ON PROPERTIES OF THE APPROXIMATELY FINITE DIMENSIONALITY FOR C*-ALGEBRAS

MASARU NAGISA

FACULTY OF ENGINEERING SCIENCE

OSAKA UNIVERSITY

1988

To my parents

Yaeko and Shuzo

Acknowledgement

It is my great pleasure to thank Professor Yoshikazu Katayama for his warm guidance and encouragements.

Special thanks go to Professor Jun Tomiyama for his valuable suggestions.

I would like to thank all the members of the groups of operator algebraists in Osaka University for their constant encouragement and valuable discussions.

But above all, my most sincere gratitude goes to Professor Osamu Takenouchi, my adviser, for his guidance, instructive discussions and constant encouragement throughout the course of this research.

Table of contents

Page

1. Introduction 1

2. Preliminaries and Notations 5

3. Hereditary C*-subalgebras of AF-algebras and their diagrams 11

4. Increasing sequences of some special AF-algebras 21

5. Examples and Applications 36

6. Decomposable norms and finite rank linear maps 47

References 58

1. Introduction

This thesis is devoted to the study of the approximately finite dimensionality for C*-algebras. We treat two classes of C*-algebras. One of them is the class of approximately finite dimensional C*-algebras (in short, AF-algebras), which are defined as an inductive limit of finite dimensional C*-algebras. AF-algebras can be characterized by their diagrams. We determine the structure of hereditary C*-subalgebra of AF-algebras based on their diagrams, and that of a special class of AF-algebras which are related to Jones' index theory ([27]) and Powers' binary shift ([37],[38],[39]).

The other class of C*-algebras we treat is that of nuclear C*-algebras, whose class is wider than the class of AF-algebras and which is an important class in the theory of operator algebras. We demonstrate that these algebras satisfy an interesting property which is related to completely positive maps and completely bounded maps of finite rank.

A UHF-algebra is a C*-algebra defined as an inductive limit of finite-dimensional full complex matrix algebras. Glimm ([23]) studied UHF-algebras in general, and he classified their isomorphic classes by indexing them with generalized integers, which is called dimension group or K_0-group in recent terminology.

Bratteli ([2]) considered AF-algebras (extended version of UHF-algebras), which are inductive limits of finite-dimensional C*-algebras. In describing their isomorphic classes, he introduced a (Bratteli) diagram of an AF-algebra and showed that a closed two-sided ideal of an AF-algebra is also AF. In particular, the diagram of a closed two-sided ideal can be assigned as a part of the diagram of the original AF-algebra. These necessary informations are supplied in section 2.

Now, a C*-subalgebra of an AF-algebra is not necessarily AF. For example, any separable commutative C*-algebra can be embedded as the center of a suitable AF-algrbra ([3]), and we remark that the typical commutative C*-algebra $C[0,1]$ is not AF, where $C[0,1]$ is the set of all the continuous functions on the unit interval $[0,1]$. Meanwhile, any hereditary C*-subalgebra of an AF-algebra is also AF according to Elliott ([20]). So we study relations between the diagram of a hereditary C*-subalgebra of an AF-algebra and the diagram of the original AF-algebra. In section 3, we will get the following result.

THEOREM. *Let B be a hereditary C*-subalgebra of an AF-algebra A. Then there exists a family E of pairwise orthogonal projections contained in $\{e_{i,i}^{(n,k)}; n \in \mathbf{N},\ 1 \leq k \leq s(n),\ 1 \leq i \leq [n,k]\}$ such that B is isomorphic to the hereditary C*-subalgebra of A generated by E.*

Accordingly, the diagram of a hereditary C*-subalgebra is obtained as a subdiagram of the original AF-algebra.

In section 4 we treat some special AF-algebras and study their structures. Let F_2 be the finite field with 2 elements $\{0,1\}$. For a sequence $\{a(n); n \in \mathbf{Z}\}$ of elements in F_2 and a sequence of unitaries $\{u_n ; n = 0, 1, 2, \cdots\}$, we assume that

$$\begin{cases} d = \sup\{n\,;a(n) = 1\} & \text{is finite,} \\ u_i u_j = (-1)^{a(i-j)} u_j u_i & \text{for any } i, j. \end{cases}$$

Then the C*-algebra P_n generated by $\{u_0, u_1, \cdots, u_n\}$ is finite-dimensional, and we can get an inductive system $\{P_n\}$ of finite dimensional C*-algebras. We can make a strict observation against this system, and see that their inclusion matrices are periodic.

In the argument of section 4, we note that F_2 can be replaced by F_p, where F_p is the finite field with p elements, p a prime number.

In section 5 we construct interesting AF-algebras, using results in section 4. These algebras appear as sequences of relative commutant algebras of Powers' binary shifts. In the earlier stage, Powers ([37]) conjectured that the sequence of relative commutant algebras was full outer conjugacy invariant of binary shifts. Recently Bures and Yin ([7]) proved that the conjugacy classes of binary shifts with a finite support coincides with their outer conjugacy classes. But, in general, Powers' conjecture is false (see [22]). As a matter of fact, we will exhibit two shifts which are not outer conjugate, whereas they have the same sequence of relative commutant algebras in the sense of the diagram.

Some properties of complete positive maps are useful to emphasize the difference of commutative C*-algebras and non-commutative C*-algebras ([42]). We characterize the order structure of a commutative C*-algebra as that of a matrix ordered Banach space of order 1 ([26]). Nuclearity of C*-algebras is characterized by the word of completely positive maps ([10],[12],[28]), that is, a C*-algebra A is nuclear if there exists a net $\{\varphi_\nu\}$ of completely positive maps of finite rank such that

$$\lim_{\nu} \| \varphi_\nu(a) - a \| = 0 \quad \text{for any } a \in A.$$

Nuclear C*-algebras are objects which are closely related to injective von Neumann algebras. Haagerup ([25]) introduced the decomposable norm for a linear map from a C*-algebra to a C*-algebra. He considered a property for a C*-algebra A that, for any linear map from $\mathbf{C}^n$ to A, its completely bounded norm is equal to its decomposable norm, and he shows that this property characterized the injectivity of von Neumann algebras. In section 6 we show that a wider class of C*-algebras than the class of nuclear C*-algebras possesses this property.

2. Preliminaries and Notations

Let $\mathcal{H}$ be a separable Hilbert space over the complex field $\mathbf{C}$ and $B(\mathcal{H})$ be the algebra of all the bounded linear operators on $\mathcal{H}$.

DEFINITION 2.1. *A C^*-algebra A is a norm closed *-subalgebra of $B(\mathcal{H})$. A C^*-algebra A is called* **AF-algebra** *if, for any $\varepsilon > 0$ and element x in A, there exist a finite-dimensional *-subalgebra F of A and an element y in F such that $\| x - y \| < \varepsilon$.*

If A is an AF-algebra, then there exists an increasing sequence $\{A_n\}$ of finite-dimensional *-subalgebras such that

$$\bigcup_{n\geq 1} A_n \quad \text{is dense in } A \quad ([2]).$$

A finite-dimensional C*-algebra is isomorphic to a direct sum of full complex matrix algebras . So A_n has the form

$$M(n,1) \oplus \cdots \oplus M(n, s(n)),$$

where $M(n,k)$ is isomorphic to the full complex matrix algebra and its dimension will be denoted by $[n,k]$. So $M(n,k)$ has a system of matrix units

$$\{e_{i,j}^{(n,k)}; 1 \leq i,j \leq [n,k]\} \quad (k = 1, 2, \cdots, s(n)).$$

Let $u_{i,j}(n)$ be the multiplicity of the inclusion from $M(n,i)$ to $M(n+1,j)$ which means that for any minimal projection p in $M(n+1,j)$, there exist orthogonal minimal projections $q(1), q(2), \cdots, q(u_{i,j}(n))$ in $M(n,i)$ such that

$$p > q(1) + q(2) + \cdots + q(u_{i,j}(n)),$$

and, for any minimal projections $r(1), r(2), \cdots, r(u_{i,j}(n)+1)$ in $M(n,i)$,

$$p \not\geq r(1) + r(2) + \cdots + r(u_{i,j}(n)+1).$$

We define the inclusion matrix $U(n)$ from A_n to A_{n+1} by

$$U(n) = (u_{i,j}(n)), \quad (i = 1, \cdots, s(n),\ j = 1, \cdots, s(n+1)).$$

Then (D, d, U) satisfies the following condition

$$(^*) \quad \sum_{k=1}^{s(n)} u_{k,l}(n) d(n,k) \leq d(n+1, l).$$

The Bratteli diagram (in short, the diagram) for an AF-algebra A is the following. Notations are due to [29]. We also add the definition of a subdiagram as it will be important in our later studies.

DEFINITION 2.2. *We call the triple (D, d, U) the* **diagram** *for A, where D is the set $\{(n,k); n \in \mathbf{N}, 1 \leq k \leq s(n)\}$, d is a map from D to $\mathbf{N}$ defined by $d(n,k) = [n,k]$ and U is the set $\{U(n); n \in \mathbf{N}\}$. Let d' be a map from D to $\mathbf{N} \cup \{0\}$ such that $d'(n,k) \leq d(n,k)$ for each $(n,k) \in D$ and $U'(n)$ be an $s(n) \times s(n+1)$ matrix, whose entries $u'_{i,j}(n)$ are non-negative integers and $u'_{i,j}(n) \leq u_{i,j}(n)$. We set $U' = \{U'(n); n \in \mathbf{N}\}$. We call a triple (D, d', U') a* **subdiagram** *of (D, d, U) if it satisfies the following four conditions,*

(1) $\qquad u'_{k,l}(n) = 0 \quad (l = 1, \cdots, s(n+1)) \quad$ *if* $d'(n,k) = 0$,

(2) $\qquad d'(n,k) = 0 \quad$ *if* $u'_{k.l}(n) = 0 \quad$ *for any* $1 \leq l \leq s(n+1)$,

(3) $\qquad \displaystyle\sum_{k=1}^{s(n)} u'_{k,l}(n) d'(n,k) \leq d'(n+1, l)$,

(4) $\qquad u'_{k,l}(n) = u_{k,l}(n) \quad$ *if* $u'_{k,l}(n) > 0$.

For a given AF-algebra A, we can construct a diagram for A. Conversely we can construct an AF-algebra whose diagram coincides with a given diagram satisfying (*). If I is a closed two-sided ideal of an AF-algebra A, then I is AF, and the diagram for I is a subdiagram of the diagram for A satisfying

$$d'(n,k) = d(n,k) \quad \text{or} \quad 0 \quad ([2]).$$

DEFINITION 2.3. *A C^*-subalgebra B of a C^*- algebra A is called* **hereditary** *if $0 < a < b, a \in A$ and $b \in B$ imply $a \in B$. Let S be a subset of A . We denote by $C^*(S)$ (resp. $HC^*(S)$) the smallest C^*-subalgebra of A (resp . the smallest hereditary C^*-subalgebra of A) containing S .*

Any hereditary C*-subalgebra of an AF-algebra is also AF . In the following section, we show that its diagram becomes a subdiagram of the diagram for the original AF-algebra .

Let T be a bounded linear map from a C*-algebra A to a C*-algebra B. We denote T_n a linear map from $A \otimes M_n$ to $B \otimes M_n$ defined by

$$T_n([a_{i,j}]) = [T(a_{i,j})]$$

for all $n \in \mathbf{N}$ and $[a_{i,j}] \in M_n(A) \cong A \otimes M_n$, where M_n is the $n \times n$ complex matrix algebra.

DEFINITION 2.4. *A linear map T from A to B is called* **positive** *(resp.* **completely positive**) *if*

$$T(A^+) \quad \textit{is contained in } B^+.$$

$$(\textit{resp. } T_n((A \otimes M_n)^+) \quad \textit{is contained in } (B \otimes M_n)^+ \quad \textit{for all } n.)$$

T is called **completely bounded** *if* $\sup\{\| T_n \|; n \in \mathbf{N}\}$ *is finite, and we call this value the* **completely bounded norm** *of T (denoted by $\| T \|_{cb}$).*

A linear map P from a C*-algebra A to a C*-subalgebra B of A is called a **projection of norm one** if P satisfies the following conditions,

$$\begin{aligned} &\|P(a)\| \leq \|a\| \quad && \text{for any } a \in A, \\ &P(b) = b && \text{for any } b \in B. \end{aligned}$$

A projection of norm one is automatically completely positive.

A linear map T is **decomposable** if it is a linear combination of completely positive maps from A to B. In another formulation, T is decomposable if and only if there exist completely positive maps S_1, S_2 from A to B such that

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}$$

is a completely positive map from A to $B \otimes M_2$.

Definition 2.5. *If T is decomposable, then we define the* **decomposable norm** *of T by*

$$\|T\|_{dec} = \inf\{\lambda;\ \lambda \geq \|S_i\| \quad \textit{for some } \ S_1, S_2 \textit{ such that}$$
$$R = \begin{pmatrix} S_1 & T^* \\ T & S_2 \end{pmatrix} \textit{ is completely positive } \}$$

Following facts are known (see, [25]),

(1) Any decomposable map is completely bounded and $\| T \|_{cb} \leq \| T \|_{dec}$.

(2) If S is a completely positive map, then $\| S \| = \| S \|_{cb} = \|S \|_{dec}$.

(3) If T_1, T_2 are decomposable, then $\| T_2 \circ T_1 \|_{dec} \leq \| T_2 \|_{dec} \| T_1 \|_{dec}$,

where T_1 is a linear map from a C*-algebra A to a C*-algebra B and T_2 is a linear map from a C*-algebra B to a C*-algebra C.

In the rest of this section, we assume that C*-algebras are unital and completely positive maps are also unital.

DEFINITION 2.6. *A C*-algebra A is called* **nuclear** *if there exist a net* $\{\varphi_\nu\}$ *of completely positive maps of finite rank such that*

$$\lim_{\nu} \| \varphi_\nu(a) - a \| = 0 \quad \textit{for any } a \in A.$$

A C*-algebra A on a Hilbert space is called **injective** if there exists a projection of norm one from $B(\mathcal{H})$ to A.

A completely positive map T from A to B is called **factorizable** if there exist a matrix algebra M_n, a completely positive map σ from A to M_n and a completely positive map τ from M_n to B such that $T = \tau \circ \sigma$. A C*-algebra A is nuclear if and only if the identity map of A is point-wise approximable by factorizable completely positive maps (the factorization property).

Definition 2.7. *Let A, B be C^*-algebras, and $A \odot B$ be the algebraic tensor product A and B. A norm β on $A \odot B$ is called a* C^***-norm** *on $A \odot B$ if it satisfies the conditions,*

$$\beta(a \otimes b) = \|a\|\|b\| \quad \text{for all } a \in A, b \in B,$$

$$\beta(x^* x) = \beta(x)^2 \quad \text{for all } x \in A \odot B.$$

The completion of $A \odot B$ with respect to this norm β is called the C^***-tensor product** *$A \otimes_\beta B$ of A and B with respect to β.*

Let A (resp. B) is a C*-algebra on a Hilbert space $\mathcal{H}$ (resp. $\mathcal{K}$). We can regard the algebraic tensor product $A \odot B$ of A and B as a subalgebra of $B(\mathcal{H} \otimes_\sigma \mathcal{K})$, where $\mathcal{H} \otimes_\sigma \mathcal{K}$ is the completion of the algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$ as a Hilbert space. The norm closure of $A \odot B$ in $B(\mathcal{H} \otimes_\sigma \mathcal{K})$ is called the **spatial tensor product** of A and B, and is denoted by $A \otimes B$. If either A or B is nuclear, then a C*-norm on $A \odot B$ is uniquely determined, that is, it coincides with the spatial tensor norm on $A \odot B$. In the case that A is finite-dimensional, A is nuclear and $A \odot B$ is already closed. So we use the notation $A \otimes B$ instead of $A \odot B$, if either A or B is finite-dimensional.

3 Hereditary C*-subalgebras of AF-algebras and their diagrams

Let A be an AF-algebra and $\{A_n\}$ be an increasing sequence of finite dimensional C*-subalgebras of A such that

$$\bigcup_n A_n \quad \text{is dense in } A \text{ with respect to the norm topology.}$$

Let E be a family of pairwise orthogonal projections in A. If we define $H(E)$ by the set of elements a such that

$$\begin{cases} a \in \bigcup_n A_n, \\ \text{there exist } n \in \mathbf{N} \text{ and } e_1, \cdots, e_n \in E \quad \text{such that } a = (\sum_{i=1}^n e_i)a(\sum_{i=1}^n e_i), \end{cases}$$

then $H(E)$ is a *-subalgebra of A.

Lemma 3.1. *$C^*(H(E))=HC^*(E)$.*

Proof: Clearly $C^*(H(E))$ is contained in $HC^*(E)$. We have to show that $C^*(H(E))$ is hereditary. We may assume that E is infinite, that is, $E = \{e_1, e_2, \cdots\}$. Suppose $0 \leq a \leq b$, $b \in C^*(H(E))$. We set $a_n = \sum_{i=1}^n e_i$. Since $C^*(H(E))$ is the norm closure of $H(E)$, there exists a sequence $\{b_n\}$ in $H(E)$ such that b_n converges to b and $a_n b_n a_n = b_n$ for each n. Then

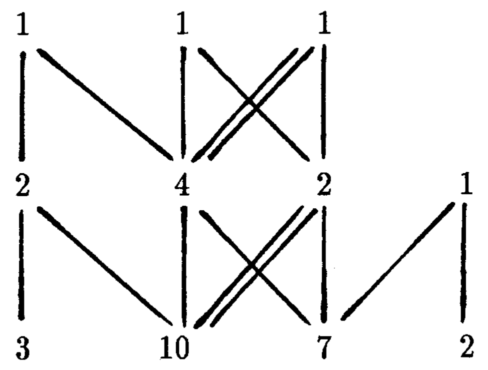
$$0 = \lim_{n\to\infty}(1-a_n)b_n(1-a_n) = \lim_{n\to\infty}(1-a_n)b(1-a_n).$$

By the assumption,

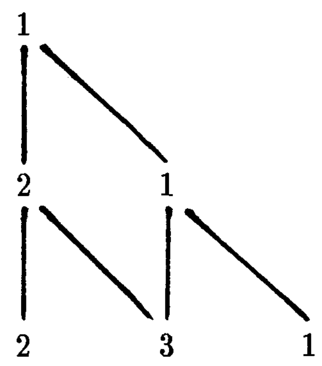
$$\lim_{n\to\infty}(a^{\frac{1}{2}} - a^{\frac{1}{2}}a_n)^*(a^{\frac{1}{2}} - a^{\frac{1}{2}}a_n) = \lim_{n\to\infty}(1-a_n)a(1-a_n) = 0,$$

so $\lim_{n\to\infty} a_n a a_n = a$. Choose a sequence $\{c_n\}$ in $\bigcup_{n=1}^\infty A_n$ which converges to a. Then we see that $\lim_{n\to\infty} a_n c_n a_n = a$ according to what we have just shown, and here $a_n c_n a_n \in H(E)$. Therefore $a \in C^*(H(E))$, that is, $C^*(H(E))$ is hereditary. ∎

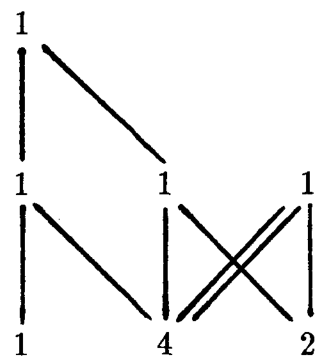
Example. Let A be an AF-algebra with a diagram as follows.



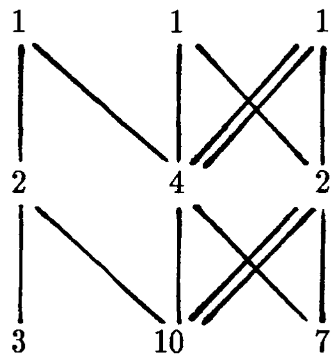
If $E = \{e_{1,1}^{(1,1)}, e_{2,2}^{(2,2)}\}$, then $HC^*(E)$ has a diagram as follows.



If $E = \{e_{1,1}^{(1,1)}, e_{2,2}^{(2,3)}\}$, then $HC^*(E)$ has a diagram as follows.



Then the smallest closed two-sided ideal of A containing $HC^*(\{e_{1,1}^{(1,1)}, e_{2,2}^{(2,2)}\})$ coincides with that of A containing $HC^*(\{e_{1,1}^{(1,1)}, e_{2,2}^{(2,3)}\})$. This ideal has a diagram as follows.



∎

Let (D, d, U) be a diagram for A. In the above example, using a subset E of $\{e_{i,i}^{(n,k)}; n \in \mathbf{N}, 1 \leq k \leq s(n), 1 \leq i \leq [n,k]\}$, we constructed a hereditary C*-subalgebra $HC^*(E)$ whose diagram coincides with a subdiagram of (D, d, U). Conversely we will see that any hereditary C*-subalgebra of A is isomorphic to such a hereditary C*-subalgebra.

Proposition 3.2. *If $E = \{e_i; i = 1, 2, \cdots\}$ and $F = \{f_i; i = 1, 2, \cdots\}$ are families of pairwise orthogonal projections in A and there exist partial isometries v_i in A such that $v_i^* v_i = e_i$ and $v_i v_i^* = f_i$ $(i = 1, 2, \cdots)$, then $HC^*(E)$ is isomorphic to $HC^*(F)$.*

Proof: Let $\mathrm{Proj}HC^*(E)/\sim$ be all the equivalence classes of projections of $HC^*(E)$ by partial isometries in $HC^*(E)$. $\mathrm{Proj}HC^*(E)/\sim$ is a local semi-group ([19]). Since $\{\sum_{i=1}^n e_i\}$ is an approximate unit for $HC^*(E)$, for each element $a \in HC^*(E)$,

$$(\sum_{i=1}^{\infty} v_i)\, a\, (\sum_{i=1}^{\infty} v_i^*)$$

is well-defined, and this element belongs to $HC^*(F)$. We define a map θ from $\mathrm{Proj}HC^*(E)/\sim$ to $\mathrm{Proj}HC^*(F)/\sim$ by

$$\theta[p] = [(\sum_{i=1}^{\infty} v_i)\, p\, (\sum_{i=1}^{\infty} v_i^*)],$$

where $[\cdot]$ denotes an equivalence class of projections. Then the map θ is an isomorphism as local semi-groups. By [29, Theorem 4.3], the map θ induces an isomorphism from $HC^*(E)$ to $HC^*(F)$. ∎

The following lemma is proved by Glimm [23] in the case of UHF-algebras. Here we prove this in the case of general C*-algebras by a way that contains fundamental techniques of the K-theory in operator algebras.

LEMMA 3.3. *Let A be a C*-algebra acting on a Hilbert space $\mathcal{H}$, B be a C*-subalgebra of A and e and f be projections in A.*

(1) If $\|e - f\| < 1$, then there exists a partial isometry v in A such that

$$v^*v = e \quad \text{and} \quad vv^* = f.$$

(2) If there exist an element a in B and a positive number ε ($< \frac{1}{16}$) such that $\|e-a\| < \varepsilon$, then there exists a projection g in B such that $\|e - g\| < 2\sqrt{\varepsilon}$.

PROOF: (1) We set $\varepsilon = 1 - \|e - f\|$ (> 0). We denote the polar decomposition of the operator fe by $v|fe|$, that is, $|fe|$ is the square root of the operator $(fe)^*(fe)$ and v is the partial isometry from the initial space of fe to the final space of fe. For any vector ξ in

$\mathcal{H}$ with $e\xi = \xi$,

$$\begin{aligned} \|fe\xi\| &\geq \|\xi\| - \|\xi - fe\xi\| \geq \|\xi\| - \|(e-f)\xi\| \\ &\geq \varepsilon\|\xi\|. \end{aligned}$$

So the initial space of fe coincides with the initial space of e. Remarking the fact $ef = (fe)^* = |fe|v^*$ and the above calculation, we can see that the final space of fe coincides with the final space of f. Therefore $v^*v = e$ and $vv^* = f$.

For any vector ξ in $\mathcal{H}$ with $e\xi = \xi$,

$$(|fe|^2\xi|\xi) = \|fe\xi\|^2 \geq \varepsilon^2\|\xi\|^2,$$

where $(\cdot|\cdot)$ is the inner product of $\mathcal{H}$. So the spectrum $Sp(|fe|)$ of the positive contraction $|fe|$ is contained in $\{0\} \cup [\varepsilon, 1]$. If we define the continuous function F_1 on $Sp(|fe|)$ by

$$F_1(x) = \begin{cases} 0 & \text{if } 0 \leq x < \varepsilon, \\ \frac{1}{x} & \text{if } \varepsilon \leq x \leq 1, \end{cases}$$

then $v = v|fe|F_1(|fe|) = feF_1(|fe|)$. Therefore v belongs to A.

(2) We may assume that a is a self-adjoint element of B. Since

$$\begin{aligned} \|a - a^2\| &\leq \|a - e\| + \|e - ea\| + \|ea - a^2\| \\ &\leq \|a - e\| + \|e - a\| + \|a\|\|e - a\| \\ &\leq 2\|a - e\| + (1 + \|a - e\|)\|a - e\| \\ &< \varepsilon(3 + \varepsilon) \\ &< 4\varepsilon \quad (< \frac{1}{4}), \end{aligned}$$

the spectrum $Sp(a)$ of a is contained in $[-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}]$. If we define the continuous function F_2 on $Sp(a)$ by

$$F_2(x) = \begin{cases} 0 & \text{if } -\frac{1}{2} \leq x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq \frac{3}{2}, \end{cases}$$

and we set $g = F_2(a)$, then g is a projection in B and $\|e - g\| < 2\sqrt{\varepsilon}$ by the inequality $\|a - a^2\| < 4\varepsilon$. ∎

LEMMA 3.4. *Let e and f be mutually orthogonal projections in an AF-algebra A. Then, for any $\varepsilon(0 < \varepsilon < 1)$, there exist $n \in \mathbf{N}$ and mutually orthogonal projections $e', f' \in A_n$ such that*

$$\max\{\| e - e' \|, \| f - f' \|, \| (e + f) - (e' + f') \|\} < \varepsilon.$$

In particular, we have $e \sim e'$, $f \sim f'$ and $e + f \sim e' + f'$ where $\sim$ denotes an equivalent relation by partial isometries in A.

PROOF: Let $0 < \delta < \varepsilon/14$. By the fact $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ and Lemma 3.3(2), there exist $n \in \mathbf{N}$ and projections $e', h \in A_n$ such that $\| e - e' \| < \delta, \| f - h \| < \delta$. By mutual orthogonality of e and f, we have

$$\|e'h\| = \|e'(h - f) + (e' - e)f\| < 2\delta.$$

If we set $h' = (1 - e')h(1 - e')$, then h' is a positive contraction and $e'h' = h'e' = 0$. Since

$$\begin{aligned}
\|h' - h'^2\| &\leq \|(1 - e')\{h - (1 - e')h(1 - e')\}h(1 - e')\| \\
&\leq \|h - h'\| \\
&= \|e'h + he' - e'he'\| \\
&< 6\delta \quad (< \frac{1}{2}),
\end{aligned}$$

the spectrum $Sp(h')$ of h' is contained in $[0, 6\delta] \cup [1 - 6\delta, 1]$. We define a continuous function F_3 on $Sp(h')$ by

$$F_3(x) = \begin{cases} 0 & \text{if } x \in [0, 6\delta), \\ x & \text{if } x \in [6\delta, 1 - 6\delta], \\ 1 & \text{if } x \in (1 - 6\delta, 1]. \end{cases}$$

If we set $f' = F_3(h')$, then f' is a projection in A_n and is orthogonal to e'. By the theory of functional calculus,

$$\begin{aligned} \|h - f'\| &\leq \|h - h'\| + \|h' - f'\| \\ &< 6\delta + \|I - F_3\|_\infty \|h'\| \\ &< 12\delta, \end{aligned}$$

where $I(x) = x$ for any $x \in Sp(h')$ and $\|\cdot\|_\infty$ is the supremum norm of continuous functions on $Sp(h')$. Then

$$\| f - f' \| \leq \| f - h \| + \| h - f' \| < 13\delta < \varepsilon,$$

$$\| (e + f) - (e' + f') \| \leq \| e - e' \| + \| f - f' \| < 14\delta < \varepsilon.$$

The last part follows from Lemma 3.3(1). ∎

The following lemma is the most technical part in this section.

LEMMA 3.5. *If $\{e_i; i = 1, 2, \cdots\}$ is a family of pairwise orthogonal projections in an AF-algebra A, then there exists a family $\{f_i; i = 1, 2, \cdots\}$ of projections in A which satisfies*

$$(*)\begin{cases} f_i \text{ is orthogonal to } f_j \quad \text{if } i \neq j, \\ f_i \sim e_i \quad \text{for each } i = 1, 2, \cdots \\ \text{and } f_i \text{ can be represented as a finite sum of elements } e_{i,i}^{(n,k)}. \end{cases}$$

PROOF: We shall construct inductively a subsequence $\{A_{n(l)}\}$ of $\{A_n\}$ and a family of projections satisfying $(*)$.

By Lemma 3.3(2), there exist $n(1) \in \mathbf{N}$ and a projection $h_1 \in A_{n(1)}$ such that $\| e_1 - h_1 \| < 1$ (in particular $e_1 \sim h_1$). Since $A_{n(1)}$ is finite dimensional, we can choose a projection $f_1 \in A_{n(1)}$, $f_1 \sim h_1$, which is represented as a finite sum of elements $e_{i,i}^{(n(1),k)}$.

We now make the following inductive hypotheses

$$
(**)\begin{cases}
(a) \quad n(1) < n(2) < \cdots < n(l), \\
\\
(b) \quad \{f_1, \cdots, f_l\} \quad \text{is a family of pairwise orthogonal projections} \\
\qquad \text{in } A_{n(l)} \text{ such that } f_j \text{ is represented as a finite sum of elements} \\
\qquad e_{i,i}^{(n,k)} \quad (j = 1, \cdots, l), \\
\\
(c) \quad \text{for } \{f_1, \cdots, f_l\} \quad \text{in (b), there exists a family } \{h_1, \cdots, h_l\} \\
\qquad \text{of pairwise orthogonal projections in } A_{n(l)} \quad \text{such that} \quad h_i \sim f_i \\
\qquad \text{in } A_{n(l)} \quad \text{and} \quad \| \sum_{i=1}^{l} h_i - \sum_{i=1}^{l} e_i \| < 1.
\end{cases}
$$

Suppose we have done up to the l-th step. Set $\| \sum_{i=1}^{l} h_i - \sum_{i=1}^{l} e_i \| = \varepsilon$, and let δ be a positive number such that $\varepsilon + \delta < 1$. Applying Lemma 3.4 to $\sum_{i=1}^{l} e_i$ and e_{l+1}, we can choose $n(l+1)$ $(> n(l))$ and projections g, h_{l+1} in $A_{n(l+1)}$ such that

$$
g \perp h_{l+1}, \; h_{l+1} \sim e_{l+1}, \; \| g - \sum_{i=1}^{l} e_i \| < \delta \text{ and } \| g + h_{l+1} - \sum_{i=1}^{l+1} e_i \| < \delta.
$$

Then $g, h_1, \cdots, h_l$ are contained in $A_{n(l+1)}$ and $\| g - \sum_{i=1}^{l} h_i \| < \varepsilon + \delta < 1$, hence we can choose ν in $A_{n(l+1)}$ such that $v^* v = g$ and $vv^* = \sum_{i=1}^{l} h_i$. Therefore $\{v^* h_1 v, \cdots, v^* h_l v, h_{l+1}\}$ is a family of pairwise orthogonal projections in $A_{n(l+1)}$. By the assumption, each $v^* h_i v$ $(i = 1, \cdots, l)$ and h_{l+1} is equivalent to e_i and h_{l+1}, respectively. Since $\{f_1, \cdots, f_l\}$ and $\{v^* h_1 v, \cdots, v^* h_l v, h_{l+1}\}$ are contained in $A_{n(l+1)}$ and $f_i \sim v^* h_i v$ $\quad (i = 1, \cdots, l)$, there exists a projection f_{l+1} such that $\{f_1, \cdots, f_{l+1}\}$ satisfies (*). Moreover

$$
\| \sum_{i=1}^{l} v^* h_i v + h_{l+1} - \sum_{i=1}^{l+1} e_i \| < \varepsilon + \delta < 1.
$$

Hence (**) is true in the case of $l + 1$. ∎

Now we can have the theorem mentioned in the introduction.

THEOREM 3.6. *Let B be a hereditary C^*-subalgebra of an AF-algebra A. Then there exists a family E of pairwise orthogonal projections contained in $\{e_{i,i}^{(n,k)}; n \in \mathbf{N}, 1 \le k \le s(n), 1 \le i \le [n,k]\}$ such that B is isomorphic to the hereditary C^*-subalgebra of A generated by E.*

PROOF: We may suppose that B is not unital. Since B is an AF-algebra, there exists a family of $\{e_1, e_2, \cdots\}$ of pairwise orthogonal projections in B such that $\{\sum_{i=1}^{l} e_i\}$ is an approximate unit for B. Hence $B = HC^*(\{e_1, e_2, \cdots\})$. By Lemma 3.5, there exists a family $\{f_1, f_2, \cdots\}$ of pairwise orthogonal projections such that $e_j \sim f_j$ and each f_j is represented as a finite sum of elements $e_{i,i}^{(n,k)}$. Set $E = \{e_{i,i}^{(n,k)}; e_{i,i}^{(n,k)}$ is a summand of some $f_j\}$. Then $HC^*(\{f_1, f_2, \cdots\}) = HC^*(E)$. By Proposition 3.2, B is isomorphic to $HC^*(E)$. ∎

Let B be a hereditary C*-subalgebra of A and I be the smallest closed two-sided ideal of A containing B. A diagram for B coincides with a subdiagram (D, d', U') of a diagram (D, d, U) for A. Set

$$d''(n,k) = \begin{cases} d(n,k) & \text{if } d'(n,k) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we get a subdiagram (D, d'', U') which satisfies (1) and (2), and this subdiagram corresponds to I. We consider the dimension group $D(B)$ $(resp. D(I))$ corresponding to the diagram (D, d', U') $(resp. (D, d'', U'))$, where $D(B)$ (resp. $D(I)$) is the Grothendieck group of the local semi-group $\mathrm{Proj}B/\sim$ (resp. $\mathrm{Proj}I/\sim$). Since a dimension group is determined by only the multiplicities (see [17],[18] and [40]), $D(B)$ coincides with $D(I)$. Therefore, by [17, Theorem 2.3] , we have the following corollary of this theorem.

COROLLARY 3.7. *B is stably isomorphic to I, that is, C^*-algebras $B \otimes K$ and $I \otimes K$ are isomorphic, where K is the C^*-algebra consisting of all the compact operators on a separable Hilbert space and a C^*-tensor product $\otimes$ means the spatial C^*-tensor product.*

This corollary can also be obtained by remarking that B is full in I, that is, B is not contained in any proper closed two-sided ideal of I ([4, Theorem 2.8]).

4. Increasing sequences of some special AF-algebras

Let $\{u_n; n = 0, 1, 2, \cdots\}$ be a sequence of unitary operators satisfying the following relations,

$$u_n^2 = 1,$$

$$u_n u_m = (-1)^{a(n-m)} u_m u_n$$

for each $n, m = 0, 1, 2, \cdots$, where $a(n) = 0$ or 1 for each $n = 0, \pm 1, \pm 2, \cdots$. We assume that $d = \sup\{|n|; a(n) = 1\}$ is finite and we consider that $a(n)$ belongs to the finite field F_2 with 2 elements $\{0, 1\}$. Let P_n be the C*-algebra generated by $\{u_i; 0 \le i \le n\}$ and Q_n be the center of P_n, that is,

$$Q_n = \{x \in P_n; xy = yx \quad \text{for any } y \in P_n\}.$$

Then $\{P_n\}$ is an increasing sequence of finite-dimensional C*-algebras. We will determine the structure of the algebra P_n and the diagram for the AF-algebra related to $\{P_n\}$. To do this, we introduce an $(n+1) \times (n+1)$ matrix $A(n)$ whose entries belong to F_2,

$$A(n) = \begin{pmatrix} a(0) & a(1) & \ldots & a(n) \\ a(-1) & a(0) & \ldots & a(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ a(-n) & a(-n+1) & \ldots & a(0) \end{pmatrix},$$

i.e., the (i, j)-th entry of $A(n)$ is $a(j - i)$. We remark that $a(i - j) = a(j - i)$, since

$$u_i u_j = (-1)^{a(i-j)} u_j u_i = (-1)^{a(i-j)+a(j-i)} u_i u_j.$$

LEMMA 4.1. $dim Q_n = 2^{dim(Ker A(n))}$.

PROOF: By the definition, Q_n is generated by the elements of the form

$$u_0^{x(0)} u_1^{x(1)} \cdots u_n^{x(n)} \quad \text{in } Q_n,$$

and these elements are linearly independent. Then we have following equivalent statements:

$$\begin{array}{ll} & u_0^{x(0)} u_1^{x(1)} \cdots u_n^{x(n)} \quad \text{belongs to } Q_n; \\ \text{if and only if} & u_0^{x(0)} u_1^{x(1)} \cdots u_n^{x(n)} \quad \text{commutes with } u_i \quad (0 \leq i \leq n); \\ \text{if and only if} & \displaystyle\sum_{k=0}^{n} a(i-k)x(k) = 0 \quad \text{for } 0 \leq i \leq n; \\ \text{if and only if} & A(n) \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n) \end{pmatrix} = 0; \end{array}$$

where we may regard that $x(i)$ belongs to F_2. Therefore the number of elements $u_0^{x(0)} u_1^{x(1)} \cdots u_n^{x(n)}$ in Q_n equals to the number of solutions X of the equation $A(n)X = 0$. This fact implies that the dimension of the algebra Q_n equals to $2^{dim(Ker(A(n))}$. ∎

PROPOSITION 4.2. *The C^*-algebra P_n is isomorphic to $M_{2^k} \otimes \mathbf{C}^{2^l}$ for some non-negative integers k, l.*

PROOF: We shall prove this by the induction on n. The algebra P_0 is generated by u_0. So

$$P_0 = \mathbf{C}\frac{1+u_0}{2} + \mathbf{C}\frac{1-u_0}{2} \cong \mathbf{C}^2.$$

We assume

$$P_n \cong M_{2^k} \otimes \mathbf{C}^{2^l} \quad \text{for some } k, l \geq 0.$$

Let α_{n+1} be an automorphism of P_n such that $\alpha_{n+1} = Ad\ u_{n+1}$. Then

$$P_{n+1} \cong P_n \times_{\alpha_{n+1}} Z_2,$$

and α_{n+1} becomes an automorphism of Q_n, where $P_n \times_{\alpha_{n+1}} Z_2$ is the C*-crossed product of P_n by the group Z_2 of order 2. Therefor we can label minimal projections $e_1, e_2, \cdots,$ $e_{2s-1}, e_{2s}, e_{2s+1}, \cdots, e_{2^l}$ in Q_n so that

$$\begin{cases} \alpha_{n+1}(e_{2i}) = e_{2i-1}, & 1 \le i \le s, \\ \alpha_{n+1}(e_i) = e_i, & 2s < i \le 2^l. \end{cases}$$

Then

$$\begin{aligned} P_n \times_{\alpha_{n+1}} Z_2 &\cong (\bigoplus_{i=1}^{s}(P_n(e_{2i-1}+e_{2i}) \times_{\alpha_{n+1}} Z_2)) \oplus (\bigoplus_{i=2s+1}^{2^l} (P_n e_i \times_{\alpha_{n+1}} Z_2)) \\ &\cong (\bigoplus_{i=1}^{s}((M_{2^k} \oplus M_{2^k}) \times_\alpha Z_2)) \oplus (\bigoplus_{i=2s+1}^{2^l} (M_{2^k} \times_\alpha Z_2)), \end{aligned}$$

where the action α of Z_2 is outer in the first direct summand and the action α of Z_2 is inner in the second direct summand. So

$$P_n \times_{\alpha_{n+1}} Z_2 \cong (M_{2^{k+1}} \otimes \mathbf{C}^s) \oplus (M_{2^k} \otimes \mathbf{C}^{2(2^l-2s)}).$$

Since Q_{n+1} is the center of $P_n \times_{\alpha_{n+1}} Z_2$, the dimension of Q_{n+1} equals to $s + 2(2^l - 2s)$. By Lemma 4.1, $dim Q_{n+1} = 2^m$ for some m. Therefore

$$s + 2(2^l - 2s) = 2^m.$$

This implies $s = 0$ or $s = 2^l$, that is,

$$P_{n+1} \cong M_{2^k} \otimes \mathbf{C}^{2^{l+1}} \quad \text{or} \quad P_{n+1} \cong M_{2^{k+1}} \otimes \mathbf{C}^{2^l}.$$

∎

LEMMA 4.3. *Let t be an element in an extended field of F_2 satisfying the identity*

$$t^n + x(n-1)t^{n-1} + \cdots + x(1)t + 1 = 0,$$

where $x(1), \cdots, x(n-1)$ are element of F_2. Then there exists a positive integer m such that $t^m + 1 = 0$.

PROOF: We put

$$p_0(t) = t^n + x(n-1)t^{n-1} + \cdots + x(1)t + 1.$$

If one of $x(i)$'s is 1, then we put $i_1 = \min\{i; x(i) = 1\}$ and

$$p_1(t) = t^{i_1}p_0(t) + p_0(t) = t^{n+i_1} + x_1(n-1)t^{n+i_1-1} + \cdots + x_1(1)t^{i_1+1} + 1.$$

If one of $x_1(i)$'s is 1, then we repeat the same procedure, that is, we put $i_2 = \min\{i; x_1(i)\}$ and

$$p_2(t) = t^{i_1+i_2}p_0(t) + p_1(t) = t^{n+i_1+i_2} + x_2(n-1)t^{n+i_1+i_2-1} + \cdots + x_2(1)t^{i_1+i_2+1} + 1.$$

If this procedure terminates after m times repetitions, then

$$\begin{aligned} p_m(t) &= (t^{i_1+i_2+\cdots+i_m} + \cdots + t^{i_1+i_2} + t^{i_1})p_0(t) \\ &= t^{n+i_1+\cdots+i_m} + 1 = 0. \end{aligned}$$

If we can repeat this procedure infinitely, then we can find two positive integers k and l $(k < l)$ such that

$$x_k(i) = x_l(i) \qquad (i = 1, \cdots, n-1).$$

Then we get $p_l(t) + p_k(t)t^{l-k} = t^{l-k} + 1$. Therefore $t^{l-k} + 1 = 0$. ∎

By the method of the above proof, we can find a positive integer m such that $t^m + 1 = 0$, and we set $D = \min\{m; t^m + 1 = 0\}$. We call the number D the degree of the sequence $\{1, x(1), \cdots, x(n-1), 1\}$.

For the polynomial $q_0(t) = t^n + x(n-1)t^{n-1} + \cdots + x(1) + 1$, we define a sequence $\{q_n(t)\}$ of polynomials as follows,

$$q_m(t) = q_{m-1}(t) + a_{m-1}(m)t^m q_0(t) \quad \text{for any} \;\; m = 1, 2, \cdots,$$

where the polynomial $q_m(t)$ is of order less than $m+n$ with coefficients in F_2 and we put

$$q_m(t) = a_m(m+n)t^{m+n} + a_{m-1}(m+n-1)t^{m+n-1} + \cdots + a_m(1)t + 1.$$

From the proof of Lemma 4.3, we see that

$$q_{D-n}(t) = t^D + 1 = 0 \quad \text{if} \quad q_0(t) = 0.$$

Then we can get the following lemma using this method.

LEMMA 4.4. *Let $\{y(i); i = 0, \cdots, D\}$ be a sequence in F_2, and satisfy the following equations,*

$$y(j) + x(1)y(j+1) + \cdots + x(n-1)y(j+n-1) + y(j+n) = 0 \quad (j = 0, \cdots, D-n),$$

where D is the degree of $\{1, x(1), \cdots, x(n-1), 1\}$. Then $y(0) = y(D)$.

We denote by Q_n the center of the algebra P_n. For the determination of the structure of Q_n, we prepare some notations. For any vector $X = {}^t(x(0), \cdots, x(n))$ in $(F_2)^{n+1}$ and a positive integer k,

$$\begin{aligned} u(X) &= u(x(0), \cdots, x(n)) = u_0^{x(0)} \cdots u_n^{x(n)}, \\ v(X) &= v(x(0), \cdots, x(n)) = \begin{cases} u(X) & \text{if} \;\; u(X)^2 = 1, \\ \sqrt{-1}u(X) & \text{otherwise}, \end{cases} \\ v_k(X) &= v(0, \cdots, 0, x(0), \cdots, x(n)) \quad \text{in} \quad P_{n+k}, \end{aligned}$$

and

$$v_{-k}(X) = \begin{cases} v(x(k), x(k+1), \cdots, x(n)) & \text{in} \quad P_{n-k} \quad \text{if} \quad n \geq k, \\ 1 & \text{if} \quad n < k. \end{cases}$$

Then $v(X)$ is a self-adjoint unitary operator. We have seen in the above that the algebra Q_n is the linear span of $\{v(X); A(n)X = 0\}$.

Proposition 4.5. *Let D be the degree of the sequence*

$$\{a(d), a(d-1), \cdots, a(1), a(0), a(1), \cdots, a(d-1), a(d)\}$$

and k $(0 \leq k < D)$ and l be non-negative integers. Then Q_{k+lD} is isomorphic to Q_k. Moreover the isomorphism $\beta_{k+lD,k}$ of Q_{k+lD} to Q_k is given by the following relation,

$$\beta_{k+lD,k}(v(X)) = v_{-lD}(X),$$

where $v(X)$ belongs to Q_{k+lD} .

Proof: At first, we consider the case $d \leq k \leq D$. Let $X = {}^t(x(0), x(1), \cdots, x(k))$ and $Y = {}^t(y(0), y(1), \cdots, y(k+lD))$ be vectors in $(F_2)^{k+1}$ and $(F_2)^{k+lD+1}$ respectively. Then $v(X)$ belongs to Q_k (i.e., $A(k)X = 0$) if and only if $x(0), x(1), \cdots, x(k)$ satisfy the following relations $(R; j)$, for any j $(-d \leq j \leq k-d)$,

$$(R;j) \qquad \sum_{s=-d}^{d} a(|s|)x(j+d+s) = 0,$$

where we put

$$\begin{aligned} x(-d) &= x(-d+1) = \cdots = x(-1) &= 0, \\ x(k+1) &= x(k+2) = \cdots = x(k+d) &= 0. \end{aligned}$$

Relations $(R;-d),\cdots,(R;k-2d)$ mean that variables $x(d),x(d+1),\cdots,x(k)$ can be represented as linear combinations of variables $x(0),x(1),\cdots,x(d-1)$, that is, there exist linear functions $\{f_j\,;d\le j\le k\}$ such that

$$x(j)=f_j(x(0),x(1),\cdots,x(d-1)) \qquad \text{for any } j \quad (d\le j\le k).$$

Therefore we can regard that relations $(R;k-2d+1),\cdots,(R;k-d)$ are binding conditions for variables $x(0),x(1),\cdots,x(d-1)$.

In a similar way, $v(Y)$ belongs to Q_{k+lD} (i.e., $A(k+lD)Y=0$) if and only if $y(0),y(1),\cdots,y(k+lD)$ satisfy the following same relation $(R;j)$, for any j $(-d\le j\le k+lD-d)$,

$$(R;j) \qquad \sum_{s=-d}^{d} a(|s|)y(j+d+s)=0,$$

where we put

$$\begin{aligned} y(-d) &= y(-d+1) &&= \cdots = y(-1) &&= 0,\\ y(k+lD+1) &= y(k+lD+2) &&= \cdots = y(k+lD+d) &&= 0. \end{aligned}$$

Relations $(R;-d),\cdots,(R;k-2d+lD)$ mean that variables $y(d),y(d+1),\cdots,y(k+lD)$ can be represented as linear combinations of variables $y(0),y(1),\cdots,y(d-1)$, that is, there exist linear functions $\{g_j\,;d\le j\le k+lD\}$ such that

$$y(j)=g_j(y(0),y(1),\cdots,y(d-1)) \quad \text{for any } j \ (d\le j\le k+lD),$$

where g_j and f_j are the same functions for any j $(d\le j\le k)$. We regard therefore that relations $(R;k+lD-2d+1),\cdots,(R;k+lD-d)$ are binding conditions for variables $y(0),y(1),\cdots,y(d-1)$. By Lemma 4.3, we have

$$y(j)=y(j+D) \qquad \text{for any } -d\le j\le k+(l-1)D+d.$$

Then the conditions $(R; k + lD - 2d + 1), \cdots, (R; k + lD - d)$ for variables $y(0)$, $y(1), \cdots, y(d-1)$ are identical to the conditions $(R; k-2d+1), \cdots, (R; k-d)$ for variables $x(0), x(1), \cdots, x(d-1)$.

For a vector $X = {}^t(x(0), x(1), \cdots, x(k))$, we define vectors

$$\begin{cases} \widetilde{X} = {}^t(x(0), x(1), \cdots, x(D-1)), \\ \widehat{X} = {}^t(x(0), x(1), \cdots, x(k+lD)), \end{cases}$$

by the following,

$$\begin{cases} x(j) = g_j(x(0), \cdots, x(d-1)) & \text{for any } k < j < D, \\ x(j) = x(j+D) & \text{for any } 0 \leq j \leq k+(l-1)D. \end{cases}$$

By the above observation, the correspondence of X and $\widehat{X}$ induces a bijection from the set $\{X; A(k)X = 0\}$ to the set $\{Y; A(k+lD)Y = 0\}$. So we can construct isomorphisms $\alpha_{k,k+lD}$ from Q_k to Q_{k+lD} and $\beta_{k+lD,k}$ from Q_{k+lD} to Q_k by the following,

$$\alpha_{k,k+lD}(v(X)) = v(\widetilde{X})v_D(\widetilde{X})\cdots v_{(l-1)D}(\widetilde{X})v_{lD}(X) = v(\widehat{X}),$$

$$\beta_{k+lD,k}(v(Y)) = v_{-lD}(Y),$$

where $v(X)$, $v(Y)$ belong to Q_k , Q_{k+lD} respectively.

In the case $0 \leq k < d$, we treat relations $(R; -d), \cdots, (R; k-d)$ as binding conditions for variables $x(0), \cdots, x(d-1)$. By the same argument as above, we can construct the isomorphism β_{k+lD} from Q_{k+lD} to Q_k. ∎

COROLLARY 4.6. *We use the same notations as in Proposition 4.5. Then,*

(1) the sequence $\{dim\ Q_n; n = 0, 1, \cdots\}$ has a period D,

(2) $dim\ Q_n = 2^d$ if $n = lD - d - 1$,

(3) $\max\{dim Q_n; n \in \mathbf{N}\} = 2^d$.

PROOF: (1) This follows from Proposition 4.5 immediately.

(2) We set $x(-d) = x(-d+1) = \cdots = x(-1) = 0$. For any elements $x(0), x(1), \cdots,$ $x(d-1)$ in F_2, we define $x(d), x(d+1), \cdots, x(lD-1)$ by the following relation,

$$(R;j) \qquad \sum_{s=-d}^{d} a(|s|)x(j+d+s) = 0,$$

for any j $(-d \leq j \leq lD - 2d - 1)$. Then we have

$$x(-d) = x(lD-d) = 0, \quad \cdots, \quad x(-1) = x(lD-1) = 0.$$

The vector $X = {}^t(x(0), x(1), \cdots, x(lD-d-1))$ becomes a solution of the equation $A(lD-d-1)X = 0$. By the proof of Proposition 4.5,

$$dim\ Q_n \leq 2^d.$$

Therefore dim $Q_{lD-d-1} = 2^d$.

(3) This follows from the proof of (2). ∎

By this corollary, if we can calculate dim Q_n for $n = 0, 1, \cdots, D-1$, we can get dim Q_n for any non-negative integer n. The dimension of Q_n is equal to the number of all solutions of the equation $A(n)X = 0$. If we can get the corank of the matrix $A(n)$, then we can get the dimension of Q_n. We can calculate the corank of the $(n+1) \times (n+1)$ matrix $A(n)$, using the corank of a $d \times d$ submatrix of a matrix, a deformation of $A(n)$. In fact, we set a $d \times 2d$ matrix A and a $2d \times 2d$ invertible matrix P as follows,

$$A = \begin{pmatrix} a(0) & a(1) & \ldots & a(d-1) & a(d) & & & \mathbf{0} \\ a(1) & \ddots & \ddots & a(d-2) & a(d-1) & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ a(d-1) & \ldots & \ldots & a(0) & a(1) & \ldots & \ldots & a(d) \end{pmatrix},$$

$$P = \begin{pmatrix} a(d-1) & a(d-2) & \dots & a(0) & a(1) & \dots & a(d-1) & a(d) \\ a(d) & & & & & & & \mathbf{0} \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ \mathbf{0} & & & & & & a(d) & \end{pmatrix},$$

i.e., the (i,j)-th entry of A is $a(|i-j|)$, and the (i,j)-th entry $P(i,j)$ of P is given by the following relation,

$$P(i,j) = \begin{cases} a(|d-j|) & \text{if } i = 1, \\ a(d) & \text{if } i - j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define a $d \times d$ matrix $B(n)$ consisting of entries of AP^{-d+n} from the $(1,1)$-th entry to the (d,d)-th entry, that is, the (i,j)-th entry of $B(n)$ is the (i,j)-th entry of AP^{-d+n} $(1 \le i,j \le n)$. Then we can see that the corank of $B(n)$ is equal to the corank of $A(n)$.

PROPOSITION 4.7. *Corank $A(n)$ = Corank $B(n)$.*

PROOF: We define infinite dimensional matrices $X_d, I_{k,l}(c)$ and a $d \times 2d$ matrix Y_d as follows,

$$(X_d)_{i,j} = \begin{cases} a(i-j) & \text{for } |i-j| \le d, \\ 0 & \text{otherwise,} \end{cases}$$

$$
(I_{k,l}(c))_{i,j} = \begin{cases} c & \text{if } (i,j) = (k,l), \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}
$$

and

$$
(Y_d)_{i,j} = (X_d)_{i,j} \quad \text{for } 1 \le i \le d, 1 \le j \le 2d,
$$

where we denote $(\ )_{i,j}$ the (i,j)-th entry of a given matrix. Inductively we construct sequences of infinite dimensional matrices $\{X_m\}_{m \geq d}$ and $d \times 2d$ matrices $\{Y_m\}_{m \geq d}$ with the following relations.

$$
\begin{aligned}
X_{m+1} = & I_{1,m+1}((X_m)_{1,m+1-d}) \\
& \times I_{2,m+1}((X_m)_{2,m+1-d}) \\
& \cdots \\
& \times I_{d,m+1}((X_m)_{d,m+1-d}) \\
& \times X_m
\end{aligned}
$$

(basic deformations of X_m with respect to its row vectors),

$$
(Y_{m+1})_{i,j} = (X_{m+1})_{i,j+m-d} \quad \text{for all } m > d.
$$

Then the matrix X_m is of the following form,

$$
\left(\begin{array}{cccc|cccc|ccc}
 & \mathbf{0} & & & & Y_m & & & & \mathbf{0} & \\
\hline
1 & \cdots & & & \cdots & 1 & & & & & \\
 & 1 & \cdots & & & \cdots & 1 & & & \mathbf{0} & \\
\mathbf{0} & & \ddots & & & & & \ddots & & & \\
 & & & 1 & \cdots & & \cdots & 1 & & & \\
\hline
 & & & & 1 & \cdots & & \cdots & 1 & & \mathbf{0} \\
 & \mathbf{0} & & & & \ddots & & & & \ddots & \\
 & & & & \mathbf{0} & & \ddots & & & & \ddots
\end{array}\right),
$$

where the zero matrix of the (1,1)-th block is a $d \times (m-d)$ matrix.

Therefore we get the next relation for entries of matrices,

$$(Y_{m+1})_{i,j} = (Y_m)_{i,j+1} + (Y_m)_{i,j}\,a(d-j) \quad \text{if } 1 \le i \le d,\ 1 \le j < 2n,$$

$$(Y_{m+1})_{i,2d} = (Y_m)_{i,1} \qquad \text{if } 1 \le i \le d.$$

These relations imply

$$Y_{m+1} = Y_m P \quad \text{for } m > d,$$

where P is a $2d \times 2d$ matrix defined as above.

Since P is invertible and $Y_d = A$, we can define matrices Y_m for $1 \le m \le d$ by $Y_m = AP^{-d+m}$.

We have already defined $d \times d$ matrices $B(m)$ by

$$(B(m))_{i,j} = (Y_m)_{i,j} \quad \text{for } 1 \le i,j \le d.$$

In the case of $m \ge d$,

$$\begin{aligned} Corank\ A(m) &= Corank\ [(X_m)_{i,j};\, 1 \le i,j \le m] \\ &= Corank\ B(m), \end{aligned}$$

since the matrix $[(X_m)_{i,j}]$ has the following form

$$\left(\begin{array}{ccc|c} & 0 & & B(m) \\ \hline 1 & & * & \\ 0 & \ddots & & * \\ & & 1 & \end{array}\right).$$

In the case of $m < d$, $B(m)$ has the following form,

$$\left(\begin{array}{cccc|c} & 0 & & & A(m) \\ \hline a(d) & a(d-1) & \cdots & a(m+1) & \\ & \ddots & \ddots & \vdots & \\ 0 & & \ddots & a(d-1) & 0 \\ & & & a(d) & \end{array}\right).$$

Therefore $Corank\ A(m) = Corank\ B(m)$. ∎

By the proof of the above proposition, the corank of $A(n)$ coincides with the corank of the following matrix,

$$\begin{pmatrix} a(d) & a(d-1) & \cdots & a(1) \\ & \ddots & \ddots & \vdots \\ & & \ddots & a(d-1) \\ 0 & & & a(d) \end{pmatrix}$$

if $n = lD - 1$ for any $l \in \mathbf{N}$. So $Q_{lD-1} = \mathbf{C}\cdot 1$. Therefore the AF-algebra $\overline{\bigcup_n P_n}$ is the UHF-algebra of type 2^∞ for any finitely supported sequence $\{a(n)\}$.

For an increasing sequence $\{i_1, \cdots, i_k\}$ of non-negative integers, we define a vector $X = {}^t\,(x(0), \cdots, x(i_k))$ in $(F_2)^{k+1}$ by

$$\begin{cases} x(j) = 1 & \text{if } j = i_l \quad \text{for some } 1 \le l \le k, \\ x(j) = 0 & \text{otherwise,} \end{cases}$$

and we denote $v(X)$ by $(i_1, \cdots, i_k)$ or (X). We decompose (X) into the difference of two projections, and we write,

$$(X) = (X)^+ - (X)^- \quad (\text{or } (i_1, \cdots, i_k) = (i_1, \cdots, i_k)^+ - (i_1, \cdots, i_k)^-),$$

that is,

$$(X)^+ = \frac{1+(X)}{2}, \quad (X)^- = \frac{1-(X)}{2}.$$

We remark that the center Q_n of P_n is a linear span of $\{v(X);\ A(n)X = 0\}$, and $\{v(X); A(n)X = 0\}$ is a group. If $\{v(X_1), \cdots, v(X_K)\}$ ($K = dim(KerA(n))$) is a generator of $\{v(X);\ A(n)X = 0\}$, then Q_n is the linear span of

$$\{(X_1)^{\delta(1)}(X_2)^{\delta(2)} \cdots (X_K)^{\delta(K)}; \delta(1), \delta(2), \cdots, \delta(K) = + \text{ or } -\},$$

and $(X_1)^{\delta(1)}(X_2)^{\delta(2)} \cdots (X_K)^{\delta(K)}$ is a minimal projection of Q_n.

THEOREM 4.8. *Let D be the degree of the sequence*

$$\{a(d), a(d-1), \cdots, a(1), a(0), a(1), \cdots, a(d-1), a(d)\}.$$

Then the inclusion matrix from P_n to P_{n+1} is equal to the inclusion matrix from P_{n+lD} to P_{n+1+lD} for any non-negative integers l and n.

PROOF: By Propositin 4.5, there exists an isomorphism $\beta_{n+lD,n}$ from Q_{n+lD} to Q_n. Then a minimal projection $(X_1)^{\delta(1)} \cdots (X_K)^{\delta(K)}$ in Q_{n+lD} is mapped to a minimal projection $(v_{-lD}(X_1))^{\delta(1)} \cdots (v_{-lD}(X_K))^{\delta(K)}$ in Q_n by this isomorphism $\beta_{n+lD,n}$, where K is the corank of $A(n)$.

The inclusion matrix from P_n to P_{n+1} is determined by the orthogonality of a minimal projection in Q_n and a minimal projection in Q_{n+1}. By Proposition 2, the algebra Q_n contains the algebra Q_{n+1} or is contained in the algebra Q_{n+1}. If $Q_n \supset Q_{n+1}$ (resp. $Q_n \subset Q_{n+1}$), then

$$\beta_{n+lD,n}|_{Q_{n+1+lD}} = \beta_{n+1+lD,n+1} \quad (\textit{resp.}\ \beta_{n+1+lD,n+1}|_{Q_{n+lD}} = \beta_{n+lD,n}).$$

Therefore the inclusion matrix from Q_n to Q_{n+1} is equal to the inclusion matrix from Q_{n+lD} to Q_{n+1+lD} with respect to the correpondence of a minimal projection in Q_n (resp. Q_{n+1}) and Q_{n+lD} (resp. Q_{n+1+lD}) by the isomorphism $\beta_{n+lD,n}$ (resp. $\beta_{n+1+lD,n+1}$).

∎

By this theorem, we can determine the Bratteli diagram of a sequence $\{Q_n\}$ of algebras for any given finitely supported sequence $\{a(n)\}$. In the next section we will try to compute that for some sequences.

5. Examples and Applications

By the argument in section 4, we can calculate the sequence $\{corankA(n); n = 1, 2, \cdots\}$, and we can get the sequence $\{\dim Q_n; n = 1, 2, \cdots\}$, according to the following relation,

$$dim\ Q_n = 2^{corank\ A(n)}.$$

Let $d = \sup\{|n|; a(n) = 1\}$. We calculate the period of $\{\text{corank } A(n)\}$ in the case of $d = 1, 2, 3, 4, 5, 6,$ as follows.

	$a(0)$	$a(1)$	$a(2)$	$a(3)$	$a(4)$	$a(5)$	$a(6)$	*period*
$d = 1$	0	1						2
$d = 2$	0	0	1					4
	0	1	1					6
$d = 3$	0	0	0	1				6
	0	0	1	1				10
	0	1	0	1				8
	0	1	1	1				12
$d = 4$	0	0	0	0	1			8
	0	0	0	1	1			14
	0	0	1	0	1			12
	0	0	1	1	1			12
	0	1	0	0	1			30
	0	1	0	1	1			24
	0	1	1	0	1			18
	0	1	1	1	1			20

	$a(0)$	$a(1)$	$a(2)$	$a(3)$	$a(4)$	$a(5)$	$a(6)$	*period*
$d = 5$	0	0	0	0	0	1		10
	0	0	0	0	1	1		18
	0	0	0	1	0	1		16
	0	0	0	1	1	1		24
	0	0	1	0	0	1		42
	0	0	1	0	1	1		34
	0	0	1	1	0	1		60
	0	0	1	1	1	1		28
	0	1	0	0	0	1		24
	0	1	0	0	1	1		40
	0	1	0	1	0	1		12
	0	1	0	1	1	1		30
	0	1	1	0	0	1		36
	0	1	1	0	1	1		12
	0	1	1	1	0	1		34
	0	1	1	1	1	1		30
$d = 6$	0	0	0	0	0	0	1	12
	0	0	0	0	0	1	1	22
	0	0	0	0	1	0	1	20
	0	0	0	0	1	1	1	30
	0	0	0	1	0	0	1	18
	0	0	0	1	0	1	1	60
	0	0	0	1	1	0	1	66
	0	0	0	1	1	1	1	36

	$a(0)$	$a(1)$	$a(2)$	$a(3)$	$a(4)$	$a(5)$	$a(6)$	*period*
$d=6$	0	0	1	0	0	0	1	16
	0	0	1	0	0	1	1	102
	0	0	1	0	1	0	1	24
	0	0	1	0	1	1	1	62
	0	0	1	1	0	0	1	62
	0	0	1	1	0	1	1	48
	0	0	1	1	1	0	1	60
	0	0	1	1	1	1	1	40
	0	1	0	0	0	0	1	70
	0	1	0	0	0	1	1	60
	0	1	0	0	1	0	1	30
	0	1	0	0	1	1	1	68
	0	1	0	1	0	0	1	68
	0	1	0	1	0	1	1	24
	0	1	0	1	1	0	1	84
	0	1	0	1	1	1	1	66
	0	1	1	0	0	0	1	102
	0	1	1	0	0	1	1	56
	0	1	1	0	1	0	1	62
	0	1	1	0	1	1	1	24
	0	1	1	1	0	0	1	120
	0	1	1	1	0	1	1	90
	0	1	1	1	1	0	1	72
	0	1	1	1	1	1	1	42

We get some examples in which, though sequences a_1 and a_2 are different, the algebra P_n for a_1 is isomorphic to the algebra P_n for a_2, for any non-negative integer n.

Example 1. (d=4)

$$\begin{cases} a_1 = (0,0,1,0,1,0,\dots), \\ a_2 = (0,0,1,1,1,0,\dots). \end{cases}$$

The period is 12. The dimension of Q_n is as follows,

$$\begin{array}{llll} dim(Q_0) = 2, & dim(Q_1) = 4, & dim(Q_2) = 2, & dim(Q_3) = 1, \\ dim(Q_4) = 2, & dim(Q_5) = 4, & dim(Q_6) = 8, & dim(Q_7) = 16, \\ dim(Q_8) = 8, & dim(Q_9) = 4, & dim(Q_{10}) = 2, & dim(Q_{11}) = 1. \end{array}$$

Example 2. (d=5)

$$\begin{cases} a_1 = (0,1,0,1,0,1,0,\dots), \\ a_2 = (0,1,1,0,1,1,0,\dots). \end{cases}$$

The period is 12. The dimension of Q_n is as follows,

$$\begin{array}{llll} dim(Q_0) = 2, & dim(Q_1) = 1, & dim(Q_2) = 2, & dim(Q_3) = 4, \\ dim(Q_4) = 8, & dim(Q_5) = 16, & dim(Q_6) = 32, & dim(Q_7) = 16, \\ dim(Q_8) = 8, & dim(Q_9) = 4, & dim(Q_{10}) = 2, & dim(Q_{11}) = 1. \end{array}$$

Example 3. (d=6)

$$\begin{cases} a_1 = (0,1,0,1,0,1,1,0,\dots), \\ a_2 = (0,1,1,0,1,1,1,0,\dots). \end{cases}$$

The period is 24. The dimension of Q_n is as follows,

$$\begin{array}{llll} dim(Q_0) = 2, & dim(Q_1) = 1, & dim(Q_2) = 2, & dim(Q_3) = 4, \\ dim(Q_4) = 8, & dim(Q_5) = 16, & dim(Q_6) = 8, & dim(Q_7) = 4, \\ dim(Q_8) = 2, & dim(Q_9) = 1, & dim(Q_{10}) = 2, & dim(Q_{11}) = 1, \\ dim(Q_{12}) = 2, & dim(Q_{13}) = 4, & dim(Q_{14}) = 8, & dim(Q_{15}) = 16, \\ dim(Q_{16}) = 32, & dim(Q_{17}) = 64, & dim(Q_{18}) = 32, & dim(Q_{19}) = 16, \\ dim(Q_{20}) = 8, & dim(Q_{21}) = 4, & dim(Q_{22}) = 2, & dim(Q_{23}) = 1. \end{array}$$

We can also determine the Bratteli diagram of algebras $\{Q_n\}$ for any given finitely supported sequence. We show the Bratteli diagrams of sequences $\{P_n; n = 0, 1, \cdots\}$ for sequences $\{0,0,1,0,1,0,\cdots\}$ and $\{0,0,1,1,1,0,\cdots\}$. We remark that their period is 12.

1. The Bratteli diagram for $\{0,0,1,0,1,0,\cdots\}$.

Each of algebras $Q_0, Q_1, \cdots, Q_{11}$ has minimal projections, which are expressed by the following form.

$$
\begin{array}{llll}
Q_0 & (0)^{\pm}, & Q_1 & (0)^{\pm}(1)^{\pm}, \\
Q_2 & (1)^{\pm}, & Q_3 & 1, \\
Q_4 & (024)^{\pm}, & Q_5 & (024)^{\pm}(135)^{\pm}, \\
Q_6 & (06)^{\pm}(135)^{\pm}(246)^{\pm}, & Q_7 & (06)^{\pm}(17)^{\pm}(246)^{\pm}(357)^{\pm}, \\
Q_8 & (17)^{\pm}(246)^{\pm}(357)^{\pm}, & Q_9 & (246)^{\pm}(357)^{\pm}, \\
Q_{10} & (357)^{\pm}, & Q_{11} & 1.
\end{array}
$$

We can see the orthogonality of the above minimal projections by the following calculation,

$$
\begin{aligned}
(024)^+ &= \frac{1}{2}\{1 + (024)\} \\
&= \frac{1}{2}\{1 - (06) + (06) + (06)(246)\} \\
&= (06)^- + (06)(246)^+ \\
&= (06)^+(246)^+ + (06)^-(246)^-.
\end{aligned}
$$

By these calculations, we get the following formulae

$$
\begin{cases}
(024)^+ &= (06)^+(246)^+ + (06)^-(246)^-, \\
(024)^- &= (06)^+(246)^- + (06)^-(246)^+, \\
(135)^+ &= (17)^+(357)^+ + (17)^-(357)^-, \\
(135)^- &= (17)^+(357)^- + (17)^-(357)^+.
\end{cases}
$$

Therefore we get the relation of the orthogonality of minimal projections as figure 1 and figure 2, and the Bratteli diagram as figure 4.

2. The Bratteli diagram for $\{0,0,1,1,1,0,\cdots\}$.

Each of algebras $Q_0, Q_1, \cdots, Q_{11}$ has minimal projections, which are expressed by the following form.

Q_0	$(0)^{\pm}$,	Q_1	$(0)^{\pm}(1)^{\pm}$,
Q_2	$(1)^{\pm}$,	Q_3	1,
Q_4	$(0134)^{\pm}$,	Q_5	$(0134)^{\pm}(1245)^{\pm}$,
Q_6	$(06)^{\pm}(1245)^{\pm}(2356)^{\pm}$,	Q_7	$(06)^{\pm}(17)^{\pm}(2457)^{\pm}(3467)^{\pm}$,
Q_8	$(17)^{\pm}(2457)^{\pm}(3467)^{\pm}$,	Q_9	$(2457)^{\pm}(3467)^{\pm}$,
Q_{10}	$(3467)^{\pm}$,	Q_{11}	1.

We can see the orthogonality of the above minimal projections by the following calculation,

$$\begin{aligned}
(0134)^{+} &= \frac{1}{2}\{1+(0134)\} \\
&= \frac{1}{2}\{1-(06)+(06)-(06)(1245)+(06)(1245)-(06)(1245)(2356)\} \\
&= (06)^{-}+(06)(1245)^{-}+(06)(1245)(2356)^{-} \\
&= (06)^{+}(1245)^{+}(2356)^{-}+(06)^{+}(1245)^{-}(2356)^{+} \\
&\quad +(06)^{-}(1245)^{+}(2356)^{+}+(06)^{-}(1245)^{-}(2356)^{-}
\end{aligned}$$

By these calculations, we get the following formulae

$$\begin{cases} (0134)^+ &= (06)^+(1245)^+(2356)^- + (06)^+(1245)^-(2356)^+ \\ & \quad +(06)^-(1245)^+(2356)^+ + (06)^-(1245)^-(2356)^- \\ (0134)^- &= (06)^+(1245)^+(2356)^+ + (06)^+(1245)^-(2356)^- \\ & \quad +(06)^-(1245)^+(2356)^- + (06)^-(1245)^-(2356)^+ \\ (1245)^+ &= (17)^+(2457)^+ + (17)^-(2457)^- \\ (1245)^- &= (17)^+(2457)^- + (17)^-(2457)^+ \\ (2356)^+ &= (2457)^+(3467)^- + (2457)^-(3467)^+ \\ (2356)^- &= (2457)^+(3467)^+ + (2457)^-(3467)^-. \end{cases}$$

Therefore we get the relation of the orthogonality of minimal projections as figure 1 and figure 3, and the Bratteli diagram as figure 4.

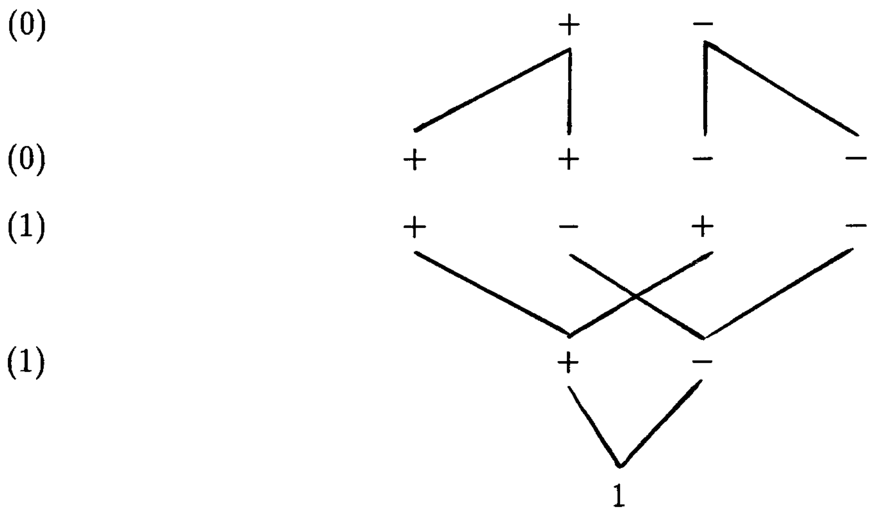


Figure 1. The orthogonality of minimal projections in Q_0, Q_1, Q_2, Q_3 for $\{0, 0, 1, 0, 1, 0, \cdots\}$ and $\{0, 0, 1, 1, 1, 0, \cdots\}$.

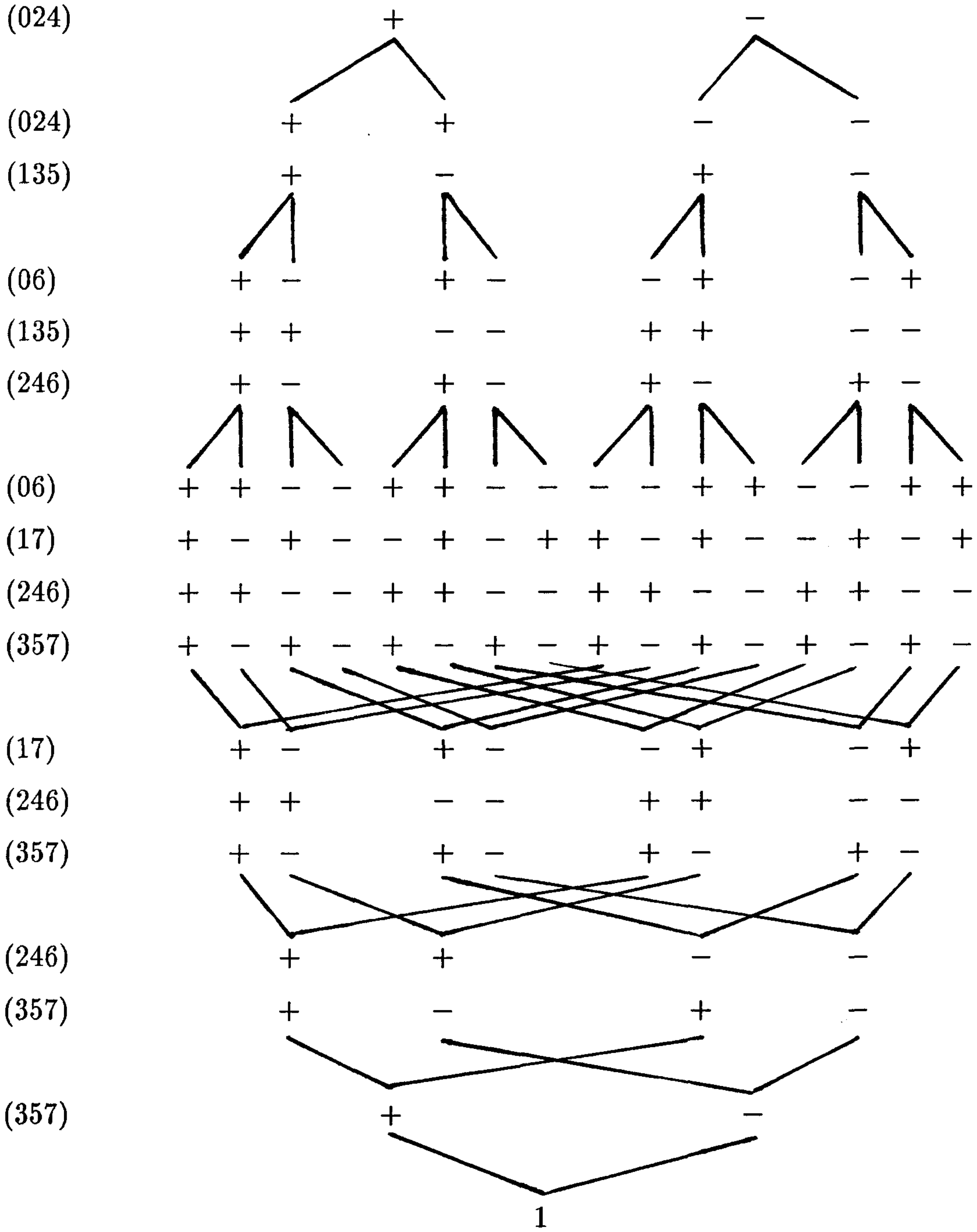


Figure 2. The orthogonality of minimal projections in Q_4, $Q_5, \cdots$, Q_{11} for $\{0, 0, 1, 0, 1, 0, \cdots\}$.

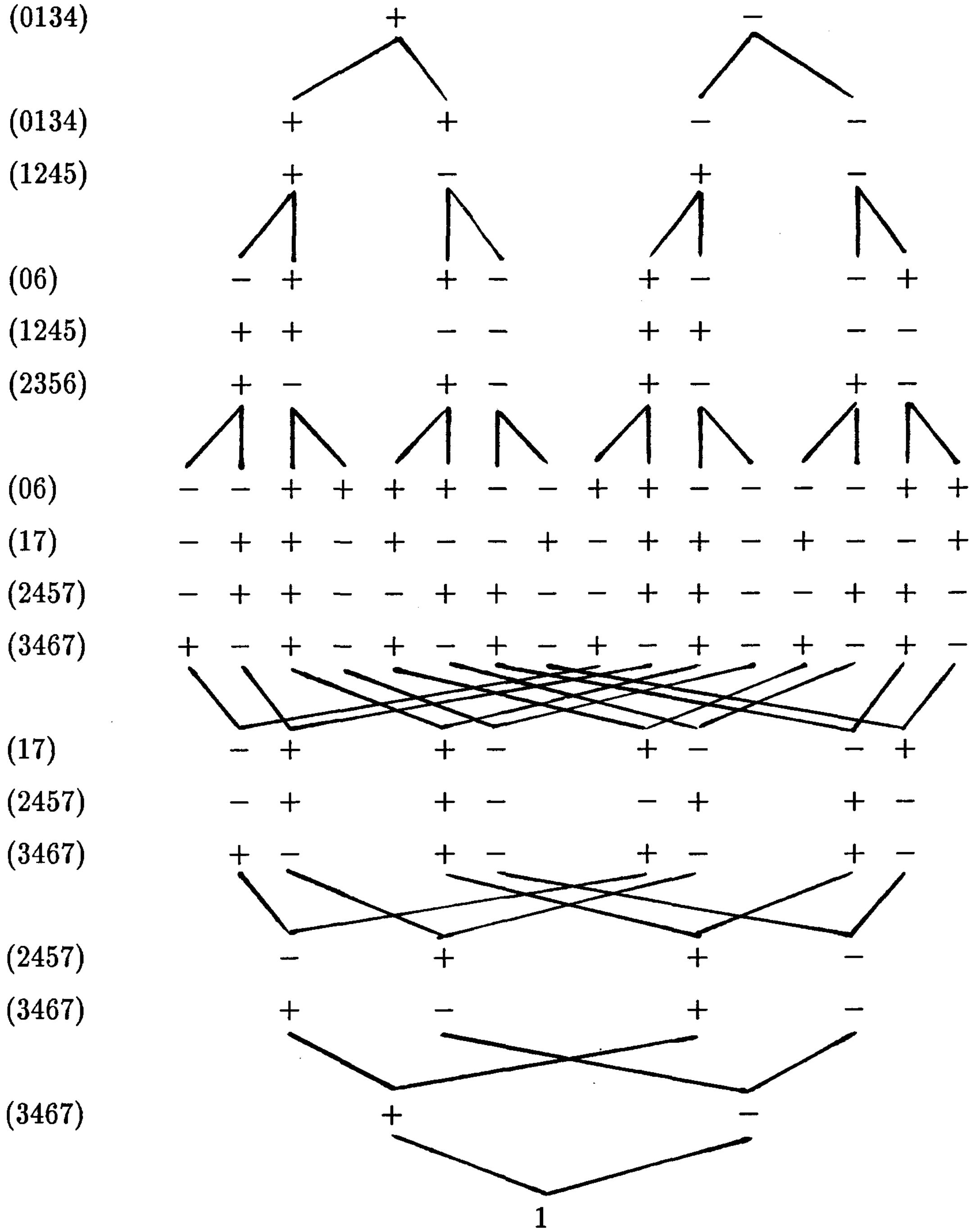


Figure 3. The orthogonality of minimal projections in $Q_4,\ Q_5, \cdots,\ Q_{11}$ for $\{0,0,1,1,1,0,\cdots\}$.

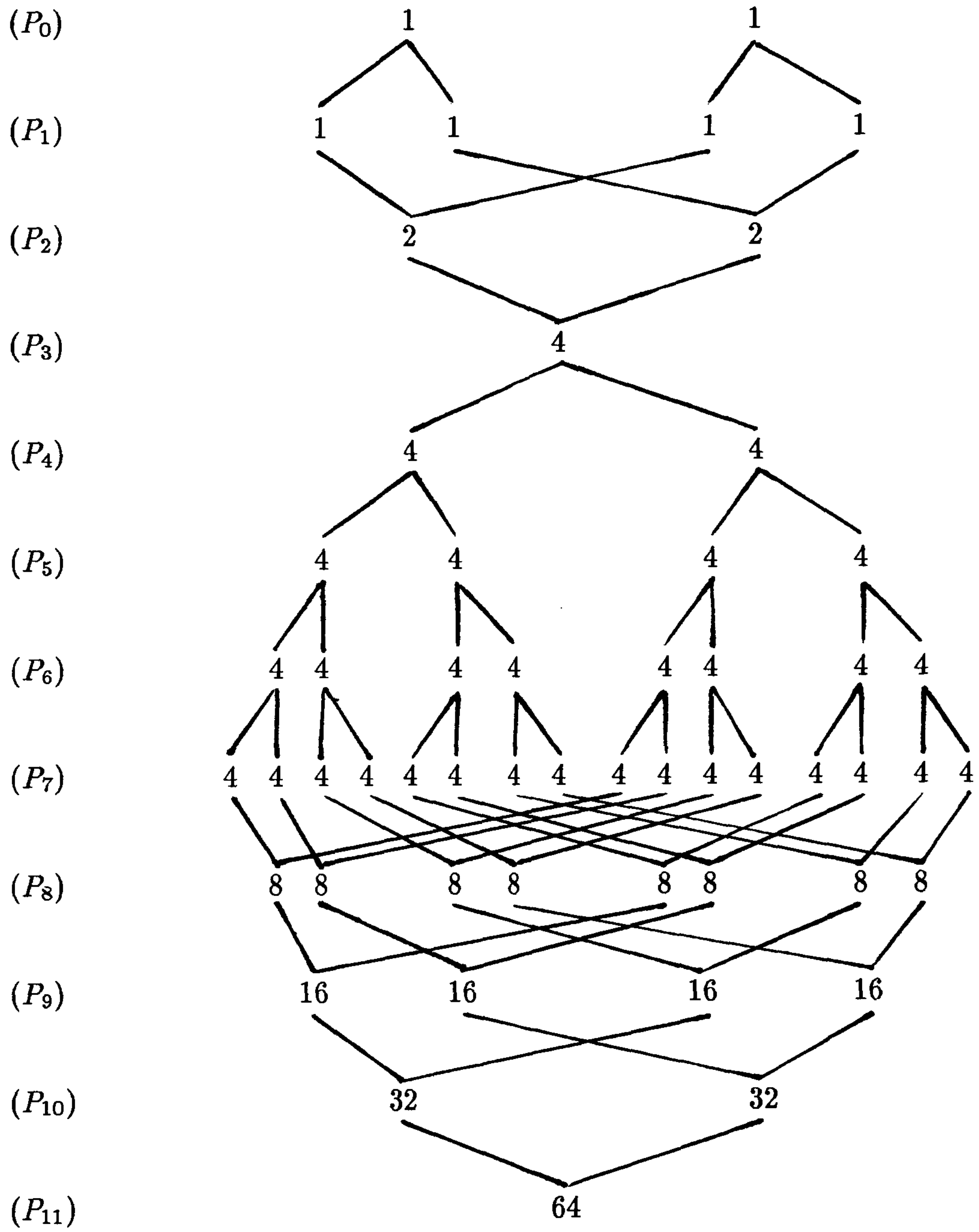


Figure 4. The Bratteli diagram of $P_0, P_1, \cdots, P_{11}$ for $\{0,0,1,0,1,0,\cdots\}$ and $\{0,0,1,1,1,0,\cdots\}$.

THEOREM. *There exist two different sequences* $\{a_1(n)\}$ *and* $\{a_2(n)\}$ *such that the Bratteli diagram of* $\overline{\bigcup_n P_n}$ *for* a_1 *coincides with the Bratteli diagram of* $\overline{\bigcup_n P_n}$ *for* a_2.

For any finitely supported sequence a, we have already seen that the AF- algebra for a is a UHF-algebra of type 2^∞. Using the canonical tracial state, we can construct a hyperfinite factor of type II_1 generated by unitaries $\{u_1, u_2, \cdots\}$. The correspondence u_i to u_{i+1} $(i = 0, 1, 2, \cdots)$ induces the binary shift σ_a of the hyperfinite factor $\mathcal{R}$ of type II_1. Then P_n is equal to the subalgebra of $\mathcal{R}$ whose element commutes with all the elements in $\sigma_a^{d+1+n}(\mathcal{R})$, where $d = sup\{|n|; a(n) = 1\}$. Bures and Yin showed that σ_{a_1} and σ_{a_2} are not outer conjugate if finitely supported sequences a_1 and a_2 are different. So the above theorem means that there exist two shifts for the hyperfinite factor of type II_1 such that they are not outer conjugate, although they have the same sequence of relative commutant algebras in the sense of the Bratteli diagram.

6. Decomposable norms and finite rank linear maps

In this section we determine the fundamental properties of completely positive maps, completely bounded maps and decomposable maps. We treat the following two conditions characterizing nuclear C*-algebras and some other C*-algebras belonging to a certain wider class than the class of nuclear C*-algebras. We donote a C*-algebra by A and the algebra consisting of all the $n \times n$ matrices by M_n.

(*) For every $n \in \mathbf{N}$ and every linear map T from M_n to A,

$$\| T \|_{dec} = \| T \|_{cb} .$$

(**) For every $n \in \mathbf{N}$ and every linear map T from $\mathbf{C}^n$ to A,

$$\| T \|_{dec} = \| T \|_{cb} .$$

LEMMA 6.1. *Let A, B and C be unital C^*-algebras and B be a C^*-subalgebra of A with the same unit. If there exists a projection P of norm one from A onto B, then*

$$\| T \|_{dec} = \| T \circ P \|_{dec}$$

for any decomposable linear map T from B to C.

PROOF: Using the fact that P is a completely positive map and the fact in section 2,

$$\| T \circ P \|_{dec} \leq \| T \|_{dec} \| P \|_{dec} = \| T \|_{dec} .$$

Conversely we must show the inequality $\| T \circ P \|_{dec} \geq \| T \|_{dec}$. By the definition of the decomposable norm, for any positive number ε, there exist completely positive maps S_1, S_2 from A to C such that

$$\| S_1 \|, \| S_2 \| < \| T \circ P \|_{dec} + \varepsilon \quad \text{and} \quad \begin{pmatrix} S_1 & T^* \circ P \\ T \circ P & S_2 \end{pmatrix}$$

is completely positive . We set $S_1' = S_1 |_B$ and $S_2' = S_2 |_B$. Since P is a projection of norm one from A onto B, for any positive element $[\, b_{i,j}]$ of $M_n(B)$,

$$\begin{pmatrix} (S_1)_n & T_n^* \circ P_n \\ T_n \circ P_n & (S_2)_n \end{pmatrix} [b_{i,j}] = \begin{pmatrix} (S_1')_n & T_n^* \\ T_n & (S_2')_n \end{pmatrix} [b_{i,j}].$$

This identitiy implies that

$$\begin{pmatrix} S_1' & T^* \\ T & S_2' \end{pmatrix}$$

is completely positive. Since $\| S_i' \| = \| S_i \|$,

$$\| T \|_{dec} \leq max\{\| S_1 \|, \| S_2 \|\} \leq \| T \circ P \|_{dec} + \varepsilon.$$

Since ε is arbitrary, it implies that $\| T \|_{dec} = \| T \circ P \|_{dec}$. ∎

Let T be a linear map from M_n to a C*-algebra A and $\{e_{i,j}\}$ be a system of matrix units of M_n. Making correspond to a linear map T an element $[Te_{i,j}]$ of $M_n(A)$, we can identify linear maps from M_n to A with $M_n(A)$. Then completely positive maps from M_n to A are identified with positive elements of $M_n(A)$ ([10]). An element of $M_n(A)$ can be represented as a linear combination of positive elements, so T is decomposable. By a similar way, we see that every linear map from $\mathbf{C}^n$ to A is decomposable.

COROLLARY 6.2. *If a C*-algebra A satisfies (*), then A satisfies (**).*

PROOF: For a linear map T from $\mathbf{C}^n$ to A, by Lemma 6.1

$$\| T \|_{dec} = \| T \circ P \|_{dec},$$

where P is a projection of norm one from M_n onto $\mathbf{C}^n$. By the assumption, it holds $\| T \circ P \|_{dec} = \| T \circ P \|_{cb}$. Since

$$\| T \|_{dec} = \| T \circ P \|_{dec} = \| T \circ P \|_{cb} \leq \| T \|_{cb},$$

this implies $\| T \|_{dec} = \| T \|_{cb}$. ∎

For a commutative C*-algebra, we can determine many properties precisely. Some of them is the property (**), the relation of positive maps and completely positive maps and the relation of bounded homomorphisms and bounded *-homomorphisms (the similarity probrem [32,33]). By the Gelfand-Naimark theorem, every commutative unital C*-algebra is isomorphic to a algebra $C(\Omega)$ of all the complex valued continuous functions on a suitable compact Hausdorff space Ω.

We consider the following special complex Banach space. Let E be a Banach space with a continuous involution *. Then $E \otimes M_n$ cannonically becomes a Banach space with a continuous involution for every $n \in \mathbf{N}$. We call E a matrix ordered Banach space if $\{E \otimes M_n, P_n(E)\}$ satisfies the following conditions,

(1) $P_n(E)$ is a closed *-invariant cone in $E \otimes M_n$ for every $n \in \mathbf{N}$,

(2) $\gamma^* P_n(E) \gamma$ is contained in $P_m(E)$ for every $\gamma \in M_{n,m}$, $m, n \in \mathbf{N}$.

We remark that the dual space of matrix ordered space is also matrix ordered with respect to the family of dual cones. We call a matrix ordered Banach space E of order 1 if P_n is the norm closure of the convex hull of $\{\gamma^* P_1 \gamma \,;\, \gamma \in M_{1,n}\}$ for every positive integer n.

Every C*-algebra is matrix ordered with respect to the family of natural cones consisting of all the positive elements. Its dual space is also matrix ordered ([Choi-Effros]). By the following lemma, we give an example of a matrix ordered Banach space of order 1.

LEMMA 6.3. *A commutative C*-algebra and its dual space are of order 1.*

PROOF: We consider the case that A is a commutative C*-algebra (we may assume A is unital), that is, $A = C(\Omega)$ where Ω is a compact Hausdorff space. For a positive element $f = [f_{i,j}] \in C(\Omega) \otimes M_n \cong C(\Omega, M_n)$ and any positive number ε, we can take a finite subset $\{x_k\}$ of Ω and a finite open covering $\{U_k\}$ of Ω such that

$$(1) \quad x_k \in U_K,$$

$$(2) \quad \|f(y) - f(x_k)\| < \varepsilon, \quad \text{if } y \in U_k.$$

Then there exists a partition of unity $\{F_p\}$ subordinate to $\{U_k\}$. We take an element y_p in $\mathrm{supp}(F_p)$, then we define an M_n-valued function g_p on Ω by $g_p = f(y_p)F_p$. By the definition, g is contained in the convex hull of $\{\gamma^* P_1 \gamma; \gamma \in M_{1,n}\}$. The property of a partition of unity implies

$$\|f - \sum_p g_p\| < \varepsilon.$$

Therefore A is of order 1.

The dual of A is identified with the predual of a commutative von Neumann algebra, and our arguments are reduced to these about a space $L^1(\Omega, \mu)$. The arguments in this case become easier than those in the above, because we can use characteristic functions instead of a partition of unity. We omit the detail. ∎

PROPOSITION 6.4. *Let E, F be matrix ordered spaces. We assume that a cone $P_n(F)$ is an intersection of $F \otimes M_n$ and a cone $P_n(F'')$ for each $n \in N$ where F'' is the dual of F'. If E or the dual of F is of order 1, then any positive, bounded linear map φ from E to F is completely positive , i.e., $\varphi_n(P_n(E))$ is contained in $P_n(F)$ for each $n \in \mathbf{N}$.*

PROOF: In the case that E is of order 1. For any $\gamma \in M_{1,n}$ and $x \in P_1(E)$, by the positivity and the linearity of φ,

$$\varphi \otimes id_n(\gamma^* x \gamma) = \gamma^* \varphi(x) \gamma \in P_n(F).$$

It follows that φ is completely positive, since E is of order 1 and φ is bounded.

In the case that the dual of F is of order 1. For any $x \in P_n(E)$, $f \in P(F')$ and $\gamma \in M_{1,n}$, by positivity of φ,

$$\begin{aligned} < \varphi \otimes id_n(x), \gamma^* f \gamma > &= <^t (\gamma^*) \varphi \otimes id_n(x)^t \gamma, f > \\ &= < \varphi(^t(\gamma^*) x^t \gamma), f > \geq 0, \end{aligned}$$

where $<,>$ means the dual pairing of the Banach space F and the its dual F'. By the assumption, φ is completely positive. ∎

The positivity of a linear map between C*-algebras naturally implies its boundedness. Moreover any C*-algebra satisfies the assumption of F in Proposition 6.4. So every positive map from a C*-algebra A to a C*-algebra B is completely positive, if one of the C*-algebras A, B is commutative. A lot of generalization of this fact are obtained by many mathematicians. The one in the above is the most generalized one among them.

PROPOSITION 6.5. *Let T be a linear map from $\mathbf{C}^n$ to $C(\Omega)$ and $\{e_1, e_2, \cdots, e_n\}$ be a* *system of orthogonal minimal projections of* $\mathbf{C}^n$. *If we define a linear map* S *from* $\mathbf{C}^n$ *to* $C(\Omega)$ *by* $S(e_i) = |T(e_i)|$, *then*

$$\begin{pmatrix} S & T^* \\ T & S \end{pmatrix}$$

is a completely positive map from $\mathbf{C}^n$ *to* $M_2(C(\Omega))$ *and*

$$\|T\|_{dec} = \|S\|.$$

PROOF: We note that an element

$$\begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix}$$

of M_2 is positive if and only if $a, c \geq 0$ and $ac \geq |b|^2$. This relation and the commutativity of $C(\Omega)$ imply the complete positivity of

$$\begin{pmatrix} S & T^* \\ T & S \end{pmatrix}.$$

We set $a_i = S_1(e_i)$ and $b_i = S_2(e_i)$ for given completely positive map R from $\mathbf{C}^n$ to $M_2(C(\Omega))$, where

$$R = \begin{pmatrix} S_1 & T^* \\ T & S_2 \end{pmatrix}.$$

Since

$$R(e_i) = \begin{pmatrix} a_i(\omega) & \overline{f_i(\omega)} \\ f_i(\omega) & b_i(\omega) \end{pmatrix}$$

is a positive element of M_2 for any $\omega \in \Omega$,

$$a_i(\omega), b_i(\omega) \geq 0 \quad \text{and} \quad a_i(\omega) + b_i(\omega) \geq 2|f_i(\omega)|.$$

Then

$$\begin{aligned}
\|S\| = \|S(1)\| &= \sup_{\omega} \sum_{i} |f_i(\omega)| \\
&\leq \sup_{\omega} \max\{\sum_{i} a_i(\omega), \sum_{i} b_i(\omega)\} \\
&= \max\{\sup_{\omega} \sum_{i} a_i(\omega), \sup_{\omega} \sum_{i} b_i(\omega)\} \\
&= \max\{\|\sum_{i} a_i\|, \|\sum_{i} b_i\|\} \\
&= \max\{\|S_1\|, \|S_2\|\}.
\end{aligned}$$

This implies $\|T\|_{dec} \geq \|S\|$. Therefore $\|T\|_{dec} = \|S\|$. ∎

COROLLARY 6.6. *For any linear map T from $\mathbf{C}^n$ to $C(\Omega)$,*

$$\|T\| = \|T\|_{cb} = \|T\|_{dec}.$$

PROOF: $\|T\| \leq \|T\|_{cb} \leq \|T\|_{dec}$ is obvious. It suffices to show that $\|T\| \geq \|T\|_{dec}$. By the above proposition,

$$\begin{aligned}
\|T\|_{dec} = \|S\| &= \|S(1)\| \\
&= \sum_{i} S(e_i)(\omega) \quad \text{for some } \omega \in \Omega.
\end{aligned}$$

There exist complex numbers α_i ($i = 1, \cdots, n$) such that $|\alpha_i| = 1$ and

$$\sum_{i} S(e_i)(\omega) = T(\sum_{i} \alpha_i e_i)(\omega) \leq \|T\|.$$

∎

By the result of Wittstock([47]) and Paulsen([35]), it is known that an injective C*-algebra satisfies the condition (*). We will show the existence of a C*-algebra which satisfies the condition (*) and is not injective.

LEMMA 6.7. *Let A be a unital C*-algebra. If T_1 and T_2 are linear maps from M_n to A such that $||T_1 - T_2|| \leq 1$, then*

$$||T_1 - T_2||_{dec} \leq n^2.$$

PROOF: We note that, for any element $a \in A$, $a \leq 1$ if and only if

$$\begin{pmatrix} 1 & a^* \\ a & 1 \end{pmatrix}$$

is positive. Let $\{e_{i,j}\}$ be a system of matrix units of M_n. We define a linear map S from M_n to A by $S(e_{i,j}) = n\delta_{i,j}1$. If we show that

$$R = \begin{pmatrix} S & (T_1 - T_2)^* \\ T_1 - T_2 & S \end{pmatrix}$$

is completely positive, then

$$||T_1 - T_2||_{dec} \leq ||S|| = ||S(1)|| = ||\sum_i S(e_{i,i})|| = n^2.$$

Therefore we may show that R is completely positive. By the assumption, since $||\ (T_1 - T_2)(e_{i,j})\ ||$ is less than 1, the norm of an element in $M_{2n}(A)$, whose (i,j)-th entry is

$$\begin{pmatrix} 0 & (T_1 - T_2)^*(e_{i,j}) \\ (T_1 - T_2)(e_{i,j}) & 0 \end{pmatrix},$$

is less than n. The canonical isomorphism between $(A \otimes M_2) \otimes M_n$ and $(A \otimes M_n) \otimes M_2$ yields a canonical rearrangement of an $n \times n$-matrix of 2×2 blocks as a 2×2-matrix of

$n \times n$ blocks, with the (i,j)-th entry of the (k,l)-th block becoming the (k,l)-th entry of the (i,j)-th block. The element of $(A \otimes M_2) \otimes M_n$, whose the (i,j)-th entry is

$$\begin{pmatrix} 0 & (T_1 - T_2)^*(e_{i,j}) \\ (T_1 - T_2)(e_{i,j}) & 0 \end{pmatrix},$$

becomes after this rearrangement, of the form,

$$\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}$$

with an element X of $A \otimes M_n$. Then $(R(e_{i,j}))$, after the same rearrangement, is of the form,

$$\begin{pmatrix} nI_n & X^* \\ X & nI_n \end{pmatrix}.$$

Since

$$\|X\| = \| \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \| \leq n,$$

we see that the matrix $[R(e_{i,j})]$ is positive. Therefore R is completely positive. ▮

We remark the following fact related to the property (*) in the above setting. If $\|T_1 - T_2\| \leq \varepsilon$ for a positive number ε, then

$$\|T_1 - T_2\|_{cb} \leq \|T_1 - T_2\|_{dec} \leq n^2\varepsilon.$$

In particular,

$$|\, \|T_1\|_{cb} - \|T_2\|_{cb}|, |\, \|T_1\|_{dec} - \|T_2\|_{dec}| \leq n^2\varepsilon.$$

THEOREM 6.8. *(1) Let A be a nuclear C^*-algebra. Then the spatial tensor product $A \otimes B(\mathcal{H})$ of A and $B(\mathcal{H})$ satisfies the condition (*), where $B(\mathcal{H})$ is the algebra of all the bounded operators on a Hilbert space $\mathcal{H}$.*

(2) Let $\{A_n\}$ be an inductive system of C-algebras and A be the inductive limit C*-algebra of $\{A_n\}$. If A_n satisfies (*) for any $n \in \mathbf{N}$, then A satisfies (*).*

PROOF: (1) For any $\varepsilon > 0$ and any linear map T from M_n to $A \otimes B(\mathcal{H})$, we need to show that $\|T\|_{dec} < \|T\|_{cb} + \varepsilon$. We set $\delta = \varepsilon/n^4$. We can take a linear map T' from M_n to the algebraic tensor product $A \odot B(\mathcal{H})$ of A and $B(\mathcal{H})$ such that $\|T(e_{i,j}) - T'(e_{i,j})\| < \delta/3$. By the nuclearity (the factorization property), there exist a positive integer k, a completely positive contraction σ from A to M_k and a completely positive contraction τ from M_k to A such that

$$\|T'(e_{i,j}) - (\tau \otimes 1)(\sigma \otimes 1)T'(e_{i,j})\| < \delta/3$$

for every i and j. Then it follows that

$$\|T(e_{i,j}) - (\tau \otimes 1)(\sigma \otimes 1)T(e_{i,j})\| < \delta.$$

Therefore we have $\|T - (\tau \otimes 1)(\sigma \otimes 1)T\| < n^2\delta$. The range of $(\sigma \otimes 1)T$ is contained in the injective C*-algebra $M_k \otimes B(\mathcal{H})$. Thus we have

$$\begin{aligned}
\|T\|_{dec} &\leq \|(\tau \otimes 1)(\sigma \otimes 1)T\|_{dec} + n^2(n^2\delta) \\
&\leq \|(\sigma \otimes 1)T\|_{dec} + \varepsilon \\
&= \|(\sigma \otimes 1)T\|_{cb} + \varepsilon \\
&\leq \|T\|_{cb} + \varepsilon.
\end{aligned}$$

This completes the proof of (1) .

(2) We may assume that $\{A_n\}$ is an increasing sequence of C*-subalgebras of A such that the union of A_n is dense in A. For any $\varepsilon > 0$ and any linear map T from M_n to A, there exist a positive integer k and a linear map T' from M_n to A_k such that $\| T - T' \| < \varepsilon/n^2$. Using Lemma 6.7 and the fact that A_k satisfies (*),

$$\|T\|_{dec} < \|T'\|_{dec} + \varepsilon = \|T'\|_{cb} + \varepsilon.$$

Since ε is arbitrary, A satisfies (*) . ∎

An example of a C*-algebra not satisfying the condition (*) is the reduced group C*-algebra $C_r^*(F(2))$, where $F(2)$ is the free group generated by two elements ([25]). We consider what a C*-algebra satisfies the condition (*). We have already known every injective C*-algebra satisfies (*). From (1) of Theorem 6.8, every nuclear C*-algebra satisfies (*), too. If C*-algebras A and B satisfy (*), then it is easily shown that $A \oplus B$ satisfies (*). If A satisfies (*) and there exists a norm one projection from A onto a C*-subalgebra B of A, then B satisfies (*). Moreover we can see the existence of a C*-algebra satisfying (*), which is neither nuclear nor injective. For example, so is $B(\mathcal{H}) \oplus K(\mathcal{H})$ or $B(\mathcal{H}) \otimes K(\mathcal{H})$, where $K(\mathcal{H})$ is the algebra of all the compact operators on a separable Hilbert space $\mathcal{H}$.

References

[1] W. Arveson, *Subalgebras of C^*-algebras*, Acta Math., 123 (1969), 141–224.

[2] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc., 171 (1972), 195-234.

[3] O. Bratteli, *The center of approximately finite-dimensional C^*-algebras*, J. Funct. Anal., 21 (1976), 195-201.

[4] L.G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math., 71 (1977), 335-348.

[5] J.W. Bunce, *The similarity problem for representations of C^*-algebras*, Proc. Amer. Math. Soc., 81 (1981), 409-414.

[6] D. Bures and H.-S. Yin, *Shifts on the hyperfinite factor of type* II_1, preprint.

[7] D. Bures and H.-S. Yin, *Outer conjugacy of shifts on the hyperfinite* II_1*-factor*, preprint.

[8] M. Choda, *Shifts on the hyperfinite* II_1*-factor*, J. Operator Theory, 17 (1987), 223–235.

[9] M.-D. Choi, *Positive linear maps on C^*-algebras*, Canad. J. Math., 24 (1972), 502–529.

[10] M.-D. Choi and E.G. Effros, *Injectivity and nuclear C^*-algebras*, Duke Math. J., 43 (1976), 309–322.

[11] M.-D. Choi and E.G. Effros, *Injectivity and operator spaces*, J. Funct. Anal., 24

(1977), 156–209.

[12] M.-D. Choi and E.G. Effros, *Nuclear C^*-algebras and the approximation property*, Amer. J. Math., 100 (1978), 61–79.

[13] E. Christensen, *On non self-adjoint representation of C^*-algebras*, Amer. J. Math., 103 (1981), 817–833.

[14] E. Christensen, *Extensions of derivations* II, Math. Scand., 50 (1982), 111–122.

[15] E. Christensen, *Similarities of* II_1 *factors with property* Γ, Kopenhavns Universitet Math. Inst., Preprint series No.16.

[16] A. Connes, *Classification of injective factors*, Ann. of Math. (2), 104 (1976), 585–609.

[17] E.G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math., no. 46, Amer. Math. Soc., Providence, 1981.

[18] E.G. Effros, D.E. Handelman and C.-L. Shen, *Dimension groups and their affine representations*, Amer. J. Math., 102 (1980), 385–407.

[19] G.A. Elliott, *On the classification of inductive limits of sequences of semi-simple finite-dimentional algebras*, J. Algebra, 38 (1976), 29–44.

[20] G.A. Elliott, *Automorphisms determined by multipliers on ideals of a C^*-algebra*, J. Funct. Anal., 23 (1976), 1–10.

[21] M. Enomoto, M. Nagisa, Y. Watatani and H. Yoshida, *Relative commutant algebras of Powers' binary shifts on the hyperfinite* II_1 *factor*, preprint.

[22] M. Enomoto and Y. Watatani, *A solution of Powers' problem on outer conjugacy of binary shifts*, preprint.

[23] J.G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc., 95 (1960), 318–340.

[24] U. Haagerup, *Solutions of the similarity problem for cyclic representations of C^*-algebras*, Ann. of Math. (2), 118 (1983), 215–240.

[25] U. Haagerup, *Injectivity and decomposition of completely bounded maps*, Lecture Notes in Math. 1132, 170-222, Springer-Verlag.

[26] T. Itoh and M. Nagisa *Completely positivity of mapping valued linear maps*, Math. Z., 197 (1988), 273–276.

[27] V. Jones, *Index for subfactors*, Invent. Math., 72 (1983), 1–25.

[28] C. Lance, *On nuclear C^*-algebras*, J. Funct. Anal,. 12 (1973), 157–176.

[29] A.J. Lazar and D.C. Taylor, *Approximately finite dimensional C^*-algebras and Bratteli diagrams*, Trans. Amer. Math. Soc., 259 (1980), 599–619.

[30] M. Nagisa, *On hereditary C^*-subalgebras of an AF-algebra*, Math. Japon., 26 (1981), 659–665.

[31] M. Nagisa, *On the decomposition of finite rank linear maps*, Math. Japon., 30 (1985), 503-510.

[32] M. Nagisa and G. Song, *On some property of bounded homomorphisms and derivations of a C^*-algebra*, Math. Japon., 32 (1987), 801-810.

[33] M. Nagisa and G. Song, *Inheritance of the solvability of the similarity problem within a C^*-algebra and its C^*-subalgebras*, to appear in Math. Japon.

[34] M. Nagisa and J. Tomiyama, *Completely positive maps in the tensor products of von Neumann algebras*, J. Math. Soc. Japan, 33 (1981), 539–550.

[35] V.I. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series 146.

[36] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. Ecole Norm. Sup. (4), 19 (1986), 57–106.

[37] R.T. Powers, *An index theory for semigroups of *-endomorphisms of $B(H)$ and type* II_1 *factors*, Canad. J. Math., 40 (1988), 86–114.

[38] G. Price, *Shifts on type* II_1 *factors*, Canad J. Math., 39 (1987), 492–511.

[39] G. Price, *Shifts of integer index on the hyperfinite* II_1 *factor*, Pacific J. Math., 132 (1988), 379-390.

[40] C.-L. Shen, *On the classification of the ordered groups associated with the approximately finite dimensional C^*-algebras*, Duke Math. J., 46 (1979), 613–633.

[41] R.R. Smith, *Completely bounded maps between C^*-algebras*, J. London Math. Soc., 27 (1983), 157–166.

[42] W.F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc., 6 (1955), 211–216.

[43] T. Takasaki and J. Tomiyama, *Stinespring type theorems for various types of completely positive maps associated to operator algebras*, Math. Japon., 27 (1982), 129–139.

[44] M. Takesaki, *Theory of operator algebras* I, Springer-Verlag, Berlin, 1979.

[45] J. Tomiyama, *Recent development of the theory of completely bounded maps between C^*-algebras*, Publ. Res. Inst. Math. Sci., 19 (1983), 1283–1303.

[46] Y. Watatani, *L'indice d'une C^*-sous-algèbre d'une C^*-algèbre simple*, C. R. Acad. Sci. Paris Ser. rm I Math., 305 (1987), 23-26.

[47] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal., 40 (1981), 127–150.

MASARU NAGISA
DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF ENGINEERING SCIENCE
OSAKA UNIVERSITY