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Soliton Cellular Automata constructed from a $U_q(D_n^{(1)})$-Crystal $B^{n,1}$ and Kirillov-Reshetikhin type bijection for $U_q(E_6^{(1)})$-Crystal $B^{6,1}$

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MARCH 2012

Soliton Cellular Automata constructed from a $U_q(D_n^{(1)})$-Crystal $B^{n,1}$ and Kirillov-Reshetikhin type bijection for $U_q(E_6^{(1)})$-Crystal $B^{6,1}$

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Abstract

In part 1 we study a class of cellular automata associated with the Kirillov-Reshetikhin crystal $B^{n,1}$ of type $D_n^{(1)}$. They have a commuting family of time evolutions and solitons of length l are labeled by $U_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l}$. The scattering rule of two solitons of lengths l_1 and l_2 ($l_1 > l_2$) including the phase shift is identified with the combinatorial R-matrix for the $U_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l_2} \otimes B_A^{2,l_1}$. In part 2 we consider the Kirrilov-Reshetikhin crystal $B^{6,1}$ for the exceptional affine type $E_6^{(1)}$. We will give a conjecture on a statistic-preserving bijection between the highest weight paths consisting of $B^{6,1}$ and the corresponding rigged configuration. The algorithm only uses the structure of the crystal graph, hence could also be applied for other exceptional types. Our $B^{6,1}$ has a different algorithm compared our $B^{1,1}$ because we must consider the element ϕ, unique element in the highest weight crystal of weight 0, in the crystal graph. We will give many examples supporting the conjecture.

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Part I

Scattering Rules in Soliton Cellular Automata Associated with $U_q(D_n^{(1)})$-Crystal $B^{n,1}$

1 Introduction

The box-ball system [33, 32] well-known as soliton cellular automata is a dynamical system of balls in a one dimensional array of boxes. The discrete KdV equation through a limiting procedure called ultradiscretization [36] was used to show the solitonic character like the KdV solitons. The rules for soliton interactions and factorization property of scattering matrices (Yang-Baxter equation) are justified by means of inverse ultradiscretization [35]. In [35] it is shown that the dynamical systems of soliton cellular automaton is described by an ultra-discrete equation obtained from extended Toda molecule equation. Later it was studied by [3] that the scattering of two solitons including the phase shift is described by isomorphism from the tensor product of two affine crystals for the quantum enveloping algebra $U_q(A^{(1)}_{n-1})$ to the other order of the tensor product. The object they used is called combinatorial R-matrix [13]. The combinatorial R-matrix has an amazing property: it satisfies the Yang-Baxter equation, which assures that the scattering of three solitions does not depend on the order of scattering of the two solitons.

The new soliton cellular automata were constructed in [8] corresponding to $U_q(\mathfrak{g}_n)$ where $\mathfrak{g}_n = A^{(2)}_{2n-1}, A^{(2)}_{2n}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, D^{(2)}_{n+1}$ and their internal degree of freedom was labeled by crystals of the smaller algebra $U_q(\mathfrak{g}_{n-1})$. Then [7] studied the scattering rule of two solitons when they collide each other. They found that the scattering rule for affine crystals corresponding to $U_q(\mathfrak{g}_n)$ can be described by combinatorial R-matrix of the smaller algebra $U_q(\mathfrak{g}_{n-1})$. The affine crystal they used is called Kirillov-Reshetikhin (KR) crystal denoted by $B^{1,1}$. (The KR crystal is parameterized by two integers. The first index corresponds to a node of the Dynkin diagram of the affine algebra except 0 and the second a positive integer.) A generalization to the KR crystal $B^{k,1}$ for $\mathfrak{g} = A^{(1)}_{n-1}$ was studied in [37] and their internal degree of freedom is given by the product of $U_q(A^{(1)}_{k-1})$-crystal $B^{k-1,l}$ and $U_q(A^{(1)}_{n-k-1})$-crystal $B^{1,l}$. A case for the exceptional algebra $\mathfrak{g} = D^{(3)}_4$ was also treated in [38]. In [38], it is shown that the scattering rule for the crystal type $U_q(D^{(3)}_4)$ is identified with the combinatorial R-matrix for $U_q(A^{(1)}_1)$-crystals and phase shifts are given by 3-times of those in the well-known box ball system.

These results can be summarized and one might find the following conjectural properties for the solitons and their scatterings of the soliton cellular automaton constructed from the KR crystal $B^{k,1}$ of the quantum affine algebra $U_q(\mathfrak{g})$. Let G be the Dynkin diagram of the corresponding finite-dimensional simple Lie algebra of $\mathfrak{g}$. Let $\breve{G}$ be the Dynkin diagram obtained by removing the node k from G and let j be the node of $\breve{G}$ that is connected to k in G. Let $\breve{\mathfrak{g}}$ be the corresponding affine algebra.

• The internal degree of freedom of soliton of length l is described by the $U_q(\breve{\mathfrak{g}})$-crystal $B^{j,l}$.
• The exchange of the internal degree of freedom by the scattering of solitons of length l_1 and l_2 ($l_1 > l_2$) is given by the crystal isomorphism $B^{j,l_1} \otimes B^{j,l_2} \xrightarrow{\sim} B^{j,l_2} \otimes B^{j,l_1}$.
• The phase shift of the scattering is described by the corresponding H function.

This property allows us to calculate the combinatorial R matrix for $B^{j,l_1} \otimes B^{j,l_2}$ just by observing the scattering of solitons in the corresponding cellular automaton. If $\breve{G}$ is not connected, then one needs to consider the tensor product as seen in [37]. Although this conjecture seems reasonable, the rigorous proof is yet to be given.

The purpose of this paper is to add another affirmative example to this conjecture. We take $\mathfrak{g} = D_n^{(1)}, k = n$. The corresponding node is a spin node and the KR crystal is $B^{n,1}$. According to the above conjecture, we have $\breve{\mathfrak{g}} = A_{n-1}^{(1)}, j = n-2$. In the crystal theory, there is a notion of the dual crystal. The dual $B^\vee$ of a crystal B is defined by setting $(e_i b)^\vee = f_i b^\vee, (f_i b)^\vee = e_i b^\vee$. Since we know $(B_1 \otimes B_2)^\vee = B_2^\vee \otimes B_1^\vee$ and $(B^{j,l})^\vee = B^{n-j,l}$ for the KR crystal of type $A_{n-1}^{(1)}$, we can expect the following property on our soliton cellular automaton.

• The internal degree of freedom of a soliton of length l is described by the $U_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l}$.
• The exchange of the internal degree of freedom by the scattering of solitons of length l_1 and l_2 ($l_1 > l_2$) is given by the crystal isomorphism $B_A^{2,l_2} \otimes B_A^{2,l_1} \xrightarrow{\sim} B_A^{2,l_1} \otimes B_A^{2,l_2}$.
• The phase shift of the scattering is described by the corresponding H function.

We check these properties in this paper, thereby obtain our main theorem (Theorem 3.16).

The paper is organized as follows. In Sec. 2, we recapitulate necessary facts from the crystal theory. In Sec. 3, we construct conserved quantities. The main theorem is given in Sec. 3, where the scattering of solitons is studied.

2 Preliminaries

In this section we review some basic definitions and facts about crystals for the $U_q'(D_n^{(1)})$-crystal $B^{n,l}$ in Section 2.1. In order to describe the crystal graphs for the finite-dimensional modules of quantum groups of classical type, Kashiwara and Nakashima introduced the analogue of semi-standard tableaux, called Kashiwara-Nakashima (KN) tableaux [16].

2.1 Crystal $B^{n,1}$

Crystal theory was introduced by Kashiwara [12] which provides a combinatorial way to study the representation theory of the quantum algebra $U_q(\mathfrak{g})$. In this paper $\mathfrak{g} = D_n^{(1)}$ is the corresponding quantum algebra. Let P be the weight lattice, $\{\alpha_i\}_{0\leq i\leq n}$ the simple roots, and $\{\Lambda_i\}_{0\leq i\leq n}$ the fundamental weights of $D_n^{(1)}$. Let $\bar{\Lambda}_i$ denote the classical part of Λ_i. The crystal B is a finite set with weight decomposition $B = \sqcup_{\lambda\in P} B_\lambda$. The Kashiwara operators e_i, f_i $(i = 0, 1, \cdots, n)$ act on B as

$$e_i : B_\lambda \longrightarrow B_{\lambda+\alpha_i} \sqcup \{0\}, \qquad f_i : B_\lambda \longrightarrow B_{\lambda-\alpha_i} \sqcup \{0\}.$$

These operators are nilpotent. By definition, we have $f_i b = b'$ if and only if $b = e_i b'$. Drawing $b \xrightarrow{i} b'$ in such case, B is endowed with the structure of colored oriented graph called *crystal graph.*

Let $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ orthonormal basis of the weight space of D_n. The simple roots and classical parts of fundamental weights for $D_n^{(1)}$ are expressed as

$$\alpha_0 = \delta - \epsilon_1 - \epsilon_2, \quad \alpha_n = \epsilon_{n-1} + \epsilon_n, \quad \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad \text{for } i = 1, 2, ..., n-1,$$

$$\bar{\Lambda}_{n-1} = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n), \quad \bar{\Lambda}_n = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n).$$

$$\bar{\Lambda}_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i \quad \text{for } i = 1, 2, ..., n-2,$$

We explain the Kirillov-Reshetikhin crystal $B^{n,l}$, $l \in \mathbb{Z}_{>0}$. We set

$\mathfrak{A} = \{1, 2, \cdots, n-1, n, \bar{n}, \overline{n-1}, \cdots, \bar{2}, \bar{1}\}$. The set of letters order on

$$\mathfrak{A} : 1 \prec 2 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \bar{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \bar{2} \prec \bar{1}.$$

where there is no order between n and $\bar{n}$. Then the crystal $B^{n,l}$ is given by

$$B^{n,1} = \left\{ \begin{array}{|c|} \hline i_n \\ \hline \vdots \\ \hline i_2 \\ \hline i_1 \\ \hline \end{array} \;\middle|\; \begin{array}{l} i_p \in \mathfrak{A}, i_1 \prec i_2 \prec \cdots \prec i_n, \\ a, \bar{a} \text{ does not coexist for any } a = 1, 2, ..., n. \\ \text{There are even number of barred letters.} \end{array} \right\}, \tag{2.1}$$

$$B^{n,l} = \left\{ \begin{array}{|c|c|c|c|} \hline c_1 & c_2 & \cdots & c_l \\ \hline \end{array} \;\middle|\; c_j \in B^{n,1}, \text{ and setting } c_j = \begin{array}{|c|} \hline c_{jn} \\ \hline \vdots \\ \hline c_{j2} \\ \hline c_{j1} \\ \hline \end{array}, \begin{array}{l} \text{where } c_{jp} \preceq c_{j+1,p} \\ \text{for } 1 \le p \le n, \\ 1 \le j < l \end{array} \right\}. \tag{2.2}$$

The weight of $b \in B^{n,1}$ is given by $wt\ b = \frac{1}{2}\sum_{j=1}^{n} \eta_j \epsilon_j$ where

$$\eta_j = \begin{cases} +1 & \text{if } j \text{ exist in } b, \\ -1 & \text{if } \bar{j} \text{ exist in } b \end{cases}$$

and that of $b = \begin{array}{|c|c|c|c|} \hline c_1 & c_2 & \cdots & c_l \\ \hline \end{array} \in B^{n,l}$ is given by $wt\ b = \sum_{j=1}^{l} wt\ c_j$.

2.2 Crystal structure on $B^{n,1}$

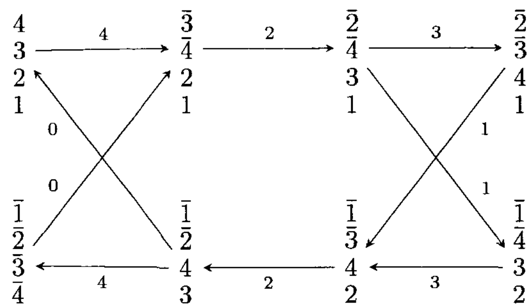
For $i = 0, 1, ..., n$

$$e_i b = \begin{cases} b' & \text{if } wt\ b' = wt\ b + \alpha_i (\text{mod } \mathbb{Z}\delta), \\ 0 & \text{if such } b' \text{ does not exist in } B^{n,1}, \end{cases}$$

$$f_i b = \begin{cases} b'' & \text{if } wt\ b'' = wt\ b - \alpha_i (\text{mod } \mathbb{Z}\delta), \\ 0 & \text{if such } b'' \text{ does not exist in } B^{n,1}. \end{cases}$$

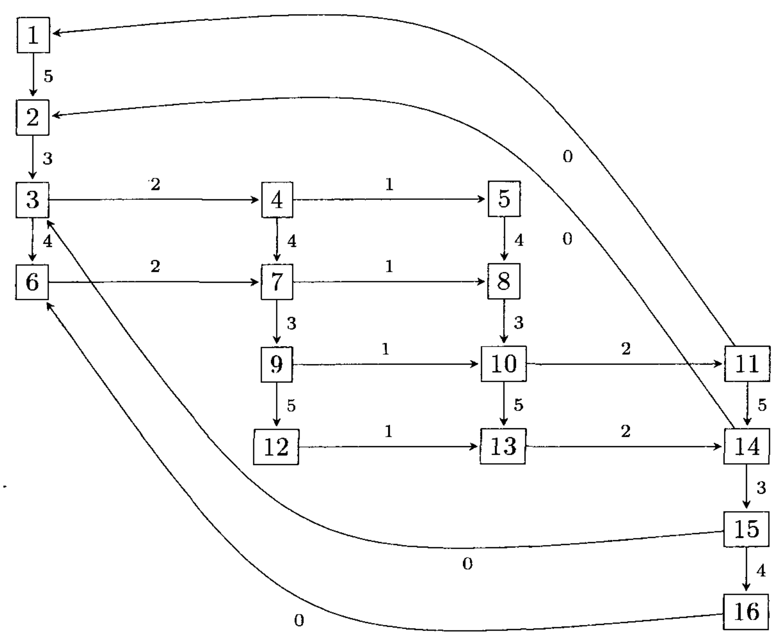
$B^{n,1}$ is the crystal base [12] of the spin representation of the quantum affine algebra $U_q'(D_n^{(1)})$.

Example 2.1 *When $n = 4$, the crystal graph of $B^{4,1}$ is depicted as follows.*



The crystal graph of $B^{1,1}$ is the same as above by interchanging the colors as $1 \leftrightarrow 4$.

Example 2.2 *When $n = 5$, the crystal graph of $B^{5,1}$ is depicted as follows.*



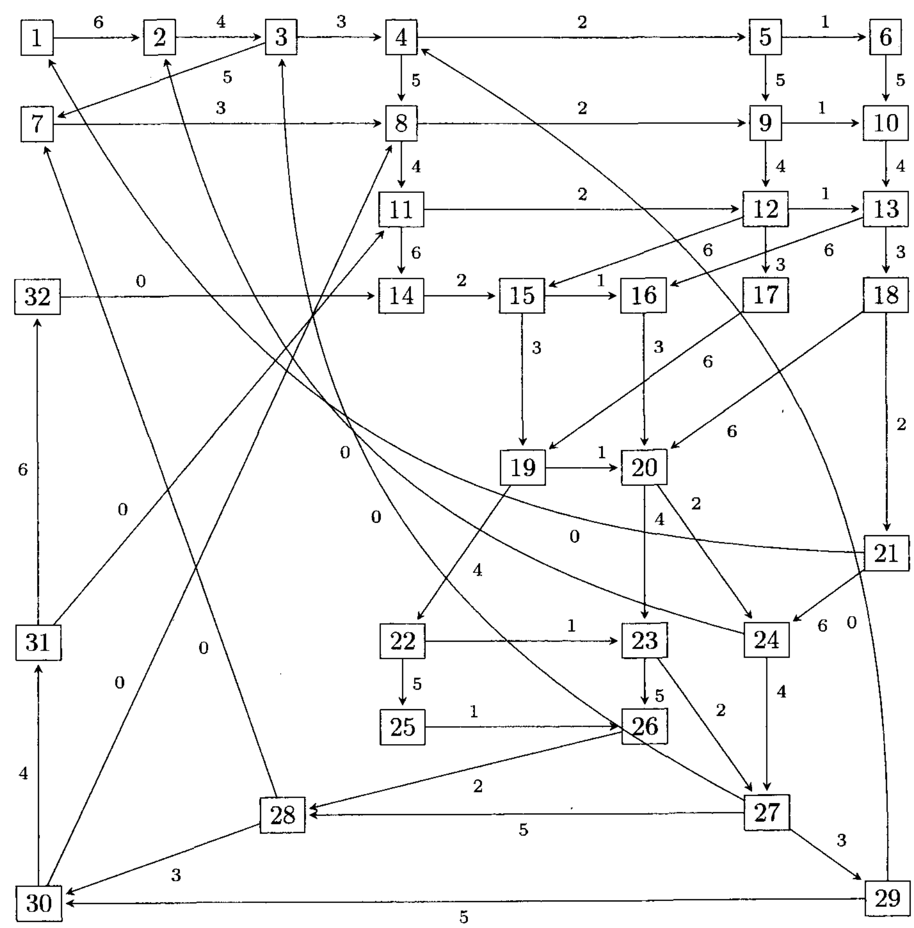
We give the correspondence between the numbers in the crystal graph with our representation of crystal elements.

$$\boxed{1} = \begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}, \quad \boxed{2} = \begin{array}{c} \bar{4} \\ \bar{5} \\ 3 \\ 2 \\ 1 \end{array}, \quad \boxed{3} = \begin{array}{c} \bar{3} \\ \bar{5} \\ 4 \\ 2 \\ 1 \end{array}, \quad \boxed{4} = \begin{array}{c} \bar{2} \\ \bar{5} \\ 4 \\ 3 \\ 1 \end{array}, \quad \boxed{5} = \begin{array}{c} \bar{1} \\ \bar{5} \\ 4 \\ 3 \\ 2 \end{array}, \quad \boxed{6} = \begin{array}{c} \bar{3} \\ \bar{4} \\ 5 \\ 2 \\ 1 \end{array},$$

$$\boxed{7} = \begin{array}{c}\bar{2}\\ \bar{4}\\ 5\\ 3\\ 1\end{array}, \quad \boxed{8} = \begin{array}{c}\bar{1}\\ \bar{4}\\ 5\\ 3\\ 2\end{array}, \quad \boxed{9} = \begin{array}{c}\bar{2}\\ \bar{3}\\ 5\\ 4\\ 1\end{array}, \quad \boxed{10} = \begin{array}{c}\bar{1}\\ \bar{3}\\ 5\\ 4\\ 2\end{array}, \quad \boxed{11} = \begin{array}{c}\bar{1}\\ \bar{2}\\ 5\\ 4\\ 3\end{array}, \quad \boxed{12} = \begin{array}{c}\bar{2}\\ \bar{3}\\ \bar{4}\\ \bar{5}\\ 1\end{array},$$

$$\boxed{13} = \begin{array}{c}\bar{1}\\ \bar{3}\\ \bar{4}\\ \bar{5}\\ 2\end{array}, \quad \boxed{14} = \begin{array}{c}\bar{1}\\ \bar{2}\\ \bar{4}\\ \bar{5}\\ 3\end{array}, \quad \boxed{15} = \begin{array}{c}\bar{1}\\ \bar{2}\\ \bar{3}\\ \bar{5}\\ 4\end{array}, \quad \boxed{16} = \begin{array}{c}\bar{1}\\ \bar{2}\\ \bar{3}\\ \bar{4}\\ 5\end{array}.$$

Example 2.3 *When $n = 6$, the crystal graph of $B^{6,1}$ is depicted as follows.*



We give the correspondence between the numbers in the crystal graph with our representation of crystal elements.

$$\boxed{1} = \begin{matrix}6\\5\\4\\3\\2\\1\end{matrix}, \quad \boxed{2} = \begin{matrix}\bar{5}\\\bar{6}\\4\\3\\2\\1\end{matrix}, \quad \boxed{3} = \begin{matrix}\bar{4}\\\bar{6}\\5\\3\\2\\1\end{matrix}, \quad \boxed{4} = \begin{matrix}\bar{3}\\\bar{6}\\5\\4\\2\\1\end{matrix}, \quad \boxed{5} = \begin{matrix}\bar{2}\\\bar{6}\\5\\4\\3\\1\end{matrix}, \quad \boxed{6} = \begin{matrix}\bar{1}\\\bar{6}\\5\\4\\3\\2\end{matrix},$$

$$\boxed{7} = \begin{matrix}\bar{4}\\\bar{5}\\6\\3\\2\\1\end{matrix}, \quad \boxed{8} = \begin{matrix}\bar{3}\\\bar{5}\\6\\4\\2\\1\end{matrix}, \quad \boxed{9} = \begin{matrix}\bar{2}\\\bar{5}\\6\\4\\3\\1\end{matrix}, \quad \boxed{10} = \begin{matrix}\bar{1}\\\bar{5}\\6\\4\\3\\2\end{matrix}, \quad \boxed{11} = \begin{matrix}\bar{3}\\\bar{4}\\6\\5\\2\\1\end{matrix}, \quad \boxed{12} = \begin{matrix}\bar{2}\\\bar{4}\\6\\5\\3\\1\end{matrix},$$

$$\boxed{13} = \begin{matrix}\bar{1}\\\bar{4}\\6\\5\\3\\2\end{matrix}, \quad \boxed{14} = \begin{matrix}\bar{3}\\\bar{4}\\\bar{5}\\\bar{6}\\2\\1\end{matrix}, \quad \boxed{15} = \begin{matrix}\bar{2}\\\bar{4}\\\bar{5}\\\bar{6}\\3\\1\end{matrix}, \quad \boxed{16} = \begin{matrix}\bar{1}\\\bar{4}\\\bar{5}\\\bar{6}\\3\\2\end{matrix}, \quad \boxed{17} = \begin{matrix}\bar{2}\\\bar{3}\\6\\5\\4\\1\end{matrix}, \quad \boxed{18} = \begin{matrix}\bar{1}\\\bar{3}\\6\\5\\4\\2\end{matrix},$$

$$\boxed{19} = \begin{matrix}\bar{2}\\\bar{3}\\\bar{5}\\\bar{6}\\4\\1\end{matrix}, \quad \boxed{20} = \begin{matrix}\bar{1}\\\bar{3}\\\bar{5}\\\bar{6}\\4\\2\end{matrix}, \quad \boxed{21} = \begin{matrix}\bar{1}\\\bar{2}\\6\\5\\4\\3\end{matrix}, \quad \boxed{22} = \begin{matrix}\bar{2}\\\bar{3}\\\bar{4}\\\bar{6}\\5\\1\end{matrix}, \quad \boxed{23} = \begin{matrix}\bar{1}\\\bar{3}\\\bar{4}\\\bar{6}\\5\\2\end{matrix}, \quad \boxed{24} = \begin{matrix}\bar{1}\\\bar{2}\\\bar{5}\\\bar{6}\\4\\3\end{matrix},$$

$$\boxed{25} = \begin{matrix}\bar{2}\\\bar{3}\\\bar{4}\\\bar{5}\\6\\1\end{matrix}, \quad \boxed{26} = \begin{matrix}\bar{1}\\\bar{3}\\\bar{4}\\\bar{5}\\6\\2\end{matrix}, \quad \boxed{27} = \begin{matrix}\bar{1}\\\bar{2}\\\bar{4}\\\bar{6}\\5\\3\end{matrix}, \quad \boxed{28} = \begin{matrix}\bar{1}\\\bar{2}\\\bar{4}\\\bar{5}\\6\\3\end{matrix}, \quad \boxed{29} = \begin{matrix}\bar{1}\\\bar{2}\\\bar{3}\\\bar{6}\\5\\4\end{matrix}, \quad \boxed{30} = \begin{matrix}\bar{1}\\\bar{2}\\\bar{3}\\\bar{5}\\6\\4\end{matrix},$$

$$\boxed{31} = \begin{matrix} \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \\ 6 \\ 5 \end{matrix}, \qquad \boxed{32} = \begin{matrix} \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \\ \bar{5} \\ \bar{6} \end{matrix}.$$

Example 2.4 *Consider the case $n = 4$.*

$$\text{Let } c_0 = \begin{matrix} 4 \\ 3 \\ 2 \\ 1 \end{matrix}, \ c^{(1)} = \begin{matrix} \bar{3} \\ \bar{4} \\ 2 \\ 1 \end{matrix}, \ c^{(2)} = \begin{matrix} \bar{2} \\ \bar{4} \\ 3 \\ 1 \end{matrix}, \ c^{(3)} = \begin{matrix} \bar{1} \\ \bar{4} \\ 3 \\ 2 \end{matrix}, \ c^{(4)} = \begin{matrix} \bar{1} \\ \bar{3} \\ 4 \\ 2 \end{matrix}, \ c^{(5)} = \begin{matrix} \bar{1} \\ \bar{2} \\ 4 \\ 3 \end{matrix}.$$

$$e_2(c^{(2)}) = c^{(1)}, \qquad f_2(c^{(2)}) = 0, \qquad f_0(c^{(2)}) = 0.$$

Example 2.5 *Consider the case $n = 5$.*

$$\text{Let } \tilde{c}_0 = \begin{matrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{matrix}, \ \tilde{c}^{(1)} = \begin{matrix} \bar{4} \\ \bar{5} \\ 3 \\ 2 \\ 1 \end{matrix}, \ \tilde{c}^{(2)} = \begin{matrix} \bar{3} \\ \bar{5} \\ 4 \\ 2 \\ 1 \end{matrix}, \ \tilde{c}^{(3)} = \begin{matrix} \bar{2} \\ \bar{5} \\ 4 \\ 3 \\ 1 \end{matrix}, \ \tilde{c}^{(4)} = \begin{matrix} \bar{1} \\ \bar{5} \\ 4 \\ 3 \\ 2 \end{matrix}, \ \tilde{c}^{(5)} = \begin{matrix} \bar{3} \\ \bar{4} \\ 5 \\ 2 \\ 1 \end{matrix}.$$

$$e_2(\tilde{c}^{(2)}) = 0, \qquad f_2(\tilde{c}^{(2)}) = \tilde{c}^{(3)}, \qquad f_0(\tilde{c}^{(2)}) = 0.$$

2.3 Crystal structure on $B^{n,l}$

Let $b \in B^{n,l}$.

$$b = \begin{array}{|c|c|c|c|} \hline c_1 & c_2 & \cdots & c_l \\ & & & \\ \hline \end{array}, \quad c_j \in B^{n,1} \tag{2.3}$$

The actions of e_i, f_i for $i \neq 0$ can be calculated by using the rule called **signature rule**. For $b \in B^{n,l}$ we associate an element $c_l \otimes c_{l-1} \otimes \cdots \otimes c_2 \otimes c_1$ of the tensor product $(B^{n,1})^{\otimes l}$ to find the indices j, j' such that

$$e_i(c_l \otimes c_{l-1} \otimes \cdots \otimes c_2 \otimes c_1) = c_l \otimes c_{l-1} \otimes \cdots \otimes e_i c_j \otimes \cdots \otimes c_2 \otimes c_1 \tag{2.4}$$

$$f_i(c_l \otimes c_{l-1} \otimes \cdots \otimes c_2 \otimes c_1) = c_l \otimes c_{l-1} \otimes \cdots \otimes f_i c_{j'} \otimes \cdots \otimes c_2 \otimes c_1 \tag{2.5}$$

With this element we associate an i-signature : $\underbrace{-\ldots-}_{\varepsilon_i(c_l)}\ \underbrace{+\ldots+}_{\varphi_i(c_l)}\ \underbrace{-\ldots-}_{\varepsilon_i(c_{l-1})}\ \underbrace{+\ldots+}_{\varphi_i(c_{l-1})}\ \ldots\ \underbrace{-\ldots-}_{\varepsilon_i(c_1)}\ \underbrace{+\ldots+}_{\varphi_i(c_1)}$.

We then reduce the signature by deleting the adjacent $+-$ pair successively. Eventually we obtain a reduced signature of the following form.

$$- - \cdots - + + \cdots +$$

Then the action e_i (resp. f_i) corresponds to changing the rightmost $-$ to $+$ (resp. leftmost $+$ to $-$). If there is no $-$ (resp. $+$) in the signature, then the action of e_i (resp. f_i) should be set to 0. The value of $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) is given by the number of $-$(resp. $+$) in the reduced signature.

Example 2.6 *Since the signature rule enables us to calculate the multiple tensor product of $B^{n,l}$'s, we consider $B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. Let $c^{(j)}(j = 1, \ldots, 5)$ as in Example 2.4.*

Consider an element $b = (c_0\ c^{(1)}c^{(1)}c^{(2)}) \otimes (c_0\ c_0\ c^{(1)}) \otimes (c^{(1)}c^{(3)}) \in B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. The 4-signature is given as follows

$$\begin{array}{llllllllllll} b = & (c^{(2)} \otimes & c^{(1)} \otimes & c^{(1)} \otimes & c_0) \otimes & (c^{(1)} \otimes & c_0 \otimes & c_0) \otimes & (c^{(3)} \otimes & c^{(1)}) \\ \eta_4 = & & - & - & + & - & + & + & & - \end{array}$$

The reduced signature is $\eta_4 = \overset{2}{-}\ \overset{3}{-}\ \overset{6}{+}$, where the upper number signifies the component of the tensor product the sign belonged to. Therefore, we have

$e_4 b = (c^{(2)} \otimes c^{(1)} \otimes e_4(c^{(1)}) \otimes c_0) \otimes (c^{(1)} \otimes c_0 \otimes c_0) \otimes (c^{(3)} \otimes c^{(1)}) = (c_0\ c_0\ c^{(1)}c^{(2)}) \otimes (c_0\ c_0\ c^{(1)}) \otimes (c^{(1)}c^{(3)})$

$f_4 b = (c^{(2)} \otimes c^{(1)} \otimes c^{(1)} \otimes c_0) \otimes (c^{(1)} \otimes f_4(c_0) \otimes c_0) \otimes (c^{(3)} \otimes c^{(1)}) = (c_0\ c^{(1)}c^{(1)}c^{(2)}) \otimes (c_0\ c^{(1)}c^{(1)}) \otimes (c^{(1)}c^{(3)})$

Example 2.7 *Since the signature rule enables us to calculate the multiple tensor product of $B^{n,l}$'s, we consider $B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. Let $\tilde{c}^{(j)} (j = 1, ..., 5)$ as in Example 2.5.*

Consider an element $b' = (\tilde{c}_0\ \tilde{c}^{(1)}\tilde{c}^{(1)}\tilde{c}^{(2)}) \otimes (\tilde{c}_0\ \tilde{c}_0\ \tilde{c}^{(1)}) \otimes (\tilde{c}^{(1)}\tilde{c}^{(3)}) \in B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. The 5-signature is given as follows

$$\begin{array}{lcccccccccccccccccc} b' = & (\tilde{c}^{(2)} & \otimes & \tilde{c}^{(1)} & \otimes & \tilde{c}^{(1)} & \otimes & \tilde{c}_0) & \otimes & (\tilde{c}^{(1)} & \otimes & \tilde{c}_0 & \otimes & \tilde{c}_0) & \otimes & (\tilde{c}^{(3)} & \otimes & \tilde{c}^{(1)}) \\ \eta_5 = & & & - & & - & & + & & - & & + & & + & & & & - \end{array}$$

The reduced signature is $\eta_5 = \overset{2}{-}\ \overset{3}{-}\ \overset{6}{+}$, *where the upper number signifies the component of the tensor product the sign belonged to. Therefore, we have*

$e_5 b' = (\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes e_4(\tilde{c}^{(1)}) \otimes \tilde{c}_0) \otimes (\tilde{c}^{(1)} \otimes \tilde{c}_0 \otimes \tilde{c}_0) \otimes (\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}) = (\tilde{c}_0\ \tilde{c}_0\ \tilde{c}^{(1)}\tilde{c}^{(2)}) \otimes (\tilde{c}_0\ \tilde{c}_0\ \tilde{c}^{(1)}) \otimes (\tilde{c}^{(1)}\tilde{c}^{(3)})$

$f_5 b' = (\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}_0) \otimes (\tilde{c}^{(1)} \otimes f_4(\tilde{c}_0) \otimes \tilde{c}_0) \otimes (\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}) = (\tilde{c}_0\ \tilde{c}^{(1)}\tilde{c}^{(1)}\tilde{c}^{(2)}) \otimes (\tilde{c}_0\ \tilde{c}^{(1)}\tilde{c}^{(1)}) \otimes (\tilde{c}^{(1)}\tilde{c}^{(3)})$

2.4 $B^{n,l}$ and $B^{n-1,l}$ of type $D_n^{(1)}$

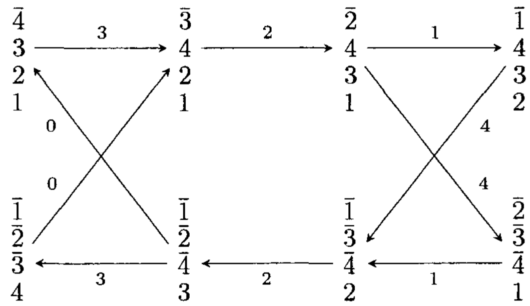
We give an affine crystal action on $B^{n,l}$. To do this we need $B^{n-1,l}$ of type $D_n^{(1)}$. $B^{n,l}$ and $B^{n-1,l}$ are associated to the spin nodes in the Dynkin diagram. As $\{1, 2, ..., n\}$-crystals we have the isomorphisms

$$B^{n,l} \cong B(l\Lambda_n), \quad B^{n-1,l} \cong B(l\Lambda_{n-1}). \tag{2.6}$$

To define the affine crystal action, we introduce an involution $\sigma : B^{n,l} \leftrightarrow B^{n-1,l}$ corresponding to the Dynkin diagram automorphism that interchanges the nodes n to $n-1$. Let $J = \{2, 3, ..., n\}$. J is the J-highest if and only if $e_i b = 0$ for every $i \in J$. By definition in [2], σ is required to commute with e_i, $f_i (i \in J)$. Hence it sufficies to define σ on J-highest elements in $B^{n,l}$ and all of the form of the LHS of (2.7) with some a, and mapped by σ as

$$\sigma : \underbrace{\boxed{\begin{array}{cc} n \cdots n & \bar{1} \cdots \bar{1} \\ \vdots & \bar{n} \cdots \bar{n} \\ 2 \cdots 2 & \vdots \\ 1 \cdots 1 & 2 \cdots 2 \end{array}}}_{a \qquad l-a} \longleftrightarrow \underbrace{\boxed{\begin{array}{cc} \bar{n} \cdots \bar{n} & \bar{1} \cdots \bar{1} \\ \vdots & n \cdots n \\ 2 \cdots 2 & \vdots \\ 1 \cdots 1 & 2 \cdots 2 \end{array}}}_{l-a \qquad a} \tag{2.7}$$

Example 2.8 *When $n = 4$, the crystal graph of $B^{3,1}$ of type $D_4^{(1)}$ is depicted as follows.*



Definition 2.9 *The action e_0 and f_0 on $B^{n,l}$ is given by*

$$e_0 = \sigma \circ e_1 \circ \sigma, \qquad f_0 = \sigma \circ f_1 \circ \sigma.$$

Example 2.10 *Consider the case $B^{n,l}$'s, where $n = 4$ and $l = 5$. Let $c^{(j)}$ $(j = 1, ..., 5)$ as in Example 2.4.*

Consider an element $b = (c_0\ c_0\ c_0\ c^{(3)}c^{(3)}) \in B^{4,5}$. We are to calculate $e_0 b$.

$$\sigma(b) = \boxed{\begin{array}{cc|ccc} \bar4 & \bar4 & \bar1 & \bar1 & \bar1 \\ 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 \end{array}} \tag{2.8}$$

$$e_1 \boxed{\begin{array}{cc|ccc} \bar4 & \bar4 & \bar1 & \bar1 & \bar1 \\ 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 \end{array}} = \boxed{\begin{array}{cc|ccc} \bar4 & \bar4 & \bar2 & \bar1 & \bar1 \\ 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 \end{array}} \tag{2.9}$$

$$f_1 \boxed{\begin{array}{cc|ccc} \bar4 & \bar4 & \bar1 & \bar1 & \bar1 \\ 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 \end{array}} = 0 \tag{2.10}$$

But, σ is required to commute with $e_i, f_i (i \in J)$. Then we apply $e_3 e_2$ to get the J-highest.

$$e_3e_2 \boxed{\begin{array}{ll} \bar{4}\,\bar{4} & \bar{2}\,\bar{1}\,\bar{1} \\ 3\,3 & 4\,4\,4 \\ 2\,2 & 3\,3\,3 \\ 1\,1 & 1\,2\,2 \end{array}} = \boxed{\begin{array}{ll} \bar{4}\,\bar{4} & \bar{4}\,\bar{1}\,\bar{1} \\ 3\,3 & 3\,4\,4 \\ 2\,2 & 2\,3\,3 \\ 1\,1 & 1\,2\,2 \end{array}} = b'. \tag{2.11}$$

Since $\sigma(f_2f_3(b')) = f_2f_3\sigma(b')$ *and* $\sigma(b') = \boxed{\begin{array}{ll} 4\,4 & \bar{1}\,\bar{1}\,\bar{1} \\ 3\,3 & \bar{4}\,\bar{4}\,\bar{4} \\ 2\,2 & 3\,3\,3 \\ 1\,1 & 2\,2\,2 \end{array}}$,

$$e_0b = f_2f_3\sigma(b') = f_2f_3 \boxed{\begin{array}{ll} 4\,4 & \bar{1}\,\bar{1}\,\bar{1} \\ 3\,3 & \bar{4}\,\bar{4}\,\bar{4} \\ 2\,2 & 3\,3\,3 \\ 1\,1 & 2\,2\,2 \end{array}} = \boxed{\begin{array}{ll} 4\,4 & \bar{1}\,\bar{1}\,\bar{1} \\ 3\,3 & \bar{4}\,\bar{4}\,\bar{2} \\ 2\,2 & 3\,3\,4 \\ 1\,1 & 2\,2\,3 \end{array}}.$$

$f_0b = 0.$

2.5 Energy function

Next consider a $\mathbb{Z}$-valued function H on $B \otimes B'$ satisfying the following property: For any $b \in B, b' \in B'$ and i such that $e_i(b \otimes b') \neq 0$

$$H(e_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0 \text{ , } \varphi_0(b) \geq \varepsilon_0(b'),\ \varphi_0(\tilde{b}') \geq \varepsilon_0(\tilde{b}), \\ H(b \otimes b') - 1 & \text{if } i = 0 \text{ , } \varphi_0(b) < \varepsilon_0(b'),\ \varphi_0(\tilde{b}') < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise.} \end{cases}$$

H is known to exist and unique up to additive constant. $\tilde{b}$ and $\tilde{b}'$ are defined from the combinatorial R matrix by $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$. The existences of the isomorphism and energy function H are guaranteed by the existence of the R matrix. See Ref [13].

Note that we normalized H so that we have $H((c_0)^l \otimes (c_0)^{l'}) = 0$. Here and later, $(c_0)^l$ means $\underbrace{c_0 \cdots c_0}_{l} \in B^{n,l}$.

Definition 2.11 *A combinatorial R matrix for the crystal $B^{n,s} \otimes B^{n,t}$ is a map*

$$R : B^{n,s} \otimes B^{n,t} \to B^{n,t} \otimes B^{n,s}$$

satisfying

(1) $R \circ e_i = e_i \circ R, \quad R \circ f_i = f_i \circ R$ *for* $i = 1, ..., n$, *and*

(2)

$$R\left(\overbrace{\boxed{c_0}\cdots\boxed{c_0}}^{s}\otimes\overbrace{\boxed{c_0}\cdots\boxed{c_0}}^{t_0}\overbrace{\boxed{c_1}\cdots\boxed{c_1}}^{t_1}\overbrace{\boxed{c_2}\cdots\boxed{c_2}}^{t_2}\cdots\overbrace{\boxed{c_{n'}}\cdots\boxed{c_{n'}}}^{t_{n'}}\right) \tag{2.12}$$

$$=\overbrace{\boxed{c_0}\cdots\boxed{c_0}}^{t}\otimes\overbrace{\boxed{c_0}\cdots\boxed{c_0}}^{s'}\overbrace{\boxed{c_1}\cdots\boxed{c_1}}^{t_1}\overbrace{\boxed{c_2}\cdots\boxed{c_2}}^{t_2}\cdots\overbrace{\boxed{c_{n'}}\cdots\boxed{c_{n'}}}^{t_{n'}} \tag{2.13}$$

$$\textit{where } c_0 = \begin{array}{c} n \\ n-1 \\ n-2 \\ n-3 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array}, \quad c_1 = \begin{array}{c} \overline{n-1} \\ \bar{n} \\ n-2 \\ n-3 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array}, \quad c_2 = \begin{array}{c} \overline{n-3} \\ \overline{n-2} \\ \overline{n-1} \\ \bar{n} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array}, \cdots$$

$$c_{n'} = \begin{array}{c} \bar{2} \\ \bar{3} \\ \cdot \\ \cdot \\ \cdot \\ \bar{n} \\ 1 \end{array} \ \textit{(if } n \textit{ is an odd number)}, \qquad c_{n'} = \begin{array}{c} \bar{1} \\ \bar{2} \\ \bar{3} \\ \cdot \\ \cdot \\ \cdot \\ \bar{n} \end{array} \ \textit{(if } n \textit{ is an even number)},$$

$n' = [\frac{n}{2}]$ *and* $t_1 + t_2 + \cdots + t_{n'} \leq s$, $\quad s' = s - (t_1 + t_2 + \cdots + t_{n'})$.

We explain how to calculate $R(b)$ for a general element b. Let $b \xrightarrow{e_{a_1}} b_1 \xrightarrow{e_{a_2}} b_2 \cdots b_{m-2} \xrightarrow{e_{a_{m-1}}} b_{m-1} \xrightarrow{e_{a_m}} \tilde{b}$, where $e_i(\tilde{b}) = 0$ for every $i \neq 0$. Namely, $\tilde{b}$ is of the form

$$b = \overbrace{\boxed{c_0}\cdots\boxed{c_0}}^{s} \otimes \overbrace{\boxed{c_0}\cdots\boxed{c_0}}^{t_0}\overbrace{\boxed{c_1}\cdots\boxed{c_1}}^{t_1}\overbrace{\boxed{c_2}\cdots\boxed{c_2}}^{t_2}\cdots\overbrace{\boxed{c_{n'}}\cdots\boxed{c_{n'}}}^{t_{n'}}$$

$R(b) = R(f_{a_1} \cdots f_{a_m} \tilde{b}) = f_{a_1} \cdots f_{a_m} R(\tilde{b}).$

Example 2.12

Set $b = \boxed{\begin{array}{ccc} \bar{3} & \bar{3} & \bar{1} \\ \bar{4} & \bar{4} & \bar{3} \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{array}} \otimes \boxed{\begin{array}{cc} \bar{2} & \bar{2} \\ \bar{4} & \bar{3} \\ 3 & 4 \\ 1 & 1 \end{array}}$ *and we calculate* $R(b)$.

$\tilde{b} = e_4 {e_2}^3 {e_4}^2 {e_3}^2 e_1 b = \boxed{\begin{array}{ccc} 4 & 4 & 4 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{array}} \otimes \boxed{\begin{array}{cc} \bar{3} & \bar{3} \\ \bar{4} & \bar{4} \\ 2 & 2 \\ 1 & 1 \end{array}}$ *and* $R(\tilde{b}) = \boxed{\begin{array}{cc} 4 & 4 \\ 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{array}} \otimes \boxed{\begin{array}{ccc} 4 & \bar{3} & \bar{3} \\ 3 & \bar{4} & \bar{4} \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{array}}$.

Since $f_1 {f_3}^2 {f_4}^2 {f_2}^3 f_4 R(\tilde{b}) = \boxed{\begin{array}{cc} \bar{3} & \bar{3} \\ \bar{4} & \bar{4} \\ 2 & 2 \\ 1 & 1 \end{array}} \otimes \boxed{\begin{array}{ccc} \bar{2} & \bar{2} & \bar{1} \\ \bar{4} & \bar{3} & \bar{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2 \end{array}}$,

we have $R\left(\boxed{\begin{array}{ccc} \bar{3} & \bar{3} & \bar{1} \\ \bar{4} & \bar{4} & \bar{3} \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{array}} \otimes \boxed{\begin{array}{cc} \bar{2} & \bar{2} \\ \bar{4} & \bar{3} \\ 3 & 4 \\ 1 & 1 \end{array}} \right) = \boxed{\begin{array}{cc} \bar{3} & \bar{3} \\ \bar{4} & \bar{4} \\ 2 & 2 \\ 1 & 1 \end{array}} \otimes \boxed{\begin{array}{ccc} \bar{2} & \bar{2} & \bar{1} \\ \bar{4} & \bar{3} & \bar{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2 \end{array}}$.

Lemma 2.13 *We give all the images of the isomorphism* $B^{n,l} \otimes B^{n,1} \simeq B^{n,1} \otimes B^{n,l}$ *that will be needed in the next section. Let the symbol*

$$\begin{array}{ccc} & \beta & \\ \alpha & + & \alpha' \\ & \beta' & \end{array}$$

signify $\alpha \otimes \beta \mapsto \beta' \otimes \alpha'$ *under the isomorphism. We also set*

$$c_a = \begin{array}{c} \overline{n-1} \\ \bar{n} \\ n-2 \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_b = \begin{array}{c} \overline{n-2} \\ \bar{n} \\ n-1 \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_d = \begin{array}{c} \overline{n-3} \\ \overline{n-2} \\ n \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_e = \begin{array}{c} \overline{n-3} \\ \overline{n-1} \\ n \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_f = \begin{array}{c} \overline{n-3} \\ \overline{n-2} \\ \overline{n-1} \\ \bar{n} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array}$$

and c_0 is the column of height n without barred letters.

We list up the cases as below where $i, j, k, l, m, p \in \mathbb{Z}_{\geq 0}$.

Case 1.

(i) $m > 0$

$$c_0^i c_a^j c_b^k c_d^l c_f^m \overset{c_0}{\underset{c_f}{+}} c_0^{i+1} c_a^j c_b^k c_d^l c_f^{m-1}$$

(ii) $l > 0$

$$c_0^i c_a^j c_b^k c_d^l \overset{c_0}{\underset{c_d}{+}} c_0^{i+1} c_a^j c_b^k c_d^{l-1}$$

(iii) $k > 0$

$$c_0^i c_a^j c_b^k \overset{c_0}{\underset{c_b}{+}} c_0^{i+1} c_a^j c_b^{k-1}$$

(iv) $j > 0$

$$c_0^i c_a^j \overset{c_0}{\underset{c_a}{+}} c_0^{i+1} c_a^{j-1}$$

(v) $i > 0$

$$c_0^i \overset{c_0}{\underset{c_0}{+}} c_0^i$$

Case 2.

(i) $i > 0$, $m > 0$

$$c_0^i c_a^j c_b^k c_d^l c_f^m \overset{c_a}{\underset{c_a}{+}} c_0^{i-1} c_a^{j+1} c_b^k c_d^{l+1} c_f^{m-1}$$

(ii) $m > 0$

$$c_a^j c_b^k c_d^l c_f^m \overset{c_a}{\underset{c_f}{+}} c_a^{j+1} c_b^k c_d^l c_f^{m-1}$$

(iii) $i > 0$

$$c_0^i c_a^j c_b^k c_d^l \overset{c_a}{\underset{c_0}{+}} c_0^{i-1} c_a^{j+1} c_b^k c_d^l$$

(iv) $l > 0$

$$c_a^j c_b^k c_d^l \overset{c_a}{\underset{c_d}{+}} c_a^{j+1} c_b^k c_d^{l-1}$$

(v) $k > 0$

$$c_a^j c_b^k \overset{c_a}{\underset{c_b}{+}} c_a^{j+1} c_b^{k-1}$$

(vi) $j > 0$

$$c_a^j \overset{c_a}{\underset{c_a}{+}} c_a^j$$

Case 3.

(i) $j > 0,\, l > 0$

$$c_0^i c_a^j c_b^k c_d^l c_f^m \underset{c_b}{\overset{c_b}{+}} c_0^i c_a^{j-1} c_b^{k+1} c_d^{l-1} c_e c_f^m$$

(ii) $j > 0$

$$c_0^i c_a^j c_b^k c_e^p c_f^m \underset{c_a}{\overset{c_b}{+}} c_0^i c_a^{j-1} c_b^{k+1} c_e^p c_f^m$$

(iii) $p > 0$

$$c_0^i c_b^k c_d^l c_e^p c_f^m \underset{c_e}{\overset{c_b}{+}} c_0^i c_b^{k+1} c_d^l c_e^{p-1} c_f^m$$

(iv) $i > 0$

$$c_0^i c_b^k c_e^p c_f^m \underset{c_a}{\overset{c_b}{+}} c_0^{i-1} c_b^{k+1} c_d c_e^p$$

(v) $m > 0,\; i > 0$

$$c_0^i c_b^k c_d^l c_f^m \underset{c_a}{\overset{c_b}{+}} c_0^{i-1} c_b^{k+1} c_d^{l+1} c_f^{m-1}$$

(vi) $l > 0$

$$c_b^k c_e^p c_d^l \underset{c_d}{\overset{c_b}{+}} c_b^{k+1} c_e^p c_d^{l-1}$$

(vii) $m > 0$

$$c_b^k c_d^l c_f^m \underset{c_f}{\overset{c_b}{+}} c_b^{k+1} c_d^l c_f^{m-1}$$

(viii) $i > 0$

$$c_0^i c_b^k c_d^l \underset{c_0}{\overset{c_b}{+}} c_0^{i-1} c_b^{k+1} c_d^l$$

(ix) $p > 0$

$$c_b^k c_e^p \underset{c_e}{\overset{c_b}{+}} c_b^{k+1} c_e^{p-1}$$

(x) $l > 0$

$$c_b^k c_d^l \underset{c_d}{\overset{c_b}{+}} c_b^{k+1} c_d^{l-1}$$

(xi) $k > 0$

$$c_b^k \underset{c_b}{\overset{c_b}{+}} c_b^k$$

Case 4.

(i) $j > 0,\, l > 0$

$$c_0^i c_a^j c_b^k c_d^l c_f^m \underset{c_d}{\overset{c_d}{+}} c_0^{i+1} c_a^{j-1} c_b^k c_d^{l-1} c_f^{m+1}$$

(ii) $j > 0,\; k > 0$

$$c_0^i c_a^j c_b^k c_f^m \underset{c_b}{\overset{c_d}{+}} c_0^{i+1} c_a^{j-1} c_b^{k-1} c_f^{m+1}$$

(iii) $k > 0$

$$c_0^i c_b^k c_d^l c_f^m \underset{c_b}{\overset{c_d}{+}} c_0^i c_b^{k-1} c_d^{l+1} c_f^m$$

(iv) $j > 0$

$$c_0^i c_a^j c_f^m \underset{c_a}{\overset{c_d}{+}} c_0^{i+1} c_a^{j-2} c_f^{m+1}$$

(v) $i > 0,\; j > 0,\; m > 0.$

$$c_0^i c_d^l c_f^m \underset{c_a}{\overset{c_d}{+}} c_0^{i-1} c_a^{j-1} c_d^3 c_f^{m-1}$$

(vi) $k > 0$

$$c_0^i c_b^k c_f^m \underset{c_b}{\overset{c_d}{+}} c_0^i c_b^{k-1} c_d c_f^m$$

(vii) $i > 0,\, m > 0$

(viii) $l > 0,\; j > 0$

(ix) $i > 0$

$c_0^i c_d^l c_f^m \overset{c_d}{\underset{c_a}{+}} c_0^{i-1} c_d^{l+2} c_f^{m-1}$ $\qquad$ $c_0^i c_a^j c_d^l \overset{c_d}{\underset{c_d}{+}} c_0^{i+1} c_a^{j-1} c_d^{l-1} c_f$ $\qquad$ $c_0^i c_d^l \overset{c_d}{\underset{c_0}{+}} c_0^{i-1} c_d^{l+1}$

(x) $j > 0,\ l > 0$

$c_a^j c_d^l c_f^m \overset{c_d}{\underset{c_d}{+}} c_0^i c_a^{j-1} c_d^{l-1} c_f^{m+1}$

(xi) $i > 0,\ m > 0$

$c_0^i c_f^m \overset{c_d}{\underset{c_a}{+}} c_0^{i-1} c_d^2 c_f^{m-1}$

(xii) $j > 0$

$c_0^i c_a c_f^m \overset{c_d}{\underset{c_a}{+}} c_0^i c_d c_f^m$

(xiii) $i > 0,\ m > 0$

$c_0^i c_d^l c_f^m \overset{c_d}{\underset{c_a}{+}} c_0^{i-1} c_d^{l+2} c_f^{m-1}$

(ivx) $m > 0$

$c_d^l c_f^m \overset{c_d}{\underset{c_f}{+}} c_d^{l+1} c_f^{m-1}$

(ix) $l > 0$

$c_d^l \overset{c_d}{\underset{c_d}{+}} c_d^l$

Case 5.

(i) $i > 0$

$c_0^i c_a^j c_b^k c_d^l c_f^m \overset{c_f}{\underset{c_0}{+}} c_0^{i-1} c_a^j c_b^k c_d^l c_f^{m+1}$

(ii) $l > 0$

$c_a^j c_b^k c_d^l c_f^m \overset{c_f}{\underset{c_d}{+}} c_a^j c_b^k c_d^{l-1} c_f^{m+1}$

(iii) $k > 0$

$c_a^j c_b^k c_f^m \overset{c_f}{\underset{c_b}{+}} c_a^j c_b^{k-1} c_f^{m+1}$

(iv) $j > 0$

$c_a^j c_f^m \overset{c_f}{\underset{c_a}{+}} c_a^{j-1} c_f^{m+1}$

(v) $m > 0$

$c_f^m \overset{c_f}{\underset{c_f}{+}} c_f^m$

Proof for Case 1 where $k > 0$.
$(c_b^k \otimes c_a^j \otimes c_0^i) \otimes c_0 \xrightarrow{e_{n-2}^k} (c_a^{j+k} \otimes c_0^i) \otimes c_0 \xrightarrow{e_n^{j+k}} c_0^{i+j+k} \otimes c_0$. Reform $c_0^{i+j+k} \otimes c_0$ to $c_0 \otimes (c_0 \otimes c_0^{i+j+k})$. Then $c_0 \otimes c_0^{i+j+k} \xrightarrow{f_n^{j+k}} c_a \otimes (c_a^{j+k-1} \otimes c_0^{i+1}) \xrightarrow{f_{n-2}^k} c_b \otimes (c_b^{k-1} \otimes c_a^j \otimes c_0^{i+1})$. Hence $R(c_0^i c_a^j c_b^k \otimes c_0) = c_b \otimes c_0^{i+1} c_a^j c_b^{k-1}$.

Definition 2.14 *A combinatorial R matrix for the crystal $B \otimes B'$ is a map $R : Aff(B) \otimes Aff(B') \to Aff(B') \otimes Aff(B)$ given by*

$$R(z^d b \otimes z^{d'} b') = z^{d' + H(b \otimes b')} \tilde{b}' \otimes z^{d - H(b \otimes b')} \tilde{b}$$

where $b \otimes b' \mapsto \tilde{b}' \otimes \tilde{b}$ under the isomorphism $B \otimes B' \xrightarrow{\sim} B' \otimes B$. The following result is a direct consequence of the ordinary (i.e. not combinatorial) Yang-Baxter equation.

2.6 Yang-Baxter equation

Let us define the affinization Aff(B) of the crystal B. We introduce an indeterminate z (the spectral parameter) and set

$$\mathrm{Aff}(B) = \{z^d b \mid d \in \mathbb{Z}, b \in B\}.$$

Thus Aff(B) is an infinite set. $z^0 b \in$Aff(B) will often be written as b. Aff(B) also admits the crystal structure by $e_i.z^d b = z^{d+\delta_{i0}}(e_i b)$, $f_i.z^d b = z^{d-\delta_{i0}}(f_i b)$.

Proposition 2.15 (*Yang-Baxter equation*). *Let $B^{n,l} = B_l$. The following equation hold on Aff$(B_l) \otimes$ Aff$(B_{l'}) \otimes$ Aff$(B_{l''})$.*

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R).$$

3 Soliton cellular automata

3.1 States and time evolutions.

Consider the crystal $(B^{n,1})^{\otimes N}$ for sufficiently large N. The elements of $(B^{n,1})^{\otimes N}$ we have in mind are of the following form:

$$\cdots \otimes c_0 \otimes \cdots \otimes c_0 \otimes c_1 \otimes \cdots \otimes c_l \otimes c_0 \otimes \cdots \otimes c_0 \otimes \cdots ,$$

Namely, relatively few elements are non c_0, and almost all are c_0. In the assertions below, we embed, if necessary, $(B^{n,1})^{\otimes N}$ into $(B^{n,1})^{\otimes N'} (N < N')$ by

$$(B^{n,1})^{\otimes N} \hookrightarrow (B^{n,1})^{\otimes N'},$$

$$b_1 \otimes \cdots \otimes b_N \mapsto b_1 \otimes \cdots \otimes b_N \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{N'-N}.$$

Lemma 3.1 *By iterating $B^{n,l} \otimes B^{n,1} \to B^{n,1} \otimes B^{n,l}$ we consider a map*

$$B^{n,l} \otimes B^{n,1} \otimes \cdots \otimes B^{n,1} \xrightarrow{\sim} B^{n,1} \otimes \cdots \otimes B^{n,1} \otimes B^{n,l},$$

$$(c_0)^l \otimes b_1 \otimes \cdots \otimes b_N \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_N \otimes \tilde{b},$$

then there exists an integer N_0 such that $\tilde{b} = (c_0)^l$ for $N \geq N_0$.

Taking sufficiently large N such that the above lemma holds, we define a map $T_l : (B^{n,1})^{\otimes N} \longrightarrow (B^{n,1})^{\otimes N}$ by $b_1 \otimes \cdots \otimes b_N \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_N$.

Lemma 3.2 *For a fixed element of $(B^{n,1})^{\otimes N}$ as a Lemma 3.1, there exists an integer l_0 such that $T_l = T_{l_0}$ for any $l \geq l_0$.*

Both lemmas are obvious from Lemma 2.13.

An element of $(B^{n,1})^{\otimes N}$ having the property described in the beginning of this subsection will be called a ***state***. Lemma 3.1 and Lemma 3.2 enable us to define an operator $T = \lim_{l\to\infty} T_l$ on the space of states. Application of T induces a transition of state. Thus it can be regarded as a certain dynamical system, in which T plays the role of 'time evolution'. By the same reason, T_l may also be viewed as another time evolution. (In this paper, time evolution means the one by T unless otherwise stated.)

3.2 Conservation laws

Fix sufficiently large N and consider a composition of the combinatorial R matrices

$$R_l = R_{NN+1} \cdots R_{23} R_{12} : \mathrm{Aff}(B^{n,l}) \otimes \mathrm{Aff}(B^{n,1})^{\otimes N} \to \mathrm{Aff}(B^{n,1})^{\otimes N} \otimes \mathrm{Aff}(B^{n,l}).$$

Here R_{ii+1} signifies that the R matrix acts on the i-th and $(i+1)$-th components of the tensor product. Applying R_l to an element $(c_0)^l \otimes p$ ($p = b_1 \otimes \cdots \otimes b_N$), we have

$$R_l((c_0)^l \otimes p) = z^{H_1} \tilde{b}_1 \otimes z^{H_2} \tilde{b}_2 \otimes \cdots \otimes z^{H_N} \tilde{b}_N \otimes z^{\tilde{E}_l(p)} (c_0)^l,$$

$$\tilde{E}_l(p) = -\sum_{j=1}^{N} H_j, \quad H_j = H(b^{(j-1)} \otimes b_j),$$

where $b^{(0)} = (c_0)^l$ and $b^{(j)}$ $(1 \leq j < N)$ is defined by

$$B^{n,l} \otimes \underbrace{B^{n,1} \otimes \cdots \otimes B^{n,1}}_{j} \simeq \underbrace{B^{n,1} \otimes \cdots \otimes B^{n,1}}_{j} \otimes B^{n,l},$$

$$(c_0)^l \otimes b_1 \otimes \cdots \otimes b_j \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_j \otimes b^{(j)}.$$

Lemma 3.3 *Let H be the energy function and $-H_j = -H(b^{(j-1)} \otimes b_j) \in \{0, 1, 2, ..., n'\}$. Then $b^{(j-1)} \otimes b_j$ commute with e_i, f_i until it will get in form $c_0^i c_d^l \otimes c_d \mapsto c_0 \otimes c_0^{i-1} c_d^{l+1}$. as in Case 4 in Lemma 2.13. Then by using the rule in [24] we will get $-H_j = 1$ if and only if $c_0^i c_d^l \otimes c_d \mapsto c_0 \otimes c_0^{i-1} c_d^{l+1}$.*

Proof. Use the definition of energy function in [31] to prove it directly.

Proposition 3.4 *For an element* $p \in (B^{n,1})^{\otimes N}$, *we have*
(1) $T_l T_{l'}(p) = T_{l'} T_l(p)$.
(2) $E_l(T_{l'}(p)) = E_l(p)$. *In particular,* $E_l(T(p)) = E_l(p)$.

We refer to [3] for the proof.

3.3 Soliton

A state of the following form is called an m soliton state of length l_1, l_2, ..., l_m,

$$...[l_1]........[l_2]..... \cdots .....[l_m]......... \tag{3.1}$$

Here $\cdots [l] \cdots$ denotes a local configuration such as

$$\cdots \otimes c_0 \otimes c_0 \otimes \underbrace{c_l \otimes c_{l-1} \otimes \cdots \otimes c_1}_{l} \otimes c_0 \otimes c_0 \otimes \cdots, \qquad (c_0 \preceq c_1 \preceq c_2 \preceq \cdots \preceq c_l)$$

where c_0 means column without barred letter and $c_j \in B^{n,1}$ with exactly two barred letters. That means we cannot form the soliton as above when the number of barred letter is more than two. $c \preceq c'$ if and only if a_j (j-th entry of c), $a_j \preceq a'_j$ for all $j = 1, 2, ..., n$.

Remark. It would be an interesting problem to consider color separation scheme in [34]: §4.7]. A reasonable choice of $B_\natural$ such as $B^{n-1,1}$ or $B^{n-2,1}$ seems to fail for $n = 4$.

Lemma 3.5 *Let* p *be a one-soliton state of length* l, *then*
(1) *The* k*th conserved quantity of* p *is given by* $E_k(p) = min(k, l)$.
(2) *The state* $T_k(p)$ *is obtained by the rightward shift by* $E_k(p)$ *lattice steps.*

Proof. (1) Recall that the conserved quantity E_k is a sum of local H functions $-H_j = -H(b^{(j-1)} \otimes b_j)$. $b^{(j-1)} \otimes b_j \mapsto \tilde{b}_j \otimes b^{(j)}$ commutes with $f_i (i \in 1, 2, ..., n-1)$ until we will get the lowest weight where $f_i b = 0$ for any $i \in \{1, 2, ..., n-1\}$. Then we apply f_0, e_0 and we will get $(c_0)^k \otimes c_0 \mapsto c_0 \otimes (c_0)^k$. By using Lemma 3.3, $-H_j = 1$ if and only if $(c_0)^i c_d^l \otimes c_d \mapsto c_0 \otimes (c_0)^{i-1} c_d^{l+1}$. $E_k(p)$ is the sum of the local $-H_j = 1$ in the tensor product in $B^{n,l}$. Hence E_k=min(k, l). Similarly, the statement (2) follows from the rule (2.3) and Lemma 3.1.

Definition 3.6 *For any state* p, *the number* $N_l = N_l(p)$ $(l = 1, 2, ...)$ *are defined by*

$$E_l = \textstyle\sum_{k \geq 1} min(k, l) N_k, \qquad E_0 = 0, \qquad N_i = -E_{l-1} + 2E_l - E_{l+1}.$$

By Lemma 3.5, we have

Proposition 3.7 *For m-soliton state* (3.1)*, N_l is the number of solitons of length l,*

$$N_l = \#\{j | l_j = l\}.$$

This proposition implies the stability of solitons, since the number $E_l(p)$, and hence $N_l(p)$, are conserved.

3.4 Type A $B^{r,s}$

In this subsection, we recall the crystal structure of $B^{r,s}$ for arbitrary r, s and the combinatorial R for $B^{r,s} \otimes B^{r',s'}$. Our reference is [31]. We use the French notation for semistandard tableau, which is upside-down from [31].

Our $U_q'(A_{n-1}^{(1)})$-crystal $B^{r,s}$ $(1 \leq r \leq n-1,\ s \in \mathbb{Z}_{>0})$ is as a set, identified with the set of semistandard tableau of rectangular shape (s^r) with letters from $\{1, 2, ..., n\}$. For an element t of $B^{r,s}$, let t_{ij} denote the letter in the i-th row from bottom and j-th column of t. We first describe the action of $e_i,\ f_i$ for $i = 1, 2, ..., n-1$. For this purpose, let us define the **Japanese reading word** of t by

$$J(t) = w^{(1)} w^{(2)} \cdots w^{(s)}, \qquad w^{(j)} = t_{1j} t_{2j} \cdots t_{rj} \qquad (j = 1, 2, ..., s).$$

We then regard $J(t)$ as an element of $(B^{1,1})^{\otimes (rs)}$. Namely, each letter is considered to be an element of $B^{1,1}$.

The bumping algorithm is defined for a pair of tableau t and single word u and depicted as $t \leftarrow u$. First, let us consider the case where t is one-row tableau. If t is empty, $t \leftarrow u$ is defined to be the tableau u with one node. Otherwise, let $t = t_{11} t_{12} \cdots t_{1m}$ and look at

$$t_{11} t_{12} \cdots t_{1m} \leftarrow u.$$

If $t_{1m} \leq u$, then define

$$t \leftarrow u = t_{11} t_{12} \cdots t_{1m} u$$

and the algorithm stops (case (a)). Otherwise, set $i_1 = \min\{i \ | t_{1i} > u\}$ and define

$$t \leftarrow u = t_{11} \cdots t_{1i_1-1} u t_{1i_1+1} \cdots t_{1m}$$

and we have the single word t_{1i_1} bumped out from t (case (b)). Now suppose we have a tableau t of l rows and let t_i be the i-th row of t. The bumping algorithm $t \leftarrow u$ proceeds as follows. Set $t'_1 = t_1 \leftarrow u$. If case (a) occurs, the algorithm stops. Otherwise, let u_1 be the letter bumped out and set $t'_2 = t_2 \leftarrow u_1$. We again devide the algorithm into the two cases. The algorithm proceeds until it stops. If case (b) still occurs in the highest row, we append the empty row above it.

Example 3.8 *Let*

$$t = \begin{matrix} 23444 \\ 11122 \end{matrix}$$

and $u = 1$. *The bumping algorithm proceeds as follows.*

$$\begin{matrix} 23444 & \\ 111\mathit{2}2 & \leftarrow 1 \end{matrix} \qquad \begin{matrix} 2\mathit{3}444 & \leftarrow 2 \\ 11112 & \end{matrix}$$

And we have the answer.

$$\begin{matrix} 3 \\ 22444 \\ 11112 \end{matrix}$$

Example 3.9 *Let*

$$t = \begin{matrix} 24555 \\ 11334 \end{matrix}$$

and $u = 2$. *The bumping algorithm proceeds as follows.*

$$\begin{matrix} 24555 & \\ 11\mathit{3}34 & \leftarrow 2 \end{matrix} \qquad \begin{matrix} 2\mathit{4}555 & \leftarrow 3 \\ 11234 & \end{matrix}$$

And we have the answer.

$$\begin{matrix} 4 \\ 23555 \\ 11234 \end{matrix}$$

For a tableau $t \in B^{r,s}$ we define the **row word** *row* t by

$$row(t) = t_r t_{r-1} \cdots t_1 \qquad t_i = t_{i1} t_{i2} \cdots t_{is} \quad (i = 1, 2, ..., r).$$

Let t be a tableau and $w = u_1u_2\cdots u_l$ a word of length l. Let $t \leftarrow w$ be a tableau obtained by applying the bumping algorithm for a single word u_j successively as

$$(\cdots((t \leftarrow u_1) \leftarrow u_2) \leftarrow \cdots) \leftarrow u_l.$$

Then we have the following proposition to obtain the combinatorial R for $B^{r,s} \otimes B^{r,m}$.

Proposition 3.10 *[31] Assume $t \in B^{r,m}$ and $t' \in B^{r,s}$. Then $t' \otimes t$ is mapped to $\tilde{t} \otimes \tilde{t}'$ by the crystal isomorphism*

$$B^{r,s} \otimes B^{r,m} \longrightarrow B^{r,m} \otimes B^{r,s} \quad \textit{if and only if} \quad t \leftarrow row(t') = \tilde{t}' \leftarrow row(\tilde{t}).$$

Moreover, the energy function $H(t' \otimes t)$ is given by the number of nodes in the shape of $t \leftarrow row(t')$ that are strictly north of the r-th row.

Note that the decomposition of $B^{r,s} \otimes B^{r,m}$ into $U_q(A^{(1)}_{n-1})$-crystals is multiplicity free. From this fact, it follows that for a given pair $t' \otimes t$ we can determine $\tilde{t}, \tilde{t}'$ uniquely. To explain the algorithm of computing $\tilde{t}, \tilde{t}'$ we prepare terminology. Let θ be a skew tableau, that is, set-theoretical difference of a Young diagram from a smaller one with letters in each node. Let τ be the shape of θ. θ is called a vertical k-strip if $|\tau| = k$ and $\tau_i \leq 1$ for any $i \geq 1$. The algorithm to obtain $\tilde{t}, \tilde{t}'$ is given as follows. Let p be the tableau obtained by the bumping algorithm $t \leftarrow row(t')$. We attach an integer from 1 to rm to each node of the skew tableau $p - p'$, where p' is the NE part of p whose shape is (s^r). The integer should be labeled in the following manner. Let θ_1 be the rightmost vertical r-strip in $p - p'$ as lower as possible. We attach integers 1 through r from lower nodes. Remove θ_1 from $p - p'$ and define the vertical r-strip θ_2 in similar manner. Continue it until we finish attaching all integers up to rm. Next we apply the reverse bumping algorithm according to the order of the labeling. Namely we find a word u_1 and a tableau p_1 whose shape is (shape of p)-(node of label 1), such that $p_1 \leftarrow u_1 = p$. (Note that such a pair p_1, u_1) is unique.) We repeat this procedure to obtain u_2 and p_2 by replacing p and the node of label 1 with p_1 and the node of label 2 and continue until we arrive at a tableau of shape (s^r). Then we have

$$\tilde{t} = ((\cdots(\phi \leftarrow u_{rm}) \leftarrow \cdots) \leftarrow u_2) \leftarrow u_1 \text{ and } \tilde{t}' = p_{rm}.$$

Note that in [31], the energy function $H(t' \otimes t)$ is given by the number of nodes in the shape of $t' \leftarrow \mathrm{row}(t)$ that are strictly east of the $\max(m, s)$-th column.

We introduce ν as a map sending an element b of the $U'_q(D^{(1)}_n)$-crystal $B^{n,l}$ to νb in the $U'_q(A^{(1)}_{n-1})$-crystal $B^{2,l}_A$. The operator ν will change $D^{(1)}_n$ to $A^{(1)}_{n-1}$ as a set, identified with the set of semistandard tableaux of rectangular shape (l^2) with letters from $\{1,2,...,n\}$. That means the element νb in $B_A^{2,l}$ is the barred letter from $U'_q(D_n^{(1)})$-crystal $B^{n,l}$ read from right to left. We give a restriction to the element b of the $U'_q(D_n^{(1)})$-crystal $B^{n,l}$ with exactly two barred letters.

Lemma 3.11 *Let* $\nu : B^{n,l} \to B_A^{2,l}$, *then* $\nu e_i b = f_i \nu b$ *where* $b \in B^{n,l}$ *and* $i \in \{1,2,...,n-1\}$.

Proof.
LHS; The action e_i for $i \neq 0$ act on $U_q(D_n^{(1)})$-crystal can be calculated by using the rule called **signature rule**. In this case e_i will change the elements in $U_q(D_n^{(1)})$-crystal $B^{n,l}$ from $\bar{i}$ (resp. $i+1$) to i (resp. $\overline{i+1}$). Finally, the operator ν acts on $e_i b$ and we can see the element i commute to $i+1$.

RHS; The map ν sends the $U_q(D_n^{(1)})$-crystal $B^{n,l}$ to $U_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l}$. That means the element $t = \nu b$ is the barred letter from $U_q(D_n^{(1)})$-crystal $B^{n,s}$ read from right to left. Note that for i-signature, letter i (resp. $i+1$) corresponds to $+$ (resp. $-$). In this case f_i is given by applying the signature rule in subsection 3.4 to $J(t)$. Hence, f_i will commute the elements i to $i+1$.

We also introduce ν as a map sending an element $b_1 \otimes b_2$ of the $U'_q(D_n^{(1)})$-crystal $B^{n,l_1} \otimes B^{n,l_2}$ to $\nu(b_2) \otimes \nu(b_1)$ in the $U'_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l_2} \otimes B_A^{2,l_1}$.

Lemma 3.12 *Let* $\nu : B^{n,l_1} \otimes B^{n,l_2} \to B_A^{2,l_2} \otimes B_A^{2,l_1}$, *then*

(1) $\nu e_i(b_1 \otimes b_2) = f_i \nu(b_1 \otimes b_2)$, (2) $\nu f_i(b_1 \otimes b_2) = e_i \nu(b_1 \otimes b_2)$,

where $b_1 \in B^{n,l_1}$, $b_2 \in B^{n,l_2}$ *and* $i \in \{1,...,n-1\}$.

Proof.
We only prove (1). Notice that

$$\nu e_i(b_1 \otimes b_2) = \begin{cases} \nu(b_2) \otimes \nu(e_i b_1) & \text{if } e_i(b_1 \otimes b_2) = e_i b_1 \otimes b_2 \\ \\ \nu(e_i b_2) \otimes \nu(b_1) & \textit{otherwise}, \end{cases}$$

$$
f_i\nu(b_1 \otimes b_2) = \begin{cases} \nu(b_2) \otimes f_i\nu(b_1) & \text{if } f_i(\nu(b_2) \otimes \nu(b_1)) = \nu(b_2) \otimes f_i\nu(b_1) \\ \\ f_i\nu(b_2) \otimes \nu(b_1) & \textit{otherwise.} \end{cases}
$$

Suppose the $i-$ singature of $b_1 \otimes b_2$ is given by

$$
\underbrace{-...-}_{a}\ \underbrace{+...+}_{b}\ \underbrace{-...-}_{c}\ \underbrace{+...+}_{d}
$$

where a, b are for b_1 and c, d are for b_2. Then by Lemma 3.11 the $i-$signature of $\nu(b_2) \otimes \nu(b_1)$ is given by

$$
\underbrace{-...-}_{d}\ \underbrace{+...+}_{c}\ \underbrace{-...-}_{b}\ \underbrace{+...+}_{a}.
$$

Thus $e_i(b_1 \otimes b_2) = e_ib_1 \otimes b_2$ if and only if $f_i(\nu(b_2) \otimes \nu(b_1)) = \nu(b_2) \otimes f_i\nu(b_1)$. The claim then follow by Lemma 3.11.

Example 3.13 *We check 3.12 (1) by an example.*

Let $b = b_1 \otimes b_2 = \boxed{\begin{matrix} \bar{3} & \bar{3} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{3} & \bar{2} \\ 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & 3 \end{matrix}} \otimes \boxed{\begin{matrix} \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \bar{4} & \bar{4} & \bar{2} & \bar{2} \\ 3 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 \end{matrix}}$ *where* $b_1 \in B^{4,6}$, $b_2 \in B^{4,4}$ *in type* $D_4^{(1)}$.

$$
LHS = (\nu \circ e_2)(b) = \nu \circ \left(\boxed{\begin{matrix} \bar{3} & \bar{3} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{3} & \bar{2} \\ 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & 3 \end{matrix}} \otimes \boxed{\begin{matrix} \bar{3} & \bar{1} & \bar{1} & \bar{1} \\ \bar{4} & \bar{4} & \bar{2} & \bar{2} \\ 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 \end{matrix}} \right) = \begin{matrix} 2244 \\ 1113 \end{matrix} \otimes \begin{matrix} 234444 \\ 111233 \end{matrix}.
$$

$$
RHS = (f_2 \circ \nu)(b) = f_2 \circ \begin{pmatrix} 2244 & \otimes & 234444 \\ 1112 & & 111233 \end{pmatrix} = \begin{matrix} 2244 \\ 1113 \end{matrix} \otimes \begin{matrix} 234444 \\ 111233 \end{matrix}.
$$

3.5 Scattering of solitons

The following is an example of the time evolution (for $t = 0, 1, ..., 6$) of a state which shows the scattering of three solitons of length 3, 2 and 1.

Example 3.14

$$
\begin{array}{l}
c^{(5)}\ c^{(4)}\ c^{(3)}\ \cdot\ \cdot\ c^{(2)}\ c^{(1)}\ \cdot\ \cdot\ c^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ c^{(5)}\ c^{(4)}\ \cdot\ \cdot\ c^{(3)}\ c^{(2)}\ c^{(1)}\ c^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ c^{(5)}\ c^{(4)}\ \cdot\ \cdot\ \cdot\ c^{(1)}\ c^{(3)}\ c^{(2)}\ c^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ c^{(5)}\ c^{(4)}\ \cdot\ \cdot\ c^{(1)}\ \cdot\ \cdot\ c^{(3)}\ c^{(2)}\ c^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ c^{(5)}\ c^{(4)}\ \cdot\ c^{(1)}\ \cdot\ \cdot\ \cdot\ \cdot\ c^{(3)}\ c^{(2)}\ c^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ c^{(5)}\ \cdot\ c^{(4)}\ c^{(1)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ c^{(3)}\ c^{(2)}\ c^{(2)}\ \cdot
\end{array}
$$

$$
\textit{where } c^{(5)} = \begin{array}{c}\bar{1}\\ \bar{2}\\ 4\\ 3\end{array},\ c^{(4)} = \begin{array}{c}\bar{1}\\ \bar{3}\\ 4\\ 2\end{array},\ c^{(3)} = \begin{array}{c}\bar{1}\\ \bar{4}\\ 3\\ 2\end{array},\ c^{(2)} = \begin{array}{c}\bar{2}\\ \bar{4}\\ 3\\ 1\end{array},\ c^{(1)} = \begin{array}{c}\bar{3}\\ \bar{4}\\ 2\\ 1\end{array}
$$

Example 3.15

$$
\begin{array}{l}
\tilde{c}^{(5)}\ \tilde{c}^{(4)}\ \tilde{c}^{(2)}\ \cdot\ \tilde{c}^{(3)}\ \tilde{c}^{(1)}\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \tilde{c}^{(5)}\ \tilde{c}^{(2)}\ \cdot\ \tilde{c}^{(4)}\ \tilde{c}^{(3)}\ \tilde{c}^{(1)}\ \cdot\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(5)}\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \tilde{c}^{(4)}\ \tilde{c}^{(1)}\ \tilde{c}^{(3)}\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(5)}\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \tilde{c}^{(2)}\ \cdot\ \tilde{c}^{(3)}\ \tilde{c}^{(3)}\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(5)}\ \tilde{c}^{(2)}\ \cdot\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(3)}\ \tilde{c}^{(3)}\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(5)}\ \cdot\ \tilde{c}^{(2)}\ \tilde{c}^{(2)}\ \cdot\ \cdot\ \cdot\ \cdot\ \tilde{c}^{(3)}\ \tilde{c}^{(3)}\ \tilde{c}^{(2)}\ \cdot
\end{array}
$$

$$
\textit{where } \tilde{c}^{(5)} = \begin{array}{c}\bar{1}\\ \bar{2}\\ 5\\ 4\\ 3\end{array},\ \tilde{c}^{(4)} = \begin{array}{c}\bar{1}\\ \bar{3}\\ 5\\ 4\\ 2\end{array},\ \tilde{c}^{(3)} = \begin{array}{c}\bar{1}\\ \bar{4}\\ 5\\ 3\\ 2\end{array},\ \tilde{c}^{(2)} = \begin{array}{c}\bar{3}\\ \bar{5}\\ 4\\ 2\\ 1\end{array},\ \tilde{c}^{(1)} = \begin{array}{c}\bar{4}\\ \bar{5}\\ 3\\ 2\\ 1\end{array}
$$

We introduce a labeling of solitons of length l using Aff($B^{n,l}$) for $U_q'(D_n^{(1)})$. Suppose there is a soliton of length $l \cdots \otimes c_l \otimes c_{l-1} \otimes \cdots \otimes c_1 \otimes \cdots (c_l \succeq c_{l-1} \succeq \cdots \succeq c_1)$ and $\cdots \otimes \tilde{c}_l \otimes \tilde{c}_{l-1} \otimes \cdots \otimes \tilde{c}_1 \otimes \cdots (\tilde{c}_l \succeq \tilde{c}_{l-1} \succeq \cdots \succeq \tilde{c}_1)$ where c_j, $\tilde{c}_j \in B^{n,l}$ in (2.2) and $(\cdot)$ means column without barred letter at time t. Say it is at position $\gamma(t)$, if (c_l) and $(\tilde{c}_l)$ is in the $\gamma(t)$th tensor component of $B^{\otimes L}$. From Proposition 3.4 (2), the position $\gamma(t)$ under the time evolution T_k is given by $\gamma(t)=\min(k,l)t+\gamma$ ($\min(k,l)$ is the velocity and γ is the phase) unless it interacts with other solitons. To such a soliton we associate an elements $z^{-\gamma}(c_1,\cdots,c_{l-1},c_l)$ and $z^{-\gamma}(\tilde{c}_1,\cdots,\tilde{c}_{l-1},\tilde{c}_l) \in$ Aff($B^{n,l}$) for $U_q'(D_n^{(1)})$.

Now consider a state of m solitons illustrated as below.

$$\ldots\ldots [l_1]\ldots\ldots.[l_2]\ldots[l_m]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .$$

We assume solitons are separated enough from each other and $l_1 > l_2 > \cdots > l_m$. Since longer solitons move faster, we can expect that the state turns out to be

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [l_m]\ldots\ldots.[l_2]\ldots.[l_1]\ldots\ldots\ldots\ldots\ldots .$$

after sufficiently many times evolutions. (The proof of this fact is the same as the main theorem.) We represent such a scattering process as

$$z^{c_1}b_1 \otimes z^{c_2}b_2 \otimes \cdots \otimes z^{c_m}b_m \mapsto z^{c'_m}b'_m \otimes \cdots \otimes z^{c'_2}b'_2 \otimes z^{c'_1}b'_1.$$

Here $z^{c_j}b_j$, $z^{c'_j}b'_j$ are elements of $\mathrm{Aff}(B^{n,l_j})$ under the identification in the previous paragraph. With these notation, the scattering process in Example 3.14 is described as

$$z^0(c^{(3)}c^{(4)}c^{(5)}) \otimes z^{-5}(c^{(1)}c^{(2)}) \otimes z^{-9}(c^{(2)}) \mapsto z^{-6}(c^{(5)}) \otimes z^{-3}(c^{(1)}c^{(4)}) \otimes \\ z^{-5}(c^{(2)}c^{(3)}c^{(3)}). \tag{3.2}$$

Let us recall some useful fact derived from representation theory. Note that $U'_q(D_n^{(1)})$ contains $U_q(A_{n-1})$ as subalgebra. This fact can be translated into the language of crystals and guarantees that

$$T_k \text{ commutes with } e_i,\ f_i\ (i = 1, 2, \cdots, n-1) \text{ on } \mathrm{Aff}(B^{n,l_1}) \otimes \cdots \otimes \mathrm{Aff}(B^{n,l_m}).$$

Here the action of e_i, f_i $(i = 1, 2, \cdots, n-1)$ on the multicomponent tensor product can be calculated using signature rule explained in Subsec. 2.3. By the actions, the power of z in an element of $\mathrm{Aff}(B^{n,l})$ is unaffected. We call this property $U_q(A_{n-1})$-invariance. This property is also used to prove our teorem. For instance, if we admit in Example 3.14, we can show that

$$e_2(z^0(c^{(3)}c^{(4)}c^{(5)}) \otimes z^{-5}(c^{(1)}c^{(2)}) \otimes z^{-9}(c^{(2)})) = z^0(c^{(3)}c^{(4)}c^{(5)}) \otimes z^{-5}(c^{(1)}c^{(1)}) \otimes z^{-9}(c^{(2)}) \mapsto e_2(z^{-6}(c^{(5)}) \otimes z^{-3}(c^{(1)}c^{(4)}) \otimes z^{-5}(c^{(2)}c^{(2)}c^{(3)})) = z^{-6}(c^{(5)}) \otimes z^{-3}(c^{(1)}c^{(4)}) \otimes z^{-5}(c^{(1)}c^{(2)}c^{(3)}).$$

Of course, this invariance is valid in the intermediate stage of scattering. To prove the theorem we need the operator ν in the Lemma 3.11 and Lemma 3.12. The operator ν as a map sending an element $b_1 \otimes b_2$ of the $U'_q(D_n^{(1)})$-crystal $B^{n,l_1} \otimes B^{n,l_2}$ to $\nu(b_2) \otimes \nu(b_1)$ in the $U'_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l_1} \otimes B_A^{2,l_2}$. The main results of this paper are the following.

Theorem 3.16

(1) *Let $b_1 \in B^{n,l_1}$, $b_2 \in B^{n,l_2}$ and the number of barred letters in each column in B^{n,l_1} and B^{n,l_2} is 2. The two body scattering $z^{c_1}b_1 \otimes z^{c_2}b_2 \mapsto z^{c_2'}\tilde{b}_2 \otimes z_1^{c_1'}\tilde{b}_1$ of solitons of length l_1 and length l_2 ($l_1 > l_2$) under the time evolution T_r ($r > l_2$) is described by the combinatorial R matrix for $U_q'(A_{n-1}^{(1)})$-crystals:*

$$Aff(B_A^{2,l_2}) \otimes Aff(B_A^{2,l_1}) \simeq Aff(B_A^{2,l_1}) \otimes Aff(B_A^{2,l_2}),$$

$$z^{c_2}t_2 \otimes z^{c_1}t_1 \mapsto z^{c_1+\delta}\tilde{t}_1 \otimes z^{c_2-\delta}\tilde{t}_2 \qquad \textit{where } \delta = H(t_2 \otimes t_1) - 2l_2.$$

Here $t_i = \nu(b_i)$, $\tilde{t}_i = \nu(\tilde{b}_i)$ for $i = 1, 2$.

(2) *The scattering of solitons is factorized into two body scatterings.*

Some remarks may be in order.

(1) The combinatorial R matrix in Theorem 3.16 has an extra $2l_2$ in the power of z. However, the Yang-Baxter equation (Proposition 2.15) holds as it is.

(2) Although we do not consider the case where there exist solitons with same length, some part of the results can be generalized to such situations. Let us consider the state

$$\ldots \underbrace{[c_k]\ldots.[c_k]}_{N_k} \ldots.. \cdots .. \underbrace{[c_b]\ldots.[c_b]}_{N_b} \ldots\ldots.. \underbrace{[c_a]\ldots.[c_a]}_{N_a} \ldots\ldots,$$

consisting og N_j solitons of length j ($1 \leq j \leq k$). By the same argument given the following proof, we can show that the scattering of these solitons is factorized into scattering of two bunches ($1 \leq j < i \leq k$)

$$\ldots \underbrace{[c_i]\ldots.[c_i]}_{N_i} \ldots \cdots \ldots.. \underbrace{[c_j]\ldots.[c_j]}_{N_j} \ldots\ldots, \; .$$

We conjecture this scattering is described by the product N_iN_j combinatorial R matrices $\mathrm{Aff}(B_A^{2,i}) \otimes \mathrm{Aff}(B_A^{2,j}) \simeq \mathrm{Aff}(B_A^{2,j}) \otimes \mathrm{Aff}(B_A^{2,i})$. This conjecture is trivially true if each solitons are separated enough.

Example 3.17 *Using the Theorem 3.16, the scattering process in Example 3.14 is calculated as*

$z^0(c^{(3)},c^{(4)},c^{(5)}) \otimes z^{-5}(c^{(1)},c^{(2)}) \otimes z^{-9}(c^{(2)}) = \{z^0(c^{(3)},c^{(4)},c^{(5)}) \otimes z^{-5}(c^{(1)},c^{(2)})\} \otimes z^{-9}(c^{(2)}) \mapsto \{z^{-5+4}(c^{(4)},c^{(5)}) \otimes z^{0-4}(c^{(1)},c^{(2)},c^{(3)})\} \otimes z^{-9}(c^{(2)}) = z^{-1}(c^{(4)},c^{(5)}) \otimes \{z^{-4}(c^{(1)},c^{(2)},c^{(3)}) \otimes z^{-9}(c^{(2)})\} \mapsto z^{-1}(c^{(4)},c^{(5)}) \otimes \{z^{-9+1}(c^{(1)}) \otimes z^{-4-1}(c^{(2)},c^{(2)},c^{(3)})\} = \{z^{-1}(c^{(4)},c^{(5)}) \otimes z^{-8}(c^{(1)})\} \otimes z^{-5}(c^{(2)},c^{(2)},c^{(3)}) \mapsto \{z^{-8+2}(c^{(5)}) \otimes z^{-1-2}(c^{(1)},c^{(4)})\} \otimes z^{-5}(c^{(2)},c^{(2)},c^{(3)})) = z^{-6}(c^{(5)}) \otimes z^{-3}(c^{(1)},c^{(4)}) \otimes z^{-5}(c^{(2)},c^{(3)},c^{(3)})$.

The result is independent of the order of the scatterings due to the Yang-Baxter equation.

Example 3.18 *Using the Theorem 3.16, the scattering process in Example 3.15 is calculated as*

$z^0(\tilde{c}^{(2)},\tilde{c}^{(4)},\tilde{c}^{(5)}) \otimes z^{-4}(\tilde{c}^{(1)},\tilde{c}^{(3)}) \otimes z^{-9}(\tilde{c}^{(2)}) = \{z^0(\tilde{c}^{(2)},\tilde{c}^{(4)},\tilde{c}^{(5)}) \otimes z^{-4}(\tilde{c}^{(1)},\tilde{c}^{(3)})\} \otimes z^{-9}(\tilde{c}^{(2)}) \mapsto \{z^{-4+3}(\tilde{c}^{(2)},\tilde{c}^{(5)}) \otimes z^{0-3}(\tilde{c}^{(1)},\tilde{c}^{(3)},\tilde{c}^{(4)})\} \otimes z^{-9}(\tilde{c}^{(2)}) = z^{-1}(\tilde{c}^{(2)},\tilde{c}^{(5)}) \otimes \{z^{-3}(\tilde{c}^{(1)},\tilde{c}^{(3)},\tilde{c}^{(4)}) \otimes z^{-9}(c^{(2)})\} \mapsto z^{-1}(\tilde{c}^{(2)},\tilde{c}^{(5)}) \otimes \{z^{-9+1}(\tilde{c}^{(2)}) \otimes z^{-3-1}(\tilde{c}^{(2)},\tilde{c}^{(3)},\tilde{c}^{(3)})\} = \{z^{-1}(\tilde{c}^{(2)},\tilde{c}^{(5)}) \otimes z^{-8}(\tilde{c}^{(2)})\} \otimes z^{-4}(\tilde{c}^{(2)},\tilde{c}^{(3)},\tilde{c}^{(3)}) \mapsto \{z^{-8+2}(\tilde{c}^{(5)}) \otimes z^{-1-2}(\tilde{c}^{(2)},\tilde{c}^{(2)})\} \otimes z^{-4}(\tilde{c}^{(2)},\tilde{c}^{(3)},\tilde{c}^{(3)})) = z^{-6}(\tilde{c}^{(5)}) \otimes z^{-3}(\tilde{c}^{(2)},\tilde{c}^{(2)}) \otimes z^{-4}(\tilde{c}^{(2)},\tilde{c}^{(3)},\tilde{c}^{(3)})$.

The result is independent of the order of the scatterings due to the Yang-Baxter equation.

Proof of Theorem 3.16. (1) Two-soliton scattering rule: Due to the $U_q(A_{n-1})$-invariance, now we check the rule for the highest weights elements $(c_a)^i \otimes ((c_a)^l, (c_b)^k, (c_d)^m) \in B^{n,i} \otimes B^{n,k+l+m}$ ($i > k+l+m$, i.e. such that $e_j(b_2 \otimes b_1) = 0$ for any $j = 1, ..., n-1$). This corresponding state is given by

$$\cdots \otimes \underbrace{c_a \cdots \otimes c_a}_{i} \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{j} \otimes \underbrace{c_d \otimes \cdots \otimes c_d}_{m} \otimes \underbrace{c_b \otimes \cdots \otimes c_b}_{k} \otimes \underbrace{c_a \otimes \cdots \otimes c_a}_{l} \otimes \cdots . \tag{3.3}$$

where,

$$c_a = \begin{array}{c} \overline{n-1} \\ \bar{n} \\ n-2 \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_b = \begin{array}{c} \overline{n-2} \\ \bar{n} \\ n-1 \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_d = \begin{array}{c} \overline{n-3} \\ \overline{n-2} \\ n \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}, \quad c_f = \begin{array}{c} \overline{n-3} \\ \overline{n-2} \\ \overline{n-1} \\ \bar{n} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array}$$

and c_0 is the column of height n without barred letters.

Now consider the time evolution by $T_{k+l+m+1}$. If $j > l$, $(c_a)^i$ moves with velocity $k+l+m+1$ and $((c_a)^l,\ (c_b)^k,\ (c_d)^m)$ with $k+l+m$. At some time,we arrive at the state

$$\cdots\otimes\underbrace{c_a\otimes\cdots\otimes c_a}_{i}\otimes\underbrace{c_0\otimes\cdots\otimes c_0}_{\max(0,\ l-m)}\otimes\underbrace{c_d\otimes\cdots\otimes c_d}_{m}\otimes\underbrace{c_b\otimes\cdots\otimes c_b}_{k}\otimes\underbrace{c_a\otimes\cdots\otimes c_a}_{l}\otimes\cdots. \tag{3.4}$$

Using Lemma 2.13, we see the state after t time units from this moment as below.

Case $m \geq l$. We separate it into several phases.

Phase 1 ($0 \leq t \leq m-l$).

$$\cdots\otimes\underbrace{c_0\otimes\cdots\otimes c_0}_{(k+l+m+1)t}\otimes\underbrace{c_a\otimes\cdots\otimes c_a}_{i-t}\otimes\underbrace{c_f\otimes\cdots\otimes c_f}_{t}\otimes\underbrace{c_d\otimes\cdots\otimes c_d}_{m-t}\otimes\underbrace{c_b\otimes\cdots\otimes c_b}_{k}\otimes \tag{3.5}$$

$$\underbrace{c_a\otimes\cdots\otimes c_a}_{l}\otimes\cdots$$

Phase 2 $(m-l \leq t \leq i-k-2l)$.

$$\cdots\otimes\underbrace{c_0\otimes\cdots\otimes c_0}_{(k+l+m+1)t}\otimes\underbrace{c_a\otimes\cdots\otimes c_a}_{i-t}\otimes\underbrace{c_f\otimes\cdots\otimes c_f}_{m-l}\otimes\underbrace{c_d\otimes\cdots\otimes c_d}_{l}\otimes\underbrace{c_b\otimes\cdots\otimes c_b}_{k}\otimes \tag{3.6}$$

$$\underbrace{c_a\otimes\cdots\otimes c_a}_{t+2l-m}\otimes\cdots.$$

Phase 3 $(\alpha \leq t \leq \alpha+m-l)$ where $\alpha = i-k-2l$.

$$\cdots\otimes\underbrace{c_0\otimes\cdots\otimes c_0}_{\alpha+(k+l+m)t}\otimes\underbrace{c_a\otimes\cdots\otimes c_a}_{t+i-2\alpha}\otimes\underbrace{c_f\otimes\cdots\otimes c_f}_{m-l+\alpha-t}\otimes\underbrace{c_d\otimes\cdots\otimes c_d}_{l+t-\alpha}\otimes\underbrace{c_b\otimes\cdots\otimes c_b}_{k}\otimes \tag{3.7}$$

$$\underbrace{c_a\otimes\cdots\otimes c_a}_{i-k-m}\otimes\cdots.$$

Phase 4 $(m-l+\alpha \leq t)$ where $\alpha = i-k-2l$.

$$\cdots\otimes\underbrace{c_0\otimes\cdots\otimes c_0}_{(k+l+m)t+\alpha}\otimes\underbrace{c_a\otimes\cdots\otimes c_a}_{m+k+l}\otimes\underbrace{c_0\otimes\cdots\otimes c_0}_{t+l-m-\alpha}\otimes\underbrace{c_d\otimes\cdots\otimes c_d}_{m}\otimes\underbrace{c_b\otimes\cdots\otimes c_b}_{k}\otimes \tag{3.8}$$

$$\underbrace{c_a \otimes \cdots \otimes c_a}_{i-k-m} \otimes \cdots .$$

After $t > m - l + \alpha$, the solitons never interact again.

Case 2 $(0 \le m < l)$. We separate it into several phases.

Phase 1 ($0 \le t \le i - k - l - m$).

$$\cdots \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{(k+l+m+1)t} \otimes \underbrace{c_a \otimes \cdots \otimes c_a}_{i-t} \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{l-m} \otimes \underbrace{c_d \otimes \cdots \otimes c_d}_{m} \otimes \underbrace{c_b \otimes \cdots \otimes c_b}_{k} \otimes \tag{3.9}$$

$$\underbrace{c_a \otimes \cdots \otimes c_a}_{l+t} \otimes \cdots$$

Phase 2 $(\beta \le t)$ where $\beta = i - k - l - m$.

$$\cdots \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{(k+l+m)t+\beta} \otimes \underbrace{c_a \otimes \cdots \otimes c_a}_{k+l+m} \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{l+t-m-\beta} \otimes \underbrace{c_d \otimes \cdots \otimes c_d}_{m} \otimes \underbrace{c_b \otimes \cdots \otimes c_b}_{k} \otimes \tag{3.10}$$

$$\underbrace{c_a \otimes \cdots \otimes c_a}_{i-k-m} \otimes \cdots$$

After $t > \beta$, the solitons never interact again. Now we have to investigate the time evolution by T_r. Note that

$$T_r^b T_{k+l+m+1}^a = T_{k+l+m+1}^a T_r^b \qquad (r > k + l + m).$$

If a and b are sufficiently large, we can reduce the observation of the scattering by T_r in the right hand side to that by $T_{k+l+m+1}$ in the left hand side, which we have just finished. Let δ_1 (resp. δ_2) be the phase shift of the soliton of length i (resp. soliton of length $(k + l + m)$). They are given by

Case 1 $(m \ge l)$.
$\delta_1 = (k + l + m)t + \alpha + (m + k + l) + (t + l - m) - \alpha - (k + l + m + 1)t = 2l + k.$

$\delta_2 = (k + l + m)t + (i - k - 2l) - i - (k + l + m)t = -(2l + k).$

Case 2 $(0 \le m < l)$.

$$\delta_1 = (k+l+m)t + \beta + (k+l+m) + (l+t-m-\beta) - (k+l+m+1)t = 2l+k.$$

$$\delta_2 = (k+l+m)t + (i-k-l-m) - i - (l-m) - (k+l+m)t = -(2l+k).$$

Let $l_2 = k+l+m$. Applying ν we see the scattering rule agree with the onegiven by the combinatorial R matrix. Use the information from (3.4) to investigate the combinatorial R matrix before two solitons collide each other. Let

$$\hat{t}_1 = \underbrace{\begin{matrix} n \cdots n \\ n-1\cdots n-1 \end{matrix}}_{i} \qquad \hat{t}_2 = \underbrace{\begin{matrix} n-2\cdots n-2 \\ n-3\cdots n-3 \end{matrix}}_{m}\underbrace{\begin{matrix} n \cdots n \\ n-2\cdots n-2 \end{matrix}}_{k}\underbrace{\begin{matrix} n \cdots n \\ n-1\cdots n-1 \end{matrix}}_{l} \tag{3.11}$$

.

Use the information from (3.8) or (3.10) to investigate the combinatorial R matrix after two solitons collide each other. Let

$$\hat{\hat{t}}_1 = \underbrace{\begin{matrix} n \cdots n \\ n-1\cdots n-1 \end{matrix}}_{k+l+m} \qquad \hat{\hat{t}}_2 = \underbrace{\begin{matrix} n-2\cdots n-2 \\ n-3\cdots n-3 \end{matrix}}_{m}\underbrace{\begin{matrix} n \cdots n \\ n-2\cdots n-2 \end{matrix}}_{k}\underbrace{\begin{matrix} n \cdots n \\ n-1\cdots n-1 \end{matrix}}_{i-k-m} \tag{3.12}$$

Then we apply the Schensted's bumping algorithm to decribe the combinatorial R matrix.

$$\hat{p} = \hat{t}_1 \leftarrow row(\hat{t}_2) \text{ (before collide)}, \qquad \hat{\hat{p}} = \hat{\hat{t}}_1 \leftarrow row(\hat{\hat{t}}_2) \text{ (after collide)}.$$

$$= \begin{array}{lllll} n \cdots n & & & & \\ n-1\cdots n-1 & n\cdots n & & & \\ n-2\cdots n-2 & n-1\cdots n-1 & n\cdots n & n\cdots n & \\ \underbrace{n-3\cdots n-3}_{m} & \underbrace{n-2\cdots n-2}_{k} & \underbrace{n-1\cdots n-1}_{i-k-m} & \underbrace{n-1\cdots n-1}_{l} & \underbrace{n\cdots n}_{k} \end{array}.$$

Here, the energy function $H(\hat{t}_2 \otimes \hat{t}_1)$ (resp. $H(\hat{\hat{t}}_2 \otimes \hat{\hat{t}}_1)$) is given by the number of nodes in the shape of $\hat{t}_1 \leftarrow row(\hat{t}_2)$ (resp. $\hat{\hat{t}}_1 \leftarrow row(\hat{\hat{t}}_2)$) that are strictly north of the 2nd row. Since $H(\hat{t}_2 \otimes \hat{t}_1) = H(\hat{\hat{t}}_2 \otimes \hat{t}_1) = 2m+k$, we have desired result.

Next we consider elements that are not neccessarily $\{1, ..., n-1\}$-highest. Let two solitons before scattering is described as

$$z^{c_1}b_1 \otimes z^{c_2}b_2.$$

Then applying $e_j(j = 1, ..., n-1)$ we get the highest weight element.

$$e_{a_1}...e_{a_m}(z^{c_1}b_1 \otimes z^{c_2}b_2) = z^{c_1}\hat{b}_1 \otimes z^{c_2}\hat{b}_2$$

The state can be reduced to

$$\underbrace{c_a \otimes \cdots \otimes c_a}_{i} \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{j} \otimes \underbrace{c_d \otimes \cdots \otimes c_d}_{m} \otimes \underbrace{c_b \otimes \cdots \otimes c_b}_{k} \otimes \underbrace{c_a \otimes \cdots \otimes c_a}_{l}.$$

Assume $\hat{b}_1 = \underbrace{c_a \otimes \cdots \otimes c_a}_{i}$ and $\hat{b}_2 = \underbrace{c_d \otimes \cdots \otimes c_d}_{m} \otimes \underbrace{c_b \otimes \cdots \otimes c_b}_{k} \otimes \underbrace{c_a \otimes \cdots \otimes c_a}_{l}$. Then

$$T_l^N e_{a_1}...e_{a_m}(z^{c_1}b_1 \otimes z^{c_2}b_2) = T_l^N(z^{c_1}\hat{b}_1 \otimes z^{c_2}\hat{b}_2) \quad \text{where } a_1, ..., a_m \in \{1, ..., n-1\}.$$

Consider the time evolution by $T_{k+l+m+1}^N(z^{c_1}\hat{b}_1 \otimes z^{c_2}\hat{b}_2)$. For case $m \geq l$, we will arrive at the state in (3.5),(3.6), (3.7) and (3.8). After $t > m - l + \alpha$ the solitons never interact again. For case $0 \leq m < l$, we will arrive at the state in (3.9) and (3.10). After $t > i - k - l$ the solitons never interact again. Then the state of solitons becomes

$$T_{k+l+m+1}^N(z^{c_1}\hat{b}_1 \otimes z^{c_2}\hat{b}_2) \mapsto z^{c_2-(2l+k)}\hat{\hat{b}}_2 \otimes z^{c_1+(2l+k)}\hat{\hat{b}}_1 \quad (\delta = -(2l+k))$$

Finally, recommute the state with $f_j(z^{c_2+\delta}\hat{\hat{b}}_2 \otimes z^{c_1-\delta}\hat{\hat{b}}_1)$ where $j = 1, ..., n-1$. Then we will get

$$f_{a_m}...f_{a_1}(z^{c_2+\delta}\hat{\hat{b}}_2 \otimes z^{c_1-\delta}\hat{\hat{b}}_1) = z^{c_2+\delta}\tilde{b}_2 \otimes z^{c_1-\delta}\tilde{b}_1.$$

Assume $T_{k+l+m+1}^N : z^{c_1}b_1 \otimes z^{c_2}b_2 \mapsto z^{c_2+\delta}\tilde{b}_2 \otimes z^{c_1-\delta}\tilde{b}_1$. Use ν in both sides to get $U_q'(A_{n-1}^{(1)})$-crystals.

$$\nu(z^{c_1}b_1 \otimes z^{c_2}b_2) \mapsto \nu(z^{c_2+\delta}\tilde{b}_2 \otimes z^{c_1-\delta}\tilde{b}_1).$$

Hence $z^{c_2}t_2 \otimes z^{c_1}t_1 \mapsto z^{c_1+\delta}\tilde{t}_1 \otimes z^{c_2-\delta}\tilde{t}_2$.

where t_1, t_2, $\tilde{t}_1$ and $\tilde{t}_2$ are barred letters from b_1, b_2, $\tilde{b}_1$ and $\tilde{b}_2$ yielded from ν. See Lemma 3.11 and 3.12 for more details about ν. Therefore we are left to show the above map is realized by the combinatorial R matrix.

Let $f_{i_l}...f_{i_1}(t_2 \otimes t_1) = \hat{t}_2 \otimes \hat{t}_1$ where $i \in \{1, ..., n-1\}$. We know $H(t_2 \otimes t_1) = H(\hat{t}_2 \otimes \hat{t}_1)$. This agree with the fact that the phase shift does not change by applying e_i $(i \in \{1, ..., n-1\})$.

Let $R(\hat{t}_2 \otimes \hat{t}_1) = \hat{\hat{t}}_1 \otimes \hat{\hat{t}}_2$. Then, $e_{i_1}...e_{i_l} R(\hat{t}_2 \otimes \hat{t}_1) = e_{i_1}...e_{i_l} \hat{\hat{t}}_1 \otimes \hat{\hat{t}}_2$

$$Re_{i_1}...e_{i_l}(\hat{t}_2 \otimes \hat{t}_1) = e_{i_1}...e_{i_l} \hat{\hat{t}}_1 \otimes \hat{\hat{t}}_2$$

$$R(t_2 \otimes t_1) = \tilde{t}_1 \otimes \tilde{t}_2.$$

Finally $z^{c_2} t_2 \otimes z^{c_1} t_1 \to z^{c_1 - \delta} \tilde{t}_1 \otimes z^{c_2 + \delta} \tilde{t}_2$ is described by the combinatorial R matrix.

(2) Factorized property: We illustrate the proof by a scattering of three solitons of length 3, 2 and 1. Other cases are similar. Now the initial state is

$$p = ...[3]..[2]..........[1]...................... .$$

Use similar technique to [1]. By applying the operator T_2^a $(a \gg 0)$ we have scattering of $[2] \otimes [1]$ and $[3] \otimes [1]$.

$$T_2^a(p) = ...[1]..............[3]...[2]........... .$$

Then by T_3^b $(b \gg 0)$ we have the third scattering of $[3] \otimes [2]$.

$$T_3^b T_2^a(p) = ...[1].........[2]........[3]........... .$$

After this the time evolution T^c act trivially. On the other hand, these evolution can be written as

$$T^c T_3^b T_2^a = T_3^b T_2^a T^c.$$

In the right hand side, we first observe three-soliton scattering caused by T^c $(c \gg 0)$ and then $T_3^b T_2^a$ act trivially. Hence, the three-soliton scattering in the right side is factorized into $3\times$(two-soliton scattering) in the left hand side.

Example 3.19 *Let*

$$c^{(1)} = \begin{array}{c}\bar{3}\\ \bar{4}\\ 2\\ 1\end{array}, \quad c^{(2)} = \begin{array}{c}\bar{2}\\ \bar{4}\\ 3\\ 1\end{array}, \quad c^{(3)} = \begin{array}{c}\bar{2}\\ \bar{3}\\ 4\\ 1\end{array}, \quad c^{(4)} = \begin{array}{c}\bar{1}\\ \bar{4}\\ 3\\ 2\end{array}, \quad c^{(5)} = \begin{array}{c}\bar{1}\\ \bar{3}\\ 4\\ 2\end{array}, \quad c^{(6)} = \begin{array}{c}\bar{1}\\ \bar{2}\\ 4\\ 3\end{array}.$$

We use Theorem 3.16 to describe the scattering process for $z^0(c^{(1)}, c^{(1)}, c^{(2)}, c^{(4)}, c^{(5)}, c^{(6)}) \otimes z^{-7}(c^{(2)}, c^{(4)}, c^{(6)}, c^{(6)})$. *Let* $b_1 \in B^{4,6}$ *and* $b_2 \in B^{4,4}$ *where* $B^{4,6}$ *and* $B^{4,4}$ *are the* $U'_q(D_n^{(1)})$*-crystal. Use* ν *in Lemma 3.11 and Lemma 3.12 to send the elements* b_1 *and* b_2 *in* $U'_q(D_n^{(1)})$*-crystal to* $U'_q(A_{n-1}^{(1)})$*-crystal. Then we will get*

$$\nu(b_1 \otimes b_2) = t_2 \otimes t_1 \quad \textit{where } t_1 = \begin{array}{c}234444\\ 111233\end{array}, \quad t_2 = \begin{array}{c}2244\\ 1112\end{array}.$$

By using Proposition 3.10 we have

$$p = t \leftarrow row(t') = \begin{array}{llllllll}4_{10} & 4_8 & & & & & & \\ 3_{12} & 3_{11} & 3_9 & & & & & \\ 2 & 2 & 2 & 2 & 4_6 & 4_4 & 4_2 & \\ 1 & 1 & 1 & 1 & 1_7 & 1_5 & 2_3 & 4_1\end{array}$$

Here, the energy function $H(t_2 \otimes t_1)$ *is given by the number of nodes in the shape of* $t_1 \leftarrow row(t_2)$ *that are strictly north of the 2nd row. So,* $H = 5$.

Since $p_{12} = \begin{array}{c}2344\\ 1233\end{array}$ *and* $u_{12}u_{11}u_{10}u_9u_8u_7u_6u_5u_4u_3u_2u_1 = 121214141424$

phase shift$=H(t_2 \otimes t_1) - 2l_2 = 5 - 2(4) = -3$.

$$\begin{array}{c}2244\\ 1112\end{array} \otimes \begin{array}{c}234444\\ 111233\end{array} \simeq \begin{array}{c}224444\\ 111112\end{array} \otimes \begin{array}{c}2344\\ 1233\end{array} \qquad \textit{where } \tilde{t}_1 = \begin{array}{c}224444\\ 111112\end{array}, \quad \tilde{t}_2 = \begin{array}{c}2344\\ 1233\end{array}.$$

Hence $z^{-7}t_2 \otimes z^0 t_1 \mapsto z^{-3}\tilde{t}_1 \otimes z^{-4}\tilde{t}_2$.

Example 3.20 *Let*

$$\tilde{c}^{(1)} = \begin{array}{c}\bar{4}\\ \bar{5}\\ 3\\ 2\\ 1\end{array}, \quad \tilde{c}^{(2)} = \begin{array}{c}\bar{3}\\ \bar{5}\\ 4\\ 2\\ 1\end{array}, \quad \tilde{c}^{(3)} = \begin{array}{c}\bar{1}\\ \bar{4}\\ 5\\ 3\\ 2\end{array}, \quad \tilde{c}^{(4)} = \begin{array}{c}\bar{1}\\ \bar{3}\\ 5\\ 4\\ 2\end{array}.$$

We use Theorem 3.16 to describe the scattering process for $z^{-3}(\tilde{c}^{(1)}, \tilde{c}^{(3)}, \tilde{c}^{(4)}) \otimes z^{-9}(\tilde{c}^{(2)})$. *Let* $b_1 \in B^{5,3}$ *and* $b_2 \in B^{5,1}$ *where* $B^{5,3}$ *and* $B^{5,1}$ *are the* $U_q'(D_n^{(1)})$*-crystal. Use* ν *in Lemma 3.11 and Lemma 3.12 to send the elements* b_1 *and* b_2 *in* $U_q'(D_n^{(1)})$*-crystal to* $U_q'(A_{n-1}^{(1)})$*-crystal. Then we will get*

$$\nu(b_1 \otimes b_2) = t_2 \otimes t_1 \quad \textit{where } t_1 = \begin{matrix} 345 \\ 114 \end{matrix}, \quad t_2 = \begin{matrix} 5 \\ 3 \end{matrix}.$$

By using Proposition 3.10 we have

$$p = t \leftarrow row(t') = \begin{matrix} 5_6 & & & \\ 3 & 4_4 & 4_2 & \\ 1 & 1_5 & 3_3 & 5_1 \end{matrix}$$

Here, the energy function $H(t_2 \otimes t_1)$ *is given by the number of nodes in the shape of* $t_1 \leftarrow row(t_2)$ *that are strictly north of the 2nd row. So,* $H = 1$.

Since $p_{12} = \begin{matrix} 5 \\ 3 \end{matrix}$ *and* $u_6u_5u_4u_3u_2u_1 = 141435$

phase shift$=H(t_2 \otimes t_1) - 2l_2 = 1 - 2(1) = -1$.

$$\begin{matrix} 345 \\ 114 \end{matrix} \otimes \begin{matrix} 5 \\ 3 \end{matrix} \simeq \begin{matrix} 5 \\ 3 \end{matrix} \otimes \begin{matrix} 445 \\ 113 \end{matrix} \quad \textit{where } \tilde{t}_1 = \begin{matrix} 5 \\ 3 \end{matrix}, \quad \tilde{t}_2 = \begin{matrix} 445 \\ 113 \end{matrix}.$$

Hence $z^{-9}t_2 \otimes z^{-3}t_1 \mapsto z^{-4}\tilde{t}_1 \otimes z^{-8}\tilde{t}_2$.

Part II

KKR TYPE BIJECTION FOR THE KIRILLOV-RESHETIKHIN CRYSTAL $B^{6,1}$ OF THE EXCEPTIONAL AFFINE ALGEBRA $E_6^{(1)}$

4 Introduction

Kerov, Kirillov and Reshetikhin [17] introduced a new combinatorial object, called rigged configuration through Bethe ansatz analysis of the Heisenberg spin chain. The bijection of Kerov, Kirillov and Reshetikhin [17, 18] is a bijection between semi-standard Young tableaux and rigged configurations and yields a fermionic formula for the Kostka-Foulkes polynomials. Subsequently, Nakayashiki and Yamada [23] have shown that the set of paths $\overline{P}(B, \lambda)$ is in bijection with the set of semi-standard Young tableaux SSYT(λ, μ) of shape λ and content $\mu = (\mu_1, \mu_2, ...)$, and the energy function corresponds to the cocharge of Lascoux and Schützenberger [21]. In [19], this bijection was generalized to $B = \otimes_j B^{r_j, s_j}$ of type $A^{(1)}_{n-1}$ and show the bijection between Littlewood-Richardson tableaux and rigged configurations.

Taking generating functions on both sides of the bijection, we obtain the so-called $X = M$ Theorem (see [29] for review for type $A^{(1)}_{n-1}$). In [6, 5] such an equality $X = M$ are generalized and conjectured for any affine type by assuming the existence of crystal bases for certain finite-dimensional modules, well-known as Kirillov-Reshetikhin (KR) modules, for quantum affine algebras. Hence, it is natural to expert that a similar bijection exists also for these generalized cases. For the X side, [14] and subsequently [2] discovered the crystal bases for KR modules of nonexceptional types. Imitating the one by KKR a bijection between rigged configurations and highest weight paths consisting of elements of KR crystals for nonexceptional affine types other than $A^{(1)}_{n-1}$ was subsequently constructed in [27, 28, 30]. These bijections have an important application for the analysis of the ultra-discrete integrable systems, also called box-ball systems [3, 4, 7].

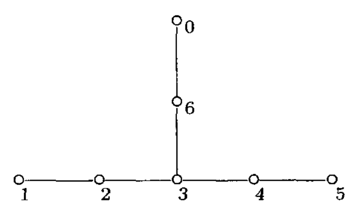


Figure 4.1: Dynkin diagram for $E^{(1)}_6$

In this paper we consider the KR crystal, named as $E^{(1)}_6$, for the exceptional affine algebra $E^{(1)}_6$. We construct a map Φ from rigged configurations to highest weight elements of $(B^{6,1})^{\otimes L}$ by executing a fundamental procedure δ repeatedly. We then conjecture that Φ is a statistic-preserving bijection (Conjecture 7.14). It is worth mentioning that our procedure only uses the crystal graph structure of the KR crystal $B^{6,1}$, hence similar constructions could be possible for other exceptional types. We remark that there is a preceding work [25] which treats $B^{1,1}$ for type $E_6^{(1)}$. Ours has a different algorithm compared to [25] because we must consider the element ϕ, unique element in the highest crystal of weight 0, in the crystal graph. We also introduced "quasi singular" where we still remove the box although it is not singular with certain condition. In this paper, we will give many examples supporting the conjecture.

5 Level 1 Perfect Crystals

In this section, we recapitulate the theory in [1]. Following [1] we give a uniform construction of level 1 perfect crystals for all affine Lie algebras.

Let $J = \{0, 1, ..., n\}$ be an index set, and let $A = (a_{i,j})_{i,j \in J}$ be a Cartan matrix of affine type. Thus, A can be characterized by the following properties: $a_{i,i} = 2$ for all $i \in J$, $a_{i,j} \in \mathbb{Z}_{\leq 0}$, and $a_{i,j} = 0$ if and only if $a_{j,i} = 0$ for all $i \neq j$ in J. The rank of A is n, and if $\nu \in \mathbb{R}^{n+1}$ and $A\nu \geq 0$ (componentwise), then $\nu > 0$ or $\nu = 0$. We assume A is indecomposable so that if $J = J' \cup J''$ where J' and J'' are nonempty, then for some $i \in J'$ and $j \in J''$, the entry $a_{i,j} \neq 0$. An affine Cartan matrix is always symmetrizable – there exists a diagonal matrix $D = \mathrm{diag}(s_i \mid i \in J)$ of positive integers such that DA is symmetric.

The free abelian group

$$Q^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d \tag{5.1}$$

is the extended coroot lattice. The linear functionals The linear α_i and Λ_i $(i \in J)$ on the complexication $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q^\vee$ of $Q^\vee$ given by

$$\langle h_j, \alpha_i \rangle := \alpha_i(h_j) = a_{j,i} \qquad \langle d, \alpha_i \rangle := \alpha_i(d) = \delta_{i,0}$$

$$\langle h_j, \Lambda_i \rangle := \Lambda_i(h_j) = \delta_{i,j} \qquad \langle d, \Lambda_i \rangle := \Lambda_i(d) = 0 \qquad (i, j \in J)$$

are the simple roots and fundamental weights, respectively. Let $\prod = \{\alpha_i \mid i \in J\}$ denote the set of simple roots and $\prod^\vee = \{h_i \mid i \in J\}$ the set of simple coroots. The weight lattice

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(Q^\vee \subset \mathbb{Z})\tag{5.2}$$

contains the set $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all $i \in J\}$ of dominant integral weights. The affine Lie algebra $\hat{\mathfrak{g}}$ attached to the data $(A, \prod, \prod^\vee, P, Q^\vee)$ has generators e_i, f_i $(i \in J)$, $h \in \mathfrak{h}$, which satisfy certain relations (see for example [12] or [10], Prop. 2.1.6]).

The canonical central element c and the null root δ are given by the expressions

$$c = c_0 h_0 + c_1 h_1 + \cdots + c_n h_n, \qquad \delta = d_0 \alpha_0 + d_1 \alpha_1 + \cdots + d_n \alpha_n, \tag{5.3}$$

where $c_0 = 1$ and $d_0 = 1$. The first term comes from the fact that the center of the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is generated by c, while the second comes from the fact that the vector $[d_0, d_1, ..., d_n]^t \in \mathbb{C}^{n+1}$ spans the null space of the cartan matrix A. We say the dominant weight $\lambda \in P^+$ has level l if $\langle c, \lambda \rangle := \lambda(c) = l$.

Since the perfect crystals reveal much about the structure of crystal bases for irreducible modules, which in turn can be used to compute their weights and characters, our goal in the subsequent sections will be to construct perfect crystals for all affine Lie algebras and to calculate the corresponding energy functions. Let $\hat{\mathfrak{g}}$ be an affine Lie algebra and let

$$\theta = d_1 \alpha_1 + \cdots + d_n \alpha_n$$

where the d_i are as in (5.3). Thus, when $\hat{\mathfrak{g}} = X_n^{(1)}$ (the so-called untwisted case), θ is the highest root of $\mathfrak{g}$.

Let $B(\theta)$ denote the crystal graph of the irreducible $U_q(\mathfrak{g})$-module $L_{\mathfrak{g}}(\theta)$. Thus, the crystal graph $B(\theta)$ corresponds to the adjoint representation of $\mathfrak{g}$(with highest weight the highest short root). In the untwisted case equality holds, $\Lambda = \Phi \cup \{0\}$.

Let Φ^+ and $\Phi^- = -\Phi^+$ denote the positive and negative roots respectively of $\mathfrak{g}$. Set $\Lambda^+ = \Lambda \cap \Phi^+$, $\Lambda^- = -\Lambda^+$, so that $\Lambda = \Lambda^+ \cup \{0\} \cup \Lambda^-$. Then we write

$$B(\theta) = \{x_\alpha \mid \alpha \in \Lambda^+\} \cup \{y_i \mid \alpha_i \in \Lambda^+\} \cup \{x_{-\alpha} \mid \alpha \in \Lambda^+\}.$$

Hence in the twisted case, $B(\theta) = \{x_{\pm\alpha} \mid \alpha \in \Phi^+\} \cup \{y_i \mid i = 1, ..., n\}$.

Set $B(0) = \{\emptyset\}$ which we identify with the crystal graph of the one-dimensional $U_q(\mathfrak{g})$-module $L_{\mathfrak{g}}(0)$. As we argue below, the set

$$B = B(\theta) \sqcup B(0) \tag{5.4}$$

can be endowed with a crystal structure as follows:

$$(i \neq 0) \quad x_\alpha \xrightarrow{i} x_\beta \Longleftrightarrow \alpha - \alpha_i = \beta \quad (\alpha, \beta \in \Lambda), \tag{5.5}$$

$$x_{\alpha_i} \xrightarrow{i} y_i \xrightarrow{i} x_{-\alpha_i} \qquad (\alpha_i \in \Lambda^+),$$

$$(i = 0) \quad x_\alpha \xrightarrow{0} x_\beta \Longleftrightarrow \alpha + \theta = \beta \quad (\alpha, \beta \neq \pm\theta), \qquad x_{-\theta} \xrightarrow{0} \phi \xrightarrow{0} x_\theta. \tag{5.6}$$

Our approach to proving this can be summarized as follows. We forget the 0-arrows in $B \otimes B$ and view it as a crystal graph for the quantum algebra $U_q(\mathfrak{g})$ associated to the simple Lie algebra $\mathfrak{g}$:

$$B \otimes B = (B(\theta) \otimes B(\theta)) \sqcup (B(\theta) \otimes B(0)) \sqcup (B(0) \otimes B(\theta)) \sqcup (B(0) \otimes B(0)). \tag{5.7}$$

Since crystals corresponding to simple modules are connected, it suffices to locate the maximal vectors ($e_i b = 0$ for all $i \in J\backslash\{0\}$) inside the components on the right and show that they are all connected to one another by various i-arrows for $i \in J$.

There are obvious maximal vectors inside $B \otimes B$,

(1) $x_\theta \otimes x_\theta$
(2) $x_\theta \otimes \phi$
(3) $\phi \otimes x_\theta$
(4) $\phi \otimes \phi$
(5) $x_\theta \otimes x_{-\theta}$

and they can be connected as displayed below:

$$\phi \otimes \phi \xrightarrow{0} \phi \otimes x_\theta \xrightarrow{*\cdots*} \phi \otimes x_{-\theta} \xrightarrow{0} x_\theta \otimes x_{-\theta} \xrightarrow{0} x_\theta \otimes \phi \xrightarrow{0} x_\theta \otimes x_\theta$$

where $\xrightarrow{*\cdots*}$ indicates that an appropriate sequence of Kashiwara operators f_i with $i \in J\backslash\{0\}$ has been applied. All other maximal vectors have the form

(6) $x_\theta \otimes x_{\theta-\alpha}$ for some $\alpha \in \Lambda^+$ or
(7) $x_\theta \otimes y_i$ for some i such that $\alpha_i \in \Lambda^+$.

6 Affine Algebra $E_6^{(1)}$ and the KR Crystal

6.1 Affine algebra $E_6^{(1)}$

We consider in this paper the exceptional affine algebra $E_6^{(1)}$. The Dynkin diagram is depicted in Figure 4.1. Note that we follow [25] for the labeling of the Dynkin nodes.

Let I be the index set of the Dynkin nodes, and let α_i, $\alpha_i^\vee$, Λ_i $(i \in I)$ be simple roots, simple coroots, fundamental weights, respectively. Following the notation in [12] we denote the projection of Λ_i onto the weight space of E_6 by $\overline{\Lambda}_i$ $(i \in I_0 := I\backslash\{0\})$ and set $\overline{P} = \bigoplus_{i \in I_0} \mathbb{Z}\overline{\Lambda}_i$, $\overline{P}^+ = \bigoplus_{i \in I_0} \mathbb{Z}_{\geq 0}\overline{\Lambda}_i$. Let $(C_{ij})_{i,j \in I}$ stand for the Cartan matrix for $E_6^{(1)}$. Namely, $C_{ij} = 2$ $(i = j)$, $-1 (i \sim j)$, 0 (otherwise), where $i \sim j$ means that i and j are adjacent in the Dynkin diagram.

6.2 KR crystal $B^{6,1}$

. Let $\mathfrak{g}$ be any affine algebra and $U_q'(\mathfrak{g})$ the corresponding quantized enveloping algebra without the degree operator. Among finite-dimensional $U_q'(\mathfrak{g})$-modules there is a distinguished family called Kirillov-Reshetikhin (KR) modules [20, 22, 9]. One of the remarkable properties of KR modules is the existence of a crystal basis [15] called a KR crystal. It was conjectured in [6, 5], and recently settled for all nonexceptional types in [26]. The KR crystal is indexed by (a, i) $(a \in I_0,\ i \in \mathbb{Z}_{>0})$ and denoted by $B^{a,i}$. For exceptional types the KR crystal is known to exist when the KR module is irreducible or the index a is adjacent to 0 [14].

The KR crystal of type $\mathfrak{g} = E_6^{(1)}$ we are interested in this paper is $B^{6,1}$ and the crystal structure is taken from [1], hence, it is the level 1 perfect crystal in §5 in the the case of $\mathfrak{g} = E_6^{(1)}$. The crystal structure of $B^{6,1}$ is depicted in Figure 6.1. Here vertices in the graph signify elements of $B^{6,1}$ and $b \xrightarrow{i} b'$ stands for $f_i b = b'$ or equivalently $b = e_i b'$. We adopt the original convention for the tensor product of crystals. Namely, if B_1 and B_2 are crystals, then for $b_1 \otimes b_2 \in B_1 \otimes B_2$ the action of e_i is defined as

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ \\ b_1 \otimes e_i b_2 & \text{if else.} \end{cases}$$

where $\varepsilon_i(b) = \max\{k|\ e_i^k b \neq 0\}$ and $\varphi_i(b) = \max\{k|\ f_i^k b \neq 0\}$.

In what follows in this paper we assume $B = B^{6,1}$. The set of classically restricted paths in $B^{\otimes L}$ and $\lambda \in \overline{P}^+$ is by definition

$$P(\lambda, L) = \{b \in B^{\otimes L}\ |\mathrm{wt}(b) = \lambda \text{ and } e_i b = 0 \text{ for all } i \in i_0\} \tag{6.1}$$

Then the following two are equivalent.

(1) b is classically restricted path of weight $\lambda \in \overline{P}^+$.

(2) $b_1 \otimes \cdots \otimes b_{L-1}$ is classically restricted path of weight $\lambda - \mathrm{wt}(b_L)$, and $\varepsilon_i(b_L) \leq \langle \lambda - \mathrm{wt}(b_L), \alpha_i^\vee \rangle$ for all $i \in I_0$.

The weight function $\mathrm{wt} : B \to \overline{P}$ is given by $\mathrm{wt}(b) = \sum_{i=I}(\varphi_i(b) - \varepsilon_i(b))\Lambda_i$. The weight function $\mathrm{wt} : B^{\otimes L} \to \overline{P}$ is defined by $\mathrm{wt}(b_1 \otimes \cdots \otimes b_L) = \sum_{j=1}^{L} \mathrm{wt}(b_j)$.

In Figure 6.1 we show the crystal graph for B_0. In this case B_0 be the subgraph obtained by ignoring the 0-arrows form $B^{6,1}$. All the 0-arrows are listed below.

$\boxed{63} \xrightarrow{0} \boxed{\phi} \xrightarrow{0} \boxed{1}$ $\quad \boxed{70} \xrightarrow{0} \boxed{3}$ $\quad \boxed{72} \xrightarrow{0} \boxed{4}$ $\quad \boxed{66} \xrightarrow{0} \boxed{5}$

$\boxed{74} \xrightarrow{0} \boxed{6}$ $\quad \boxed{76} \xrightarrow{0} \boxed{7}$ $\quad \boxed{67} \xrightarrow{0} \boxed{8}$ $\quad \boxed{78} \xrightarrow{0} \boxed{9}$ $\quad \boxed{68} \xrightarrow{0} \boxed{10}$

$\boxed{69} \xrightarrow{0} \boxed{11}$ $\quad \boxed{53} \xrightarrow{0} \boxed{18}$ $\quad \boxed{55} \xrightarrow{0} \boxed{19}$ $\quad \boxed{56} \xrightarrow{0} \boxed{20}$ $\quad \boxed{57} \xrightarrow{0} \boxed{22}$

$\boxed{58} \xrightarrow{0} \boxed{24}$ $\quad \boxed{59} \xrightarrow{0} \boxed{27}$ $\quad \boxed{60} \xrightarrow{0} \boxed{29}$ $\quad \boxed{61} \xrightarrow{0} \boxed{31}$ $\quad \boxed{62} \xrightarrow{0} \boxed{36}$

$\boxed{48} \xrightarrow{0} \boxed{2}$ $\quad \boxed{51} \xrightarrow{0} \boxed{17}$

The correspondence between the number in the graph and the crystal elements is given below, where $\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$.

$\boxed{1} = x_{\theta}$, $\boxed{2} = x_{\theta-\alpha_6}$, $\boxed{3} = x_{\theta-\alpha_6-\alpha_3}$, $\boxed{4} = x_{\theta-\alpha_6-\alpha_3-\alpha_4}$, $\boxed{5} = x_{\theta-\alpha_6-\alpha_3-\alpha_4-\alpha_5}$,

$\boxed{6} = x_{\theta-\alpha_6-\alpha_3-\alpha_2}$, $\boxed{7} = x_{\theta-\alpha_6-\alpha_3-\alpha_4-\alpha_2}$, $\boxed{8} = x_{\theta-\alpha_6-\alpha_3-\alpha_4-\alpha_2-\alpha_5}$,

$\boxed{9} = x_{\theta-\alpha_6-\alpha_3-\alpha_4-\alpha_2-\alpha_3}$, $\boxed{10} = x_{\theta-\alpha_6-2\alpha_3-\alpha_4-\alpha_2-\alpha_5}$, $\boxed{11} = x_{\theta-\alpha_6-2\alpha_3-2\alpha_4-\alpha_2-\alpha_5}$,

$\boxed{12} = x_{\theta-2\alpha_6-\alpha_3-\alpha_4-\alpha_2-\alpha_3}$, $\boxed{13} = x_{\theta-2\alpha_6-2\alpha_3-\alpha_4-\alpha_2-\alpha_5}$, $\boxed{14} = x_{\theta-2\alpha_6-2\alpha_3-2\alpha_4-\alpha_2-\alpha_5}$,

$\boxed{15} = x_{\theta-2\alpha_6-3\alpha_3-2\alpha_4-\alpha_2-\alpha_5}$, $\boxed{16} = x_{\theta-2\alpha_6-3\alpha_3-2\alpha_4-2\alpha_2-\alpha_5}$, $\boxed{17} = x_{\theta-\alpha_6-\alpha_3-\alpha_2-\alpha_1}$,

$\boxed{18} = x_{\theta-\alpha_6-\alpha_3-\alpha_4-\alpha_2-\alpha_1}$, $\boxed{19} = x_{\theta-\alpha_6-\alpha_3-\alpha_4-\alpha_2-\alpha_5-\alpha_1}$, $\boxed{20} = x_{\theta-\alpha_6-2\alpha_3-\alpha_4-\alpha_2-\alpha_1}$,

$\boxed{21} = x_{\theta-2\alpha_6-2\alpha_3-\alpha_4-\alpha_2-\alpha_1}$, $\boxed{22} = x_{\theta-\alpha_6-\alpha_5-2\alpha_3-\alpha_4-\alpha_2-\alpha_1}$,

$\boxed{23} = x_{\theta-2\alpha_6-\alpha_5-2\alpha_3-\alpha_4-\alpha_2-\alpha_1}$, $\boxed{24} = x_{\theta-\alpha_6-\alpha_5-2\alpha_3-2\alpha_4-\alpha_2-\alpha_1}$,

$\boxed{25} = x_{\theta-2\alpha_6-\alpha_5-2\alpha_3-2\alpha_4-\alpha_2-\alpha_1}$, $\boxed{26} = x_{\theta-2\alpha_6-\alpha_5-3\alpha_3-2\alpha_4-\alpha_2-\alpha_1}$,

$\boxed{27} = x_{\theta-\alpha_6-2\alpha_3-\alpha_4-2\alpha_2-\alpha_1}$, $\boxed{28} = x_{\theta-2\alpha_6-2\alpha_3-\alpha_4-2\alpha_2-\alpha_1}$,

$\boxed{29} = x_{\theta-\alpha_6-\alpha_5-2\alpha_3-\alpha_4-2\alpha_2-\alpha_1}$, $\boxed{30} = x_{\theta-2\alpha_6-\alpha_5-2\alpha_3-\alpha_4-2\alpha_2-\alpha_1}$,

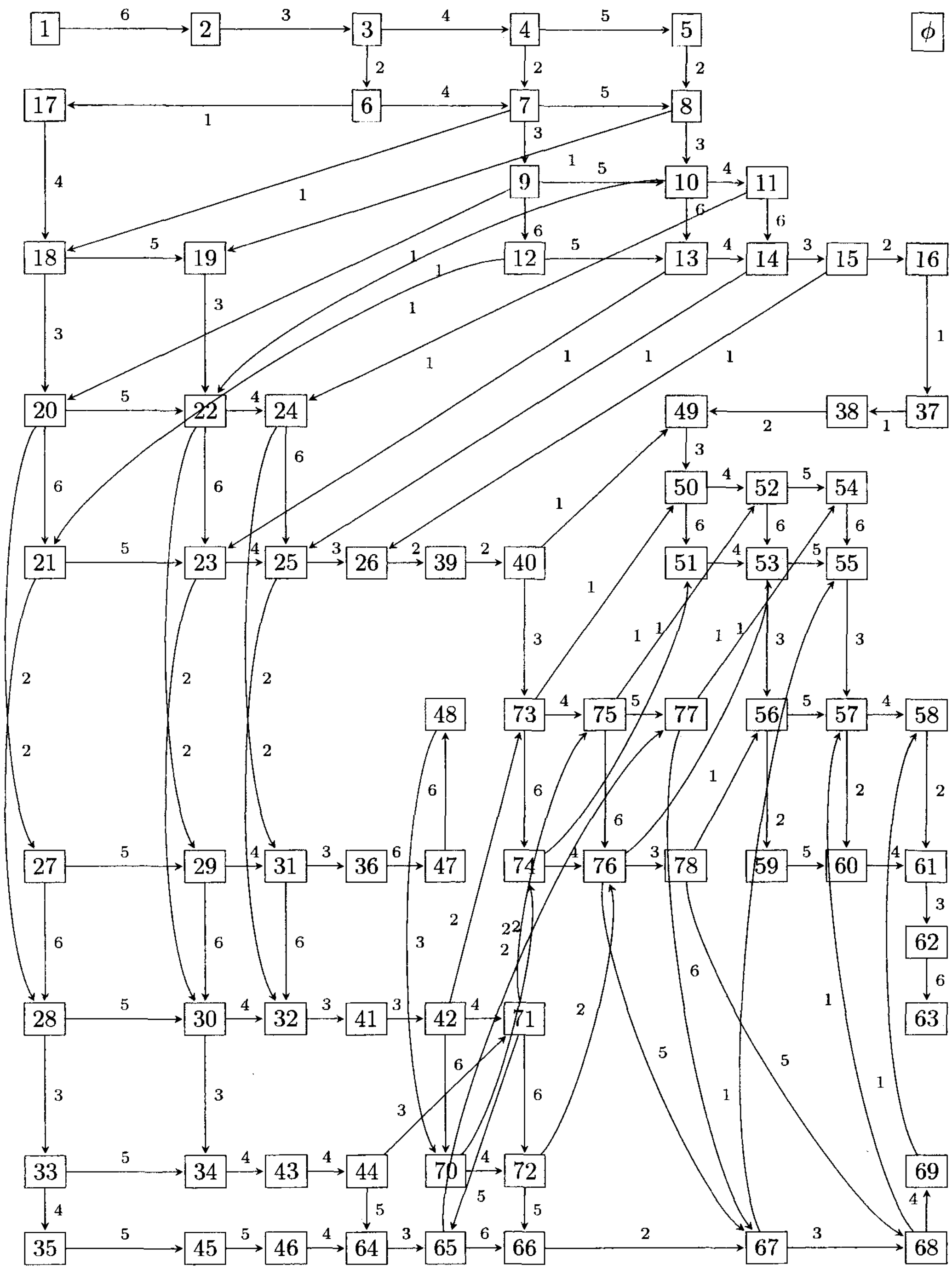


Figure 6.1: Crystal graph of B_0 for $B^{6,1}$

$\boxed{31} = x_{\theta-\alpha_6-\alpha_5-2\alpha_3-2\alpha_4-2\alpha_2-\alpha_1}$, $\boxed{32} = x_{\theta-2\alpha_6-\alpha_5-2\alpha_3-2\alpha_4-2\alpha_2-\alpha_1}$,

$\boxed{33} = x_{\theta-2\alpha_6-3\alpha_3-\alpha_4-2\alpha_2-\alpha_1}$, $\boxed{34} = x_{\theta-2\alpha_6-\alpha_5-3\alpha_3-\alpha_4-2\alpha_2-\alpha_1}$,

$\boxed{35} = x_{\theta-2\alpha_6-3\alpha_3-2\alpha_4-2\alpha_2-\alpha_1}$, $\boxed{36} = x_{\theta-\alpha_6-\alpha_5-3\alpha_3-2\alpha_4-2\alpha_2-\alpha_1}$,

$\boxed{37} = y_1$, $\boxed{38} = x_{-\alpha_1}$, $\boxed{39} = y_2$, $\boxed{40} = x_{-\alpha_2}$, $\boxed{41} = y_3$, $\boxed{42} = x_{-\alpha_3}$,

$\boxed{43} = y_4$, $\boxed{44} = x_{-\alpha_4}$, $\boxed{45} = y_5$, $\boxed{46} = x_{-\alpha_5}$, $\boxed{47} = y_6$, $\boxed{48} = x_{-\alpha_6}$,

$\boxed{49} = x_{-\alpha_1-\alpha_2}$, $\boxed{50} = x_{-\alpha_1-\alpha_2-\alpha_3}$, $\boxed{51} = x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_6}$,

$\boxed{52} = x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$, $\boxed{53} = x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_6}$, $\boxed{54} = x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5}$,

$\boxed{55} = x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}$, $\boxed{56} = x_{-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_6}$,

$\boxed{57} = x_{-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6}$, $\boxed{58} = x_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$,

$\boxed{59} = x_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_6}$, $\boxed{60} = x_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6}$,

$\boxed{61} = x_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$, $\boxed{62} = x_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$,

$\boxed{63} = x_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-\alpha_5-2\alpha_6}$, $\boxed{64} = x_{-\alpha_4-\alpha_5}$, $\boxed{65} = x_{-\alpha_4-\alpha_5-\alpha_3}$,

$\boxed{66} = x_{-\alpha_4-\alpha_5-\alpha_3-\alpha_6}$, $\boxed{67} = x_{-\alpha_2-\alpha_4-\alpha_5-\alpha_3-\alpha_6}$, $\boxed{68} = x_{-\alpha_2-\alpha_4-\alpha_5-2\alpha_3-\alpha_6}$,

$\boxed{69} = x_{-\alpha_2-2\alpha_4-\alpha_5-2\alpha_3-\alpha_6}$, $\boxed{70} = x_{-\alpha_3-\alpha_6}$, $\boxed{71} = x_{-\alpha_3-\alpha_4}$,

$\boxed{72} = x_{-\alpha_3-\alpha_4-\alpha_6}$, $\boxed{73} = x_{-\alpha_2-\alpha_3}$, $\boxed{74} = x_{-\alpha_2-\alpha_3-\alpha_6}$,

$\boxed{75} = x_{-\alpha_2-\alpha_3-\alpha_4}$, $\boxed{76} = x_{-\alpha_2-\alpha_3-\alpha_4-\alpha_6}$, $\boxed{77} = x_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}$,

$\boxed{78} = x_{-\alpha_2-2\alpha_3-\alpha_4-\alpha_6}$, $\boxed{\phi} = \phi$.

Example 6.1 *The element*

$$b = \boxed{1} \cdot \boxed{2} \cdot \boxed{3} \cdot \boxed{7} \cdot \boxed{9} \cdot \boxed{2} \cdot \boxed{46}$$

of $B^{\otimes 7}$ *is a classically restricted path of weight* $2\bar{\Lambda}_1 + \bar{\Lambda}_3 + \bar{\Lambda}_4$. *The dot* $\cdot$ *signifies* $\otimes$.

6.3 One-dimensional sums

The energy function $D : B^{\otimes L} \to \mathbb{Z}$ gives the grading on $B^{\otimes L}$. In our case where a path is an element of the tensor product of a single KR crystal it takes a simple form. Due to the existence of the universal R-matrix and the fact that $B \otimes B$ is connected, by [13] there is a unique (up to global additive constant) function $H : B \otimes B \to \mathbb{Z}$ called the local energy function, such that

$$H(e_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0 \text{ , and } e_0(b \otimes b') = e_0 b \otimes b', \\ H(b \otimes b') - 1 & \text{if } i = 0 \text{ , and } e_0(b \otimes b') = b \otimes e_0 b', \\ H(b \otimes b') & \text{otherwise.} \end{cases} \tag{6.2}$$

We normalize H by the condition

$$H\left(\boxed{1} \otimes \boxed{1}\right) = 0. \tag{6.3}$$

More specifically, the value of H is calculated as follows. Firstly, one knows the crystal graph of $B_0 \otimes B_0$ decomposes into five connected components as

$$B_0 \otimes B_0 = B(2\bar{\Lambda}_6) \oplus B(\bar{\Lambda}_3 + \bar{\Lambda}_6) \oplus B(\bar{\Lambda}_1 + \bar{\Lambda}_5 + \bar{\Lambda}_6) \oplus 3B(\bar{\Lambda}_6) \oplus 2B(0)$$

where $B(\lambda)$ stands for the highest weight E_6-crystal of highest weight λ and the highest weight vector is given by $\boxed{1} \otimes \boxed{1}$, $\boxed{1} \otimes \boxed{2}$, $\boxed{1} \otimes \boxed{12}$, $\boxed{1} \otimes \boxed{47}$ and $\boxed{1} \otimes \boxed{63}$. H is constant on each component, and takes the value $H(\boxed{1} \otimes \boxed{1}) = 0$, $H(\boxed{1} \otimes \boxed{2}) = -1$ and $H = -2$ for the rest. One can confirm it from the fact that $e_0^2\left(\boxed{1} \otimes \boxed{1}\right) = \boxed{1} \otimes \boxed{63}$ and $e_0\left(\boxed{1} \otimes \boxed{2}\right) = \boxed{1} \otimes \boxed{48}$ belong to the first and second component. Before this, we also need the value of $H = -1$ for all $H(1 \otimes \phi)$, $H(\phi \otimes 1)$, $H(\phi \otimes \phi)$. With this H the energy function D is defined by

$$D(b_1 \otimes \cdots \otimes b_L) = \sum_{j=0}^{L-1} (L - j) H(b_j \otimes b_{j+1}). \tag{6.4}$$

where $b_0 = \boxed{1}$. Define the one-dimensional sum $X(\lambda, L; q) \in \mathbb{Z}_{\geq 0}[q^{-1}]$ by

$$X(\lambda, L; q) = \sum_{b \in P(\lambda, L)} q^{D(b)} \tag{6.5}$$

7 Rigged Configuration and The Bijection

7.1 The fermionic formula

This subsection reviews the defination of the fermionic formula from [5, 6]. We at first provide the definition that is valid for any simply laced affine type $\mathfrak{g}$ and datum $\mathbf{L}$, and then restrict $\mathfrak{g}$ and $\mathbf{L}$ to $E_6^{(1)}$ and the case corresponding to paths we consider in this paper. Fix $\lambda \in \bar{P}^+$ and a matrix $\mathbf{L} = (L_i^{(a)})_{a\in I_0, i\in\mathbb{Z}_{>0}}$ of nonnegative integers, almost all zero. Let $\nu = (m_i^{(a)})$ be another auch matrix. Say that ν is an admissible configuration if it satisfies

$$\sum_{a\in I_0, i\in\mathbb{Z}_{>0}} i m_i^{(a)} \alpha_a = \sum_{a\in I_0, i\in\mathbb{Z}_{>0}} i L_i^{(a)} \bar{\Lambda}_a - \lambda \tag{7.1}$$

and

$$p_i^{(a)} \geq 0 \qquad \text{for all a} \in I_0 \text{ and } i \in \mathbb{Z}_{>0}, \tag{7.2}$$

where

$$p_i^{(a)} = \sum_{j\in\mathbb{Z}_{>0}} \left(L_j^{(a)} \min(i,j) - \sum_{b\in I_0} (\alpha_a|\alpha_b) \min(i,j) m_j^{(b)} \right). \tag{7.3}$$

Write $C(\lambda, \mathbf{L})$ for the set of admissible configuration for $\lambda \in \overline{P}^+$ and $\mathbf{L}$. Define the charge of a configuration ν by

$$c(\nu) = \frac{1}{2} \sum_{a,b\in I_0} \sum_{j,k\in\mathbb{Z}_{>0}} (\alpha_a|\alpha_b) \min(j,k) m_j^{(a)}) m_k^{(b)} - \sum_{a\in I_0} \sum_{j,k\in\mathbb{Z}_{>0}} \min(j,k) L_j^{(a)} m_k^{(a)}. \tag{7.4}$$

Using (7.3), $c(\nu)$ is written as

$$c(\nu) = -\frac{1}{2} \left(\sum_{a\in I_0, i\in\mathbb{Z}_{>0}} p_i^{(a)} m_i^{(a)} + \sum_{a\in I_0, j,k\in\mathbb{Z}_{>0}} \min(j,k) L_j^{(a)} m_k^{(a)} \right) \tag{7.5}$$

The fermionic formula is then defined by

$$M(\lambda, \mathbf{L}; q) = \sum_{\nu \in C(\lambda, \mathbf{L})} q^{c(\nu)} \prod_{a \in I_0} \prod_{i \in \mathbb{Z}_{>0}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix} \tag{7.6}$$

We now set $\mathfrak{g} = E_6^{(1)}$ and

$$L_i^{(a)} = L\delta_{a6}\delta_{i1} \qquad (a \in I_0, i \in \mathbb{Z}_{>0}) \tag{7.7}$$

The latter restriction corresponds to considering paths in $(B^{6,1})^{\otimes L}$. By abuse of notation we denote the fermionic formula under the restriction (7.7) by $M(\lambda, L; q)$. Then the $X = M$ conjecture of [6, 5] states in this particular case that

$$X(\lambda, L, q) = M(\lambda, L; q). \tag{7.8}$$

7.2 Rigged configuration

The fermionic formula $M(\lambda, \mathbf{L}; q)$ can be interpreted using combinatorial objects called rigged configuration. These objects are a direct combinatorialization of the fermionic formula $M(\lambda, \mathbf{L}; q)$. Let $\nu = (m_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$ be an admissible configuration. We identify ν with a sequence of partitions $\{\nu^{(a)}\}_{a \in I_0}$ such that $\nu^{(a)} = (1^{m_1^{(a)}} 2^{m_2^{(a)}} \cdots)$. Let $J = \{J^{a,i}\}_{(a,i) \in I_0 \times \mathbb{Z}_{>0}}$ be a double sequence of partitions. Then a rigged configuration is a pair (ν, J) subjek to the restriction (7.1) and the requirement that $J = \{J^{a,i}\}$ be a partition contained in $m_i^{(a)} \times p_i^{(a)}$ rectangle.

For a partition μ and $i \in \mathbb{Z}_{>0}$, define

$$Q_i(\mu) = \sum_j \min(\mu_j, i), \tag{7.9}$$

the area of μ in the first i columns. Then setting $Q_i^{(a)} = Q_i(\nu^{(a)})$ the vacancy number (7.3) under the restriction (7.7) is rewritten as

$$p_i^{(a)} = L\delta_{a6} - 2Q_i^{(a)} + \sum_{b \sim a} Q_i^{(b)}, \tag{7.10}$$

where $b \sim a$ stands for $C_{ba} = -1$ as desired in subsection 6.1. The set of rigged configurations for fixed X and $\mathbf{L}$ is denoted by $\mathrm{RC}(\lambda, L)$. Then (7.6) is equivalent to

$$M(\lambda, L; q) = \sum_{(\nu, J) \in \mathrm{RC}(\lambda, L)} q^{c(\nu, J)}. \tag{7.11}$$

where

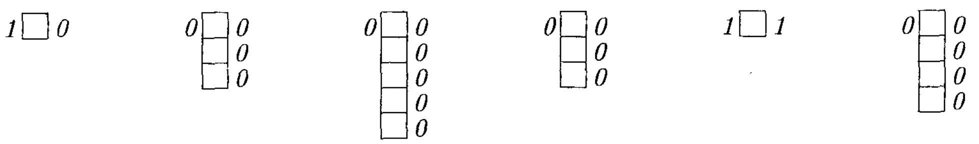
$$c(\nu, J) = c(\nu) + |J| \tag{7.12}$$

with $c(\nu)$ as in (7.4) and $|J| = \sum_{(a,i)\in I_0\times\mathbb{Z}_{>0}} |J^{(a,i)}|$. The set $\mathrm{RC}(\lambda, L)$ with the restriction (7.7) is denoted by $\mathrm{RC}(\lambda, L)$.

Example 7.1 *A rigged configuration in $RC(2\overline{\Lambda}_6, 3)$ is illustrated below.*

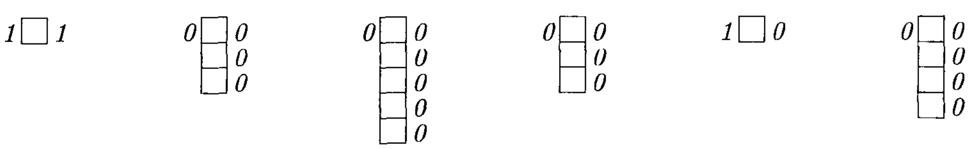
We have $c(\nu) = -5$, $c(\nu, J) = -2$.

Example 7.2 *A rigged configuration in $RC(\overline{\Lambda}_1 + \overline{\Lambda}_5, 3)$ is illustrated below.*



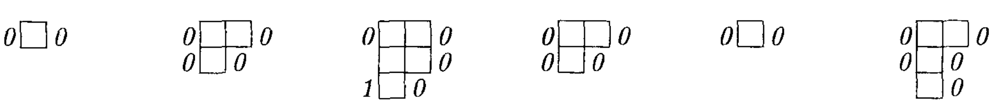
We have $c(\nu) = -7$, $c(\nu, J) = -6$.

Example 7.3 *A rigged configuration in $RC(\overline{\Lambda}_1 + \overline{\Lambda}_5, 3)$ is illustrated below.*



We have $c(\nu) = -7$, $c(\nu, J) = -6$.

Example 7.4 *A rigged configuration in $RC(\overline{\Lambda}_1 + \overline{\Lambda}_5, 3)$ is illustrated below.*

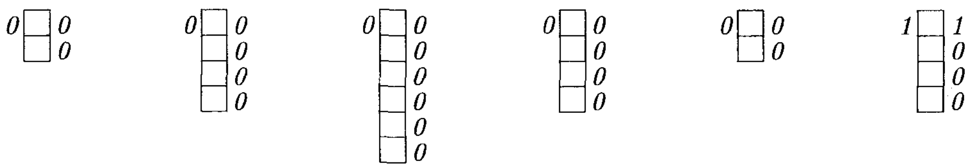


We have $c(\nu) = -5$, $c(\nu, J) = -5$.

Example 7.5 *A rigged configuration in $RC(\overline{\Lambda}_6, 3)$ is illustrated below.*

We have $c(\nu) = -5$, $c(\nu, J) = -5$.

Example 7.6 *A rigged configuration in $RC(\overline{\Lambda}_6, 3)$ is illustrated below.*

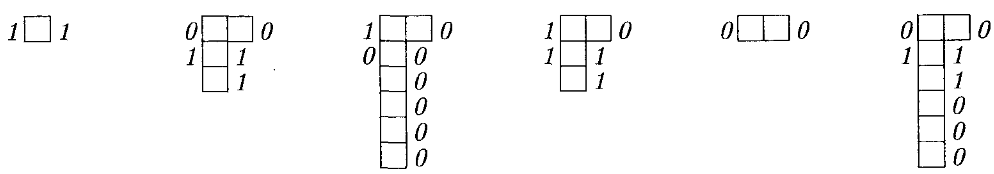


We have $c(\nu) = -8$, $c(\nu, J) = -7$.

Example 7.7 *A rigged configuration in $RC(\overline{\Lambda}_3, 3)$ is illustrated below.*

We have $c(\nu) = -6$, $c(\nu, J) = -3$.

Example 7.8 *A rigged configuration in $RC(2\overline{\Lambda}_1 + \overline{\Lambda}_3 + \overline{\Lambda}_4, 7)$ is illustrated below.*

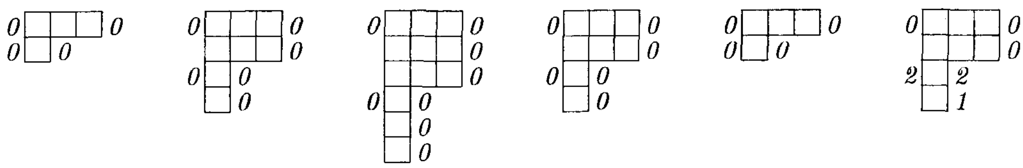


We have $c(\nu) = -27$, $c(\nu, J) = -20$.

Example 7.9 *A rigged configuration in $RC(\overline{\Lambda}_3, 3)$ is illustrated below.*

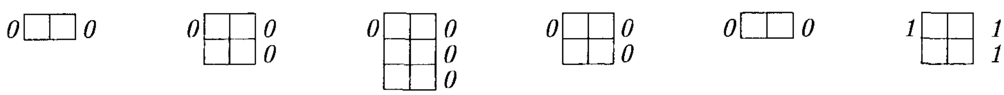
We have $c(\nu) = -6$, $c(\nu, J) = -4$.

Example 7.10 *A rigged configuration in* $RC(0, 4)$ *is illustrated below.*



We have $c(\nu) = -10$, $c(\nu, J) = -7$.

Example 7.11 *A rigged configuration in* $RC(\bar{\Lambda}_6, 3)$ *is illustrated below.*



We have $c(\nu) = -4$, $c(\nu, J) = -2$.

Example 7.12 *A rigged configuration in* $RC(2\bar{\Lambda}_6, 3)$ *is illustrated below.*

We have $c(\nu) = -4$, $c(\nu, J) = -4$.

Example 7.13 *A rigged configuration in* $RC(\bar{\Lambda}_3, 3)$ *is illustrated below.*

We have $c(\nu) = -4$, $c(\nu, J) = -4$.

7.3 The bijection from RCs to paths

We now describe the bijection $\Phi : \mathrm{RC}(\lambda, L) \to P(\lambda, L)$. Let $(\nu, J) \in \mathrm{RC}(\lambda, L)$. We shall define a map $\gamma : \mathrm{RC}(\lambda, L) \to B$ which associates to (ν, J) an element of B. Denote by $\mathrm{RC}_b(\lambda, L)$ the elements of $\mathrm{RC}(\lambda, L)$ such that $\lambda(\nu, J) = b$. We shall define a bijection $\delta : \mathrm{RC}_b(\lambda, L) \to \mathrm{RC}(\lambda - \mathrm{wt}(b), L-1)$. The disjoint union of these bijections then defines a bijection $\delta : \mathrm{RC}(\lambda, L) \to \bigsqcup_{b \in B} \mathrm{RC}(\lambda - \mathrm{wt}(b), L-1)$.

The bijection Φ is defined recursively as follows. For $L = 0$ the bijection Φ sends the empty rigged configuration (the only element of the set $\mathrm{RC}(\lambda, L)$) to the empty path (the only element of $P(\lambda, L)$). For $L > 0$ define Φ recursively by

$$\Phi(\nu, J) = \Phi(\delta(\nu, J)) \otimes \gamma(\nu, J). \tag{7.13}$$

Here follows the main conjecture.

Conjecture 7.14 *$\Phi : RC(\lambda, L) \to P(\lambda, L)$ is a bijection such that*

$$c(\nu, J) = D(\Phi(\nu, J)) \qquad \text{for all } (\nu, J) \in RC(\lambda, L). \tag{7.14}$$

8 The Procedure δ and Examples

In this section, for $(\nu, J) \in \mathrm{RC}(\lambda, L)$, an algorithm is given which defines $b = \gamma(\nu, J)$, the new smaller rigged configuration $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$ such that $(\tilde{\nu}, \tilde{J}) \in \mathrm{RC}(\rho, L-1)$ where $\rho = \lambda - \mathrm{wt}(b)$.

Before explaining the algorithm δ we prepare some terminologies. We call a row in $\nu^{(a)}$ singular (resp. quasi-singular) if its rigging (number on the right) is equal to the corresponding vacancy number $P_i^{(a)}$ (resp. $P_i^{(a)} - 1$). We also set $(c_1, \cdots, c_6) = (1, 2, 3, 2, 1, 2)$.

Lemma 8.1 *Let $\hat{\delta} = \sum_{i=0}^{6} c_i \alpha_i$ is the Kac label.*

Then $(c_1, ..., c_6) = (1, 2, 3, 2, 1, 2)$

8.1 Algorithm δ

First check wheather there are c_i singular rows of length 1 in $\nu^{(a)}$ for any $1 \leq a \leq 6$. If this is true, we define ν' to be the sequence of diagrams obtained by removing all these singular rows from $\nu^{(a)}$, and set $b = \boxed{\phi}$.

Otherwise, we start from $b = \boxed{1}$ in the crystal graph B_0 and set $\ell_0 = 1$. Repeat the following process for $j = 1, 2, ...$ until stopped. From b proceed by one step through an arrow of color a. Find the minimal integer $i \geq \ell_{j-1}$ such that $\nu^{(a)}$ has a singular row of length i and set $\ell_j = i$, reset b to be the sink of the arrow. If there is no such integer, then set $\ell_j = \infty$ and stop. If there are two or three arrows sourcing from b, compare the minimal integers i and take the smaller one. If the integers are the same, either one can be taken. If there is no such integers, then set $\ell_j = \infty$ and stop.

Suppose the process has not stopped until we arrive at either $b = \boxed{16}$, $\boxed{26}$, $\boxed{32}$, $\boxed{34}$, $\boxed{35}$ or $\boxed{36}$. Search a quasi-singular row in $\nu^{(a)}$ of minimal length i such that $i \geq \ell_{j-1}$. If there is no such row, set $\ell_j = \infty$ and stop. Otherwise, set $\ell_j = i$. Then there are two possibilities.

(S) $\ell_j \geq 2$ and the corresponding row is singular.
(Q) $\ell_j \geq 1$ and the corresponding row is quasi-singular.

In either case reset b to be the sink of the arrow. Then $b = \boxed{37}, \boxed{39}, \boxed{41}, \boxed{43}, \boxed{45}$ or $\boxed{47}$. If (Q) occured, continue the same process as before, namely, continue to search the minimal singular row of length i such that $i \geq \ell_{j-1}$. If (S) occured, we call this row of length ℓ_{j-1} "doubled", set $\ell_j = \ell_{j-1}$ and b to be the sink of second arrow of color a. Next, let a' be the color of the arrow sourcing from b. Search an integer k such that $k < j-1$, $\ell_k = \ell_{j-1}$. If such k exists, then we call this row "doubled", set $\ell_j = \ell_{j-1}$ and b to be the sink of the arrow of color a'. Otherwise, search a singular row of length i such that $i \geq \ell_{j-1}$ and i is minimal. If such i exists, set $\ell_j = i$, b to be the sink of the arrow sourcing from a', and stop calling this row "doubled".

We also introduce the notation $\ell_k^{(a)} (= \ell_j)$ if at the j-th step the arrow has color a and it is the k-th one having color a from the beginning.

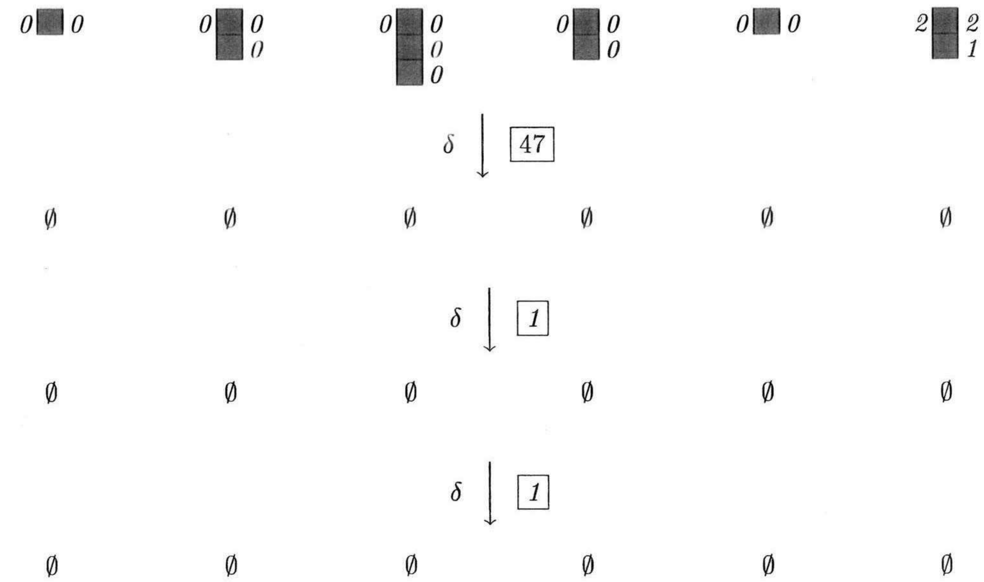
8.2 New Rigged Configuration

The new configuration $\tilde{\nu} = (\tilde{m}_i^{(a)})$ is changed to

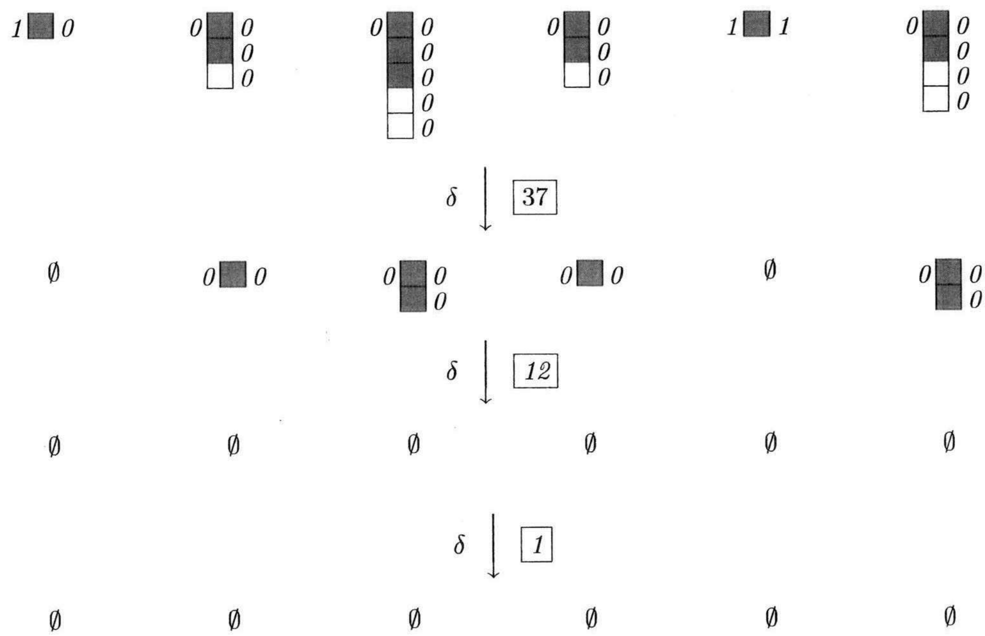
$$\tilde{m}_i^{(a)} = m_i^{(a)} - \sum_{k=1}^{k_a} (\delta_{i,\ell_k^{(a)}} - \delta_{i,\ell_k^{(a)}-1}) \tag{8.1}$$

where k_a is the maximum of k such that $\ell_k^{(a)}$ is finite. This is equivalent to say that we remove a box from each chosen during the process in δ. If a row is declared to be doubled, remove two boxes. The rigging remains equal if no box is removed from the row. If a box is removed, the new rigging is declared to be singular, except when a box of a singular row next to case (Q) is removed, in which case declared to be quasi-singular.

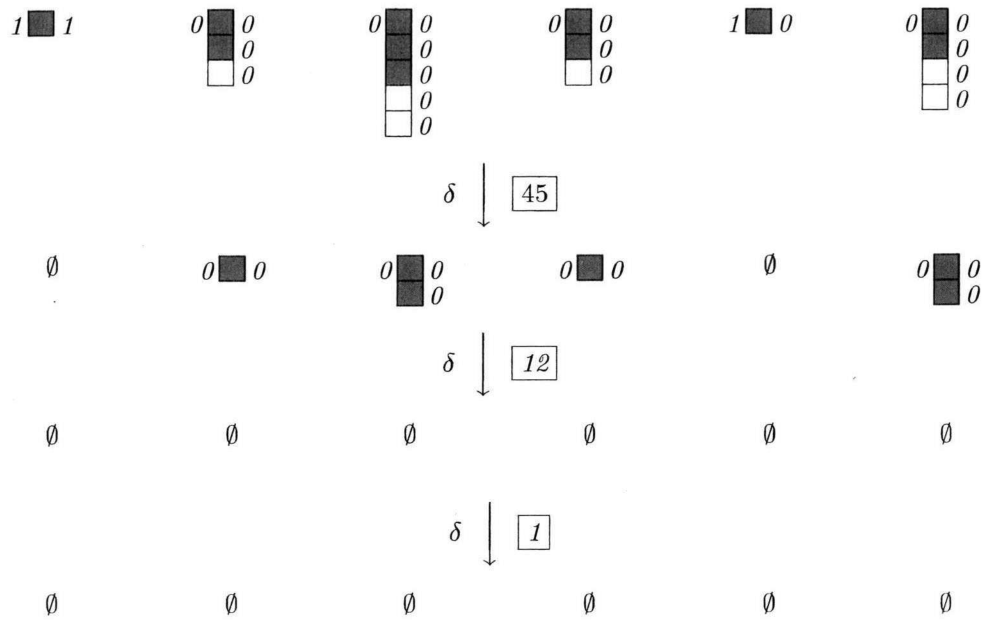
Example 8.2 *The algorithm Φ for the rigged configuration in Example 7.1 is described at each step δ below.*



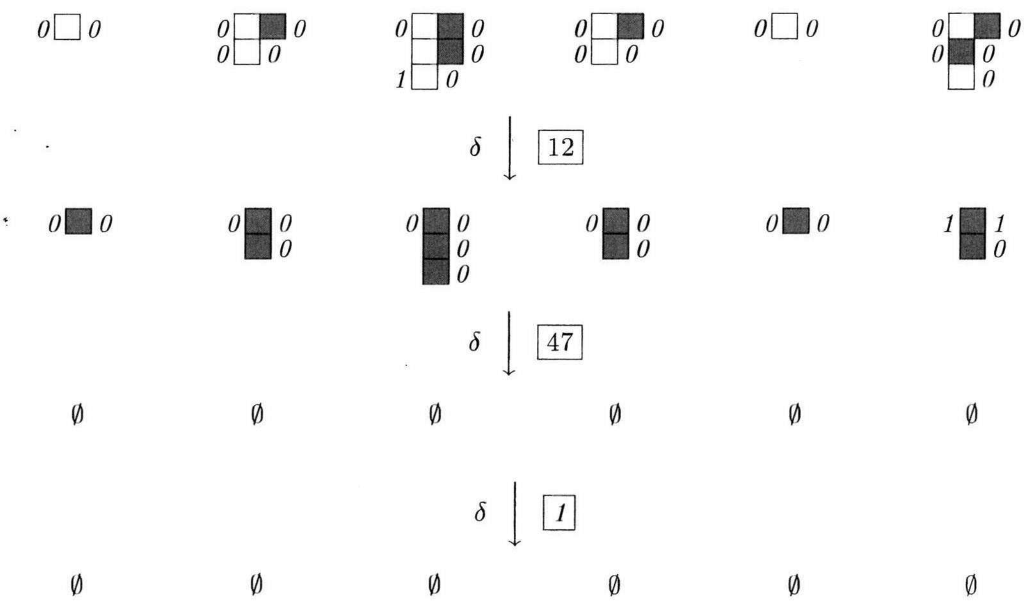
Example 8.3 *The algorithm Φ for the rigged configuration in Example 7.2 is described at each step δ below.*



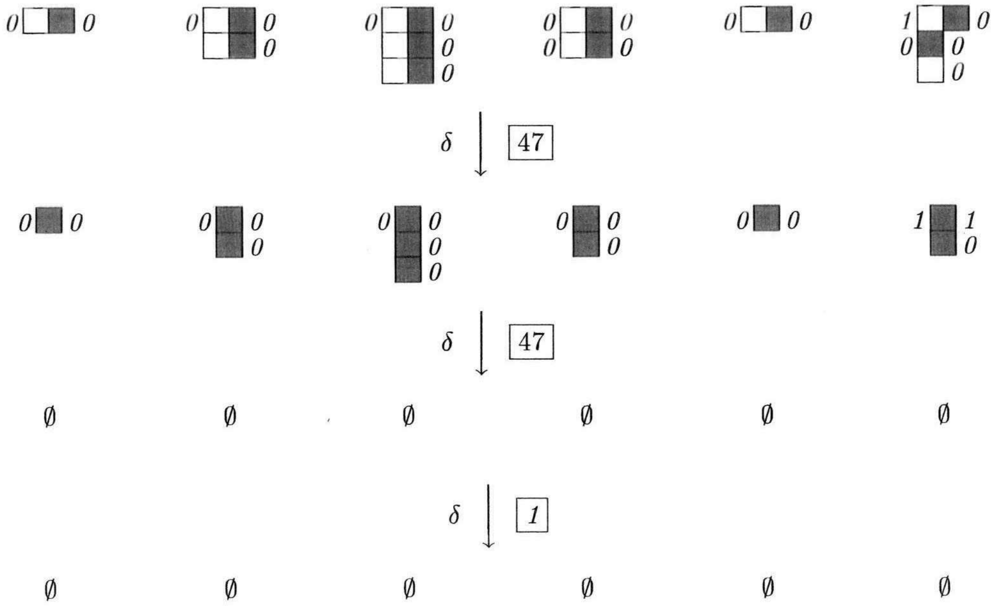
Example 8.4 *The algorithm Φ for the rigged configuration in Example 7.3 is described at each step δ below.*



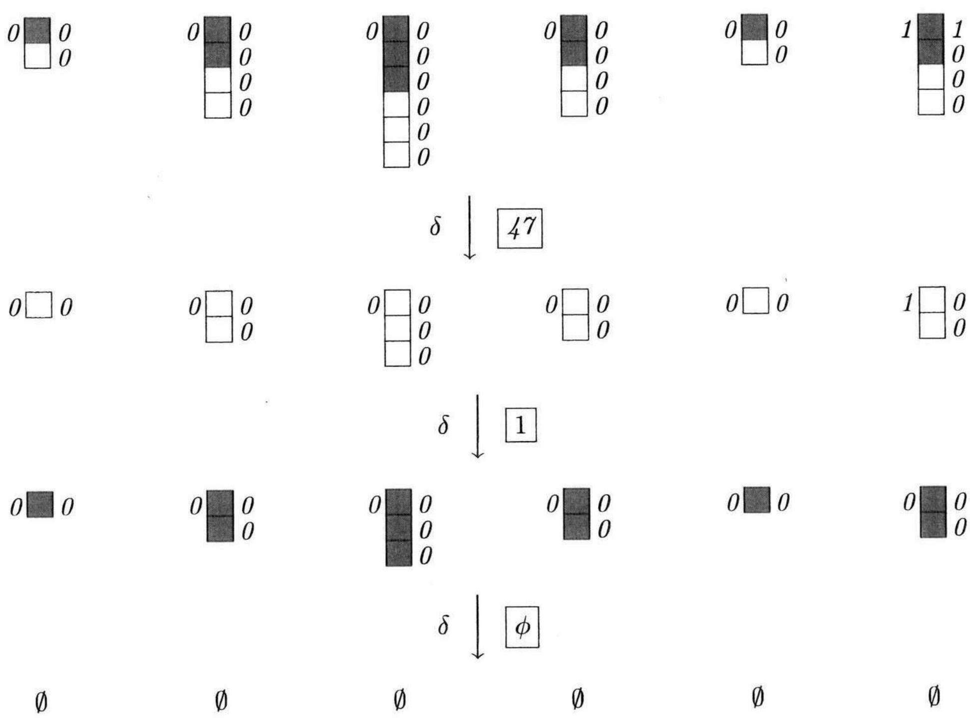
Example 8.5 *The algorithm Φ for the rigged configuration in Example 7.4 is described at each step δ below.*



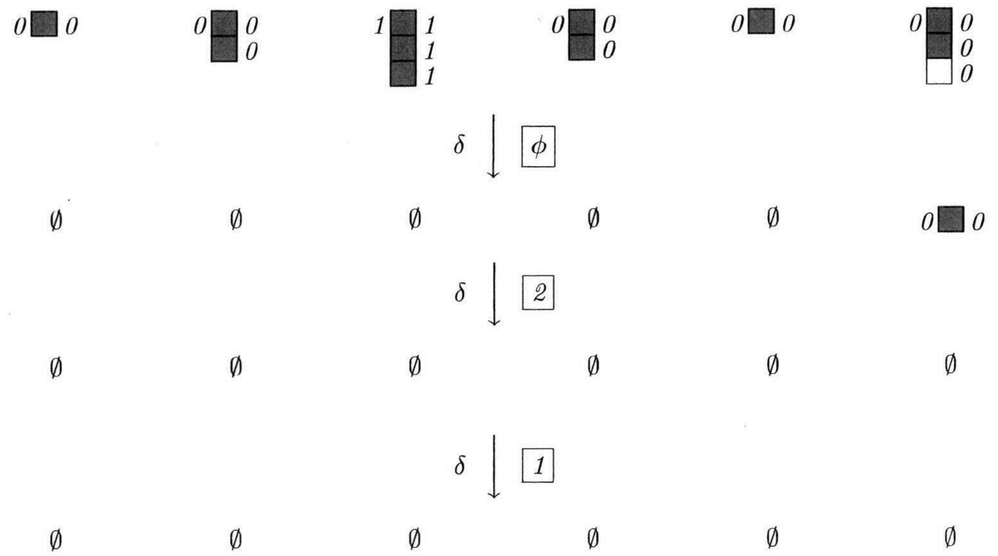
Example 8.6 *The algorithm Φ for the rigged configuration in Example 7.5 is described at each step δ below.*



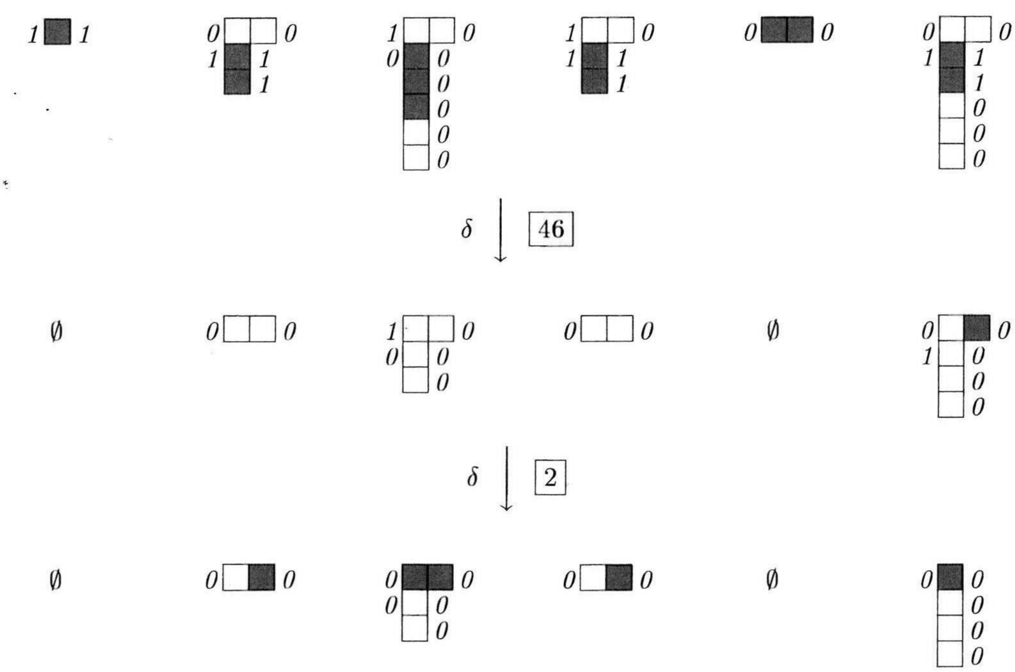
Example 8.7 *The algorithm Φ for the rigged configuration in Example 7.6 is described at each step δ below.*

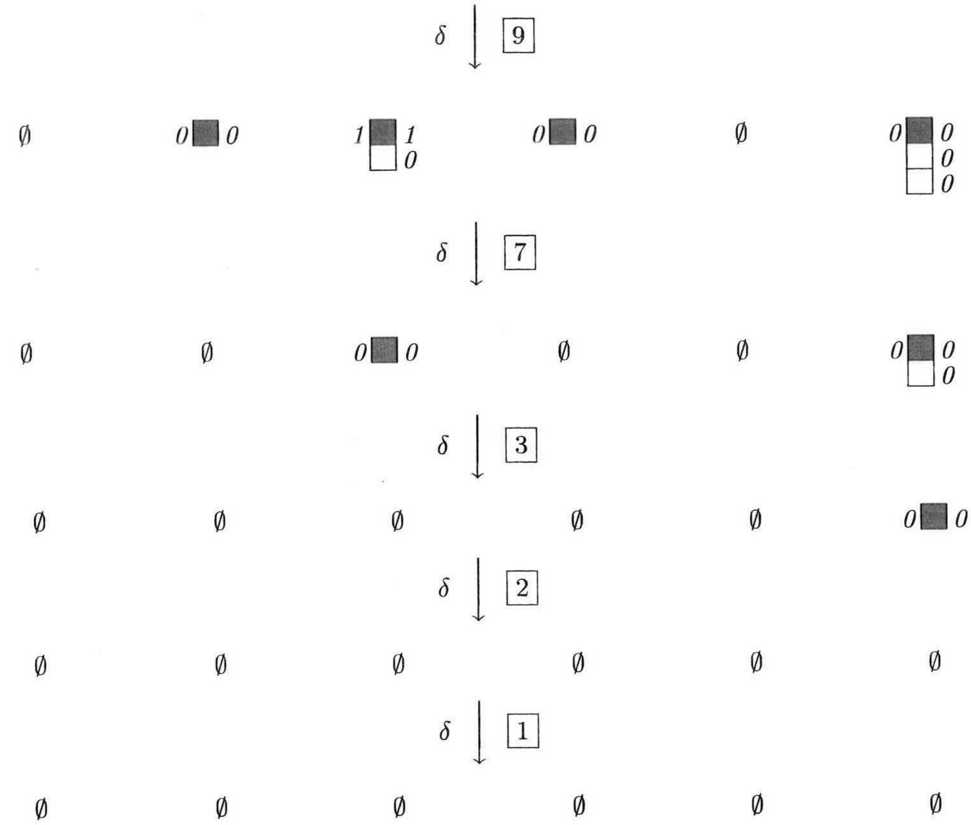


Example 8.8 *The algorithm Φ for the rigged configuration in Example 7.7 is described at each step δ below.*



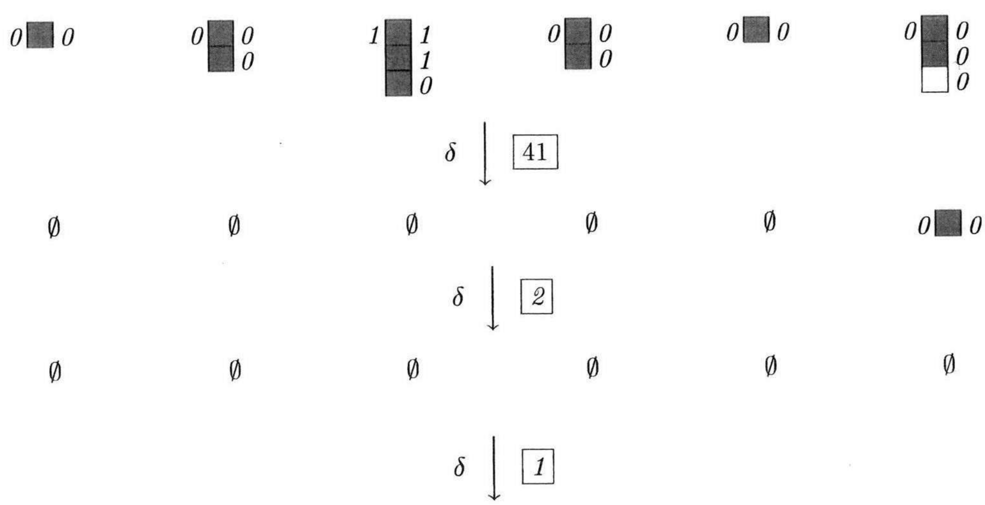
Example 8.9 *The algorithm Φ for the rigged configuration in Example 7.8 is described at each step δ below.*





Hence this rigged configuration corresponds to the path in Example 6.1 by Φ.

Example 8.10 *The algorithm Φ for the rigged configuration in Example 7.9 is described at each step δ below.*



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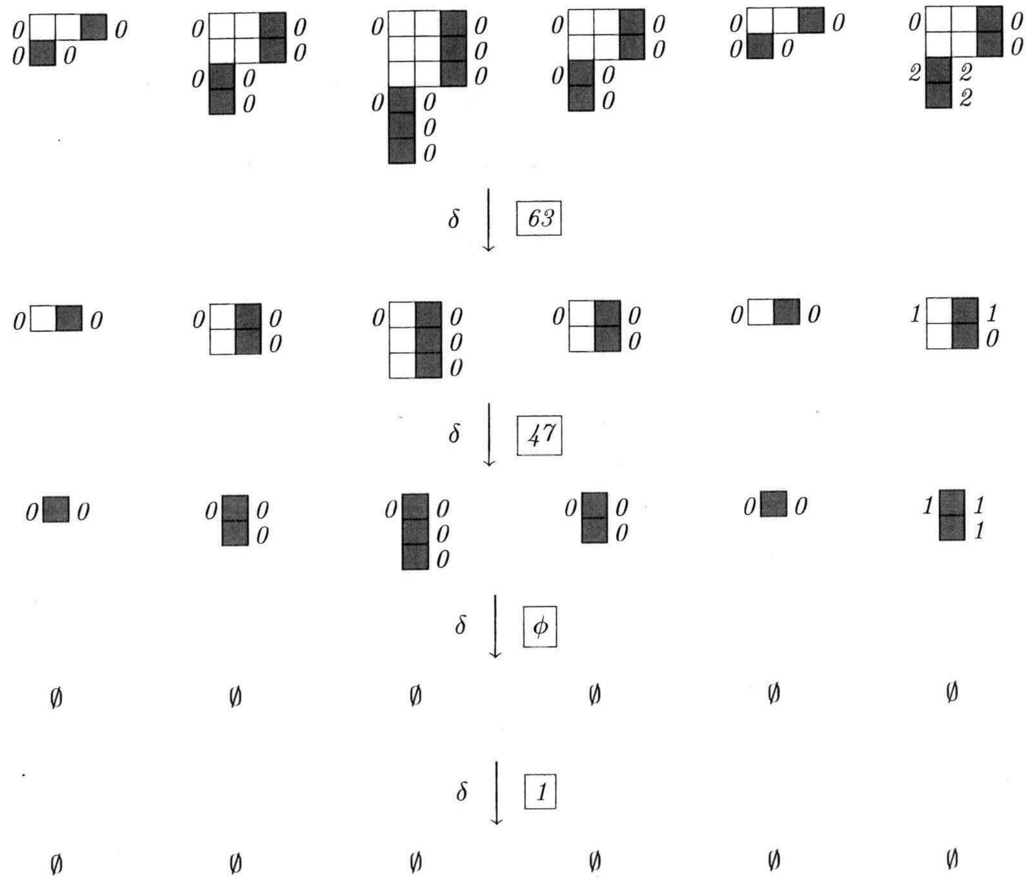
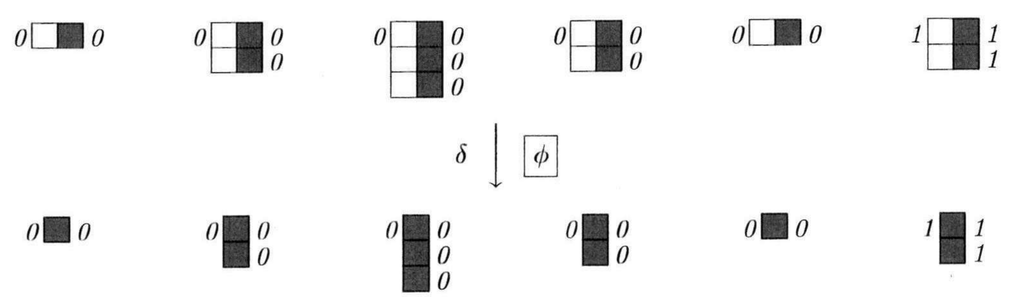
Example 8.11 *The algorithm Φ for the rigged configuration in Example 7.10 is described at each step δ below.*



Example 8.12 *The algorithm Φ for the rigged configuration in Example 7.11 is described at each step δ below.*



$\delta \downarrow \boxed{\phi}$

$\emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset$

$\delta \downarrow \boxed{1}$

$\emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset$

Example 8.13 *The algorithm Φ for the rigged configuration in Example 7.12 is described at each step δ below.*

$\delta \downarrow \boxed{1}$

$\delta \downarrow \boxed{47}$

$\emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset$

$\delta \downarrow \boxed{1}$

$\emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset$

Example 8.14 *The algorithm Φ for the rigged configuration in Example 7.13 is described at each step δ below.*

$\delta \downarrow \boxed{2}$

$$\delta \Big\downarrow \boxed{47}$$

$$\emptyset \qquad \emptyset \qquad \emptyset \qquad \emptyset \qquad \emptyset \qquad \emptyset$$

$$\delta \Big\downarrow \boxed{\mathit{1}}$$

$$\emptyset \qquad \emptyset \qquad \emptyset \qquad \emptyset \qquad \emptyset \qquad \emptyset$$

Conclusions

In Part 1, we study the behaviour of the soliton cellular automata associated with the Kirillov-Reshetikhin crystal $B^{n,1}$ of type $D_n^{(1)}$. They have a commuting family of the time evolutions and solitons of length l which are labeled by $U_q(A_{n-1}^{(1)})$-crystal $B_A^{2,l}$. In my thesis, we give an alternative way to calculate the scattering of two solitons including phase shift and it can be identified with combinatorial R-matrix for the $U_q(A_{n-1}^{(1)})$-crystal $B_A^{1,l_2} \otimes B_A^{2,l_1}$. In Part 2, we give a conjecture on statistic-preserving bijection between the highest weight path consisting of $B^{6,1}$ and corresponding rigged configuration. In this part, we give many examples supporting the conjecture. Our work is different from Okado and Sano [25] because we must consider a unique element ϕ in the crystal graph. To prove this conjecture is more complicated campared with [25]. In my future work, we want to prove this conjecture and make this as a theorem.

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List of Publications

List of Peer-Reviewed Publications

[1] Mahathir bin Mohamad, Scattering Rules in Soliton Cellular Automata Associated with $U_q(D_n^{(1)})$-Crystal $B^{n,1}$, Journal of Physics A: Mathematical and Theoretical **45**, (2012) 22 pages.

[2] K.C. Misra, M. Mohamad, M. Okado, Zero Action on Perfect Crystals for $U_q(G_2^{(1)}$, Symmetry, integrability and Geometry: Method and applications **6**, (2010) 022, 12 pages.

Seminar/ Workshop

[1] "Infinite Analysis 11 - Frontier of Integrability -", Poster Presentation, 2011.

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