APPLICATION OF HADAMARD VARIATION TO A FREE BOUNDARY PROBLEM

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Abstract

The study of variational inequalities and free boundary problems finds application in a wide variety of disciplines including physics, engineering and economics as well as potential theory and geometry. In this study, we consider an application in physics and engineering where the steady state of the fluid flow through flow media rises to free boundary problem for linear elliptic equations. This problem has been widely considered in literature by Baiocchi [7], Brezis [9], Chipot [12] etc. At first, we deal with quantities defined on a domain. If the domain is perturbed, the quantities are perturbed. Such variation with respect to domain perturbation is called Hadamard variation. In this research, we present Hadamard variation as an iterative scheme for computing solutions of a free boundary problem. We also combine this scheme with the other iterative scheme, traction method. The iteration converges smoothly, beginning from a suitably defined initial guess. We then study the Hadamard variational formula theoretically. Particularly, we obtain Hadamard second variational formula which is an extension of Garabedian-Schiffer's formula, developing a simple methodology which provide a clearer understanding of Hadamard variational formula.

Contents

Introduction

Filtration Problem

Filtration or dam problem is a mathematical problem of understanding the fluid behaviour solving free boundary problem. In general, such a free boundary is not a priori known and the main problem is determining it. The dam is occupies by fluid that is; water. In addition, the pressure and velocities which occupies the domain also need to be determined. Besides the equations, an initial and boundary condition on the fixed boundary, the problem description also requires an initial configuration of the liquid domain as well as some conditions which hold along the free boundary. Here, steady flows in which water seeps through the dam built by porous media from a higher reservoir to a lower reservoir will be consider. The water does not flow through the entire dam because of gravity, and the lower reservoir side of the dam is dry. The interface separating the dry and wet regions of the dam is a free boundary. The fluid can flow out of the domain through the lower reservoir side, while a free-slip condition was imposed on the reservoir side and at the bottom. Numerically, the former approximation of the wet region of the domain will be derived. The wet region was set before doing analysis to the filtration problem. Since the dam has a free surface, an assumption to the dam shape has to be made which will be assumed as a rectangular shape. There are many problems in tackling the free boundary. Therefore, we define the problem on the fixed domain with the wet portion is in the admissible domain of the dam.

Mathematical Background

One method used to solve mathematical models of free and moving boundary flow is the fixed domain method which can be classified into two groups:

1. variational inequality.
2. extended pressure head method.

Fixed domain methods of variational problems can be divided into three parts:

1. variational inequality / quasi-variational inequality formulation.

2. general inequality formulations - set solved in fixed domains.

3. free residual flow.

1. **Variational Inequality Formulation or Quasi- Variational Inequality Formulation:** Baiocchi [5] was the first to apply the variational inequality method. This method is one of the classical methods to solve free boundary problems. It also uses an extension of the pressure head but adds an integral transformation. This method possesses a beautiful theoretical structure and yields simple numerical algorithms.

2. **General Inequality Formulation:** It was introduced by Brezis et. al. (1970's) and Alt [1]. It was a new approach and formulation which required no integral transformation. Variational inequalities compete well with direct method for solving the seepage problem numerically.

   More complicated, the solution is less regular but allowed more general cases especially regarding to the geometric domain. Alt's formulation allowed the possibility of partially saturated flow fields. General formulation was treated from a numerical point of view. However, its framework was restricted to a certain types of finite elements. Fixed boundaries also could not be modelled precisely. Nevertheless, it was able, for steady state cases, to predict zones of saturation and partial saturation in seepage flow fields.

3. **Free Residual Flow:** It was introduced by Dasai (1976), and he used finite element in conjunction to see the essence of the approach.

Becoming a model of the variational theory of boundary value problems for partial differential equations, the theory of variational inequality was developed very fast. Its theory represents a truly natural generalization of the theory of the boundary value problems. It allows us to consider new problems arising from many fields of applied mathematics such as mechanics and physics. Variational inequality also one of the well-known and strong tool to deal with many free boundary problems arising from natural phenomena, especially fluid dynamics. From these facts, we use variational inequality to solve the filtration problem. One of the methods in variational inequality is the Hadamard variational formula.

The Hadamard variational formula is a famous formula for simplification of Laplace eigenvalue under Dirichlet boundary conditions including Neumann and mixed boundary conditions. Hadamard variation is a process where the boundary is modified gradually where we want to know how the boundary values will vary under the perturbation of the domain (or the boundary). This process is also an optimization process. We present Hadamard variation in this research as an iterative scheme for solving the filtration problem. Mathematical basis of the above materials is listed in Appendices of this thesis.

Chapter 1

Analytic Methods to the Filtration Problem

1.1 Variational Inequality

We consider an incompressible stationary flow of water through an inhomogeneous porous media (say $\mathscr{Dam}$). The dam is assumed to be with a parallel vertical walls, with the distance a apart, separated by two reservoirs of fluid (water) and two water levels denoted by height, $x_2 = h_1$ and $x_2 = h_2$ respectively $(0 \leq h_2 < h_1 \leq x_2)$. Let $\mathscr{Dam}$ be the region of porous dam whose boundary is Lipschitz continuous. We assume that the boundary of the dam, $\partial(\mathscr{Dam})$ contains three parts, B_1, the impervious part, B_2, the part contacting with air, and B_3, the part contacting with water reservoirs. Let Ω denote the portion of water in $\mathscr{Dam}$ and let the boundary $\partial(\mathscr{Dam})$ consist of the following:

$\Gamma_1 = B_1$	the impervious part
$\Gamma_2 \subset \mathscr{Dam}$	the free boundary
$\Gamma_3 = B_3$	the part contacting with water reservoirs
$\Gamma_4 \subset B_2$	the part contacting with air

This filtration problem has a free boundary part which connects with the water reservoirs. See Figure 1.1 below.

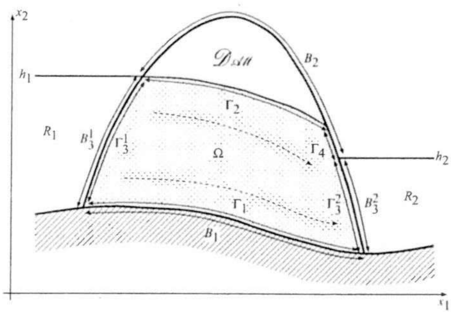

Figure 1.1: The configuration of the dam.

We assume that there are two disjoint reservoirs, R_j, $(j=1,2)$ and both parts touch the impervious basement. Let $h_j(h_1>h_2)$ be the height of the water in R_j, $(j=1,2)$. Set $B_3^j := \partial R_j \cap B_3$ which implies $B_3 = B_3^1 \cup B_3^2$. Let p be the hydrostatic pressure of the fluid and piezometric head, u defines by

$$u(x) = p(x) + x_2 \tag{1.1}$$

for $x=(x_1,x_2)$ in Ω. Since the pressure p on B_3^j is given by $h_j - x_2$, it holds that $u = h_j$ on B_3^j, $(j=1,2)$. We then introduce a subset of $\mathcal{DAM}$. Let ζ_2 be the point where the left surface of the contacts $\partial(\mathcal{DAM})$, that is, $\zeta_2 = \overline{B}_3^1 \cap \overline{B}_2$. Suppose that $\eta > 0$ is the angle between B_3^1 and l and the other is on B_1. Then $\mathcal{DAM}^0 \subset \mathcal{DAM}$ is defined using B_3^1 and l (See Figure1.2). Set

$$\mathcal{DAM}^1 = \{x=(x_1,x_2) \in \ \mathcal{DAM} - \mathcal{DAM}^0 | x_2 \geq h_2\}$$
$$\mathcal{DAM}^2 = \{x=(x_1,x_2) \in \ \mathcal{DAM} - \mathcal{DAM}^0 | x_2 < h_2\}.$$

We define $u^0 \in H^1(\mathcal{DAM})$ by

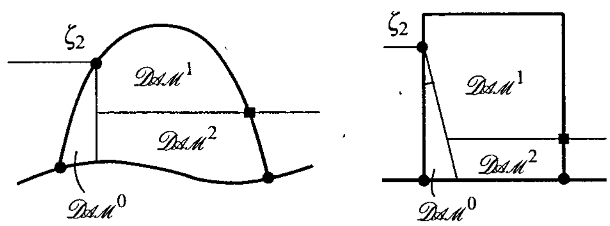

Figure 1.2: $\mathcal{DAM}^j$ $(j=0,1,2)$.

$$u^0(x) = \begin{cases} h_1, & \text{on } B_3^1; \\ x_2, & \text{in } \mathcal{DAM}^1; \\ h_2, & \text{in } \mathcal{DAM}^2. \end{cases} \tag{1.2}$$

Define also

$$\mathbb{K} := \{\zeta \in H^1(\mathcal{DAM}) | \zeta \geq 0 \ \text{ on } \ B_2, \zeta = 0 \ \text{ on } \ B_3\}. \tag{1.3}$$

Then the problem is formulated to find the wet region $\Omega \subset \mathcal{DAM}$ and the piezometric function $u \in H^1(\Omega)$ that defined on Ω which satisfies

$$\begin{aligned} &\textstyle\int_\Omega \nabla u \cdot \nabla \zeta \ dx \leq 0, && \forall \zeta \in \mathbb{K}, \\ &u = u^0 && \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4. \end{aligned} \tag{1.4}$$

Remark: We consider the variational problem; finding u such that

$$u \in \widehat{\mathbb{K}}, \qquad D(u) = \min_{v \in \widehat{\mathbb{K}}} D(v), \tag{1.5}$$

where $\widehat{\mathbb{K}} = \{\zeta \in \mathbb{K} \mid |\zeta = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\}$. Since $\widehat{\mathbb{K}}$ is convex, if u is a solution to (1.5), then for any $v \in \widehat{\mathbb{K}}$ and $0 < \varepsilon \ll 1$, $u + \varepsilon(v-u) = u(1-\varepsilon) + \varepsilon v \in \widehat{\mathbb{K}}$. Therefore, it holds that

$$D(u + (\varepsilon(v-u)) \geq D(u). \tag{1.6}$$

Hence,

$$\begin{aligned} D(u+\varepsilon(v-u)) &= \int_\Omega \frac{1}{2}|\nabla(u+\varepsilon(v-u))|^2 \; dx \\ &\geq D(u) \\ &= \int_\Omega \frac{1}{2}|\nabla u|^2 \; dx, \qquad 0 < \forall \varepsilon \ll 1. \end{aligned}$$

Then, taking $\varepsilon \downarrow 0$, we get

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq 0, \qquad \forall v \in \widehat{\mathbb{K}}, \;\; u \in \widehat{\mathbb{K}}. \tag{1.7}$$

Applying Green's formula, we obtain,

$$-\int_\Omega \Delta u \cdot (v-u) \; dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(v-u) \; ds \geq 0, \quad \forall v \in \widehat{\mathbb{K}} \tag{1.8}$$

if u is sufficiently regular. Then, we obtain

$$\int_\Omega \Delta u \cdot (v-u) \; dx \leq \int_{\partial\Omega} \frac{\partial u}{\partial n}(v-u) \; ds, \qquad \forall v \in \widehat{\mathbb{K}}. \tag{1.9}$$

By choosing $v = u + \varepsilon\zeta$, $0 < \varepsilon \ll 1$, for any $\zeta \in C_0^\infty(\Omega)$, we get $v \in \widehat{\mathbb{K}}$. Obviously $v \in H^1(\Omega)$. Next, $\gamma v = \gamma(u + \varepsilon v) = \gamma u + \varepsilon \gamma v = \gamma u = \gamma u^0$. Finally, $v = u + \varepsilon\zeta \geq u$ in Ω. Then, since $v - u = 0$ on $\partial\Omega$, we have

$$\int_\Omega \nabla u \cdot (u + \varepsilon\zeta - u) \; dx \leq 0$$

$$\int_\Omega \nabla u \cdot \nabla\zeta \leq 0, \quad \forall \zeta \in C_0^\infty(\Omega)$$

We can conclude that

$$\int_\Omega (\Delta u)\zeta \; dx = 0, \qquad \forall \zeta \in C_0^\infty(\Omega)$$

and therefore,

$$\Delta u = 0 \qquad \text{in } \Omega. \tag{1.10}$$

However, (1.4) is not equivalent to (1.5). We should note that Γ_2 in (1.4) is also unknown. That is why (1.4) has not been formulated by the variational inequality described in Appendix B.

1.2 PDE Formulation

Suppose that u is a solution to (1.4). We then extend $\xi = 0$ outside Ω which implies $\pm\xi \in \mathbb{K}, \forall \xi \in C_0^\infty(\Omega)$. Hence

$$\int_\Omega \nabla u \cdot \nabla \xi \, dx = \int_\Omega (-\Delta u) \cdot \xi \, dx = 0 \tag{1.11}$$

Therefore, if u is regular, we obtain

$$\Delta u = 0 \qquad \text{in } \Omega. \tag{1.12}$$

Generally, when $\zeta \in \mathbb{K}$, we get

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla \zeta \, dx &= \int_\Omega (-\Delta u) \cdot \zeta \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \zeta \, ds \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \zeta \, ds \leq 0. \end{aligned} \tag{1.13}$$

Given $x_0 \in \Gamma_1 \setminus (endpoints)$. Let $\widehat{\Omega}$ be a small ball with the center x_0. Take $\xi \in C_0^\infty(\widehat{\Omega})$, and set $\zeta = \pm\xi \mid_{\overline{\Omega}} \in \mathbb{K}$. Since ξ and x_0 is an arbitrary, (1.13) implies

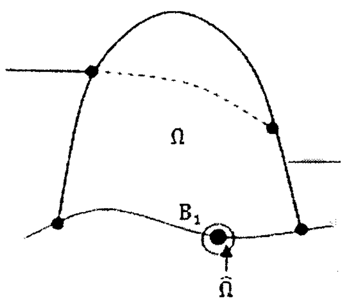

Figure 1.3: Support in Ω

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma_1. \tag{1.14}$$

Similarly, $\frac{\partial u}{\partial n} = 0$ on Γ_2. Since $\frac{\partial u}{\partial n} = 0$ on Γ_1 and Γ_2 is proven, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \zeta \, ds = \int_{\Gamma_3 \cup \Gamma_4} \frac{\partial u}{\partial n} \cdot \zeta \leq 0, \qquad \forall \zeta \in \mathbb{K}.$$

Since $\zeta = 0$ on Γ_3 from (1.3), it holds that

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \zeta \, ds = \int_{\Gamma_4} \frac{\partial u}{\partial n} \cdot \zeta \leq 0, \qquad \forall \zeta \in \mathbb{K}. \tag{1.15}$$

Take $x_0 \in \Gamma_4 \setminus (endpoints)$ and a small ball $\widehat{\Omega}$ with center x_0. Taking $\xi \in C_0^\infty(\widehat{\Omega})$, $\forall \xi \geq 0$, we have $\zeta = \xi\Big|_{\overline{\Omega}} \in \mathbb{K}$. Similarly to (1.14)

$$\frac{\partial u}{\partial n} \leq 0 \qquad \text{on } \Gamma_4. \tag{1.16}$$

We then extend the boundary value of u to $\mathscr{D}\mathscr{M}$:

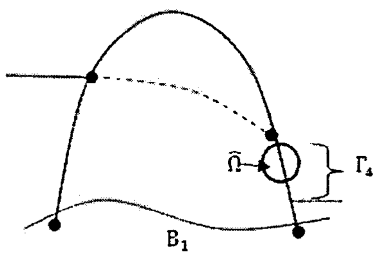

Figure 1.4: Support at Γ_4.

$$\begin{array}{llll} \Delta u = 0 & & & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & & & \text{on } \Gamma_1, \\ u = u^0 & \text{and} & \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2, \\ u = u^0 & & & \text{on } \Gamma_3, \\ u = u^0 & \text{and} & \frac{\partial u}{\partial n} \leq 0 & \text{on } \Gamma_4 \end{array} \tag{1.17}$$

where $n := (n_1, n_2)$ is the unit outer normal vector on $\partial\Omega$. Here, Γ_2 is the free boundary part, and we impose it with both Dirichlet's and Neumann's conditions, while Γ_4 is the part where the water comes out of the dam. In fact, (1.4) is a weak form of (1.17).

Remark: Darcy's law claims that u is the velocity potential of fluid:

$$\mathbf{v} = -k\nabla u$$

where k is the permeability coefficient which is assumed to be constant. We assume that the density of the fluid is a constant. Then, $|u|^2_{H^1(\Omega)} = 2D_\Omega(u)$ (where $|u|_{H^1(\Omega)}$ is a seminorm) is equal to the kinetic energy of the fluid up to a constant, where $D_\Omega(v)$ is the Dirichlet integral defined by

$$D_\Omega(v) = \frac{1}{2}\int_\Omega |\nabla v(x)|^2 \; dx \qquad \text{for } v \in \mathbb{K}.$$

1.3 A New Variational Formulation

In this part, we define a new variational formulation of the dam problem.

Assumption 1.3.1. *On the configuration of the Lipschitz domain $\mathscr{D}_{\mathscr{AM}}$, we suppose the following conditions:*

1. *There are two reservoirs of water (one of them maybe empty) separated by the dam. Assuming without loss of generality, water level of the left-hand side reservoir is higher than the other side.*
2. *Each reservoir contacts the impervious base.*
3. *$B_1 \subset \partial(\mathscr{D}_{\mathscr{AM}})$ (impervious part) and $B_2 \cup B_3 \subset \partial(\mathscr{D}_{\mathscr{AM}})$ (air and water parts) are continuous, piecewise C^2 curves, both are graphs in the direction of x_2, and $B_2 \cup B_3$ lies above B_1. Particularly, $B_2 \cup B_3$ is of C^2 around the point $\overline{\Gamma}_3^2 \cap \overline{\Gamma}_4$ (where B_2 meets the surface of the right hand reservoir), if $\overline{\Gamma}_3^2 \neq 0$.*
4. *For the exact solution $\Omega \subset \mathscr{D}_{\mathscr{AM}}$, $\Gamma_4 = \partial\Omega \cap B_2$ is a connected non-empty interval in B_2.*

We consider a set of admissible domains, which are candidates for the solution of the filtration problem. Let $\Omega \subset \mathscr{D}_{\mathscr{AM}}$ be a set which looks like Figure 1.1. As shown in Figure 1.1, we denote each portion of $\partial\Omega$ by

$$
\begin{aligned}
&\Gamma_1 = B_1,\\
&\Gamma_2 = \partial\Omega \cap \mathscr{D}_{\mathscr{AM}},\\
&\Gamma_3^j = B_3^j \ (j = 1,2) \text{ and } \Gamma_3 = \Gamma_3^1 \cup \Gamma_3^2,\\
&\Gamma_4 = \partial\Omega \cup B_2.
\end{aligned}
$$

Then, let $\mathscr{K} = \mathscr{K}(\Omega) \in H^1(\Omega)$ and $\mathscr{K}^* = \mathscr{K}^*(\Omega)$ be its dual cone of $\mathscr{K}(\Omega)$:

$$
\begin{aligned}
\mathscr{K} &= \{v \in H^1(\Omega) | v = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, v \geq 0 \ \text{ on } \ \Gamma_4\},\\
\mathscr{K}^* &= \{v \in H^1(\Omega) | (\nabla v, \nabla \chi) \leq 0, \forall \chi \in K(\Omega)\}.
\end{aligned}
$$

Here, the boundary value of $v \in H^1(\Omega)$ is taken in the sense of trace. Also, for $\chi \in C^{0,1}(\Omega) \cap C^2(\Omega)$ with $\Delta\chi = 0$ in Ω, we have

$$
\chi \in \mathscr{K}^*(\Omega) \quad \Leftrightarrow \quad \frac{\partial \chi}{\partial n} \leq 0 \ \text{ on } \ \Gamma_4.
$$

In fact,

$$
(\nabla v, \nabla \chi) = \int_{\partial\Omega} v \cdot \frac{\partial \chi}{\partial n} \ ds - \int_{\Omega} v \cdot \Delta\chi \ dx \leq 0
$$

Since $\Delta\chi = 0$ and $v \in \mathscr{K}(\Omega)$ is arbitrary, we get

$$
\int_{\Gamma_4} v \cdot \frac{\partial \chi}{\partial n} \leq 0,
$$

therefore,

$$
\frac{\partial \chi}{\partial n} \leq 0 \qquad \text{on } \Gamma_4. \tag{1.18}
$$

Then we define $\mathscr{A}(\Omega) \in H^1(\Omega)$ by

$$\mathscr{A}(\Omega) = \{v \in \mathscr{K}^*(\Omega) | v = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\}, \tag{1.19}$$

where u^0 is defined by (1.2). This set $\mathscr{A}(\Omega)$ may be "wild" as the free boundary Γ_2. Assume that this Γ_2 is continuous. If $\mathscr{A}(\Omega) \neq 0$, there exists a unique solution of $u_\Omega \in \mathscr{A}(\Omega)$ which attains the minimum value of the Dirichlet integral:

$$D_\Omega(u_\Omega) = \inf_{v \in \mathscr{A}(\Omega)} D_\Omega(v).$$

Hence,

$$D(u_\Omega + s\zeta) \geqq D(u_\Omega), \qquad \forall \zeta \in C_0^\infty(\Omega)$$

where ζ is an arbitrary, $|s| \ll 1$ and $v = u_\Omega + s$, $\zeta \in \mathscr{A}(\Omega)$. Particularly, the function

$$s \mapsto D(u_\Omega + s\zeta)$$

attains its minimum at $s = 0$, so that

$$\frac{d}{ds} D_\Omega(u_\Omega + s\zeta)\Big|_{s=0} = 0.$$

Therefore,

$$\frac{d}{ds} D_\Omega(u_\Omega + s\zeta)\Big|_{s=0} = \int_\Omega \nabla u_\Omega \cdot \nabla \zeta \ dx$$

i.e.

$$\int_\Omega \nabla u_\Omega \cdot \nabla \zeta \ dx = 0, \qquad \forall \zeta \in \mathscr{K}^*(\Omega), \qquad v = u^0 \text{ on } \partial\Omega. \tag{1.20}$$

If u_Ω is sufficiently regular on $\overline{\Omega}$, then $\Delta u_\Omega = 0$ in Ω. The relation

$$\frac{\partial u_\Omega}{\partial n} = 0 \qquad \text{on } \Gamma_1 \tag{1.21}$$

is obtained similarly. Thus u_Ω satisfies

$$\begin{aligned} &\Delta u_\Omega = 0 \text{ in } \Omega, \quad u_\Omega = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ &\frac{\partial u_\Omega}{\partial n} \leq 0 \text{ on } \Gamma_4, \quad \frac{\partial u_\Omega}{\partial n} = 0 \text{ on } \Gamma_1. \end{aligned} \tag{1.22}$$

Although, $\frac{\partial u_\Omega}{\partial n} \neq 0$ on Γ_2 in general, u_Ω may be regarded as velocity potential of fluid whose region is Ω. The condition $\mathscr{A}(\Omega) \neq \emptyset$ says that the free boundary Γ_2 is sufficiently smooth so that the flow was determined by the potential u_Ω that has the kinetic energy $D(u_\Omega)$.

Definition 1.3.2 (Admissible domain). *Let $\Omega_0 \subset \mathscr{D}_{\mathscr{M}}$ be the exact solution of the dam problem, then from the Assumption 1.3.1(4), $\Gamma_4 = \partial\Omega_0 \cap B_2$ is a dis-connected interval with the lower end point $\zeta_4 = \overline{\Gamma}_3^2 \cap \overline{\Gamma}_4$.*

We set a nonempty interval $b_2 \subset \partial\Omega_0 \cup B_2$ with $\zeta_4 \in \overline{b}_2$. Take and fix M_0 as sufficiently large positive number.

Definition 1.3.3. *Under the Assumption 1.3.1, a subset $\Omega \subset \mathscr{D}_{\mathscr{M}}$ is called an admissible domain if Ω satisfies the following:*

1. *Ω is a Lipschitz domain.*

2. *$\partial\Omega \supset B_1 \cup B_3$.*

3. *$\partial\Omega \setminus \overline{B_1 \cup B_3}$ is a $C^{0,1}$ curve and is a monotone decreasing graph in the direction x_2.*

4. *$\mathscr{A}(\Omega) \neq \emptyset$ and $\inf_{v \in A(\Omega)} D_\Omega(v) \leq M_0$.*

5. *$\partial\Omega \cap B_2$ is a connected interval in B_2 and $b_2 \subset \partial\Omega \cap B_2$.*

We define $\mathscr{A}_{\mathscr{D}}$ as the set of an admissible domain.

Assume $\Gamma_4 \neq 0$ as the exact solution of Ω of the filtration problem from the Definition 1.3.3(5) associated with Assumption 1.3.1 which are the conditions for lower semicontinuinity of J. Corresponding to $\mathscr{A}(\Omega)$ that defined by (1.19), we define the subset for each $\Omega \in \mathscr{A}_{\mathscr{D}}$:

$$\mathscr{B}(\Omega) = \{v \in H^1(\Omega) | v = u^0 \text{ on } \Gamma_3 \cup \Gamma_4\}. \tag{1.23}$$

Obviously, $\mathscr{A}(\Omega) \subset \mathscr{B}(\Omega) \neq \emptyset$ for each $\Omega \subset \mathscr{A}_{\mathscr{D}}$. Using the Dirichlet integral D_Ω, we define a functional $a(\Omega)$, $b(\Omega)$ and $J(\Omega)$ by

$$a(\Omega) = \inf_{v \in \mathscr{A}(\Omega)} D(v), \quad b(\Omega) = \inf_{v \in \mathscr{B}(\Omega)} D(v), \quad J(\Omega) = a(\Omega) - b(\Omega). \tag{1.24}$$

Since $\mathscr{A}(\Omega) \subset \mathscr{B}(\Omega)$, we can see that $J(\Omega) \geq 0$. Note that $\mathscr{A}(\Omega)$ and $\mathscr{B}(\Omega)$ are closed convex subsets in $H^1(\Omega)$. Both $a(\Omega)$ and $b(\Omega)$ are attained by some $u_\Omega \in \mathscr{A}(\Omega)$ and $w_\Omega \in \mathscr{B}(\Omega)$, respectively

$$\Delta w_\Omega = 0 \text{ in } \Omega, \quad w_\Omega = u^0 \text{ on } \Gamma_3 \cup \Gamma_4, \quad \frac{\partial w_\Omega}{\partial n} = 0 \text{ on } \Gamma_1 \cup \Gamma_2. \tag{1.25}$$

In fact,

$$\int_\Omega \nabla v \cdot \nabla w_\Omega \; dx = \int_\Omega (-\Delta w_\Omega) \cdot v \; dx = 0 \tag{1.26}$$

Since $\pm v \in \Omega$,

$$\Delta w_\Omega = 0 \qquad \text{in } \Omega. \tag{1.27}$$

Then,

$$\begin{aligned}\int_{\Omega} \nabla v \cdot \nabla w_{\Omega} \; dx &= \int_{\Omega} (-\Delta w_{\Omega}) \cdot v \; dx + \int_{\partial\Omega} \frac{\partial w_{\Omega}}{\partial n} \; ds \\ &= \int_{\partial\Omega} \frac{\partial w_{\Omega}}{\partial n} \; ds \leq 0.\end{aligned} \tag{1.28}$$

Since v is arbitrary,

$$\int_{\Gamma_1 \cup \Gamma_2} \frac{\partial w_{\Omega}}{\partial n} = 0 \tag{1.29}$$

is obtained. Hence, we get (1.25).

Let $\{\Omega_n\} \subset \mathscr{A}_{\mathscr{D}}$ be a sequence of admissible domains, which converges to $\Omega \subset \mathscr{A}_{\mathscr{D}}$ uniformly. Let $\zeta_2 = \overline{\Gamma}_3^1 \cup \Gamma_2$. The point ζ_2 is where the $\partial(\mathscr{DM})$ contacts the surface of the left reservoir. We need to consider boundary domain, which is vertical in the neighbourhood of ζ_2. The domain Ω, may have a cusp at point ζ_2, and it will be excluded in an explicit way.

Definition 1.3.4. *If $\Omega \in \mathscr{A}_{\mathscr{D}}$ has a vertical wall around ζ_2, then, the following condition is satisfied:*

Let a sufficiently small $\eta > 0$ be fixed. Consider the cones defined by a straight lines crossing at ζ_2 with angle η and a connected subset of Γ_1. Then $\overline{\Omega}$ contains such a cone.

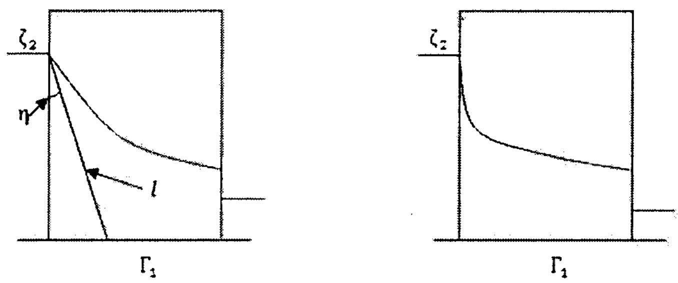

Figure 1.5: The cone condition and a domain with a cusp

The exact solution Ω contacts $\partial(\mathscr{DM})$ at the right angles of ζ_2. Thus, we confirm that $\Omega \subset \mathscr{A}_{\mathscr{S}}$. As for the left vertical wall around, ζ_2, the cone condition of Definition 1.3.4 is also essential for the compactness of $\mathscr{A}_{\mathscr{S}}$.

Problem 1.3.5. *We want to find $\Omega \in \mathscr{A}_{\mathscr{S}}$ such that there exists a unique $u \in H^1(\Omega)$ which satisfies (1.4) and (1.17).*

Theorem 1.3.6. *Under Assumption 1.3.1, Definition 1.3.3 and Definition 1.3.4 we have $\inf_{\mathscr{A}_{\mathscr{D}}} J = 0$ for the functional $J : \mathscr{A}_{\mathscr{S}} \to \mathbb{R}$ defined by (1.24). Moreover, an admissible domain $\Omega \in \mathscr{A}_{\mathscr{D}}$ is a solution of Problem 1.3.5 if and only if $J(\Omega) = \inf_{\mathscr{A}_{\mathscr{D}}} J = 0$.*

1.4 Conformal Mapping

In this section, we will provide the theory and the convergence of our new variational formulation. Suppose that the configuration of a $\mathscr{DM}$ satisfies Assumption 1.3.1. Let $z_j = e^{2\pi\sqrt{-1}(j-1)/3}$, $j = 1, 2, 3$. Given $\Omega \in \mathscr{A}_{\mathscr{D}}$, we take

$\zeta_1 = \overline{\Gamma}_1 \cap \overline{\Gamma}_3^2$, $\zeta_2 = \overline{\Gamma}_3^1 \cap \overline{\Gamma}_2$ and $\zeta_3 = \overline{\Gamma}_1 \cap \overline{\Gamma}_3^1$.

From Riemann's mapping theorem there exists a unique mapping φ_Ω for each $\Omega \in \mathscr{A}_\mathscr{D}$ admitting

$$\varphi_\Omega : B \to \Omega : \text{conformal}$$

$$\varphi_\Omega : B \to \overline{\Omega} : \text{homeomorphic}$$

$$\varphi_\Omega(z_j) = \zeta_j, \quad j = 1,2,3$$

where $B = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ denotes the unit disk for any simply connected admissible domain $\Omega \in \mathscr{A}_\mathscr{D}$. For each $\Omega \in \mathscr{A}_\mathscr{D}$ there exists a conformal mapping $\varphi_\Omega : B \to \Omega$.

Definition 1.4.1. *Let $\Gamma \subset \mathbb{R}^2$ be a continuous image of an open interval in $\mathbb{R}$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projection defined by $\pi((x_1,x_2)) = x_1$. Then, Γ is a graph in the direction of x_2, if $(\pi|_\Gamma)^{-1}(x_1)$ is connected for all $x_1 \in \pi(\Gamma)$.*

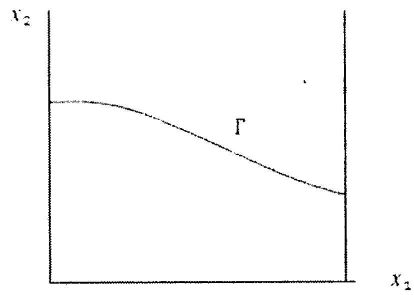

Figure 1.6: Canonical projection of a dam.

From Definition 1.4.1, we define $\widetilde{\Gamma}_j \subset \partial B$ by $\widetilde{\Gamma}_j = \varphi^{-1}(\Gamma_j)$, $(j = 1,2,4)$ and for $\widetilde{\Gamma}_3^i = \varphi^{-1}(\widetilde{\Gamma}_3^i)$ $(i = 1,2)$ for each $\Omega \in \mathscr{A}_\mathscr{D}$. Here, $\widetilde{\Gamma}_3^1$ and $\widetilde{\Gamma}_1$ are not depend on φ_Ω, while the others do. Let also $\tilde{u}^0 = u^0 \circ \varphi_\Omega$. Therefore, (1.17) is transformed into

$$\begin{aligned} \Delta\tilde{u} &= 0 && \text{in } B, \\ \frac{\partial \tilde{u}}{\partial n} &= 0 && \text{on } \widetilde{\Gamma}_1, \\ \tilde{u} = \tilde{u}^0 \text{ and } \frac{\partial \tilde{u}}{\partial n} &= 0 && \text{on } \widetilde{\Gamma}_2, \\ \tilde{u} &= \tilde{u}^0 && \text{on } \widetilde{\Gamma}_3, \\ \tilde{u} = \tilde{u}^0 \text{ and } \frac{\partial \tilde{u}}{\partial n} &\leq 0 && \text{on } \widetilde{\Gamma}_4. \end{aligned} \tag{1.30}$$

Proposition 1.4.2. *If $\tilde{u}$ is regular, then it satisfies (1.30) if and only if $u = \tilde{u}^0 \circ \varphi_\Omega^{-1}$ satisfies (1.17).*

Next we proceed to the problem that is defined from the conformal mapping.

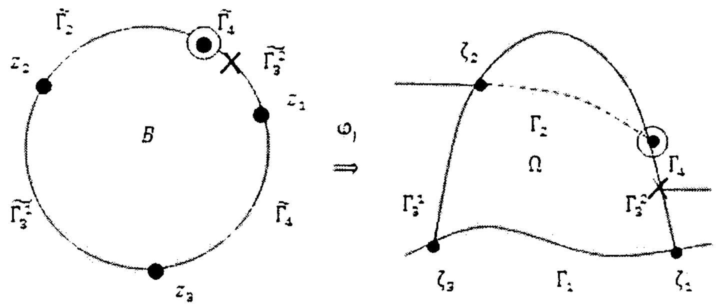

Figure 1.7: Dam defined by the unit disk.

Problem 1.4.3 (Filtration problem or dam problem). *Suppose that Assumption 1.3.1 holds. Find $\Omega \in \mathscr{A}_{\mathscr{D}}$ such that there exists a unique $\tilde{u} \in H^1(B)$ which satisfies (1.30).*

Using the function spaces $\mathscr{A}(\Omega)$ and $\mathscr{B}(\Omega)$ that defined in (1.19) and (1.23) respectively, we set

$$\tilde{\mathscr{A}}(\Omega) = \{\tilde{v} = v \circ \varphi_\Omega \in H^1(B) \mid v \in \mathscr{A}(\Omega)\},$$

$$\tilde{\mathscr{B}}(\Omega) = \{\tilde{v} = v \circ \varphi_\Omega \in H^1(B) \mid v \in \mathscr{B}(\Omega)\}.$$

We then define the functional $\tilde{a}(\Omega), \tilde{b}(\Omega)$ and $\tilde{J}(\Omega)$ like before,

$$\tilde{a}(\Omega) = \inf_{v \in \tilde{A}(\Omega)} D_B(\tilde{v}), \qquad \tilde{b}(\Omega) = \inf_{v \in \tilde{B}(\Omega)} D_B(\tilde{v}), \qquad \tilde{J}(\Omega) = \tilde{a}(\Omega) - \tilde{b}(\Omega).$$

Recall that $u_\Omega \in \mathscr{A}(\Omega)$ and $w_\Omega \in \mathscr{B}(\Omega)$ which attain $a(\Omega) = \inf_{v \in \mathscr{A}(\Omega)} D(v)$ and $b(\Omega) = \inf_{v \in \mathscr{B}(\Omega)} D(v)$ satisfy (1.22) and (1.25), respectively. Then we define $\tilde{u}_\Omega = u_\Omega \circ \varphi_\Omega \in \tilde{\mathscr{A}}(\Omega)$ and $\tilde{w}_\Omega = w_\Omega \circ \varphi_\Omega \in \tilde{\mathscr{B}}(\Omega)$ which satisfy

$$\Delta u_\Omega = 0 \text{ in } B, \quad u_\Omega = u^0 \text{ on } \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3 \cup \tilde{\Gamma}_4,$$
$$\tfrac{\partial u_\Omega}{\partial n} \le 0 \text{ on } \tilde{\Gamma}_4, \quad \tfrac{\partial u_\Omega}{\partial n} = 0 \text{ on } \tilde{\Gamma}_1$$

and

$$\Delta w_\Omega = 0 \text{ in } B, \quad w_\Omega = u^0 \text{ on } \tilde{\Gamma}_3 \cup \tilde{\Gamma}_4, \quad \tfrac{\partial w_\Omega}{\partial n} = 0 \text{ on } \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$$

according to (1.22) and (1.25), respectively. On the other hand, we have

$$D_B(v \circ \varphi_\Omega) = \int_B |\nabla(v \circ \varphi_\Omega)|^2 \, d\tilde{x} = D_\Omega(v), \qquad v \in H^1(\Omega)$$

for any conformal mapping $\varphi_\Omega : B \to \Omega$. Therefore, if $\tilde{u}_\Omega$ attains $D_B(\tilde{u}_\Omega) = \tilde{a}(\Omega)$ and $\tilde{w}_\Omega$ attains $D_B(\tilde{w}_\Omega) = \tilde{b}(\Omega)$, it holds that

$$\Delta \tilde{u}_\Omega = 0 \qquad \text{in } B, \quad \tilde{u}_\Omega = \tilde{u}^0 \qquad \text{on } \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3 \cup \tilde{\Gamma}_4,$$

$$\frac{\partial \tilde{u}_\Omega}{\partial n} \leq 0 \qquad \text{on } \widetilde{\Gamma}_4, \qquad \frac{\partial \tilde{u}_\Omega}{\partial n} = 0 \qquad \text{on } \widetilde{\Gamma}_1, \tag{1.31}$$

$$\Delta \tilde{w}_\Omega = 0 \qquad \text{in } B, \quad \tilde{w}_\Omega = \tilde{u}^0 \qquad \text{on } \widetilde{\Gamma}_3 \cup \widetilde{\Gamma}_4, \qquad \frac{\partial \tilde{w}_\Omega}{\partial n} = 0 \qquad \text{on } \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2. \tag{1.32}$$

Theorem 1.4.4. *For the functional $\tilde{J} : \mathscr{A}_\mathscr{D} \to \mathbb{R}$, we have $\inf_{\mathscr{A}_\mathscr{D}} \tilde{J} = 0$ and an admissible domain $\Omega \in \mathscr{A}_\mathscr{D}$ is a solution to Problem 1.4.3 if and only if $\tilde{J}(\Omega) = \inf_{\mathscr{A}_\mathscr{D}} \tilde{J} = 0$.*

Let us recall that each $\Omega \in \mathscr{A}_\mathscr{D}$ is identified with the conformal mapping $\varphi_\Omega : B \to \Omega$ satisfying $\varphi_\Omega(z_j) = \zeta_j$, $j = 1,2,3$. Therefore, we can define the distance in $\mathscr{A}_\mathscr{D}$ by

$$dist(\Omega_1, \Omega_2) = \|\varphi_{\Omega_1} - \varphi_{\Omega_2}\|_\infty$$

where $\|\cdot\|_\infty$ is the maximum norm defined on B. Let γ be a sufficiently smooth curve with the length parameter. The norm of $H^{1/2}(\gamma)$ is then defined by $\|v\|_{H^{1/2}(\gamma)} = (\|v\|^2_{L^2(\gamma)} + \mid v \mid_{1/2(\gamma)^2})^{1/2}$, where $\mid v \mid_{H^{1/2}(\gamma)} = \int_\gamma \int_\gamma \frac{|v(s)-v(t)|^2}{|s-t|^2}\, dsdt < \infty$. $H^{1/2}(\gamma)$ becomes a Hilbert space with the corresponding inner product. In the Lipschitz domain Ω, if $v \in H^1(\Omega)$, then $v\mid_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Conversely, if $n \in H^{1/2}(\partial\Omega)$ then there is $u \in H^1(\Omega)$ such that $u\mid_{\partial\Omega} = n$. The infimum of $\|\nabla u\|_2$ of such u is attained if $\Delta u = 0$.

Definition 1.4.5. *When a sequence $\{\varphi_{\Omega_n}\}$ converges uniformly to φ_Ω, we say that $\{\Omega_n\}$ converges uniformly to Ω where Ω_n, $\Omega \in \mathscr{A}_\mathscr{D}$, $n = 1,2,\ldots$.*

Corollary 1.4.6. *Suppose that $\{\Omega_n\} \subset \mathscr{A}_\mathscr{D}$ converges uniformly to $\Omega \subset DAM$ and $\mathscr{A}_\mathscr{D} \subset C(\overline{B};\mathbb{R}^2)$. Then $\Omega \in \mathscr{A}_\mathscr{D}$.*

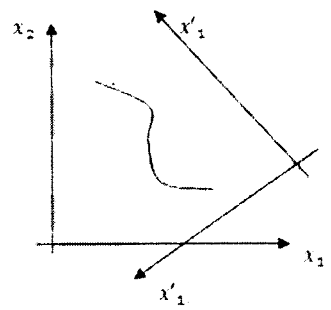

Figure 1.8: Monotone decreasing graphs are Lipschitz.

Proof. We have Definition 1.3.3(4) for Ω. Since $\varphi_{\Omega_n}\mid_{\partial B}$ converges uniformly to $\varphi_\Omega\mid_{\partial B}$, Definition 1.3.3(2),(5) and Definition 1.3.4 hold for Ω. We show that Ω is a Lipschitz domain. Recall that, a domain $D \subset \mathbb{R}^2$ is called Lipschitz if there exist finite open sets $U_i \subset \mathbb{R}^2$, $i = 1,\cdots,N$ such that $\partial D \subset \cup_{i=1}^N U_i$ where ∂D denote as the boundary of the domain D and each $\partial D \cap U_i$ can be regarded as a graph of a Lipschitz function. Therefore, from Assumption 1.3.1, we can define an open set and change of coordinates on $\partial\Omega \cap \overline{B_1 \cup B_3}$. Since at the left hand side of $\partial\Omega \setminus \overline{B_1 \cup B_3}$ cannot contain a cusp because we excluded it, $\partial\Omega$ is Lipschitz around the point.

From Definition 1.3.3(3), we know that $\partial\Omega_n \setminus \overline{B_1 \cup B_3}$ is monotone decreasing in the direction x_2, so $\partial\Omega \setminus \overline{B_1 \cup B_3}$ is also monotone decreasing in the x_2 direction. Rotate the coordinates (x_1, x_2) by $3\pi/4$ radian through two points on a decreasing graph of direction x_2. The absolute value of its slope is less than or equal to 1 on the new coordinate. This means that $\partial\Omega \setminus \overline{B_1 \cup B_3}$ can be regarded as graph of Lipschitz function on the new coordinates. Therefore, Ω is Lipschitz and satisfies Definition 1.3.3(1), (3). Therefore, $\Omega \in \mathscr{A}_{\mathscr{D}}$. □

We claim that if a sequence of admissible domains in $\mathscr{A}_{\mathscr{D}}$ converge uniformly then they converge in $H^1(B;\mathbb{R}^2)$.

Lemma 1.4.7. *Suppose that a sequence $\{\Omega_n\} \subset \mathscr{A}_{\mathscr{D}}$ converges uniformly to $\Omega \subset DAM$. Then we have*

$$\lim_{n\to\infty} \|\varphi_{\Omega_n} - \varphi_\Omega\|_{H^1(B;\mathbb{R}^2)} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\varphi_{\Omega_n}|_{\partial B} - \varphi_\Omega|_{\partial B}\|_{H^{1/2}(\partial B;\mathbb{R}^2)} = 0. \tag{1.33}$$

Proof. It is obvious that the first convergence of (1.33) implies the second convergence of (1.33). Since φ_{Ω_n} converges uniformly to φ_Ω, we have $\lim_{n\to\infty} \|\varphi_{\Omega_n} - \varphi_\Omega\|_{L^2(B;\mathbb{R}^2)} = 0$. We will use the dominated convergence theorem of the Lebesgue integral to show $\|D_B(\varphi_{\Omega_n})\| \to \|D_B(\varphi_\Omega)\|$. We denote $|\Omega|$ as the Lebesgue measure for $\Omega \subset \mathscr{A}_{\mathscr{D}}$. Then, we take $\chi_{\Omega_n}, \chi_\Omega$ be the characteristic functions of Ω_n and Ω. Since $|\partial\Omega| = 0$, we conclude that $\chi_{\Omega_n} \to \chi_\Omega$ for a.a. in B as $n \to \infty$. From Lebesgue's Dominated Convergence Theorem, we know that $|\Omega_n| = \int_{\mathbb{R}^2} \chi_{\Omega_n}\, dx \to |\Omega| = \int_{\mathbb{R}^2} \chi_\Omega\, dx$ as $n \to \infty$. Then we know that $|\Omega_n| = D_B(\varphi_{\Omega_n})$ and $|\Omega| = D_B(\varphi_\Omega)$. Since Ω is measurable, therefore, $\|\varphi_{\Omega_n}\|$ is bounded with $|\Omega_n| = (2D_B(\varphi_{\Omega_n}))^{1/2}$. Thus, from φ_{Ω_n} we can abstract a subsequence $\varphi_{\Omega_{n_i}}$ such that

$$\varphi_{\Omega_{n_i}} \rightharpoonup \varphi_\Omega \text{ weakly in } H^1(B;\mathbb{R}^2), \qquad \lim_{i\to\infty} \|D_B(\varphi_{\Omega_{n_i}})\| = \lim_{i\to\infty} |\Omega_{n_i}| = |\Omega| = D_B(\varphi_\Omega)$$

which means that $\varphi_{\Omega_{n_i}}$ converges to φ_Ω in $H^1(B;\mathbb{R}^2)$. Therefore, $\{\varphi_{\Omega_n}\}$ converges to φ_Ω in $H^1(B;\mathbb{R}^2)$ and (1.33) holds . □

Lemma 1.4.8. *Suppose that $\{\Omega_n\} \subset \mathscr{A}_{\mathscr{D}}$ converges uniformly to $\Omega \subset \mathscr{DAM}$. Then we have,*

$$\lim_{n\to\infty} \inf \tilde{a}(\Omega_n) \geq \tilde{a}(\Omega). \tag{1.34}$$

Thus, the functional $\tilde{a}(\Omega)$ is lower semicontinuous with respect to uniform convergence.

Proof. Let $\tilde{u}_{\Omega_n} \in \tilde{\mathscr{A}}(\Omega_n)$ attains the minimum value of D_B in $\tilde{\mathscr{A}}(\Omega_n)$, that is $D_B(\tilde{u}_{\Omega_n}) = \tilde{a}(\Omega_n)$ and $\tilde{u}_{\Omega_n}$ satisfies (1.31) for $\Omega = \Omega_n$. Define $\tilde{u}_\Omega$ as the solution of

$$\Delta \tilde{u}_\Omega = 0 \text{ in } B, \qquad \tilde{u}_\Omega = u^o \circ \varphi_\Omega \text{ on } \widetilde{\Gamma}_2 \cup \widetilde{\Gamma}_3 \cup \widetilde{\Gamma}_4, \qquad \frac{\partial \tilde{u}_\Omega}{\partial n} = 0 \text{ on } \widetilde{\Gamma}_1. \tag{1.35}$$

Then, we have $\tilde{u}_\Omega \in \tilde{\mathscr{A}}(\Omega)$, where $\tilde{\mathscr{A}} \neq 0$. From the above assumption and maximum principle, $\{\tilde{u}_{\Omega_n}\}$ converges uniformly to $\tilde{u}_\Omega$ in $C(\overline{B})$. Since Dirichlet integral is a lower semi-continuity with respect to weak convergence, we have from the Definition 1.3.3(4),

$$M_0 \geq \liminf_{n\to\infty} \tilde{a}(\Omega_n) = \liminf_{n\to\infty} D_B(\tilde{u}_{\Omega_n}) \geq D_B(\tilde{u}_\Omega) \tag{1.36}$$

and $\tilde{u}_\Omega \in H^1(B)$. As we know, $\{\tilde{u}_{\Omega_n}\}$ is bounded in $H^1(B)$. Therefore, it admits a subsequence $\{\tilde{u}_{\Omega_{n_i}}\}$,

$$\tilde{u}_{\Omega_{n_i}} \rightharpoonup \tilde{u}_{\Omega_n} \quad \text{weakly in } H^1(B),$$

$$\tilde{u}_{\Omega_{n_i}}|_{\partial B} \rightharpoonup \tilde{u}_{\Omega_n}|_{\partial B} \quad \text{weakly in } H^{1/2}(\partial B).$$

Then we define $H^{-1/2}(\partial B)$ as the dual space of $H^{1/2}(\partial B)$. We denote the duality pairing as $\langle\cdot,\cdot\rangle_{H^{-1/2}(\partial B),H^{1/2}(\partial B)}$. We know that $u \in H^1(B)$, such that $\Delta u = 0$ in B. Thus, $\eta \in H^{1/2}(\partial B)$. Therefore, $u = \eta \in H^{1/2}(\partial B)$. Generally, $\frac{\partial u}{\partial n} \in H^{-1/2}(\partial B)$ and the map $H^{1/2}(B) \ni \eta \mapsto \frac{\partial u}{\partial n} \in H^{-1/2}(\partial B)$ is continuous. Hence, since $\tilde{u}_{\Omega_n}|_{\partial B}$ converges weakly to $u_\Omega|_{\partial B}$ in $H^{1/2}(\partial B)$, $\frac{\partial \tilde{u}_{\Omega_n}}{\partial n}$ converges weakly to $\frac{\partial \tilde{u}_\Omega}{\partial n}$ in $H^{-1/2}(\partial B)$. This also means that for $\rho \in H^{1/2}(\partial B)$,

$$\lim_{n\to\infty}\left\langle \frac{\partial \tilde{u}_{\Omega_n}}{\partial n}, \rho \right\rangle_{H^{-1/2}(\partial B),H^{1/2}(\partial B)} = \left\langle \frac{\partial \tilde{u}_\Omega}{\partial n}, \rho \right\rangle_{H^{-1/2}(\partial B),H^{1/2}(\partial B)}$$

where $\rho > 0$ is in $H^{1/2}(\partial B)$ of which support is denoted as $supp\rho$. This $supp\rho$ is also included in $\widetilde{\Gamma}_4$ for Ω and Ω_n. Since $\frac{\partial \tilde{u}_{\Omega_n}}{\partial n} \leq 0$ on $\widetilde{\Gamma}_4$, we obtain

$$\left\langle \frac{\partial \tilde{u}_\Omega}{\partial n}, \rho \right\rangle_{H^{-1/2}(\partial B),H^{1/2}(\partial B)} \leq 0$$

from the support definition. Since ρ is arbitrary, we can conclude that $\frac{\partial u_\Omega}{\partial n} \leq 0$ on $\widetilde{\Gamma}_4$ for Ω, $\tilde{u}_\Omega \in \tilde{\mathscr{A}}(\Omega) \neq \emptyset$. From the definition, $\tilde{u}_\Omega$ satisfy (1.31) and its Dirichlet integral attains the minimum value $\tilde{a}(\Omega) \in \tilde{\mathscr{A}}(\Omega)$. Therefore, $D_B(\tilde{u}_\Omega) = \tilde{a}(\Omega)$. Hence, (1.34) indicates the lower semicontinuity and (1.36) follows. □

We summarize the basic properties of space $H^{1/2}(\gamma)$.

Lemma 1.4.9. *Let $\gamma = (a,b) \subset \mathbb{R}$ be a bounded interval. Let $f \in C^{0,1}(\overline{\gamma})$ and $g \in H^{1/2}(\gamma)$.*

1. *We have the following estimates:*

$$\|f\|_{H^{1/2}(\gamma)} \leq C(\gamma)\|f\|_{C^{0,1}(\overline{\gamma})},$$

$$\|fg\|_{H^{1/2}(\gamma)} \leq C(\gamma)\|f\|_{C^{0,1}(\overline{\gamma})}\|g\|_{C^{0,1}(\gamma)}.$$

2. *If g satisfies, with some $\varepsilon > 0$,*

$$g(x) = O((x-a)^{\varepsilon}), \quad x \searrow a, \qquad\qquad g(x) = ((b-x)^{\varepsilon}), \quad x \nearrow b,$$

then its O-extension $\tilde{g}$ is defined by

$$\tilde{g}(x) = \begin{cases} g(x), & x \in \varepsilon, \\ 0, & x \in \mathbb{R} - \gamma \end{cases}$$

belongs to $H^{1/2}(\mathbb{R})$; $g(x) \in H^{1/2}(\mathbb{R})$.

Lemma 1.4.10. *Let $\gamma = (0,1)$. Suppose that there is a sequence $\{\eta_n\} \subset H^{1/2}(\gamma) \cap C(\overline{\gamma})$ satisfying the following:*

1. *$\{\eta_n\}$ converges to $\eta \in H^{1/2}(\gamma) \cap C(\overline{\gamma})$ in both $H^{1/2}(\gamma)$ and $C(\overline{\gamma})$ norms.*
2. *There exists a sequence $\{\alpha_n \subset \gamma\}$ such that*
   - $\lim\limits_{n\to\infty} \alpha_n = \alpha \in \gamma$.
   - *$\eta_n(\alpha_n) = \eta(\alpha) = h$ for any n, where h is a constant.*
3. *There exists a subinterval $\gamma_0 \subset \gamma$ with $\alpha \in \gamma_0$ such that $\eta_n\,|_{\gamma_0} \in C^1(\gamma_0)$, $\eta\,|_{\gamma_0} \in C^1(\gamma_0)$ and $\eta_n\,|_{\gamma_0}$ converges to $\eta\,|_{\gamma_0}$ in $C^1(\gamma_0)$.*

Then, if we define ρ_n and ρ by

$$\rho_n(t) = \begin{cases} \eta_n(t) & t \in [0,\alpha_n] \\ h & t \in [\alpha_n,1], \end{cases} \qquad \rho(t) = \begin{cases} \eta(t) & t \in [0,\alpha] \\ h & t \in [\alpha,1], \end{cases}$$

we have $\rho, \rho_n \in H^{1/2}(\gamma)$ for sufficiently large n and $\lim\limits_{n\to\infty} \|\rho_n - \rho\|_{H^{1/2}(\gamma)} = 0$.

Lemma 1.4.11. *Let $\{\Omega_n\} \subset \mathscr{A}_{\mathscr{D}}$. Suppose that $\{\Omega_n\}$ converges uniformly to $\Omega \subset DAM$. Let $\gamma = (z_1, z_2) \subset \partial B$ be the curve between z_1 and z_2. Let $\tilde{u}_{\Omega_n} \in \tilde{\mathscr{A}}(\Omega_n)$, $\tilde{u}_{\Omega} \in \tilde{\mathscr{A}}(\Omega)$ be the functions satisfying (1.31). Then we have $\tilde{u}_{\Omega_n}\,|_{\gamma} \to \tilde{u}_{\Omega}\,|_{\gamma}$ in $H^{1/2}(\gamma)$ as $n \to \infty$.*

Proof. Let $\zeta_4 = \overline{\Gamma}_3^2 \cap \overline{\Gamma}4$. Let $z_4^n = \varphi_{\Omega_n}^{-1}(\zeta_4)$ and $z_4 = \varphi_{\Omega}^{-1}(\zeta_4)$. Then, the boundary value on γ of $\tilde{u}_{\Omega_n}$ and $\tilde{u}_{\Omega}$ are

$$\begin{aligned} \tilde{u}_{\Omega_n} &= h_2 \text{ on } (z_1, z_4^n) \subset \gamma, \quad \tilde{u}_{\Omega_n} = \varphi_{\Omega_n}^2 \text{ on } (z_4^n, z_2) \subset \gamma, \\ \tilde{u}_{\Omega} &= h_2 \text{ on } (z_1, z_4) \subset \gamma, \quad \tilde{u}_{\Omega} = \varphi_{\Omega}^2 \text{ on } (z_4, z_2) \subset \gamma, \end{aligned} \tag{1.37}$$

where $\varphi_{\Omega_n} = (\varphi_{\Omega_n}^1, \varphi_{\Omega_n}^2)$ and $\varphi_{\Omega} = (\varphi_{\Omega}^1, \varphi_{\Omega}^2)$. We know that $\varphi_{\Omega_n}\,|_{\gamma} \to \varphi_{\Omega}\,|_{\gamma}$ in $C(\overline{\gamma};\mathbb{R}^2)$, hence, $\lim\limits_{n\to\infty} z_4^n = z_4$. If the right hand side of water reservoir is empty, which means that $\Gamma_3^2 = \emptyset$, then we have $\zeta_1 = \zeta_4$. From Lemmas 3.2.7

and 1.4.10 the right hand reservoir is not empty and $\zeta_4 \neq \zeta_1$. From Assumption 1.3.1(3), $\partial(DAM)$ is a C^2 curve at z_4. Therefore, the boundary regularities of conformal mappings yield φ_{Ω_n}, φ_Ω of $C^{1,\varepsilon}$ around z_4^n, z_4 for any ε, $0 < \varepsilon < 1$. From Lemma 1.4.10, there exists a subinterval $\gamma_0 \subset \gamma$ such that $z_4, z_4^n \in \gamma_0$ and $\varphi_\Omega|_{\gamma_0}, \varphi_{\Omega_n}|_{\gamma_0} \in C^{1,\varepsilon}(\gamma_0;\mathbb{R}^2)$ for sufficiently large n. Since $\|\varphi_{\Omega_n}\|_{C^{1,\varepsilon}(\gamma_0;\mathbb{R}^2)}$ is bounded and $C^{1,\varepsilon}(\gamma_0;\mathbb{R}^2) \subset C^1(\gamma_0;\mathbb{R}^2)$ is compact. Therefore, we can extract a subsequence $\{\varphi_{\Omega_{n_j}}\}$ such that

$$\varphi_{\Omega_{n_j}}|_{\gamma_0} \to \varphi_\Omega|_{\gamma_0} \quad \text{in } C^1(\gamma_0;\mathbb{R}^2), \qquad n_j \to \infty.$$

Hence,

$$\tilde{u}_{\Omega_{n_j}}|_\gamma \to \tilde{u}_\Omega|_\gamma \quad \text{in } H^1(\gamma), \qquad n_j \to \infty.$$

Therefore, we can conclude that $\{\tilde{u}_{\Omega_n}\}$ converges in $H^{1/2}(\gamma)$. □

Theorem 1.4.12. *Let $\{\Omega_n\} \subset \mathscr{A}_{\mathscr{D}}$. Suppose that $\{\Omega_n\}$ converges uniformly to $\Omega \subset DAM$. Then we have*

$$\lim_{n\to\infty} \tilde{b}(\Omega_n) = \tilde{b}(\Omega) \tag{1.38}$$

and consequently

$$\liminf_{n\to\infty} \tilde{J}(\Omega_n) \geq \tilde{J}(\Omega), \qquad \liminf_{n\to\infty} J(\Omega_n) \geq J(\Omega). \tag{1.39}$$

Proof. Recall that $\tilde{w}_{\Omega_n}$ and $\tilde{w}_\Omega$ attain the minimum values of the Dirichlet integral in $\tilde{\mathscr{B}}_{\Omega_n}$ and $\tilde{\mathscr{B}}_\Omega$ and satisfying boundary value problems:

$$\begin{aligned}
&\Delta\tilde{w}_{\Omega_n} \text{ in } B, \quad \tilde{w}_{\Omega_n} = \tilde{u}_{\Omega_n} \text{ on } \widetilde{\Gamma}_3 \cup \widetilde{\Gamma}_4, \quad \frac{\partial \tilde{w}_{\Omega_n}}{\partial n} = 0 \text{ on } \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2,\\
&\Delta\tilde{w}_\Omega = 0, \qquad \tilde{w}_\Omega = \tilde{u}_\Omega \text{ on } \widetilde{\Gamma}_3 \cup \widetilde{\Gamma}_4, \qquad \frac{\partial \tilde{w}_\Omega}{\partial n} = 0 \text{ on } \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2.
\end{aligned}$$

Let ζ_5 and ζ_5^n be the points contacting at the air part of the free boundary on $\mathscr{DAM}$ for Ω_n and Ω. Let $z_5^n = \varphi_{\Omega_n}^{-1}(\zeta_5^n)$ and $z_5 = \varphi_\Omega^{-1}(\zeta_5)$. As before, from the theory of conformal mappings, there exists conformal mappings of ψ_n and ψ which map B to a rectangle, $R_n = (0,k_n)\times(0,1)$ and $R = (0,k)\times(0,1)$ where k_n and k are positive constants with $\lim_{n\to\infty} k_n = k$ and

$$\psi_n = \psi(z_2) = (0,1), \quad \psi_n(z_3) = \psi(z_3) = (0,0),$$

$$\psi_n = (k_n,1), \quad \psi(z_5) = (k,1), \quad \psi_n(z_1) = (k_n,0), \quad \psi(z_n) = (k,0).$$

Then we denote the boundaries of the rectangles as ∂R_n and ∂R. Denote $\tilde{v}_n = \tilde{w}_{\Omega_n} \circ \psi_n^{-1}$ and $\tilde{v} = \tilde{w}_\Omega \circ \psi^{-1}$ which satisfy

$$\begin{aligned}
&\Delta\tilde{v}_n = 0 \text{ in } R_n \quad \tilde{v}_n = \rho_n \text{ on } \gamma_2^n \cup \gamma_4, \quad \frac{\partial \tilde{v}_n}{\partial n} = 0 \text{ on } \gamma_1^n \cup \gamma_3^n,\\
&\Delta\tilde{v} = 0 \text{ in } R, \qquad \tilde{v} = \rho \text{ on } \gamma_2 \cup \gamma_4, \qquad \frac{\partial \tilde{v}}{\partial n} = 0 \text{ on } \gamma_1 \cup \gamma_3,
\end{aligned}$$

where $\rho_n = \tilde{u}_{\Omega_n} \circ \psi_n^{-1}$ and $\rho = \tilde{u}_\Omega \circ \psi^{-1}$. Consider $\overline{v}_n = (x_1, x_2) = \tilde{v}_n(\alpha_n x_1, x_2)$ where $\alpha_n = k_n/k$. Then, $\overline{v}_n$ satisfies

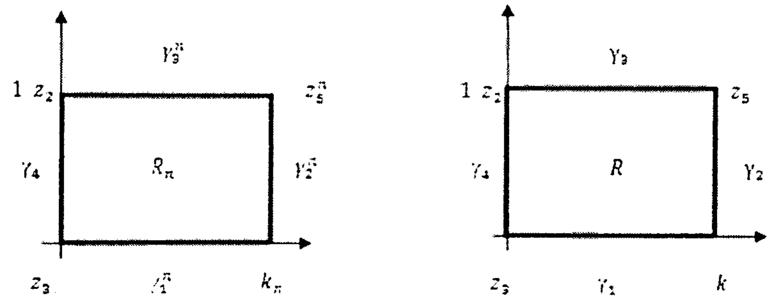

Figure 1.9: The rectangles and their boundaries.

$$\Delta \overline{v}_n = (1-\alpha_n^2) f_n \text{ in } R, \quad \overline{v}_n = \rho_n \text{ on } \gamma_2 \cup \gamma_4, \quad \frac{\partial \overline{v}_n}{\partial n} = 0 \text{ on } \gamma_1 \cup \gamma_3,$$

where $f_n = \partial^2 \tilde{v}_n(\alpha_n x_1, x_2)/\partial x_2^2 \in H^{-1}(R)$, where $H^{-1}(R)$ is the dual space of $H^1(R)$. From Lemma 1.4.11, we know that $\rho_n \to \rho$ in $H^{1/2}(\gamma_2 \cup \gamma_4)$, hence $\lim_{n\to\infty} \alpha_n = 1$ and $\|f_n\|_{H^{-1}(R)}$ is bounded. Therefore, we can conclude that $\tilde{v} \to v$ in $H^1(R)$. This means that

$$\lim_{n\to\infty} \|\tilde{w}_{\Omega_n} - \tilde{w}_\Omega\|_{H^1(B)} = 0, \qquad \lim_{n\to\infty} \tilde{b}(\Omega_n) = \tilde{b}(\Omega).$$

Therefore, combining (1.34), this theorem is proven. □

Corollary 1.4.13. *Let $\{\Omega_n\} \subset \mathscr{A}_\mathscr{D}$ converge to $\Omega \in \mathscr{A}_\mathscr{D}$ such that $\lim_{n\to\infty} J(\Omega_n) = 0$. Then we have $J(\Omega) = 0$ and Ω is the solution of the filtration problem.*

Remark: Now we know that $\tilde{b}(\Omega_n)$ is continuous with respect to uniform convergence of $\{\Omega_n\}$, while the function $\tilde{a}(\Omega_n)$ is only lower semicontinuous. We clarify why they are like this. Let $\gamma = (z_1, z_2)$, $\gamma_1 = (z_1, z_2] \cup (z_2, z_3) \subset \partial B$ be the arcs between z_1, z_2 and z_3. Recall that $\tilde{w}_{\Omega_n}$ and $\tilde{w}_\Omega$ satisfy the boundary conditions

$$\tilde{w}_{\Omega_n} = \tilde{u}_{\Omega_n} \text{ on } (z_1, z_5^n) \subset \gamma, \quad \frac{\partial \tilde{w}_{\Omega_n}}{\partial n} = 0 \text{ on } (z_5^n, z_2), \quad \tilde{w}_{\Omega_n} = h_1 \text{ on } (z_2, z_3),$$
$$\tilde{w}_\Omega = \tilde{u}_\Omega \text{ on } (z_1, z_5) \subset \gamma, \quad \frac{\partial \tilde{w}_\Omega}{\partial n} = 0 \text{ on } (z_5, z_2), \quad \tilde{w}_\Omega = h_1 \text{ on } (z_2, z_3),$$

Hence, using the fact that $\tilde{u}_{\Omega_n}|_\gamma \to \tilde{u}_\Omega|_\gamma$ in $H^{1/2}(\gamma)$ we have shown that the functional $\tilde{b}(\Omega_n)$ is continuous with respect to uniform convergence of Ω_n. While for $\tilde{u}_\Omega$ and $\tilde{u}_{\Omega_n}$ the boundary conditions are

$$\tilde{u}_{\Omega_n} = h_2 \text{ on } (z_1, z_4^n), \quad \tilde{u}_{\Omega_n} = \varphi_{\Omega_n}^2 \text{ on } (z_4^n, z_2), \quad \tilde{u}_{\Omega_n} = h_1 \text{ on } (z_2, z_3)$$
$$\tilde{u}_\Omega = h_2 \text{ on } (z_1, z_4), \quad \tilde{u}_\Omega = \varphi_\Omega^2 \text{ on } (z_4, z_2), \quad \tilde{u}_\Omega = h_1 \text{ on } (z_2, z_3).$$

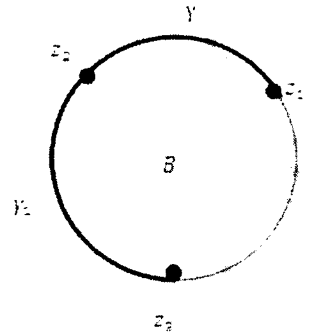

Figure 1.10: γ and γ_1 be the arcs between z_1, z_2 and z_3.

Generally, we cannot claim that $\tilde{u}_{\Omega_n}|_{\gamma_1} \to \tilde{u}_{\Omega}|_{\gamma_1}$ in $H^{1/2}(\gamma)$ although we have $\tilde{u}_{\Omega_n}$, $\tilde{u}_{\Omega} \in H^{1/2}(\gamma_1)$ and Lemma 1.4.11. That is why we cannot claim that $\tilde{a}(\Omega_n)$ is continuous.

1.5 The Compactness of $\mathscr{A}_{\mathscr{D}}$ in $C(\overline{B};\mathbb{R}^2)$

In this section, we want to show that $\mathscr{A}_{\mathscr{D}}$ is sequentially compact in $C(\overline{B};\mathbb{R}^2)$. We need to show that for any sequence $\{\Omega_n\}_{n=0}^{\infty} \subset \mathscr{A}_{\mathscr{D}}$, there exists a subsequence $\{\Omega_{n_i}\}$, and an admissible domain $\Omega \in \mathscr{A}_{\mathscr{D}}$ such that $\{\Omega_{n_i}\}$ converges uniformly to Ω.

We then define

$$B_R(z_0) = \{w : |w - z_0| < r\}, \qquad S_r(z_0) = B \cap B_r(z_0),$$

$$C_r(z_0) = \overline{B} \cap \partial B_r(z_0).$$

If $z_0 \in \partial B$, then $C_r(z_0)$ can be rewritten as

$$C_r(z_0) = \{z_0 + re\sqrt{-1}\theta : \theta_1(r) \le \theta \le \theta_2(r)\}, \quad 0 < \theta_2(r) - \theta_1(r) < \pi.$$

Theorem 1.5.1. *[14]. Every $X \in H^1(B;\mathbb{R}^d)$ possesses a representative $Z(r,\theta)$ of $X(z_0 + re^{\sqrt{-1}\theta})$, $z_0 \in \partial B$, which is absolutely continuous with respect to θ for a.a $r \in (0,2)$ and which has the following property:*

For every $\delta(0,R^2)$, $0 < R < 1$, there is a measurable subset $I \subset (\delta, \sqrt{\delta})$, whose 1-dimensional Lebesgue measure is positive, such that

$$|Z(\rho,\theta) - Z(\rho,\theta')| \le \int_{\theta}^{\theta'} |Z_\theta(\rho,\theta)| d\theta \le \eta(\delta,R)|\theta - \theta'|^{\frac{1}{2}}$$

holds for a.a $\rho \in I$ *and* $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$, *where*

$$\eta(\delta,R) = \left\{ \frac{4}{log(1/\delta)} \int_{S_R(z_0)} |\nabla X|^2 \right\}^{\frac{1}{2}}.$$

Then we have the following theorem.

Theorem 1.5.2. *The functions* $\{\varphi_\Omega|_{\partial B}\}_{\Omega\in\mathscr{A}_\mathscr{D}}$ *are equicontinuous on* ∂B.

Proof. We claim that there exists for all $\varepsilon > 0$, a number $\lambda(\varepsilon) > 0$ which does not depend on Ω with the following property: any pair of points $P, Q \in \partial\Omega$ with

$$0 < |P - Q| < \lambda(\varepsilon)$$

decompose into two arcs $Y_1(P,Q)$ and $Y_2(P,Q)$ such that

$$dist Y_1(P,Q) < \varepsilon$$

holds. Hence $0 < \varepsilon < \varepsilon_0 \equiv \min_{j\neq k} |\zeta_j - \zeta_k|$, then $Y_1(P,Q)$ could contain at most one point ζ_j. Consider the following case,

Case 1 $P,Q \in \Gamma_3^i$, $i = 1,2$ or $P,Q \in \Gamma_1$,

Case 2 $P,Q \in \Gamma_2 \cup \Gamma_4$,

Case 3 "$P \in \Gamma_3^1$, $Q \in \Gamma_1$" or "$P \in \Gamma_1$, $Q \in \Gamma_3^2$" or "$P \in \Gamma_3^2$, $Q \in \Gamma_2 \cup \Gamma_4$",

Case 4 $P \in \Gamma_2 \cup \Gamma_4$, $Q \in \Gamma_3^1$.

Since $\Gamma_2 \cup \Gamma_4$ is monotone, from the definition, it holds that *Case 2* with $\lambda(\varepsilon) = \varepsilon$. Since we extended the cusp at ζ_2, we also holds for *Case 4*. While for other cases,

Let $\delta_0 \in (0,1)$ be a fixed point number with

$$2\sqrt{\delta_0} < \min_{j\neq k} |z_j - z_k| = \sqrt{3}.$$

For arbitrary ε, $0 < \varepsilon < \varepsilon_0$, we take some number $\delta = \delta(\varepsilon) > 0$ such that there exist

$$\left(\frac{4\pi M}{\log(1/\delta)}\right)^{1/2} < \lambda(\varepsilon) \quad \text{and} \quad \delta < \delta_0, \tag{1.40}$$

where $M = 2|\mathscr{D}\mathscr{M}| > \sup_{\Omega\in\mathscr{A}_\mathscr{D}} D_B(\varphi_\Omega)$. From Courant-Lebesgue Lemma, for any $z_0 \in \partial B$, there exists $\rho \in (\delta, \sqrt{\delta})$ such that

$$|\varphi_\Omega(z) - \varphi_\Omega(z')| \le \left(\frac{4\pi M}{\log(1/\delta)}\right)^{1/2},$$

where $z, z' \in \partial B$ is the intersection points of ∂B and $\partial B_\rho(z_0)$. From (1.40), we infer $\sup|\varphi_\Omega(z) - \varphi_\Omega(z')| < \lambda(\varepsilon)$, whence

$$dist Y_1(\varphi_\Omega(z) - \varphi_\Omega(z')) < \varepsilon.$$

Therefore,

$$|\varphi_\Omega(w) - \varphi_\Omega(w')| < \varepsilon \quad \text{for any } w, w' \in \partial B \quad with \ |w - w'| < 2\delta.$$

This means the equicontinuity of $\{\varphi_\Omega|_{\partial B}\}$ on ∂B. □

Theorem 1.5.3. *The set of admissible domains $\mathscr{A}_\mathscr{D} \subset C(\overline{B};\mathbb{R}^2)$ is sequentially compact. That is, for any sequence $\{\Omega_n\} \subset \mathscr{A}_\mathscr{D}$, there exists a subsequence $\{\Omega_{n_i}\}$ which converges to an admissible domain $\Omega \in \mathscr{A}_\mathscr{D}$ uniformly.*

Proof. Let $\{\Omega_n\}$ be any sequence of admissible domain. By Theorem 1.5.2, $\{\varphi_{\Omega_n}|_{\partial B}\}$ is equicontinuous on ∂B. We also know that $\{\varphi_\Omega|_{\partial B}\}$ is uniformly bounded. By Ascoli-Arzela' theorem, we assure the conclusion that there exists a subsequence $\{\Omega_{\Omega_{n_i}}|_{\partial B}\}$ and $\eta \in C(\partial B)$ that $\{\Omega_{\Omega_{n_i}}|_{\partial B}\}$ converges to η on ∂B as $n_i \to \infty$. From the maximum principle of harmonic functions, $\varphi_{\Omega_{n_i}}$ converges uniformly to φ in $\overline{B}$. Therefore, $\varphi \in \mathscr{A}_\mathscr{D}$ and the proof is complete. □

Corollary 1.5.4. *Suppose that a sequence $\{\Omega_n\} \subset \mathscr{A}_\mathscr{D}$ satisfies $\lim_{n\to\infty} J(\Omega_n) = 0$. Then $\{\Omega_n\}$ converges uniformly to the exact solution $\Omega \in \mathscr{A}_\mathscr{D}$ of the filtration problem.*

Chapter 2

Numerical Methods to the Filtration Problem

In this chapter, we deal with quantities defined on a domain. If the domain is perturbed, the quantities are also perturbed. Such a variation with respect to domain perturbation is called Hadamard's variation. In this chapter, we present an iterative scheme for computing solutions of a free boundary problem based on Hadamard's variation. Since the iterative scheme is based on rigorous mathematics, it is stable and effective. A numerical example shows its usefulness.

2.1 Introduction

Suppose that there are two disjoint water reservoirs separated by a dam made of the porous medium (earth, for example). There exists a flow of water in the dam (see Figure 1). Our task here is to find the flow region inside the dam and the velocity potential function. This problem is called the **filtration problem** (or the **seepage problem**, the **dam problem**, etc.), and has been one of the most typical examples of free boundary problems. The filtration problem is treated in many text books, see [7], [15], [18]. In this chapter, we denote the region of the dam by $\mathscr{DAM}$. We assume that $\mathscr{DAM}$ is a Lipschitz domain in $\mathbb{R}^2$.

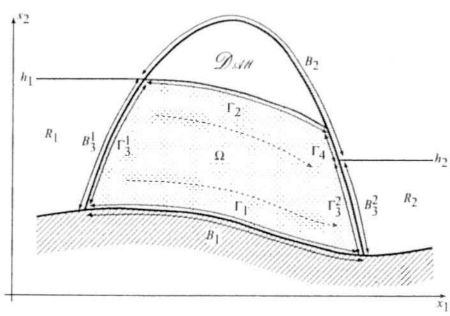

Figure 1: The configuration of the dam.

Our aim is to present a new iterative scheme for computing the flow region. In [19], a new variational principle of the filtration problem is introduced (Section 3). With the variational principle, our iterative scheme is designed as an optimization procedure. By such an iterative process, the boundary of the possible-flow region is modified gradually. Therefore, it is important to know how much quantities related to the variational principle would vary under domain perturbation. In this chapter, we give the first variation of the function which governs the filtration problem (Section 4). Using the first variation, we present a new iterative scheme (Section 5). It is called the traction method and the basis of the traction method are given by Azegami [4]. Azegami presented the traction method for optimal shape design. The authors have found that the traction method is also very effective for computing numerical solutions of free boundary problems. Although, many authors have present iterative schemes for the filtration problem, this is the first one which uses the first variation of any variational principle. A numerical example shows its effectiveness.

2.2 Definition of the filtration problem

In this section, we formulate the filtration problem. For more details, see the above mentioned text books and [19]. We assume that the boundary $\partial(\mathscr{DAM})$ consists of three parts: B_1, the impervious part; B_2, the part in contact with the air; and $B_3 = B_3^1 \cup B_3^2$, the part in contact with the water reservoirs R_1 and R_2. We already assumed that the levels of the water reservoirs (denoted by h_1 and h_2, $h_1 > h_2$) are different and that there exists a steady water flow inside $\mathscr{DAM}$. We denote by Ω the portion of water flow in $\mathscr{DAM}$ which is not *a priori* known. The boundary $\partial\Omega$ consists of four parts:

$$
\begin{array}{ll}
\Gamma_1 = B_1 & \text{(the impervious part)} \\
\Gamma_2 \subset \mathscr{DAM} & \text{(the free boundary)} \\
\Gamma_3^i = B_3^i & \text{(the part in contact with} \\
 & \text{water reservoir} R_i,\ i = 1,2) \\
\Gamma_3 = \Gamma_3^1 \cup \Gamma_3^2 & \\
\Gamma_4 \subset B_2 & \text{(the part in contact with air)}
\end{array}
$$

Let $\Gamma \subset \mathbb{R}^2$ be a curve. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projection defined by $\pi((x_1,x_2)) := x_1$. In this chapter, we say that Γ is a **graph** in the direction of x_2, if $(\pi|_\Gamma)^{-1}(x_1)$ is connected for all $x_1 \in \pi(\Gamma)$. For the configuration of the Lipschitz domain, $\mathscr{DAM}$, we assume the following properties in this paper:

(1) There are two reservoirs of water (one of them may be empty) separated by the dam. We assume without loss of generality that the water level of the left-hand side reservoir is higher than that of the other.

(2) Each reservoir contacts the impervious base.

(3) $B_1 \subset \partial(\mathcal{DAM})$ (impervious part) and $B_2 \cup B_3 \subset \partial(\mathcal{DAM})$ (air and water parts) are continuous, piecewise C^2 curves, both are graphs in the direction of x_2, and $B_2 \cup B_3$ lies above B_1.

The problem is to find the flow region Ω and the velocity potential function u of the flow. To define the boundary value of u we introduce the following subsets of $\mathcal{DAM}$. Let ζ_2 be the point where the surface of the left reservoir contacts $\partial(\mathcal{DAM})$. That is $\zeta_2 = \overline{B_3^1} \cap \overline{B_2}$. Let sufficiently small $\eta > 0$ be taken and fixed.

Let us assume that $\mathcal{DAM}$ splits into two connected subsets by a segment $l \subset \mathcal{DAM}$ such that the end points are ζ_2 and the other is on B_1. Suppose that the angle between B_3^1 and l is η. Then, $\mathcal{DAM}^0 \subset \mathcal{DAM}$ is defined as the region between B_3^1 and l (see Figure 2). Set

$$\mathcal{DAM}^1 := \{x = (x_1, x_2) \in \mathcal{DAM} - \mathcal{DAM}^0 \mid x_2 \geq h_2\},$$
$$\mathcal{DAM}^2 := \{x = (x_1, x_2) \in \mathcal{DAM} - \mathcal{DAM}^0 \mid x_2 < h_2\}.$$

We then define $u^0 \in H^1(\mathcal{DAM})$ by

$$u^0(x) := \begin{cases} h_1 & \text{on } B_3^1, \\ x_2 & \text{in } \mathcal{DAM}^1, \\ h_2 & \text{in } \mathcal{DAM}^2. \end{cases}$$

(In $\mathcal{DAM}^0$, u^0 is defined in an appropriate way.)

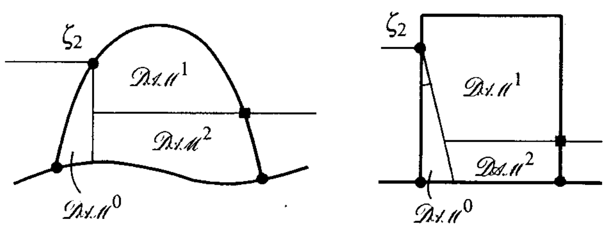

Figure 2: $\mathcal{DAM}^j$ $(j = 0, 1, 2)$.

Then, the **filtration problem** is to find the flow region $\Omega \subset \mathcal{DAM}$ and the piezometric function (velocity potential) u defined on Ω which satisfies

$$\begin{aligned}
&\Delta u = 0 && \text{in } \Omega, \\
&\frac{\partial u}{\partial n} = 0 && \text{on } \Gamma_1, \\
&u = u^0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 && \text{on } \Gamma_2, \\
&u = u^0 && \text{on } \Gamma_3, \\
&u = u^0 \quad \text{and} \quad \frac{\partial u}{\partial n} \leq 0 && \text{on } \Gamma_4,
\end{aligned} \tag{2.1}$$

where $n := (n_1, n_2)$ is the unit outer normal vector of $\partial\Omega$. Note that on the free boundary Γ_2, both Dirichlet's and Neumann's conditions are imposed.

2.3 A Variational Principle of the Filtration Problem

For both mathematical analysis and numerical computation, it would be nice if we have a variational principle of the filtration problem. In this section, we explain a variational principle introduced in [19]. The idea is simple. Let $\Omega \subset \mathscr{D}_{\mathscr{M}}$ be a candidate of the exact solution of the filtration problem (that is, the true flow region). Let u_Ω, $w_\Omega \in H^1(\Omega)$ be two harmonic functions with

$$u_\Omega = u^0, \quad \frac{\partial w_\Omega}{\partial n} = 0 \quad \text{on } \Gamma_2.$$

We suppose that u_Ω, w_Ω satisfy the boundary conditions of (2.1) on $\Gamma_1 \cup \Gamma_3 \cup \Gamma_4$. If Ω is the exact solution, u_Ω must be equal to w_Ω. If Ω is not the exact solution, the "difference between u_Ω and w_Ω" should represent the distance between Ω and the exact solution in some way. Although, one may take any norm to measure the "difference between u_Ω and w_Ω", we measure the difference in the following manner.

At first, we define the subsets $\mathscr{A}(\Omega)$, $\mathscr{B}(\Omega) \subset H^1(\Omega)$ by

$$\mathscr{A}(\Omega) := \left\{ v \in \mathscr{K}^*(\Omega) \mid v = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \right\},$$
$$\mathscr{B}(\Omega) := \left\{ v \in H^1(\Omega) \mid v = u^0 \quad \text{on} \quad \Gamma_3 \cup \Gamma_4 \right\},$$

where $\mathscr{K}^*(\Omega)$ is defined by

$$\mathscr{K}(\Omega) := \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \quad v \geq 0 \text{ on } \Gamma_4 \right\},$$
$$\mathscr{K}^*(\Omega) := \left\{ v \in H^1(\Omega) \mid (\nabla v, \nabla \chi) \leq 0, \quad \forall \chi \in \mathscr{K}(\Omega) \right\}.$$

Note that for a harmonic function $\chi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$, χ belongs to $\mathscr{K}^*$ if and only if $\partial\chi/\partial n \leq 0$ on Γ_4 in the variational sense. Take and fix M_0, a sufficiently large positive number. Let also D_Ω denotes the Dirichlet integral on Ω:

$$D_\Omega(v) := \frac{1}{2} \int_\Omega |\nabla v|^2.$$

Definition 2.3.1. *Under the setting defined so far, a subset $\Omega \subset \mathscr{D}_{\mathscr{M}}$ is called* **admissible** *if Ω satisfies the following conditions:* (1) *Ω is a Lipschitz domain.* (2) *$\partial\Omega \supset B_1 \cup B_3$.* (3) *$\partial\Omega \setminus \overline{B_1 \cup B_3}$ is a $C^{0,1}$ curve and is a monotonously decreasing graph in the direction x_2.* (4) *$\mathscr{A}(\Omega) \neq \emptyset$ and $\inf_{v \in \mathscr{A}(\Omega)} D_\Omega(v) \leq M_0$. We denote by $\mathscr{A}_{\mathscr{D}}$ the set of all admissible domains.*

The functional $a(\Omega), b(\Omega), J(\Omega) : \mathscr{A}_{\mathscr{G}} \to \mathbb{R}$ are defined by

$$a(\Omega) := \inf_{v \in \mathscr{A}(\Omega)} D_\Omega(v), \quad b(\Omega) := \inf_{v \in \mathscr{B}(\Omega)} D_\Omega(v),$$
$$J(\Omega) := a(\Omega) - b(\Omega).$$

Since $\mathscr{A}(\Omega) \subset \mathscr{B}(\Omega)$, we have $J(\Omega) \geq 0$.

From the Dirichlet's principle, we know that the values $a(\Omega)$ and $b(\Omega)$ are attained by the harmonic functions u_Ω and w_Ω (that is, $a(\Omega) = D_\Omega(u_\Omega)$ and $b(\Omega) = D_\Omega(w_\Omega)$), respectively, which satisfy the boundary conditions

$$\begin{cases} u_\Omega = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \frac{\partial u_\Omega}{\partial n} \leq 0 \text{ on } \Gamma_4, \quad \frac{\partial u_\Omega}{\partial n} = 0 \text{ on } \Gamma_1, \end{cases} \tag{2.2}$$

$$w_\Omega = u^0 \text{ on } \Gamma_3 \cup \Gamma_4, \quad \frac{\partial w_\Omega}{\partial n} = 0 \text{ on } \Gamma_1 \cup \Gamma_2. \tag{2.3}$$

We have the following variational principle for the filtration problem:

Theorem 2.3.2. *We have* $\inf_{\mathscr{A}_{\mathscr{G}}} J = 0$ *for the functional* $J : \mathscr{A}_{\mathscr{G}} \to \mathbb{R}$. *Moreover, an admissible domain* $\Omega \in \mathscr{A}_{\mathscr{G}}$ *is a solution of the filtration problem if and only if* $J(\Omega) = \inf_{\mathscr{A}_{\mathscr{G}}} J = 0$.

2.4 The Hadamard Variations of $a(\Omega)$ and $b(\Omega)$

By Theorem 2.3.2, the filtration problem may be solved (in particular, numerically) by an optimization process. In an optimization process, the boundary would be modified gradually and, therefore, it is very important to know how $a(\Omega)$ and $b(\Omega)$ would vary under perturbation of the domain (or the boundary). Such variations with respect to domain perturbation are called the **Hadamard variations**. In this section, we give the first variations of $a(\Omega)$ and $b(\Omega)$ with respect to domain perturbation obtained in [20]. In the next section, we present an iterative scheme using the obtained first variations.

Suppose that we have $\Omega \in \mathscr{A}_{\mathscr{G}}$ and try to modify it. Let a vector field $\mathsf{S} \in W^{1,\infty}(\mathscr{D}_{\mathscr{M}})$ be given. We consider the ordinary equation

$$\frac{dc}{dt}(t) = \mathsf{S}(c(t)), \quad t \geq 0,$$
$$c(0) = x, \quad x \in \mathscr{D}_{\mathscr{M}}.$$

Then, for each $x \in \mathscr{D}_{\mathscr{M}}$ the solution $c(t)$ forms an integral curve. Then, $\mathscr{T}_t(x) := c(t)$ satisfies the following:

- $\mathscr{T}_0(x) = x, \forall x \in \mathscr{D}_{\mathscr{M}}$.

- $\mathcal{T}_t$ is a diffeomorphism of $\mathscr{D}\mathscr{A}\mathscr{M}$ for sufficiently small $t > 0$.
- $\mathcal{T}_t$ is smooth with respect to t.
- $\mathcal{T}_t$ has the Taylor expansion

$$\mathcal{T}_t(x) = x + t\mathsf{S}(x) + o(t).$$

We use this $\mathcal{T}_t$ as perturbations of $\mathscr{D}\mathscr{A}\mathscr{M}$.

Now, let $\Omega \in \mathscr{A}_{\mathscr{D}}$ be a candidate of the solution of the filtration problem. To consider a perturbation of Ω, only the free boundary Γ_2 and Γ_4 would be moved. Hence, we may assume that

$$\mathrm{supp}\mathsf{S} \cap \Omega \subset \mathscr{D}\mathscr{A}\mathscr{M}^1 \;\text{ or }\; \mathrm{supp}\mathsf{S} \cap \partial\Omega \subset \Gamma_2 \cup \Gamma_4. \tag{2.4}$$

The harmonic function $u_\Omega \in \mathscr{A}(\Omega)$ is a solution of the boundary value problem (2.2). Its weak form is

$$(\nabla u_\Omega, \nabla v)_{0,\Omega} = 0, \quad \forall v \in V_0(\Omega),$$
$$u_\Omega = u^0 \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

where

$$V_0(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\}.$$

For a sufficiently small $t > 0$, let $\Omega_t := \mathcal{T}_t(\Omega)$ and suppose that $\Omega_t \in \mathscr{A}_{\mathscr{D}}$. We also consider the harmonic function u_{Ω_t} such that $a(\Omega_t) := D_{\Omega_t}(u_{\Omega_t})$. Setting $\Gamma_2^t := \partial\Omega_t \cap \mathscr{D}\mathscr{A}\mathscr{M}$, $\Gamma_4^t := \partial\Omega_t \cap B_2$ and

$$V_0(\Omega_t) := \{v \in H^1(\Omega_t) \mid v = 0 \text{ on } \Gamma_2^t \cup \Gamma_3 \cup \Gamma_4^t\},$$

u_{Ω_t} satisfies a similar weak form as above, and we have

$$\tilde{v} \in V_0(\Omega_t) \Longleftrightarrow \tilde{v} \circ \mathcal{T}_t \in V_0(\Omega). \tag{2.5}$$

To show the theorem below, (2.5) plays an important role. Let $\langle \cdot, \cdot \rangle_{\Gamma_2 \cup \Gamma_4}$ denote the duality pair of $H^{-1/2}(\Gamma_2 \cup \Gamma_4)$ and $H^{1/2}(\Gamma_2 \cup \Gamma_4)$. The first variation $\delta a(\Omega)$ is defined by

$$\delta a(\Omega) := \lim_{t \to +0} \frac{a(\Omega_t) - a(\Omega)}{t},$$

and we have the following theorem.

Theorem 2.4.1. *Let $\Omega \in \mathscr{A}_{\mathscr{D}}$ be an admissible domain. Suppose that the perturbation $\mathcal{T}_t(x) = x + t\mathsf{S}(x) + o(t)$ satisfies that $\Omega_t := \mathcal{T}_t(\Omega) \in \mathscr{A}_{\mathscr{D}}$ for all sufficiently small $t > 0$ and (2.4). Then, the first variation $\delta a(\Omega)$ is written* *by*

$$\delta a(\Omega) = \frac{1}{2}\left\langle 1 - \left(\frac{\partial p_\Omega}{\partial n}\right)^2, \delta\rho \right\rangle_{\Gamma_2\cup\Gamma_4},$$

where $p := u_\Omega - x_2$ *and* $\delta\rho := \mathrm{S}\cdot n$ *is the normal component of* S.

Remark: In the classical sense, if Γ_2 is sufficiently smooth, $\partial u_\Omega/\partial x_1$ and $\partial u_\Omega/\partial x_2$ exist at almost all points on Γ_2. Then, the first variation is written as a usual integral over $\Gamma_2 \cup \Gamma_4$:

$$\delta a(\Omega) = \frac{1}{2}\int_{\Gamma_2\cup\Gamma_4}\left(1 - \left(\frac{\partial p_\Omega}{\partial n}\right)^2\right)\delta\rho ds.$$

The harmonic function $w_\Omega \in \mathscr{B}(\Omega)$ is a solution of the boundary value problem (2.3). Its weak form is

$$(\nabla w_\Omega, \nabla v)_{0,\Omega} = 0, \quad \forall v \in V_1(\Omega),$$

$$w_\Omega = u^0 \quad \text{on } \Gamma_3 \cup \Gamma_4,$$

where

$$V_1(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \ \text{ on } \Gamma_3 \cup \Gamma_4\}.$$

We now consider the harmonic function $w_{\Omega_t} \in \mathscr{B}(\Omega_t)$ which satisfies a similar weak form with

$$V_1(\Omega_t) := \{v \in H^1(\Omega_t) \mid v = 0 \ \text{ on } \Gamma_3 \cup \Gamma_4^t\},$$

where $\Gamma_4^t := \partial\Omega_t \cap B_2$. The difficulty here comes from the fact that

$$\tilde{v} \in V_1(\Omega_t) \Longleftrightarrow \tilde{v}\circ\mathscr{T}_t \in V_1(\Omega) \tag{2.6}$$

is *not* valid in general since the boundary point $\zeta_5 := \overline{\Gamma}_2 \cap \overline{\Gamma}_4$ may be "peeled off" by the perturbation (see Figure 3).

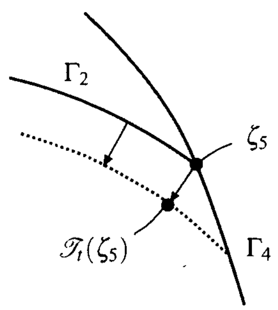

Figure 3: The boundary point ζ_5 may be "peeled off" by the perturbation.

Therefore, we need to impose an additional assumption on perturbation. If (2.6) holds for all sufficiently small $t \geq 0$, the perturbation $\mathscr{T}_t$ of $\Gamma_2 \cup \Gamma_4$ is said to satisfy the **NPO condition**. (The term "NPO" stands for

"Non-Peeling-Off.") Let $b(\Omega_t) := D_{\Omega_t}(w_{\Omega_t})$. Then, the first variation $\delta b(\Omega)$ of $b(\Omega)$ is defined by

$$\delta b(\Omega) := \lim_{t \to 0+} \frac{b(\Omega_t) - b(\Omega)}{t}.$$

Theorem 2.4.2. *Let $\Omega \in \mathscr{A}_{\mathscr{D}}$ be an admissible domain and $w_\Omega \in \mathscr{B}(\Omega)$ be such that $b(\Omega) = D_\Omega(w_\Omega)$. Suppose that the perturbation $\mathscr{T}_t(x) = x + t\mathsf{S}(x) + o(t)$ satisfies that $\Omega_t := \mathscr{T}_t(\Omega) \in \mathscr{A}_{\mathscr{D}}$ for all sufficiently small $t > 0$ and (2.4). Moreover, we assume that the NPO condition (2.6) holds. Then, the first variation $\delta b(\Omega)$ of the functional $b(\Omega)$ is written by*

$$\delta b(\Omega) = \frac{1}{2} \left\langle \left(\frac{\partial w_\Omega}{\partial s} \right)^2, \delta\rho \right\rangle_{\Gamma_2},$$

where $\partial/\partial s$ is tangential derivative along Γ_2 and $\delta\rho := \mathsf{S} \cdot n$ is the normal component of S.

Corollary 2.4.3. *Suppose that all assumptions of Theorem 3 and 4 hold. Then, the first variation $\delta J(\Omega)$ of the functional $J(\Omega) := a(\Omega) - b(\Omega)$ is written by*

$$\begin{aligned} \delta J(\Omega) &:= \lim_{t \to 0+} \frac{J(\Omega_t) - J(\Omega)}{t} \\ &= \frac{1}{2} \left\langle 1 - \left(\frac{\partial p_\Omega}{\partial n} \right)^2 - \left(\frac{\partial w_\Omega}{\partial s} \right)^2, \delta\rho \right\rangle_{\Gamma_2}. \end{aligned} \tag{2.7}$$

Moreover, $\delta J(\Omega) = 0$ for any sufficiently small $\delta\rho$ if and only if $\Omega \in \mathscr{A}_{\mathscr{D}}$ is the solution of the filtration problem.

2.5 The Traction Method — an Iterative Scheme

From this previous section, we present Hadamard variation as an iterative scheme in this section. Suppose that we are trying to obtain the flow region Ω and the potential u by an iterative scheme. Let $\Omega^{(k)}$ and $\Gamma_2^{(k)} \subset \partial\Omega^{(k)}$ be k-th guess of the flow region and the free boundary, respectively. Since the first variation of the functional $J : \mathscr{A}_{\mathscr{D}} \to \mathbb{R}$ is

$$\delta J(\Omega^{(k)}) = \left\langle 1 - \left(\frac{\partial p_{\Omega^{(k)}}}{\partial n} \right)^2 \left(\frac{\partial w_{\Omega^{(k)}}}{\partial s} \right)^2, \delta\rho \right\rangle_{\Gamma_2^{(k)}},$$

an intuitive iterative scheme is defined by

$$\begin{aligned} FV(x) &:= 1 - \left(\frac{\partial p_{\Omega^{(k)}}}{\partial n} \right)^2 - \left(\frac{\partial w_{\Omega^{(k)}}}{\partial s} \right)^2, \\ \Gamma_2^{(k+1)} &:= \left\{ x + \varepsilon FV(x) n(x) \,\middle|\, x \in \Gamma_2^{(k)} \right\}, \end{aligned} \tag{2.8}$$

for $x \in \Gamma_2^{(k)}$, where ε is a positive dumping parameter, and $n(x)$ is the unit outer normal vector at $x \in \Gamma_2^{(k)}$. This scheme (2.8) might be called a *steepest descent method*. However, even when ε is set very small, numerical experiments show that this scheme does not work at all.

After several iterations, $\Gamma_2^{(k)}$ becomes "jagged" and computation cannot be carried out any more.

We next propose another iterative scheme which is defined in the following way. Let $z^{(k)} \in H^1(\Omega^{(k)})$ be the solution of the boundary value problem:

$$\begin{aligned}
&\Delta z^{(k)} = 0 \qquad \text{in } \Omega^{(k)},\\
&z^{(k)} = 0 \qquad \text{on } \Gamma_3 \cup \Gamma_4^{(k)},\\
&\frac{\partial z^{(k)}}{\partial n} = 0 \quad \text{on } \Gamma_1,\\
&\frac{\partial z^{(k)}}{\partial n} = FV \quad \text{on } \Gamma_2^{(k)}.
\end{aligned} \tag{2.9}$$

Then, the iteration is defined by

$$\Gamma_2^{(k+1)} := \left\{ x - z^{(k)}(x)n(x) \;\middle|\; x \in \Gamma_2^{(k)} \right\}.$$

The method is called the **traction method** and was presented by Azegami (see [4] [17] and the references therein) as a numerical iterative scheme for optimal shape design. Numerical experiments show that the traction method works very well for the filtration problem. Beginning from a suitably defined initial guess, the iteration converges smoothly to a numerical solution.

In the following, we point out the two significant natures of the traction method. Firstly, the traction method decreases the value of $J(\Omega)$ in its iterative process. Let $\Omega \subset \mathscr{D}_{\mathscr{A}\mathscr{M}}$ be an admissible domain. Suppose that the perturbed domain Ω_τ is defined by the traction method

$$\Gamma_2^\tau := \left\{ x - \tau z(x)n(x) \;\middle|\; x \in \Gamma_2 \right\}, \quad \tau > 0,$$

where $z(x)$ is a solution of the boundary value problem similar to (2.9). Letting $\delta\rho := -z$ and $FV := \partial z/\partial n$ on Γ_2 and $\delta\rho := FV := 0$ elsewhere on $\partial\Omega$, we have

$$\begin{aligned}
\langle FV, \delta\rho \rangle_{\Gamma_2} &= \left\langle \frac{\partial z}{\partial n}, (-z) \right\rangle_{\Gamma_2}\\
&= \left\langle \frac{\partial z}{\partial n}, (-z) \right\rangle_{\Omega} = -\int_\Omega |\nabla z|^2 dx
\end{aligned}$$

and

$$\begin{aligned} J(\Omega_\tau) &= J(\Omega) + \tau\, \delta J(\Omega) + o(\tau) \\ &= J(\Omega) + \tau \langle FV, \delta\rho \rangle_{\Gamma_2} + o(\tau) \\ &= J(\Omega) - \tau \int_\Omega |\nabla z|^2 dx + o(\tau). \end{aligned}$$

Therefore, we may expect

$$J(\Omega_\tau) < J(\Omega)$$

at each step of the traction method. This nature of the traction method is already pointed out by Kaizu and Azegami [17] in a different context.

Secondly, numerical experiments suggest that the traction method seems to have a stabilizing and smoothing effect of the free boundary. Although, the mechanism effect of the traction method is not understood completely at this point, we here give a partial explanation. Let the k-th guess $\Gamma_2^{(k)}$ of the free boundary is in $C^{1,\alpha}$ class $(0 < \alpha < 1)$. Then, it follows from the regularity theory of linear elliptic partial differential equations(PDE) that

$$\begin{aligned} & u_{\Omega^{(k)}}, p_{\Omega^{(k)}}, w_{\Omega^{(k)}} \in C^{1,\alpha}(\Omega^{(k)} \cup \Gamma_2^{(k)}) \quad \text{and} \\ & FV := 1 - \left(\frac{\partial p_{\Omega^{(k)}}}{\partial n}\right)^2 - \left(\frac{\partial w_{\Omega^{(k)}}}{\partial s}\right)^2 \in C^{0,\alpha}(\Gamma_2^{(k)}). \end{aligned}$$

Since the Neumann-Dirichlet map

$$C^{0,\alpha}(\Gamma_2) \ni FV = \frac{\partial z^{(k)}}{\partial n} \mapsto z^{(k)} \in C^{1,\alpha}(\Gamma_2)$$

is used in the iterative procedure of the traction method, the updated $\Gamma_2^{(k+1)}$ is in $C^{1,\alpha}$ class again. Hence, the traction method at least preserves the smoothness of the free boundary. If $\Gamma_2^{(k)}$ is updated by the "steepest descend method" (2.8), however, $\Gamma_2^{(k+1)}$ is only in $C^{0,\alpha}$. Probably, this is a reason of the unstable behaviour of the scheme (2.8).

2.6 A Numerical Example

In this section, we give a numerical example which show the effectiveness of the traction method. Let positive numbers $h_1 > h_2 > 0$, $a > 0$ be given. As $\mathscr{D}\mathscr{A}\mathscr{M}$ we take a rectangle $\mathscr{D}\mathscr{A}\mathscr{M} := (0, h_1) \times (0, a)$ (see Figure 4).

In the case that $\mathscr{D}\mathscr{A}\mathscr{M}$ is rectangle, the filtration problem is reformulated as a variational inequality, and the existence and the uniqueness are proved nicely (see Appendix). First, we compute the solution of the variational inequality and use it to obtain the initial guess for iterative scheme. We set the values $a = 1.62$, $h_1 = 3.22$,

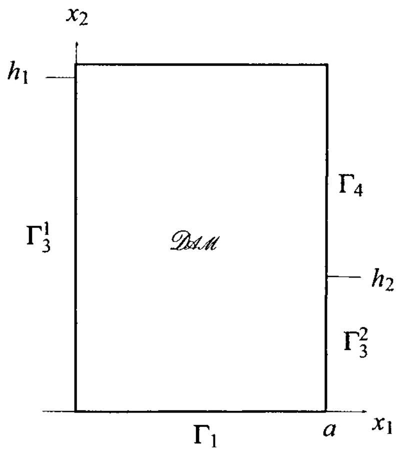

Figure 4: A rectangle dam.

$h_2 = 0.84$. In Figure 5, we show the numerical solution of the variational inequality. Suppose that $w \in H^1(\mathscr{DAM})$ is the solution of the variational inequality. Then the flow region Ω is represented as $\Omega = \{x \in \mathscr{DAM} : w(x) > 0\}$. In Figure 5, therefore, we draw all triangle elements on which the finite element solution is positive. The union of such elements can be regarded as a numerical approximation of the flow region.

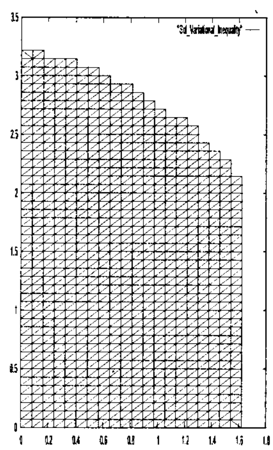

Figure 5: The numerical solution of the variational inequality.

Then, we use the approximated region as an initial region for the traction method (Figure 6).

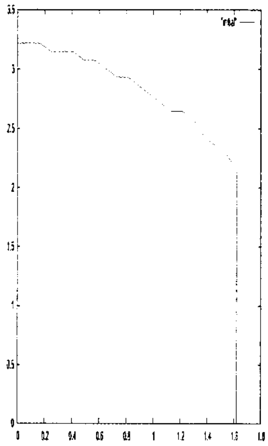

Figure 6: The initial guess made by the variational inequality.

After several steps, the traction method converges smoothly, and we obtain a numerical solution.

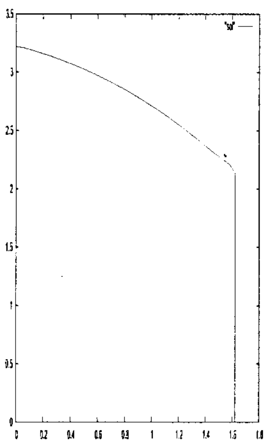

Figure 7: The numerical solution of the traction method.

2.7 Numerical Example (continued) – Level Set approaches

There are two types of level-set approaches; one of them reformulates the problem into a variational inequality when the dam $\mathscr{DAM}$ is a rectangle. In the other level-set approach, the filtration problem is reformulated into a problem to find a pair (p,γ), where p is the pressure of the flow and γ is the characteristic function of the flow region. In this section, we explain them briefly.

In the case that the dam is a rectangle, Baiocchi [5] transformed the problem into a variational inequality, and showed the existence and uniqueness of the solution of the filtration problem (see also [18]). Let $a > 0$, $h_1 > h_2 > 0$ be positive constants. Let $\mathscr{DAM} := (0,a) \times (0,h_1)$ be a rectangle dam. Let the function $g \in H^{2,\infty}(\mathscr{DAM})$ be defined by

$$g(x_1,x_2) := \begin{cases} \frac{a-x_1}{2a}(h_1-x_2)^2 + \frac{x_1}{2a}(h_2-x_2)^2, & 0 < x_2 < h_2 \\ \frac{a-x_1}{2a}(h_1-x_2)^2, & h_2 < x_2 < h_1 \end{cases}$$

Let $K \subset H^1(\mathscr{D_{AM}})$ be defined by

$$K := \{v \in H^1(\mathscr{D_{AM}}) : v \geq 0 \text{ in } \mathscr{D_{AM}},\ v = g \text{ on } \partial(\mathscr{D_{AM}})\}$$

From the theory of variational inequalities (see [18]), there exists a unique solution $w \in K$ of the variational inequality

$$\int_{\mathscr{D_{AM}}} \nabla w \cdot \nabla(v-w) \geq -\int_{\mathscr{D_{AM}}} (v-w),\ \forall v \in K.$$

Then, the domain $\Omega := \{x \in \mathscr{D_{AM}} : w > 0\}$ is the desired flow region and $u := x_2 - \partial w/\partial x_2$ is the desired velocity potential. Unfortunately, this beautiful theory works only for the cases that the both sides walls of the dam is vertical.

Later, Alt [1] and Brezis-Kinderlehrer-Stampacchia [9] gave different approaches which can treat general situations, and proved the existence of a solution of the filtration problem. The uniqueness of the solution was proved by Alt-Gilardi [3] and Carrillo-Chipot [11]. In this approach, the filtration problem is formulated to find a pair (p,γ) as is stated in the following. Let $e := (0,1)$.

Problem 2.7.1. *Find a pair (p,γ), where $p \in H^1(\mathscr{D_{AM}})$, $\gamma \in L^\infty(\mathscr{D_{AM}})$ such that*

$$0 \leq \gamma \leq 1, \quad \gamma = 1 \text{ on } \{p \geq 0\}$$

$p = p^0$ on $B_2 \cup B_3$ such that

$$\int_{\mathscr{D_{AM}}} \nabla\zeta \cdot (\nabla p + \gamma e) \leq 0, \quad \forall \zeta \in H^1(\mathscr{D_{AM}})$$

with $\zeta \geq 0$ on B_2 and $\zeta = 0$ on B_3.

As stated above, it has been proved that there exists a unique solution (p,γ). Also, it is shown that γ is the characteristic function of the region $\{p > 0\}$: $\gamma = \chi_{\{p>0\}}$, and $u := p + x_2$ is the velocity potential of flow inside $\mathscr{D_{AM}}$ (see [15] for detail).

As long as we know, any level-set approaches for the filtration problem so far are modifications of one of the above mentioned theorems.

Chapter 3

Unified Approach to the Hadamard Variation

In this chapter, we present a new variational formula for Hadamard variation. We use and derive Hadamard first variational formula and obtain a new Hadamard second variational formula which also an extension of Garabedian-Schiffer's formula. We develop a new methodology which provides a much clearer understanding of Hadamard variational formula. We also obtain a simple proof of Hadamard variational formula.

3.1 General Domain Perturbation

Let $\Omega \subset \mathbb{R}$ be a bounded Lipschitz domain and let $\mathscr{T}_t : \Omega \to \mathscr{T}_t\Omega = \Omega_t$ be a bi-Lipschitz homeomorphism. Assume that domain deformation $\mathscr{T}_t x$ takes the Taylor expansion

$$\mathscr{T}_t x = x + t\mathsf{S}x + \frac{1}{2}t^2\mathsf{T}x + o(t^2).$$

Thus it holds that

$$\left.\frac{\partial \mathscr{T}_t x}{\partial t}\right|_{t=0} = \mathsf{S}x, \qquad \mathscr{T}_0 x = x, \quad \forall x \in \Omega. \tag{3.1}$$

We then define,

$$\begin{aligned}\delta\rho &= \left.\frac{\partial \mathscr{T}_t x}{\partial t}\right|_{t=0} \cdot n \\ &= \mathsf{S}\cdot n, \qquad x \in \partial\Omega\end{aligned}$$

We differentiate again (3.1) and obtain

$$\left.\frac{\partial^2 \mathscr{T}_t x}{\partial t^2}\right|_{t=0} = \mathsf{T}x \tag{3.2}$$

$$
\begin{aligned}
\delta^2\rho &= \left.\frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\right|_{t=0} \cdot n \\
&= [(\mathsf{S}\cdot\nabla)\mathsf{S}]\cdot n\big|_{t=0} \\
&= \mathsf{T}x\cdot n.
\end{aligned}
$$

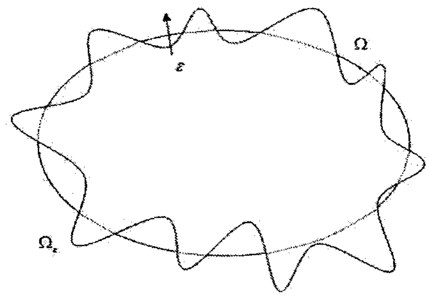

Figure 3.1: Domain perturbation.

3.2 Remarks:

3.2.1 Lie Perturbation

Let $v = v(x)$ be a $C^{0,1}$ vector field in $\widetilde{\Omega}$, $\Omega \subset\subset \widetilde{\Omega}$. For all $x \in \widetilde{\Omega}$, we have the ordinary differential equation,

$$
\frac{dc(t)}{dt} = v(c(t)), \qquad c(0) = x
$$

where the integral curve $c : (-\varepsilon, \varepsilon) \to \widetilde{\Omega}$ is determined. Let

$$
c(t) = \mathcal{T}_t x.
$$

Hence

$$
\mathcal{T}_t x\big|_{t=0} = x, \qquad \frac{\partial \mathcal{T}_t x}{\partial t} = v(\mathcal{T}_t x)
$$

$$
\left.\frac{\partial \mathcal{T}_t x}{\partial t}\right|_{t=0} = v(x).
$$

$$
\left.\frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\right|_{t=0} = \nabla v(\mathcal{T}_t x)\cdot \left.\frac{\partial \mathcal{T}_t x}{\partial t}\right|_{t=0} = (v\cdot\nabla)v(x)
$$

Therefore, we have the Taylor expansion as

$$
\mathcal{T}_t(x) = x + t v(x) + \frac{t^2}{2}(v\cdot\nabla)v(x) + o(t^2).
$$

Therefore, we have the Taylor expansion as

$$\mathscr{T}_t(x) = x + tv(x) + \frac{t^2}{2}(v\cdot\nabla)v(x) + o(t^2).$$

Hence

$$\mathrm{S}x = v(x), \qquad \mathrm{T}x = (v\cdot\nabla)v(x).$$

3.2.2 Non Peeling off Condition

Given a part $\gamma \subset \Gamma = \partial\Omega$. We say that non peeling off (N.P.O) condition on γ is achieved if $S\cdot n = 0$ on γ. This will not be satisfied for the normal perturbation described below.

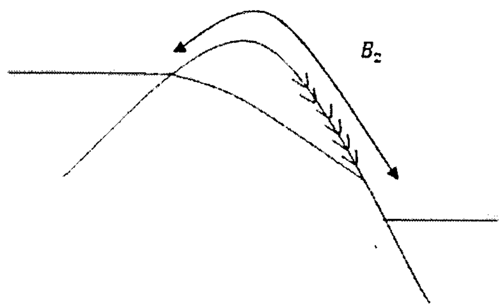

Figure 3.2: Non peeling of condition at B_2.

3.2.3 Normal Perturbation

If $\partial\Omega$ is C^1, we take a C^1 deformation of $\Gamma = \partial\Omega$. Let $\Gamma_t = \partial\Gamma_t : x + t(\delta\rho)(x)\cdot n_x$, $x \in \partial\Omega$ where $\delta\rho$ is a given function on $\partial\Omega$. There exist a C^1 diffeomorphism $\mathscr{T}_t : \Omega \to \Omega_t$, $|t| \ll 1$. Therefore,

$$\delta\rho = \left.\frac{\partial \mathscr{T}_t x}{\partial t}\right|_{t=0} \cdot n, \quad x \in \partial\Omega$$

where $\delta^2\rho = 0$ and $\{s_1, \ldots, s_{n-1}\}$ is an orthonormal basis of $\partial\Omega$. This deformation does not work if $\partial\Omega$ has a corner.

3.3 Dirichlet Problem

3.3.1 Chain Rule

Given a $C^{1,1}$ scalar field $f(x)$, the Hesse matrix

$$Il_x f = H_x f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}$$

is a symmetric matrix of the second order tensor and let $H_x f \cdot (\mathsf{S})^2 = ((H_x f)\mathsf{S},\mathsf{S})_{\mathbb{R}^n}$, S is a vector. Combining the functional differential equations, we obtain

$$\frac{du}{dt}(\mathcal{T}_t x,t) = \frac{\partial u}{\partial t}(\mathcal{T}_t x,t) + \left(\frac{\partial \mathcal{T}_t x}{\partial t}\right)\cdot \nabla u(\mathcal{T}_t x,t) \tag{3.3}$$

and

$$\begin{aligned} \frac{d^2u}{dt^2}(\mathcal{T}_t x,t) &= \frac{\partial^2 u}{\partial t^2}(\mathcal{T}_t x,t) + 2\left(\frac{\partial \mathcal{T}_t x}{\partial t}\right)\cdot \nabla\left(\frac{\partial u}{\partial t}(\mathcal{T}_t x,t)\right) \\ &+ \left(\frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\right)\cdot \nabla u(\mathcal{T}_t x,t) + \left((H_x u(\mathcal{T}_t x,t))\frac{\partial \mathcal{T}_t x}{\partial t},\frac{\partial \mathcal{T}_t x}{\partial t}\right). \end{aligned} \tag{3.4}$$

In fact, from the right hand side of (3.3),

$$\frac{d^2u}{dt^2}(\mathcal{T}_t x,t) = \frac{d}{dt}\left\{\frac{\partial u}{\partial t}(\mathcal{T}_t x,t) + \frac{\partial \mathcal{T}_t x}{\partial t}\cdot \nabla u(\mathcal{T}_t x,t)\right\}.$$

$$\frac{d^2u}{dt^2}(\mathcal{T}_t x,t) = \frac{\partial^2 u}{\partial t^2}(\mathcal{T}_t x,t) + \frac{d}{dt}\left\{\frac{\partial \mathcal{T}_t x}{\partial t}\cdot \nabla\frac{\partial u}{\partial t}(\mathcal{T}_t x,t)\right\} \tag{3.5}$$

and from the right hand side of second last equation of (3.5),

$$\begin{aligned} \frac{d}{dt}\left\{\frac{\partial \mathcal{T}_t x}{\partial t}\cdot \nabla u(\mathcal{T}_t x,t)\right\} &= \frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\cdot \nabla u(\mathcal{T}_t x,t) + \frac{\partial \mathcal{T}_t x}{\partial t}\cdot \frac{d}{dt}\nabla u(\mathcal{T}_t x,t) \\ &= \frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\cdot \nabla u(\mathcal{T}_t x,t) + \frac{\partial \mathcal{T}_t x}{\partial t}\cdot \left\{\frac{\partial u}{\partial t}\nabla(\mathcal{T}_t x,t) + \frac{\partial \mathcal{T}_t x}{\partial t}\cdot \nabla^2 u(\mathcal{T}_t x,t)\right\} \\ &= \frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\cdot \nabla u(\mathcal{T}_t x,t) + \frac{\partial \mathcal{T}_t x}{\partial t}\cdot \nabla\frac{\partial u}{\partial t}(\mathcal{T}_t x,t) + H_x u(\mathcal{T}_t x,t)\frac{\partial \mathcal{T}_t x}{\partial t}\cdot\frac{\partial \mathcal{T}_t x}{\partial t} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d^2u}{dt^2}(\mathcal{T}_t x,t) &= \frac{\partial^2 u}{\partial t^2}(\mathcal{T}_t x,t) + 2\left(\frac{\partial \mathcal{T}_t x}{\partial t}\right)\cdot \nabla\left(\frac{\partial u}{\partial t}(\mathcal{T}_t x,t)\right) \\ &+ \frac{\partial^2 \mathcal{T}_t x}{\partial t^2}\cdot \nabla u(\mathcal{T}_t x,t) + H_x u(\mathcal{T}_t x,t)\frac{\partial \mathcal{T}_t x}{\partial t}\cdot\frac{\partial \mathcal{T}_t x}{\partial t} \end{aligned}$$

3.3.2 Green's Function

Given $\varphi \in C^{1,1}(\tilde{\Omega})$. Let $u = u(x,t) \in C^{1,1}(\overline{\Omega}_t)$ be the solution of the boundary value problem

$$\begin{cases} \Delta u(\cdot,t) = 0 & \text{in } \Omega_t \\ u(\cdot,t) = \varphi & \text{on } \partial\Omega_t. \end{cases} \tag{3.6}$$

Here, $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is Laplacian and $x = (x_1, \ldots, x_n)$. Let $G = G(x,y)$ be Green's function of $-\Delta$:

$$-\Delta G(\cdot,y) = \delta(\cdot - y) \qquad G(\cdot,y)|_{\partial\Omega} = 0.$$

Proposition 3.3.1. *If $f \in C^2(\overline{\Omega})$,*

$$f(y) = -\int_\Omega G(x,y)\Delta f(x)\ dx - \int_{\partial\Omega} f(x)\frac{\partial G(x,y)}{\partial n}\ ds_x, \qquad y \in \Omega. \tag{3.7}$$

Proof. Let $y \in \Omega$. For $r > 0$ of $\overline{B_r(y)} \subset \Omega$, let $B_r(y) = \{z \in \mathbb{R}^n : |z-y| < r\}$. Define also $\Omega_r = \Omega \setminus B_r(y)$. Consider the fundamental solution of $\Gamma(x)$ for Laplacian Δ

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} log|x|, & n = 2, \\ \frac{1}{(n-2)\omega_n}|x|^{2-n}, & n \geq 3, \end{cases}$$

where $\omega_n = |S^{n-1}|$, $S^{n-1} = \partial B(0,1)$. In $\Omega \setminus \{y\}$, $\Gamma(x-y)$ is harmonic. Let f be a smooth function. We obtain

$$-\int_{\Omega_r} \Gamma(x-y)\Delta f(x)\ dx + \int_{\partial\Omega_r} \frac{\partial f}{\partial n}(x)\Gamma(x-y)\ ds = \int_{\partial\Omega_r} f(x)\frac{\partial}{\partial n}\Gamma(x-y)\ ds \tag{3.8}$$

by Green's Formula where ds is the area element. On the boundary $\partial B_r(y)$,

$$\frac{\partial}{\partial n_x}\Gamma(x-y) = \frac{d}{dr}(-\Gamma(|x-y|)) = \frac{1}{\omega_n r^{n-1}}, \qquad \int_{B_r(y)} [\cdot]\ ds = \int_{S^{n-1}} [\cdot] r^{n-1}\ d\omega.$$

At the right hand side of (3.8)

$$\begin{aligned} \int_{\partial\Omega_r} f(x)\frac{\partial}{\partial n}\Gamma(x-y)\ ds_x &= \int_{\partial\Omega} f(x)\frac{\partial}{\partial n}\Gamma(x-y)\ ds + \int_{\partial B_r(y)} f(y+s)\frac{1}{\omega_n r^{n-1}}\ ds \\ &= \int_{\partial\Omega} f(x)\frac{\partial}{\partial n}\Gamma(x-y)\ ds + \frac{1}{\omega_n}\int_{S^{n-1}} f(y+r\omega)\ d\omega \\ &\to \int_{\partial\Omega} f(x)\frac{\partial}{\partial n}\Gamma(x-y)\ ds + f(y) \qquad \text{as } r \to 0+. \end{aligned}$$

Hence,

$$-\int_\Omega \Gamma(x-y)\Delta f(x)\ dx + \int_{\partial\Omega} \frac{\partial f(x)}{\partial n}\Gamma(x-y)\ ds = \int_{\partial\Omega} f(x)\frac{\partial}{\partial n_x}\Gamma(x-y)\ ds + f(y). \tag{3.9}$$

It holds that

$$\Delta u = 0 \quad \text{on } \Omega, \qquad u(x) = -\Gamma(x-y), \quad x \in \partial\Omega \tag{3.10}$$

where

$$G(x,y) = \Gamma(x-y) + u(x).$$

In fact, fix $y \in \Omega$.

$$-\Delta G(x,y) = \delta(x-y) \quad \text{in } \Omega, \qquad G\big|_{x\in\partial\Omega} = 0 \quad \text{on } \partial\Omega.$$

$$-\Delta\Gamma(x-y) = \delta(x-y) \quad \text{in } \Omega,$$

$$u(x) \equiv G(x,y) - \Gamma(x-y).$$

Then,

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u(x)\big|_{x\in\partial\Omega} = -\Gamma(x-y) \quad \text{on } \partial\Omega$$

Hence, functions u and f in Green's formula satisfy

$$-\int_{\Omega} u(x)\Delta f(x)\ dx + \int_{\partial\Omega} \frac{\partial f(x)}{\partial n} u(x)\ ds = \int_{\partial\Omega} f(x)\frac{\partial u(x)}{\partial n}\ ds. \tag{3.11}$$

$$-\int_{\Omega} \Gamma(x-y)\Delta f(x)\ dx + \int_{\partial\Omega} \frac{\partial f(x)}{\partial n}\Gamma(x-y)\ ds = \int_{\partial\Omega} f(x)\frac{\partial}{\partial n}\Gamma(x-y)\ ds + f(y) \tag{3.12}$$

$$-\int_{\Omega} u(x)\Delta f(x)\ dx + \int_{\partial\Omega} \frac{\partial f(x)}{\partial n} u(x)\ ds = \int_{\partial\Omega} f(x)\frac{\partial u(x)}{\partial n}\ ds \tag{3.13}$$

From (3.9) and (3.13) we get (3.7). □

Taking a C^2 family as a subdomains such that by taking a small $\Omega_k \subset\subset \Omega$, $\Omega_k \uparrow \Omega$, we obtain the following.

Lemma 3.3.2. *Let Ω be $C^{0,1}$ and $f \in C^{0,1}(\overline{\Omega})$.*

$$\Delta f = 0 \ \ in \ \ \Omega$$

then

$$f(y) = -\int_{\partial\Omega} f(x)\frac{\partial G}{\partial n}(x,y)\ ds_x, \qquad y \in \Omega. \tag{3.14}$$

3.4 Liouville's Theorem

In this section, we discuss about Liouville's theorem which play an important role in Hadamard second derivational formula. Following Garabedian [16] where S and T are components of outer normal vectors and recall that

$$\delta\rho = \left.\frac{\partial \mathscr{T}_t}{\partial t}\right|_{t=0} \cdot n = \mathsf{S}\cdot n, \qquad \delta^2\rho = \left.\frac{\partial^2 \mathscr{T}_t}{\partial t^2}\right|_{t=0} \cdot n = \mathsf{T}\cdot n. \tag{3.15}$$

Theorem 3.4.1. *Consider that we have*

$$\left.\frac{\partial}{\partial t} det(J\mathscr{T}_t x)\right|_{t=0} = \nabla\cdot\mathsf{S} \tag{3.16}$$

and

$$\left.\frac{\partial^2}{\partial t^2} det(J\mathscr{T}_t x)\right|_{t=0} = \nabla\cdot\mathsf{T}(x) + 2\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}). \tag{3.17}$$

Proof. Consider Jacobi matrix of $\mathscr{T}_t x$:

$$J\mathscr{T}_t x = \begin{pmatrix} 1+tS_{1x_1}+\frac{1}{2}t^2\mathrm{T}_{1x_1} & tS_{1x_2}+\frac{1}{2}t^2\mathrm{T}_{1x_2} & \cdots & tS_{1x_n}+\frac{1}{2}t^2\mathrm{T}_{1x_n} \\ tS_{2x_1}+\frac{1}{2}t^2\mathrm{T}_{2x_1} & 1+tS_{2x_2}+\frac{1}{2}t^2\mathrm{T}_{2x_2} & \cdots & tS_{2x_n}+\frac{1}{2}t^2\mathrm{T}_{2x_n} \\ \vdots & \vdots & \cdots & \vdots \\ tS_{nx_1}+\frac{1}{2}t^2\mathrm{T}_{nx_1} & tS_{nx_2}+\frac{1}{2}t^2\mathrm{T}_{nx_2} & \cdots & 1+tS_{nx_n}+\frac{1}{2}t^2\mathrm{T}_{nx_n} \end{pmatrix}. \tag{3.18}$$

Then it holds that

$$\det J\mathscr{T}_t x = 1 + t\sum_{i<j}(S_{jx_j}+S_{ix_i}) + t^2\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}) + \frac{t^2}{2}\sum_i \mathrm{T}_{ix_i}$$

and have the first derivation of $\det J\mathscr{T}_t x$,

$$\begin{aligned}\left.\frac{\partial}{\partial t}(\det J\mathscr{T}_t x)\right|_{t=0} &= \sum_{i<j}(S_{jx_j}+S_{ix_i}) + 2t\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}) \\ &= \sum_{i<j}(S_{jx_j}+S_{ix_i}) \\ &= \nabla\cdot\mathsf{S}\end{aligned}$$

and also the second derivation of $\det J\mathscr{T}_t x$,

$$\begin{aligned}\left.\frac{\partial^2}{\partial t^2}(\det J\mathscr{T}_t x)\right|_{t=0} &= \sum_{i<j}(S_{jx_j}+S_{ix_i}) + 2t\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}) \\ &= \nabla\cdot\mathsf{T}\cdot + 2\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i})\end{aligned}$$

□

Corollary 3.4.2. *(First Derivation of Liouville's Theorem) Consider $C^{0,1}$ function $c(x,t)$, $|t| < \varepsilon$, $x \in \widetilde{\Omega}$, and let*

$\Omega_t = \mathscr{T}_t\Omega$. *Then,*

$$\frac{d}{dt}\left(\int_{\Omega_t} c(x,t)\ dx\right)\bigg|_{t=0} = \int_\Omega (c_t(x,0) + \nabla\cdot(c(x,0)\mathsf{S}x)\ dx \\ = \int_\Omega c_t(x,0)\ dx + \int_{\partial\Omega} c(x,0)\delta\rho\ ds. \tag{3.19}$$

Proof. It also holds that

$$\int_{\Omega_t} c(x,t)\ dx = \int_\Omega c(\mathscr{T}_t x,t) det(J\mathscr{T}_t x)\ dx \tag{3.20}$$

where $J\mathscr{T}_t x$ is the Jacobi matrix of $\mathscr{T}_t x$. Using (3.3), we have

$$\frac{d}{dt}\int_{\partial\Omega} c(x,t)\ dx = \int_\Omega \left(c_t(\mathscr{T}_t x,t) + \nabla c(\mathscr{T}_t x,t)\cdot\frac{\partial\mathscr{T}_t x}{\partial t}\right) det(J\mathscr{T}_t x)\ dx \\ + \int_\Omega c(\mathscr{T}_t x,t)\frac{\partial}{\partial t} det(J\mathscr{T}_t x)\ dx. \tag{3.21}$$

If $t \to 0$, then

$$\frac{d}{dt}\int_{\partial\Omega} c(x,t)\ dx\bigg|_{t=0} = \int_\Omega (c_t(x,0) + \nabla c(x,0)\cdot\frac{\partial\mathscr{T}_t x}{\partial t}\ dx \\ + \int_\Omega c(x,0)\nabla\cdot\mathsf{S}x)\ dx.$$

□

From the first derivation of Liouville's Theorem, we try to obtain second derivation of Liouville's Theorem. Assume that $\partial\Omega$, c, and S is sufficiently smooth. Therefore,

Corollary 3.4.3. *(Second Derivation of Liouville's Theorem) Let* $c(x,t), c_t(x,t) \in C^{0,1}$ *and* $\Omega_t = \mathscr{T}_t(\Omega)$. *Then, with* $\delta\rho = \mathsf{S}\cdot n$,

$$\frac{d^2}{dt^2}\left(\int_{\Omega_t} c(x,t)\ dx\right)\bigg|_{t=0} = \int_\Omega c_{tt}(x,0)\ dx + \langle 2c_t(\cdot,0) + \nabla\cdot(c(\cdot,0)\mathsf{S}), \delta\rho\rangle_{\partial\Omega}. \tag{3.22}$$

Proof. Using (3.4) and (3.21), we differentiate it again by t and get

$$\frac{d^2}{dt^2}\left(\int_{\Omega_t} c(x,t)\ dx\right) = \int_\Omega [c_{tt}(\mathscr{T}_t x,t) + 2\nabla_x c_t(\mathscr{T}_t x,t)\mathsf{S} + H_x c(\mathscr{T}_t x,t)\cdot\mathsf{S}\cdot\mathsf{S}] det(J\mathscr{T}_t x)\ dx \\ + \int_\Omega [\nabla_x c(\mathscr{T}_t x,t)\cdot\mathsf{T}] det(J\mathscr{T}_t x)\ dx + \int_\Omega c(\mathscr{T}_t x,t)\frac{\partial^2}{\partial t^2} det(J\mathscr{T}_t x)\ dx \\ +2\int_\Omega [c_t(\mathscr{T}_t x,t) + \nabla_x c(\mathscr{T}_t x,t)\cdot\mathsf{S}]\ \frac{\partial}{\partial t} det(J\mathscr{T}_t x)\ dx.$$

when $t \to 0+$,

$$
\begin{aligned}
&= \int_{\Omega} [c_{tt}(x,0) + 2\nabla_x c_t(x,0)\cdot \mathrm{S} + H_x c(x,0)\mathrm{S}\cdot\mathrm{S}]\ dx \\
&\qquad + \int_{\Omega} [\nabla_x c(x,0)\cdot \mathrm{T}]\ dx + \int_{\Omega} c(x,0)[\nabla\cdot\mathrm{T} + 2\sum S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}]\ dx \\
&\qquad + 2\int_{\Omega} [c_t(x,0) + \nabla_x c(x,0)\cdot\mathrm{S}]\nabla\cdot\mathrm{S}\ dx \\
&= \int_{\Omega} [c_{tt}(x,0) + 2\nabla_x c_t(x,0)\cdot\mathrm{S} + \nabla_x c(x,0)\cdot\mathrm{T} + H_x c(x,0)\mathrm{S}\cdot\mathrm{S}]\ dx \\
&\qquad + 2\int_{\Omega} [c_t(x,0) + \nabla_x c(x,0)\cdot\mathrm{S}]\nabla\cdot\mathrm{S}\ dx \\
&\qquad + \int_{\Omega} c(x,0)[\nabla\cdot\mathrm{T} + 2\sum_{i<j} S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}]\ dx.
\end{aligned}
$$

Using (3.17), therefore,

$$
\begin{aligned}
\frac{d^2}{dt^2}\left(\int_{\Omega_t} c(x,t)\ dt\right)\Bigg|_{t=0} &= \int_{\Omega} c_{tt}(x,0)\ dx + \int_{\Omega} H_x c(x,0)\mathrm{S}\cdot\mathrm{S}\ dx + 2\int_{\partial\Omega} c_t(x,0)\delta\rho\ ds \\
&\quad + \int_{\partial\Omega} c(x,0)\delta^2\rho\ ds + 2\int_{\Omega}(\nabla_x c(x,0)\cdot\mathrm{S})(\nabla\cdot\mathrm{S})\ dx \\
&\quad + 2\int_{\Omega} c(x,0)\sum_{i<j}(S_{ix_j}S_{jx_i} - S_{ix_i}S_{jx_j})\ dx.
\end{aligned} \tag{3.23}
$$

For vector S in (3.23), using Divergence theorem we got Jacobi matrix $J(\mathrm{S})$ as

$$
\begin{aligned}
\int_{\Omega}(\nabla_x c(x,0)\cdot\mathrm{S})(\nabla\cdot\mathrm{S})\ dx &= 2\int_{\partial\Omega}(\nabla_x c(x,0)\cdot\mathrm{S})\delta\rho\ ds \\
&\quad - \int_{\Omega} H_x c(x,0)\mathrm{S}\cdot\mathrm{S})\ dx - \int_{\Omega}(J(\mathrm{S})\mathrm{S})\cdot\nabla_x c(x,0)\ dx.
\end{aligned}
$$

We then substitute into (3.23),

$$
\begin{aligned}
\frac{d^2}{dt^2}\left(\int_{\Omega_t} c(x,t)\ dx\right)\Bigg|_{t=0} &= \int_{\Omega} c_{tt}(x,0)\ dx + 2\int_{\partial\Omega} c_t(x,0)\delta\rho\ ds + \int_{\partial\Omega} c(x,0)\delta^2\rho\ ds \\
&\quad - \int_{\Omega} H_x c(x,0)\mathrm{S}\cdot\mathrm{S}\ dx - 2\int_{\Omega}(J(\mathrm{S})(\mathrm{S}))\cdot\nabla_x c(x,0)\ dx \\
&\quad + 2\int_{\Omega} c(x,0)\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{jx_i}S_{ix_j})\ dx + 2\int_{\partial\Omega} c_t(x,0)\delta\rho\ ds.
\end{aligned} \tag{3.24}
$$

Here the left hand side of (3.24) can be written as

$$W = 2\int_{\partial\Omega} (\nabla_x c(x,0)\cdot \mathsf{S})\delta\rho \ \ ds$$

$$X = -\int_{\Omega} H_x c(x,0)\mathsf{S}\cdot\mathsf{S} \ \ dx$$

$$Y = -2\int_{\Omega} (J(\mathsf{S})\mathsf{S})\cdot\nabla_x c(x,0) \ \ dx$$

$$Z = 2\int_{\Omega} c(x,0)\sum_{i<j}(S_{ix_i}S_{jx_j} - S_{ix_j}S_{jx_i}) \ \ dx$$

Then, we 'differentiate' the last dx and get

$$\begin{aligned} X+Y &= -\sum_{i,j}\int_{\Omega} c_{x_ix_j}S_iS_j - 2\sum_{i,j} c_{x_i}S_{ix_j}S_j \ \ dx \\ &= -\int_{\partial\Omega}(\nabla c\cdot\mathsf{S})\delta\rho + \sum_{i,j}\int_{\Omega} c_{x_i}S_iS_{jx_j} - \sum_{i,j}\int_{\Omega} c_{x_i}S_{ix_j}S_j \ \ dx \\ &= -\int_{\partial\Omega}(\nabla c\cdot\mathsf{S})\delta\rho + \sum_{i,j}\int_{\Omega} c_{x_i}(S_iS_{jx_j} - S_{ix_j}S_j) \ \ dx \end{aligned} \tag{3.25}$$

$$\begin{aligned} Z &= 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_j}n_i - 2\sum_{i<j}\int_{\Omega} c_{x_i}S_iS_{jx_i} - 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_i}n_j + 2\sum_{i<j}\int_{\Omega} c_{x_j}S_iS_{jx_i} \\ &= 2\sum_{i<j}\int_{\partial\Omega} cS_i(S_iS_{jx_j}n_i - S_iS_{jx_i}n_j) \ \ ds - 2\sum_{i<j}\int_{\Omega}(c_{x_i} - c_{x_j})S_iS_{jx_i}. \end{aligned}$$

Hence

$$\begin{aligned} X+Y+Z = &-\int_{\partial\Omega}(\nabla_x c(x,0)\cdot\mathsf{S})\delta\rho + 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_j}n_i - 2\sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_i}n_j \\ &-\sum_{i<j}\int_{\Omega} c_{x_i}S_iS_{jx_j} + \sum_{i>j}\int_{\Omega} c_{x_i}S_iS_{jx_j} + \sum_{i<j}\int_{\Omega} c_{x_j}S_iS_{jx_j} - \sum_{i>j}\int_{\Omega} c_{x_j}S_iS_{jx_i}. \end{aligned} \tag{3.26}$$

Therefore,

$$\begin{aligned} X+Y+Z = &-\int_{\partial\Omega}(\nabla_x c(x,0)\cdot\mathsf{S})\delta\rho + \sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_j}n_i - \sum_{i<j}\int_{\partial\Omega} cS_iS_{jx_i}n_j \\ &+\sum_{i<j}\int_{\partial\Omega} cS_{ix_i}S_jn_j - \sum_{i<j}\int_{\partial\Omega} cS_{ix_j}S_jn_i. \end{aligned} \tag{3.27}$$

Hence, changing i and j,

$$X+Y+Z = -\int_{\partial\Omega}(\nabla_x C(x,0)\cdot\mathsf{S})\delta\rho + c(\nabla\cdot\mathsf{S})\delta\rho - c(x,0)\delta^2\rho \ \ ds. \tag{3.28}$$

Using (3.15), $W+X+Y+Z$,

$$W+X+Y+Z = \int_{\partial\Omega}(\nabla_x c(x,0)\cdot\mathsf{S})\delta\rho + c(x,0)(\nabla\cdot\mathsf{S})\delta\rho - c(x,0)\delta^2\rho \ \ ds. \tag{3.29}$$

Therefore,

$$\frac{d^2}{dt^2}\left(\int_{\Omega_t} c(x,t)\right)\Bigg|_{t=0} = \int_\Omega c_{tt}(x,0)\ dx + \langle 2c_t(x,0) + \nabla\cdot(c(x,0)\mathrm{S}), \delta\rho\rangle_{\partial\Omega}. \tag{3.30}$$

□

3.5 First Variational Formula

First, we will discuss about the first variation $\delta G(x,y)$ concerning the perturbation domain of Green's function $G(x,y)$ of Laplacian Δ. Recall that $\mathscr{T}_t : \Omega \to \Omega_t = \mathscr{T}_t\Omega$ be bi-Lipschitz. Let $G_t(x,y)$ be Green's function on Ω_t and $u(x,t) = G_t(x,y) - \Gamma(x-y)$.

Let $\Omega \subset \mathbb{R}^n$ and $G(x,y,t)$ be Green's function on Ω_t and

$$G(x,y,t) = \Gamma(x-y) + u(x,t).$$

Then the boundary value problem define as,

$$\Delta u(x,t) = 0 \qquad \text{on } \Omega_t, \qquad\qquad u(x,t) = -\Gamma(x-y), \qquad x \in \partial\Omega_t \tag{3.31}$$

We have $G(x,y,0) = G(x,y)$, $u(x,0) = u(x)$. Let $y \in \Omega$ is the inner point and take t small so that $y \in \Omega_t$. There exists $\frac{\partial G}{\partial t}(\mathscr{T}_t^{-1}(x),y)\big|_{t=0}$ local uniformly in $\overline{\Omega}\times\Omega$. Hence we obtain the existence of

$$\delta G(x,y) = \frac{\partial G_t(x,y)}{\partial t}\bigg|_{t=0} = \frac{\partial u(x)}{\partial t}\bigg|_{t=0} = \dot{u}(x).$$

For $x \in \partial\Omega$, $u(\mathscr{T}_t x,t) = -\Gamma(\mathscr{T}_t x - y)$. Hence from (3.3),

$$\frac{\partial u}{\partial t}(\mathscr{T}_t x,t) + \frac{\partial \mathscr{T}_t x}{\partial t}\cdot\nabla u(\mathscr{T}_t x,t) = -\frac{\partial \mathscr{T}_t x}{\partial t}\cdot\nabla\Gamma(\mathscr{T}_t x - y). \tag{3.32}$$

Therefore,

$$\begin{aligned} &\Delta\dot{u} = 0 && \text{in } \Omega \\ &\dot{u} = -\mathrm{S}\cdot\nabla(u+\Gamma(\cdot - y)) = -\mathrm{S}\cdot\nabla G(\cdot,y) = -\delta\rho\frac{\partial G(\cdot,y)}{\partial n} && \text{on } \partial\Omega \end{aligned} \tag{3.33}$$

where $\delta\rho = \mathrm{S}\cdot n$. This implies the first variational formula.

Theorem 3.5.1. *Let Ω be $C^{0,1}$ and $\mathscr{T}_t : \Omega \to \mathscr{T}_t\Omega$ be bi-Lipschitz. Then*

$$\delta G(x,y) = -\left\langle \frac{\partial G(\cdot,x)}{\partial n}, \delta\rho\frac{\partial G(\cdot,y)}{\partial n}\right\rangle \tag{3.34}$$

Proposition 3.5.2. *Let $\partial\Omega$ be $C^{0,1}$ and g is harmonic in Ω. Then,*

$$(\nabla\delta G(\cdot,x),\nabla g) = -\left\langle \delta\rho\frac{\partial G(\cdot,x)}{\partial n},\frac{\partial g}{\partial n}\right\rangle, \qquad x\in\Omega. \tag{3.35}$$

Proof. There exists $\tilde{g}$ be $C^{0,1}$ in $\widetilde{\Omega}$ with $\tilde{g}|_{\Omega} = g$. We take smooth $\Omega_k \downarrow \Omega$ and we define g_k by

$$\Delta g_k = 0 \quad \text{in } \Omega_k, \qquad g_k = \tilde{g} \quad \text{on } \partial\Omega_k.$$

It holds that $g_k \to g$ in $C^{0,1}$ on $\overline{\Omega}$. Given k, we have

$$(\nabla G_t(\cdot,x),\nabla g_k)_t = \int_{\partial\Omega_t} G_t(\cdot,x)\frac{\partial g_k}{\partial n}\ ds - \int_{\Omega_t} G_t(\cdot,x)\Delta g_k\ dx = 0 \tag{3.36}$$

for $|t| \ll 1$. Then for $\frac{d}{dt}\cdot\big|_{t=0}$, $k\to\infty$; it holds that

$$(\nabla\delta G(\cdot,x),\nabla g) + \int_{\partial\Omega}\nabla\cdot(S\nabla G(\cdot,x)\cdot\nabla g)\ ds = 0. \tag{3.37}$$

Hence, for $\Delta g = 0$ in Ω,

$$\begin{aligned}(\nabla\delta G(\cdot,x),\nabla g) = -\int_{\partial\Omega}\nabla(S\nabla G(\cdot,x)\cdot\nabla g)\ ds &= -\int_{\partial\Omega}[\nabla G(\cdot,x)\cdot\nabla g]\,\delta\rho\ ds \\ &= -\left\langle \delta\rho\frac{\partial G(\cdot,x)}{\partial n},\frac{\partial g}{\partial n}\right\rangle.\end{aligned}$$

□

Lemma 3.5.3. *Given $y\in\Omega$, let*

$$h(x) \equiv (\nabla\delta G(\cdot,x),\nabla\delta G(\cdot,y)). \tag{3.38}$$

Then

$$\Delta h = 0, \qquad h = -\delta\rho\frac{\partial\dot{u}}{\partial n} \quad \text{on } \partial\Omega, \tag{3.39}$$

where

$$\dot{u} = \delta G(\cdot,y).$$

Proof. We apply Proposition 3.5.2 for $g = \delta G(\cdot,y) = \dot{u}$. Since

$$\Delta\dot{u} = 0 \quad \text{on } \Omega$$

it holds that

$$\begin{aligned} h(x) &\equiv (\nabla \delta G(\cdot,x), \nabla \delta G(\cdot,y)) \\ &= -\left\langle \delta\rho \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial \delta G(\cdot,y)}{\partial n} \right\rangle, \qquad x,y \in \Omega \\ &= -\left\langle \frac{\partial G(\cdot,x)}{\partial n}, \delta\rho \frac{\partial \delta G(\cdot,y)}{\partial n} \right\rangle. \end{aligned} \tag{3.40}$$

Therefore,

$$\Delta h = 0 \quad \text{on } \Omega \qquad h = -\delta\rho \frac{\partial \dot{u}}{\partial n} \quad \text{on } \partial\Omega \tag{3.41}$$

by Proposition 3.5.2. □

3.6 Second Variational Formula

In this section, we compute the second variation of the Green's function with respect to domain perturbation. Let $u(x,t)$ be a solution of the Dirichlet problem (3.31). Consider the second variation of Green's function, $\delta^2 G(x,y)$. Let $\delta G(x,y) = \dot{u}(x), y \in \Omega$. It holds that

$$\delta G(x,y) = -\delta\rho \frac{\partial G(x,y)}{\partial n_x}, \qquad x \in \partial\Omega. \tag{3.42}$$

We define the second variation of the Green function, $\delta^2 G(x,y)$ by

$$\delta^2 G(x,y) = \left. \frac{\partial^2 G(x,y,t)}{\partial t^2} \right|_{t=0} = \ddot{u}(x). \tag{3.43}$$

Recall that the harmonic functionis a solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = -\Gamma(\cdot - y) \quad \text{on } \partial\Omega$$

and we have $u(\cdot,y) + \Gamma(\cdot,y) = G(\cdot,y)$. Then $u_t = G_t(\cdot,y) - \Gamma(\cdot,y)$ satisfies

$$\Delta u_t = 0 \quad \text{in } \Omega_t, \qquad u_t = -\Gamma(\cdot - y) \quad \text{on } \partial\Omega_t.$$

Similarly to the first variation, we use $u(\mathscr{T}_t x, t) = -\Gamma(\mathscr{T}_t x - y), x \in \partial\Omega$ to obtain the following lemma.

Lemma 3.6.1. *Given $y \in \Omega$, define*

$$H(x) \equiv \ddot{u}(x) + 2(\nabla \delta G(\cdot,x), \nabla \delta G(\cdot,y)) = \ddot{u}(x) + 2h(x).$$

Then,

$$\begin{aligned} \Delta H &= 0 && \textit{in } \Omega \\ H(x) &= -\mathsf{T}\cdot\nabla G(\cdot,y) - H_x G(\cdot,y)\mathsf{S}\cdot\mathsf{S} - 2\mathsf{S}\cdot\nabla_x \delta G(\cdot,y) + 2h && \textit{on } \partial\Omega. \end{aligned} \tag{3.44}$$

Proof. First, it holds that

$$\Delta \ddot{u} = 0 \quad \text{in } \Omega. \tag{3.45}$$

By (3.4), we have

$$\ddot{u} + 2\mathsf{S}\cdot\nabla_x \delta G(\cdot,y) + \mathsf{T}\cdot\nabla G(\cdot,y) + H_x G(\cdot,y)\mathsf{S}\cdot\mathsf{S} = 0 \qquad \text{on } \partial\Omega. \tag{3.46}$$

From Lemma 3.5.3, it follows that

$$\begin{aligned} H &= \ddot{u} + 2h \\ &= -\mathsf{T}\cdot G(\cdot,y) - H_x G(\cdot,y)\mathsf{S}\cdot\mathsf{S} - 2\mathsf{S}\cdot\nabla_x \delta G(\cdot,y) + 2h \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.47}$$

□

Lemma 3.6.2. *Let $\{s_1,\ldots,s_{n-1},n\}$ be the orthogonal system of the tangential space $\partial\Omega$ and n be the unit normal. Let f be a C^2 generic function. Assume that ∇f is Lipschitz continuous on $\overline{\Omega}$. Then the Hesse matrix of Hf, $Hf = \nabla(\nabla f)^t$ is given by*

$$\begin{aligned} Hf = &\sum_{i=1}^{n-1} (\nabla s_i)\frac{\partial f}{\partial s_i} + (\nabla n)\frac{\partial f}{\partial n} + \sum_{i,j=1}^{n-1} s_i \otimes s_j \frac{\partial^2 f}{\partial s_i \partial s_j} \\ &+ \sum_{i=1}^{n-1} (s_i\otimes n + n\otimes s_i)\frac{\partial^2 f}{\partial s_i \partial n} + n\otimes n \frac{\partial^2 f}{\partial n^2}. \end{aligned} \tag{3.48}$$

Proof. Here we have,

$$\begin{aligned} Hf &= \nabla \otimes \left(\sum_{i=1}^{n-1} s_i \frac{\partial f}{\partial s_i} + n\frac{\partial f}{\partial n} \right) \\ &= \sum_{i=1}^{n-1} (\nabla s_i)\frac{\partial f}{\partial s_i} + (\nabla n)\frac{\partial f}{\partial n} + \sum_{i=1}^{n-1} s_i \otimes \nabla\left(\frac{\partial f}{\partial s_i}\right) + n\otimes\nabla\left(\frac{\partial f}{\partial n}\right). \end{aligned}$$

$$\begin{aligned} Hf = &\sum_{i=1}^{n-1} (\nabla s_i)\frac{\partial f}{\partial s_i} + (\nabla n)\frac{\partial f}{\partial n} + \sum_{i,j=1}^{n-1} s_i \otimes s_j \frac{\partial^2 f}{\partial s_i \partial s_j} \\ &+ \sum_{i=1}^{n-1} (s_i\otimes n + n\otimes s_i)\frac{\partial^2 f}{\partial s_i \partial n} + n\otimes n \frac{\partial^2 f}{\partial n^2}. \end{aligned}$$

□

Hence

$$\begin{aligned} HG = &\sum_{i=1}^{n-1} (\nabla s_i)\frac{\partial G}{\partial s_i} + (\nabla n)\frac{\partial G}{\partial n} + \sum_{i,j=1}^{n-1} s_i \otimes s_j \frac{\partial^2 G}{\partial s_i \partial s_j} \\ &+ \sum_{i=1}^{n-1} (s_i\otimes n + n\otimes s_i)\frac{\partial^2 G}{\partial s_i \partial n} + n\otimes n \frac{\partial^2 G}{\partial n^2}. \end{aligned} \tag{3.49}$$

Proposition 3.6.3. *If f, $\frac{\partial f}{\partial n}$ are Lipschitz on $\overline{\Omega}$, $f\mid_{\partial\Omega}=0$, then*

$$\Delta f = (\nabla\cdot n)\frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2} \qquad on\ \partial\Omega$$

Proof. Let $\{s_1,\ldots,s_{n-1},n\}$ be the orthonormal system of the tangential space of $\partial\Omega$. Then $f\mid_{\partial\Omega}=0$ implies $\frac{\partial f}{\partial s_i}\mid_{\partial\Omega} = \frac{\partial^2 f}{\partial s_i \partial s_j}\mid_{\partial\Omega} = 0$. Then it implies from (3.48)

$$Hf = (\nabla n)\frac{\partial f}{\partial n} + \sum_{i=1}^{n-1}(s_i\otimes n + n\otimes s_i)\frac{\partial^2 f}{\partial s_i \partial n} + n\otimes n\frac{\partial^2 f}{\partial n^2}.$$

Here we have

$$s_k\otimes n + n\otimes s_k = (s_k^i n^j + s_k^j n^i)_{ij}.$$

Hence,

$$tr(s_k\otimes n + n\otimes s_k) = \sum_{i,j}(s_k^i n^j + n^i s_k^j) = 2s_k\cdot n = 0$$

and

$$tr(n\otimes n) = \sum_{i,j} n^i n^j = |n|^2 = 1.$$

Then we have

$$\Delta f = tr(Hf) = \nabla\cdot n\frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2} \qquad \text{on } \partial\Omega.$$

□

By Proposition 3.6.3, we have

$$\frac{\partial^2 G(\cdot,y)}{\partial n^2} = -(\nabla\cdot n)\frac{\partial G(\cdot,y)}{\partial n} \tag{3.50}$$

by

$$\Delta G(\cdot,y) = \frac{\partial^2 G(\cdot,y)}{\partial n^2} + (\nabla\cdot n)\frac{\partial G(\cdot,y)}{\partial n} = 0. \tag{3.51}$$

Then we define sectional curvatures of $\partial\Omega$ that denoted by κ_i, $i = 1,2,\ldots,N-1$ as

$$\frac{\partial n}{\partial s_i} = \kappa_i s_i. \tag{3.52}$$

We define also

$$a\otimes b = (a_i b_j)_{ij} = ab^t,$$

where $a = (a_i)_i$ and $b = (b_i)_i$,

$$\nabla s_i = \left(\frac{\partial s_i^k}{\partial s_j}\right)_{jk} \qquad \nabla n = \left(\frac{\partial n^k}{\partial x_j}\right)_{jk}$$

with $s_i = (s_i^k)_k$ and $n = (n^k)_k$,

$$\nabla n = \sum_{i=}^{n-1} \kappa_i s_i \otimes s_i \tag{3.53}$$

and

$$tr(\kappa_i s_i \otimes s_i) = \sum_{i=1}^{n-1} k_i = \nabla \cdot n.$$

Lemma 3.6.4. *For $H = H(x)$ defined from Lemma 3.6.1, it holds that*

$$H = -2h + \chi \frac{\partial G(\cdot,y)}{\partial n} \qquad on\ \partial\Omega$$

where

$$\chi = \delta^2\rho + \sum_{i=1}^{n-1} \left[\tilde{k}((\mu_i)^2 - (\delta\rho)^2) - 2(\mu_i)\frac{\partial(\delta\rho)}{\partial s_i} \right].$$

Proof. We have

$$H = -\mathsf{T} \cdot \nabla G(\cdot,y) - \mathsf{S} \cdot [\nabla^2 G(\cdot,y)]\mathsf{S} - 2\mathsf{S} \cdot \nabla \dot{u} + 2h \qquad \text{on } \partial\Omega$$

where $G(\cdot,y) = \frac{\partial G(\cdot,y)}{\partial s_i} = 0$, $\dot{u} = -\delta\rho \frac{\partial G(\cdot,y)}{\partial n}$ on $\partial\Omega$, and $\frac{\partial^2}{\partial s_i \partial n} = \frac{\partial^2}{\partial n \partial s_i}$. Define

$$\mathsf{S} = \frac{\partial \mathscr{T}_t x}{\partial t}\bigg|_{t=0} = (\delta\rho)n + \sum_{i=1}^{n-1} (\mathsf{S} \cdot s_i)s_i \tag{3.54}$$

by $\delta\rho = \frac{\partial \mathscr{T}_t x}{\partial t}\Big|_{t=0} \cdot n$. Putting $\mathsf{S} \cdot s_i = \mu_i$, we obtain

$$\mathsf{S} = (\delta\rho)n + \sum_{i=1}^{n-1} \mu_i s_i \tag{3.55}$$

and hence

$$\mathsf{S} \cdot \mathsf{S} = \sum_{i=1}^{n-1} \mu_i \mu_j (s_i, s_j) + 2\delta\rho(n, \sum_{i=1}^{n-1} \mu_i s_i) + (\delta\rho)^2(n,n). \tag{3.56}$$

Here,

$$\mathsf{T} \cdot \nabla G(\cdot,y) = \delta^2\rho \frac{\partial G(\cdot,y)}{\partial n}, \qquad \delta^2\rho = \mathsf{T} \cdot n.$$

Since

$$[n \otimes n]s \cdot s = \sum_{l,k} n^k n^l s_l s_k = (n \cdot s)(n \cdot s) = 0$$

$$[n \otimes n]n \cdot s = \sum_{l,k} n^k n^l n_l s_k = (n \cdot n)(n \cdot s) = 0$$

$$[n \otimes n]n \cdot n = \sum_{l,k} n^k n^l n_l n_k = (n \cdot n)(n \cdot n) = 1,$$

it holds that

$$n\otimes n\frac{\partial^2 G(\cdot,y)}{\partial n^2}(\mathsf{S}\cdot\mathsf{S}) = (\delta\rho)^2\frac{\partial^2 G(\cdot,y)}{\partial n^2} = (\nabla\cdot n)(\delta\rho)^2\frac{\partial G(\cdot,y)}{\partial n}$$

by (3.50). We have also

$$\begin{aligned}&(\nabla n)\frac{\partial G(\cdot,y)}{\partial n}(\mathsf{S}\cdot\mathsf{S})\\ &= \left(\sum_{i=1}^{n-1}\mu_i\mu_j(\nabla n)s_i\cdot s_j + 2\sum_{i=1}^{n-1}\mu_i\delta\rho(\nabla n)s_i\cdot n + (\delta\rho)^2(\nabla n)n\cdot n\right)\frac{\partial G(\cdot,y)}{\partial n}\end{aligned}$$

and

$$(s_i\otimes n + n\otimes s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}(\mathsf{S}\cdot\mathsf{S}) = 2(\delta\rho)\sum_{i=1}^{n-1}\mu_i\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}$$

by (3.55). Therefore,

$$\begin{aligned}\mathsf{S}\cdot[\nabla^2 G(\cdot,y)]\,\mathsf{S} &= H_x G(\cdot,y)\mathsf{S}\cdot\mathsf{S}\\ &= n\otimes n\frac{\partial^2 G(\cdot,y)}{\partial n^2}(\mathsf{S}\cdot\mathsf{S}) + (\nabla n)\frac{\partial G(\cdot,y)}{\partial n}(\mathsf{S}\cdot\mathsf{S})\\ &+\sum_{i=1}^{n-1}(s_i\otimes n + n\otimes s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}(\mathsf{S}\cdot\mathsf{S}).\end{aligned}$$

Since $\nabla n = \sum_{i=1}^{n-1}\kappa_i s_i\otimes s_i$ and $\sum_{i=1}^{n-1}\kappa_i = \nabla\cdot n$. It holds that

$$[\nabla n]s_i\cdot s_j = (\sum_{m=1}^{n-1}\kappa_m s_m\otimes s_m)s_i\cdot s_j = \kappa_k\delta_{kl}$$

where

$$[s_i\otimes s_i]s_k\cdot s_l = \sum_{i,j}s_m^i s_m^j s_k^j s_l^i = (s_m\cdot s_k)(s_m\cdot s_l) = \delta_{mk}\delta_{ml}.$$

Thus,

$$\begin{aligned}\mathsf{S}\cdot[\nabla^2 G(\cdot,y)]\,\mathsf{S} &= 2\delta\rho\sum_{i=1}^{n-1}\mu_i\frac{\partial^2 G}{\partial s_i\partial n}(\cdot,y) - (\nabla\cdot n)(\delta\rho)^2\frac{\partial G(\cdot,y)}{\partial n}\\ &+\left(\sum_{i=1}^{n-1}\mu_i\mu_i(\nabla n)s_i\cdot s_j + 2\sum_{i=1}^{n-1}(\mu_i)\delta\rho(\nabla n)s_i\cdot n + (\delta\rho)^2(\nabla n)n\cdot n\right)\frac{\partial G(\cdot,y)}{\partial n}\\ &= 2\delta\rho\sum_{i=1}^{n-1}\mu_i\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} + \left(\sum_{i=1}^{n-1}\kappa_i((\mu_i)^2-(\delta\rho)^2)\right)\frac{\partial G(\cdot,y)}{\partial n}.\end{aligned}\tag{3.57}$$

Here we have,

$$2\mathsf{S}\cdot\nabla\delta G(\cdot,y) = 2\sum_{i=1}^{n-1}\mu_i\frac{\partial\delta G(\cdot,y)}{\partial s_i} + 2\delta\rho\frac{\partial\delta G(\cdot,y)}{\partial n}.\tag{3.58}$$

where $\dot{u} = -\delta\rho\frac{\partial G(\cdot,y)}{\partial n}$ on $\partial\Omega$ and (3.39), we have

$$2\mathsf{S}\cdot\nabla\delta G(\cdot,y) = 2\sum_{i=1}^{n-1}\mu_i\frac{\partial\delta G(\cdot,y)}{\partial s_i} - 2h \qquad \text{on } \partial\Omega.$$

Hence,

$$\begin{aligned}2\mathsf{S}\cdot\nabla\delta G(\cdot,y) + H_x G(\cdot,y)\mathsf{S}\cdot\mathsf{S} = -2h + \sum_{i=1}^{n-1}\kappa_i((\mu_i)^2-(\delta\rho)^2)\frac{\partial G(\cdot,y)}{\partial n} \\ +2\sum_{i=1}^{n-1}\mu_i\frac{\partial\delta G(\cdot,y)}{\partial s_i} + 2\delta\rho\sum_{i=1}^{n-1}\mu_i\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}.\end{aligned} \tag{3.59}$$

From (3.42),

$$\frac{\partial}{\partial s_i}\left(\delta G(\cdot,y) + \delta\rho\frac{\partial G(\cdot,y)}{\partial n}\right) = \frac{\partial\delta G(\cdot,y)}{\partial s_i} + \frac{\partial(\delta\rho)}{\partial s_i}\frac{\partial G(\cdot,y)}{\partial n} + \delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}.$$

It follows that

$$\frac{\partial\delta G(\cdot,y)}{\partial s_i} + \delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} = -\frac{\partial(\delta\rho)}{\partial s_i}\frac{\partial G(\cdot,y)}{\partial n}. \tag{3.60}$$

Also from (3.59), it holds that

$$2\sum_{i=1}^{n-1}\mu_i\left[\frac{\partial\delta G(\cdot,y)}{\partial s_i} + \delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}\right] = -2\sum_{i=1}^{n-1}\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\frac{\partial G(\cdot,y)}{\partial n}.$$

Then (3.59) implies

$$\begin{aligned}&2S\cdot\nabla\delta G(\cdot,y) + H_x G(\cdot,y)\mathsf{S}\cdot\mathsf{S} \\ &\quad = -2h + \sum_{i=1}^{n-1}\left[\kappa_i((\mu_i)^2-(\delta\rho)^2) - 2\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\right]\frac{\partial G(\cdot,y)}{\partial n}.\end{aligned} \tag{3.61}$$

Thus,

$$\begin{aligned}H &= -2h + \delta^2\rho\frac{\partial G(\cdot,y)}{\partial n} + \sum_{i=1}^{n-1}\left[\kappa_i((\mu_i)^2-(\delta\rho)^2) - 2\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\right]\frac{\partial G(\cdot,y)}{\partial n} \\ &= -2h + \chi\frac{\partial G(\cdot,y)}{\partial n} \qquad \text{on } \partial\Omega\end{aligned}$$

where

$$\chi = \delta^2\rho + \sum_{i=1}^{n-1}\left[\kappa_i((\mu_i)^2-(\delta\rho)^2) - 2\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\right]. \tag{3.62}$$

□

Remark that $\frac{\partial n}{\partial s_i} = \kappa_i s_i$,

$$\sum_{i-1}^{n-1}\kappa_i(\mathsf{S}\cdot s_i)(\mathsf{S}\cdot s_i) = \sum_{i=1}^{n-1}(\mathsf{S}\cdot s_i)(\mathsf{S}\cdot\kappa_i s_i) = (\mathsf{S}\cdot s_i)\left(\mathsf{S}\cdot\frac{\partial n}{\partial s_i}\right).$$

As we know,

$$\frac{\partial}{\partial s_i}(\mathsf{S}\cdot n) = \left(\frac{\partial \mathsf{S}}{\partial s_i}\right)\cdot n + \mathsf{S}\cdot\left(\frac{\partial n}{\partial s_i}\right).$$

Therefore,

$$\sum_{i=1}^{n-1}\kappa_i(\mathsf{S}\cdot s_i)(\mathsf{S}\cdot s_i) = \sum_{i=1}^{n-1}(\mathsf{S}\cdot s_i)\left(\frac{\partial(\delta\rho)}{\partial s_i} - \left(\frac{\partial \mathsf{S}}{\partial s_i}\right)\cdot n\right). \tag{3.63}$$

where $\sum_{i=1}^{n-1}\kappa_i = \nabla\cdot n$. Recall that $\delta\rho = \mathsf{S}\cdot n$, $\delta^2\rho = \mathsf{T}\cdot n$ and $\frac{\partial n}{\partial n} = 0$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n-1}\mathsf{S}\cdot s_i\left(\frac{\partial \mathsf{S}}{\partial s_i}\cdot n\right) = \sum_{i=1}^{n-1}\mathsf{S}\cdot s_i((\nabla\mathsf{S}\cdot s_i)\cdot n) &= \nabla\mathsf{S}(\sum_{i=1}^{n-1}\mu_i s_i)\cdot n \\ &= ((\nabla\mathsf{S}\cdot(\mathsf{S}-\delta\rho n))\cdot n) \\ &= \delta^2\rho - \delta\rho\left(\frac{\partial \mathsf{S}}{\partial n}\cdot n\right) \\ &= \delta^2\rho - \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial n}.\end{aligned} \tag{3.64}$$

At the same time,

$$\begin{aligned}\sum_{i=1}^{n-1}\mathsf{S}\cdot s_i\frac{\partial(\delta\rho)}{\partial s_i} = \nabla(\delta\rho)\cdot(\sum_{i=1}^{n-1}\mu_i\cdot s_i) &= \nabla(\delta\rho)\cdot(\mathsf{S}-\delta\rho\cdot n) \\ &= \mathsf{S}\cdot\nabla(\delta\rho) - \delta\rho\nabla(\delta\rho)\cdot n \\ &= (\mathsf{S}\cdot\nabla - \delta\rho\frac{\partial}{\partial n})(\delta\rho) \\ &= (\mathsf{S}\cdot\nabla)\delta\rho - \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial n}\end{aligned} \tag{3.65}$$

where $\delta\rho\frac{\partial(\delta\rho)}{\partial n} = \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial n}$. By normal boundary perturbation $\mathsf{S}\cdot s_i = 0$, $i = 1,\dots,n-1$,

$$\delta^2\rho = (\mathsf{S}\cdot\nabla)\delta\rho = \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial n}. \tag{3.66}$$

From Proposition 3.6.3,

$$\frac{\partial^2 G(\cdot,y)}{\partial n^2} = -\tilde{\kappa}\frac{\partial G(\cdot,y)}{\partial n}$$

where $\tilde{\kappa} = \sum_{i=1}^{n-1}\kappa_i = \nabla\cdot n$.

Theorem 3.6.5. *Let $\delta\rho = \mathsf{S}\cdot n$ and $\delta^2\rho = \mathsf{T}\cdot n$. Therefore,*

$$\begin{aligned}\ddot{u} = \delta^2 G(x,y) &= H - 2(\nabla\delta G(\cdot,x),\nabla\delta G(\cdot,y)) \\ &= -2(\nabla\delta G(\cdot,x),\nabla\delta G(\cdot,y)) \\ &\quad + \left\langle\left[-\tilde{\kappa}(\delta\rho)^2 - (S\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n}\right]\frac{\partial G(\cdot,x)}{\partial n},\frac{\partial G(\cdot,y)}{\partial n}\right\rangle\end{aligned} \tag{3.67}$$

where $\delta\rho = \frac{\partial \mathscr{T}_t}{\partial t}\Big|_{t=0} \cdot n$, $\delta^2\rho = \frac{\partial^2 \mathscr{T}_t}{\partial t^2}\Big|_{t=0} \cdot n$, $\mathsf{S} = \frac{\partial \mathscr{T}_t}{\partial t}\Big|_{t=0}$, *and* $\tilde{\kappa} = \sum_{i=1}^{n-1} \kappa_i = \nabla \cdot n$. *Recall that*

$$h \equiv (\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))$$

Proof. Let $\chi = \delta^2\rho + \sum_{i=1}^{n-1}\left[\kappa_i((\mathsf{S}\cdot s_i)^2 - (\delta\rho)^2) - 2(\mathsf{S}\cdot s_i)\frac{\partial(\delta\rho)}{\partial s_i}\right]$. Then, using (3.64) and (3.65)

$$\begin{aligned}
\chi &= \delta^2\rho + \left[\mathsf{S}\cdot s_i\frac{\partial(\delta\rho)}{\partial s_i} - \mathsf{S}\cdot s_i\left(\frac{\partial \mathsf{S}}{\partial s_i}\cdot n\right) - \tilde{\kappa}(\delta\rho)^2 - 2\mathsf{S}\cdot s_i\frac{\partial(\delta\rho)}{\partial s_i}\right] \\
&= \delta^2\rho + \left[-\mathsf{S}\cdot s_i\frac{\partial(\delta\rho)}{\partial s_i} - \mathsf{S}\cdot s_i\left(\frac{\partial \mathsf{S}}{\partial s_i}\cdot n\right) - \tilde{\kappa}(\delta\rho)^2\right] \\
&= -\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta^2 G(x,y) &= H - 2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y)) \\
&= -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y)) + \left\langle\left[-\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n}\right]\frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle
\end{aligned}$$

□

Let Ω is a Lipschitz domain with $W^{1,\infty}$ and $\mathscr{T}_t x$ is a perturbation of the Green's function $G(\cdot,y)$ of second variation $\delta^2 G(\cdot,y)$. Therefore,

$$\delta^2 G(x,y) = \left\langle \chi\frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle_{\partial\Omega} - 2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega, \tag{3.68}$$

where

$$\chi = -\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n},$$

Theorem 3.6.6 (General Garabedian-Schiffer). *Given* $S\cdot s_i = 0$, $i = 1,\cdots,n-1$ *as the normal boundary perturbation,*

$$\delta^2 G(x,y) = \left\langle (\delta^2\rho - \tilde{\kappa}(\delta\rho)^2)\frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle_{\partial\Omega} - 2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega \tag{3.69}$$

called as the General Garabedian-Schiffer's formula.

Proof. Given $\mathsf{S}\cdot s_i = 0$, $i = 1,\cdots,n-1$, where $(\mathsf{S}\cdot\nabla)\delta\rho = \delta^2\rho$. Therefore,

$$\delta^2 G(x,y) = -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y)) + \left\langle\left[-\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n}\right]\frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle. \tag{3.70}$$

Given normal perturbation $\mathsf{S}\cdot s_i = 0$, $i = 1, \cdots, n-1$. Hence,

$$(\mathsf{S}\cdot\nabla)\delta\rho = \delta\rho\frac{\partial}{\partial n}\delta\rho = \frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial n}.$$

Therefore

$$\begin{aligned}\delta^2 G(x,y) &= -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y)) + \left\langle \left[-\tilde{\kappa}(\delta\rho)^2 - (\mathsf{S}\cdot\nabla)\delta\rho + 2(\mathsf{S}\cdot\nabla)\delta\rho\right] \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle \\ &= -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y)) + \left\langle \left[-\tilde{\kappa}(\delta\rho)^2 + \delta^2\rho\right] \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle.\end{aligned} \tag{3.71}$$

□

3.7 Case Study

In this section, we consider the dimensional of $n = 2$, and $\Omega = B(0,1)$.

3.7.1 Material Derivative

Let $Sx = x$ and $\mathscr{T}_t x = e^t x$. Given $G_t = \frac{1}{2\pi}\log\frac{e^t}{|x|}$ and $\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$. Therefore,

$$u(x,t) = G_t(x,0) - \Gamma(x) = \frac{t}{2\pi}.$$

Hence,

$$\delta G(x,0) = \dot{u}(x,0) = \frac{1}{2\pi}, \qquad \delta^2 G(x,0) = \ddot{u}(x,0) = 0, \ \ x\in\Omega.$$

Given also unit tangent vector, $S = \frac{x}{|x|}$, $\delta\rho = n\cdot x = |x| = 1$, $\frac{1}{2}\frac{\partial(\delta\rho)^2}{\partial n} = 1$ and $\tilde{\kappa} = \nabla\cdot n = 1$. Therefore,

$$\delta^2\rho = \mathsf{T}\cdot n = x\cdot\frac{x}{|x|} = |x|$$

Then, we get

$$\tilde{\kappa}(\delta\rho)^2 + \delta^2\rho - \frac{\partial}{\partial n}(\delta\rho)^2 = 1\cdot 1 + 1 - 2 = 0$$

which implies

$$(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y)) = 0. \tag{3.72}$$

Consider

$$G(x,y) = \frac{1}{4\pi}\log\frac{1 - 2x\cdot y + |x|^2|y|^2}{|x-y|^2}$$

We then get

$$G(e^{-t}x,e^{-t}y) = \frac{1}{4\pi}\log\frac{1-2(e^{-t}x)\cdot(e^{-t}y)+|e^{-t}x|^2|e^{-t}y|^2}{|e^{-t}x-e^{-t}y|^2}$$
$$= \frac{1}{2\pi}\log\frac{e^{2t}-2x\cdot y+e^{-2t}|x|^2|y|^2}{|x-y|^2}.$$

Then we obtain

$$\delta G(x,y) = \frac{\partial G_t(x,y)}{\partial t} = \frac{1}{2\pi}\frac{1-|x|^2|y|^2}{1-2x\cdot y+|x|^2|y|^2}$$

which implies

$$\nabla_x \delta G(x,y) = \frac{1}{\pi}\frac{2|y|^2(-1+x\cdot y)x+(1-|x|^2|y|^2)y}{(1-2x\cdot y+|x|^2|y|^2)^2}.$$

We have then

$$(\nabla\delta G(\cdot,x),\nabla\delta G(\cdot,y)) = \frac{1}{\pi^2}\int_\Omega \frac{2|x|^2(-1+z\cdot x)z+(1-|z|^2|x|^2)x}{(1-2z\cdot x+|z|^2|x|^2)^2} \cdot\frac{2|y|^2(-1+z\cdot y)z+(1-|z|^2|y|^2)y}{(1-2z\cdot y+|z|^2|y|^2)^2}\,dz$$

which then implies (3.72)

3.7.2 Normal Boundary Perturbation

Let $\Gamma_t : x+t\delta\rho(x)n_x, x\in\Gamma$. Given $\delta\rho(x)=1$, $n=\frac{x}{|x|}=x$ and $\Gamma_t=(1+t)\Gamma$. Hence,

$$G_t(x,y) = G\left((1+t)^{-1}x,(1+t)^{-1}y\right)$$

$$G_t(x,0) = \frac{1}{2\pi}\log\frac{1+t}{|x|}$$

$$u(x,t) = G_t(x,0)-\Gamma(x) = \frac{1}{2\pi}\log(1+t)$$

$$\delta G(x,0) = \dot{u}(x) = \frac{1}{2\pi}$$

$$\delta^2 G(x,0) = -\frac{1}{2\pi}$$

It holds that

$$G_t(x,y) = G_t((1+t)^{-1}x,(1+t)^{-1}y)$$
$$= \frac{1}{4\pi}\log\frac{1+2((1+t)^{-1}x\cdot(1+t)^{-1}y)+(1+t)^{-1}|x|^2(1+t)^{-1}|y|^2}{|(1+t)^{-1}x-(1+t)^{-1}y|^2}$$
$$= \frac{1}{4\pi}\log\frac{(1+t)^2+2x\cdot y+(1+t)^{-2}|x|^2|y|^2}{|x-y|^2}.$$

Hence, the derivative of $G(x,y)$ is given by

$$\nabla_x G(x,y) = \frac{1}{4\pi}\cdot\frac{-2y+2|y|^2x}{1-2x\cdot y+|x|^2|y|^2} - \frac{1}{4\pi}\frac{2(x-y)}{|x-y|^2}.$$

When $x \cdot \nabla_x G(x,y)$,

$$x \cdot \nabla_x G(x,y) = \frac{1}{2\pi} \frac{|x|^2|y|^2 - x \cdot y}{1 - 2x \cdot y + |x|^2|y|^2} - \frac{1}{2\pi} \frac{|x|^2 - x \cdot y}{|x-y|^2}.$$

Therefore,

$$\begin{aligned} x \cdot \nabla_x G(x,y)\big|_{|x|=1} = \frac{\partial G(x,y)}{\partial n} &= \frac{1}{2\pi} \frac{|y|^2 - x \cdot y}{1 - 2x \cdot y + |y|^2} - \frac{1}{2\pi} \frac{1 - x \cdot y}{|x-y|^2} \\ &= \frac{1}{2\pi} \frac{|y|^2 - 1}{|x-y|^2} \end{aligned}$$

Here,

$$\left\langle \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n} \right\rangle = \frac{1}{4\pi^2} \int_{|z|=1} \frac{(1-|x|^2)(1-|y|^2)}{|z-x|^2|z-y|^2} \, dS_z.$$

We then obtain

$$\left\langle \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,0)}{\partial n} \right\rangle = \frac{1}{2\pi}.$$

3.8 Mixed Problem

3.8.1 First Variational Formula

Let $\partial\Omega = \overline{\gamma^0} \cup \overline{\gamma^1}$, where $\gamma^0 \cap \gamma^1 = \emptyset$ is a piecewise smooth. Let $\gamma_t^0 = \mathcal{T}_t(\gamma^0)$ and $\gamma_t^1 = \mathcal{T}_t(\gamma^1)$. We consider the Green's function as a boundary value solution

$$-\Delta G(\cdot,y)_t = \delta y \quad \text{in } \Omega_t, \quad G_t(\cdot,y) = 0, \quad \text{on } \gamma_t^0 \quad \frac{\partial G_t(\cdot,y)}{\partial n_t} = 0 \quad \text{on } \gamma_t^1. \tag{3.73}$$

Consider $y \in \Omega_t$, then we fix $u(x,t) = G_t(x,y) - \Gamma(x-y)$. Therefore

$$\Delta u(\cdot,t) = 0 \quad \text{in } \Omega_t, \quad u(\cdot,t) = -\Gamma(\cdot - y) \text{ on } \gamma_t^0, \quad \frac{\partial u}{\partial n_t}(\cdot,t) = \frac{\partial}{\partial n_t}\Gamma(\cdot - y) \text{ on } \gamma_t^1. \tag{3.74}$$

If $\Delta f = 0$, in Ω, then it holds that

$$\begin{aligned} f(y) &= \int_{\partial\Omega} -f(y)\frac{\partial G(x,y)}{\partial n_x} + \frac{\partial f(x)}{\partial n_x} G(x,y) \; dS_x \\ &= -\left\langle f, \frac{\partial G(\cdot,y)}{\partial n} \right\rangle_{\gamma^0} + \left\langle \frac{\partial f}{\partial n}, G(\cdot,y) \right\rangle_{\gamma^1}, \quad y \in \Omega \end{aligned}$$

where $G = G_0(x,y)$. Then $u(x,t) = G_t(x,y) - \Gamma(x-y), y \in \Omega$ satisfies

$$\begin{aligned} \Delta u(\cdot,t) = 0 \text{ in } \Omega_t \qquad & u(\cdot,t) = -\Gamma(\cdot - y) \text{ on } \gamma_t^0 \\ \frac{\partial u}{\partial n_t}(\cdot,t) = -\frac{\partial \Gamma}{\partial n_t}(\cdot - y) \text{ on } \gamma_t^1. & \end{aligned} \tag{3.75}$$

Let $\dot{u} = \frac{\partial u}{\partial t}\Big|_{t=0}$. Therefore,

$$\Delta \dot{u} = 0 \quad \text{in } \Omega, \quad \dot{u} = -\delta\rho \frac{\partial G(\cdot,y)}{\partial n} \quad \text{on } \gamma^0. \tag{3.76}$$

For $x \in \gamma^1$, we consider $x = \mathscr{T}_t x + s n_t$.

$$\frac{\partial}{\partial s}\left\{u(\mathscr{T}_t x + s n_t, t) + \Gamma(\mathscr{T}_t x + s n_t - y)\right\}\Big|_{s=0} = 0$$

which implies

$$\frac{\partial^2}{\partial s \partial t}\left\{u(\mathscr{T}_t x + s n_t, t) + \Gamma(\mathscr{T}_t x + s n_t - y)\right\}\Big|_{s=0} = 0.$$

Similar to (3.32),

$$\begin{aligned} &\frac{\partial u}{\partial t}(\mathscr{T}_t x + s n_t, t) + \frac{\partial}{\partial t}(\mathscr{T}_t x + s n_t) \cdot \nabla u(\mathscr{T}_t x + s n_t, t) \\ &\qquad = -\frac{\partial}{\partial t}(\mathscr{T}_t x + s n_t) \cdot \nabla \Gamma(\mathscr{T}_t x + s n_t - y). \end{aligned} \tag{3.77}$$

We differentiate (3.77) by s, then at the right hand side we get

$$\begin{aligned} &-\frac{d}{ds}\left\{\frac{\partial}{\partial t}(\mathscr{T}_t x + s n_t) \cdot \nabla \Gamma(\mathscr{T}_t x + s n_t - y)\right\}_{s=0,t=0} \\ &= -\frac{d}{ds}\left\{(\mathscr{T}_t x + s n_t) \cdot \frac{\partial}{\partial t}(\nabla \Gamma(\mathscr{T}_t x + s n_t - y)) + \frac{\partial}{\partial t}(\mathscr{T}_t x + s n_t) \cdot \Gamma(\mathscr{T}_t x + s n_t - y)\right\}_{s=0,t=0} \\ &= -\left\{\frac{\partial n_t}{\partial t} \cdot \nabla \Gamma(\mathscr{T}_t x - y) + \left[\nabla^2 \Gamma(\mathscr{T}_t x - y)\right] \frac{\partial \mathscr{T}_t x}{\partial t} \cdot n_t\right\}_{t=0} \\ &= -\left\{\frac{\partial n_t}{\partial t} \cdot \nabla \Gamma(\mathscr{T}_t x - y) + \left[\nabla^2 \Gamma(\mathscr{T}_t x - y)\right] \mathsf{S} \cdot n_t\right\}_{t=0} \\ &= -\frac{\partial n}{\partial t}\Big|_{t=0} \cdot \nabla \Gamma(x-y) + \left[\nabla^2 \Gamma(x-y)\right] \mathsf{S} \cdot n. \end{aligned} \tag{3.78}$$

While at the left hand side of (3.77),

$$\begin{aligned} &\frac{d}{ds}\left\{\frac{\partial}{\partial t}(\mathscr{T}_t x + s n_t, t) \cdot \nabla u(\mathscr{T}_t x + s n_t, t) + \frac{\partial u}{\partial t}(\mathscr{T}_t x + s n_t, t)\right\}\Big|_{s=0,t=0} \\ &= \frac{d}{ds}\left\{(\mathscr{T}_t x + s n_t)\frac{\partial}{\partial t}(\nabla u(\mathscr{T}_t x + s n_t, t)) + \frac{\partial}{\partial t}(\mathscr{T}_t x + s n_t)\nabla u(\mathscr{T}_t x + s n_t, t) + n_t \cdot \nabla \frac{\partial u}{\partial t}(\mathscr{T}_t x, t)\right\}_{s=0,t=0} \\ &= \left\{\frac{\partial n_t}{\partial t} \cdot \nabla u(\mathscr{T}_t x, t) + [\nabla^2 u(\mathscr{T}_t x, t)]\frac{\partial \mathscr{T}_t x}{\partial t} \cdot n_t + n_t \cdot \nabla \frac{\partial u}{\partial t}(\mathscr{T}_t x, t)\right\}_{t=0} \\ &= \frac{\partial n_t}{\partial t}\Big|_{t=0} \cdot \nabla u + [\nabla^2 u]\mathsf{S} \cdot n + n \cdot \nabla \dot{u}. \end{aligned} \tag{3.79}$$

Therefore,

$$\begin{aligned}\frac{\partial \dot{u}}{\partial n} &= -\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot \nabla u - \mathsf{S}\cdot[\nabla^2 u]n \\ &= -\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot \nabla(u-\Gamma(\cdot - y)) - \mathsf{S}\cdot \nabla^2[u-\Gamma(\cdot-y)]n \\ &= -\frac{\partial n_t}{\partial t}\cdot \nabla G(\cdot,y) - \mathsf{S}\cdot[\nabla^2 G(\cdot,y)]n\end{aligned} \tag{3.80}$$

where $G(\cdot,y) = u-\Gamma(\cdot,y)$. At the last part of the right hand side of the (3.80), using (3.49) where $\frac{\partial G}{\partial n} = \frac{\partial^2 G}{\partial s_i \partial n} = 0$, we obtain

$$HG = \sum_{i=1}^{n-1} \nabla s_i \frac{\partial G}{\partial s_i} + \sum_{i=1}^{n-1} s_i \otimes s_j \frac{\partial^2 G}{\partial s_i \partial s_j} + n\otimes n \frac{\partial^2 G}{\partial n^2}.$$

As we know,

$$\mathsf{S} = \sum_{i=1}^{n-1} \mu_i s_i + (\delta\rho)n. \tag{3.81}$$

Therefore,

$$[\nabla^2 G(\cdot,y)]\,\mathsf{S}\cdot n = \left(\nabla s_i \frac{\partial G(\cdot,y)}{\partial s_i} + \sum_{i=1}^{n-1} s_i\otimes s_j \frac{\partial^2 G(\cdot,y)}{\partial s_i \partial s_j} + n\otimes n \frac{\partial^2 G(\cdot,y)}{\partial n^2}\right)\left(\sum_{i=1}^{n-1}\mu_i s_i\cdot n + (\delta\rho)n\cdot n\right) \tag{3.82}$$

We rewrite at the right hand side of (3.82) as

$$X = \left(s_i\otimes s_j \frac{\partial^2 G(\cdot,y)}{\partial s_i \partial s_j}\right)\left(\sum_{i=1}^{n-1}\mu_i s_i\cdot n + (\delta\rho)n\cdot n\right) \tag{3.83}$$

$$Y = \left(n\otimes n \frac{\partial^2 G(\cdot,y)}{\partial n^2}\right)\left(\sum_{i=1}^{n-1}\mu_i s_i\cdot n + (\delta\rho)n\cdot n\right). \tag{3.84}$$

$$Z = \left(\nabla s_i \frac{\partial G(\cdot,y)}{\partial s_i}\right)\left(\sum_{i=1}^{n-1}\mu_i s_i\cdot n + (\delta\rho)n\cdot n\right). \tag{3.85}$$

It holds that

$$[s_k\otimes s_l](s_m\cdot n) = \sum_{i,j} s_k^i s_l^j s_m^j n^i = (s_l\cdot s_m)(s_k\cdot n) = 0$$

$$[s_k\otimes s_l](n\cdot n) = \sum_{i,j} s_k^i s_l^j n^j n^i = (s_l\cdot n)(s_k\cdot n) = 0$$

$$[s_i\otimes s_j](\delta\rho) = (s_i s_j^t)\mathsf{S}\cdot n = 0$$

$$[n\otimes n](s_k\cdot n) = \sum_{i,j} n^i n^j s_k^j n^i = |n|^2 s_k\cdot n = 0$$

$$[n\otimes n](n\cdot n) = \sum_{i,j} n^i n^j n^j n^i = |n|^2|n|^2 = 1$$

$$[n\otimes n](\delta\rho) = (nn^t)(\mathsf{S}\cdot n) = \delta\rho$$

$$[\nabla s_i](s_l \cdot n) = \sum_{k,j} \frac{\partial s_i^k}{\partial x}(s_l^j n^k) = [(s_i \cdot \nabla)s_l] \cdot n = \frac{\partial s_i}{\partial s_l} \cdot n$$
$$= \frac{\partial}{\partial s_l}(s_i \cdot n) - s_i \frac{\partial n}{\partial s_l} = -k_i \delta_{il}$$

$$[\nabla s_i](n \cdot n) = \sum \frac{\partial s_i^k}{\partial x}(n^j n^i) = [(s_i \cdot \nabla)n] \cdot n = \frac{\partial s_i}{\partial n} \cdot n$$
$$= \frac{\partial}{\partial n}(s_i \cdot n) - s_i \frac{\partial n}{\partial n} = 0.$$

Hence,

$$[\nabla s_i]\, \mathsf{S} \cdot n = -(\mathsf{S} \cdot s_i)(\nabla \cdot n) = -\sum_{i=1}^{n-1} \mu_i \kappa_i.$$

Therefore, we get

$$X = \left(s_i \otimes s_j \frac{\partial^2 G(\cdot,y)}{\partial s_i \partial s_j} \right) (\sum_{i=1}^{n-1} \mu_i s_i \cdot n + (\delta\rho) n \cdot n) = 0.$$

$$Y = \left(n \otimes n \frac{\partial^2 G(\cdot,y)}{\partial n^2} \right) (\sum_{i=1}^{n-1} \mu_i s_i \cdot n + (\delta\rho) n \cdot n) = \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2}.$$

$$Z = \left(\nabla s_i \frac{\partial G(\cdot,y)}{\partial s_i} \right) (\sum_{i=1}^{n-1} \mu_i s_i \cdot n + (\delta\rho) n \cdot n) = -\sum_{i=1}^{n-1} \mu_i \kappa_i \frac{\partial G(\cdot,y)}{\partial s_i}.$$

Thus,

$$X + Y + Z = [\nabla^2 G(\cdot,y)]\, \mathsf{S} \cdot n = \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2} - \sum_{i=1}^{n-1} \mu_i \kappa_i \frac{\partial G(\cdot,y)}{\partial s_i} \qquad \text{on } \gamma^1. \tag{3.86}$$

Therefore we can have the following theorem.

Theorem 3.8.1. *It holds that*

$$\begin{aligned} &\Delta \dot{u} = 0 \quad \text{in } \Omega, \qquad \dot{u} = -\delta\rho \frac{\partial G(\cdot,y)}{\partial n} \quad \text{on } \gamma^0 \\ &\frac{\partial \dot{u}}{\partial n} = -\frac{\partial n_t}{\partial t}\Big|_{t=0} \cdot \nabla G(\cdot,y) - \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2} + \sum_{i=1}^{n-1} \mu_i \kappa_i \frac{\partial G(\cdot,y)}{\partial s_i} \qquad \text{on } \gamma^1 \end{aligned} \tag{3.87}$$

where $\dot{u} = \delta G(x,y)$.

Let Ω *be Lipschitz and* $g \in C^{0,1}(\overline{\Omega})$ *be harmonic in* Ω. *Hence, from (3.87)*

$$\sum_{i=1}^{n-1} \mu_i \int_{\gamma^1} \kappa_i \frac{\partial G(\cdot,y)}{\partial s_i} = \int_{\gamma^0} \left(\delta G(\cdot,y) + \delta\rho \frac{\partial G(\cdot,y)}{\partial n} \right) \frac{\partial g}{\partial n}\, ds$$
$$+ \int_{\gamma^1} \left(\frac{\partial \delta G(\cdot,y)}{\partial n} + \frac{\partial n_t}{\partial t}\Big|_{t=0} \cdot \nabla G(\cdot,y) + \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2} \right) g\, ds.$$

Next, from **Lie Perturbation**, we have

$$\tilde{x} = \tilde{x}(\xi_1, \cdots, \xi_{n-1}) \in \gamma^1, \qquad \xi = (\xi_1, \cdots, \xi_{n-1})$$

$$s_i = \frac{\partial \tilde{x}}{\partial \xi_i}, \qquad s_i(t) = \frac{\partial \mathscr{T}_t x}{\partial t}$$

where $\tilde{x}(t) = \mathscr{T}_t \tilde{x} \in \gamma_t^1$ and $n_t \cdot s_i(t) = 0$. Given that

$$\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot s_i + n \cdot \frac{\partial s_i(t)}{\partial \xi_i} = 0$$

$$\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot s_i = -n \cdot \frac{\partial s_i(t)}{\partial \xi_i} = -n \cdot \frac{\partial \mathsf{S}\tilde{x}}{\partial \xi}.$$

Therefore,

$$\frac{\partial(\delta\rho)}{\partial s_i} = \frac{\partial(\mathsf{S}\cdot n)}{\partial s_i} = [\nabla \mathsf{S}]\, n \cdot s_i + [\nabla n]\, s_i \cdot \mathsf{S} = [(s_i \cdot \nabla)]\, \mathsf{S} \cdot n + \kappa_i \mu_i.$$

We define

$$\nabla G(\cdot,y) = \sum_{i=1}^{n-1} s_i \frac{\partial G(\cdot,y)}{\partial s_i}.$$

Then we obtain,

$$\begin{aligned}
\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot \nabla G(\cdot,y) &= \left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot \sum_{i=1}^{n-1} s_i \frac{\partial G(\cdot,y)}{\partial s_i} \\
&= -\frac{\partial S\tilde{x}}{\partial \xi_i} \cdot n \sum_{i=1}^{n-1} \frac{\partial G(\cdot,y)}{\partial s_i} \\
&= -\sum_{i=1}^{n-1} \left(\mu_i \kappa_i - \frac{\partial \delta\rho}{\delta s_i} \right) \frac{\partial G(\cdot,y)}{\partial s_i}.
\end{aligned}$$

Therefore, from (3.87)

$$\begin{aligned}
\frac{\partial \dot{u}}{\partial n} &= -\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot \nabla G(\cdot,y) + \sum_{i=1}^{n-1} \kappa_i \mu_i \frac{\partial G(\cdot,y)}{\partial s_i} - \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2} \\
&= \sum_{i=1}^{n-1} \left(\frac{\partial \delta\rho}{\partial s_i} - \mu_i \kappa_i \right) \frac{\partial G(\cdot,y)}{\partial s_i} + \sum_{i=1}^{n-1} \mu_i \kappa_i \frac{\partial G(\cdot,y)}{\partial s_i} - \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2} \\
&= \sum_{i=1}^{n-1} \frac{\partial \delta\rho}{\partial n} \frac{\partial G(\cdot,y)}{\partial s_i} - \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial n^2}.
\end{aligned} \tag{3.88}$$

Now we apply (3.49),

$$-\frac{\partial^2 G(\cdot,y)}{\partial n^2} = \sum_{i=1}^{n-1} \left\{ \frac{\partial^2 G(\cdot,y)}{\partial s_i^2} + \nabla s_i \frac{\partial G(\cdot,y)}{\partial s_i} \right\}.$$

Here

$$tr\{\nabla s_i\} = 0$$

where

$$\begin{aligned}
\frac{\partial}{\partial n}(s_i s_j) &= \frac{\partial s_i}{\partial n} \cdot s_j + s_i \frac{\partial s_j}{\partial n} = 0 \\
\frac{\partial}{\partial s_k}(s_i s_j) &= \frac{\partial s_i}{\partial s_k} \cdot s_j + s_i \frac{\partial s_j}{\partial s_k} = 0
\end{aligned}$$

for $i,j,k = 1,\ldots,n-1$. Then (3.88) can be simplified as

$$\begin{aligned}\frac{\partial \dot{u}}{\partial n} &= \sum_{i=1}^{n-1} \frac{\partial \delta\rho}{\partial s_i}\frac{\partial G(\cdot,y)}{\partial s_i} + \sum_{i=1}^{n-1} \delta\rho \frac{\partial^2 G(\cdot,y)}{\partial s_i^2} \\ &= \sum_{i=1}^{n-1} \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial G(\cdot,y)}{\partial s_i}\right).\end{aligned} \tag{3.89}$$

Therefore using Stokes Theorem, the first variational formula can be derived as follows:

Theorem 3.8.2. *If Ω is a $C^{0,1}$ bounded domain and $\mathcal{T}_t : \Omega \to \mathcal{T}_t(\Omega) = \Omega$, $|t| \ll 1$ is a family of bi-Lipschitz homeomorphisms satisfying the domain deformation, then it holds that*

$$\begin{aligned}\delta G(\cdot,y) &= \left\langle \frac{\partial G(\cdot,x)}{\partial n}, \delta\rho \frac{\partial G(\cdot,y)}{\partial n}\right\rangle_{\gamma^0} - \sum_{i=1}^{n-1}\left\langle \frac{\partial G(\cdot,x)}{\partial s_i}, \delta\rho\frac{\partial G(\cdot,y)}{\partial s_i}\right\rangle_{\gamma^1} \\ &+ \ll G(\cdot,x), \delta\rho \frac{\partial G(\cdot,y)}{\partial s_i} \gg, \quad x,y \in \Omega.\end{aligned} \tag{3.90}$$

3.8.2 Second Variational Formula

Let

$$\Delta \ddot{u} = 0 \qquad \text{in } \Omega, \quad \ddot{u} + 2\delta\rho\frac{\partial \dot{u}}{\partial n} + \chi\frac{\partial G(\cdot,y)}{\partial n} = 0 \quad \text{on } \gamma^0$$

where

$$\chi = \delta^2\rho - (\delta\rho)^2\nabla\cdot n + \sum_{i=1}^{n-1}\mu_i\left(\mu_i\kappa_i - 2\frac{\partial\delta\rho}{\partial n}\right), \quad \mu_i = \mathsf{S}\cdot s_i.$$

Let $x \in \gamma^1$,

$$\frac{d}{ds}\left\{\frac{\partial}{\partial t}(\mathcal{T}_t x + s n_t)\cdot\nabla u(\mathcal{T}_t x + s n_t, t) + u_t(\mathcal{T}_t x + s n_t, t)\right\} = -\frac{d}{ds}\left\{\frac{\partial}{\partial t}(\mathcal{T}_t x + s n_t)\nabla\Gamma(\mathcal{T}_t x + s n_t - y)\right\}. \tag{3.91}$$

Hence, the left hand side of (3.91),

$$\frac{d}{ds}\left\{\frac{\partial}{\partial t}(\mathscr{T}_t x+sn_t)\cdot\nabla u(\mathscr{T}_t x+sn_t,t)+u_t(\mathscr{T}_t x+sn_t,t)\right\}\Bigg|_{s=0,t=0}$$

$$=\frac{d}{ds}\left\{\frac{\partial^2}{\partial t^2}(\mathscr{T}_t x+sn_t)\cdot\nabla u(\mathscr{T}_t x+sn_t,t)\right.$$

$$+\frac{\partial}{\partial t}(\mathscr{T}_t x+sn_t)\cdot\nabla\left[\frac{\partial}{\partial t}(\mathscr{T}_t x+sn_t)\cdot\nabla u(\mathscr{T}_t x+sn_t,t)+u_t(\mathscr{T}_t x+sn_t,t)\right]$$

$$\left.+\frac{\partial}{\partial t}(\mathscr{T}_t x+sn_t)\cdot\nabla u_t(\mathscr{T}_t x+sn_t,t)+u_{tt}(\mathscr{T}_t x+sn_t,t)\right\}\Bigg|_{s=0}$$

$$=\frac{\partial^2 n_t}{\partial t^2}\bigg|_{t=0}\cdot\nabla u+\mathsf{T}\cdot\left[\nabla^2 u\right]n+\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla\left[\mathsf{S}\cdot\nabla u+\dot{u}\right]$$

$$+\mathsf{S}\cdot\nabla\left[\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla u+\mathsf{S}\cdot(\nabla^2 u)n+\nabla\dot{u}\cdot n\right]+\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla u+\mathsf{S}\cdot\nabla\frac{\partial\dot{u}}{\partial n}+\frac{\partial\ddot{u}}{\partial n}.$$

While at the right hand side of (3.91),

$$-\frac{d}{ds}\left\{\frac{\partial}{\partial t}(\mathscr{T}_t x+sn_t)\nabla\Gamma(\mathscr{T}_t x+sn_t-y)\right\}\Bigg|_{s=0,t=0}$$

$$=-\frac{d}{ds}\left\{\frac{\partial^2}{\partial t^2}(\mathscr{T}_t+sn_t)\cdot\nabla\Gamma(\mathscr{T}_t x+sn_t-y)+\frac{\partial}{\partial t}(\mathscr{T}_t+sn_t)\cdot\nabla\left[\frac{\partial}{\partial t}(\mathscr{T}_t x+sn_t)\nabla\Gamma(\mathscr{T}_t+sn_t-y)\right]\right\}\Bigg|_{s=0}$$

$$=-\left\{\frac{\partial^2 n_t}{\partial t^2}\bigg|_{t=0}\cdot\nabla\Gamma(x-y)+\mathsf{T}\cdot\nabla^2[\cdot\Gamma(\cdot-y)]n+\nabla\ddot{u}\cdot n\right.$$

$$\left.+\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla\left[\mathsf{S}\cdot\nabla\Gamma(\cdot-y)\right]+\mathsf{S}\cdot\nabla\left[\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\Gamma(\cdot-y)+\mathsf{S}\cdot\nabla^2(\Gamma(\cdot-y))n\right]\right\}.$$

Hence

$$\frac{\partial^2 n_t}{\partial t^2}\bigg|_{t=0}\cdot\nabla G(\cdot,y)+\mathsf{T}\cdot\left[\nabla^2 G(\cdot-y)\right]n+\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla\left[\mathsf{S}\cdot\nabla G(\cdot,y)+\dot{u}\right]$$

$$+\mathsf{S}\cdot\nabla\left[\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla G(\cdot,y)+\mathsf{T}\cdot\left[\nabla^2 G(\cdot,y)\right]n+\frac{\partial\dot{u}}{\partial n}\right]$$

$$+\frac{\partial n_t}{\partial t}\bigg|_{t=0}\cdot\nabla\dot{u}+\mathsf{S}\cdot\nabla\frac{\partial\dot{u}}{\partial n}+\frac{\partial\ddot{u}}{\partial n}=0\qquad\text{on }\gamma^1.$$

Before we proceed to the second variation, we need to define the second variation $\ddot{u}$, we need to calculate $\frac{\partial^2 n_t}{\partial t^2}\,|_{t=0}$. First we use $|n_t|^2=1$,

$$\frac{\partial n_t}{\partial t}\cdot n_t=0\quad\frac{\partial^2 n_t}{\partial t^2}\cdot n_t=-\left|\frac{\partial n_t}{\partial t}\right|^2.$$

Thus,

$$\frac{\partial n_t}{\partial t}\cdot n_t=-\sum_{n-1}^{i=1}\left(\frac{\partial\mathsf{S}\tilde{x}}{\partial\xi_i}\cdot n\right)^2.\tag{3.92}$$

Then the relation $n_t \cdot s_i(t) = 0$ implies

$$\frac{\partial n_t}{\partial t} \cdot s_i(t) + n_t \cdot \frac{\partial s_i(t)}{\partial t}.$$

We differentiate again and we get

$$\frac{\partial}{\partial t}\left(\frac{\partial n_t}{\partial t} \cdot s_i(t) + n_t \cdot \frac{\partial s_i(t)}{\partial t}\right) = \frac{\partial^2 n_t}{\partial t^2} \cdot s_i(t) + 2\frac{\partial n_t}{\partial t} \cdot \frac{\partial s_i(t)}{\partial t} + n \cdot \frac{\partial^2 s_i(t)}{\partial t^2} = 0. \tag{3.93}$$

Hence

$$\frac{\partial^2 n_t}{\partial t^2} \cdot s_i(t) = -2\frac{\partial n_t}{\partial t} \cdot \frac{\partial s_i(t)}{\partial t} - n \cdot \frac{\partial^2 s_i(t)}{\partial t^2}$$

$$\begin{aligned} \left.\frac{\partial^2 n_t}{\partial t^2}\right|_{t=0} \cdot s_i &= -2\sum_{j=1}^{n-1}\left(\frac{\partial n_t}{\partial t} \cdot s_j\right) s_j \cdot s_i - n \cdot \ddot{s}_i \\ &= -2\sum_{j=1}^{n-1}\left(\frac{\partial \mathsf{S}\tilde{x}}{\partial \xi_j} \cdot n\right)\left(s_j \cdot \frac{\partial \mathsf{S}\tilde{x}}{\partial \xi_i}\right) - n \cdot \frac{\partial \mathcal{T}_t \tilde{x}}{\partial \xi_i}. \end{aligned} \tag{3.94}$$

Then we get

$$\begin{aligned} \left.\frac{\partial^2 n_t}{\partial t^2}\right|_{t=0} = &-\sum_{i=1}^{n-1}\left[\sum_{j=1}^{n-1} 2\left(\frac{\partial \mathsf{S}\tilde{x}}{\partial \xi_j} \cdot n\right)\left(n \cdot \frac{\partial \mathsf{S}\tilde{x}}{\partial \xi_i}\right) - n \cdot \frac{\partial \mathcal{T}_t \tilde{x}}{\partial \xi_i}\right] s_i \\ &-\sum_{i=1}^{n-1}\left[\left(\frac{\partial \mathsf{S}\tilde{x}}{\partial \xi_i} \cdot n\right)^2\right] n. \end{aligned} \tag{3.95}$$

Then the second variation $\ddot{u}$ can be defined by (3.92), (3.94) and (3.95). Since the value $\frac{\partial \dot{u}}{\partial n}$ on γ^1 is prescribed in the first variational formula, the unknown data involved by $\dot{u}$ are reduced to

$$\begin{aligned} &\Delta H = 0 \quad \text{on } \Omega, \quad H = -2\delta\rho\frac{\partial \dot{u}}{\partial n} \quad \text{on } \gamma^0, \\ &\frac{\partial H}{\partial n} = -2\left.\frac{\partial n_t}{\partial t}\right|_{t=0} \cdot \nabla \dot{u} - 2\delta\rho\frac{\partial^2 \dot{u}}{\partial n^2} \quad \text{on } \gamma^1 \end{aligned} \tag{3.96}$$

where $H = -2h$.

Theorem 3.8.3. *Assume*

$$\delta\rho = \mathsf{S} \cdot n = 0 \qquad on\ \sigma \tag{3.97}$$

and $\delta_0 G(x,y)$ be the first variation of $G(x,y)$ concerning deformations

$$\mathcal{T}_t^0 x = x + t\mathsf{S}_0 x + \frac{t^2}{2}\mathsf{T}_0 x + o(t^2).$$

S_0 should satisfy

$$\mathsf{S}_0 \cdot n = \begin{cases} \delta\rho & on\ \gamma^0 \\ 0 & on\ \gamma^1. \end{cases}$$

Therefore,

$$h = (\nabla \delta G(\cdot,x), \nabla \delta G(\cdot,y)) + (\nabla \delta_0 G(\cdot,x), \nabla \delta G(\cdot,y)) + (\nabla \delta G(\cdot,x), \nabla \delta_0 G(\cdot,y)). \tag{3.98}$$

Proof. From the first variation, $\dot{u}_0^* = \delta_0 G(\cdot,x)$ should satisfy

$$\begin{aligned} &\Delta \dot{u}^* = 0 \quad \text{on } \Omega, \quad \dot{u}^* = -\delta\rho \frac{\partial G(\cdot,x)}{\partial n} \quad \text{on } \gamma^0, \\ &\frac{\partial \dot{u}^*}{\partial n} = \sum_{i=1}^{n-1} \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial G(\cdot,x)}{\partial s_i}\right) \quad \text{on } \gamma^1. \end{aligned} \tag{3.99}$$

From Lemma 3.5.3, we have,

$$\begin{aligned} (\nabla \delta_0 G(\cdot,x), \nabla \delta G(\cdot,y)) = (\nabla \dot{u}_0^*, \nabla \dot{u}) &= \left\langle \frac{\partial \dot{u}_0^*}{\partial n}, \dot{u} \right\rangle \\ &= \left\langle \frac{\partial G(\cdot,x)}{\partial n}, \delta\rho \frac{\partial \dot{u}}{\partial n} \right\rangle. \end{aligned} \tag{3.100}$$

Since $\dot{u}^* = \delta G(\cdot,x)$ satisfies

$$\begin{aligned} &\Delta \dot{u}^* = 0 \quad \text{in } \Omega, \quad \dot{u}^* = -\delta\rho \frac{\partial G(\cdot,x)}{\partial n} \quad \text{on } \gamma^0 \\ &\frac{\partial \dot{u}^*}{\partial n} = \sum_{n-1}^{i=1} \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial G(\cdot,x)}{\partial s_i}\right) \quad \text{on } \gamma^1. \end{aligned}$$

$$\begin{aligned} (\nabla \delta G(\cdot,x), \nabla \delta G(\cdot,y)) &= (\nabla \dot{u}^*, \nabla \dot{u}) \\ &= \left\langle \dot{u}^*, \frac{\partial \dot{u}}{\partial n} \right\rangle \\ &= -\left\langle \frac{\partial G(\cdot,x)}{\partial n}, \delta\rho \frac{\partial \dot{u}}{\partial n} \right\rangle_{\gamma^0} + \sum_{i=1}^{n-1} \left\langle \dot{u}^*, \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial G(\cdot,y)}{\partial s_i}\right) \right\rangle_{\gamma^1} \\ &= -\left\langle \frac{\partial G(\cdot,x)}{\partial n}, \delta\rho \frac{\partial \dot{u}}{\partial n} \right\rangle_{\gamma^0} + \sum_{i=1}^{n-1} \left\langle \delta G(\cdot,y), \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial G(\cdot,x)}{\partial s_i}\right) \right\rangle_{\gamma^1}. \end{aligned} \tag{3.101}$$

Then we have

$$(\nabla \delta_0 G(\cdot,x) + \nabla \delta G(\cdot,x), \nabla \delta G(\cdot,y)) = \sum_{i=1}^{n-1} \left\langle \delta G(\cdot,y), \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial G(\cdot,x)}{\partial s_i}\right) \right\rangle_{\gamma^1} \tag{3.102}$$

and

$$(\nabla \delta_0 G(\cdot,y) + \nabla \delta G(\cdot,y), \nabla \delta G(\cdot,x)) = \sum_{i=1}^{n-1} \left\langle \delta G(\cdot,x), \frac{\partial}{\partial s_i}\left(\delta\rho \frac{\partial \dot{u}}{\partial s_i}\right) \right\rangle_{\gamma^1}. \tag{3.103}$$

Thus, we obtain

$$
\begin{aligned}
h &= (\nabla\delta G(\cdot,x),\nabla\delta G(\cdot,y)) + (\nabla\delta_0 G(\cdot,x),\nabla\delta G(\cdot,y)) + (\nabla\delta G(\cdot,x)+\nabla\delta_0 G(\cdot,y)) \\
&= (\nabla\delta_0 G(\cdot,x),\nabla\delta G(\cdot,y)) + (\nabla\delta G(\cdot,x),\nabla\delta_0 G(\cdot,y)+\nabla\delta G(\cdot,y)) \\
&= \left\langle \frac{\partial G(\cdot,x)}{\partial n},\delta\rho\frac{\partial \dot{u}}{\partial n}\right\rangle_{\gamma^0} + \left\langle \delta G(\cdot,x),\frac{\partial}{\partial s_i}\left(\delta G(\cdot,x)\frac{\partial \dot{u}}{\partial s_i}\right)\right\rangle_{\gamma^1}.
\end{aligned}
\tag{3.104}
$$

□

Then we recalculate again Lemma 3.6.2 and Proposition 3.6.3, and we obtain,

$$
\begin{aligned}
\Delta f = \nabla\cdot\nabla f &= \nabla\cdot\left(\sum_{i=1}^{n-1} s_i\frac{\partial f}{\partial s_i} + n\frac{\partial f}{\partial n}\right) \\
&= \sum_{i=1}^{n-1}\nabla s_i\frac{\partial f}{\partial s_i} + \nabla n\frac{\partial f}{\partial n} + s_i\otimes\nabla\frac{\partial f}{\partial s_i} + n\otimes\nabla\frac{\partial f}{\partial n} \\
&= \sum_{i=1}^{n-1}\left((\nabla s_i)\frac{\partial f}{\partial s_i} + s_i\otimes\nabla\frac{\partial f}{\partial s_i}\right) + (\nabla\cdot n)\frac{\partial f}{\partial n} + n\otimes\nabla\frac{\partial f}{\partial n} \\
&= (\nabla\cdot n)\frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2} + \sum_{i=1}^{n-1}\left((\nabla\cdot s_i)\frac{\partial f}{\partial s_i} + \frac{\partial^2 f}{\partial s_i^2}\right).
\end{aligned}
\tag{3.105}
$$

Recall that

$$
\frac{\partial n}{\partial n} = 0, \quad \frac{\partial s_i}{\partial n} = 0 \quad and \quad \frac{\partial n}{\partial s_i} = \kappa_i s_i
$$

and from Frenet-Serret formula,

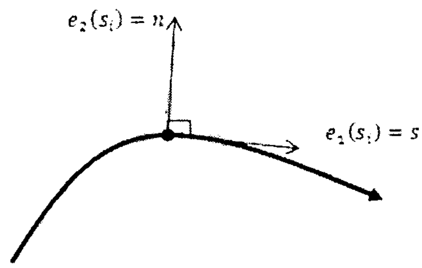

Figure 3.3: Curve S with component $\{e_2(s_i), e_1(s)\}$

$$
\begin{aligned}
&e_1'(s_i) = \kappa e_2(s_i) \quad e_2'(s_i) = -\kappa e_1(s_i) \\
&\frac{\partial s_i}{\partial s_i} = e_1' = \kappa e_2 = -\kappa n \quad \frac{\partial n}{\partial s_i} = -e_2' = -\kappa e_1 = \kappa s_i.
\end{aligned}
$$

Hence,

$$\nabla n = \frac{\partial n}{\partial s_i} \otimes s_i + \frac{\partial n}{\partial n} \otimes n = \kappa s_i \otimes s_i \quad \nabla s_i = \frac{\partial s_i}{\partial s_i} \otimes s_i + \frac{\partial s_i}{\partial n} \otimes n = -\kappa n \otimes s_i$$

$$\nabla \cdot n = tr(\nabla n) = \sum_{i=1}^{n-1} \kappa_i, \quad \nabla \cdot s_i = tr(\nabla s_i) = 0.$$

Therefore, from Lemma 3.6.2 and (3.105) we get

$$\begin{aligned} Hf &= \sum_{i=1}^{n-1} \left[(-\kappa_i n \otimes s_i) \frac{\partial f}{\partial s_i} + (\kappa_i s_i \otimes s_i) \frac{\partial f}{\partial n} \right] + \sum_{i,j=1}^{n-1} s_i \otimes s_j \frac{\partial^2}{\partial s_i \partial s_j} \\ &= \sum_{i=1}^{n-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 f}{\partial s_i \partial n} + n \otimes n \frac{\partial^2 f}{\partial n^2} \end{aligned} \tag{3.106}$$

and

$$\begin{aligned} \Delta f &= \sum_{i=1}^{n-1} \kappa_i \frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2} + \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial s_i^2} \\ &= \tilde{\kappa} \frac{\partial f}{\partial n} + \frac{\partial^2}{\partial n^2} + \sum_{i=1}^{n-1} \frac{\partial^2 s_i}{\partial s_i^2}. \end{aligned} \tag{3.107}$$

Recall from (3.46)

$$\ddot{u} + 2\mathsf{S} \cdot \nabla_x \delta G(\cdot, y) + \mathsf{T} \cdot \nabla G(\cdot, y) + H_x G(\cdot, y) \mathsf{S} \cdot \mathsf{S} = 0 \qquad \text{on } \partial\Omega \tag{3.108}$$

and since $\Delta G(\cdot, y) = 0$,

$$\frac{\partial^2 G(\cdot, y)}{\partial n^2} = -\tilde{\kappa} \frac{\partial G(\cdot, y)}{\partial n}.$$

Now we proceed to calculate (3.108). By using (3.54),

$$\mathsf{T} \cdot \nabla G(\cdot, y) = \delta^2 \rho \frac{\partial G(\cdot, y)}{\partial n}. \tag{3.109}$$

While

$$\begin{aligned} H_x G(\cdot, y) &= \sum_{i=1}^{n-1} \kappa_i \left((s_i \otimes s_i) \frac{\partial G(\cdot, y)}{\partial n} - (n \otimes s_i) \frac{\partial G(\cdot, y)}{\partial s_i} \right) + \sum_{i,j=1}^{n-1} s_i \otimes s_j \frac{\partial G(\cdot, y)}{\partial s_i \partial s_j} \\ &+ \sum_{i=1}^{n-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} + n \otimes n \frac{\partial^2 G(\cdot, y)}{\partial n^2} \\ &= \sum_{i=1}^{n-1} (\kappa_i s_i \otimes s_i) \frac{\partial G(\cdot, y)}{\partial n} + \sum_{i=1}^{n-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} + n \otimes n \frac{\partial^2 G(\cdot, y)}{\partial n^2} \\ &= \sum_{i=1}^{n-1} (\kappa_i s_i \otimes s_i) \frac{\partial G(\cdot, y)}{\partial n} - \sum_{i=1}^{n-1} \kappa_i (n \otimes n) \frac{\partial G(\cdot, y)}{\partial n} + \sum_{i=1}^{n-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} \\ &= \sum_{i=1}^{n-1} \kappa_i [(s_i \otimes s_i) - (n \otimes n)] \frac{\partial G(\cdot, y)}{\partial n} + \sum_{i=1}^{n-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n}. \end{aligned} \tag{3.110}$$

Therefore,

$$(H_x G(\cdot,y))\mathsf{S} = \left[\sum_{i=1}^{n-1} \kappa_i[(s_i\otimes s_i)-(n\otimes n)]\frac{\partial G(\cdot,y)}{\partial n} + \sum_{i=1}^{n-1}(s_i\otimes n+n\otimes s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i \partial n}\right]\left(\sum_{i=1}^{n-1}\mu_i s_i + (\delta\rho)n\right). \tag{3.111}$$

Now we consider at the first right hand side of (3.111).

$$[\kappa_i s_i\otimes s_i - \kappa_i n\otimes n]\frac{\partial G(\cdot,y)}{\partial n}\left(\sum_{i=1}^{n-1}\mu_i s_i + (\delta\rho)n\right).$$

It holds that

$$\sum_{i=1}^{n-1}(\kappa_i s_i\otimes s_i)\sum_{i=1}^{n-1}\mu_i s_i = \sum_{i=1}^{n-1}(\kappa_i s_i\otimes s_i)(\mathsf{S}-(\delta\rho)n)$$
$$\sum_{i=1}^{n-1}(\kappa_i n\otimes n)\sum_{i=1}^{n-1}\mu_i s_i = \sum_{i=1}^{n-1}(\kappa_i n\otimes n)(\mathsf{S}-(\delta\rho)n)$$
$$\sum_{i=1}^{n-1}(\kappa_i s_i\otimes s_i)(\delta\rho) = 0$$
$$\sum_{i=1}^{n-1}(\kappa_i n\otimes n)(\delta\rho)n = 0.$$

Hence,

$$\begin{aligned}[\kappa_i s_i\otimes s_i - \kappa_i n\otimes n]\frac{\partial G(\cdot,y)}{\partial n}&\left(\sum_{i=1}^{n-1}\mu_i s_i + (\delta\rho)n\right)\\ &= \sum_{i=1}^{n-1}\kappa_i(\mu_i s_i-(\delta\rho)n)\frac{\partial G(\cdot,y)}{\partial n}.\end{aligned} \tag{3.112}$$

While at second right hand side of (3.111).

$$\left[\sum_{i=1}^{n-1}(s_i\otimes n+n\otimes s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i \partial n}\right]\left(\sum_{i=1}^{n-1}\mu_i s_i + (\delta\rho)n\right).$$

Then it also holds that

$$\sum_{i=1}^{n-1}(s_i\otimes n)(\mu_i s_i) = \sum_{i=1}^{n-1}s_i\otimes n(\mathsf{S}-(\delta\rho)n) = \sum_{i=1}^{n-1}\mu_i n$$
$$\sum_{i=1}^{n-1}(n\otimes s_i)(\mu_i s_i) = \sum_{i=1}^{n-1}(n\otimes s_i)(\mathsf{S}-(\delta\rho)n) = (\delta\rho)s_i$$
$$\sum_{i=1}^{n-1}(s_i\otimes n)(\delta\rho)n = 0$$
$$\sum_{i=1}^{n-1}(n\otimes s_i)(\delta\rho)n = 0.$$

Hence

$$\begin{aligned}\left[\sum_{i=1}^{n-1}(s_i\otimes n+n\otimes s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i \partial n}\right]&\left(\sum_{i=1}^{n-1}\mu_i s_i + (\delta\rho)n\right)\\ &= \sum_{i=1}^{n-1}(\mu_i n+(\delta\rho)s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i \partial n}.\end{aligned} \tag{3.113}$$

Therefore

$$
\begin{aligned}
(H_x G(\cdot,y))\mathsf{S} &= \sum_{i=1}^{n-1} \kappa_i(\mu_i s_i - (\delta\rho)n)\frac{\partial G(\cdot,y)}{\partial n} + \sum_{i=1}^{n-1}(\mu_i n + (\delta\rho)s_i)\frac{\partial^2 G(\cdot,y)}{\partial s_i \partial n} \\
&= \sum_{i=1}^{n-1}\left(\delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} + \kappa_i\mu_i\frac{\partial G(\cdot,y)}{\partial n}\right)s_i + \sum_{i=1}^{n-1}\left(\delta\rho\frac{\partial G(\cdot,y)}{\partial n} + \mu_i\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}\right).
\end{aligned}
\tag{3.114}
$$

Using (3.106) and we obtain

$$
\begin{aligned}
2\nabla\delta G(\cdot,y) &= 2\sum_{i=1}^{n-1}\frac{\partial\delta G(\cdot,y)}{\partial s_i\partial n}s_i + 2\frac{\partial\delta G(\cdot,y)}{\partial n}n \\
&= -2\sum_{i=1}^{n-1}\left(\delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} + \frac{\partial G(\cdot,y)}{\partial n}\frac{\partial(\delta\rho)}{\partial s_i}\right)s_i + 2\frac{\partial G(\cdot,y)}{\partial n}n.
\end{aligned}
\tag{3.115}
$$

Hence

$$
\begin{aligned}
2\nabla G(\cdot,y) + (H_x G(\cdot,y))\mathsf{S} &= \sum_{i=1}^{n-1}\left(-2\delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} - 2\frac{\partial(\delta\rho)}{\partial s_i}\frac{\partial G(\cdot,y)}{\partial n} + \delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} - \kappa_i\mu_i\frac{\partial G(\cdot,y)}{\partial n}\right)s_i \\
&\quad + \sum_{i=1}^{n-1}\left(\mu_i\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} - \kappa_i(\delta\rho)\frac{\partial G(\cdot,y)}{\partial n}\right)n + 2\frac{\partial G(\cdot,y)}{\partial n}n \\
&= \sum_{i=1}^{n-1}\left(\mu_i\kappa_i\frac{\partial G(\cdot,y)}{\partial n} - 2\frac{\partial(\delta\rho)}{\partial s_i}\frac{\partial G(\cdot,y)}{\partial n}s_i - \delta\rho\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n}\right)s_i \\
&\quad + \sum_{i=1}^{n-1}\left(\mu_i\frac{\partial^2 G(\cdot,y)}{\partial s_i\partial n} - \kappa_i(\delta\rho)\frac{\partial G(\cdot,y)}{\partial n}\right)n + 2\frac{\partial G(\cdot,y)}{\partial n}n
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathsf{S}\cdot[2\nabla G(\cdot,y) + (H_x G(\cdot,y))\mathsf{S}] &= \sum_{i=1}^{n-1}\left(\kappa_i(\mu_i)^2 - \kappa_i(\delta\rho)^2 - 2\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\right)\frac{\partial G(\cdot,y)}{\partial n} + 2(\delta\rho)\frac{\partial G(\cdot,y)}{\partial n} \\
&= \sum_{i=1}^{n-1}\left(\kappa_i((\mu_i)^2 - (\delta\rho)^2) - 2\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\right)\frac{\partial G(\cdot,y)}{\partial n} + 2(\delta\rho)\frac{\partial G(\cdot,y)}{\partial n}.
\end{aligned}
\tag{3.116}
$$

Similar to Lemma 3.6.4, it holds that

$$
\ddot{u} = \delta^2 G(\cdot,y) = -2h + \chi\frac{\partial G(\cdot,y)}{\partial n} \tag{3.117}
$$

where

$$
\chi = \delta^2\rho + \sum_{i=1}^{n-1}\left[\kappa_i((\mu_i)^2 - (\delta\rho)^2) - 2\mu_i\frac{\partial(\delta\rho)}{\partial s_i}\right]. \tag{3.118}
$$

Therefore, we can proof the second variational formula of Hadamard variation for mixed problem similar to the

Hadamard variational formula in Section 3.6 where

$$\chi = -\tilde{\kappa}(\delta\rho)^2 - (\mathrm{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)}{\partial s_i}, \quad \tilde{\kappa} = \sum_{i=1}^{n-1}\kappa_i.$$

From (3.104) and (3.117), we obtain the following theorem,

Theorem 3.8.4. *Let Ω is a Lipschitz domain with $W^{1,\infty}$ and $\mathcal{T}_t x$ is a perturbation of Green's function of second variation $\delta^2 G(\cdot,y)$. Therefore, we have*

$$\begin{aligned}\delta^2 G(\cdot,y) &= 2(\nabla\delta_0 G(\cdot,x),\nabla\delta G(\cdot,y)) - 2(\nabla\delta G(\cdot,x),\nabla\delta_0 G(\cdot,y)+\nabla\delta G(\cdot,y))\\ &+\left\langle\left[-\tilde{\kappa}(\delta\rho)^2 - (\mathrm{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n}\right]\frac{\partial G(\cdot,x)}{\partial n},\frac{\partial G(\cdot,y)}{\partial n}\right\rangle.\end{aligned} \tag{3.119}$$

Particularly, given that $\mathrm{S}\cdot s_i = 0$, $i = 1,\ldots,n-1$ as the normal boundary perturbation,

$$\begin{aligned}\delta^2 G(\cdot,y) &= 2(\nabla\delta_0 G(\cdot,x),\nabla\delta G(\cdot,y)) - 2(\nabla\delta G(\cdot,x),\nabla\delta_0 G(\cdot,y)+\nabla\delta G(\cdot,y))\\ &+\left\langle(\delta^2\rho - \tilde{\kappa}(\delta\rho)^2)\frac{\partial G(\cdot,x)}{\partial n},\frac{\partial G(\cdot,y)}{\partial n}\right\rangle.\end{aligned} \tag{3.120}$$

Chapter 4

Conclusions and Future Works

This chapter discusses the work which has been done and overall conclusion and future works to complete this study.

The study of variational inequalities and free boundary problems find application in a wide variety of disciplines in physics, engineering, economics etc.. In this thesis we study about the Hadamard variational formula that includes in one of the topics in the study of variational inequalities. We implement our Hadamard variational formula to one of the dam or filtration problem. Dam problem is one of the famous problem in a free boundary problem. We implement our first variational formula to the dam problem by doing numerical analysis. For this numerical simulation, we used level-set approach and we changed Hadamard variational formula into a boundary trial method along with the finite element. Then to get an optimal shape design, we used the approximated region as an initial region for the traction method. As a result we found that the Hadamard first variational formula was suitable to solve the free boundary problem of the dam problem.

In this thesis also we include our study on the mathematical analysis of the Hadamard variational formula. We develop a method that gives a simple and clear proof to understand the Hadamard variational formula. We also obtain Hadamard second variational formula that is also an extension of the Garabedian-Schiffer formula.

For now, we only consider the Dirichlet problem for the Hadamard variational formula. For further investigation, we would like also to consider this Hadamard variational formula for the Neumann problem. Furthermore, we also want to implement our second variational formula to the dam problem as for the numerical analysis. Two methods that we still consider for solving our dam problem are: finite element method and Newton method for the optimal shape design. We consider this Newton method as our next method in order to implement our second Hadamard variational formula because it is simple and more efficient to solve our free boundary problem.

Appendix A

Functional Analysis

A.1 Sobolev Spaces

Definition A.1.1. *Let Ω be an open subset $\mathbb{R}^n$. The Lebesque space is defined as*

$$L^p(\Omega)(or\ L_p(\Omega)) = \{v : \Omega \to \mathbb{R} | \int_\Omega |v(x)|^p \ dx \leq \infty\}.$$

This is a Banach space with norm defined by

$$\|v\|_{L^p(\Omega)} = (\int_\Omega |v(x)|^p)^{\frac{1}{p}}.$$

When $p = 2$, the space $L^2(\Omega)$ is a Hilbert space.

There are two phase requirements of functional space need to be consider Sobolev spaces.

Definition A.1.2. *Let k a nonnegative integer, $p \geq 1$. The function*

$$\{u \in W^k(\Omega); D^\alpha u \in L^p(\Omega),\ \text{for any}\ \alpha\ \text{with}\ |\alpha| \leq k\}$$

endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p \ dx\right)^{\frac{1}{p}}$$

is called Sobolev space, denoted by $W^{k,p}(\Omega)$. Sobolev space also can be noted as

$$H^1(\Omega) = W^{1,2}(\Omega)$$

where $k = 1$ and $p = 2$ which is a Hilbert space with the inner product

$$(u,v)_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u \cdot D^\alpha v \; dx, \qquad u,v \in H^k(\Omega).$$

Let $\frac{\partial v}{\partial x_j} \in L^2(\Omega)$ $(1 \leq j \leq n)$ is taken for $v \in L^2(\Omega)$. Therefore, $v \in H^1(\Omega)$ if $v \in L^2(\Omega)$ and

$$\int_\Omega v\frac{\partial \phi}{\partial x_j} = -\int_\Omega v_j \phi \qquad (\phi \in C_0^\infty(\Omega)) \tag{A.1}$$

for $v_j \in L^2(\Omega)$, $(1 \leq j \leq n)$. Here, $C_0^\infty(\Omega)$ is a set of functions support on compact set in Ω are infinitely differentiable function.

A.2 Banach and Hilbert Spaces

Let Ω be a real vector space. A function

$$\|\cdot\| : \Omega \to \mathbb{R} \qquad v \mapsto \|v\|$$

is called norm on Ω if it satisfies $v \in \Omega$ and all $\lambda \in \mathbb{R}$,

- $\|v\| \geq 0; \|0\| = 0$ *iff* $x = 0$
- $\|\lambda v\| = |\lambda|\|v\|$
- $\|\lambda + v\| \geq \|\lambda\| + \|v\|$.

This norm is complete when from definition,

Definition A.2.1. *Let Ω be a normed vector space with norm $\|\cdot\|$. A sequence $\{v_k\}$ in Ω is said to be convergence to $v \in \Omega$ if*

$$\lim_{k\to\infty} \|v_k - v_n\| = 0.$$

A sequence $\{v_k\}$ in Ω is called a Cauchy sequence if $\forall \varepsilon > 0$ exist $N(\varepsilon)$ such that

$$n,k \geq N(\varepsilon) \Rightarrow \|v_k - v_n\| < \varepsilon.$$

It is also called a Banach space when the norm is a complete normed space i.e., only Cauchy sequence converges. An inner product space is a normed space. Then, it is a Hilbert space if it is a Banach space which respect to this norm. It is easy to say that the norm is induced by the inner product satisfies parallelogram law:

$$\begin{aligned}\|v+u\|^2+\|v-u\|^2 &= <v+u,v+u> + <v-u,v-u> \\ &= \|v\|^2 + <v,u> + <u,v> + \|u\|^2 + \|v\|^2 - <v,u> - <u,v> + \|u\|^2 \\ &= 2\|v\|^2 + 2\|u\|^2 + 2\|v\|\|u\| - 2\|v\|\|u\| \\ &= 2(\|v\|^2 + \|u\|^2).\end{aligned}$$

A Hilbert space is a Banach space where it has a subsequence of weak convergence for arbitrary bounded sequence.

Theorem A.2.2. *For every bounded linear functional F on the Hilbert space H, there is a uniquely determined element $u \in H$ such that $F(v) = (v,u)$ for all $v \in H$ and $\|F\| = \|u\|$.*

Theorem A.2.3. *Let M be a closed subspace of Hilbert space H. Then for every $x \in H$ we have $x + y = z$ where $y \in M$ and $z \in M^{\perp}$.*

Proposition A.2.4. *Let H be a Hilbert space and $L : H \longrightarrow F$ is a linear functional. The following statements are equivalent.*

1. *L is continuous.*
2. *L is continuous at 0.*
3. *L is continuous at some point.*
4. *There is a constant $c > 0$ such that $|L(u)| \leq \|u\|$ for every u in H.*

Definition A.2.5. *A bounded linear functional L on H is a linear functional for which there is a constant $C > 0$ such that $|L(u)| \leq C\|u\|$ for all u in H.*

Define $L : H \to F$,

$$\|L\| = \sup\{|L(u)| : \|L(u)\| \leq 1\},$$

for a bounded linear functional where $\|L\| < \infty$ and $\|L\|$ is the norm of L.

Proposition A.2.6. *If L is bounded linear functional, then*

$$\begin{aligned}\|L\| &= \sup\{|L(u)| : \|u\| = 1\} \\ &= \sup\{|L(u)|/\|u\| : u \in H, u \neq 0\} \\ &= \inf\{C > 0 : |L(u)| \leq C\|u\|, u \in H\}.\end{aligned}$$

$|L(u)| \leq \|L\|\|u\|, \forall u \in H.$

A.3 Monotone Operators

Let K be a closed convex set in real Hilbert space H, and let y_0 be a point in H. Then there exists $x_0 \in K$ (called the projection of y_0 on K) which is nearest to y_0, that is,

$$\|x_0 - y_0\| \leq \|x - y_0\|, \qquad \forall x \in K.$$

Rewrite the above equation into

$$(x_0 - y_0, x - x_0) \geq 0, \qquad \forall x \in K \qquad (*)$$

Let X be a reflexive Banach space with X' as its duality. The pairing $\langle , \rangle$, between X and X' and K is a closed convex set of X. Hence,

Definition A.3.1. *A (nonlinear) operator $A : K \to X'$ is called monotone if*

$$\langle Au - Av, u - v \rangle \geq 0, \qquad \forall u, v \in K. \tag{A.2}$$

It is strictly monotone if

$$\langle Au - Av, u - v \rangle = 0$$

implies for $u = v$.

When K is convex, A is called hemicontinuous if given $u, v \in K$, the mapping

$$[0,1] \ni t \to \langle A(tu + (1-t)v), u - v \rangle$$

is continuous.

Definition A.3.2. *The operator $A : k \to X'$ is continuous on finite dimensional subspaces if for every finite-dimensional subspace M of X the mapping $A : K \cap M \to X'$ is weakly continuous.*

Theorem A.3.3. *Suppose that $A : X \to X'$ is monotone and hemicontinuous [K=X]. Then for any bounded closed convex subset K of X there exists a $u_0 \in K$ such that*

$$\langle Au_0, v - u_0 \rangle \geq 0, \qquad \forall v \in K. \tag{A.3}$$

The above theorem is a direct consequence of the following

Theorem A.3.4. *Let K be a bounded closed convex subset of X and suppose that $A : K \to X'$ is monotone and A is continuous on finite-dimensional subspaces of K. Then there exist a $u_0 \in K$ such that (A.3) holds.*

Proof of these two theorem depends on *Minty's Lemma*:

Lemma A.3.5 (Minty's Lemma). *Let K be a nonempty closed convex set of X and let $A : K \to X'$ be monotone and hemicontinuous. Then u_0 satisfies (A.3) if only if*

$$\langle Av, v - u_0 \rangle \geq 0, \qquad \forall v \in K. \tag{A.4}$$

Proof. By monotocity of A,

$$0 \leq \langle Av - Au_0, v - u_0 \rangle = \langle Av, v - u_0 \rangle - \langle Au_0, v - u_0 \rangle,$$

Thus,

$$0 \leq \langle Au_0, v - u_0 \rangle \leq \langle Av, v - u_0 \rangle$$

so that (A.3) implies (A.4). Conversely, for any $w \in K$,

$$v = tw + (1-t)u_0 = u + t(w - u_0) \text{ in } K \text{ if } 0 < t < 1.$$

Then from (A.4),

$$\langle A(u + t(w - u_0)), w - u_0 \rangle \geq 0.$$

Taking $t \to 0$, we obtain (A.3) for any $v = w \in K$. □

Theorem A.3.6 (Brouwer's Fixed-Point Theorem). *If $1 \leq d < \infty$, B = the closed unit ball of $\mathbb{R}^d$ and $T : B \to B$ is a continuous map, then there is a point u in B such that $T(u_0) = u_0$.*

Corollary A.3.7. *Let K be a compact convex bounded subset of a Banach space, X, and $T : K \to K$ is completely continuous mapping. Then T has a fixed point u_0 in K that is $Tu_0 = u_0$.*

Proof. (Theorem A.3.3) Let M be any finite-dimensional subspace of X and define $j : M \to X$ and dual map $j^* : X' \to M$. Then

$$j^*Aj \text{ maps bounded sets of } M \text{ into bounded sets of } M'. \tag{A.5}$$

Otherwise there exists a sequence $v_n \in M$, $0 < \|v_n\| \leq C$ such that $\|j^*Av_n\| \to \infty$. The monotocity of A implies that

$$\langle j^*Av_n - j^*Au, v_n - u \rangle \geq 0, \qquad \forall u \in M.$$

Hence

$$\left\langle y_n - \frac{j^*Au}{\|j^*Av_n\|}, v_n - u \right\rangle \geq 0, \quad \text{where } y_n = \frac{j^*Av_n}{\|j^*Av_n\|}.$$

Since y_n are elements of M', there exist a subsequence $y_n \to y, \|y\| = 1$. Also, suppose that $v_n \to v$. But then $\langle y, v - u\rangle \geq 0, \forall u \in M$ which gives $y = 0$ a contradiction. □

Definition A.3.8. *A is coercive on K if there exist an element $\phi \in K$ such that*

$$\frac{1}{\|u-\phi\|}\langle Au - A\phi, u - \phi\rangle \to +\infty, \quad u \in K, \|u\| \to \infty. \tag{A.6}$$

Theorem A.3.9. *Let K be unbounded closed convex set and A be as in Theorem A.3.3 or A.3.4. If A is coercive, then there exist a solution of*

$$\langle Au_0, v - u_0\rangle \geq 0, \qquad \forall v \in K. \tag{A.7}$$

Proof. Let $R > 0$, there exist at least one solution of the variational inequality

$$\langle Au_R, v - u_R\rangle \geq 0, \qquad \forall u_R, v \in K.$$

We introduce the bounded convex set

$$K_R = K \cap \{\|u\| \leq R\}$$

Since

$$\|u_R\| < R, \qquad \text{for some } R > 0, \tag{A.8}$$

given $\forall v \in K$, there exists an $\varepsilon > 0$ sufficiently small so that

$$w = u_R + \varepsilon(v - u_R)$$

belongs to K_R. Hence

$$\begin{aligned} 0 &\leq \langle Au_R, w - u_R\rangle \\ &= \langle Au_R, \varepsilon(v - u_R)\rangle \\ &= \varepsilon\langle Au_R, v - u_R\rangle \end{aligned}$$

and (A.7) follows for $u_0 = u_R$ with any $v \in K$. Then it follows that for any $C > 0$, there is an $R > 0$ such that $R > \|\phi\|$ and

$$\langle Av - A\phi, v - \phi\rangle \geq C\|v - \phi\|, \qquad \forall v \in K, \ \|v\| \geq R.$$

Take $C > \|A\phi\|$. Then

$$\begin{aligned}\langle Av, v-\phi\rangle &\geq C\|v-\phi\| + \langle A\phi, v-\phi\rangle \\ &\geq C\|v-\phi\| - \|A\phi\|\|v-\phi\| \\ &\geq (C-\|A\phi\|)\|v-\phi\| \\ &\geq (C-\|A\phi\|)(\|v\|-\|\phi\|) \\ &\geq 0.\end{aligned}$$

For $\|u_R\| = R$ and upon taking $v = u_R$, the inequalities become

$$\langle Au_R, u_R - \phi\rangle > 0.$$

which is a contradiction to the variational inequality that satisfied u_R. □

A.4 Schauder's Fixed Point Theorem

Schauder's fixed point theorem is the extension of Brouwer's fixed point theorem to a topological vector space which may be of infinite dimension.

Definition A.4.1. *If X is a normed space and $E \subseteq X$, a function $T: E \to X$ is said to be compact if T is continuous and $T(M)$ is compact whenever M is a bounded subset of E.*

If E is itself a compact subset of X, then every continuous function from E into X is compact.
The following lemma is needed in the proof of Schauder's theorem.

Theorem A.4.2 (Schauder's Fixed Point Theorem). *Let K be a compact convex subset of Banach space of X. If $T: K \to K$ is continuous , then T has a fixed point in K.*

Corollary A.4.3. *Let K be a closed and bounded convex subset of a Banach space X and $T: K \to K$ a completely continuous mapping. Then T has a fixed point in K.*

Proof. Let K' = the closed convex hull of $T(k)$. It follows that K' is compact and thus T mapping K' into K' has a fixed point in K'. □

A.5 Compact operators

Definition A.5.1. *Let X and Y be normed linear spaces. Suppose T is a linear operator with domain and range in Y. We say that T is compact if for each bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ contains a subsequence converging to some limit in Y.*

A.6 Variational Inequality in Hilbert Spaces

Let V be a real Hilbert space and

$$V' = \{f : V \to \mathbb{R} | \text{bounded linear}\} \tag{A.9}$$

denotes the dual of it. We set $(,)$ the inner product on V and its norm $\|\cdot\|$, K is a closed convex set in V, and

$$V' \times V \to \mathbb{R}.$$

$$f, x \to \langle f, x \rangle$$

the pairing between V and V'. Let $a(u,v)$ be a bilinear form on $V \times V$ which is bounded, that is,

$$| a(u,v) | \leq C\|u\|\|v\|, \qquad \forall u, v \in V. \tag{A.10}$$

Definition A.6.1. *The bilinear form $a(u,v)$ is coercive on V if there exists $\alpha > 0$ such that*

$$a(u,u) \geq \alpha\|u\|^2, \qquad \forall u \in V. \tag{A.11}$$

Example: $A(u,v) = \int_\Omega \nabla u \cdot \nabla v \ dx, u, v \in V = H_0^1(\Omega)$ in (B.21) is a bounded bilinear form. In fact,

$$\begin{aligned} | A(u,v) | &= \int_\Omega | \nabla u \cdot \nabla v | \ dx \\ &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &= \|u\|\|v\|, \end{aligned}$$

where $\|u\| = \|\nabla u\|_{L^2(\Omega)}$ is a norm of V by Poincaré's Inequality, see Theorem B.1.1.

Proposition A.6.2. *The above $A(u,v)$ is coercive.*

Proof. In fact,

$$A(u,u) = \int_\Omega | \nabla u |^2 \ dx = \frac{1}{2}\int_\Omega | \nabla u |^2 \ dx + \frac{1}{2}\int_\Omega | \nabla u |^2 \ dx.$$

Using Poincaré's inequality, we have $\|u\|^2 \leq \gamma\|\nabla u\|^2, \forall u \in V$ hence

$$\begin{aligned} a(u,u) &\geq \frac{1}{2}\int_\Omega | \nabla u |^2 \ dx + C\|u\|^2 \\ &\geq C'\|u\|^2 \qquad \forall \ u \in V, \end{aligned}$$

where $C = \frac{1}{\gamma}, C' = \min(\frac{1}{2}, \frac{1}{2C})$ are positive constants. □

Theorem A.6.3. *If $K \neq 0$ is a closed convex set in V, and $a(u,v)$ is a symmetric bounded coercive bilinear form in V, there exists a unique solution u of the variational inequality*

$$u \in K, \quad a(u, v-u) \geq \langle f, v-u \rangle, \qquad \forall v \in K. \tag{A.12}$$

Further, the map $f \mapsto u$ is Lipschitz, that is, if u_1, u_2 are solutions to (A.12), the corresponding to $f_1, f_2 \in V'$, then

$$\|u_1 - u_2\|_V \leq \frac{1}{\alpha}\|f_1 - f_2\|_{V'}, \tag{A.13}$$

Proof. The existence of the solution to (A.12) is similar to the obstacle problem in Section 2.4. There exists a solution to the variational problem $\frac{1}{2}\inf_{v \in K} a(u,v)$ which satisfies (A.12). The uniqueness follows from (A.13). Suppose that there exist $u_1, u_2 \in V'$ solution of the variational inequalities

$$u_i \in K: \; a(u_i, v-u_i) \geq \langle f_i, v-u_i \rangle, \qquad \forall v \in K, i = 1,2.$$

Adding $v = u_2$ for u_1 and $v = u_1$ for u_2, we obtain upon adding

$$a(u_1 - u_2, u_1 - u_2) \leq \langle f_1 - f_2, u_1 - u_2 \rangle.$$

Hence by the coerciveness of a,

$$\alpha\|u_1 - u_2\|_V^2 \leq \langle f_1 - f_2, u_1 - u_2 \rangle \leq \|f_1 - f_2\|_{V'} \cdot \|u_1 - u_2\|.$$

Therefore,

$$\|u_1 - u_2\|_V \leq \frac{1}{\alpha}\|f_1 - f_2\|_{V'}.$$

□

Generally, if $a(u,v)$ is symmetric, then a scalar product in V is defined by $((u,v)) = a(u,v)$. Setting $\langle f, v \rangle = ((\tilde{\tilde{f}}, v))$, (A.12) means that the solution u is the projection of $\tilde{\tilde{f}}$ on K. If $a(u,v)$ is not symmetric, we introduce the coercive bilinear form

$$a_t(u,v) = s(u,v) + t\sigma(u,v), \qquad 0 \leq t \leq 1 \tag{A.14}$$

where

$$s(u,v) = \frac{1}{2}(a(u,v) + a(v,u)),$$

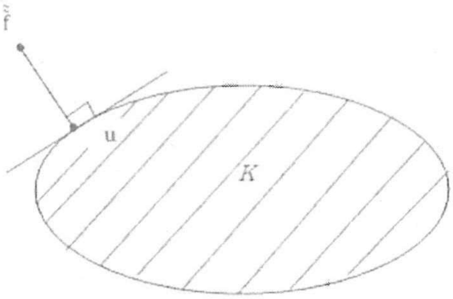

Figure A.1: Projection of $\tilde{\tilde{f}}$ on K with u as the solution.

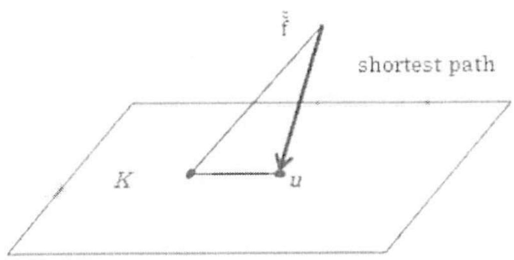

Figure A.2: Projection $\tilde{\tilde{f}}$ on K with shortest path to u.

and

$$\sigma(u,v) = \frac{1}{2}(a(u,v) - a(v,u))$$

which are symmetric and skew-symmetric of parts $a(u,v)$ and set $a_\tau(u,v) = s(u,v) + \tau\sigma(u,v)$ for $a \leq \tau \leq 1$. Since if $t = 0$ the existence of the solution to

$$u \in K, a_\tau(u, v-u) \geq < f, v-u > \quad \forall v \in K \tag{A.15}$$

has been already established, here, we proceed to extend the existence into t-intervals step by step, until $t = 1$, using (A.13) and a contraction mapping theorem.

Definition A.6.4. *Let $F : X \to X$ be a metric space to itself. A point $a \in X$ is called a fixed point of F if $F(a) = a$.*

Definition A.6.5. *Let (X, d_X) and (Y, d_Y) be a metric spaces. A map $\phi : X \to Y$ is called a contraction if there exists a positive number $c < 1$ such that*

$$d_Y(\phi(x), \phi(y)) \leq c d_X(x,y)$$

for all $x, y \in X$.

Theorem A.6.6 (Contraction Mapping Theorem). *Let X be complete metric space and let $F : X \to X$ be a contraction mapping. Then there exist a unique fixed point of F.*

Proof. By the definition, there exists a number $c \in (0,1)$ such that

$$d(\phi(x), \phi(y)) \leq c d(x,y) \tag{A.16}$$

Let $a_0 \in X$ be an arbitrary point. Define a sequence a_n inductively by setting $a_{n+1} = \phi(a_n)$. Here, we claim that $\{a_n\}$ is a Cauchy sequence. First, note that for any $n \geq 1$, from (A.16), $d(a_{n+1}, a_n) = d(\phi(a_n), \phi(a_{n-1})) \leq cd(a_n, a_{n-1})$. Then from the induction,

$$d(a_{n+1}, a_n) \leq c^n d(a_1, a_0) \tag{A.17}$$

for all $n \geq 1$.

$$d(a_{n+1}, a_n) \leq cd(a_{n-1}, a_n) \leq c^2 d(a_{n-2}, a_{n-1}) \leq \ldots \leq c^n(a_n, a_{n-1}).$$

This implies that a_n's are Cauchy's. Therefore, for $m > n \geq 1$,

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, a_{m-1}) + d(a_{m-1}, a_{m-2}) + \ldots + d(a_{n+1}, a_n) \\ &\leq (c^{m-1} + c^{m-2} + \ldots + c^n) d(a_1, a_0) \\ &\leq \frac{c^n}{1-c} d(a_1, a_0). \end{aligned}$$

This show that $d(a_m, a_n) \to 0$ as $n, m \to \infty$.

Since (X, d) is complete, there exists $a \in X$ such that $a_n \to a$. Being a contraction, ϕ is continuous. Hence,

$$\phi(a) = \lim_{n \to \infty} \phi(a) = \lim_{n \to \infty} a_{n+1} = a. \tag{A.18}$$

Thus a is a fixed point of ϕ.

If $b \in X$ is also a fixed point of ϕ, then

$$d(a,b) = d(\phi(a), \phi(b)) \leq c d(a,b)$$

which implies, since $c < 1$, that $d(a,b) = 0$ and hence that $a = b$. Thus the fixed point is unique. □

Suppose that we have already proved the existence for all $0 \leq t \leq t_j$ and set $\tau = t_j$. We rewrite

$$a_t(u, v-u) \geq \langle f, v-u \rangle$$

in the form

$$a_\tau(u, v-u) \geq F_u(v-u)$$

where

$$F_u(v) = \langle f, v \rangle + (t-\tau)\sigma(u,v)$$

In fact,

$$a_\tau(u, v-u) \geq \langle f, v-u \rangle + (t-\tau)\sigma(u, v-u)$$

Taking that $t = \tau$, we get

$$a_\tau(u, v-u) \geq \langle f, v-u \rangle$$

This mapping $v \mapsto F_u(v)$ is a bounded linear functional on V and thus can be written as $\langle f_u, v \rangle, f_u \in V'$.

For any $w \in V$, consider the variational inequality

$$v \in K, \qquad a_\tau(z, v-z) \geq \langle f_w, v-z \rangle, \qquad \forall v \in K. \tag{A.19}$$

From our assumption, (A.19) has a unique solution z. Let $z = Tw$. Given $u_1 = Tw_1$ and $u_2 = Tw_2$,

$$\|Tw_1 - Tw_2\| \leq \frac{1}{\alpha}\|f_{w_1} - f_{w_2}\|_{V'} \leq C|t-\tau|\|w_1 - w_2\|.$$

Taking $|t-\tau| \leq \frac{1}{2C}$, we conclude that T is a contraction mapping in V and admits a unique fixed point u. For this $u = w$,

$$u \in K, \;\; a_t(u, v-u) \geq \langle f, v-u\rangle \quad \text{for } \; v \in K.$$

If t is a finite number of times, and if $t = 1$ then it admits a solution to (A.12).

When the convex set K varies, we formulate it in the setting of Theorem A.6.3 for simplicity. We set condition

$$K_n \text{ are closed convex sets}, \;\; K = w - \lim K_n. \tag{A.20}$$

This condition means that

1. if $x \in K$ then there exist $x_n \in K_n$ with $\|x_n - x\| \to 0$,
2. the weak limit of any sequence $x_{n'} (x_{n'} \in K_{n'})$ is in K.

The condition 2. above is satisfied, for instance, if $K_n \subset K$.

Theorem A.6.7. *Let $a(u,v)$ and K be as in Theorem A.6.3, and let K_n satisfy (A.20). Let $f_n \in V'$, $f_n \to f \in V'$, and denote by u_n the solution of*

$$u_n \in K_n, \;\; a(u_n, v-u_n) \geq \langle f_n, v-u_n\rangle, \qquad \forall v \in K_n. \tag{A.21}$$

Then $u_n \to u$ is weakly convergence in V.

Proof. Let $v \in K_n$

$$\begin{aligned}\alpha\|u_n - v\|^2 &\leq a(u_n - v, u_n - v)\\ &= a(u_n, u_n - v) - a(v, u_n - v)\end{aligned}$$

Thus, from Theorem A.6.3,

$$\begin{aligned}\alpha\|u_n - v\|^2 &\leq \langle f_n, u_n - v\rangle + C\|v\|\|u_n - v\|\\ &\leq (C\|v\| + \|f_n\|_{V'})\|u_n - v\|\end{aligned}$$

By taking $v = v_n \to v^* \in K$, the equation above become $\|u_v\| \leq C$, with C is a constant. Hence, u_n has a weakly convergence. This means that $u_n \to w$ weakly implies that w is the unique solution of (A.12). Since $u \to a(v,u)$ is continuous,

$$a(v, u_n) \to a(v, w), \tag{A.22}$$

By Minty's lemma,

$$a(v_n, v_n - u_n) \geq \langle f_n, v_n - u_n \rangle$$

for any $v \in K$. Take any $v \in K$ and v_n such that $\|v_n - v\| \to 0$, then

$$\begin{aligned}|a(v, v - u_n) - a(v_n, v_n - u_n)| &= |a(v - v_n, v - u_n) + a(v_n, v - u_n)| \\ &\leq C\|v - v_n\|\|v - u_n\| + C\|v_n\|\|v - u_n\| \\ &\leq C\|v - u_n\| \qquad \to 0.\end{aligned}$$

It follows that

$$a(v, v - u_n) \geq \langle f_n, v_n - u_n \rangle + \varepsilon_n, \qquad \varepsilon_n \to 0.$$

Therefore,

$$\langle f_n, v_n - u_n \rangle \to \langle f, v - w \rangle$$

and using (A.22) we obtain,

$$a(v, v - w) \geq \langle f, v - w \rangle, \qquad \forall v \in K.$$

Since $w \in K$, w is the unique solution of the variational inequality (A.12) □

Appendix B

Elliptic Problems and Calculus of Variation

B.1 Laplace Equation

Dirichlet Principle

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and

$$J(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 \; dx, \tag{B.1}$$

where $v \in H^1(\Omega)$. Given $u^0 \in H^1(\Omega)$, we want to attain

$$\min F(v), \qquad v \in H^1(\Omega), \qquad v = u^0 \text{ on } \partial\Omega. \tag{B.2}$$

If u attains this minimum, then it is a weak solution to

$$\Delta u = 0 \qquad on \ \Omega, \qquad u = u^0 \text{ on } \partial\Omega, \tag{B.3}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots \frac{\partial^2}{\partial x_n^2}$, $x = (x_1, x_2, \dots, x_n) \in \Omega$ is the Laplacian.
To see this, let $v \in H_0^1(\Omega)$ be arbitrary, where

$$H_0^1(\Omega) = \{v \in H^1(\Omega) | v = 0 \text{ on } \partial\Omega\}.$$

Then $w = u + \varepsilon v \in H^1(\Omega)$, $w = u^0$ on $\partial\Omega$, where $\varepsilon \in \mathbb{R}$ is arbitrary. Since u attains (B.2), it holds that $F(u) \leq F(w)$, or equivalently,

$$F(u+\varepsilon v) \geq F(u), \qquad \forall \varepsilon \in \mathbb{R}.$$

In particular, the function

$$\varepsilon \mapsto f(\varepsilon) = F(u+\varepsilon v).$$

attains the minimum at $\varepsilon = 0$, so that

$$0 = \frac{d}{d\varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0}.$$

Here,

$$\begin{aligned} f(\varepsilon) &= F(u+\varepsilon v) \\ &= \frac{1}{2}\int_\Omega |\nabla(u+\varepsilon v)|^2 \; dx \\ &= \frac{1}{2}\int_\Omega |\nabla u + \varepsilon \nabla v|^2 \; dx \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2 \; dx \end{aligned}$$

and hence

$$f'(\varepsilon) = \int_\Omega \nabla \cdot \nabla v + \varepsilon |\nabla v|^2 \; dx.$$

In particular

$$0 = \frac{d}{d\varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0} = f'(0) = \int_\Omega \nabla u \cdot \nabla v \; dx$$

i.e.

$$\int_\Omega \nabla u \cdot \nabla v \; dx = 0, \qquad \forall v \in H_0^1(\Omega), \qquad u = u^0 \;\text{ on } \partial\Omega. \tag{B.4}$$

This means that u is a *weak solution* to (2). In fact, if $u = u(x)$ is sufficiently regular on $\bar{\Omega}$, then

$$\int_\Omega \nabla u \cdot \nabla v \; dx = -\int_\Omega \Delta u \cdot v \; dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot v \; ds = \int_\Omega \Delta u \cdot v \; dx = 0 \tag{B.5}$$

for $v \in H_0^1(\Omega)$, where ν denotes the unit normal vector on $\partial\Omega$. Since $v \in H_0^1(\Omega)$ is arbitrary, u solves $\Delta u = 0$ in Ω at (B.2) if u is regular.

Trace to the Boundary

We have seen that, if u attains (B.2), then it is a weak solution to (B.3). Here, we will have the following problem:

1. Existence of the solution for variation of problem (B.2).

2. The regularity of u, the minimizer of (B.2).

For the first problem, we note that $H^1(\Omega)$ with norm

$$\|v\|_{H^1} = \left\{ \int_\Omega (|\nabla v|^2 + |v|^2) dx \right\}^{\frac{1}{2}} \tag{B.6}$$

is a Hilbert space. Every element in $H^1(\Omega)$ may take the boundary value. For example, if Ω satisfies the "restricted cone property" the operator,

$$\gamma : v \in C(\overline{\Omega}) \quad \mapsto \quad v|_{\partial\Omega} \in C(\partial\Omega) \tag{B.7}$$

is extended as a bounded linear operator: $H^1(\Omega) \to L^2(\partial\Omega)$ denoted by the symbol γ. It is called the trace operator.

Proof. We shall examine this property for the case $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N, x_N > 0\}$, a half space.
In fact, let $v \in C_0^\infty(\overline{\mathbb{R}^N_+})$. With the notation $x = (x', x_N)$, we have

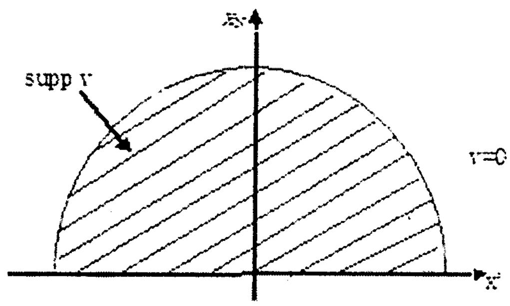

Figure B.1: Trace

$$|v(x',0)|^2 = -\int_0^{+\infty} \frac{\partial}{\partial x_N} v(x',x_N)^2 dx_N = -2\int_0^{+\infty} v(x',x_N)\frac{\partial v}{\partial x_N}(x',x_N)\ dx_N.$$

Using the inequality $2ab \le a^2 + b^2$, we obtain

$$|v(x',0)|^2 \le \int_0^{+\infty} (|v(x',x_N)|^2 + |\frac{\partial v}{\partial x_N}(x',x_N)|^2)\ dx_N.$$

By integration in x',

$$\int_{R^{N-1}} |v(x',0)|^2 dx' \le \int_{R^N_+} (|v(x)|^2 + |\frac{\partial v}{\partial x_N}(x)|^2)\ dx$$

that is $\|v\|_{L^2(\partial\mathbb{R}^N_+)} \le \|v\|_{H^1(\mathbb{R}^N_+)}$. By the density of $C_0^\infty(\overline{\mathbb{R}}^N_+)$ in $H^1(\mathbb{R}^N_+)$, therefore, the result has been shown. □

Conversely, given a function $u^0(\xi)$ on $\partial\Omega$. Its extension over $\overline{\Omega}$ will be useful. For simplicity, we assume that $u^0(\xi)$, $\xi \in \partial\Omega$ is a trace of an element in $H^1(\Omega)$, denoted by the same symbol u^0. A domain Ω is called Lipschitz if its boundary $\partial\Omega$ is Lipschitz continuous. The Lipschitz domain can take a corner on the boundary, but it satisfies the restricted cone property. If Ω is a Lipschitz domain, then,

$$v \in H_0^1(\Omega) \qquad \Leftrightarrow \qquad v \in H_0^1(\Omega), \quad \gamma v = 0, \tag{B.8}$$

where $H_0^1(\Omega)$ is a Hilbert space defined as a closure $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

Poincaré Inequality

Henceforth, $\|\cdot\|_p$ is a standard norm $L^p, 1 \leq p \leq \infty$.

Theorem B.1.1. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain then there is $C = C(\Omega) > 0$ such that*

$$\|v\|_2 \leq C\|\nabla v\|_2 \qquad (v \in H_0^1(\Omega)). \tag{B.9}$$

Proof. It suffices to show (B.9) for $v \in C_0^\infty(\Omega)$. In fact by the definition, for every $v \in H_0^1(\Omega)$. There is a sequence $\{v_k\} \subset C_0^\infty(\Omega)$ satisfy

$$\|v_k - v\| \to 0,$$

Recall (B.6). In particular,

$$\|\nabla v_k\|_2 \to \|\nabla v\|_2, \qquad \|v_k\|_2 \to \|v\|_2.$$

Then (B.9) will follow from

$$\|v_k\|_2 \leq C\|\nabla v_k\|_2$$

with $C > 0$ independent of $k = 1, 2, \cdots$.

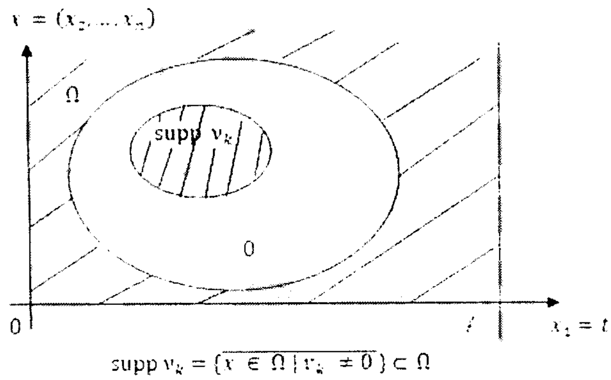

Figure B.2: Support of v_k

Let $v \in C_0^\infty(\Omega)$. Since Ω is bounded there exists a constant $\ell > 0$:

$$\Omega \subset \{x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n | 0 < x_1 < \ell\}.$$

Define $v = 0$ outside Ω. For any $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $1 \le p < \infty$, it holds that $v(x) = \int_0^{x_1} D_1 v(t, x_2, \cdots, x_n) dt$ and there,

$$\begin{aligned}
|v(x)|^p &= |\int_0^{x_1} D_1 v(t, x_2, \ldots, x_n)\ dt|^p \\
&\le (\int_0^\ell |D_1 v(t, x_2, \ldots, x_n)|\ dt)^p \\
&\le \int_0^\ell |D_1 v(t, x_2, \ldots, x_n)|^p\ dt \cdot \ell^{\frac{p}{p'}} \quad \text{where} \frac{p}{p'} = p - 1 \\
&= \ell^{p-1} \cdot \int_0^\ell |D_1 v(t, x_2, \ldots, x_n)|^p\ dt.
\end{aligned}$$

Integrating over Ω and putting $p = 2$,

$$\begin{aligned}
\|v\|_2^2 &= \int_\Omega |v(x)|^2\ dx \\
&= \int_{\mathbb{R}^n} |v(x)|^2\ dx \\
&\le \ell \int_0^\ell\ dt \int_{\mathbb{R}^n} |D_1 v(t, x_2, \ldots, x_n)|^2\ dx \\
&= \ell \int_0^t dt \cdot \int_0^\ell dx_1 \cdot \int_{\mathbb{R}^{n-1}} |D_1 v(t, x_2, \cdots, x_n)|^2 dx_2 \ldots dx_n \\
&= \ell^2 \int_0^\ell\ dt \int_{\mathbb{R}^{n-1}} |D_1 v(t, x_1, x_2, \ldots, x_n)|^2\ dx \\
&= \ell^2 \|D_1 v\|^2 \\
&\le \ell^2 \|\nabla v\|^2.
\end{aligned}$$

Therefore, (B.9) is shown for $v \in C_0^\infty(\Omega)$. □

Existence of the Weak Solution

We shall show the existence of the weak solution to (B.3) by the Dirichlet principle. Let Ω be a Lipschitz domain with boundary $\partial\Omega$ and $u^0 \in H^1(\Omega)$.

Theorem B.1.2. *The variational problem (B.2), i.e.* $\min_{v \in E} J(v)$,

$$E = \{v \in H^1(\Omega) | v = u^0 \ \ on \ \ \partial\Omega\} \tag{B.10}$$

is attained.

Proof. Since $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx \ge 0$, there is $\{v_k\} \subset E, k = 1, 2, \ldots$ such that $J(v_k) \to j = \inf_{v \in E} J(v)$. Since

$v_k - u^0 \in H_0^1(\Omega)$, we have

$$\|v_k - u^0\|_2 \leq C\|\nabla(v_k - u^0)\|_2 \tag{B.11}$$

with a constant $C = C(\Omega) > 0$.

Therefore

$$\|v_k\|_2 \leq \|u^0\|_2 + C\{\|\nabla v_k\|_2 + \|\nabla u^0\|_2\} \leq C_1 \tag{B.12}$$

by $J(v_k) \leq C_2$ where C_1, C_2 are constant. This inequality implies $\|v_k\|_2 + \|\nabla v_k\|_2 \leq C_2$ with a constant C_2, and hence $\{v_k\} \subset E \subset H^1(\Omega)$ is bounded.

There exists a subsequence, denoted by the same symbol, and $u \in H^1(\Omega)$ such that

$$v_k \rightharpoonup u \quad in\ H^1(\Omega). \tag{B.13}$$

Since the operator $\gamma : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$ is continuous, (B.13) implies $v_k \mid_{\partial\Omega} \rightharpoonup u \mid_{\partial\Omega}$. Therefore, $u = u^0$ on $\partial\Omega$ by $v_k \in E$. Thus we obtain $u \in E$. Since

$$v \in H^1(\Omega) \longmapsto J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx. \tag{B.14}$$

is weakly lower semi-continuous, (B.13) implies

$$\varliminf_{k\to\infty} J(v_k) \geq J(u). \tag{B.15}$$

Thus $J[u] = \inf_{v\in E} J[v]$ and hence this minimum is attained by $v = u$. □

B.2 Poisson Equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. We consider

$$-\Delta u = f(x) \in L^2(\Omega) \tag{B.16}$$

in Ω. The Dirichlet problem for (B.16) with homogeneous boundary condition

$$u \mid_{\partial\Omega} = 0 \tag{B.17}$$

is further imposed.

Assume $u \in C^2(\Omega)$ is a solution of (B.16), and $v \in C_0^\infty(\Omega)$ is arbitrary. We multiply v and integrate over Ω:

$$-\int_\Omega \Delta u \cdot v \; dx = \int_\Omega fv \; dx. \tag{B.18}$$

By integrating by parts the left hand side of (B.18),

$$-\int_\Omega \Delta u \cdot v \; dx = \int_\Omega \nabla u \cdot \nabla v \; dx = -\int_\Omega u \cdot \Delta v \; dx.$$

The equation (B.18) then becomes

$$-\int_\Omega u \cdot \Delta v \; dx = \int_\Omega fv \; dx \tag{B.19}$$

which denotes that u is a solution to (B.16) in the distributional sense.

Denote that (f,v) and $A(u,v)$ the inner product of L^2 and the Dirichlet form, respectively,

$$(f,v) = \int_\Omega fv \; dx, \tag{B.20}$$

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx. \tag{B.21}$$

We say that $u \in H_0^1(\Omega)$ a weak solution to (B.16)-(B.17) if

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx = (f,v) \quad \forall v \in H_0^1(\Omega) \tag{B.22}$$

We shall use Riesz's representation theorem to guarantee the weak solution to (B.16)-(B.17).

Theorem B.2.1. *Given $f \in L^2(\Omega)$, we have a unique weak solution to (B.16)-(B.17).*

Proof. We show that

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx$$

casts the inner product on $H_0^1(\Omega)$. It suffices to note that by the Poincaré inequality, $A(u,u) = 0$ in $H_0^1(\Omega)$ implies that $u = 0$.

Clearly, for any $f \in L^2(\Omega)$,

$$F(v) = \int_\Omega fv \; dx, \qquad v \in H_0^1(\Omega)$$

is a bounded linear functional in $H_0^1(\Omega)$ by $|F(v)| \leq \|f\|_2 \|v\|_2 \leq C\|f\| \|\nabla v\|_2$. From Theorem B.2.1, there exists a unique $u \in H_0^1(\Omega)$ such that

$$A(u,v) = F(v) = \int_\Omega fv \ dx, \qquad \forall v \in H_0^1(\Omega)$$

i.e.

$$\int_\Omega \nabla u \cdot \nabla v \ dx = \int_\Omega fv \ dx, \qquad \forall v \in H_0^1(\Omega)$$

This shows the unique existence of the weak solution of Dirichlet problem of (B.16) and (B.17). □

Instead of Riesz representation theorem, we can show the existence of the solution to the Poisson equation by the direct method of calculus of variation. Let

$$J[v] = \frac{1}{2}\int_\Omega |\nabla v|^2 \ dx - \int_\Omega fv \ dx \tag{B.23}$$

If $u \in H_0^1(\Omega)$ attain the minimum of $J[v]$ in $H_0^1(\Omega)$, then for any $\varphi \in H_0^1(\Omega)$, as for a function of ε,

$$\begin{aligned} F(\varepsilon) &= J[u+\varepsilon v] \\ &= \frac{1}{2}\int_\Omega |\nabla(u+\varepsilon v)|^2 \ dx - \int_\Omega f(u+\varepsilon v) \ dx \\ &= \frac{1}{2}\int_\Omega^2 |\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2 \ dx - \int_\Omega f(u+\varepsilon v) \ dx \end{aligned}$$

achieves its minimum at $\varepsilon = 0$ and hence $F'(0) = 0$. Since

$$F'(\varepsilon) = \int_\Omega (\nabla u + \varepsilon \nabla v) \cdot \nabla v \ dx - \int_\Omega fv \ dx.$$

$$F'(0) = \int_\Omega \nabla u \cdot \nabla v \ dx - \int_\Omega fv \ dx$$

$F'(0) = 0$ implies (B.19) for any $v \in H_0^1(\Omega)$. We say that $u \in H_0^1(\Omega)$ is an extremal of $J[v]$ if $\frac{d}{d\varepsilon}F(u+\varepsilon v)\Big|_{\varepsilon=0} = 0$ for any $v \in H_0^1(\Omega)$.

Proposition B.2.2. *If $u \in H_0^1(\Omega)$ is an extremal of the functional $J[v]$ in $H_0^1(\Omega)$, then u is a weak solution of the Dirichlet problem of B.16 and B.17.*

To proof the existence of a minimizer of the corresponding functional is as follows.

Lemma B.2.3. *For any $f \in L^2(\Omega)$, the functional $J[v]$ is bounded from below in $H_0^1(\Omega)$.*

Proof. By Poincaré's inequality Theorem B.2.1 and Cauchy inequality with $\varepsilon > 0$, for any $v \in H_0^1(\Omega)$

$$J[v] = \frac{1}{2\mu}\int_\Omega v^2\ dx - \int_\Omega \left(\frac{\varepsilon}{2}v^2 + \frac{1}{2\varepsilon}f^2\right)\ dx$$
$$= \frac{1}{2}\left(\frac{1}{\mu} - \varepsilon\right)\int_\Omega v^2\ dx - \frac{1}{2\varepsilon}\int_\Omega f^2\ dx$$

where $\mu > 0$ is the Poincaré constant:

$$\mu\|\nabla v\|_2^2 \geq \|v\|_2^2, \qquad v \in H_0^1(\Omega).$$

Taking $\varepsilon > 0$ such that $\varepsilon \leq \frac{1}{\mu}$, therefore

$$J[v] \geq -\frac{1}{2\varepsilon}\int_\Omega f^2\ dx$$

and the boundedness from below of $J[v]$ in $H_0^1(\Omega)$ is proved. □

By Lemma B.2.3, $\inf\limits_{v\in H_0^1(\Omega)} J[v]$ is a finite number. Then it implies that there exists $u_k \in H_0^1(\Omega)$ such that

$$\lim_{k\to\infty} J[u_k] = \inf_{H_0^1(\Omega)} J[v]$$

This $\{u_k\}$ is called a minimizing sequence of $J[v] \in H_0^1(\Omega)$. The existence of the limit $\lim\limits_{k\to\infty} J[u_k]$ implies the boundedness of $J[u_k]$, i.e. for some constant M,

$$|J[u_k]| \leq M, \qquad k = 1,2,\ldots$$

Lemma B.2.4. *For any $v \in H_0^1$ and $f \in L^2(\Omega)$,*

$$\int_\Omega |\nabla v|^2\ dx \leq 4\mu\int_\Omega f^2\ dx + 4J[v] \tag{B.24}$$

$$\int_\Omega v^2\ dx \leq 4\mu^2\int_\Omega f^2\ dx + 4\mu J[v] \tag{B.25}$$

where $\mu > 0$ is the Poincaré constant.

Proof. Using the Cauchy inequality and the Poincaré inequality,

$$\int_\Omega |\nabla v|^2\ dx \leq \varepsilon\int_\Omega (v^2 + \frac{1}{\varepsilon}f^2)\ dx + 2J[v]$$
$$\leq \varepsilon\mu\int_\Omega |\nabla v|^2\ dx + \frac{1}{\varepsilon}\int_\Omega f^2 + 2J[v],$$

where $\varepsilon > 0$. Choosing $\varepsilon = \frac{1}{2\mu}$,

$$\int_\Omega |\nabla v|^2 \, dx \le \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx + 2\mu \int_\Omega f^2 \, dx + 2J[v]$$
$$\int_\Omega |\nabla v|^2 \, dx \le 4\mu \int_\Omega f^2 \, dx + 4J[v].$$

Using the Poincaré inequality, we have

$$\int_\Omega |\nabla v|^2 \, dx \le 4\mu^2 \int_\Omega f^2 \, dx + 4\mu J[v]$$
$$\int_\Omega v^2 \, dx \le 4\mu^2 \int_\Omega f^2 \, dx + 4\mu J[v].$$

□

From these two inequalities, it can be shown that $\{u_k\}$ is bounded in $H_0^1(\Omega)$ i.e. $\{u_k\}$ and $\{\nabla u_k\}$ are bounded in $L^2(\Omega)$, which implies the existence of a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ and a function $u \in H_0^1(\Omega)$ such that $u_{k_i} \rightharpoonup u$, $\nabla u_{k_i} \rightharpoonup \nabla u \ \ (i \to \infty)$ in $L^2(\Omega)$. In particular,

$$\lim_{i\to\infty} \int_\Omega f u_{k_i} \, dx = \int_\Omega f u \, dx.$$

Moreover from

$$\int_\Omega |\nabla (u_{k_i} - u)|^2 \, dx \ge 0$$

i.e.

$$\int_\Omega |\nabla u_{k_i}|^2 \, dx - 2\int_\Omega \nabla u_{k_i} \cdot \nabla u \, dx + \int_\Omega |\nabla u|^2 \, dx \ge 0$$
$$\int_\Omega |\nabla u_{k_i}|^2 \, dx \ge 2\int_\Omega \nabla u_{k_i} \cdot \nabla u \, dx - \int_\Omega |\nabla u|^2 \, dx$$

it follows

$$\begin{aligned} \underline{\lim}_{i\to\infty} \int_\Omega |\nabla u_{k_i}|^2 \, dx &\ge 2\lim_{i\to\infty} \int_\Omega \nabla u_{k_i} \cdot \nabla u \, dx - \int_\Omega |\nabla u|^2 \, dx \\ &= 2\int_\Omega |\nabla u|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx \\ &= \int_\Omega |\nabla u|^2 \, dx \end{aligned}$$

from $\nabla u_{k_i} \rightharpoonup \nabla u$.

So,

$$\begin{aligned} \underline{\lim}_{i\to\infty} J[u_{k_i}] &= \frac{1}{2}\underline{\lim}_{i\to\infty} \int_\Omega |\nabla u_{k_i}|^2 \, dx - \lim_{i\to\infty} \int_\Omega f u_{k_i} \, dx \\ &\ge \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx \\ &= J[u] \end{aligned}$$

Hence,

$$\inf_{H_0^1(\Omega)} J[v] \leq \underline{\lim}_{i\to\infty} J[u_{k_i}] = \lim_{i\to\infty} J[u_{k_i}] = \inf_{H_0^1(\Omega)} J[v]$$

Thus $J[u] = \inf_{H_0^1(\Omega)} J[v]$ i.e. u is a minimizer of $J[v]$ in $H_0^1(\Omega)$.

Proposition B.2.5. *For any $f \in L^2(\Omega)$, the function $J[v]$ admits a minimum in $H_0^1(\Omega)$. Combining Propositions B.2.1 and Lemma B.2.4, we obtain again the existence of the weak solution to (B.16) and (B.17).*

Uniqueness of the weak solution can be shown in the following way. Let $u_1, u_2 \in H_0^1(\Omega)$ be weak solutions of (B.16) and (B.17). Then by definition of weak solutions,

$$\int_\Omega \nabla u_i \cdot \nabla v \; dx = \int_\Omega fv \; dx, \qquad v \in C_0^\infty(\Omega) \;\; (i = 1,2).$$

Hence

$$\int_\Omega \nabla u_i \cdot \nabla v \; dx = \int_\Omega fv \; dx, \qquad v \in H_0^1(\Omega) \;\; (i = 1,2)$$

Denote $u = u_1 - u_2$. Then,

$$\int_\Omega \nabla u \cdot \nabla v \; dx = 0, \qquad v \in H_0^1(\Omega).$$

Therefore,

$$\int_\Omega \nabla u \cdot \nabla u \; dx = \int_\Omega |\nabla u|^2 = 0$$

Thus, $\nabla u = 0$ in Ω and using the homogeneous boundary value conditions yields $u = 0$ in Ω. □

B.3 $W^{2,p}$ Regularity for the General Elliptic Problem

Hölder Continuity

Denote by $C^\alpha(\overline{\Omega})$ where $0 < \alpha < 1$, Ω is an open set in $\mathbb{R}^n$, the space of functions which are Hölder continuous with exponent α, that is, $u \in C^\alpha(\overline{\Omega})$ if and only if

$$H_\alpha(u) \equiv \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty. \tag{B.26}$$

This $C^\alpha(\overline{\Omega})$ is a Banach space with the norm

$$\|u\|_\alpha = \|u\|_\infty + H_\alpha(u),$$

where $\|u\|_\infty$ is the maximum norm of $u(x)$

$$\|u\|_\infty = \sup_{x\in\Omega} |u(x)|.$$

If (B.26) is finite for $\alpha = 1$, then u is called a *Lipschitz function*: $|u(x) - u(y)| \leq C|x-y|$, where C is a constant. Similarly, $u \in C^{m+\alpha}(\overline{\Omega})$ (m is a positive integer) if

$$\|u\|_{m+\alpha} \equiv \|u\|_m + \sum_{|\beta|=m} H_\alpha(D^\beta u) < \infty,$$

where

$$\|u\|_m = \sum_{|\beta|\leq m} \|D^\beta u\|_\infty.$$

Here, $D^\beta = (\frac{\partial}{\partial x_1})^{\beta_1}, \ldots, (\frac{\partial}{\partial x_n})^{\beta_n}$, $\beta = (\beta_1, \ldots, \beta_n)$ is the multi index, and $|\beta| = \beta_1 + \cdots + \beta_n$.

Consider the operator

$$Au \equiv -\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \qquad a_{ij}(x) = a_{ji}(x). \tag{B.27}$$

Definition B.3.1. *A is said to be elliptic in Ω if its coefficient matrix $(a_{ij}(x))$ is positive definite as a symmetric matrix, that is if*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda_x|\xi|^2 \qquad (\lambda_x > 0) \tag{B.28}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$; it is uniformly elliptic if $\lambda_x > \exists\lambda > 0$ for all $x \in \Omega$.

Excercise 1. $y(u) = x_+^{\frac{1}{2}}$ is a Hölder continuous with exponent $\alpha = \frac{1}{2}$ on $x \in [-1, 1]$ the graph is,

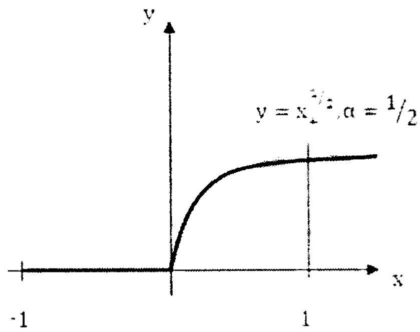

Figure B.3: Graph $y = x_+^{\frac{1}{2}}$.

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < +\infty. \tag{B.29}$$

but for the case of $\alpha > \frac{1}{2}$,

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} = +\infty. \tag{B.30}$$

Exercise 2. $\Delta = \frac{\partial^2}{\partial x_0^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ in $\mathbb{R}^n$ is uniformly elliptic:

Δ is a uniformly elliptic operator is replaced by $Au = a_{ij}u_{ij}$, where $a_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$. A is elliptic at a point $x \in \Omega$ if the coefficient matrix $[a_{ij}]$ is positive; that is if $\lambda_{\min}$ in (B.28) denote as minimum eigenvalues of $[a_{ij}]$, $\lambda_{\min} > 0$. In this case we have $\alpha_{\min} = 1 \equiv \lambda$. Then, λ is bounded on Ω and a_{ij} is bounded. Therefore, A is uniformly elliptic in Ω.

Schauder's Boundary Estimates

Suppose that $\partial\Omega$ is locally $C^{2+\alpha}$, $f \in C^{\alpha}(\overline{\Omega})$, $\phi \in C^{2+\alpha}(\overline{\Omega})$, $0 < \alpha < 1$ and

$$\sum \|a_{ij}\|_\alpha + \sum \|b_i\|_\alpha + \|c\|_\alpha \leq K,$$

$$\sum a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in R^n \;\; (\lambda > 0).$$

If $u \in C^{2+\alpha}(\overline{\Omega})$ and

$$Au = f \qquad in \;\; \Omega,$$

$$u = \phi \qquad in \;\; \partial\Omega,$$

then

$$\|u\|_{2+\alpha} \leq C(\|f\|_\alpha + \|u\|_\infty + \|\phi\|_{2+\alpha})$$

where C is a constant depending only on λ, K and Ω.

The Schauder interior estimates involve norms $\overline{\|u\|}_{m+\alpha}$ which defined as follows:

$$\overline{H}_{j,\alpha}(u) = \sup_{x,y \in \Omega} d_{x,y}^{j+\alpha} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \qquad \overline{H}_\alpha = \overline{H}_{0,\alpha},$$

where $d_x = \text{dist}(x, \partial\Omega)$, $d_{x,y} = \min(d_x, d_y)$,

$$\overline{\|u\|}_\alpha = \|u\|_\infty + \overline{H}_\alpha(u),$$

$$\overline{\|u\|}_{m+\alpha} = \sum_{|\beta| \leq m} \sup_\Omega |d_x^{|\beta|} D^\beta u(x)| + \sum_{|\beta| = m} \overline{H}_{m,\alpha}(D^\alpha u).$$

If $\overline{\|u\|}_{m+\alpha} < \infty$, then we say that u belongs to $\overline{C}^{m+\alpha}(\Omega)$.

Schauder's Interior Estimates

Theorem B.3.2. *Let Ω be an open subset of $\mathbb{R}^n$ with domain diameter $\leq D$, $f \in \overline{C}^\alpha(\Omega)$ be a bounded solution and a_{ij}, b_{ij}, c are measurable*

$$\sum \overline{\|a_{ij}\|}_\alpha + \sum \overline{\|b_i\|}_\alpha + \overline{\|c\|}_\alpha \leq K,$$

$$\sum a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega,\ \ \xi \in R^n\ \ (\lambda > 0).$$

Let $u \in C^2(\Omega) \cap L^\infty(\Omega)$ be a bounded solution in Ω of

$$Au = f \qquad \text{in } \Omega,$$

then

$$\overline{\|u\|}_{2+\alpha} \leq C(\overline{\|f\|}_\alpha + \|u\|_\infty),$$

where C is depending only λ, K, and D.

If $u \in C^{m+\alpha}(\overline{\Omega}_0)$ for any open subset Ω_0 with $\overline{\Omega}_0 \subset \Omega$. Therefore, $u \in C^{m+\alpha}$. Then it can be deduced that if a_{ij}, b_i, c and f belong to $C^{m+\alpha}$, then u belongs to $C^{m+2+\alpha}(\Omega)$. L^p estimates where $1 < p < \infty$ is needed here with assumptions on A, f is weaker:

$$|a_{ij}(x) - a_{ij}(y)| \leq \omega(|x-y|) \qquad [\omega(t) \to 0 \text{ if } t \to 0], \tag{B.31}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \ \ \forall x \in \Omega, \quad \xi \in R^n \ \ (\lambda > 0) \tag{B.32}$$

$$\sum |a_{ij}| + \sum |b_i| + |c| \leq K. \tag{B.33}$$

The L^p Estimates

Theorem B.3.3. *Let $\partial\Omega$ belong to C^2 locally, $f \in L^p(\Omega)$, $\phi \in W^{2,p}(\Omega)$, $1 < p < \infty$. If $u \in W^{2,p}(\Omega)$,*

$$Au = f \qquad \text{in } \Omega,$$

$$u - \phi \in H_0^1(\Omega),$$

then

$$|u|_{2,p} \leq C(|f|_p + |\phi|_{2,p}),$$

where C is constant depending only on λ, K, the modulus of continuity ω and domain Ω.

The interior L^p elliptic estimates with G is any compact subdomain of Ω and C is a constant depending only on λ, K, ω, Ω and G are in the form of

$$|u|_{2,p}^G \leq C(|f|_p + |u|_p),$$

$|u|_{2,p}^G$ stands for $W^{2,p}$ norm of u in G. If $\partial G \cap \partial\Omega$ is nonempty, we need local L^p estimates in a subdomain G of Ω.

Theorem B.3.4. *Suppose that G_1 is an open set, $G \subset G_1 \subset \Omega$, $\partial G \cap \partial\Omega$ is contained in the interior of $\partial G_1 \cap \partial\Omega$, $\partial G \cap \Omega \subset G_1$. If*

$$Au = f \qquad \text{in } G_1,$$

$$\zeta(u-\phi) \in H_0^1(G_1)$$

for any $\zeta \in C^\infty(R^n)$, $\zeta = 0$ in a neighbourhood of $\partial G_1 \cap \Omega$, then

$$|u|_{2,p}^{G} \le C(|f|_p^{G_1} + |u|_p^{G_1} + |\phi|_{2,p}^{G_1})$$

Strong Maximum Principle

Let u be a function in $H^2(\Omega) \cap C(\Omega)$ satisfying $Au \le 0$ in Ω. If at some point x^0 in Ω, u assumes to be positive maximum, then $u \equiv$ constant in Ω (then $c = 0$ a.e.). If u is not necessarily continuous in Ω; "maximum" of u is replaced by "essential supremum" ($ess\sup$) of u: If $ess\sup_\Omega u$ is positive and coincides with $ess\sup_B u$ for any ball center x_0 and arbitrarily small radius, then $u = const$. This implies:

$$\text{If } u \in H^2(\Omega) \cap H_0^1(\Omega), \quad Au \le 0 \text{ a.e. in } \Omega \text{, then } u \le 0 \text{ a.e. in } \Omega. \tag{B.34}$$

The Schauder boundary estimates can be used to solve the Dirichlet Problem

Theorem B.3.5. *Let Ω be a $C^{2+\alpha}$ domain in $\mathbb{R}^n$, and let A be strictly elliptic in Ω with coefficients in $C^\alpha(\overline{\Omega})$ and with $c \ge 0$, then,*

$$\begin{aligned} Au &= f && \text{in } \Omega \\ u &= \phi && \text{on } \partial\Omega, \end{aligned} \tag{B.35}$$

where A, f, ϕ and Ω are as in the statement of the estimates have a unique $C^{2+\alpha}(\overline{\Omega})$ solution for all such f and ϕ.

L^p estimates also can be used to solve Dirichlet problem.

B.4 Obstacle Problem

Here, we consider an example of the variational inequality.

Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega fu \, dx \tag{B.36}$$

and the closed convex set in $H^1(\Omega)$,

$$K = \{u; u - u^0 \in H_0^1(\Omega), u \ge \phi \text{ a.e.}\} \tag{B.37}$$

where $f \in L^2(\Omega)$, $\phi(x)$ is a continuous function in $\overline{\Omega}$, and $u^0 \in H^1(\Omega)$. Here we note the following lemma.

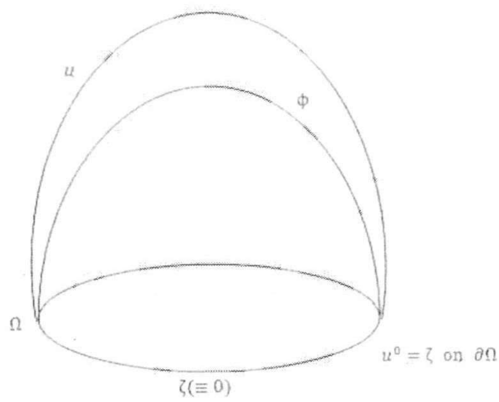

Figure B.4: Obstacle problem.

Lemma B.4.1. *K is closed and convex in* $H^1(\Omega)$.

Proof. To show that K is closed, suppose that $u_j \to u \in H^1(\Omega)$ for $u_j \in K$. Since the boundary $\partial\Omega$ is sufficiently regular, we can uniquely define the trace γu of $u \in H^1(\Omega)$ on $\partial\Omega$, and $\gamma u \in H^{\frac{1}{2}}(\partial\Omega)$. The mapping $u \to \gamma u$ is linear continuous: $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$. By $\zeta = \gamma u^0$, the set K is equal to:

$$K = \{u \mid u \in H^1(\Omega),\ \ \gamma u = \zeta, u \geq \phi \text{ a.e. in } \Omega\}.$$

Since $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is continuous, $u_j \to u$ in $H^1(\Omega)$ implies $\gamma u_j \to \gamma u$ in $H^{\frac{1}{2}}(\partial\Omega)$. But $\gamma u_j = \zeta$ so that $\gamma u = \zeta$ follows. Thus we have only to show $u \geq \phi$ a.e. in Ω. In fact, $u_j \to u$ in $L^2(\Omega)$ by $u_j \to u$ in $H^1(\Omega)$, therefore there exists $\{u'_j\} \subset \{u_j\}$ such that $u'_j \to u$ a.e. in Ω. We shall write $u'_j = u_j$ for simplicity. Since $u_j \geq \phi$ a.e. in Ω, there exists a Lebesgue measureable set $A_j, \mid A_j \mid = 0$ such that,

$$u_j(x) \geq \phi(x), \qquad\qquad \forall x \in \Omega \setminus A_j.$$

While for $u'_j \to u$ a.e. in Ω, there exists $A_0, \mid A_0 \mid = 0$, such that

$$u_j(x) \to u(x), \qquad\qquad \forall x \in \Omega \setminus A_0.$$

We put $A = \cup_{j=1}^{\infty} A_j \cup A_0$ which implies $\mid A \mid \leq \sum_{j=1}^{\infty} \mid A_j \mid + \mid A_0 \mid = 0$. Here, $\mid A \mid = 0$. We obtain,

$$u_j(x) \geq \phi(x), \qquad\qquad \forall x \in \Omega \setminus A$$

and

$$u_j(x) \to u(x), \qquad \forall x \in \Omega \setminus A.$$

Then it follows that, $u(x) \geq \phi(x), \forall x \in \Omega \setminus A$. Therefore $u \geq \phi$ a.e. in Ω.

To show that K is convex, we consider $0 \leq \varepsilon \leq 1$, take $u, v \in K$, and put $w = \varepsilon v + (1-\varepsilon)u$. Then $\varepsilon\zeta + (1-\varepsilon)\zeta \in H^1(\Omega)$ as $H^1(\Omega)$ is a vector space. Then it follows that

$$\gamma w = \varepsilon \gamma v + (1-\varepsilon)\gamma u = \varepsilon\zeta + (1-\varepsilon)\zeta = \zeta$$

while

$$\varepsilon v + (1-\varepsilon)u \geq \varepsilon\phi + (1-\varepsilon)\phi = \phi \qquad \text{a.e. on } \Omega$$

because $u, v \geq \phi$ a.e. in Ω. Thus $w = \varepsilon v + (1-\varepsilon)u \in K$ and hence, K is convex. □

Assume that $u^0 \geq \phi$; then K is nonempty because $u^0 \in K$. Suppose that u is a solution to

$$u \in K, \ \ J(u) = \min_{v \in K} J(v). \tag{B.38}$$

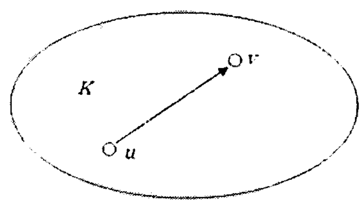

Figure B.5: K is a closed convex set.

We consider a variational problem; finding u such that

$$u \in K, \qquad J(u) = \min_{v \in K} J(v). \tag{B.39}$$

If u is a solution of the problem (B.39), then for any $v \in K$ and $0 < \varepsilon < 1$, $u + \varepsilon(v-u) = (1-\varepsilon)u + \varepsilon v$ in K because K is convex. Therefore, for

$$J(u + \varepsilon(v-u)) \geq J(u).$$

Hence, one has

$$J(u+\varepsilon(v-u)) = \int_\Omega \frac{1}{2} \mid \nabla(u+\varepsilon(v-u)) \mid^2 - f(u+\varepsilon(v-u)) \ dx$$
$$\geq J(u) = \int_\Omega \frac{1}{2} \mid \nabla u \mid^2 - fu \ dx, \qquad 0 < \forall \varepsilon < 1.$$

Then

$$\int_\Omega \frac{1}{2} \mid \nabla u \mid^2 + 2 \cdot \frac{\varepsilon}{2} \nabla u \cdot \nabla(v-u) + \frac{\varepsilon^2}{2} \mid \nabla(v-u) \mid^2 - fu - \varepsilon f(v-u) \ dx \geq \int_\Omega \frac{1}{2} \mid \nabla u \mid^2 - fu \ dx$$

$$\int_\Omega \varepsilon \nabla u \cdot \nabla(v-u) + \frac{\varepsilon^2}{2} \mid \nabla(v-u) \mid^2 - \varepsilon f(v-u) \ dx \geq 0$$

$$\int_\Omega \nabla u \cdot \nabla(v-u) + \frac{\varepsilon}{2} |\nabla(v-u)|^2 \ dx \geq \int_\Omega f(v-u) \ dx.$$

Taking $\varepsilon \downarrow 0$, we get

$$\int \nabla u \cdot \nabla(v-u) \ dx \geq \int f(v-u) \ dx \quad \forall v \in K; \ u \in K. \tag{B.40}$$

If $u \in H^2(\Omega)$, by applying Green's formula to the left hand-side of (B.40), we obtain

$$\int_\Omega -\Delta u(v-u) \ dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(v-u) \ ds \geq \int_\Omega f(v-u) \ dx \qquad \forall v \in K. \tag{B.41}$$

We then obtain

$$\int (\Delta u + f)(v-u) \ dx \leq \int_{\partial\Omega} \frac{\partial u}{\partial n}(v-u) \ ds \quad \forall v \in K. \tag{B.42}$$

By choosing $v = u + \varepsilon\zeta, \zeta \geq 0, 0 < \varepsilon \ll 1, \forall \zeta \in C_0^\infty(\Omega)$, we get $v \in K$. In fact, obviously $v \in H^1(\Omega)$. Next, $\gamma v = \gamma(u+\varepsilon\zeta) = \gamma u + \varepsilon\gamma\zeta = \gamma u = \gamma u^0$. Finally, $v = u + \varepsilon\zeta \geq u \geq \phi$ a.e. in Ω. Then since $v - u = 0$ on $\partial\Omega$, we have

$$\int_\Omega (\Delta u + f)\zeta \ dx \leq 0.$$

Since $\zeta \geq 0, \zeta \in C_0^\infty(\Omega)$ is arbitrary, it follows that

$$\Delta u + f \leq 0 \qquad \text{in } \Omega. \tag{B.43}$$

Assuming the existence of the continuous solution u to (B.39), the set

$$N = \{x \in \Omega; u(x) > \phi(x)\} \tag{B.44}$$

is open, while

$$\Lambda = \{x \in \Omega; u(x) = \phi(x)\} \tag{B.45}$$

is closed, where N is called the *noncoincidence set* and Λ is called the *coincidence set*. For any $\zeta \in C_0^\infty(N)$ the

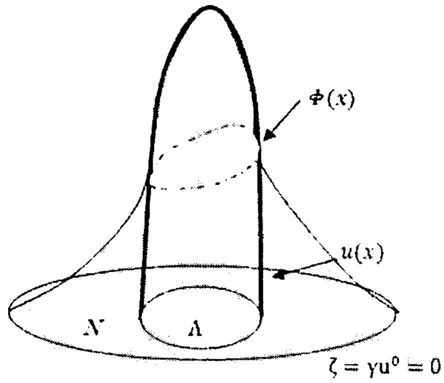

Figure B.6: Coincidence and noncoincidence set

function $v = u \pm \varepsilon\zeta$ in K, $0 < \varepsilon \ll 1$. In fact, obviously $v = u \pm \varepsilon\zeta \in H^1(\Omega)$. Next, $\gamma v = \gamma(u \pm \varepsilon\zeta) = \gamma u \pm \varepsilon\gamma\zeta = \gamma u = \gamma u^0$. Finally, since $\operatorname{supp}\zeta \subset N$, it holds that $u \pm \varepsilon\zeta \geq \phi$ a.e., provided that $0 < \varepsilon \ll 1$. Then from (B.40), we obtain

$$\pm\varepsilon \int_\Omega (\nabla u \cdot \nabla\zeta + f\zeta)\ dx = \pm\varepsilon \int_N (\Delta u + f)\zeta\ dx \leq 0, \qquad \forall\zeta \in C_0^\infty(N)$$

whence we can conclude that

$$\int_\Omega (\Delta u + f)\zeta\ dx = 0, \qquad \forall\zeta \in C_0^\infty(N)$$

and, therefore

$$\Delta u + f = 0 \qquad in\ N = \{u(x) > \phi(x)\}, \tag{B.46}$$

Therefore if u is the solution to (B.39) which belongs to $H^2(\Omega) \cap C(\Omega)$, then,

$$\Delta u + f \leq 0 \text{ in } \Omega,$$

$$u \geq \phi \text{ in } \Omega,$$

$$(\Delta u + f)(u - \phi) = 0 \text{ in } \Omega,$$

$$u - u^0 \in H_0^1(\Omega).$$

The third equality holds by (B.46). In fact, if $x \in N$ then $\Delta u(x) + f(x) = 0$, and hence $(\Delta u + f) \cdot (u - \phi) = 0$ at x. In the other case of $x \notin N$, it holds that $x \in \Lambda$, so that $u(x) - \phi(x) = 0$ and hence $(\Delta u + f) \cdot (u - \phi) = 0$.

The set K is called the constraint set, and in the case of (B.37), ϕ is an *obstacle* and (B.39) is the *obstacle problem*.

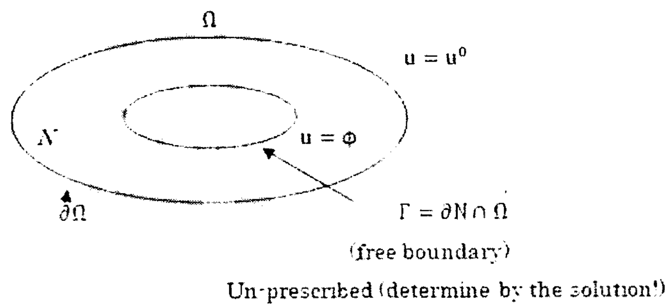

Figure B.7: Coincidence and noncoincidence set from above view.

The boundary of the noncoincidence set in Ω,

$$\Gamma = \partial N \cap \Omega,$$

is called *free boundary*. In fact, since the function $u - \phi$ attains its minimum value zero at any point in Λ, if $u - \phi$ has first derivatives continuous, one has $u_x = \phi_x$ and $u_y = \phi_y$ on Γ. It follows that

$$u - \phi = 0, \qquad \nabla(u - \phi) = 0 \quad \text{on } \Gamma. \tag{B.47}$$

The above u may be regarded as the solution of the Dirichlet problem,

$$\begin{aligned} \Delta u + f = 0 \qquad & \text{in } N, \\ u = u^0 \qquad & \text{on } \partial N \cap \partial\Omega, \\ u = \phi \qquad & \text{on } \partial N \cap \Gamma, \end{aligned} \tag{B.48}$$

with additional condition

$$\nabla u = \nabla\phi \qquad \text{on } \partial N \cap \Gamma \tag{B.49}$$

compensating that Γ is not a priori unknown.

Consider the special case $n = 1, f = 0$. Then the variational inequality (B.37), (B.38) is to minimize

$$\int_a^b [u'(x)]^2 \, dx$$

with $u(a) = u_1, u(b) = u_2$ and $u(x) \geq \phi(x)$. Take $u_1 > \phi(a), u_2 > \phi(b)$. Assume that $\phi(x)$ is strictly convex, and from (B.48) and (B.49) we deduce that the curve $y = u(x)$ consists of three arcs:

1. A line segment l_1 connecting (a, u_1) to a point $(a', \phi(a'))$, tangent to $y = \phi(x)$ at $x = a'$.

2. An arc $\gamma : y = \phi(x), a' < x < b'$.

3. A line segment l_2 connecting $(b', \phi(b'))$ to (b, u_2), tangent to $y = \phi(x)$ at $x = b'$.

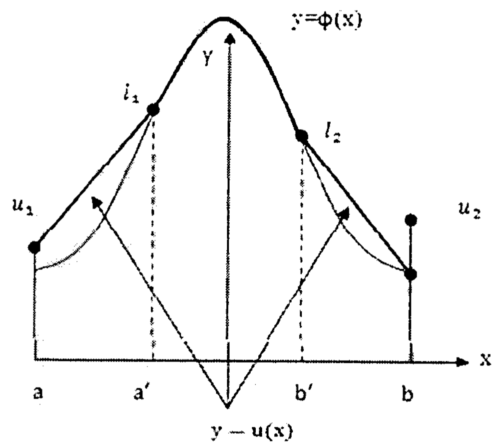

Figure B.8: A graph consists three arcs

The free boundary consists two points $(a', \phi(a'))$ and $(b', \phi(b'))$. Thus, $u''(x)$ has a jump discontinuity at $x = a'$. Let $\Omega \subset \mathbb{R}^n$ be an open set with (B.21) and let $\phi(x)$ (the obstacle) be the function that satisfying

$$\phi \in C^2(\overline{\Omega}) \tag{B.50}$$

and assume that

$$\begin{aligned} &\textit{the coefficients of } A \textit{ are in } C^{\alpha}(\overline{\Omega}), \\ &A \textit{ is uniformly elliptic in } \Omega,\ c(x) \geq 0; \end{aligned} \tag{B.51}$$

$$\begin{cases} \partial\Omega & \text{is in } C^{2+\alpha}, \\ f \in C^{\alpha}(\overline{\Omega}), & u^0 \in C^{2+\alpha}(\overline{\Omega}), \\ u^0 \geq 0 & \text{on } \partial\Omega. \end{cases} \tag{B.52}$$

Theorem B.4.2. *Assume that (B.50)-(B.52) holds. Then there exists a solution u of the variational inequality*

$$\left.\begin{aligned} Au - f &\geq 0 \\ u &\geq \phi \\ (Au - f)(u - \phi) &= 0 \end{aligned}\right\} a.e. \ in \ \Omega, \tag{B.53}$$

$$u = u^0 \qquad on \ \Omega,$$

and $u \in W^{2,p}(\Omega)$ *for any* $1 < p < \infty$.

Theorem B.4.3. *Let u_1, u_2 be a solution to $W^{2,2}(\Omega) \cap C(\overline{\Omega})$ of the variational inequality (B.53) corresponding to f_1 and f_2. If $f_1 \geq f_2$, then $u_1 \geq u_2$.*

Theorem B.4.4. *Under the assumptions of (B.50)-(B.52), the variational inequality (B.53) is unique.*

Equation (B.50) can be replace by

$$\begin{cases} \phi \in C^{0,1}(\Omega_0), \\ \frac{\partial^2 \phi}{\partial \xi} \geq -C \quad \text{in } D'(\Omega_0), \quad \text{for any direction } \xi, \end{cases} \tag{B.54}$$

where Ω_0 is a neighbourhood of $\overline{\Omega}$.

Theorem B.4.5. *Let (B.51), (B.52) and (B.54) hold, then the assertion of Theorem B.4.2 is valid.*

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List of Publications

1. Nuha Loling Othman, Takashi Suzuki, Takuya Tsuchiya, "A Combined Scheme for Computing Numerical Solutions of a Free Boundary Problem", International Journal of Mathematics and Computers in Simulation, Issue 1, Vol.5, pp 53-60, 2011.

2. Nuha Loling Othman, Takashi Suzuki, Takuya Tsuchiya, "Application of Hadamard's Variation to Numerical Solutions of a Free Boundary", Advances in Mathematical and Computational Methods in Science and Engineering, pp 114-121, Nov 2010.

3. Nuha Loling Othman, Takashi Suzuki, Takuya Tsuchiya, "A Simple Derivation of Hadamard's Variational Formula", RIMS Koukyuroku 1733, pp 127-141, March 2011.