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EXTENDED AFFINIZATION OF INVARIANT AFFINE REFLECTION ALGEBRAS

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Abstract

The class of invariant affine reflection algebras is the most general known extension of the class of affine Kac–Moody Lie algebras, introduced in 2008. We develop a method known as "affinization" for the class of invariant affine reflection algebras, and show that starting from an algebra belonging to this class together with a certain finite order automorphism, and applying the so called "affinization method", we obtain again an invariant affine reflection algebra. This can be considered as an important step towards the realization of invariant affine reflection algebras.

1. Introduction

The class of affine Kac–Moody Lie algebras has been of great interest in the past fifty years, mostly for its applications to various areas of Mathematics and Theoretical Physics. This has been a strong motivation for mathematicians to extend this class. Among such extensions, the most important ones are the class of *extended affine Lie algebras* [1], the class of *toral type extended affine Lie algebras* [11, 21], the class of *locally extended affine Lie algebras* [18] and the most recent one which covers all of the previous ones, the class of *invariant affine reflection algebras* (IARA's for short), introduced in 2008 by E. Neher [19].

One of the central concepts of the theory of affine Kac–Moody Lie algebras and its extensions, which has captured the interest of many mathematicians, is the concept of "realization". Historically, the most popular way of realizing affine Lie algebras and their generalizations is a developed version of a method known as "*affinization*", due to V. Kac [16, Chapter 8]. Roughly speaking, the method of affinization can be described as follows. Let $\mathfrak{g}$ be a Lie algebra from a class $\mathcal{T}$, $\mathcal{A}$ the ring of Laurent polynomials, and σ a finite order automorphism of $\mathfrak{g}$. Then applying the affinization method to these data, one obtains another element $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus C \oplus D$ of the class $\mathcal{T}$, where $\tilde{\mathfrak{g}}$ is a subalgebra of the loop algebra $\mathfrak{g} \otimes \mathcal{A}$, C is a subspace contained in the center and D consists of certain derivations.

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One knows that affine Kac–Moody Lie algebras, which are extended affine Lie algebras of nullity one (see [4]), are obtained through the method of affinization starting form finite dimensional simple Lie algebras, which are extended affine Lie algebras of nullity zero. It is therefore natural to ask "whether it is possible to obtain (to realize) extended affine Lie algebras of higher nullity from the ones with lower nullity, through the affinization method. This question was positively answered by U. Pollmann [20], where she realized extended affine Lie algebras of nullity 2, up to derivations and central extensions, starting from the ones of nullity one. In the past two decades, there have been several other attempts of applying the affinization method, either directly or indirectly by using a closely related method, in order to realize extended affine Lie algebras; see for example [8, 9, 10, 22]. In [5], the method of affinization was defined in a general setting, in fact this setting provides a framework of producing new Lie algebras from the old ones in a prescribed way. The mentioned work was led to realization of almost all centerless Lie tori (see [6, 2, 3]), a class of Lie algebras characterizing the core modulo center of extended affine Lie algebras.

In this paper, we consider the method of affinization for the class of IARA's, in an extended way. Namely, in our method, the ring of Laurent polynomials is replaced with a certain associative algebra, and moreover, the way of inserting the central elements and derivations to the construction allows us to produce IARA's of arbitrary higher nullity from the ones we start with. So our work extends the results of [5], and in part [13].

The paper is organized as follows. In Section 1, we gather preliminary definitions and results needed throughout the work. In Section 2, we study two special types of gradings imposed by certain automorphisms on the underlying Lie algebras. In Sections 3 and 4, we study the effect of these gradings on so-called toral pairs in general and on IARA's in particular. In the latter case, it is shown that if the corresponding toral subalgebra is replaced with its degree zero homogeneous subspace, one gets a new IARA with a generally different root system. In Section 5, as a by-product of the results in earlier sections, we show that the fixed point subalgebra of an IARA under a certain finite order automorphism is again an IARA. This gives a new perspective to an old question, going back to [15], concerning the structure of fixed point subalgebras. Finally, Sections 6 and 7 are devoted to our results on affinization of IARA's. Roughly speaking, we show that the outcome of "affinization" of an IARA under a certain automorphism is again an IARA. We consider this as an important step towards realization of IARA's. We use our method to give examples of IARA's which are neither locally extended affine Lie algebras nor toral type extended affine Lie algebras.

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2. Preliminaries

In this section, we gather preliminary definitions and results which we need throughout the paper. In this work, all vector spaces are considered over a field $\mathbb{F}$ of characteristic zero. For any vector space W, we denote its dual space by $W^\star$. For a nonempty set S, by id_S, we mean the identity map on S and by $|S|$ the cardinal number of S. If R is an integral domain with the field of fractions Q, A an R-module and S a subset of A, we denote by $\langle S\rangle$, the R-span of S. A map $(\,\cdot\,,\,\cdot\,)\colon A\times A\to Q$ is called a *symmetric bihomomorphism* if $(\,\cdot\,,\,\cdot\,)$ is an R-module homomorphism on each component and $(a,b)=(b,a)$ for all $a,b\in A$. For a symmetric bihomomorphism $(\,\cdot\,,\,\cdot\,)\colon A\times A\to Q$, the set $A^0:=\{a\in A\mid (a,b)=0;\ \text{for all } b\in A\}$ is called the *radical* of the form $(\,\cdot\,,\,\cdot\,)$. We also set

$$S^0:=S\cap A^0 \quad\text{and}\quad S^\times:=S\setminus S^0.$$

The elements of S^0 (resp. $S^\times$) are called *isotropic* (resp. *nonisotropic*) elements of S. A subset S of A is called *indecomposable* or *connected* if $S^\times$ cannot be written as a disjoint union of two its nonempty orthogonal subsets with respect to $(\,\cdot\,,\,\cdot\,)$. In the special case when $R=\mathbb{Z}$, the bihomomorphism $(\,\cdot\,,\,\cdot\,)$ is called a *positive definite form* (resp. *positive semidefinite form*) if $(a,a)>0$ (resp. $(a,a)\geq 0$) for all nonzero $a\in A$. For a subset S of A equipped with a positive semidefinite form $(\,\cdot\,,\,\cdot\,)$, we have

$$S^0=\{\alpha\in S\mid(\alpha,\alpha)=0\}\quad\text{and}\quad S^\times=\{\alpha\in S\mid(\alpha,\alpha)\neq 0\}.$$

DEFINITION 2.1. Let $\mathfrak{g}$ be a Lie algebra and $T\subseteq\mathfrak{g}$ a subalgebra, we call T a *toral* subalgebra or an *ad-diagonalizable* subalgebra if

$$\mathfrak{g}=\bigoplus_{\alpha\in T^\star}\mathfrak{g}_\alpha(T) \tag{2.2}$$

where for any $\alpha\in T^\star$,

$$\mathfrak{g}_\alpha(T):=\{x\in\mathfrak{g}\mid[t,x]=\alpha(t)x,\ \text{for all } t\in T\}.$$

In this case $(\mathfrak{g},T)$ is called a *toral pair*, the decomposition (2.2) the *root space decomposition* of $(\mathfrak{g},T)$ and $R:=\{\alpha\in T^\star\mid\mathfrak{g}_\alpha(T)\neq 0\}$ the *root system* of $(\mathfrak{g},T)$. We will usually abbreviate $\mathfrak{g}_\alpha(T)$ by $\mathfrak{g}_\alpha$. Since any toral subalgebra is abelian, $T\subseteq\mathfrak{g}_0$ and so $0\in R$ unless $T=\{0\}=\mathfrak{g}$. A toral subalgebra is called a *splitting Cartan subalgebra* if $T=\mathfrak{g}_0$, in this case $(\mathfrak{g},T)$ is called a *split* toral pair.

Now let $(\mathfrak{g},T)$ be a toral pair with root system R, namely $\mathfrak{g}=\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$. Suppose that $\mathfrak{g}$ satisfies the following two axioms:
(IA1) $\mathfrak{g}$ has an invariant nondegenerate symmetric bilinear form $(\cdot,\cdot)$ whose restriction to T is nondegenerate.

(IA2) For each $\alpha \in R \setminus \{0\}$, there exist $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $0 \neq [e_\alpha, f_\alpha] \in T$. One can see that for each $\alpha \in R$, there exists a unique $t_\alpha \in T$ which represents α via $(\,\cdot\,,\,\cdot\,)$ (i.e. $\alpha(t) = (t_\alpha, t)$ for all $t \in T$) and that the map $\nu\colon T \to T^\star$ given by $\nu(t) = (t, \,\cdot\,)$ is a monomorphism whose image contains $\operatorname{span}_{\mathbb{F}} R$. Now it follows that the bilinear form on T can be transferred to a bilinear form on $\operatorname{span}_{\mathbb{F}} R$ defined by

$$(\alpha, \beta) = (t_\alpha, t_\beta), \text{ for all } \alpha, \beta \in \operatorname{span}_{\mathbb{F}} R.$$

Here, we record the definition of an invariant affine reflection algebra, the main object of this study.

DEFINITION 2.3 ([19, Section 6.7]). Let $(\mathfrak{g}, T)$ be a toral pair with root system R. Assume $\mathfrak{g} \neq 0$. The pair $(\mathfrak{g}, T)$ (or simply $\mathfrak{g}$) is called an *invariant affine reflection algebra* (IARA for short) if it satisfies (IA1), (IA2) as above and (IA3) below:
(IA3) For every $\alpha \in R$ with $(\alpha, \alpha) \neq 0$ and for all $x_\alpha \in \mathfrak{g}_\alpha$, the adjoint map $\operatorname{ad} x_\alpha$ is locally nilpotent on $\mathfrak{g}$.

We call an invariant affine reflection algebra $(\mathfrak{g}, T)$ *division*, if (IA2) is replaced with the stronger axiom (IA2)$'$ below:
(IA2)$'$ For each $\alpha \in R \setminus \{0\}$ and any $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, there exists $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $0 \neq [e_\alpha, f_\alpha] \in T$.

REMARK 2.4. (i) In this work, we always assume for a toral pair $(\mathfrak{g}, T)$ satisfying (IA1), the corresponding root system is not the zero set.
(ii) If $(\mathfrak{g}, T)$ is a split toral pair, then axiom (IA1) implies (IA2)$'$, in particular any invariant affine reflection algebra with a splitting Cartan subalgebra is division. To see this, one can combine Lemma 2.7 and (3.1) below.

Let us also recall the definition of an affine reflection system. This notion is due to E. Neher [19, Chapter 3] but here we state an equivalent definition given in [12, Definition 1.3].

DEFINITION 2.5. Let A be an abelian group equipped with a nontrivial symmetric positive semidefinite form $(\cdot,\cdot)$ and R be a subset of A. The triple $(A,(\cdot,\cdot),R)$, or R if there is no confusion, is called an *affine reflection system* if it satisfies the following 3 axioms:
(R1) $R = -R$,
(R2) $\langle R \rangle = A$,
(R3) for $\alpha \in R^\times$ and $\beta \in R$, there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$(\beta + \mathbb{Z}\alpha) \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\} \quad \text{and} \quad d - u = (\beta, \alpha^\vee).$$

Each element of R is called a *root*. Elements of $R^\times$ (resp. R^0) are called *non-isotropic roots* (resp. *isotropic roots*).

The affine reflection system R is called *irreducible*, if
(R4) $R^\times$ is indecomposable.
Moreover, R is called *tame*, if
(R5) $R^0 \subseteq R^\times - R^\times$ (elements of R^0 are *non-isolated*).

A *locally finite root system* is, by definition, an affine reflection system for which $A^0 = \{0\}$, see [17, 12].

REMARK 2.6. It is shown in [19] that the root system R of an IARA $(\mathfrak{g}, T)$ is an affine reflection system in the $\mathbb{Z}$-span of R. We note that as in this case $R \subseteq T^\star$ and $\mathbb{F}$ is of characteristic zero,the $\mathbb{Z}$-span of R is a torsion free abelian group.

Lemma 2.7. *Let* $(\mathfrak{g}, T)$ *be a toral pair, with root system* R, *satisfying* (IA1) *and* (IA2). *If* $\alpha \in R$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ *and* $[x, y] \in T$, *then* $[x, y] = (x, y)t_\alpha$.

Proof. We will show that $[x, y] - (x, y)t_\alpha$ is an element of the radical of the form on T; then we are done as $(\cdot\,,\,\cdot)$ is nondegenerate on T. For this, suppose $t \in T$ is arbitrary. Then

$$\begin{aligned}([x, y] - (x, y)t_\alpha, t) &= ([x, y], t) - (x, y)(t_\alpha, t)\\ &= (x, [y, t]) - (x, y)\alpha(t)\\ &= (x, \alpha(t)y) - (x, y)\alpha(t) = 0.\end{aligned}$$

□

We recall that an algebra $\mathcal{A}$ is called *G-graded*, G an abelian group, if $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$, where each $\mathcal{A}^g$ is a subspace of $\mathcal{A}$, such that $\mathcal{A}^g\mathcal{A}^h \subseteq \mathcal{A}^{g+h}$ for all $g, h \in G$. We will usually indicate this by saying "Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$ be a G-graded algebra". Each $\mathcal{A}^g$, $g \in G$, is called a *homogeneous space* and each element of $\mathcal{A}^g$ a *homogeneous element*. A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called a *graded subalgebra* if $\mathcal{B} = \bigoplus_{g\in G}(\mathcal{B} \cap \mathcal{A}^g)$. The *support* of a G-graded algebra $\mathcal{A}$ is the set $\operatorname{supp}_G \mathcal{A} := \{g \in G \mid \mathcal{A}^g \neq \{0\}\}$. We usually use superscripts to indicate homogeneous spaces, however, when $\mathcal{A}$ admits two gradings, we use subscripts to distinguish two gradings, namely $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$ and $\mathcal{A} = \bigoplus_{q\in Q} \mathcal{A}_q$. In this case, we say $\mathcal{A}$ admits a *compatible* (G, Q)*-grading* if for all $g \in G$, $\mathcal{A}^g = \bigoplus_{q\in Q} \mathcal{A}^g_q$ where $\mathcal{A}^g_q := \mathcal{A}^g \cap \mathcal{A}_q$. A bilinear form $(\cdot\,,\,\cdot)$ on a G-graded algebra $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$ is called *G-graded*, if $(\mathcal{A}^g, \mathcal{A}^h) = \{0\}$ for $g, h \in G$ with $g + h \neq 0$.

DEFINITION 2.8. Let $\mathcal{A}$ be a unital associative algebra. An element $a \in \mathcal{A}$ is called *invertible* if there exists a unique element $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = a^{-1}a = 1$. Suppose $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$ is G-graded, then it is called

- *predivision G-graded*, if every nonzero homogeneous space contains an invertible element;
- *division G-graded*, if every nonzero homogeneous element is invertible;

- an *associative G-torus*, if $\mathcal{A}$ is predivision graded and $\dim \mathcal{A}^g \leq 1$ for all $g \in G$.

We close this section by recalling some facts from representation theory of finite groups.

Let G be an arbitrary finite group. By $\mathbb{F}[G]$, we mean the group algebra of G over $\mathbb{F}$. Let $\{\chi_1, \ldots, \chi_n\}$ be the set of all irreducible characters of G in which χ_i corresponds to an irreducible module V_i. Assume $\mathbb{F}$ contains all eigenvalues of all $g \in G$ acting on V_i, $1 \leq i \leq n$. For each $1 \leq i \leq n$, define an element e_i in $\mathbb{F}[G]$, by

$$e_i := \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g, \tag{2.9}$$

in which by $|G|$ we mean the order of the group G. It follows that $\{e_1, \ldots, e_n\}$ forms a complete set of orthogonal idempotents in $\mathbb{F}[G]$, i.e. $e_i e_j = \delta_{ij} e_i$ and $e_1 + \cdots + e_n = 1$. So if M is any $\mathbb{F}[G]$-module, then

$$M = \bigoplus_{j=1}^{n} e_j \cdot M. \tag{2.10}$$

Now if $\pi_j \colon M \to e_j \cdot M$ is the projection onto $e_j \cdot M$, then $\sum_{j=1}^{n} \pi_j = \mathrm{id}$ and $\pi_i \pi_j = \delta_{ij} \pi_i$.

Suppose now that G is a finite cyclic group of order m, say $G = \{1, \sigma, \ldots, \sigma^{m-1}\}$. Assume that $\mathbb{F}$ contains an m-th primitive root of unity ζ. Since G is abelian, any finite dimensional irreducible G-module is one dimensional. Now it follows that for

$$\chi_j \colon G \to \mathbb{F}, \quad \sigma^i \mapsto \zeta^{ij}; \quad (0 \leq i, j \leq m-1),$$

$\chi_0, \ldots, \chi_{m-1}$ form a complete set of irreducible characters of G. Therefore, if M is any $\mathbb{F}[G]$-module, we have $M = \bigoplus_{j=0}^{m-1} M_j$, where $M_j := \{x \in M \mid \sigma(x) = \zeta^j x\}$, and

$$\pi_j = \frac{1}{m} \sum_{i=0}^{m-1} \zeta^{-ji} \sigma^i. \tag{2.11}$$

3. Gradings induced by automorphisms

In this section, we consider two gradings induced by a finite order automorphism on a toral pair, and study their basic properties. Let m be a fixed positive integer and suppose $\mathbb{F}$ contains an m-th primitive root of unity ζ. Throughout this section, we assume $(\mathfrak{g}, T)$ is a toral pair, with root system R, satisfying axioms (IA1) and (IA2) of an IARA. Then $\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where for each $\alpha \in R$,

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x, \text{for all } t \in T\}.$$

Also, by (IA1), $\mathfrak{g}$ is equipped with an invariant nondegenerate symmetric bilinear form $(\cdot\,,\,\cdot)$, such that the form restricted to T is nondegenerate. It is easy to see that for any $\alpha, \beta \in R$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ if $\alpha + \beta \notin R$. Also as the form is invariant, one sees that

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\} \quad \text{unless} \quad \alpha + \beta = 0, \quad (\alpha, \beta \in R), \tag{3.1}$$

and concludes that

$$(\cdot\,,\,\cdot) \text{ restricted to } \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},\ \alpha \in R, \text{ is nondegenerate.} \tag{3.2}$$

In addition, by (IA1) and (IA2) for each $\alpha \in R$, there exists a unique element $t_\alpha \in T$ such that $\alpha(t) = (t, t_\alpha)$ for all $t \in T$.

Now let σ be an automorphism of $\mathfrak{g}$ satisfying
(A1) $\sigma^m = \mathrm{id}_{\mathfrak{g}}$,
(A2) $\sigma(T) = T$,
(A3) $(\sigma(x), \sigma(y)) = (x, y)$ for all $x, y \in \mathfrak{g}$.
For $i \in \mathbb{Z}$, let $\bar{i}$ be the image of i in $\mathbb{Z}_m$ under the canonical map (for the simplicity of notation, we always denote $\bar{0}$ by 0). Then setting

$$\mathfrak{g}^{\bar{i}} := \{x \in \mathfrak{g} \mid \sigma(x) = \zeta^i x\} \tag{3.3}$$

for each $i \in \mathbb{Z}$, it is easy to see that $\mathfrak{g} = \bigoplus_{\bar{i} \in \mathbb{Z}_m} \mathfrak{g}^{\bar{i}}$ which defines a $\mathbb{Z}_m$-grading on $\mathfrak{g}$. Also by (A2), one can define a similar grading $T = \bigoplus_{\bar{i} \in \mathbb{Z}_m} T^{\bar{i}}$ on T, making T into a graded subalgebra of $\mathfrak{g}$. Using σ, we may define an automorphism, denoted again by σ, on the vector space $T^\star$ by $\sigma(\alpha) := \alpha \circ \sigma^{-1}$, $\alpha \in T^\star$. Then $\sigma^m = \mathrm{id}_{T^\star}$ and so σ induces a $\mathbb{Z}_m$-grading on $T^\star$ as above. One can easily see that for each $\alpha \in R$, $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$. Thus

$$\sigma(R) = R. \tag{3.4}$$

Note that, if $\bar{i}, \bar{j} \in \mathbb{Z}_m$, $x \in \mathfrak{g}^{\bar{i}}$ and $y \in \mathfrak{g}^{\bar{j}}$, then by (A3), $(x, y) = (\sigma(x), \sigma(y)) = (\zeta^i x, \zeta^j y) = \zeta^{i+j}(x, y)$. Thus $(x, y) = 0$ if $\overline{i+j} \neq 0$. Consequently

$$(\cdot\,,\,\cdot) \text{ is a } \mathbb{Z}_m\text{-graded bilinear form on } \mathfrak{g}. \tag{3.5}$$

For $\alpha \in R$, we define $\pi(\alpha)$ to be the restriction of α to T^0. Since we may consider any element $\beta \in (T^0)^\star$ as an element of $T^\star$ by $\beta(\sum_{\bar{i} \neq 0} T^{\bar{i}}) = 0$, we can consider $\pi(\alpha)$ as an element of $T^\star$.

For $j \in \mathbb{Z}$, let $\pi_j \colon \mathfrak{g} \to \mathfrak{g}^{\bar{j}}$ be the projection of $\mathfrak{g}$ onto $\mathfrak{g}^{\bar{j}}$ with respect to the grading $\mathfrak{g} = \sum_{\bar{j} \in \mathbb{Z}_m} \mathfrak{g}^{\bar{j}}$. We use the same notation π_j for the projection of T onto $T^{\bar{j}}$,

and $T^\star$ onto $(T^\star)^{\bar{j}}$, with respect to the $\mathbb{Z}_m$-gradings on T and $T^\star$, respectively. One observes that

$$\sigma \circ \pi_j = \pi_j \circ \sigma = \zeta^j \pi_j. \tag{3.6}$$

Since the group $\{1, \sigma, \ldots, \sigma^{m-1}\}$ acts on $\mathfrak{g}$, T and $T^\star$, the following lemma follows immediately from (2.11).

Lemma 3.7. *For any $j \in \mathbb{Z}$, we have $\pi_j = (1/m) \sum_{i=0}^{m-1} \zeta^{-ji} \sigma^i$.*

For $\alpha \in T^\star$, define

$$\mathfrak{g}_{\pi(\alpha)} := \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x, \text{ for all } t \in T^0\}. \tag{3.8}$$

Then we have $\mathfrak{g} = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathfrak{g}_{\pi(\alpha)}$ and

$$\mathfrak{g}_{\pi(\alpha)} = \sum_{\{\beta\in R \mid \pi(\beta)=\pi(\alpha)\}} \mathfrak{g}_\beta; \quad \alpha \in R. \tag{3.9}$$

Lemma 3.10. *For $\alpha \in T^\star$, $\pi(\alpha) = \pi_0(\alpha)$.*

Proof. Suppose $0 \le j \le m-1$ and $t \in T^{\bar{j}}$. Then by Lemma 3.7, we have

$$\begin{aligned}
\pi_0(\alpha)(t) &= \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha)(t) \\
&= \frac{1}{m} \sum_{i=0}^{m-1} \alpha(\sigma^{-i}(t)) \\
&= \frac{1}{m} \left(\sum_{i=0}^{m-1} \zeta^{-ji} \right) \alpha(t).
\end{aligned}$$

Now since ζ is a primitive m-th root of unity, we have $\sum_{i=0}^{m-1} \zeta^{-ji} = 0$ unless $j = 0$. Thus $\pi_0(\alpha)(t) = \alpha(t)$ for $t \in T^0$ and $\pi_0(\alpha)(t) = 0$ for $t \in \sum_{\bar{j}\neq 0} T^{\bar{j}}$. Therefore by the way $\pi(\alpha)$ is defined, we have $\pi(\alpha) = \pi_0(\alpha)$. □

We note that $\sigma(\mathfrak{g}_{\pi(\alpha)}) = \mathfrak{g}_{\sigma(\pi(\alpha))} = \mathfrak{g}_{\pi(\alpha)}$, $\alpha \in R$. Thus for $\alpha \in R$ and $j \in \mathbb{Z}$,

$$\pi_j(\mathfrak{g}_\alpha) \subseteq \pi_j(\mathfrak{g}_{\pi(\alpha)}) = \mathfrak{g}_{\pi(\alpha)}^{\bar{j}}. \tag{3.11}$$

Thanks to Lemma 3.10, we have $\pi(\alpha) = \pi_0(\alpha) = (1/m)\sum_{i=0}^{m-1} \sigma^i(\alpha)$ for $\alpha \in T^\star$, so from now on and for the simplicity of notation, we denote all projections $\mathfrak{g} \to \mathfrak{g}^0$, $T \to T^0$ and $T^\star \to (T^\star)^0$, with respect to the corresponding $\mathbb{Z}_m$-gradings, by π, that is

$$\pi = \pi_0 = \frac{1}{m}\sum_{i=0}^{m-1} \sigma^i. \tag{3.12}$$

Lemma 3.13. *Let $\gamma \in \operatorname{span}_{\mathbb{F}} R$. Then $\pi(t_\gamma) = t_{\pi(\gamma)}$ and it is the unique element in T^0 satisfying $\pi(\gamma)(t) = (t, t_{\pi(\gamma)})$ for all $t \in T^0$.*

Proof. First, we note that by (3.4) and (3.12), $\pi(\gamma) \in \operatorname{span}_{\mathbb{F}} R$. Now for $t \in T$ and $\alpha \in \operatorname{span}_{\mathbb{F}} R$, we have

$$(\sigma(t_\alpha), t) = (t_\alpha, \sigma^{-1}(t)) = \alpha(\sigma^{-1}(t)) = \sigma(\alpha)(t).$$

Thus $t_{\sigma(\alpha)} = \sigma(t_\alpha)$. Using this, we are immediately done. □

Now (3.9) together with Lemma 3.13 and the same argument as in Lemma 2.7, gives the following result.

Proposition 3.14. *The pair $(\mathfrak{g}, T^0)$ is a toral pair, with root system $\pi(R)$, satisfying axiom (IA1) of an IARA. Moreover, if $\alpha \in R$, $x \in \mathfrak{g}_{\pi(\alpha)}$, $y \in \mathfrak{g}_{-\pi(\alpha)}$ and $[x, y] \in T^0$, then $[x, y] = (x, y)t_{\pi(\alpha)}$.*

Recall that we now have two gradings on $\mathfrak{g}$, namely the $\mathbb{Z}_m$-grading induced from automorphism σ and the one induced from the set $\pi(R)$. For $\alpha \in R$ and $h \in \mathbb{Z}_m$, set

$$\mathfrak{g}^h_{\pi(\alpha)} := \mathfrak{g}^h \cap \mathfrak{g}_{\pi(\alpha)}.$$

Since the adjoint action of T^0 stabilizes $\mathfrak{g}^h$ we have

$$\mathfrak{g}^h = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathfrak{g}^h_{\pi(\alpha)}. \tag{3.15}$$

Thus the following is established.

Lemma 3.16. *The Lie algebra $\mathfrak{g}$ admits a compatible $(\langle\pi(R)\rangle, \mathbb{Z}_m)$-grading*

$$\mathfrak{g} = \bigoplus_{\gamma\in\langle\pi(R)\rangle,\, h\in\mathbb{Z}_m} \mathfrak{g}^h_\gamma$$

such that for any $h \in \mathbb{Z}_m$, $\mathfrak{g}^h_\gamma = \{0\}$ whenever $\gamma \notin \pi(R)$.

Lemma 3.17. *Let $\alpha, \beta \in R$ and $h, k \in \mathbb{Z}_m$.*
(i) *If $\pi(\alpha) + \pi(\beta) \neq 0$ then $(\mathfrak{g}_{\pi(\alpha)}, \mathfrak{g}_{\pi(\beta)}) = \{0\}$.*
(ii) *If $(\mathfrak{g}^h_{\pi(\alpha)}, \mathfrak{g}^k_{\pi(\beta)}) \neq \{0\}$, then $h + k = 0$ and $\pi(\alpha) + \pi(\beta) = 0$.*

Proof. (i) Since the form $(\cdot, \cdot)$ is invariant, a standard argument as in the finite dimensional theory, gives the result.

(ii) It follows from part (i) together with the fact that the form on $\mathfrak{g}$ is $\mathbb{Z}_m$-graded and nondegenerate. □

Next, we use Lemma 3.13 to define a bilinear form on the $\mathbb{F}$-span of $\pi(R)$ by

$$(\pi(\alpha), \pi(\beta)) := (t_{\pi(\alpha)}, t_{\pi(\beta)}) = (\pi(t_\alpha), \pi(t_\beta)).$$

We conclude this section with the following useful result which will be used in the sequel. In the following lemma, in addition to (IA1) and (IA2), we suppose that $(\mathfrak{g}, T)$ satisfies (IA3).

Lemma 3.18. *Let $(\mathfrak{g}, T)$ be an invariant affine reflection algebra. If R is indecomposable, then $\pi(R) := \{\pi(\alpha) \mid \alpha \in R\}$ is indecomposable.*

Proof. We first note that by Remark 2.6, R is an affine reflection system. So by [12, Theorem 1.13], for $\alpha \in R$, $\mathbb{Z}\alpha \subseteq R$ if and only if $\alpha \in R^0$. Therefore $\sigma(\mathrm{span}_{\mathbb{F}}(R^0)) \subseteq \mathrm{span}_{\mathbb{F}}(R^0)$. Now one only needs to adjust the proof of [13, Proposition 2.6 (ii)] to our situation. □

4. Toral pairs and automorphisms

In this section, we use the same notation as in previous sections. As in Section 3, we assume that $(\mathfrak{g}, T)$ is a toral pair, with root system R, satisfying axioms (IA1) and (IA2). We also assume that σ is an automorphism of $\mathfrak{g}$ which in addition to axioms (A1)–(A3) satisfies the following axiom:
(A4) $C_{\mathfrak{g}^0}(T^0) := \{x \in \mathfrak{g}^0 \mid [t, x] = 0;\ \text{for all } t \in T^0\} \subseteq \mathfrak{g}_0$.

Recall that, we have

$$\mathfrak{g} = \sum_{\alpha \in R} \mathfrak{g}_\alpha = \sum_{\alpha \in R} \mathfrak{g}_{\pi(\alpha)} = \sum_{h \in \mathbb{Z}_m} \mathfrak{g}^h = \sum_{\alpha \in R, h \in \mathbb{Z}_m} \mathfrak{g}_{\pi(\alpha)} \cap \mathfrak{g}^h,$$

and $T = \sum_{h \in \mathbb{Z}_m} T^h$.

For $\alpha \in R$, let $l_\sigma(\alpha)$ be the least positive integer such that $\sigma^{l_\sigma(\alpha)}(\alpha) = \alpha$, then $l_\sigma(\alpha) \mid m$ and we have the following lemma which gives an equivalent condition to (A4). The proof of this lemma is essentially similar to the proof of [5, Proposition 3.25], however for the convenience of the reader, we provide a proof here.

Lemma 4.1. *(A4) is equivalent to (A4)′ below*:
(A4)′ For $\alpha \in R \setminus \{0\}$, either $\pi(\alpha) \neq 0$ or $\{x \in \mathfrak{g}_\alpha \mid \sigma^{l_\sigma(\alpha)}(x) = x\} = \{0\}$.
Moreover if m is prime, then (A4) and (A4)′ are equivalent to
(A4)″ $\pi(\alpha) \neq 0$ for every $\alpha \in R \setminus \{0\}$.

Proof. Suppose (A4) holds but (A4)′ fails, then there exist $\alpha \in R \setminus \{0\}$ and $0 \neq x \in \mathfrak{g}_\alpha$ such that $\pi(\alpha) = 0$ and $\sigma^{l_\sigma(\alpha)}(x) = x$. Abbreviate $l_\sigma(\alpha)$ by l and let $y := x + \sigma(x) + \cdots + \sigma^{l-1}(x)$, then $\sigma(y) = y$ and so $y \in \mathfrak{g}^0$. Also since the elements $\sigma^i(x)$ $(0 \leq i \leq l-1)$ belong to different root spaces, $y \neq 0$. In addition $y \in \mathfrak{g}_{\pi(\alpha)} = \mathfrak{g}_{\pi(0)} = C_{\mathfrak{g}}(T^0)$, so $y \in C_{\mathfrak{g}^0}(T^0) \subseteq \mathfrak{g}_0$ which is a contradiction as $y \notin \mathfrak{g}_0$.

Conversely, assume (A4)′ holds and let $x = \sum_{\alpha \in R} x_\alpha \in C_{\mathfrak{g}^0}(T^0)$, where $x_\alpha \in \mathfrak{g}_\alpha$. Since $\sigma(x) = x$, $\sigma(x_\alpha) = x_{\sigma(\alpha)}$ for any $\alpha \in R$, therefore $\sigma^{l_\sigma(\alpha)}(x_\alpha) = x_\alpha$. Thus by (A4)′, for any $0 \neq \alpha \in R$ with $\pi(\alpha) = 0$, $x_\alpha = 0$. On the other hand, for every $t \in T^0$, $0 = [t, x] = \sum_{\alpha \in R} \pi(\alpha)(t) x_\alpha$. Hence $x_\alpha = 0$ for any $\alpha \in R \setminus \{0\}$ with $\pi(\alpha) \neq 0$ and so $x = x_0 \in \mathfrak{g}_0$.

Finally, suppose that m is a prime number. Clearly it suffices to show that (A4)′ implies (A4)″. Suppose to the contrary that $\pi(\alpha) = 0$ for some nonzero $\alpha \in R$. By Lemma 3.7, $\sigma(\alpha) \neq \alpha$, so $l_\sigma(\alpha) \neq 1$. Now as $l_\sigma(\alpha)$ divides m and m is prime, we have $l_\sigma(\alpha) = m$. Hence $\sigma^{l_\sigma(\alpha)}(x_\alpha) = x_\alpha$ for all $x_\alpha \in \mathfrak{g}_\alpha$ which contradicts (A4)′. □

Lemma 4.2. *Suppose $\alpha, \beta \in R$ with $\alpha \neq \beta$ and $\pi(\alpha) = \pi(\beta)$. If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\beta}$, then $\pi([x, y]) = 0$.*

Proof. If $\alpha - \beta \notin R$, there is nothing to prove, so suppose $\alpha - \beta \in R$. We have

$$[x, y] \in \mathfrak{g}_{\alpha-\beta} \subseteq \mathfrak{g}_{\pi(\alpha-\beta)} = \mathfrak{g}_{\pi(0)} = C_{\mathfrak{g}}(T^0).$$

Therefore, $\sigma^i([x, y]) \in C_{\mathfrak{g}}(T^0)$, for all i, and so $\pi([x, y]) \in C_{\mathfrak{g}^0}(T^0)$. Thus by (A4), $\pi([x, y]) \in \mathfrak{g}_0$. On the other hand, $\sigma^i([x, y]) \in \mathfrak{g}_{\sigma^i(\alpha-\beta)}$, for all i, also as $\alpha - \beta \neq 0$, we have $\sigma^i(\alpha - \beta) \neq 0$. So $\pi([x, y])$ is a sum of elements, each belongs to a root space corresponding to a nonzero root. But since $\pi([x, y]) \in \mathfrak{g}_0$, this can happen only if $\pi([x, y]) = 0$. □

Lemma 4.3. (i) *For $x, y \in \mathfrak{g}$ and $j, k \in \mathbb{Z}$, we have*

$$[\pi_j(x), \pi_k(y)] = \pi_{j+k}([x, \pi_k(y)]).$$

In particular,

$$[\pi_j(x), \pi_{-j}(y)] = \pi([x, \pi_{-j}(y)]) = \frac{1}{m} \sum_{i=0}^{m-1} \pi([x, \zeta^{ji} \sigma^i(y)]).$$

(ii) *If* $\alpha \in R$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ *and* $l := l_\sigma(\alpha)$, *then for* $j \in \mathbb{Z}$,

$$[\pi_j(x), \pi_{-j}(y)] = \frac{1}{m} \sum_{i=0}^{(m/l)-1} \pi([x, \zeta^{jil}\sigma^{il}(y)]).$$

Proof. (i) It is clear, since π_j is the projection onto $\mathfrak{g}^{\bar{j}}$ with respect to $\mathbb{Z}_m$-gradation of $\mathfrak{g}$.

(ii) Assume that α, x, y and l are as in the statement. By part (i),

$$[\pi_j(x), \pi_{-j}(y)] = \frac{1}{m}\left(\sum_{\{0 \le t \le m-1 \colon l|t\}} \pi([x, \zeta^{jt}\sigma^t(y)]) + \sum_{\{0 \le t \le m-1 \colon l \nmid t\}} \pi([x, \zeta^{jt}\sigma^t(y)]) \right).$$

So it is enough to show that $\pi([x, \sigma^t(y)]) = 0$ for all $0 \le t \le m-1$ with $l \nmid t$. Assume that $l \nmid t$. Then $\alpha - \sigma^t(\alpha) \neq 0$, $x \in \mathfrak{g}_\alpha$, $\sigma^t(y) \in \mathfrak{g}_{-\sigma^t(\alpha)}$ and $\pi(\alpha) = \pi(\sigma^t(\alpha))$, thus by Lemma 4.2, $\pi([x, \sigma^t(y)]) = 0$. □

Next, consider $\mathrm{Aut}(\mathfrak{g})$, the automorphism group of $\mathfrak{g}$. One knows that the subgroup (σ) of $\mathrm{Aut}(\mathfrak{g})$ generated by σ, acts naturally on R. We call any orbit of this action, a *σ-orbit*. Then two roots α, β belong to the same σ-orbit if and only if $\sigma^i(\alpha) = \beta$, for some i. Fix a set $\mathrm{orb}(R)$ of distinct representatives for all σ-orbits, namely $R = \biguplus_{\alpha \in \mathrm{orb}(R)} (\sigma) \cdot \alpha$. The following two lemma is of great importance for our goal.

Lemma 4.4. *Let* $0 \le j \le m-1$.
(i) *If* $\alpha, \beta \in R$ *belong to the same* σ*-orbit, then* $\pi_j(\mathfrak{g}_\alpha) = \pi_j(\mathfrak{g}_\beta)$.
(ii) *For* $\alpha \in R$,

$$\mathfrak{g}^{\bar{j}}_{\pi(\alpha)} = \sum_{\{\beta \in \mathrm{orb}(R) \mid \pi(\beta) = \pi(\alpha)\}} \pi_j(\mathfrak{g}_\beta).$$

(iii) *Let* α, β *belong to distinct* σ*-orbits of* R *with* $\pi(\alpha) = \pi(\beta)$. *If* $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\beta}$, *then* $[\pi_j(x), \pi_{-j}(y)] = 0$.

Proof. (i) Suppose $\beta = \sigma^n(\alpha)$, $n \in \mathbb{Z}$. By (3.6), $\pi_j \circ \sigma^n = \zeta^{nj}\pi_j$. Therefore

$$\pi_j(\mathfrak{g}_\beta) = \pi_j(\mathfrak{g}_{\sigma^n(\alpha)}) = \pi_j\sigma^n(\mathfrak{g}_\alpha) = \zeta^{nj}\pi_j(\mathfrak{g}_\alpha) = \pi_j(\mathfrak{g}_\alpha).$$

(ii) By Lemma 3.16, for every $1 \le j \le m-1$ and every $\alpha \in R$ we have $\mathfrak{g}^{\bar{j}}_{\pi(\alpha)} = \pi_j(\mathfrak{g}_{\pi(\alpha)})$. Now this together with (3.9) implies that

$$\mathfrak{g}^{\bar{j}}_{\pi(\alpha)} = \sum_{\{\beta \in R \mid \pi(\beta) = \pi(\alpha)\}} \pi_j(\mathfrak{g}_\beta). \tag{4.5}$$

and so the result follows immediately from part (i).

(iii) By part (i) of Lemma 4.3, $[\pi_j(x), \pi_{-j}(y)] = (1/m)\sum_{i=0}^{m-1} \pi([x, \zeta^{ji}\sigma^i(y)])$. By the assumption, for any i, $\alpha - \sigma^i(\beta) \neq 0$ and so by Lemma 4.2, $\pi([x, \zeta^{ij}\sigma^i(y)] = 0$, hence $[\pi_j(x), \pi_{-j}(y)] = 0$. □

Let $\alpha \in R$, $l := l_\sigma(\alpha)$ and $j \in \mathbb{Z}$. For $x \in \mathfrak{g}_\alpha$, we set

$$\bar{x}_j := \sum_{i=0}^{(m/l)-1} \zeta^{-jil}\sigma^{il}(x) \in \mathfrak{g}_\alpha. \tag{4.6}$$

Note that the implication $\bar{x}_j \in \mathfrak{g}_\alpha$ follows from the fact that $\sigma^l(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma^l(\alpha)} = \mathfrak{g}_\alpha$. The following observation is a key result for the rest of the work.

Lemma 4.7. *Suppose $\alpha \in R$, $l := l_\sigma(\alpha)$, $x \in \mathfrak{g}_\alpha$ and $j \in \mathbb{Z}$. Then*
(i) $\pi_j(x) = (1/m)\sum_{i=0}^{l-1} \zeta^{-ji}\sigma^i(\bar{x}_j)$,
(ii) $\pi_j(x) \neq 0$ *if and only if* $\bar{x}_j \neq 0$.

Proof. Set $k := (m/l) - 1$. Using Lemma 3.7, we have

$$\begin{aligned}
m\pi_j(x) &= \sum_{i=0}^{m-1} \zeta^{-ji}\sigma^i(x) \\
&= \sum_{i=0}^{l-1} \zeta^{-ji}\sigma^i(x) + \sum_{i=l}^{2l-1} \zeta^{-ji}\sigma^i(x) + \cdots + \sum_{i=kl}^{m-1} \zeta^{-ji}\sigma^i(x) \\
&= \sum_{s=0}^{k}\sum_{i=0}^{l-1} \zeta^{-j(sl+i)}\sigma^{sl+i}(x) \\
&= \sum_{i=0}^{l-1} \zeta^{-ji}\sigma^i\left(\sum_{s=0}^{k} \zeta^{-jsl}\sigma^{sl}(x)\right) \\
&= \sum_{i=0}^{l-1} \zeta^{-ji}\sigma^i(\bar{x}_j).
\end{aligned}$$

This proves (i).

(ii) Since for each $0 \le i \le l-1$, $\sigma^i(\bar{x}_j) \in \mathfrak{g}_{\sigma^i(\alpha)}$ and $\alpha, \sigma(\alpha), \ldots, \sigma^{l-1}(\alpha)$ are distinct roots, we concluded that $\sum_{i=0}^{l-1} \zeta^{-ji}\sigma^i(\bar{x}_j) = 0$ if and only if $\bar{x}_j = 0$. Therefore using part (i), we are done. □

5. Division IARA's and automorphisms

In this section, we use the same notation as in previous sections. We also assume that $(\mathfrak{g}, T)$ is a division IARA with root system R, that is, $(\mathfrak{g}, T)$ satisfies axioms (IA1),

(IA2)$'$ and (IA3). Further suppose that σ is an automorphism of $\mathfrak{g}$ satisfying (A1)–(A4). In Section 4, we saw hat $(\mathfrak{g}, T^0)$ is a toral pair satisfying axiom (IA1), and established several other properties of $(\mathfrak{g}, T^0)$. Our main aim in this section is to show that $(\mathfrak{g}, T^0)$ is an IARA with root system $\pi(R)$. This in particular implies that $\pi(R)$ is an affine reflection system.

Lemma 5.1. *Let $\alpha \in R$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. If $j \in \mathbb{Z}$ and $\bar{x}_j$ is defined as in* (4.6), *then*
(i) $[\pi_j(x), \pi_{-j}(y)] = (1/m)\pi([\bar{x}_j, y])$,
(ii) $(\pi_j(x), \pi_{-j}(y)) = (1/m)(\bar{x}_j, y)$.

Proof. (i) Let $k := (m/l) - 1$. By Lemma 4.3, replacing j with $-j$, α with $-\alpha$ and x with y, we have

$$\begin{aligned}
[\pi_j(x), \pi_{-j}(y)] &= \frac{1}{m}\sum_{i=0}^{k} \pi([\zeta^{-jli}\sigma^{li}(x), y]) \\
&= \frac{1}{m}\pi\left(\left[\sum_{i=0}^{k} \zeta^{-jli}\sigma^{li}(x), y\right]\right) \\
&= \frac{1}{m}\pi([\bar{x}_j, y]).
\end{aligned}$$

(ii) By Lemma 4.7 (i), $\pi_j(x) = (1/m)\sum_{i=0}^{l-1} \zeta^{-ji}\sigma^i(\bar{x}_j)$ and $\pi_{-j}(y) = (1/m)\times \sum_{i=0}^{l-1} \zeta^{ji}\sigma^i(\bar{y}_{-j})$. Also, using the definition of $l := l_\sigma(\alpha)$ and (3.1), we see that $(\mathfrak{g}_{\sigma^i(\alpha)}, \mathfrak{g}_{\sigma^j(-\alpha)}) = \{0\}$, if $0 \le i \ne j \le l-1$. Hence

$$\begin{aligned}
(\pi_j(x), \pi_{-j}(y)) &= \frac{1}{m^2}\sum_{i=0}^{l-1}(\sigma^i(\bar{x}_j), \sigma^i(\bar{y}_{-j})) \\
&= \frac{1}{m^2} l(\bar{x}_j, \bar{y}_{-j}) \\
&= \frac{1}{m^2} l \sum_{i=0}^{k}(\bar{x}_j, \zeta^{jil}\sigma^{il}(y)) \\
&= \frac{1}{m^2} l \sum_{i=0}^{k} \zeta^{jil}(\sigma^{-il}(\bar{x}_j), y) \\
&= \frac{1}{m^2} l \sum_{i=0}^{k}(\bar{x}_j, y) \\
&= \frac{1}{m}(\bar{x}_j, y).
\end{aligned}$$

□

Lemma 5.2. *Let $\alpha \in R$ with $\pi(\alpha) \neq 0$. Suppose $x \in \mathfrak{g}_\alpha$ and $\pi_j(x) \neq 0$, for some $j \in \mathbb{Z}$. Then there exists $y \in \mathfrak{g}_{-\alpha}$ such that $0 \neq [\pi_j(x), \pi_{-j}(y)] \in T^0$.*

Proof. Contemplating (4.6), Lemma 4.7 implies that $\bar{x}_j$ is a nonzero element of $\mathfrak{g}_\alpha$. Since by our assumption, the axiom (IA2)$'$ holds for $(\mathfrak{g}, T)$, there exists $y \in \mathfrak{g}_{-\alpha}$ such that $0 \neq [\bar{x}_j, y] \in T$. Therefore, by Lemma 2.7,

$$(\bar{x}_j, y) \neq 0. \tag{5.3}$$

Now combining this, Lemmas 5.1, 2.7 and 3.13, we get

$$[\pi_j(x), \pi_{-j}(y)] = \frac{1}{m}(\bar{x}_j, y)t_{\pi(\alpha)} \in T^0.$$

But as $\pi(\alpha) \neq 0$, we have $t_{\pi(\alpha)} \neq 0$, and so we are done by (5.3). □

Lemma 5.4. *Let $\alpha \in R$ with $\pi(\alpha) \neq 0$, and $j \in \mathbb{Z}$. Then for every $0 \neq e^{\bar{j}}_{\pi(\alpha)} \in \mathfrak{g}^{\bar{j}}_{\pi(\alpha)}$ there exists $f^{\bar{j}}_{\pi(\alpha)} \in \mathfrak{g}^{-\bar{j}}_{-\pi(\alpha)}$ such that $0 \neq [e^{\bar{j}}_{\pi(\alpha)}, f^{\bar{j}}_{\pi(\alpha)}] \in T^0$. In particular, axiom* (IA2) *holds for the toral pair* $(\mathfrak{g}, T^0)$.

Proof. By Lemma 4.4, $e^{\bar{j}}_{\pi(\alpha)} = \pi_j(x_1) + \cdots + \pi_j(x_n)$ where $x_i \in \mathfrak{g}_{\alpha_i}$ for some α_i's belong to distinct σ-orbits of R, satisfying $\pi(\alpha_i) = \pi(\alpha)$, for all i. Thus for some $1 \leq i \leq n$, $\pi_j(x_i) \neq 0$, and by Lemma 5.2, there exists $y \in \mathfrak{g}_{-\alpha_i}$ such that $0 \neq [\pi_j(x), \pi_{-j}(y)] \in T^0$. So using Lemma 4.3 (iii), we have

$$[e^{\bar{j}}_{\pi(\alpha)}, \pi_{-j}(y)] = [\pi_j(x_i), \pi_{-j}(y)] \in T^0 \setminus \{0\}.$$

Now setting $f^{\bar{j}}_{\pi(\alpha)} := \pi_{-j}(y)$, we get the first assertion as by (3.11), $\pi_{-j}(y) \in \mathfrak{g}^{-j}_{-\pi(\alpha)}$. To see the final assertion in the statement, let $\alpha \in R$ with $\pi(\alpha) \neq 0$. As $0 \neq \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\pi(\alpha)} = \sum_{\bar{j} \in \mathbb{Z}_m} \mathfrak{g}^{\bar{j}}_{\pi(\alpha)}$, we have $\mathfrak{g}^{\bar{j}}_{\pi(\alpha)} \neq 0$ for some $\bar{j}$. Now by the first part of the statement, there exist $e^{\bar{j}}_{\pi(\alpha)} \in \mathfrak{g}^{\bar{j}}_{\pi(\alpha)}$ and $f^{\bar{j}}_{\pi(\alpha)} \in \mathfrak{g}^{-\bar{j}}_{-\pi(\alpha)}$ such that $0 \neq [e^{\bar{j}}_{\pi(\alpha)}, f^{\bar{j}}_{\pi(\alpha)}] \in T^0$. This means that (IA2) holds for $(\mathfrak{g}, T^0)$. □

We are now ready to state the main result of this section, which extends [13, Theorem 3.4] to a rather larger class.

Theorem 5.5. *Let $(\mathfrak{g}, T)$ be a division IARA with corresponding root system R and bilinear form $(\cdot, \cdot)$. Suppose σ is an automorphism of $\mathfrak{g}$ satisfying* (A1)–(A4), *and T^0 is the set of fixed points of σ on T. For $\alpha \in R$, let $\pi(\alpha)$ be the restriction of α to T^0. Then $(\mathfrak{g}, T^0)$ is an IARA with root system $\pi(R) := \{\pi(\alpha) \mid \alpha \in R\}$. In particular, $\pi(R)$ is an affine reflection system. Moreover, if R is indecomposable, then so is $\pi(R)$.*

Proof. We have shown in Lemma 3.14 that $(\mathfrak{g}, T^0)$ is a toral pair such that $\mathfrak{g} = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathfrak{g}_{\pi(\alpha)}$, and that axiom (IA1) of Definition 2.3 holds for $(\mathfrak{g}, T^0)$. Also by Lemma 5.4, (IA2) holds for $\mathfrak{g}$. So it remains to prove (IA3).

Let $\alpha, \beta \in R$ with $(\pi(\alpha), \pi(\alpha)) \neq 0$, $x \in \mathfrak{g}_{\pi(\alpha)}$ and $y \in \mathfrak{g}_{\pi(\beta)}$. We must show that $\mathrm{ad}(x)^n(y) = 0$ for some n. We know that $\mathrm{ad}(x)^n(y) \in \mathfrak{g}_{n\pi(\alpha)+\pi(\beta)}$, so

$$[t_{\pi(\alpha)}, (\mathrm{ad}\, x)^n(y)] = (n\pi(\alpha) + \pi(\beta))(t_{\pi(\alpha)})(\mathrm{ad}\, x)^n(y).$$

Therefore if $(\mathrm{ad}\, x)^n(y)$ is nonzero, it is an eigenvector for $\mathrm{ad}\, t_{\pi(\alpha)}$ with eigenvalue $(n\pi(\alpha) + \pi(\beta))(t_{\pi(\alpha)})$. But for distinct values of n, the scalers $(n\pi(\alpha) + \pi(\beta))(t_{\pi(\alpha)})$ are distinct, so it is enough to show that $\mathrm{ad}\, t_{\pi(\alpha)}$ has a finite number of eigenvalues as an operator on $\mathfrak{g}$. One knows that each eigenvalue of $\mathrm{ad}\, t_{\pi(\alpha)}$ on $\mathfrak{g}$ is of the form $\pi(\gamma)(t_{\pi(\alpha)})$ for some $\gamma \in R$, and by Lemma 3.13,

$$\pi(\gamma)(t_{\pi(\alpha)}) = (\pi(\gamma), \pi(\alpha)) = (\gamma, \pi(\alpha)) \subseteq \frac{1}{m}(\underbrace{A + A + \cdots + A}_{m\text{-times}})$$

where $A := \{(\gamma, \beta) \mid \gamma, \beta \in R\}$. Now since R is an affine reflection system, the set A is finite; see [19, Sections 3.7, 3.8] and [17, Theorem 8.4]. Therefore $\mathrm{ad}\, t_{\pi(\alpha)}$ has only a finite number of eigenvalues. These all together show that $(\mathfrak{g}, T^0)$ is an IARA. Thus its root system $\pi(R)$ is an affine reflection system, by [19, Theorem 6.8]. The final assertion of the statement follows from Lemma 3.18. □

REMARK 5.6. Suppose $\pi(\alpha) \in \pi(R)^\times$ and $h \in \mathbb{Z}_m$. By Lemmas 5.4 and 2.7, we may choose $e^h_{\pi(\alpha)} \in \mathfrak{g}^h_{\pi(\alpha)}$ and $f^h_{\pi(\alpha)} \in \mathfrak{g}^h_{-\pi(\alpha)}$ such that $[e^h_{\pi(\alpha)}, f^h_{\pi(\alpha)}] = (e^h_{\pi(\alpha)}, f^h_{\pi(\alpha)})t_{\pi(\alpha)} \neq 0$. So multiplying $f^h_{\pi(\alpha)}$ by $2/((e^h_{\pi(\alpha)}, f^h_{\pi(\alpha)})(\pi(\alpha), \pi(\alpha)))$ we have

$$[e^h_{\pi(\alpha)}, f^h_{\pi(\alpha)}] = \frac{2t_{\pi(\alpha)}}{(\pi(\alpha), \pi(\alpha))}.$$

Thus setting $h_{\pi(\alpha)} := 2t_{\pi(\alpha)}/(\pi(\alpha), \pi(\alpha))$, the triple $\{e^h_{\pi(\alpha)}, h_{\pi(\alpha)}, f^h_{\pi(\alpha)}\}$ forms an $\mathfrak{sl}_2$-triple.

Lemma 5.7. *Let $j \in \mathbb{Z}$, $\alpha \in R \setminus \{0\}$, $\pi(\alpha) = 0$ and $\pi_j(\mathfrak{g}_\alpha) \neq \{0\}$.*
(i) *For each $x \in \mathfrak{g}_\alpha$ with $\pi_j(x) \neq 0$, there exists $y \in \mathfrak{g}_{-\alpha}$ such that $[\pi_j(x), \pi_{-j}(y)] = 0$, but $(\pi_j(x), \pi_{-j}(y)) \neq 0$.*
(ii) *There exists $e \in \mathfrak{g}^{\bar{j}}_{\pi(0)}$ and $f \in \mathfrak{g}^{-\bar{j}}_{\pi(0)}$ such that $[e, f] = 0$ but $(e, f) \neq 0$.*

Proof. (i) Let $x \in \mathfrak{g}_\alpha$ and $\pi_j(x) \neq 0$. By Lemma 4.7 (ii), we have $0 \neq \bar{x}_j \in \mathfrak{g}_\alpha$. Since $(\mathrm{IA2})'$ holds for $(\mathfrak{g}, T)$, there exists $y \in \mathfrak{g}_{-\alpha}$ such that $0 \neq [\bar{x}_j, y] \in T$. Therefore, by Lemma 2.7, $(\bar{x}_j, y) \neq 0$. Now this, together with Lemma 5.1 (ii), gives

$$(\pi_j(x), \pi_{-j}(y)) = \frac{1}{m}(\bar{x}_j, y) \neq 0.$$

On the other hand, combining Lemmas 5.1 (i), 2.7 and 3.13, we obtain

$$m[\pi_j(x), \pi_{-j}(y)] = \pi_j([\bar{x}_j, y]) = (\bar{x}_j, y)\pi(t_\alpha) = (\bar{x}_j, y)t_{\pi(\alpha)} = (\bar{x}_j, y)t_0 = 0.$$

(ii) By assumption, $\pi_j(\mathfrak{g}_\alpha) \neq 0$. So $\pi_j(x) \neq 0$ for some $x \in \mathfrak{g}_\alpha$. Now taking $e := \pi_j(x) \in \mathfrak{g}^{\bar{j}}_{\pi(0)}$ and $f := \pi_{-j}(y) \in \mathfrak{g}^{-\bar{j}}_{\pi(0)}$ as in part (i), we are done. □

As it will be revealed from the sequel, if $\mathfrak{g}_0$ is abelian, the axioms (A1)–(A4) imposed on the automorphism σ, are enough for our purposes in this work. However, this is not the case for a general IARA. To be more precise, we note that the main difference of the class of invariant affine reflection algebras with extended affine Lie algebras or locally extended affine Lie algebras, is that in the latter ones, the subspaces T and $\mathfrak{g}_0$ coincide, while in an IARA, T might be a proper subspace of $\mathfrak{g}_0$. This in particular, forces $\mathfrak{g}_0$ not to be necessarily abelian. In this case, to have a control on the action of σ on the pair $(\mathfrak{g}_0, T)$, we need the following "tameness condition" whose offshoot is given in Lemma 5.8.

(A5) If $\{0\} \neq \mathfrak{g}^{\bar{j}}_{\pi(0)} \subseteq \mathfrak{g}_0$, then $T^{\bar{j}} \neq \{0\}$, $j \in \mathbb{Z}$.

Lemma 5.8. *Suppose σ satisfies* (A1)–(A4). *Also suppose that $\mathfrak{g}_0$ is abelian or* (A5) *holds for σ. If $j \in \mathbb{Z}$ and $\mathfrak{g}^{\bar{j}}_{\pi(0)} \neq \{0\}$, then there exist $e \in \mathfrak{g}^{\bar{j}}_{\pi(0)}$ and $f \in \mathfrak{g}^{-\bar{j}}_{\pi(0)}$ such that $[e, f] = 0$, but $(e, f) \neq 0$.*

Proof. Assume $j \in \mathbb{Z}$ and $\mathfrak{g}^{\bar{j}}_{\pi(0)} \neq \{0\}$. By (3.9), $\mathfrak{g}^{\bar{j}}_{\pi(0)} = \sum_{\{\alpha \in R \mid \pi(\alpha)=0\}} \pi_j(\mathfrak{g}_\alpha)$. If $\pi_j(\mathfrak{g}_\alpha) \neq 0$ for some nonzero root α with $\pi(\alpha) = 0$, we are done by Lemma 5.7. Otherwise, $\{0\} \neq \mathfrak{g}^{\bar{j}}_{\pi(0)} = \mathfrak{g}^{j}_{0} = \pi_j(\mathfrak{g}_0) \subseteq \mathfrak{g}_0$. Now if $\mathfrak{g}_0$ is abelian, then since $(\cdot, \cdot)$ is nondegenerate and $\mathbb{Z}_m$-graded on $\mathfrak{g}_0$, there exists $e \in \mathfrak{g}^{\bar{j}}_0$ and $f \in \mathfrak{g}^{-\bar{j}}_0$ such that $(e, f) \neq 0$ but as $\mathfrak{g}_0$ is abelian $[e, f] = 0$. If (A5) holds, then, $T^{\bar{j}} = \pi_j(T) \neq 0$. Since $(\cdot, \cdot)$ is nondegenerate and $\mathbb{Z}_m$-graded on T, there exist $e \in T^{\bar{j}}$ and $f \in T^{-\bar{j}}$ such that $(e, f) \neq 0$ but as T is abelian $[e, f] = 0$. □

Assumption (A5) (Lemma 5.8) will be used to prove condition (IA2) holds for a Lie algebra $\hat{\mathfrak{g}}$ which will be introduced in Section 7.

6. Fixed point subalgebras of IARA's

An interesting subject of research on algebras is the study of subalgebra of points which are fixed by certain types of automorphisms. The starting point of such a study, in our context, is the work of Borel and Mostow [15] on semisimple Lie algebras. They showed that the subalgebra of fixed points of a finite order automorphism of a semisimple Lie algebra is a reductive Lie algebra. Motivated by this work, in [9], the authors showed that the fixed point subalgebra of an extended affine Lie algebra is a

sum of extended affine Lie algebras (up to existence of some isolated root spaces), a subspace of the center and a subspace which is contained in the centralizer of the core. They also showed that the core of the fixed point subalgebra modulo its center is isomorphic to the direct sum of the cores modulo centers of the involved summands. In [10], the authors did a similar study on the fixed points of a Lie torus under certain automorphism and obtained some similar results. In [22], the author considered the same study in a rather more general context, namely root graded Lie algebras. She proved that the core of the subalgebra of fixed points of a root graded Lie algebra under a suitable automorphism is the sum of a root graded Lie algebra $\mathcal{L}$ and a subspace $\mathcal{K}$ whose normalizer contains $\mathcal{L}$.

We now consider the same question for an IARA, namely what is the structure of fixed points of a division IARA $(\mathfrak{g}, T)$ under an automorphism σ satisfying axioms (A1)–(A4). We will show, using the results of the previous sections, that this subalgebra is a division IARA with toral subalgebra T^0. Since conditions (A1)–(A4) introduced in [9] and [10] coincide with conditions (A1)–(A4) given here, the following theorem generalizes and at the same time gives a new perspective to some of the results there.

Theorem 6.1. *Let $(\mathfrak{g}, T)$ be a division IARA with corresponding root system R and bilinear form $(\cdot,\cdot)$. Suppose σ is an automorphism of $\mathfrak{g}$ satisfying* (A1)–(A4) *and $\mathfrak{g}^0$ (resp. T^0) is the set of fixed points of σ on $\mathfrak{g}$ (resp. T). Then $(\mathfrak{g}^0, T^0)$ is a division IARA with root system*

$$R^\sigma := \{\pi(\alpha) \mid \alpha \in R,\ \mathfrak{g}^0_{\pi(\alpha)} \neq 0\}. \tag{6.2}$$

In particular, R^σ is an affine reflection system.

Proof. By Lemma 3.16,

$$\mathfrak{g}^0 = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathfrak{g}^0_{\pi(\alpha)} = \bigoplus_{\tilde{\alpha}\in R^\sigma} \mathfrak{g}^0_{\tilde{\alpha}}$$

where R^σ is given by (6.2). So $(\mathfrak{g}^0, T^0)$ is a toral pair. In addition, since by (3.5) the form $(\cdot,\cdot)$ is $\mathbb{Z}_m$-graded on $\mathfrak{g}$, it is nondegenerate on both $\mathfrak{g}^0$ and T^0, therefore (IA1) holds. Also (IA2)$'$ holds by Lemma 5.4. Next let $\alpha \in R^\sigma$ with $(\pi(\alpha), \pi(\alpha)) \neq 0$, and $x \in \mathfrak{g}^0_{\pi(\alpha)}$. By Theorem 5.5, $(\mathfrak{g}, T^0)$ is an IARA and so (IA3) holds for $(\mathfrak{g}, T^0)$. Therefore as $\mathfrak{g}^0_{\pi(\alpha)} \subseteq \mathfrak{g}_{\pi(\alpha)}$, $\operatorname{ad} x$ is locally nilpotent on $\mathfrak{g}$ and so on $\mathfrak{g}^0$. This shows that (IA3) holds for $(\mathfrak{g}^0, T^0)$ and so $(\mathfrak{g}^0, T^0)$ is a division IARA. Now R^σ as the root system of an IARA is an affine reflection system. □

REMARK 6.3. By Theorems 5.5 and 6.1, both $\pi(R)$ and R^σ are affine reflection systems with $R^\sigma \subseteq \pi(R)$. It is shown in [9] that R^σ might be a proper subset of $\pi(R)$,

and in fact in many examples this is the case. It is worth mentioning that R^σ and $\pi(R)$ might not be necessarily of the same type, see [9, Example 3.70].

7. Extended Affinization

In this section, we study *extended affinization*, a process in which starting from an IARA $\mathfrak{g}$ with root system R and a finite order automorphism of $\mathfrak{g}$, we get a new IARA whose root system is an extension of $\pi(R)$ (see (3.12)). The notion of *affinization* was initiated by V. Kac [16] in order to realize affine Kac–Moody Lie algebras. Since then, this method has been used by different authors to realize certain generalizations of affine Lie algebras, e.g. in [5], the authors use this method to realize extended affine Lie algebras, also in [2] and [3], using this method, the authors realize Lie tori.

Throughout this section, $(\mathfrak{g}, T)$ is an IARA with root system R, σ an automorphism of $\mathfrak{g}$ satisfying (A1)–(A3), and T^0 the set of fixed points of σ on T. We recall that for $\alpha \in R$, $\pi(\alpha)$ is the restriction of α to T^0 and that we have a $(\langle \pi(R) \rangle, \mathbb{Z}_m)$-grading on $\mathfrak{g}$ as in Lemma 3.16. Suppose Λ is a torsion free abelian group and let $\rho\colon \Lambda \to \mathbb{Z}_m$ be a group epimorphism. For $\lambda \in \Lambda$, we take $\bar{\lambda} := \rho(\lambda)$.

Suppose $\mathcal{A}$ is a unital commutative associative algebra. In addition, suppose $\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}^\lambda$ is predivision Λ-graded. It is easy to see that in this case $\operatorname{supp}_\Lambda(\mathcal{A})$ is a subgroup of Λ. Since the Λ-grading of $\mathfrak{g}$ depends only on $\operatorname{supp}_\Lambda(\mathcal{A})$, we may assume without loss of generality that $\Lambda = \operatorname{supp}_\Lambda(\mathcal{A})$, that is,

$$\mathcal{A}^\lambda \neq \{0\} \quad \text{for all} \quad \lambda \in \Lambda. \tag{7.1}$$

Further assume that $\mathcal{A}$ admits a Λ-graded invariant nondegenerate symmetric bilinear form ϵ, where "invariant" means $\epsilon(ab, c) = \epsilon(a, bc)$ for all $a, b, c \in \mathcal{A}$. In addition, we assume that

$$\epsilon(1, 1) \neq 0. \tag{7.2}$$

One gets using this that $\epsilon(a, a^{-1}) \neq 0$ for all invertible elements $a \in \mathcal{A}$ as the form is invariant. We now consider the Lie algebra $\mathfrak{g} \otimes \mathcal{A}$ with multiplication defined by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for every $x, y \in \mathfrak{g}$ and $a, b \in \mathcal{A}$. Now define a form on $\mathfrak{g} \otimes \mathcal{A}$ by linear extension of

$$(x \otimes a, y \otimes b) = (x, y)\epsilon(a, b), \tag{7.3}$$

for $x, y \in \mathfrak{g}$ and $a, b \in \mathcal{A}$. It is easy to see that this form is a Λ-graded invariant symmetric bilinear form on $\mathfrak{g} \otimes \mathcal{A}$.

The following is a slight generalization of [2, Definition 3.1.1].

DEFINITION 7.4. The subalgebra

$$\tilde{\mathfrak{g}} := L_\rho(\mathfrak{g}, \mathcal{A}) := \bigoplus_{\lambda \in \Lambda} (\mathfrak{g}^{\bar{\lambda}} \otimes \mathcal{A}^\lambda)$$

of $\mathfrak{g} \otimes \mathcal{A}$ is called the *loop algebra* of $\mathfrak{g}$ relative to ρ and $\mathcal{A}$. In the case that $\rho = 0$, we denote $L_\rho(\mathfrak{g}, \mathcal{A})$ by $L(\mathfrak{g}, \mathcal{A})$ and note that $L(\mathfrak{g}, \mathcal{A}) = \mathfrak{g} \otimes \mathcal{A}$.

From definition, it is clear that $\tilde{\mathfrak{g}}$ is a Λ-graded Lie algebra with homogenous spaces $\tilde{\mathfrak{g}}^\lambda := \mathfrak{g}^{\bar{\lambda}} \otimes \mathcal{A}^\lambda$, $\lambda \in \Lambda$.

In the following lemma, we make use of a fact from linear algebra, namely if V is a vector space equipped with a nondegenerate symmetric bilinear from and W a finite dimensional subspace of V, then there is a finite dimensional subspace U of V containing W such that the form restricted to U is nondegenerate (for a proof see [18, Lemma 3.6]).

Lemma 7.5. *The form on $\mathfrak{g} \otimes \mathcal{A}$ restricted to $\tilde{\mathfrak{g}}$ is a Λ-graded invariant nondegenerate symmetric bilinear form.*

Proof. As we have seen above the form on $\mathfrak{g} \otimes \mathcal{A}$ is Λ-graded symmetric and invariant. So it remains to prove the nondegeneracy of the form. Since $(\cdot, \cdot)$ is Λ-graded on $\tilde{\mathfrak{g}}$, it is enough to show that for fixed $\lambda \in \Lambda$ and $0 \neq \tilde{x} \in \mathfrak{g}^{\bar{\lambda}} \otimes \mathcal{A}^\lambda$, there exists $\tilde{y} \in \mathfrak{g}^{-\bar{\lambda}} \otimes \mathcal{A}^{-\lambda}$ such that $(\tilde{x}, \tilde{y}) \neq 0$. Now we may write $\tilde{x} = \sum_{i=1}^n x_i \otimes a_i$, where $\{a_1, \ldots, a_n\}$ is a linearly independent subset of $\mathcal{A}^\lambda$ and $x_i \in \mathfrak{g}^{\bar{\lambda}}$ for $1 \leq i \leq n$. Since ϵ is nondegenerate on $\mathcal{A}^\lambda \oplus \mathcal{A}^{-\lambda}$, there exists a finite dimensional subspace X of $\mathcal{A}^\lambda \oplus \mathcal{A}^{-\lambda}$ such that $\{a_1, \ldots, a_n\} \subseteq X$ and that the form restricted to X is nondegenerate. Extend $\{a_1, \ldots, a_n\}$ to a basis $\{a_1, \ldots, a_n, a_{n+1}, \ldots, a_m\}$ of X. Now as ϵ is nondegenerate on X, there exist $b_1, \ldots, b_m \in X$ such that $\epsilon(a_i, b_j) = \delta_{ij}$ for all i, j. For $1 \leq j \leq n$, let $\bar{b}_j$ be the projection of b_j into $\mathcal{A}^{-\lambda}$ with respect to the decomposition $\mathcal{A}^\lambda \oplus \mathcal{A}^{-\lambda}$. Since ϵ is Λ-graded and $a_1, \ldots, a_n \in \mathcal{A}^\lambda$, we have

$$(a_i, \bar{b}_j) = (a_i, b_j) = \delta_{i,j} \quad \text{for all} \quad 1 \leq i, j \leq n.$$

Now $x_j \neq 0$ for some j, as $\tilde{x} \neq 0$. Since $(\cdot, \cdot)$ is nondegenerate and $\mathbb{Z}_m$-graded on $\mathfrak{g}^{\bar{\lambda}} \oplus \mathfrak{g}^{-\bar{\lambda}}$, there exists $y_j \in \mathfrak{g}^{-\bar{\lambda}}$ such that $(x_j, y_j) \neq 0$. So, setting $\tilde{y} := y_j \otimes \bar{b}_j$, we have

$$\begin{aligned}
(\tilde{x}, \tilde{y}) &= \left(\sum_{i=1}^n x_i \otimes a_i, y_j \otimes \bar{b}_j \right) \\
&= \sum_{i=1}^n (x_i, y_j) \epsilon(a_i, \bar{b}_j) \\
&= (x_j, y_j) \epsilon(a_j, \bar{b}_j) \\
&= (x_j, y_j) \neq 0,
\end{aligned}$$

as required. This shows that the form on $\tilde{\mathfrak{g}}$ is nondegenerate. $\square$

Next suppose $\lambda \in \Lambda$, then by Proposition 3.14 and (3.15), we have

$$\tilde{\mathfrak{g}}^{\lambda} = \mathfrak{g}^{\bar{\lambda}} \otimes \mathcal{A}^{\lambda} = \bigoplus_{\pi(\alpha)\in\pi(R)} (\mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^{\lambda}). \tag{7.6}$$

Now we set

$$\tilde{T} := T^0 \otimes 1.$$

Then for $\alpha \in R$, $\pi(\alpha)$ can be considered as an element of $\tilde{T}^{\star}$ by linear extension of $\pi(\alpha)(t \otimes 1) = \alpha(t)$ for $t \in T^0$. We consider the adjoint action of $\tilde{T}$ on $\tilde{\mathfrak{g}}$. Suppose $t \in T^0$, $x \in \mathfrak{g}^{\bar{\lambda}}$ and $a \in \mathcal{A}^{\lambda}$, for some $\lambda \in \Lambda$. We have

$$[t \otimes 1, x \otimes a] = [t, x] \otimes a \in \mathfrak{g}^{\bar{\lambda}} \otimes \mathcal{A}^{\lambda}.$$

So the adjoint action of $\tilde{T}$ on $\tilde{\mathfrak{g}}$ stabilizes $\mathfrak{g}^{\bar{\lambda}} \otimes \mathcal{A}^{\lambda}$. Define, for $\alpha \in R$,

$$\tilde{\mathfrak{g}}_{\pi(\alpha)} := \{x \in \tilde{\mathfrak{g}} \mid [\tilde{t}, x] = \pi(\alpha)(\tilde{t})x \text{ for all } \tilde{t} \in \tilde{T}\}.$$

Then it is easy to check that $\mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^{\lambda} \subseteq \tilde{\mathfrak{g}}_{\pi(\alpha)}$ for $\alpha \in R$ and $\lambda \in \Lambda$. So by (7.6),

$$\begin{aligned} \tilde{\mathfrak{g}} &= \bigoplus_{\lambda\in\Lambda} \tilde{\mathfrak{g}}^{\lambda} = \bigoplus_{\lambda\in\Lambda} \bigoplus_{\pi(\alpha)\in\pi(R)} (\mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^{\lambda}) \\ &= \bigoplus_{\pi(\alpha)\in\pi(R)} \bigoplus_{\lambda\in\Lambda} (\mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^{\lambda}) \\ &\subseteq \bigoplus_{\pi(\alpha)\in\pi(R)} \tilde{\mathfrak{g}}_{\pi(\alpha)} \subseteq \tilde{\mathfrak{g}}. \end{aligned} \tag{7.7}$$

Thus we have

$$\tilde{\mathfrak{g}} = \bigoplus_{\pi(\alpha)\in\pi(R)} \tilde{\mathfrak{g}}_{\pi(\alpha)} \tag{7.8}$$

with

$$\tilde{\mathfrak{g}}_{\pi(\alpha)} = \bigoplus_{\lambda\in\Lambda} (\mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^{\lambda}). \tag{7.9}$$

Therefore we have the following lemma.

Lemma 7.10. *$\tilde{\mathfrak{g}}$ admits a compatible $(\langle\pi(R)\rangle, \Lambda)$-grading*

$$\tilde{\mathfrak{g}} = \bigoplus_{\lambda\in\Lambda, \gamma\in\langle\pi(R)\rangle} \mathfrak{g}^{\bar{\lambda}}_{\gamma} \otimes \mathcal{A}^{\lambda}$$

where for any $\lambda \in \Lambda$, $\tilde{\mathfrak{g}}_\gamma^\lambda = \{0\}$ *if* $\gamma \notin \pi(R)$.

Consider the $\mathbb{F}$-vector space

$$\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} \Lambda. \tag{7.11}$$

Since Λ is torsion free, we may identify Λ with the subspace $1 \otimes \Lambda$ of $\mathcal{V}$. Now as Λ spans $\mathcal{V}$, it contains a basis $\{\lambda_i \mid i \in I\}$ of $\mathcal{V}$. For any $i \in I$, set $d_i \in \mathcal{V}^\star$ by $d_i(\lambda_j) := \delta_{ij}$, $j \in I$, and let $\mathcal{V}^\dagger$ be the restricted dual of $\mathcal{V}$ with respect to the basis $\{\lambda_i \mid i \in I\}$, namely

$$\mathcal{V}^\dagger := \operatorname{span}_{\mathbb{F}} \{d_i \mid i \in I\} \subseteq \mathcal{V}^\star. \tag{7.12}$$

Define

$$\hat{\mathfrak{g}} := \widehat{L_\rho(\mathfrak{g}, \mathcal{A})} := \tilde{\mathfrak{g}} \oplus \mathcal{V} \oplus \mathcal{V}^\dagger \quad \text{and} \quad \hat{T} := \tilde{T} \oplus \mathcal{V} \oplus \mathcal{V}^\dagger = (T^0 \otimes 1) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger. \tag{7.13}$$

If $\rho = 0$, we denote $\hat{\mathfrak{g}}$ by $\widehat{L(\mathfrak{g}, \mathcal{A})}$. We make $\hat{\mathfrak{g}}$ into a Lie algebra by letting the Lie bracket be

$$\begin{aligned} &[d, x] = d(\lambda)x, \quad d \in \mathcal{V}^\dagger,\ x \in \tilde{\mathfrak{g}}^\lambda,\ \lambda \in \Lambda, \\ &[\mathcal{V}, \hat{\mathfrak{g}}] = \{0\}, \\ &[x, y] = [x, y]_{\tilde{\mathfrak{g}}} + \sum_{i \in I} ([d_i, x], y)\lambda_i, \quad x, y \in \tilde{\mathfrak{g}}, \end{aligned} \tag{7.14}$$

where by $[\,\cdot\,,\,\cdot\,]_{\tilde{\mathfrak{g}}}$ and $(\,\cdot\,,\,\cdot\,)$, we mean the Lie bracket and the bilinear form on $\tilde{\mathfrak{g}}$, respectively. Note that for each $x, y \in \tilde{\mathfrak{g}}$, $\sum_{i \in I}([d_i, x], y)\lambda_i$ makes sense as $[d_i, x] = 0$, for all but a finite number of $i \in I$. We next extend the form on $\tilde{\mathfrak{g}}$ to a bilinear form on $\hat{\mathfrak{g}}$ by

$$\begin{aligned} &(\mathcal{V}, \mathcal{V}) = (\mathcal{V}^\dagger, \mathcal{V}^\dagger) = (\mathcal{V}, \tilde{\mathfrak{g}}) = (\mathcal{V}^\dagger, \tilde{\mathfrak{g}}) := \{0\}, \\ &(v, d) = (d, v) := d(v), \quad d \in \mathcal{V}^\dagger,\ v \in \mathcal{V}. \end{aligned} \tag{7.15}$$

The above form is clearly nondegenerate on $\hat{\mathfrak{g}}$. For any $\lambda \in \Lambda$, define $\delta_\lambda \in \hat{T}^\star$ by

$$\delta_\lambda((T \otimes 1) \oplus \mathcal{V}) = \{0\}, \quad \delta_\lambda(d) = (\lambda, d), \quad d \in \mathcal{V}^\dagger.$$

Then the assignment $\lambda \mapsto \delta_\lambda$ affords an embedding of Λ into $\hat{T}^\star$, by the nondegeneracy of $(\,\cdot\,,\,\cdot\,)$. So we may identify λ with δ_λ and suppose that $\Lambda \subseteq \hat{T}^\star$.

For $\alpha \in R$, one can extend $\pi(\alpha) \in \pi(R)$ to $\hat{T}^\star$ by defining $\pi(\alpha)(\mathcal{V} \oplus \mathcal{V}^\dagger) := \{0\}$. Now let $x \in \tilde{\mathfrak{g}}^\lambda_{\pi(\alpha)}$, $\lambda \in \Lambda$, $\alpha \in R$, and $(t \otimes 1) + v + \bar{v} \in \hat{T}$, $t \in T^0$, $v \in \mathcal{V}$, $\bar{v} \in \mathcal{V}^\dagger$, then

$$\begin{aligned}[(t \otimes 1) + v + \bar{v}, x] &= [t \otimes 1, x]_{\tilde{\mathfrak{g}}} + \bar{v}(\lambda)x \\ &= (\pi(\alpha)(t \otimes 1) + \bar{v}(\lambda))x \\ &= (\pi(\alpha) + \lambda)((t \otimes 1) + v + \bar{v})x.\end{aligned}$$

This shows that

$$\hat{\mathfrak{g}} = \bigoplus_{\tilde{\alpha} \in \hat{T}^\star} \hat{\mathfrak{g}}_{\tilde{\alpha}}, \tag{7.16}$$

where

$$\hat{\mathfrak{g}}_{\tilde{\alpha}} := \{x \in \hat{\mathfrak{g}} \mid [\hat{t}, x] = \tilde{\alpha}(\hat{t})x \text{ for all } \hat{t} \in \hat{T}\}.$$

That is, $(\hat{\mathfrak{g}}, \hat{T})$ is a toral pair. Moreover, if $\hat{R}$ is the root system of $(\hat{\mathfrak{g}}, \hat{T})$, then

$$\hat{R} \subseteq \pi(R) \oplus \Lambda, \tag{7.17}$$

and for $\alpha \in R$ and $\lambda \in \Lambda$,

$$\hat{\mathfrak{g}}_{\pi(\alpha)+\lambda} = \begin{cases} \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^\lambda & \text{if} \quad \pi(\alpha) + \lambda \neq 0, \\ (\mathfrak{g}^0_{\pi(0)} \otimes \mathcal{A}^0) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger & \text{if} \quad \pi(\alpha) + \lambda = 0. \end{cases} \tag{7.18}$$

Next for $\lambda \in \Lambda$, we put

$$R_{\bar{\lambda}} := \{\alpha \in R \mid \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \neq \{0\}\}, \tag{7.19}$$

then it follows from Lemma 3.16, (7.16), (7.18), (7.8), (7.9) and (7.1) that

$$R = \bigcup_{\lambda \in \Lambda} R_{\bar{\lambda}} \quad \text{and} \quad \hat{R} = \bigcup_{\lambda \in \Lambda} (\pi(R_{\bar{\lambda}}) + \lambda). \tag{7.20}$$

Now we can prove the main theorem of this section which is a rather comprehensive extension of [5, Theorem 3.63].

Theorem 7.21. *Let $(\mathfrak{g}, T)$ be a division IARA with corresponding root system R. Suppose σ is an automorphism of $\mathfrak{g}$ satisfying* (A1)–(A4). *Assume further that either* (A5) *holds or $\mathfrak{g}_0$ is abelian. Suppose Λ is a torsion free abelian group and $\rho\colon \Lambda \to \mathbb{Z}_m$ a group epimorphism. In addition, let $\mathcal{A}$ be a unital commutative associative predivision Λ-graded algebra, with* $\operatorname{supp}_\Lambda(\mathcal{A}) = \Lambda$. *Then $(\hat{\mathfrak{g}} = \widehat{L_\rho(\mathfrak{g}, \mathcal{A})}, \hat{T})$ is an IARA with root system $\hat{R} = \bigcup_{\lambda \in \Lambda}(\pi(R_{\bar{\lambda}}) + \lambda)$. Moreover, if R is indecomposable then so is $\hat{R}$. Finally, if T is a splitting Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{A}^0 = \mathbb{F}$, then $\hat{T}$ is also a splitting Cartan subalgebra of $\hat{\mathfrak{g}}$.*

Proof. We have already seen that $(\hat{\mathfrak{g}}, \hat{T})$ is a toral pair, so it remains to verify conditions (IA1)–(IA3) of Section 2. We know that the form introduced by (7.15) on $\hat{\mathfrak{g}}$ is nondegenerate on both $\hat{\mathfrak{g}}$ and $\hat{T}$ and so (IA1) holds for $\hat{\mathfrak{g}}$.

We next show that (IA2) holds. Assume that $\alpha \in R$, $\lambda \in \Lambda$, $\pi(\alpha) + \lambda \neq 0$ and $\hat{\mathfrak{g}}_{\pi(\alpha)+\lambda} \neq 0$. By (7.18), $\hat{\mathfrak{g}}_{\pi(\alpha)+\lambda} = \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^{\lambda}$, so $\mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \neq 0$ and $\mathcal{A}^{\lambda} \neq 0$. As $\mathcal{A}$ is predivision Λ-graded, there exists $a \in \mathcal{A}^{\lambda}$ and $b \in \mathcal{A}^{-\lambda}$ such that $ab = 1$. To proceed with the proof, we divide the argument into two cases $\pi(\alpha) \neq 0$ and $\pi(\alpha) = 0$. Assume first that $\pi(\alpha) \neq 0$, then by Lemma 5.4, there exist $0 \neq x \in \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)}$ and $0 \neq y \in \mathfrak{g}^{-\bar{\lambda}}_{-\pi(\alpha)}$ such that $0 \neq [x, y] \in T^0$, and thus

$$[x \otimes a, y \otimes b] = ([x, y] \otimes 1) + \sum_{i \in I} ([d_i, x \otimes a], y \otimes b)\lambda_i \in \hat{T} \setminus \{0\},$$

as required.

Next, assume $\pi(\alpha) = 0$, then by Lemma 5.8, there exist $x \in \mathfrak{g}^{\bar{\lambda}}_{\pi(0)}$ and $y \in \mathfrak{g}^{-\bar{\lambda}}_{\pi(0)}$ such that $[x, y] = 0$ but $(x, y) \neq 0$. So we have

$$\begin{aligned}[x \otimes a, y \otimes b] &= ([x, y] \otimes 1) + \sum_{i \in I} ([d_i, x \otimes a], y \otimes b)\lambda_i \\ &= 0 + \sum_{i \in I} \epsilon(a, b)\, d_i(\lambda)(x, y)\lambda_i.\end{aligned}$$

This is a nonzero element of $\hat{T}$ since $(x, y) \neq 0$, $\epsilon(a, b) = \epsilon(1, 1) \neq 0$ and as $\lambda = 0 + \lambda = \pi(\alpha) + \lambda \neq 0$, $d_i(\lambda) \neq 0$ for some $i \in I$. This means that (IA2) holds for $\hat{\mathfrak{g}}$.

Finally, we consider (IA3). Let $\alpha \in R$, $\lambda \in \Lambda$ and $(\pi(\alpha) + \lambda, \pi(\alpha) + \lambda) \neq 0$. As $(\lambda, \lambda) = (\lambda, \pi(\alpha)) = 0$, we have $(\pi(\alpha), \pi(\alpha)) \neq 0$. Since by Theorem 5.5, $\pi(R)$ is an affine reflection system, one can use a similar technique as in the proof of Theorem 5.5 to show that $\mathrm{ad}(x)$ is locally nilpotent for any $x \in \hat{\mathfrak{g}}_{\pi(\alpha)+\lambda}$. So $\hat{\mathfrak{g}}$ satisfies (IA3) and $\hat{\mathfrak{g}}$ is an IARA. Moreover, the root system $\hat{R}$ of $(\hat{\mathfrak{g}}, \hat{T})$ satisfies $\hat{R} = \bigcup_{\lambda \in \Lambda} (\pi(R_{\bar{\lambda}}) + \lambda)$, by (7.20).

Next, suppose R is indecomposable. Since Λ is contained in the radical of the form, $\hat{R}$ is indecomposable if and only if $\cup_{\lambda \in \Lambda} \pi(R_{\bar{\lambda}})$ is indecomposable. But by (7.20) this union is $\pi(R)$ which is indecomposable by Lemma 3.18.

To see the final assertion of the theorem, we note that if $\mathfrak{g}_0 = T$, then by (A4), $C_{\mathfrak{g}^0}(T^0) = T^0$. Therefore as $\mathcal{A}^0 = \mathbb{F}$, using (7.18), we have

$$\begin{aligned}\hat{\mathfrak{g}}_0 &= ((\mathfrak{g}_{\pi(0)} \cap \mathfrak{g}^0) \otimes \mathcal{A}^0) \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger} \\ &= (C_{\mathfrak{g}^0}(T^0) \otimes 1) \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger} \\ &= (T^0 \otimes 1) \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger} = \hat{T}.\end{aligned}$$

Thus $\hat{T}$ is a splitting Cartan subalgebra of $\hat{\mathfrak{g}}$ as required. □

Corollary 7.22. *Let* $(\mathfrak{g}, T)$ *be an IARA with corresponding root system* R *and bilinear from* $(\cdot\,,\,\cdot)$. *Let* Λ *be a torsion free abelian group and* $\mathcal{A}$ *be a unital commutative associative predivision* Λ*-graded algebra, with* $\mathrm{supp}_\Lambda(\mathcal{A}) = \Lambda$. *Define* $\hat{\mathfrak{g}} := (\mathfrak{g} \otimes \mathcal{A}) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$ *and* $\hat{T} := (T \otimes 1) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$, *where* $\mathcal{V}$ *and* $\mathcal{V}^\dagger$ *are defined as in* (7.11) *and* (7.12), *respectively. Then* $(\hat{\mathfrak{g}}, \hat{T})$ *is an IARA with root system* $\hat{R} = R \oplus \Lambda$. *Moreover, if* R *is indecomposable then so is* $\hat{R}$.

Proof. Taking σ to be the identity automorphism and recalling from Remark 2.4 that $T \neq \{0\}$, it is apparent that σ satisfies conditions (A1)–(A5). Therefore, if $(\mathfrak{g}, T)$ is division, we are done by Theorem 7.21. Now a close look at the proof of Theorem 7.21, shows that the division property, that is (IA2)$'$, guarantees the existence of nonzero elements $x \in \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)}$ and $y \in \mathfrak{g}^{-\bar{\lambda}}_{-\pi(\alpha)}$ ($\alpha \in R$ and $\lambda \in \Lambda$ with $\pi(\alpha) + \lambda \neq 0$) such that

$$\begin{cases} [x, y] \in T^0 \setminus \{0\} & \text{if} \quad \pi(\alpha) \neq 0, \\ [x, y] = 0 \text{ and } (x, y) \neq 0 & \text{if} \quad \pi(\alpha) = 0. \end{cases} \tag{7.23}$$

However when σ is the identity automorphism, (7.23) clearly holds with the weaker axiom (IA2). Finally, since σ is the identity automorphism, it follows immediately from Theorem 7.21 that $\hat{R} = R \oplus \Lambda$ □

Corollary 7.24. *Let* $(\mathfrak{g}, T)$ *be a division IARA with corresponding root system* R. *Suppose* σ *is an automorphism of* $\mathfrak{g}$ *satisfying* (A1)–(A4). *Assume further that* $\mathfrak{g}_0$ *is abelian. Suppose* Λ *is a torsion free abelian group and* $\rho\colon \Lambda \to \mathbb{Z}_m$ *a group epimorphism. In addition, let* $\mathcal{A}$ *be a commutative associative* Λ*-torus, with* $\mathrm{supp}_\Lambda(\mathcal{A}) = \Lambda$. *Then* $(\hat{\mathfrak{g}} = \widehat{L_\rho(\mathfrak{g}, \mathcal{A})}, \hat{T})$ *is a division IARA, with root system* $\hat{R}$.

Proof. By Theorem 7.21, $(\hat{\mathfrak{g}}, \hat{T})$ is an IARA. So the only condition which we should verify is (IA2)$'$. Suppose $\alpha \in R$, $\lambda \in \Lambda$, $\pi(\alpha) + \lambda \neq 0$ and $\hat{\mathfrak{g}}_{\pi(\alpha)+\lambda} = \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)} \otimes \mathcal{A}^\lambda \neq \{0\}$. Since $\mathcal{A}$ is a Λ-torus, $\mathcal{A}^\lambda$ is one dimensional, say $\mathcal{A}^\lambda = \mathrm{span}_{\mathbb{F}}\{a\}$, where a is invertible with inverse b. Then any element of $\hat{\mathfrak{g}}_{\pi(\alpha)+\lambda}$ is of the form $x \otimes a$ for some $0 \neq x \in \mathfrak{g}^{\bar{\lambda}}_{\pi(\alpha)}$. Now fix a nonzero element $x \otimes a \in \hat{\mathfrak{g}}_{\pi(\alpha)+\lambda}$. If $\pi(\alpha) \neq 0$, then by Lemma 5.4, there exists $y \in \mathfrak{g}^{-\bar{\lambda}}_{-\pi(\alpha)}$ such that $0 \neq [x, y] \in T^0$. So as $[x, y] \neq 0$, we have

$$[x \otimes a, y \otimes b] = ([x, y] \otimes 1) + \sum_{i \in I} ([d_i, x], y)\lambda_i \in \hat{T} \setminus \{0\}.$$

Now suppose $\pi(\alpha) = 0$. We claim that there exists $y \in \mathfrak{g}^{-\bar{\lambda}}_{\pi(0)}$ such that $[x, y] = 0$ and $(x, y) \neq 0$. For this, take $j \in \mathbb{Z}$ such that $\bar{j} = \bar{\lambda}$. By Lemma 4.4 (ii), we have

$$\mathfrak{g}^{\bar{\lambda}}_{\pi(0)} = \mathfrak{g}^{\bar{j}}_{\pi(0)} = \sum_{\{\beta \in \mathrm{orb}(R) \mid \pi(\beta) = 0\}} \pi_j(\mathfrak{g}_\beta).$$

Therefore, $x = \pi_j(x_1)+\cdots+\pi_j(x_n)$ where $x_i \in \mathfrak{g}_{\alpha_i}$ for some $\alpha_1,\ldots,\alpha_n$ belong to distinct σ-orbits of R and $\pi(\alpha_i) = 0$, $1 \le i \le n$. As $x \neq 0$, $\pi_j(x_k) \neq 0$ for some $\mathbb{F}$. Now if $\alpha_k \neq 0$, then by Lemma 5.7, there exists $y_k \in \mathfrak{g}_{-\alpha_k}$ such that $(\pi_j(x_k), \pi_{-j}(y_k)) \neq 0$ and $[\pi_j(x_k), \pi_{-j}(y_k)] = 0$. Set $y := \pi_{-j}(y_k)$, then by Lemma 4.3 (iii), we have $[x, y] = [\pi_j(x_k), \pi_{-j}(y_k)] = 0$. Also by Lemma 5.1 (ii), we have

$$(x, y) = \sum_{i=1}^{n}(\pi_j(x_i), \pi_{-j}(y_k)) = \frac{1}{m}\sum_{i=1}^{n}((\overline{x_i})_j, y_k) = (\pi_j(x_k), \pi_{-j}(y_k)) \neq 0,$$

where considering (4.6), we note that for $i \neq k$, $((\overline{x_i})_j, y_k) \in (\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_k}) = \{0\}$, as $\alpha_i - \alpha_k \neq 0$. So we are done in the case $\alpha_k \neq 0$.

Next, suppose that $\alpha_k = 0$. Then $0 \neq \pi_j(x_k) \in \pi_j(\mathfrak{g}_0) \subseteq \mathfrak{g}_{\pi(0)}^{\bar{j}}$. Since $(\,\cdot\,,\,\cdot\,)$ is nondegenerate on $\mathfrak{g}_0$, there exists $y \in \mathfrak{g}_0$ such that $(\pi_j(x_k), y) \neq 0$. But as $(\,\cdot\,,\,\cdot\,)$ is $\mathbb{Z}_m$-graded, we may assume that $y = \pi_{-j}(y) \in \mathfrak{g}_0^{-\bar{j}} \subseteq \mathfrak{g}_{\pi(0)}^{-\bar{j}}$. Then $[\pi_j(x), y] = 0$, as by assumption $\mathfrak{g}_0$ is abelian. Now repeating the same argument as in the case $\alpha_k \neq 0$ (using Lemmas 4.3 (iii) and 5.1), we get $(x, y) \neq 0$ and $[x, y] = 0$. This completes the proof of the claim. Now we note that $\lambda \neq 0$ as $\pi(\alpha) + \lambda \neq 0$. So $d_j(\lambda) \neq 0$ for some $j \in I$. Therefore we have

$$[x \otimes a, y \otimes b] = 0 + \sum_{i\in I}([d_i, x \otimes a], y \otimes b)\lambda_i = \sum_{i\in I} d_i(\lambda)\epsilon(a, b)(x, y)\lambda_i \in \hat{T},$$

which is nonzero, as $\epsilon(a, b) \neq 0$, $(x, y) \neq 0$ and $d_j(\lambda) \neq 0$. □

Suppose that $(\mathfrak{g}, T)$ is an IARA with root system R and σ is an automorphism of $\mathfrak{g}$ satisfying (A1)–(A4) such that the order of σ is prime. As we have already seen, the automorphism σ induces a linear isomorphism $\sigma\colon T^\star \to T^\star$ with $\sigma(R) = R$. In fact σ is an automorphism of R in the sense of [19]. The following lemma shows that $\sigma(\delta) = \delta$ for each $\delta \in R^0$. In particular, one gets that an automorphism of an IARA, satisfying the above conditions, preserves each isotropic root space. This is a nontrivial fact that one should consider in constructing suitable automorphisms of IARA's.

Lemma 7.25. *Suppose $(A,(\cdot,\cdot),R)$ is a tame affine reflection system. In addition, suppose that A is 2-torsion free, and σ is an automorphism of A with $\sigma(R) = R$ (a root system automorphism) of period m such that $\pi(\delta) := (1/m)\sum_{i=1}^{m-1}\sigma^i(\delta) \neq 0$ for any nonzero $\delta \in R^0$. Then $\sigma(\delta) = \delta$ for each $\delta \in R^0$.*

Proof. Since R is tame, it follows from [12, Theorem 1.13] that

$$R^0 + 2\langle R^0\rangle \subseteq R^0. \tag{7.26}$$

Now suppose $\delta \in R^0$. Then by (7.26), $n\delta \in R^0$ for all $n \in \mathbb{Z}$ and so $n\sigma(\delta) \in R$ for all n. But this can happen only if $\sigma(\delta) \in R^0$ [12, Theorem 1.13]. Now again from (7.26),

we have $2\delta - \sigma(2\delta) \in R^0$. But $\pi(2\delta - \sigma(2\delta)) = 0$ and so by assumption $2(\delta - \sigma(\delta)) = 0$. Now since A is 2-torsion free, we get $\sigma(\delta) = \delta$ as required. $\square$

REMARK 7.27. In this remark, we discuss the structure of a commutative associative predivision Λ-graded algebra $\mathcal{A}$, Λ an abelian group. We refer the reader to [19, Section 4.5] for a more general discussion. As we have already mentioned, $\mathrm{supp}_\Lambda(\mathcal{A})$ is a subgroup of Λ, and so without loss of generality we may suppose that $\mathrm{supp}_\Lambda(\mathcal{A}) = \Lambda$. Suppose $\{u_\lambda \mid \lambda \in \Lambda\}$ is a family of invertible elements $u_\lambda \in \mathcal{A}^\lambda$. Put $B := \mathcal{A}^0$, then $\mathcal{A}^\lambda = Bu_\lambda$ for all λ and $\{u_\lambda\}_{\lambda\in\Lambda}$ is a free basis for the B-module $\mathcal{A}$ and the multiplication on $\mathcal{A}$ is uniquely determined by

$$u_\lambda u_\mu = \tau(\lambda, \mu)u_{\lambda+\mu} \quad \text{and} \quad u_\lambda b = bu_\lambda \quad (b \in B), \tag{7.28}$$

where $\tau\colon \Lambda \times \Lambda \to U(B)$ is a function, $U(B)$ being the group of units of B. Associativity and commutativity of $\mathcal{A}$ leads to

$$\tau(\lambda, \mu)\tau(\lambda + \mu, \nu) = \tau(\mu, \nu)\tau(\lambda, \mu + \nu), \quad \tau(\lambda, \mu) = \tau(\mu, \lambda), \tag{7.29}$$

for $\lambda, \mu, \nu \in \Lambda$. In other words, $\tau\colon \Lambda \times \Lambda \to U(B)$ is a *symmetric* 2-*cocycle*. Conversely, given any unital commutative associative $\mathbb{F}$-algebra B and a symmetric 2-cocycle $\tau\colon \Lambda \times \Lambda \to U(B)$, one can define a commutative associative predivision Λ-graded $\mathbb{F}$-algebra by (7.28). To be more precise, let $\mathcal{A}$ be the free B-module with basis $\{u_\lambda\}_{\lambda\in\Lambda}$, namely $\mathcal{A} := \bigoplus_{\lambda\in\Lambda} Bu_\lambda$. Then, identifying B with Bu_0 through $b \mapsto b\tau(0, 0)^{-1}u_0$, $b \in B$, and using (7.28) as the multiplication rule on $\mathcal{A}$, we get the desired algebra. A commutative associative algebra arising in this way is called a *twisted group algebra* and is denoted by $B^t[\Lambda]$. To summarize, any commutative associative predivision graded algebra $\mathcal{A}$ with support Λ is graded isomorphic to a twisted group algebra $B^t[\Lambda]$. It follows that, $\mathcal{A}$ is division graded if and only if B is a field, and is an associative Λ-torus if and only if $B = \mathbb{F}$.

8. Examples

In this section, we illustrate extended affinization through some examples. In the first example, using extended affinization process, we construct a generalization of the class of *toroidal* Lie algebras. In the second example, starting from a certain IARA, we show that we can iterate extended affinization process to get a series of IARA's. Finally, in the last example, we apply extended affinization starting from an IARA of type A and ending up with an IARA of type BC. Before going to the main body of this section, we make a convention that in each example,

($\star$) B is a unital associative algebra over $\mathbb{F}$ admitting an invariant nondegenerate symmetric bilinear form ϵ such that $\epsilon(1, 1) \neq 0$.

EXAMPLE 8.1. Suppose $(\mathfrak{g}, T)$ is an IARA with corresponding root system R and bilinear form $(\cdot,\cdot)$. Assume B is commutative and let Λ be a torsion free abelian group. Consider the twisted group algebra $B^t[\Lambda] := \bigoplus_{\lambda\in\Lambda} Bz^\lambda$ and recall that we have $bz^\lambda cz^\mu = bc\tau(\lambda,\mu)z^{\lambda+\mu}$, $z^\lambda b = bz^\lambda$ for $b, c \in B$, $\lambda, \mu \in \Lambda$ where $\tau\colon \Lambda\times\Lambda \to U(B)$ is a symmetric 2-cocycle. We extend ϵ to $B^t[\Lambda]$ by linear extension of

$$\epsilon(bz^\lambda, cz^\mu) := \begin{cases} \epsilon(b,c) & \lambda+\mu = 0, \\ 0 & \lambda+\mu \neq 0. \end{cases} \tag{8.2}$$

Set $B^t[\Lambda]^\lambda := Bz^\lambda$, $\lambda \in \Lambda$. Then by Remark 7.27, $B^t[\Lambda]$ is a commutative associative predivision Λ-graded algebra over $\mathbb{F}$ and one can easily verify that ϵ is a Λ-graded invariant nondegenerate symmetric bilinear form on $B^t[\Lambda]$.

Define $\hat{\mathfrak{g}}$ and $\hat{T}$ as in (7.13) with $\rho = 0$. Namely

$$\hat{\mathfrak{g}} = \widehat{L(\mathfrak{g}, B^t[\Lambda])} = (\mathfrak{g}\otimes B^t[\Lambda]) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger \quad \text{and} \quad \hat{T} = (T\otimes 1)\oplus \mathcal{V}\oplus\mathcal{V}^\dagger,$$

with corresponding Lie bracket and bilinear form defined by (7.14) and (7.15), respectively. Then by Corollary 7.22, $(\hat{\mathfrak{g}}, \hat{T})$ is an IARA with root system $\hat{R} = R\oplus\Lambda$. We note that this structure in fact generalizes the well known structure of *toroidal Lie algebras*.

EXAMPLE 8.3. We continue with the same notations as in Example 8.1, in particular $\hat{\mathfrak{g}} = \widehat{L(\mathfrak{g}, B^t[\Lambda])}$. Set $\mathcal{A} := B^t[\Lambda]$ and suppose $\sigma \in \mathrm{Aut}(\mathfrak{g})$ satisfies axioms (A1)–(A4). Let $\mu\colon \Lambda \to \mathbb{Z}$ be a group homomorphism. The map μ induces an automorphism of $\mathcal{A}$, denoted again by μ, defined by $\mu(x) := \zeta^{\mu(\lambda)}x$ for any $x \in \mathcal{A}^\lambda$, where ζ is a primitive m-th root of unity. Both σ and μ can be considered as automorphisms of $\hat{\mathfrak{g}}$ by

$$\sigma = \sigma\otimes \mathrm{id} \text{ on } \mathfrak{g}\otimes\mathcal{A} \text{ and } \sigma = \mathrm{id} \text{ on } \mathcal{V}\oplus\mathcal{V}^\dagger,$$
$$\mu = \mathrm{id}\otimes\mu \text{ on } \mathfrak{g}\otimes\mathcal{A} \text{ and } \mu = \mathrm{id} \text{ on } \mathcal{V}\oplus\mathcal{V}^\dagger.$$

Set $\hat{\sigma} := \sigma\mu \in \mathrm{Aut}(\hat{\mathfrak{g}})$. We claim that $\hat{\sigma}$ satisfies (A1)–(A4). Since σ and μ commute and both are of period m over $\hat{\mathfrak{g}}$, (A1) holds. Also (A2) holds since σ and μ stabilize $T\otimes 1$, $\mathcal{V}$ as well as $\mathcal{V}^\dagger$, and (A3) holds since σ preserves the form $(\cdot,\cdot)$ on $\mathfrak{g}$ and μ preserves the form ϵ on $\mathcal{A}$. For (A4), first note that

$$\hat{\mathfrak{g}}^0 = \left(\sum_{\lambda\in\Lambda} \mathfrak{g}^{-\overline{\mu(\lambda)}}\otimes\mathcal{A}^\lambda\right)\oplus\mathcal{V}\oplus\mathcal{V}^\dagger \quad \text{and} \quad \hat{T}^0 = (T^0\otimes 1)\oplus\mathcal{V}\oplus\mathcal{V}^\dagger.$$

Also

$$C_{\hat{\mathfrak{g}}}(\hat{T}^0) = (C_{\mathfrak{g}}(T^0)\otimes\mathcal{A}^0)\oplus\mathcal{V}\oplus\mathcal{V}^\dagger,$$

so

$$C_{\hat{\mathfrak{g}}^0}(\hat{T}^0) = (C_{\mathfrak{g}^0}(T^0)\otimes\mathcal{A}^0)\oplus\mathcal{V}\oplus\mathcal{V}^\dagger.$$

Since σ satisfies (A4), $C_{\mathfrak{g}^0}(T^0) \subseteq \mathfrak{g}_0$. Thus

$$C_{\hat{\mathfrak{g}}^0}(\hat{T}^0) \subseteq (\mathfrak{g}_0 \otimes \mathcal{A}^0) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger = \hat{\mathfrak{g}}_0$$

and (A4) holds for $\hat{\sigma}$.

We now further assume that $\mathfrak{g}$ is division, $\mathfrak{g}_0$ is abelian and $B = \mathbb{F}$. Then by Remark 7.27 and Corollary 7.24, $(\hat{\mathfrak{g}}, \hat{T})$ is a division IARA. In addition, $\hat{\mathfrak{g}}_0 = (\mathfrak{g}_0 \otimes B) \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$ is abelian. Therefore $(\hat{\mathfrak{g}}, \hat{T})$ and $\hat{\sigma}$ satisfy conditions of Theorem 7.21, with $\hat{\mathfrak{g}}$, $\hat{T}$ and $\hat{\sigma}$ in place of $\mathfrak{g}$, T and σ, respectively. Now let Λ' be a torsion-free abelian group, $\rho' \colon \Lambda' \to \mathbb{Z}_m$ a group epimorphism and $\mathcal{A}'$ a suitable Λ'-graded commutative associative algebra. Then starting from $(\hat{\mathfrak{g}}, \hat{T})$ and $\hat{\sigma}$, one can use Theorem 7.21 to construct a new IARA $\widehat{L_{\rho'}(\hat{\mathfrak{g}}, \mathcal{A}')}$. This process can be iterated using suitable inputs.

EXAMPLE 8.4. Suppose J is a nonempty index set, with a fixed total ordering, and $\mathbf{q} = (q_{ij})$ is a $J \times J$ matrix over $\mathbb{F}$ such that $q_{ij} = \pm 1$, $q_{ji} = q_{ij}$ and $q_{ii} = 1$, for all $i, j \in J$. We recall that B and ϵ are as in $(\star)$. Let $\mathcal{A} := B_{\mathbf{q}}[z_j^{\pm 1}]_{j \in J}$ be the unital associative algebra generated by $\{z_j, z_j^{-1}, b \mid j \in J,\ b \in B\}$ subject to the relations

$$z_j z_j^{-1} = z_j^{-1} z_j = 1, \quad z_i z_j = q_{ij} z_j z_i \quad \text{and} \quad z_i b = b z_i, \quad (i, j \in J,\ b \in B). \tag{8.5}$$

Take $\Lambda := \mathbb{Z}^{|J|}$ and for $\lambda = (\lambda_j)_{j \in J} \in \Lambda$. Set $z^\lambda := \Pi_{j \in J} z_j^{\lambda_j} \in \mathcal{A}$, where product makes sense with respect to the total ordering on J. Then $\mathcal{A}$ is a predivision Λ-graded associative algebra with $\mathcal{A}^\lambda = B z^\lambda$ for each $\lambda \in \Lambda$. Moreover, a similar argument as in [14, Proposition 2.44] shows that

$$\mathcal{A} = [\mathcal{A}, \mathcal{A}] \oplus Z(\mathcal{A}). \tag{8.6}$$

Let K be a nonempty index set and denote by $K^\pm$, the set $K \uplus \{0\} \uplus (-K)$ where $-K$ is a copy of K whose elements are denoted by $-k, k \in K$. Let $\mathcal{K}$ be the Lie subalgebra $\mathfrak{sl}_{K^\pm}(\mathcal{A})$ of all finitary $K^\pm \times K^\pm$ matrices over $\mathcal{A}$ generated by the elementary matrices ae_{ij}, $i \neq j \in K^\pm$, $a \in \mathcal{A}$ (for details the reader is referred to [19, Section 7]). One knows that there is a unique Λ-grading on $\mathcal{K}$ such that for each $i \neq j \in K^\pm$ and $a \in \mathcal{A}^\lambda$, $ae_{ij} \in \mathcal{K}^\lambda$.

One can extend ϵ from B to $\mathcal{A}$ as in (8.2), then we can define a Λ-graded invariant nondegenerate symmetric bilinear form on the set of finitary $K^\pm \times K^\pm$-matrices by linear extension of

$$(ae_{ij}, be_{ks})_{\mathcal{K}} := \delta_{i,s}\delta_{j,k}\epsilon(a, b) \quad \text{for} \quad a, b \in \mathcal{A},\ i, j, k, s \in K^\pm.$$

By [19, Section 7.10] the restriction of this form to $\mathcal{K}$ is nondegenerate if and only if $Z(\mathcal{K}) = \{0\}$. Also by [19, Section 7.4], $Z(\mathcal{K}) = \{0\}$ if $|K| = \infty$, and

$$Z(\mathcal{K}) = \{z I_{2n+1} \mid z \in Z(\mathcal{A}),\ (2n+1)z \in [\mathcal{A}, \mathcal{A}]\}, \tag{8.7}$$

if $|K| = n < \infty$, where by I_{2n+1} we mean the identity $(2n+1) \times (2n+1)$ matrix. Therefor by (8.6), $Z(\mathcal{K}) = \{0\}$ in this case too. So the restriction of the form to $\mathcal{K}$ is Λ-graded and nondegenerate.

Suppose $\dot{T} := \mathrm{span}_{\mathbb{F}}\{e_{ii} - e_{jj} \mid i \neq j \in K^{\pm}\}$. For $i \in K^{\pm}$, define $\varepsilon_i \in \dot{T}^{\star}$ by $\varepsilon_i(e_{jj} - e_{kk}) := \delta_{ij} - \delta_{ik}$, for $j \neq k \in K^{\pm}$. For $i, j \in K^{\pm}$, put $\dot{\alpha}_{ij} := \varepsilon_i - \varepsilon_j$ and $\dot{R} := \{\dot{\alpha}_{ij} \mid i, j \in K^{\pm}\}$. For $\dot{\alpha} \in \dot{R}$, set $\mathcal{K}_{\dot{\alpha}} := \{x \in \mathcal{K} \mid [t, x] = \dot{\alpha}(t)x \text{ for all } t \in \dot{T}\}$. Then $\mathcal{K} = \bigoplus_{\dot{\alpha} \in \dot{R}} \mathcal{K}_{\dot{\alpha}}$. We note that $\mathcal{K}_0$ consists of diagonal elements of $\mathcal{K}$. In addition, if we assume that $\epsilon(1, 1) = 1$, then for any $i, j, s, k \in \mathcal{K}$ with $i \neq j$ and $s \neq k$, we have

$$\begin{aligned}(e_{ii} - e_{jj}, e_{kk} - e_{ss})_{\mathcal{K}} &= \delta_{ik} - \delta_{is} - (\delta_{jk} - \delta_{js}) \\ &= \epsilon_i(e_{kk} - e_{ss}) - \epsilon_j(e_{kk} - e_{ss}) \\ &= \dot{\alpha}_{ij}(e_{kk} - e_{ss}).\end{aligned}$$

Thus $t_{\dot{\alpha}_{ij}} := e_{ii} - e_{jj}$ is the unique element in $\dot{T}$ representing $\dot{\alpha}_{ij}$ via $(\cdot\,,\,\cdot)_{\mathcal{K}}$.

Next, consider the $\mathbb{F}$-vector space $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} \Lambda$, identify Λ as a subset of $\mathcal{V}$ and fix a basis $\{\lambda_j \mid j \in J\}$ for $\mathcal{V}$. Define the vector space $\mathcal{V}^{\dagger} := \sum_{j \in J} \mathbb{F} d_j \subseteq \mathcal{V}^{\star}$ as in (7.12). Set

$$\mathfrak{g} := \mathcal{K} \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger} \quad \text{and} \quad T := \dot{T} \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger}.$$

Define the Lie bracket on $\mathfrak{g}$ as in (7.14), and extend the form $(\cdot\,,\,\cdot)_{\mathcal{K}}$ on $\mathcal{K}$ to a form $(\cdot\,,\,\cdot)$ on $\mathfrak{g}$ as in (7.15). Then it is clear that $(\cdot\,,\,\cdot)$ is nondegenerate both on $\mathfrak{g}$ and T.

We note that each $\dot{\alpha} \in \dot{R}$ can be considered as an element of $T^{\star}$ by requiring $\dot{\alpha}(\mathcal{V}) = \dot{\alpha}(\mathcal{V}^{\dagger}) := \{0\}$. One can easily see that $t_{\dot{\alpha}}$ represents $\dot{\alpha}$ via $(\cdot\,,\,\cdot)$ for each $\dot{\alpha} \in \dot{R}$. Also we can consider any $\lambda \in \Lambda$ as an element of $T^{\star}$ by $\lambda(\dot{T}) = \lambda(\mathcal{V}) := \{0\}$ and $\lambda(d) := d(\lambda)$ for any $d \in \mathcal{V}^{\dagger}$. Then clearly $t_{\lambda} = \lambda$. If for $\alpha \in T^{\star}$ we define $\mathfrak{g}_{\alpha}$ in the usual manner, then it is easy to verify that for any $\lambda \in \Lambda$,

$$\begin{aligned}&\mathfrak{g}_{\dot{\alpha}_{ij}+\lambda} = \mathcal{A}^{\lambda} e_{ij}, \quad (\dot{\alpha}_{ij} \neq 0), \\ &\mathfrak{g}_{\lambda} = \text{the set of diagonal matrices in } \mathcal{K} \text{ with enteries from } \mathcal{A}^{\lambda}, \quad (\lambda \neq 0), \\ &\mathfrak{g}_0 = (\text{the set of diagonal matrices in } \mathcal{K} \text{ with enteries from } \mathcal{A}^0 \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger}.\end{aligned} \tag{8.8}$$

So $\mathfrak{g} = \bigoplus_{\dot{\alpha} \in \dot{R}, \lambda \in \Lambda} \mathfrak{g}_{\dot{\alpha}+\lambda}$. Therefore $(\mathfrak{g}, T)$ is a toral pair with root system

$$R = \dot{R} + \Lambda,$$

and (IA1) holds for $\mathfrak{g}$. We next show that (IA2) holds. Fix $\lambda \in \Lambda$ and choose an invertible element $a \in \mathcal{A}^{\lambda}$, then for $i \neq j$ we have

$$\begin{aligned}&[ae_{ij}, a^{-1}e_{ji}] = e_{ii} - e_{jj} + \sum_{s \in J}([d_s, ae_{ij}], a^{-1}e_{ji})\lambda_s \in T, \\ &[a(e_{ii} - e_{jj}), a^{-1}(e_{ii} - e_{jj})] = 2\sum_{s \in J} d_s(\lambda)\epsilon(a, a^{-1})\lambda_s \in T,\end{aligned} \tag{8.9}$$

where the first equality is always nonzero and the second equality is nonzero if $\lambda \neq 0$. Note that if $i \neq j$, then $a^{\pm 1}e_{ij} \in \mathfrak{g}_{\dot{\alpha}_{ij} \pm\lambda}$, and $a^{\pm 1}(e_{ii} - e_{jj}) \in \mathfrak{g}_{\pm\lambda}$. So (IA2) holds. Finally, since $\dot{R}$ is a locally finite root system of type $\dot{A}_{K^{\pm}}$, (IA3) holds by a similar argument as in the proof of Theorem 5.5. Consequently, $(\mathfrak{g}, T)$ is an IARA.

We further show that $\mathfrak{g}$ is division if and only if $\mathcal{A}$ is division graded. Using the fact that the elements of $\dot{T}$ are diagonal matrices with trace zero, it is not difficult to see that if $\mathfrak{g}$ is division, then $\mathcal{A}$ is division graded. Assume now that $\mathcal{A}$ is division graded. We must show that (IA2)$'$ holds for $\mathfrak{g}$. Let $\lambda \in \Lambda$, $0 \neq a \in \mathcal{A}^{\lambda}$ and $i \neq j \in K^{\pm}$, then

$$[ae_{ij}, a^{-1}e_{ji}] = e_{ii} - e_{jj} + \sum_{s\in I}([d_s, ae_{ij}], a^{-1}e_{ji})\lambda_s \in T \setminus \{0\}$$

as required. Also if $\sum_{i\in K_0^{\pm}} a_i e_{ii} \in \mathfrak{g}_{\lambda}$, for a finite subset $K_0^{\pm}$ of $K^{\pm}$, where $0 \neq a_i \in \mathcal{A}^{\lambda}$ and $\lambda \neq 0$, then

$$\begin{aligned}\left[\sum_{i\in K_0^{\pm}} a_i e_{ii}, \sum_{i\in K_0^{\pm}} a_i^{-1} e_{ii}\right] &= \sum_{s\in J} d_s(\lambda)\left(\sum_{i\in K_0^{\pm}} \epsilon(a_i, a_i^{-1})\right)\lambda_s \\ &= \sum_{s\in J} d_s(\lambda)|K_0^{\pm}|\lambda_s \in T \setminus \{0\}\end{aligned}$$

as required. Therefore $\mathfrak{g}$ is division if and only if $\mathcal{A}$ is division graded. Indeed by [19, Section 4.5], $\mathfrak{g}$ is division if and only if B is division. So from now on, *we assume that B is division*.

There exists an involution $\bar{\ }$ (a self-inverting anti-automorphism) on $\mathcal{A}$ (see [7, Section 2]) such that $\bar{z_j} = z_j$, for any $j \in J$ and $\bar{b} = b$ for all $b \in B$. By definition, it is clear that $\epsilon(\bar{a}, \bar{b}) = \epsilon(a, b)$ for any $a, b \in \mathcal{A}$. Using the involution $\bar{\ }$, we can define an involution * on $\mathcal{K}$ by $(ae_{ij})^* = \bar{a}e_{-j,-i}$. I t is straightforward to see that the linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\sigma(x) = -x^* \quad \text{for} \quad x \in \mathcal{K} \quad \text{and} \quad \sigma(x) = x \quad \text{for} \quad x \in \mathcal{V} \oplus \mathcal{V}^{\dagger},$$

is a Lie algebra automorphism.

We will show that σ satisfies (A1)–(A5). Clearly $\sigma^2(x) = x$ for any $x \in \mathfrak{g}$, thus σ satisfies (A1) with $m = 2$. Also it is clear from definition that σ satisfies (A2). In addition, observe that

$$\begin{aligned}(\sigma(ae_{ij}), \sigma(be_{ks})) &= ((ae_{ij})^*, (be_{ks})^*) \\ &= (\bar{a}e_{-j,-i}, \bar{b}e_{-s,-k}) \\ &= \delta_{jk}\delta_{is}\epsilon(\bar{a}, \bar{b}) \\ &= \delta_{jk}\delta_{is}\epsilon(a, b) \\ &= (ae_{ij}, be_{ks}).\end{aligned}$$

So (A3) holds for σ. Since $m = 2$ is prime, instead of (A4) we will show that σ satisfies the equivalent condition (A4)$''$ (see Lemma 4.1) namely, we show that for $0 \neq \alpha \in R$, $\pi(\alpha) \neq 0$. Recall from Section 3 that since σ satisfies (A2), it induces an automorphism on $T^\star$, denoted again by σ. We now note that * maps diagonal matrices to diagonal matrices and $\bar{\ }$ preserves homogeneous subspaces of $\mathcal{A}$. Thus by (8.8) for any $\lambda \in \Lambda$, $\sigma(\mathfrak{g}_\lambda) = \mathfrak{g}_\lambda$, implying $\sigma(\lambda) = \lambda$. Consequently, $\pi(\lambda) = \lambda$ and so if $\lambda \neq 0$, then so is $\pi(\lambda)$. On the other hand, we have $\sigma^{-1} = \sigma$ so for any $t \in T$ and $i \neq j \in K^\pm$,

$$\begin{aligned}\pi(\dot{\alpha}_{ij})(t) &= \frac{1}{2}(\dot{\alpha}_{ij} + \sigma(\dot{\alpha}_{ij})(t)) \\ &= \frac{1}{2}(\dot{\alpha}_{ij}(t) + \dot{\alpha}_{ij}(\sigma(t))) \\ &= \frac{1}{2}(\dot{\alpha}_{ij}(t + \sigma(t))).\end{aligned}$$

Using this, we see that for $i \neq j \in K^\pm$ and $t := e_{ii} - e_{jj}$,

$$\pi(\dot{\alpha}_{ij})(t) = \begin{cases} 1 & -j \neq i \text{ and } i, j \neq 0, \\ \dfrac{1}{2} & -j \neq i,\ i = 0 \text{ or } j = 0, \\ 2 & -j = i. \end{cases}$$

Consequently, $\pi(\alpha) \neq 0$ for any $0 \neq \alpha \in R$. In particular (A4)$''$ holds.

We next show that (A5) holds. Let $i \neq j \in K \cup \{0\}$, then we have

$$0 \neq e_{ii} - e_{-i-i} \in T^0 \quad \text{and} \quad 0 \neq e_{ii} - e_{jj} + e_{-i-i} - e_{-j-j} \in T^{\bar{1}}.$$

In particular (A5) holds. Therefore $(\mathfrak{g}, T)$ and σ satisfy all requirements of Theorem 7.21 and so we can construct a new IARA $(\hat{\mathfrak{g}}, \hat{T})$.

Note that by (8.8),

$$\mathfrak{g}_0 = (\text{the set of diagonal matrices in } \mathcal{K} \text{ with entries from } \mathcal{A}^0) \oplus \mathcal{V} \oplus \mathcal{V}^0.$$

So, $\mathfrak{g}_0$ is abelian if and only if $\mathcal{A}^0 = B$ is abelian, indeed, if and only if B is a field.

Now that we have a suitable automorphism on $\mathfrak{g}$, choosing a torsion-free abelian group Λ', a group epimorphism $\rho\colon \Lambda' \to \mathbb{Z}_2$ and a predivision Λ'-graded commutative associative algebra $\mathcal{A}'$, we can use Theorem 7.21 to construct another IARA, $\hat{\mathfrak{g}}$ with a root system $\hat{R}$.

It is now interesting to have a discussion on the type of $\hat{\mathfrak{g}}$. Note that we have

$$R = \{\epsilon_i - \epsilon_j + \lambda \mid i \neq j \in K^\pm,\ \lambda \in \Lambda\}.$$

By definition of σ one can easily check that for any $i \in K$, $\sigma(\epsilon_i) = -\epsilon_{-i}$, and as we have already seen $\sigma(\lambda) = \lambda$ for any $\lambda \in \Lambda$. Therefore

$$\begin{aligned}
\pi(R) &= \left\{ \frac{1}{2}(\epsilon_i - \epsilon_j + \epsilon_{-j} - \epsilon_{-i}) + \lambda \;\middle|\; i \neq j \in K^{\pm},\ \lambda \in \Lambda \right\} \\
&= \left\{ \pm\frac{1}{2}(\epsilon_i - \epsilon_{-i}) + \lambda \;\middle|\; i \in K, \lambda \in \Lambda \right\} \\
&\quad \cup \left\{ \pm\frac{1}{2}((\epsilon_i - \epsilon_{-i}) \pm (\epsilon_j - \epsilon_{-j})) + \lambda \;\middle|\; i \neq j \in K, \lambda \in \Lambda \right\} \\
&\quad \cup \{\pm(\epsilon_i - \epsilon_{-i}) + \lambda \mid i \in K,\ \lambda \in \Lambda\}.
\end{aligned}$$

This makes it clear that $\pi(R)$ is an affine reflection system of type BC. But by (7.20), $\hat{R}$ and $\pi(R)$ have the same type. Thus $\hat{\mathfrak{g}}$ is an IARA of type BC.

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