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UNIVERSALITY OF SOME FUNCTIONS RELATED TO ZETA-FUNCTIONS OF CERTAIN CUSP FORMS

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Abstract

It is well-known that the zeta-function $\zeta(s, F)$ attached to a normalized Hecke-eigen cusp form F of weight κ is universal in the sense that their shifts $\zeta(s+i\tau, F)$ with appropriate $\tau \in \mathbb{R}$ approximate any analytic function uniformly on compact subsets of the strip $\{s = \sigma + it \in \mathbb{C} \colon \kappa/2 < \sigma < (\kappa+1)/2\}$ with any prescribed accuracy. In this paper we consider some classes of operators Φ such that the function $\Phi(\zeta(s, F))$ is universal in the above sense. In particular, this implies the universality of the functions, for example, $\zeta(s, F)^N$ (N-th power) and $\zeta^{(N)}(s, F)$ (N-th derivative) with $N \in \mathbb{N}$, $e^{\zeta(s,F)}$, $\sin\zeta(s, F)$, and $\cos\zeta(s, F)$.

1. Introduction and statement of results

Approximation theory of analytic functions has a long and rich history. One fundamental result in this field is Mergelyan's theorem [21], see also [29]. It asserts that every continuous function $f(s)$ on a compact set $K \subset \mathbb{C}$ with connected complement which is analytic in the interior of K can be approximated to any accuracy uniformly on K by polynomials in $s = \sigma + it$. That is, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Examples show that the requirements on $f(s)$ and K cannot be weakened. Thus, Mergelyan's theorem gives a necessary and sufficient condition for approximation of analytic functions by polynomials. It turned out that not only polynomials, but also some other type of functions come into play in the above approximation problem: there exists a function $g(s)$ whose shifts $g(s + i\tau)$, $\tau \in \mathbb{R}$, approximate every analytic target function uniformly on compact subsets of a certain region. The first explicit example of such a function is the Riemann zeta-function $\zeta(s)$; its approximation property is called universality and was discovered by Voronin [28]. The modern version of Voronin's theorem is the following statement (and can be found, for example, in [12]):

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Theorem 1. *Let K be a compact subset of the strip $\{s = \sigma + it \in \mathbb{C} \colon 1/2 < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] \colon \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Here we denote by $\operatorname{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

This universality theorem of Voronin has immediate consequences. For instance, a straightforward application yields many results concerning the distribution of values of $\zeta(s)$ due to the school of Harald Bohr in the first half of the twentieth century, though their approach had been limited on approximation of constants rather than functions (e.g., [7]). Another corollary is the hypertranscendence of the zeta-function, meaning that there is no non-trivial algebraic differential equation having $\zeta(s)$ as a solution; this was already known by Hilbert (as follows from his famous address at the ICM in Paris in 1900), however, the first published proof seems to date back to Stadigh (cf. [25]). Actually, with functional independence a much stronger property can be deduced from universality: Given continuous functions $F_0, F_1, \ldots, F_N$, all defined on $\mathbb{C}^{n+1}$ and suppose that not all identically zero, then

$$\sum_{k=0}^{N} s^k F_k(\zeta(s), \zeta(s)', \ldots, \zeta^{(n)}(s)) \neq 0$$

for some $s \in \mathbb{C}$ (for a proof we refer to [28, 27]). Andersson [1] applied the universality theorem to disprove certain conjectures of Ramachandra. There is even an application of the universality theorem to mathematical physics (see [6]).

Voronin himself and others like Bagchi, Gonek, Reich, have extended and generalized Theorem 1 in various ways. In the meantime universality theorems like the above, or variations thereof, are known for many different zeta- or L-functions; e.g., Dedekind zeta-functions, Lerch zeta-functions, Dirichlet L-functions, to mention only a few; for results, problems and references we refer to [2], [3], [10], [12], [16], [27] as well as to the survey papers [13], [19], and [20].

The universality of zeta-functions without Euler products (such as Lerch zeta-functions) has interesting applications to the distribution of zeros of those zeta-functions (see [16]). The connection between the universality and the distribution of zeros of certain multiple zeta-functions has recently been studied by Nakamura ([23], [24]). An application of the universality to the class number problem can be found in a paper of Mishou and Nagoshi ([22]).

Maybe the most spectacular statement related to universality is that the Riemann hypothesis ($\zeta(s) \neq 0$ for $\operatorname{Re} s > 1/2$) is true if, and only if, the zeta-function can approximate itself inside the strip $1/2 < \operatorname{Re} s < 1$ (in the sense of Voronin's theorem). The analogue for Dirichlet L-functions to primitive characters of this equivalence was already proved by Bohr [8]; Bagchi [2] [4] used the universality in order to extend this result to the Riemann zeta-function as stated in the previous sentence. Meanwhile, this phenomenon and further examples of applications of universality have been studied for numerous generalizations of the Riemann zeta-function and Dirichlet L-functions. It seems that universality is an outstanding and important feature of Dirichlet series.

Let $F(z)$ be a normalized Hecke-eigen cusp form of weight κ for the full modular group with Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi i m z}, \quad c(1) = 1.$$

The associated zeta-function $\zeta(s, F)$ is defined by

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s};$$

this Dirichlet series converges for $\sigma > (\kappa + 1)/2$ and $\zeta(s, F)$ can be analytically continued to an entire function. In [17] it has been shown that $\zeta(s, F)$ is universal; in this case, K needs to be a compact subset with connected complement of the strip $D = D_F = \{s \in \mathbb{C} \colon \kappa/2 < \sigma < (\kappa + 1)/2\}$.

In [14], it was shown that the derivative $\zeta'(s, F)$ is also universal. That is, the operator $\Phi \colon g \to g'$ preserves the universality property for $\zeta(s, F)$. This phenomenon suggests the problem of finding some set of operators Φ, as large as possible, for which the function $\Phi(\zeta(s, F))$ satisfies the universality property. In the case of the Riemann zeta-function $\zeta(s)$ the problem has been discussed in [15].

A sufficiently wide class of operators Φ with the universality property for $\Phi(\zeta(s,F))$ can be described as follows. For a region G on the complex plane, denote by $H(G)$ the space of analytic functions on G endowed with the topology of uniform convergence on compacta. Let $\alpha > 0$. We say that the operator $\Phi \colon H(D) \to H(D)$ belongs to the class $\mathrm{Lip}(\alpha)$ if the following hypotheses are satisfied:

1° For each polynomial $p = p(s)$, and every compact subset $K \subset D$ with connected complement, we can find an element $q \in \Phi^{-1}\{p\} \subset H(D)$ such that $q(s) \neq 0$ on K;

2° For every compact subset $K \subset D$ with connected complement, there exist a constant $c > 0$ and a compact subset $K_1 \subset D$ with connected complement, for which

$$\sup_{s\in K} |\Phi(g_1(s)) - \Phi(g_2(s))| \leq c \sup_{s\in K_1} |g_1(s) - g_2(s)|^{\alpha}$$

holds for all $g_1, g_2 \in H(D)$.

Theorem 2. *If* $\Phi \in \mathrm{Lip}(\alpha)$, *then the function* $\Phi(\zeta(s, F))$ *is universal in the following sense*: *Let* K *be a compact subset of the strip* D *with connected complement, and let*

$f(s)$ be a function, continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \sup_{s\in K} |\Phi(\zeta(s + i\tau, F)) - f(s)| < \varepsilon\right\} > 0. \tag{1}$$

Note that it is not necessary to assume that $f(s)$ is non-vanishing on K. The proof of this theorem will be given in the next section, where we will also prove that the aforementioned operator $\Phi\colon g \to g'$ is an element of Lip(1).

Now we introduce some other classes of operators Φ for which the function $\Phi(\zeta(s, F))$ is universal. Let

$$S_F = \left\{g \in H(D)\colon \frac{1}{g(s)} \in H(D) \text{ or } g(s) \equiv 0\right\}.$$

Denote by U_F the class of continuous operators $\Phi\colon H(D) \to H(D)$ such that, for every open set $G \subset H(D)$, the set $(\Phi^{-1}G) \cap S_F$ is non-empty.

Theorem 3. *For any $\Phi \in U_F$, the function $\Phi(\zeta(s, F))$ is universal, in the same sense as in the statement of* Theorem 2.

The next theorem is similar to Theorem 3, but it is more convenient. Let V be an arbitrary positive number. Define $D_V = D_{F,V} = \{s \in \mathbb{C}\colon \kappa/2 < \sigma < (\kappa + 1)/2,\ |t| < V\}$ and

$$S_{F,V} = \left\{g \in H(D_V)\colon \frac{1}{g(s)} \in H(D_V) \text{ or } g(s) \equiv 0\right\}.$$

Consider the class $U_{F,V}$ of continuous operators $\Phi\colon H(D_V) \to H(D_V)$ such that, for each polynomial $p = p(s)$, the set $(\Phi^{-1}\{p\}) \cap S_{F,V}$ is non-empty. An example of $\Phi \in U_{F,V}$ will be given in the last section.

Theorem 4. *Let K and $f(s)$ be the same as in* Theorem 2. *Suppose that $V > 0$ is such that $K \subset D_V$, and that $\Phi \in U_{F,V}$. Then* (1) *holds for every $\varepsilon > 0$.*

Next we introduce a certain subset of $H(D)$, whose elements can be approximated by the shifts $\Phi(\zeta(s + i\tau, F))$. For $a_1, \dots, a_r \in \mathbb{C}$, let

$$H_{\Phi(0);a_1,\dots,a_r}(D) = \{g \in H(D)\colon (g(s) - a_j)^{-1} \in H(D),\ j = 1, \dots, r\} \cup \{\Phi(0)\}.$$

Denote by $U_{F;a_1,\dots,a_r}$ the class of continuous operators $\Phi\colon H(D) \to H(D)$ such that

$$\Phi(S_F) \supset H_{\Phi(0);a_1,\dots,a_r}(D). \tag{2}$$

Theorem 5. *Suppose that* $\Phi \in U_{F;a_1,\dots,a_r}$.
(i) *Let* $K \subset D$ *be an arbitrary compact subset, and* $f(s) \in H_{\Phi(0);a_1,\dots,a_r}(D)$. *Then* (1) *holds for every* $\varepsilon > 0$.
(ii) *When* $r = 1$, *let* $K \subset D$ *be a compact subset with connected complement, and let* $f(s)$ *be a continuous and* $\neq a_1$ *function on* K *which is analytic in the interior of* K. *Then* (1) *holds for every* $\varepsilon > 0$.

The universality of the functions $\zeta(s,F)^N$ $(N \in \mathbb{N})$, $e^{\zeta(s,F)}$, $\sin(\zeta(s,F))$, $\cos(\zeta(s,F))$ etc. can be shown as special cases of this theorem. This point will be discussed in the last section.

In general case, we have the following statement.

Theorem 6. *Let* $\Phi\colon H(D) \to H(D)$ *be a continuous operator,* $K \subset D$ *be a compact subset, and* $f(s) \in \Phi(S_F)$. *Then* (1) *holds for every* $\varepsilon > 0$.

The universality theorem for $\zeta(s, F)$ proved in [17] was extended to the case of newforms in [18], and further to the case of certain subclass of Selberg class by [26], [27]. It is plausible that the theorems in the present paper could be generalized to such situations.

2. Proof of Theorem 2

Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous nonvanishing function on K which is analytic in the interior of K, and $\varepsilon > 0$. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3}$$

By hypothesis 1° of the class $\mathrm{Lip}(\alpha)$, there exists an element $q \in \Phi^{-1}\{p\} \subset H(D)$ satisfying $q(s) \neq 0$ on K. Let $\tau \in \mathbb{R}$ satisfy the inequality

$$\sup_{s\in K_1} |\zeta(s + i\tau, F) - q(s)| < c^{-1/\alpha} \left(\frac{\varepsilon}{2}\right)^{1/\alpha}, \tag{4}$$

where c and K_1, corresponding to the set K, are those in hypothesis 2° of the class $\mathrm{Lip}(\alpha)$. Then, by 2°, for τ satisfying (4),

$$\sup_{s\in K} |\Phi(\zeta(s + i\tau, F)) - p(s)| \le c \sup_{s\in K_1} |\zeta(s + i\tau, F) - q(s)|^{\alpha} < \frac{\varepsilon}{2}. \tag{5}$$

Since $q(s) \neq 0$ on K, we can apply the universality theorem for $\zeta(s, F)$ ([17]) to find that the set of those τ satisfying (4) is of positive lower density. Therefore, combining with (5) we obtain

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] \colon \sup_{s\in K} |\Phi(\zeta(s + i\tau, F)) - p(s)| < \frac{\varepsilon}{2}\right\} > 0.$$

From this and (3), we find that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] \colon \sup_{s\in K} |\Phi(\zeta(s + i\tau, F)) - f(s)| < \varepsilon\right\} > 0.$$

This completes the proof of Theorem 2.

Next we show that the operator $\Phi \colon g \to g'$ is in the class Lip(1). Let K be a compact subset of the strip D with connected complement. Choose an open set G and a compact subset K_1 of D with connected complement such that $K \subset G \subset K_1$. We take a simple closed contour L lying in $K_1 \setminus G$ and enclosing the set K. Then, by Cauchy integral formula

$$\Phi(g_1(s)) - \Phi(g_2(s)) = \frac{1}{2\pi i} \int_L \frac{g_1(z) - g_2(z)}{(z - s)^2}\, \mathrm{d}z,$$

we find that, for all $s \in K$,

$$\Phi(g_1(s)) - \Phi(g_2(s)) \leq c \sup_{s\in L} |g_1(s) - g_2(s)| \leq c \sup_{s\in K_1} |g_1(s) - g_2(s)|$$

with some $c = c(K, L) > 0$. Thus, hypothesis 2° is true with $\alpha = 1$. Obviously, hypothesis 1° is also satisfied.

3. Lemmas

Now we prepare several lemmas necessary for the proof of Theorems 3 to 6. Voronin's proof [28] of the universality for the Riemann zeta-function is analytic and based on the approximation of $\zeta(s)$ in mean by finite Euler products. There exists another way for the proof of universality theorems due to Bagchi [2] which applies probabilistic limit theorems in the space of analytic functions to zeta-functions. Denote by $\mathcal{B}(X)$ the class of Borel sets of the space X, and let P and P_n, $n \in \mathbb{N}$, be probability measures on $(X, \mathcal{B}(X))$. We remind that P_n is said to converge weakly to P as $n \to \infty$ if, for every real bounded continuous function h on X,

$$\lim_{n\to\infty} \int_X h\, \mathrm{d}P_n = \int_X h\, \mathrm{d}P.$$

There exist some equivalents of the weak convergence of probability measures. The following one will be useful for us.

Lemma 7. *P_n converges weakly to P as $n \to \infty$ if and only if, for every open set $G \subset X$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G).$$

This lemma is a part of Theorem 2.1 from [5].

In [17], the universality of the function $\zeta(s, F)$ has been obtained by using the weak convergence of the probability measure

$$P_{T,V}(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas}\{\tau \in [0, T]\colon \zeta(s + i\tau, F) \in A\}, \quad A \in \mathcal{B}(H(D_V)),$$

as $T \to \infty$. We will also deduce the universality for $\Phi(\zeta(s, F))$ from the weak convergence of

$$P_{T,\Phi}(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas}\{\tau \in [0, T]\colon \Phi(\zeta(s + i\tau, F)) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$.

Now we remind a simple but useful result from the theory of weak convergence of probability measures. Let X_1 and X_2 be two metric spaces, and let $h\colon X_1 \to X_2$ be a $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable function, i.e.,

$$h^{-1}\mathcal{B}(X_2) \subset \mathcal{B}(X_1).$$

Then every probability measure P on $(X_1, \mathcal{B}(X_1))$ induces the unique probability measure Ph^{-1} on $(X_2, \mathcal{B}(X_2))$ defined by $(Ph^{-1})(A) = P(h^{-1}A)$, $A \in \mathcal{B}(X_2)$. Note that if the function h is continuous, then it is $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable.

Lemma 8. *Suppose that P and P_n, $n \in \mathbb{N}$, are probability measures on $(X_1, \mathcal{B}(X_1))$, and that the function $h\colon X_1 \to X_2$ is continuous. If P_n converges weakly to P as $n \to \infty$, then also $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

This lemma is a corollary of a more general Theorem 5.1 of [5].

Denote by γ the unit circle $\{s \in \mathbb{C}\colon |s| = 1\}$, and define the torus

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. By the Tikhonov theorem, with the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Thus, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ exists, and we have a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ denote the projection of $\omega \in \Omega$ to the coordinate space γ_p. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega, F)$ by the Euler product

$$\zeta(s, \omega, F) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p) + \beta(p) = c(p)$. In view of the Deligne estimates

$$|\alpha(p)| \leq p^{(\kappa-1)/2}, \quad |\beta(p)| \leq p^{(\kappa-1)/2},$$

the latter infinite product, for almost all $\omega \in \Omega$, converges uniformly on compact subsets of the strip D ([11]), and thus, it defines an $H(D)$-valued random element. Denote by P_ζ the distribution of $\zeta(s, \omega, F)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega\colon \zeta(s, \omega, F) \in A), \quad A \in \mathcal{B}(H(D)).$$

Lemma 9. *The probability measure*

$$P_T(A) \overset{\text{def}}{=} \frac{1}{T} \operatorname{meas}\{\tau \in [0, T]\colon \zeta(s + i\tau, F) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ *as* $T \to \infty$.

Proof of the lemma is given in [11].

Lemma 10. *Suppose that* $\Phi\colon H(D) \to H(D)$ *is a continuous operator. Then the probability measure* $P_{T,\Phi}$ *converges weakly to the distribution* $P_{\zeta,\Phi}$ *of the random element* $\Phi(\zeta(s, \omega, F))$ *as* $T \to \infty$.

Proof. This lemma is an immediate consequence of Lemmas 9 and 8. In fact, we have that $P_{T,\Phi} = P_T\Phi^{-1}$. Therefore, the continuity of Φ, and Lemmas 9 and 8 imply the weak convergence of $P_T\Phi^{-1}$ to $P_\zeta\Phi^{-1}$ as $T \to \infty$. On the other hand, by the definition of $P_\zeta\Phi^{-1}$, we find that, for $A \in \mathcal{B}(H(D))$,

$$\begin{aligned}(P_\zeta\Phi^{-1})(A) = P_\zeta(\Phi^{-1}A) &= m_H(\omega \in \Omega\colon \zeta(s, \omega, F) \in \Phi^{-1}A) \\ &= m_H(\omega \in \Omega\colon \Phi(\zeta(s, \omega, F)) \in A),\end{aligned}$$

and thus, $P_\zeta\Phi^{-1}$ is the distribution of the random element $\Phi(\zeta(s, \omega, F))$. □

For $V > 0$, denote by $P_{T,V}$ and $P_{\zeta,V}$ the restrictions to the space $(H(D_V), \mathcal{B}(H(D_V)))$ of the measures P_T and P_ζ, respectively.

Lemma 11. *For every $V > 0$, the probability measure $P_{T,V}$ converges weakly to $P_{\zeta,V}$ as $T \to \infty$.*

This lemma is proved in [17].

For $V > 0$, denote by $P_{T,\Phi,V}$ and $\zeta_V(s,\omega,F)$ the restrictions to the space $(H(D_V), \mathcal{B}(H(D_V)))$ of the measure $P_{T,\Phi}$ and of the random element $\zeta(s,\omega,F)$, respectively.

Lemma 12. *Suppose that $\Phi\colon H(D_V) \to H(D_V)$ is a continuous function. Then the probability measure $P_{T,\Phi,V}$ converges weakly to the distribution of the random element $\Phi(\zeta_V(s,\omega,F))$ as $T \to \infty$.*

Proof. This lemma is deduced from Lemmas 11 and 8 in the same way as the proof of Lemma 10. $\square$

The next step of the probabilistic method for the proof of universality is an identification of supports of the limit measures in limit theorems in the space of analytic functions. Let S be a separable metric space, and P be a probability measure on $(S, \mathcal{B}(S))$. We remind that the minimal closed set $S_P \subset S$ such that $P(S_P) = 1$ is called the support of the measure P. The support of the distribution of a random element X is called the support of that element and is written as S_X.

Lemma 13. *The support of the random element $\zeta_V(s,\omega,F)$ is the set $S_{F,V}$.*

This lemma is proved in [17]. It is to be noted that a new method (the positive density method) was applied for the proof of this lemma.

Lemma 14. *The support of the random element $\zeta(s,\omega,F)$ is the set S_F.*

Proof. In [17], for the proof of Lemma 13, the space $H(D_V)$ (with notation D_N in place of D_V) is used only in the proof of Lemma 6 from [17]. This is applied for the estimate

$$|\rho(\pm iy)| \le \mathrm{e}^{Vy} \int_{\mathbb{C}} |\mathrm{d}\mu h^{-1}(s)|, \quad y \to \infty,$$

where

$$\rho(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz}\, \mathrm{d}\mu h^{-1}(s), \quad z \in \mathbb{C},$$

and μh^{-1} is a complex measure with compact support contained in $\{s \in \mathbb{C}\colon 1/2 < \sigma < 1,\ |t| < V\}$. Since the support of μh^{-1} is compact, there exists a finite number $V_1 > 0$ such that

$$|\rho(\pm iy)| \le \mathrm{e}^{V_1 y} \int_{\mathbb{C}} |\mathrm{d}\mu h^{-1}(s)|.$$

The remaining part of the proof is the same as that in the proof of Lemma 13. □

Lemma 15. *Suppose that* $\Phi \in U_F$. *Then the support of the random element* $\Phi(\zeta(s, \omega, F))$ *is the whole of* $H(D)$.

Proof. Let g be an arbitrary element of $H(D)$, and G be an open neighbourhood of g. In view of the continuity of Φ, we have that the set $\Phi^{-1}G$ is open as well. The definition of the class U_F shows that there exists an element $g_1 \in S_F$ which is an element of $\Phi^{-1}G$. This means that $\Phi^{-1}G$ is an open neighbourhood of the element g_1. Since the support of the random element $\zeta(s, \omega, F)$ consists of all elements g_1 such that, for every open neighbourhood G of g_1, the inequality $P_\zeta(G) > 0$ is satisfied, we have, by Lemma 14, that

$$m_H(\omega \in \Omega\colon \zeta(s, \omega, F) \in \Phi^{-1}G) > 0.$$

Therefore,

$$m_H(\omega \in \Omega\colon \Phi(\zeta(s, \omega, F)) \in G) = m_H(\omega \in \Omega\colon \zeta(s, \omega, F) \in \Phi^{-1}G) > 0.$$

Since g and G are arbitrary, this proves the lemma. □

Lemma 16. *Suppose that* $\Phi \in U_{F,V}$. *Then the support of the random element* $\Phi(\zeta_V(s, \omega, F))$ *is the whole of* $H(D_V)$.

Proof. We argue similarly to the proof of Lemma 15. Let $\hat{g}$ be an arbitrary element of $H(D_V)$, and G be any open neighbourhood of $\hat{g}$. Then $\Phi^{-1}G$ is also an open set. We have to prove that the set $(\Phi^{-1}G) \cap S_{F,V}$ is non-empty.

The space $H(D_V)$ is metrisable. It is known (see, for example, [9]) that there exists a sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of D_V such that

$$D_V = \bigcup_{n=1}^{\infty} K_n,$$

$K_n \subset K_{n+1}$ for all $n \in \mathbb{N}$, and if K is a compact subset of D_V, then $K \subset K_n$ for some $n \in \mathbb{N}$. Then we have that

$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{s \in K_n} |f(s) - g(s)|}{1 + \sup_{s \in K_n} |f(s) - g(s)|}, \quad f, g \in H(D_V),$$

is a metric on $H(D_V)$ which induces the topology of uniform convergence on compacta.

Let $\varepsilon > 0$ be an arbitrary fixed number. We fix $n_0 \in \mathbb{N}$ such that

$$\sum_{n > n_0} 2^{-n} < \frac{\varepsilon}{2}. \tag{6}$$

Now if

$$\sup_{s\in K_{n_0}} |f(s)-g(s)| < \frac{\varepsilon}{2},$$

then, in view of the relation $K_n \subset K_{n+1}$, $n \in \mathbb{N}$, we have that

$$\sup_{s\in K_n} |f(s)-g(s)| < \frac{\varepsilon}{2}$$

for all $n = 1, \dots, n_0$. This and (6) show that

$$\rho(f, g) = \sum_{n=1}^{n_0} 2^{-n} \frac{\sup_{s\in K_n} |f(s)-g(s)|}{1+\sup_{s\in K_n} |f(s)-g(s)|} + \sum_{n>n_0} 2^{-n} < \varepsilon.$$

Thus, the function g approximates a function f with a given accuracy in $H(D_V)$ if g approximates f with a suitable accuracy uniformly on K_n for sufficiently large n. Clearly, the sets K_n can be chosen so as to be with connected complements. Therefore, in the space $H(D_V)$, we can restrict our consideration only to approximation on compact subsets with connected complements.

Let $K \subset D_V$ be a compact subset with connected complement. Then, by the Mergelyan theorem, there exists a polynomial $p = p(s)$ which approximates $\hat{g}$ uniformly on K with desired accuracy. Therefore, since $\hat{g} \in G$, we may assume that $p \in G$, too. Since $\Phi \in U_{F,V}$, we have that the set $(\Phi^{-1}\{p\}) \cap S_{F,V}$ is non-empty. Therefore, $(\Phi^{-1}G) \cap S_{F,V} \neq \varnothing$. Hence, using Lemma 13, we find that

$$m_H(\omega \in \Omega\colon \Phi(\zeta_V(s, \omega, F)) \in G) > 0,$$

and the lemma is proved. □

Lemma 17. *Suppose that $\Phi \in U_{F;a_1,\dots,a_r}$. Then the support $S_{\zeta,\Phi}$ of the random element $\Phi(\zeta(s, \omega, F))$ includes the closure of the set $H_{\Phi(0);a_1,\dots,a_r}(D)$.*

Proof. The definition (2) of the class $U_{F;a_1,\dots,a_r}$ shows that, for each $g \in H_{\Phi(0);a_1,\dots,a_r}(D)$, there exists an element $g_1 \in S_F$ such that $\Phi(g_1) = g$. Hence, for every open neighbourhood G of $g \in H_{\Phi(0);a_1,\dots,a_r}(D)$, in view of Lemma 14,

$$m_H(\omega \in \Omega\colon \Phi(\zeta(s, \omega, F)) \in G) = m_H(\omega \in \Omega\colon \zeta(s, \omega, F) \in \Phi^{-1}G) > 0.$$

This implies that $g \in S_{\zeta,\Phi}$, so $H_{\Phi(0);a_1,\dots,a_r}(D)$ is, and hence its closure is, a subset of $S_{\zeta,\Phi}$. □

REMARK. If, instead of (2), the stronger condition

$$\Phi(S_F) = H_{\Phi(0);a_1,\dots,a_r}(D) \tag{7}$$

holds, then $S_{\zeta,\Phi}$ coincides with the closure of $H_{\Phi(0);a_1,\dots,a_r}(D)$. In fact, by (7) we have $S_F \subset \Phi^{-1}(H_{\Phi(0);a_1,\dots,a_r}(D))$, and hence

$$m_H(\omega \in \Omega\colon \Phi(\zeta(s,\omega,F)) \in H_{\Phi(0);a_1,\dots,a_r}(D)) \geq m_H(\omega \in \Omega\colon \zeta(s,\omega,F) \in S_F),$$

but the right-hand side is equal to 1 by Lemma 14, hence

$$m_H(\omega \in \Omega\colon \Phi(\zeta(s,\omega,F)) \in H_{\Phi(0);a_1,\dots,a_r}(D)) = 1.$$

Because of the minimality of the support, this implies that $S_{\zeta,\Phi}$ is a subset of the closure of $H_{\Phi(0);a_1,\dots,a_r}(D)$.

4. Proofs of Theorems 3 to 6

Proof of Theorem 3. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{8}$$

We take the open subset

$$\mathcal{G} = \left\{h \in H(D)\colon \sup_{s\in K}|h(s) - p(s)| < \frac{\varepsilon}{2}\right\}$$

of the space $H(D)$. Recall that $P_{\zeta,\Phi}$ denotes the distribution of the random element $\Phi(\zeta(s,\omega,F))$. We find, by Lemmas 10 and 7, that

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\{\tau \in [0,T]\colon \Phi(\zeta(s+i\tau,F)) \in \mathcal{G}\} = \liminf_{T\to\infty} P_{T,\Phi}(\mathcal{G}) \geq P_{\zeta,\Phi}(\mathcal{G}). \tag{9}$$

In virtue of Lemma 15, the polynomial $p(s)$ belongs to the support of the random element $\Phi(\zeta(s,\omega,F))$. Therefore, since $\mathcal{G}$ is an open neighbourhood of $p(s)$, a property of the support implies the inequality $P_{\zeta,\Phi}(\mathcal{G}) > 0$. This together with (9) shows that

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\left\{\tau \in [0,T]\colon |\Phi(\zeta(s+i\tau,F)) - p(s)| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (8) proves the theorem. □

Proof of Theorem 4. We use Lemmas 12, 7, and 16, and repeat the arguments of the proof of Theorem 3. □

Proof of Theorem 5. (i) Put

$$\mathcal{G}_1 = \left\{g \in H(D)\colon \sup_{s\in K}|g(s) - p(s)| < \varepsilon\right\}.$$

By the hypotheses of the theorem, the function $f(s)$ belongs to $H_{\Phi(0);a_1,\ldots,a_r}(D)$, thus, by Lemma 17, it is an element of the support of the measure $P_{\zeta,\Phi}$. Therefore, $P_{\zeta,\Phi}(\mathcal{G}_1) > 0$, and Lemmas 10 and 7 show that

$$\begin{aligned}&\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K}|\Phi(\zeta(s+i\tau,F)) - f(s)| < \varepsilon\right\}\\ &\geq P_{\zeta,\Phi}(\mathcal{G}_1) > 0.\end{aligned}$$

(ii) Now suppose that $r = 1$. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{10}$$

Since $f(s) \neq a_1$ on K, we have that $p(s) \neq a_1$ on K as well if ε is small enough. Therefore, we can define a continuous branch of $\log(p(s) - a_1)$ which will be an analytic function in the interior of K. By the Mergelyan theorem again, there exists a polynomial $p_1(s)$ such that

$$\sup_{s\in K}|p(s) - a_1 - e^{p_1(s)}| < \frac{\varepsilon}{4}. \tag{11}$$

Let

$$h_{a_1}(s) = e^{p_1(s)} + a_1.$$

Then $h_{a_1}(s) \in H(D)$, and $h_{a_1}(s) \neq a_1$. Therefore, by Lemma 17, the function $h_{a_1}(s)$ is an element of the support of the random element $\Phi(\zeta(s,\omega,F))$. Moreover, it follows from (10) and (11) that

$$\sup_{s\in K}|f(s) - h_{a_1}(s)| < \frac{\varepsilon}{2}. \tag{12}$$

Define

$$\mathcal{G}_2 = \left\{g \in H(D)\colon \sup_{s\in K}|g(s) - h_{a_1}(s)| < \frac{\varepsilon}{2}\right\}.$$

Then we have that $P_{\zeta,\Phi}(\mathcal{G}_2) > 0$, and hence Lemmas 10 and (10) yield

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon |\Phi(\zeta(s+i\tau,F)) - h_{a_1}(s)| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (12), we obtain the assertion. □

Proof of Theorem 6. It is not difficult to see that the support $S_{\zeta,\Phi}$ of the measure $P_{\zeta,\Phi}$ is the closure of $\Phi(S_F)$. In fact, if g is an arbitrary element of $\Phi(S_F)$, and G is its any open neighbourhood, then by Lemma 14 we have that

$$m_H(\omega \in \Omega\colon \zeta(s,\omega,F) \in \Phi^{-1}G) > 0.$$

Hence,

$$m_H(\omega \in \Omega\colon \Phi(\zeta(s,\omega,F)) \in G) > 0.$$

Moreover, by Lemma 14,

$$m_H(\omega \in \Omega\colon \Phi(\zeta(s,\omega,F)) \in \Phi(S_F)) = m_H(\omega \in \Omega\colon \zeta(s,\omega,F) \in S_F) = 1.$$

Thus the support $S_{\zeta,\Phi}$ is the closure of the set $\Phi(S_F)$.

The rest of the proof is the same as that of Theorem 5 (i). □

5. Examples

We conclude this paper with giving several explicit examples of operators satisfying the assumptions of our theorems.

(I) First we give an example of $\Phi \in U_{F,V}$, hence Theorem 4 can be applied to this Φ.

For $g \in H(D_V)$, let

$$\Phi(g) = c_1 g' + \cdots + c_r g^{(r)}, \quad c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\}.$$

In virtue of the Cauchy integral formula, the function Φ is continuous. Moreover, for each polynomial $p = p(s)$, there exists a polynomial $q = q(s)$, $q \in \Phi^{-1}\{p\}$, and $q(s) \neq 0$ for $s \in D_V$. In fact, let

$$p(s) = a_0 + a_1 s + \cdots + a_k s^k, \quad a_k \neq 0.$$

We take

$$q(s) = b_0 + b_1 s + \cdots + b_{k+1} s^{k+1}, \quad b_{k+1} \neq 0.$$

Then, in the case $r \le k+1$,

$$\begin{cases} q'(s) = b_1 + 2b_2 s + \cdots + (k+1) b_{k+1} s^k, \\ q''(s) = 2b_2 + \cdots + (k+1)k b_{k+1} s^{k-1}, \\ \cdots \\ q^{(r)}(s) = r!\, b_r + \cdots + (k+1)k \cdots (k-r+2) b_{k+1} s^{k-r+1}. \end{cases}$$

Thus, the equation $\Phi(q) = p$ implies

$$\begin{aligned} &c_1 b_1 + 2c_1 b_2 s + \cdots + (k+1) c_1 b_{k+1} s^k \\ &+ 2c_2 b_2 + \cdots + (k+1)k c_2 b_{k+1} s^{k-1} + \cdots \\ &+ r!\, c_r b_r + \cdots + (k+1)k \cdots (k-r+2) c_r b_{k+1} s^{k-r+1} \\ &= a_0 + a_1 s + \cdots + a_k s^k. \end{aligned}$$

Hence,

$$\begin{cases} (k+1)c_1 b_{k+1} = a_k, \\ kc_1 b_k + (k+1)kc_2 b_{k+1} = a_{k-1}, \\ \cdots \\ c_1 b_1 + 2c_2 b_2 + \cdots + r!\, c_r b_r = a_0. \end{cases}$$

The case $r > k+1$ is considered similarly. So, in any case, we may determine the coefficients $b_1, \dots, b_{k+1}$. After this procedure, we may choose b_0 to be $|b_0|$ large enough, so that $q(s) \neq 0$ for $s \in D_V$.

(II) Next we give several explicit examples of Theorems 5 and 6. First, consider the operator $\Phi\colon g \to g^N$ ($N \in \mathbb{N}$). Then $\Phi(0) = 0$. If $f \in H(D)$ is non-vanishing on D, then we can find a solution $g \in S_F$ such that $\Phi(g) = f$, that is, $g = \sqrt[N]{f}$. Therefore $\Phi \in U_{F;0}$. Applying Theorem 5 with $r = 1$, $a_1 = 0$, we obtain the universality of $\zeta(s, F)^N$.

The second example is $\Phi\colon g \to gg'$. In this case we have to solve the equation $(g^2(s))' = f(s)$ ($f \in H(D)$) in $g \in S_F$. Thus, $g^2(s) = f_1(s)$, where $f_1(s) \in H(D)$ is a primitive function of $f(s)$. If $f_1(s) \neq 0$ on D, we can solve this equation as $g(s) = \sqrt{f_1(s)}$. Therefore $f \in \Phi(S_F)$ and Theorem 6 can be applied to obtain the universality of $\zeta(s, F)\zeta'(s, F)$.

Thirdly consider $\Phi\colon g \to e^g$. Then $\Phi(0) = 1$. We apply Theorem 5 with $r = 2$, $a_1 = 0$, $a_2 = 1$. If $f \in H(D)$ does not take the values 0, 1 on D, then $\Phi(g) = f$ has the solution $g = \log f$ (where we may choose an arbitrary branch of the logarithm, but we require that it is continuous on D), and g is non-vanishing on D. Hence $g \in S_F$, and so $\Phi \in U_{F;0,1}$. Therefore Theorem 5 gives the universality of $e^{\zeta(s,F)}$ for $f \in H_{1;0,1}(D)$.

Finally we discuss trigonometric functions. Consider Φ: $g \to \cos g = (e^{ig} + e^{-ig})/2$. Then $\Phi(0) = 1$. We apply Theorem 5 with $r = 2$, $a_1 = 1$, $a_2 = -1$. If $f \in H_{1;1,-1}(D)$, we can choose the solution

$$g = \frac{1}{i} \log(f + (f^2 - 1)^{1/2})$$

for $\Phi(g) = f$. In fact, since f does not take the values ± 1, the term $(f^2 - 1)^{1/2}$ is well-defined. Also $f + (f^2 - 1)^{1/2} \neq 0$ (because if $= 0$, then $(f^2 - 1)^{1/2} = -f$, so $f^2 - 1 = f^2$, a contradiction). Therefore g is well-defined. Since $e^{ig} = f + (f^2 - 1)^{1/2}$ and

$$e^{-ig} = \frac{1}{f + (f^2 - 1)^{1/2}} = f - (f^2 - 1)^{1/2},$$

we find $\cos g = f$. Moreover $f + (f^2 - 1)^{1/2} \neq 1$ (because if $= 1$, then $(f^2 - 1)^{1/2} = 1 - f$, hence $f = 1$, which contradicts the assumption), so $g \in S_F$. Therefore $\Phi \in U_{F;1,-1}$, and $\cos(\zeta(s, F))$ has the universality property for $f \in H_{1;1,-1}(D)$. Similarly we can prove the universality of $\sin(\zeta(s, F))$, and also of $\sinh(\zeta(s, F))$ and $\cosh(\zeta(s, F))$.

REMARK. (i) In the case $\Phi\colon g \to g^N$, the stronger condition (7) holds (see the remark at the end of Section 3).
(ii) In the case $\Phi\colon g \to e^g$, to apply Theorem 5 we assume in the above that $f(s) \neq 0, 1$ on D. However, if we restrict our consideration to the case of compact K with connected complement, it is possible to remove the latter assumption $f(s) \neq 1$. In fact, instead of $g = \log f$ we can choose $g = \log f + 2\pi ik$ ($k \in \mathbb{N}$) as a solution of $\Phi(g) = f$. For any compact K, it is possible to find $k = k(K)$ for which $\log f + 2\pi ik \neq 0$ on K. Since K is with connected complement, by the universality theorem ([17]) we find τ for which $|\zeta(s + i\tau, F) - (\log f(s) + 2\pi ik)|$ is small on K, hence $|e^{\zeta(s+i\tau,F)} - f(s)|$ is also small on K.

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References

[1] J. Andersson: *Disproof of some conjectures of K. Ramachandra*, Hardy–Ramanujan J. **22** (1999), 2–7.
[2] B. Bagchi: *The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series*, Ph.D. Thesis, Indian Statistical Institute, Calcutta, 1981.
[3] B. Bagchi: *A joint universality theorem for Dirichlet L-functions*, Math. Z. **181** (1982), 319–334.
[4] B. Bagchi: *Recurrence in topological dynamics and the Riemann hypothesis*, Acta Math. Hungar. **50** (1987), 227–240.
[5] P. Billingsley: Convergence of Probability Measures, Wiley, New York, 1968.
[6] K.M. Bitar, N.N. Khuri and H.C. Ren: *Path integrals and Voronin's theorem on the universality of the Riemann zeta function*, Ann. Physics **211** (1991), 172–196.
[7] H. Bohr: *Zur Theorie der Riemannschen Zetafunktion im kritischen Streifen*, Acta Math. **40** (1915), 67–100.
[8] H. Bohr: *Über eine quasi-periodische Eigenschaft Dirichletscher Reihen mit Anwendung auf die Dirichletschen L-Funktionen*, Math. Ann. **85** (1922), 115–122.
[9] J.B. Conway: Functions of One Complex Variable, Springer, New York, 1973.
[10] S.M. Gonek: *Analytic properties of zeta and L-functions*, Ph.D. Thesis, University of Michigan, 1979.
[11] A. Kačėnas and A. Laurinčikas: *On Dirichlet series related to certain cusp forms*, Liet. Mat. Rink. **38** (1998), 82–97, (Russian), Lithuanian Math. J. **38** (1998), 64–76.
[12] A. Laurinčikas: Limit Theorems for the Riemann Zeta-Function, Kluwer Acad. Publ., Dordrecht, 1996.
[13] A. Laurinčikas: *The universality of zeta-functions*, Acta Appl. Math. **78** (2003), 251–271.
[14] A. Laurinčikas: *On the derivatives of zeta-functions of certain cusp forms*, Glasg. Math. J. **47** (2005), 87–96.
[15] A. Laurinčikas: *Universality of the Riemann zeta-function*, J. Number Theory **130** (2010), 2323–2331.
[16] A. Laurinčikas and R. Garunkštis: The Lerch Zeta-Function, Kluwer Acad. Publ., Dordrecht, 2002.

[17] A. Laurinčikas and K. Matsumoto: *The universality of zeta-functions attached to certain cusp forms*, Acta Arith. **98** (2001), 345–359.

[18] A. Laurinčikas, K. Matsumoto and J. Steuding: *The universality of L-functions associated with new forms*, Izv. Ross. Akad. Nauk Ser. Mat. **67** (2003), 83–98, (Russian), Izv. Math. **67** (2003), 77–90.

[19] K. Matsumoto: *Probabilistic value-distribution theory of zeta-functions*, Sūgaku **53** (2001), 279–296, (Japanese), Sugaku Expositions **17** (2004), 51–71.

[20] K. Matsumoto: *Some problems on mean values and the universality of zeta and multiple zeta-functions*; in Analytic and Probabilistic Methods in Number Theory (Palanga, 2001), TEV, Vilnius, 2002, 195–199.

[21] S.N. Mergelyan: *Uniform approximations of functions of a complex variable*, Uspehi Matem. Nauk (N.S.) **7** (1952), 31–122, (Russian).

[22] H. Mishou and H. Nagoshi: *Functional distribution of $L(s, \chi_d)$ with real characters and denseness of quadratic class numbers*, Trans. Amer. Math. Soc. **358** (2006), 4343–4366.

[23] T. Nakamura: *Zeros and the universality for the Euler–Zagier–Hurwitz type of multiple zeta-functions*, Bull. Lond. Math. Soc. **41** (2009), 691–700.

[24] T. Nakamura: *The universality for linear combinations of Lerch zeta functions and the Tornheim–Hurwitz type of double zeta functions*, Monatsh. Math. **162** (2011), 167–178.

[25] A. Ostrowski: *Über Dirichletsche Reihen und algebraische Differentialgleichungen*, Math. Z. **8** (1920), 241–298.

[26] J. Steuding: *On the universality for functions in the Selberg class*; in Proceedings of the Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften **360**, Univ. Bonn, Bonn, 2003, 22 pp.

[27] J. Steuding: Value-Distribution of L-Functions, Lecture Notes in Mathematics **1877**, Springer, Berlin, 2007.

[28] S.M. Voronin: *A theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 475–486, (Russian), Math. USSR Izv. **9** (1975), 443–453.

[29] J.L. Walsh: Interpolation and Approximation by Rational Functions in the Complex Domain, Third edition. American Mathematical Society Colloquium Publications **20**, Amer. Math. Soc., Providence, RI, 1960.

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