -projective resolution of  $T$ . Since  $f \in \text{Hom}(\tilde{I}_p, \widetilde{\text{Hom}}_l(\mathcal{R}, B))$  is decided by  $f(1, \sigma_1, \dots, \sigma_p)(g) \in B_g$ , we have

$$\text{Hom}(\tilde{I}_p, \widetilde{\text{Hom}}_l(\mathcal{R}, B)) = \prod_{\sigma_1, \dots, \sigma_p, g \in G} B_{\sigma_1, \dots, \sigma_p, g},$$

where  $B_{\sigma_1, \dots, \sigma_p, g}$  is a copy of  $B$ .  $\partial_p = \text{Hom}(\tilde{\varphi}_p, \widetilde{\text{Hom}}_l(\mathcal{R}, B))$  satisfies

$$(i) \quad \partial_p((b_{\sigma_1, \dots, \sigma_p, g})_{\sigma_1, \dots, \sigma_p, g \in G}) = \left( \sigma_1 b_{\sigma_1^{-1} \sigma_2, \dots, \sigma_1^{-1} \sigma_{p+1}, \sigma_1^{-1} g} + \sum_{i=1}^{p+1} (-1)^i b_{\sigma_1, \dots, \check{\sigma}_i, \dots, \sigma_{p+1}, g} \right)_{\sigma_1, \dots, \sigma_{p+1}, g \in G}.$$

By the definition of  $\text{Ext}$ ,  $\text{Ext}_{\mathcal{R}}^p(T, \widetilde{\text{Hom}}_l(\mathcal{R}, B)) = \text{Ker} \partial_p / \text{Im} \partial_{p-1}$ . For every  $(b_{\sigma_1, \dots, \sigma_p, g})_{\sigma_1, \dots, \sigma_p, g \in G} \in \text{Ker} \partial_p$  and in the case of  $p \geq 1$ , we have  $c_{\sigma_1, \dots, \sigma_{p-1}, g} = (-1)^p b_{\sigma_1, \dots, \sigma_{p-1}, g, g}$ . This satisfies

$$(ii) \quad \partial_{p-1}(c_{\sigma_1, \dots, \sigma_{p-1}, g})_{\sigma_1, \dots, \sigma_{p-1}, g \in G} = (b_{\sigma_1, \dots, \sigma_p, g})_{\sigma_1, \dots, \sigma_p, g \in G}.$$

In fact,

$$(iii) \quad \begin{aligned} & \partial_{p-1}(c_{\sigma_1, \dots, \sigma_{p-1}, g})_{\sigma_1, \dots, \sigma_{p-1}, g \in G} \\ &= \left( \sigma_1 c_{\sigma_1^{-1} \sigma_2, \dots, \sigma_1^{-1} \sigma_p, \sigma_1^{-1} g} + \sum_{i=1}^p (-1)^i c_{\sigma_1, \dots, \check{\sigma}_i, \dots, \sigma_p, g} \right)_{\sigma_1, \dots, \sigma_p, g \in G} \\ &= \left( (-1)^p \sigma_1 b_{\sigma_1^{-1} \sigma_2, \dots, \sigma_1^{-1} \sigma_p, \sigma_1^{-1} g, \sigma_1^{-1} g} + \sum_{i=1}^p (-1)^{p+i} b_{\sigma_1, \dots, \check{\sigma}_i, \dots, \sigma_p, g, g} \right)_{\sigma_1, \dots, \sigma_p, g \in G}. \end{aligned}$$

By substituting  $g$  for  $\sigma_{p+1}$  in (i), we have

$$(iv) \quad \sigma_1 b_{\sigma_1^{-1} \sigma_2, \dots, \sigma_1^{-1} \sigma_p, \sigma_1^{-1} g, \sigma_1^{-1} g} + \sum_{i=1}^p (-1)^i b_{\sigma_1, \dots, \check{\sigma}_i, \dots, \sigma_p, g, g} + (-1)^{p+1} b_{\sigma_1, \dots, \sigma_p, g} = 0.$$

(ii) is obtained by (iii) and (iv). Therefore  $\text{Ext}_{\mathcal{R}}^p(T, \widetilde{\text{Hom}}_l(\mathcal{R}, B)) = 0$  (if  $p > 0$ ) is proved. This shows  $\text{Ext}_{\mathcal{R}}^p(T, \widetilde{\text{Hom}}_l(P, B)) = 0$  (if  $p > 0$ ) for every projective left  $\mathcal{R}$ -module  $P$ .

Second, for left  $\mathcal{R}$ -modules  $X$  and  $Y$ ,  $\rho: \text{Hom}_{\mathcal{R}}(T, \widetilde{\text{Hom}}_l(X, Y)) \rightarrow \text{Hom}_{\mathcal{R}}(X, Y)$  defined by  $f \mapsto f(1)$  is an isomorphism.

Let  $\mathbf{X}$  be a projective resolution of  $T$  and  $\mathbf{Y} = \widetilde{\text{Hom}}_l(\mathcal{C}, B)$ .  $\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})$  is a double complex, and hence we have two spectral sequences with the same limit:

$$I_2^{p,q} = H^p(H^q(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y}))) \Rightarrow_p \text{Tot}^n(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})),$$

$$II_2^{p,q} = H^q(H^p(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y}))) \Rightarrow_q \text{Tot}^n(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})).$$

By the first assertion above  $H^p(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})) = \text{Hom}_{\mathcal{R}}(T, Y)$  (if  $p = 0$ ) and  $H^p(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})) = 0$  (otherwise). We have  $II_2^{p,q} = H^q(\text{Hom}_{\mathcal{R}}(T, Y)) = H^q(\mathcal{C}, B) = \text{Ext}_{\mathcal{R}}^q(A, B)$  (if  $p = 0$ ) and  $II_2^{p,q} = 0$  (otherwise) by the second assertion above. Therefore  $\text{Tot}^n(\text{Hom}_{\mathcal{R}}(\mathbf{X}, \mathbf{Y})) = \text{Ext}_{\mathcal{R}}^n(A, B)$ . Since each term  $X_p$  of  $\mathbf{X}$  is projective,

$\text{Hom}_{\mathcal{R}}(X_p, -)$  is exact.

$$\begin{aligned} \text{H}^q(\text{Hom}_{\mathcal{R}}(X_p, \mathbf{Y})) &= \text{H}^q(\text{Hom}_{\mathcal{R}}(X_p, \widetilde{\text{Hom}}_l(\mathcal{C}, B))) \\ &= \text{Hom}_{\mathcal{R}}(X_p, \text{H}^q(\widetilde{\text{Hom}}_l(\mathcal{C}, B))) \end{aligned}$$

shows that  $I_2^{p,q} = \text{Ext}_{\mathcal{R}}^p(T, \text{H}^q(\widetilde{\text{Hom}}_l(\mathcal{C}, B)))$ .  $\square$

**Corollary 5.3.**

$$\text{pd}_{\mathcal{R}(G, \mathbf{k})}(T) \leq \text{l.gl.dim}(\mathcal{R}(G, \mathbf{k})) \leq \text{pd}_{\mathcal{R}(G, \mathbf{k})}(T) + \text{l.gl.dim}(l^f(G, \mathbf{k})).$$

We can also prove the Tor version of Lemma 5.2, and hence

$$\text{fd}_{\mathcal{R}(G, \mathbf{k})}(T) \leq \text{wd}(\mathcal{R}(G, \mathbf{k})) \leq \text{fd}_{\mathcal{R}(G, \mathbf{k})}(T) + \text{wd}(l^f(G, \mathbf{k})).$$

We estimate  $\text{pd}_{\mathcal{R}(G, \mathbf{k})}(T)$  and  $\text{fd}_{\mathcal{R}(G, \mathbf{k})}(T)$  by the homological dimensions of  $G$ .

**Lemma 5.4.** (1)  $\text{pd}_{\mathcal{R}(G, \mathbf{k})}(T) \leq \text{cd}_{\mathbf{k}}(G)$ , where  $\text{cd}_{\mathbf{k}}(G) = \text{pd}_{G\mathbf{k}}(\mathbf{k})$ .  
 (2)  $\text{fd}_{\mathcal{R}(G, \mathbf{k})}(T) \leq \text{hd}_{\mathbf{k}}(G)$ , where  $\text{hd}_{\mathbf{k}}(G) = \text{fd}_{G\mathbf{k}}(\mathbf{k})$ .  
 (3) If  $\text{cd}_{\mathbf{k}}(G) < \infty$ , then  $\text{cd}_{\mathbf{k}}(G) = \text{pd}_{\mathcal{R}(G, \mathbf{k})}(T)$ .  
 (4) If  $\text{hd}_{\mathbf{k}}(G) < \infty$ , then  $\text{hd}_{\mathbf{k}}(G) = \text{fd}_{\mathcal{R}(G, \mathbf{k})}(T)$ .

Proof. Since the functor  $\mathcal{R}(G, \mathbf{k}) \otimes_{G\mathbf{k}} -$  is exact by Lemma 5.1, a projective resolution (flat resolution) of  $\mathbf{k}$  is mapped to a projective resolution (flat resolution) of  $T$  by  $\mathcal{R}(G, \mathbf{k}) \otimes_{G\mathbf{k}} -$ . Therefore (1) and (2) are obtained.

(3) and (4) are proved by the same argument as [18, Section 4].  $\square$

We estimate  $\text{l.gl.dim}(l^f(G, \mathbf{k}))$  and  $\text{wd}(l^f(G, \mathbf{k}))$ .

**Lemma 5.5.** Let  $\Delta$  be a countably infinite set.

- (1) If  $\mathbf{k}$  is a field, then  $\text{wd}(l^f(\Delta, \mathbf{k})) = 0$ .
- (2) If  $\mathbf{k}$  is  $\mathbf{Z}$ , then  $\text{wd}(l^f(\Delta, \mathbf{k})) = 1$ .

If the continuum hypothesis is true, then

- (3) if  $\mathbf{k}$  is a field, then  $\text{l.gl.dim}(l^f(\Delta, \mathbf{k})) = 2$ ,
- (4) if  $\mathbf{k}$  is  $\mathbf{Z}$ , then  $\text{l.gl.dim}(l^f(\Delta, \mathbf{k})) \leq 3$ .

Proof. (1) We see that for every  $F \in l^f(\Delta, \mathbf{k})$ ,  $F \in F \cdot l^f(\Delta, \mathbf{k}) \cdot F$  is satisfied. Therefore  $l^f(\Delta, \mathbf{k})$  is von-Neumann regular [11, xviii, the third paragraph]. This implies that  $\text{wd}(l^f(\Delta, \mathbf{k})) = 0$  [11, (5.62a) p. 185].

(2)  $l^f(\Delta, \mathbf{k})$  is not von-Neumann regular since  $2 \notin 2 \cdot l^f(\Delta, \mathbf{k}) \cdot 2$ . Every ideal of  $l^f(\Delta, \mathbf{k})$  is generated by projective modules with the form  $l^f(\Delta, \mathbf{k}) \cdot \chi_{Xn}$ , and hence every ideal of  $l^f(\Delta, \mathbf{k})$  is flat. This implies  $\text{wd}(l^f(\Delta, \mathbf{k})) = 1$  [11, (5.69) p. 187].

(3) By the theorem of Osofsky [15, Corollary 2.47] for every ring  $R$  if every left ideal of  $R$  is generated by  $\aleph_h$  elements, then

$$\text{l.gl.dim}(R) \leq \text{wd}(R) + h + 1.$$

Every ideal of  $l^f(\Delta, \mathbf{k})$  is generated by characteristic functions on  $\Delta$ . Therefore if the continuum hypothesis is true, then  $\text{l.gl.dim}(l^f(\Delta, \mathbf{k})) \leq 2$ . Since  $l^f(\Delta, \mathbf{k})$  has a non-projective ideal ( $l^f(\Delta, \mathbf{k})$  is not semi-simple and not hereditary),  $\text{l.gl.dim}(l^f(\Delta, \mathbf{k})) = 2$  [11, (5.14) p. 169].

(4) It is proved by the theorem of Osofsky and (2).  $\square$

We remark that  $\text{wd}(l^\infty(\Delta, \mathbf{C})) \geq 1$ . Indeed, for  $s_1, \dots, s_n, \dots \in \Delta$  there exists a function  $f \in l^\infty(G, \mathbf{C})$  such that  $f(s_n) = \exp(-n)$ , and  $f \notin f \cdot l^\infty(\Delta, \mathbf{C}) \cdot f$ .

**Theorem 5.6.** *If  $\mathbf{k}$  is a field, then*

- (1) *if  $\text{hd}_\mathbf{k}(G) < \infty$ , then  $\text{wd}(\mathcal{R}(G, \mathbf{k})) = \text{hd}_\mathbf{k}(G)$ ,*  
*and if the continuum hypothesis is true, then*
- (2) *if  $\text{cd}_\mathbf{k}(G) < \infty$ , then  $\text{cd}_\mathbf{k}(G) \leq \text{l.gl.dim}(\mathcal{R}(G, \mathbf{k})) \leq \text{cd}_\mathbf{k}(G) + 2$ .*

Proof. The assertions (1) and (2) follow from Corollary 5.3, Lemmas 5.4 and 5.5.  $\square$

## 6. Geometric description of $\mathcal{R}(G, \mathbf{k})$

In this section, we need some theorems of groupoid theory and category theory. Everything needed in this section is in Sections 2.5, 2.6 and 2.7. Let  $G$  be a finitely generated group with the identity element  $e$ , and  $\mathcal{G} = G \ltimes \beta G$  an étale groupoid. We consider  $\underline{\text{Mod}}_\mathbf{k}(\mathcal{G})$ . We define the characteristic object  $U \in \underline{\text{Mod}}_\mathbf{k}(\mathcal{G})$ .

**DEFINITION 6.1.** We define  $U = \beta G \times G\mathbf{k}$ , where  $G\mathbf{k}$  has the discrete topology. An element  $(x, \alpha) \in \beta G \times G\mathbf{k}$  is denoted by  ${}_x(\alpha)$ . We also define

$$\begin{aligned} p_0: U &\rightarrow \beta G \quad \text{by} \quad p_0({}_x(\alpha)) = x, \\ p_1: (G \ltimes \beta G) \times_{p_0} U &\rightarrow U \quad \text{by} \quad p_1((g, x), {}_x(\alpha)) = {}_{gx}(g\alpha), \\ M: U \oplus U &\rightarrow U \quad \text{by} \quad M({}_x(\alpha), {}_x(\beta)) = {}_x(\alpha + \beta) \quad (x \in \beta G), \\ \mathcal{U}: \Theta &\rightarrow U \quad \text{by} \quad \mathcal{U}(x) = {}_x(0) \quad (x \in \beta G), \\ v: U &\rightarrow U \quad \text{by} \quad v({}_x(\alpha)) = {}_x(-\alpha) \quad (x \in \beta G), \\ \mathcal{M}: \mathbf{k} \oplus U &\rightarrow U \quad \text{by} \quad \mathcal{M}(k, {}_x(\alpha)) = {}_x(k\alpha) \quad (x \in \beta G). \end{aligned}$$

$U = ((U, p_0, p_1), M, \mathcal{U}, v, \mathcal{M})$  is an object of  $\underline{\text{Mod}}_\mathbf{k}(\mathcal{G})$ .  $p_0, p_1, M, \mathcal{U}, v$  and  $\mathcal{M}$  are open maps.  $\{{}_x(e) \mid x \in \beta G\}$  can be identified with  $\beta G$  by  $p_0$ .

A morphism from the characteristic object  $U$  is determined on  $\beta G \subseteq U$ :

**Lemma 6.2.** *For every object  $S = ((S, q_0, q_1), M_S, \mathcal{U}_S, v_S, \mathcal{M}_S)$  of  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  a continuous map  $w: \beta G \rightarrow S$  satisfying  $q_0 \circ w = \text{id}_{\beta G}$  defines the unique morphism  $f: U \rightarrow S$  such that  $f|_{\beta G} = w$ .*

Proof. We define  $f$  by  $f(x(\sum_{i=1}^n g_i k_i)) = \sum_{i=1}^n k_i \cdot (g_i, g_i^{-1}x) \cdot w(g_i^{-1}x)$  for every  $x \in \beta G$ ,  $g_i \in G$  and  $k_i \in \mathbf{k}$ . Therefore if  $f$  is a morphism, then  $f$  is uniquely defined. Since  $q_0 \circ w = \text{id}_{\beta G}$ ,  $f$  satisfies  $q_0 \circ f = p_0$ . In order to prove  $f$  is a morphism we may prove that  $f$  is continuous, but this is a routine.  $\square$

Let  $\mathcal{A}$  be an A.B.3 category.  $U \in \text{Ob}(\mathcal{A})$  is called a *projective object* if for every epi  $b \in \text{Hom}(B, C)$  and morphism  $a \in \text{Hom}(U, C)$  there exists  $c \in \text{Hom}(U, B)$  such that  $a = b \circ c$ . A projective object  $U \in \text{Ob}(\mathcal{A})$  is called a *projective generator* if every non-zero  $A \in \text{Ob}(\mathcal{A})$  satisfies  $\text{Hom}(U, A) \neq 0$ .  $U \in \text{Ob}(\mathcal{A})$  is said to be *small* if every morphism from  $U$  into a coproduct  $s: U \rightarrow \bigoplus_{\lambda \in \Lambda} A_{\lambda}$  factors as  $U \rightarrow \bigoplus_{\lambda \in J} A_{\lambda} \rightarrow \bigoplus_{\lambda \in \Lambda} A_{\lambda}$  where  $J$  is a finite subset of  $\Lambda$  and morphism between the coproducts is the one which preserves injections.

**Theorem 6.3** ([12, Theorem 3.1, p. 631]). *Let  $\mathcal{A}$  be an A.B.3 category with a small projective generator  $U$  and  $\text{End}_{\mathcal{A}}(U)$  denote the endomorphism ring of  $U$ . Then the functor  $T: \mathcal{A} \rightarrow \text{Mod-End}_{\mathcal{A}}(U)$  defined by  $T(A) = \text{Hom}(U, A)$  is an additive equivalence.*

To see that Theorem 6.3 is applied to  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  we prove that the characteristic object  $U$  is a small projective generator of  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$ .

**Lemma 6.4.** (1)  $U \in \text{Ob}(\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G}))$  is a projective object.  
 (2)  $U \in \text{Ob}(\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G}))$  is a projective generator.  
 (3)  $U \in \text{Ob}(\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G}))$  is small.

Proof. (1) For every  $B = (B, p_{0,B}, p_{1,B})$ ,  $C = (C, p_{0,C}, p_{1,C}) \in \text{Ob}(\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G}))$ , a morphism  $a: U \rightarrow C$  and an epi  $b: B \rightarrow C$  there exists an open set  $V_x \subseteq C$  such that  $a(x(e)) \in V_x$  and  $p_{0,C}|_{V_x}$  is a homeomorphism for each  $x \in \beta G$ . Since  $b: B \rightarrow C$  is an epi,  $b$  is surjective. Therefore there exists  $y_x \in B$  such that  $b(y_x) = a(x(e))$ .  $b$  is continuous, and hence  $y_x \in b^{-1}(V_x)$  is open. We have an open set  $L'_x \subseteq b^{-1}(V_x)$  such that  $y_x \in L'_x$  and  $p_{0,B}|_{L'_x}$  is a homeomorphism. We also have a clopen set  $L''_x \subseteq p_{0,B}(L'_x) \cap p_{0,C}(V_x)$  since the topology of  $\beta G$  is generated by clopen sets (Lemma 2.5 (2)).  $b$  satisfies  $p_{0,C} \circ b = p_{0,B}$ , and hence  $L_x = (p_{0,B}|_{L'_x})^{-1}(L''_x)$  satisfies  $L_x \subseteq b^{-1}(V_x)$ ,  $y_x \in L_x$  and  $b|_{L_x}$  is a homeomorphism. Clopen sets  $W_x = a^{-1}(b(L_x))$  satisfy  $\bigcup_{x \in \beta G} W_x = \beta G$ .  $\beta G$  is compact, and hence  $\bigcup_{j=1}^m W_{x_j} = \beta G$ . We have a refinement  $\{A_i\}_{i=1}^n$  of  $\{W_{x_j}\}_{j=1}^m$  such that  $\beta G = \bigsqcup_{i=1}^n A_i$ . We can choose  $k_i$  for each  $i$  such that  $A_i \subseteq W_{x_{k_i}}$ , and hence

we define a continuous map  $w$  by  $w|_{A_i} = (b|_{L_{x_{k_i}}})^{-1} \circ a|_{A_i}$ . By Lemma 6.2 there exists a morphism  $c$  such that  $c|_{\beta G} = w$ . This  $c$  satisfies  $b \circ c = a$ .

(2) By (1)  $U$  is a projective object. We will prove that every non-zero object  $A = (A, p_{0,A}, p_{1,A}) \in \text{Ob}(\underline{\text{Mod}}_k(\mathcal{G}))$  satisfies  $\text{Hom}(U, A) \neq 0$ . Since  $A$  is non-zero, there exists  $a \neq 0 \in A$ . For  $x = p_{0,A}(a) \in \beta G$  since  $p_{0,A}$  is a local homeomorphism and by Lemma 2.5 (2) the topology of  $\beta G$  is generated by clopen sets, there exists clopen  $W_x \subseteq A$  such that  $a \in W_x$  and  $p_{0,A}|_{W_x}$  is a homeomorphism. We define a continuous map  $w$  by  $w|_{p_{0,A}(W_x)} = (p_{0,A}|_{W_x})^{-1}$  and  $w|_{\beta G - p_{0,A}(W_x)} = 0$ . By Lemma 6.2 there exists a morphism  $f$  such that  $f|_{\beta G} = w$ .  $f(x) = a \neq 0$  shows that  $\text{Hom}(U, A) \neq 0$ .

(3) For every morphism from  $U$  into a coproduct  $s: U \rightarrow \bigoplus_{\lambda \in \Lambda} A_\lambda$  there exists a finite set  $\Lambda_x \subseteq \Lambda$  such that  $s(x) \in \bigoplus_{\lambda \in \Lambda_x} A_\lambda \subseteq \bigoplus_{\lambda \in \Lambda} A_\lambda$  for each  $x \in \beta G$ .  $\bigoplus_{\lambda \in \Lambda_x} A_\lambda$  is open in  $\bigoplus_{\lambda \in \Lambda} A_\lambda$ .  $s$  is continuous and the topology of  $\beta G$  is generated by clopen sets (Lemma 2.5 (2)), and hence there exists clopen  $W_x \subseteq \beta G$  such that  $s(W_x) \subseteq \bigoplus_{\lambda \in \Lambda_x} A_\lambda$  and  $x \in W_x$ .  $\beta G$  is compact, and hence there exist  $x_1, \dots, x_m$  such that  $\bigcup_{j=1}^m W_{x_j} = \beta G$ . Therefore  $s(\beta G) \subseteq \bigoplus_{\lambda \in \bigcup_{j=1}^m \Lambda_{x_j}} A_\lambda$ . This shows that  $s: U \rightarrow \bigoplus_{\lambda \in \Lambda} A_\lambda$  factors as  $U \rightarrow \bigoplus_{\lambda \in J} A_\lambda \rightarrow \bigoplus_{\lambda \in \Lambda} A_\lambda$ , where  $J = \bigcup_{j=1}^m \Lambda_{x_j}$  and morphism between the coproducts is the one which preserves injections.  $\square$

Since  $U$  is a small projective generator, by Theorem 6.3 the functor  $T: \underline{\text{Mod}}_k(\mathcal{G}) \rightarrow \text{Mod-End}_{\underline{\text{Mod}}_k(\mathcal{G})}(U)$  defined by  $T(A) = \text{Hom}(U, A)$  is an additive equivalence. We describe the ring  $\text{End}_{\underline{\text{Mod}}_k(\mathcal{G})}(U)$ .

**Lemma 6.5.** (1) A morphism  $f: U \rightarrow U$  is determined by a finite decomposition  $G = \bigsqcup_{i=1}^n L_i$  and  $\alpha_i \in G\mathbf{k}$  such that  $f(x(e)) = {}_x(\alpha_i)$  for every  $x \in \hat{L}_i$ .  
(2)  $\text{End}_{\underline{\text{Mod}}_k(\mathcal{G})}(U)$  is isomorphic to the ring  $\mathcal{R}(G, \mathbf{k})^{op}$ .

Proof. (1) By Lemma 6.2 a continuous map  $w = f|_{\beta G}: \beta G \rightarrow U$  uniquely defines  $f$ . Since  $U = \beta G \times G\mathbf{k}$ , we have a continuous projection  $\pi_2: \beta G \times G\mathbf{k} \rightarrow G\mathbf{k}$ . The topology of  $G\mathbf{k}$  is discrete, and hence for  $\alpha \in G\mathbf{k}$ ,  $\{\alpha\}$  is clopen. For clopen  $W_\alpha = (\pi_2 \circ w)^{-1}(\{\alpha\})$ ,  $\beta G = \bigsqcup_{\alpha \in G\mathbf{k}} W_\alpha$ .  $\beta G$  is compact, and hence  $\beta G = \bigsqcup_{i=1}^m W_{\alpha_i}$ . By Lemma 2.5 (3),  $W_{\alpha_i} = \hat{L}_i$  by some  $L_i \subseteq G$ , and  $G = \bigsqcup_{i=1}^m L_i$ . Since  $\pi_2 \circ w|_{\hat{L}_i} = \alpha_i$ ,  $w(x) = {}_x(\alpha_i)$  for every  $x \in \hat{L}_i$ .

(2) We define a map  $\theta: \text{End}_{\underline{\text{Mod}}_k(G \times \beta G)}(A) \rightarrow \mathcal{R}(G, \mathbf{k})$  by

$$\theta(f) = \sum_{i=1}^n \chi_{L_i} \alpha_i,$$

where  $L_i$  and  $\alpha_i$  are determined by (1). Since  $\theta$  is bijective by (1), we will prove that it is a ring homomorphism. For every  $f$  and  $g \in \text{End}_{\underline{\text{Mod}}_k(G \times \beta G)}(A)$  such that  $\theta(f) = \sum_{i=1}^n \chi_{L_i} \alpha_i$  and  $\theta(g) = \sum_{j=1}^m \chi_{M_j} \beta_j$  we have an expression  $\alpha_i = \sum_{l=1}^{n_i} h_{i,l} k_{i,l}$  by  $h_{i,l} \in$

$G$  and  $k_{i,l} \in \mathbf{k}$ . We have the following equations:

$$\begin{aligned}\theta(f)\theta(g) &= \left( \sum_{i=1}^n \chi_{L_i} \alpha_i \right) \left( \sum_{j=1}^m \chi_{M_j} \beta_j \right) \\ &= \left( \sum_{i=1}^n \chi_{L_i} \sum_{l=1}^{n_i} h_{i,l} k_{i,l} \right) \left( \sum_{j=1}^m \chi_{M_j} \beta_j \right) \\ &= \left( \sum_{i=1}^n \sum_{l=1}^{n_i} \sum_{j=1}^m \chi_{L_i \cap h_{i,l} M_j} h_{i,l} k_{i,l} \beta_j \right).\end{aligned}$$

On the other hand, we have  $G = \bigsqcup_{i,j} (L_i \cap h_{i,l} M_j)$ , and for every  $x \in \widehat{(L_i \cap h_{i,l} M_j)}$

$$\begin{aligned}g \circ f(x(e)) &= g(x(\alpha_i)) = g\left(x\left(\sum_{l=1}^{n_i} h_{i,l} k_{i,l}\right)\right) \\ &= \sum_{l=1}^{n_i} k_{i,l} g((h_{i,l}, h_{i,l}^{-1} x) \cdot {}_{h_{i,l}^{-1} x}(e)) \\ &= \sum_{l=1}^{n_i} k_{i,l} (h_{i,l}, h_{i,l}^{-1} x) \cdot {}_{h_{i,l}^{-1} x}(\beta_j) \\ &= x\left(\sum_{l=1}^{n_i} h_{i,l} k_{i,l} \beta_j\right).\end{aligned}$$

Therefore  $\theta(f)\theta(g) = \theta(g \circ f)$ . □

By Theorem 6.3, Lemmas 6.4 and 6.5, we have

**Theorem 6.6.** *The functor  $T': \underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G}) \rightarrow \mathcal{R}(G, \mathbf{k})\text{-Mod}$  defined by  $T'(A) = (\text{Res } \theta^{-1})(\text{Hom}(U, A))$  is an (additive) equivalence, where  $\theta$  is an isomorphism defined in Lemma 6.5 (2).*

## 7. Appendix

**7.1. An alternative proof of Theorem 1.** Let  $G$  and  $G'$  be quasi-isometric finitely generated groups, we have Diagram 1.

Q.I. means quasi-isometric and W.E. means weak equivalent: let  $\mathcal{G}$  and  $\mathcal{H}$  be étale groupoids,  $\mathcal{G}$  and  $\mathcal{H}$  are said to be weakly equivalent (Morita equivalent) if there exists an étale groupoid  $\mathcal{K}$  and there exist essential morphisms  $\Phi: \mathcal{K} \rightarrow \mathcal{G}$ ,  $\Psi: \mathcal{K} \rightarrow \mathcal{H}$  (about precise definitions see [4, 1.4, 1.5] or [14, Section 5]). The following theorems for a quasi-isometry are known.

Diagram 1.

$$\begin{array}{ccc}
G & & G' \\
& \simeq_{\text{Q.I.}} & \\
& \Downarrow \text{(i)} & \\
G \ltimes \beta G & \simeq_{\text{W.E.}} & G' \ltimes \beta G' \\
& \Downarrow \text{(ii)} & \\
\underline{\text{Sh}}(G \ltimes \beta G) & \simeq_{\text{as topos}} & \underline{\text{Sh}}(G' \ltimes \beta G') \\
& \Downarrow \text{(iii)} & \\
\underline{\text{Mod}}_{\mathbf{k}}(G \ltimes \beta G) & \simeq_{\text{additive e.q.}} & \underline{\text{Mod}}_{\mathbf{k}}(G' \ltimes \beta G') \\
\wr \text{Theorem 6.6} & & \wr \text{Theorem 6.6} \\
\mathcal{R}(G, \mathbf{k})\text{-Mod} & & \mathcal{R}(G', \mathbf{k})\text{-Mod}
\end{array}$$

**Theorem 7.1** ([19, Corollary 3.6, p. 820]). *Let  $G$  and  $G'$  be finitely generated groups. If  $G$  and  $G'$  are quasi-isometric, then  $G \ltimes \beta G$  and  $G' \ltimes \beta G'$  are weakly equivalent.*

Theorem 7.1 is proved by the notion of the coarse space. The converse of the theorem is also true:

**Theorem 7.2.** *Let  $G$  and  $G'$  be finitely generated groups. If  $G \ltimes \beta G$  and  $G' \ltimes \beta G'$  are weakly equivalent, then  $G$  and  $G'$  are quasi-isometric.*

Proof. If  $G \ltimes \beta G$  and  $G' \ltimes \beta G'$  are weakly equivalent, then  $G$  and  $G'$  have a topological coupling  $\Omega$ . Gromov's dynamical criterion [8, 0.2.C<sub>2</sub>'] shows that  $G$  and  $G'$  are quasi-isometric.  $\square$

(i) in Diagram 1 is obtained by Theorems 7.1 and 7.2. For an étale groupoid  $\mathcal{G}$  the category of left étale  $\mathcal{G}$ -spaces  $\underline{\text{Sh}}(\mathcal{G})$  is in fact a (Grothendieck) topos (see [13], and about toposes see [10]) and its equivalence class is a weak equivalence invariant of an étale groupoid, and also  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  is a weak equivalence invariant of an étale groupoid:

**Theorem 7.3** ([4, Section 2.3]). *Let  $\mathcal{G}$  and  $\mathcal{G}'$  be étale groupoids. If  $\mathcal{G}$  and  $\mathcal{G}'$  are weakly equivalent, then  $\underline{\text{Sh}}(\mathcal{G})$  and  $\underline{\text{Sh}}(\mathcal{G}')$  are equivalent as (Grothendieck) toposes, and hence  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G})$  and  $\underline{\text{Mod}}_{\mathbf{k}}(\mathcal{G}')$  are additively equivalent.*

We have (iii) in Diagram 1. For  $\underline{\text{Sh}}(\mathcal{G})$  the converse is also true:

**Theorem 7.4** ([13, 7.7 Theorem]). *Let  $\mathcal{G}$  and  $\mathcal{G}'$  be étale groupoids.  $\mathcal{G}$  and  $\mathcal{G}'$  are weakly equivalent if and only if  $\underline{\text{Sh}}(\mathcal{G})$  and  $\underline{\text{Sh}}(\mathcal{G}')$  are equivalent as (Grothendieck) toposes.*

(ii) in Diagram 1 is obtained by Theorem 7.4. We lose some information about quasi-isometry classes of finitely generated groups by (iii) in Diagram 1. Thus, we have the following problem:

PROBLEM 7.5. Let  $G$  and  $G'$  be finitely generated groups. Is it true that if  $\mathcal{R}(G, \mathbf{k})$  and  $\mathcal{R}(G', \mathbf{k})$  are Morita equivalent, then  $G$  and  $G'$  are quasi-isometric? If not, give a counter-example.

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