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# Nonperturbative solutions for canonical quantum gravity*

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**Abstract**

In this paper we will make a survey of solutions to the Wheeler-De Witt equation which have been found up to now in Ashtekar's formulation for canonical quantum gravity. Roughly speaking they are classified into two categories, namely, Wilson-loop solutions and topological solutions. While the program of finding solutions which are composed of Wilson loops is still in its infancy, it is expected to be developed in the near future. Topological solutions are the only solutions at present which we can give their interpretation in terms of spacetime geometry. While the analysis made here is formal in the sense that we do not deal with rigorously regularized constraint equations, these topological solutions are expected to exist even in the fully regularized theory and they are considered to yield vacuum states of quantum gravity. We also make an attempt to review the spin network states as intuitively as possible. In particular, the explicit formulae for two kinds of measures on the space of spin network states are given.

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# 1 Introduction

## 1.1 Motivations for quantum gravity

By a large amount of empirical evidence in astrophysics, it turns out that general relativity describes almost (more than 99 percent) correctly classical behavior of the gravitational interaction. According to the singularity theorem demonstrated by Penrose and Hawking[47], however, general relativity necessarily leads to spacetime singularity after the stellar collapse or at the beginning of the cosmological evolution as long as the energy momentum tensor satisfies certain positivity conditions. This intrinsic pathology suggests that general relativity is in fact the effective theory of a certain more fundamental theory of gravity, which is called the theory of quantum gravity. It is natural to think that, in the very early period of the universe in which the characteristic scale of the system is the Planck length $L_P \equiv (G\hbar/c^3)^{1/2} \sim 10^{-33}cm$, or the Planck energy scale $\Lambda_P \equiv \hbar c/L_P \sim 10^{19}GeV$, nonperturbative effects of quantum gravity spoil the validity of general relativity and therefore the initial singularity is circumvented. This is the main motivation for quantum gravity from the perspective of general relativitists.

Particle physicists have other motivations. Most quantum field theories including the Standard model are plagued with the problem of ultraviolet divergences. Among these UV divergences some relatively tamable ones are handled by the renormalization prescription. Here a natural hope arises that the UV divergences may be removed by taking account of quantum gravity, because it is conceivable in quantum gravity that the physics at the Planck scale is quite different from that at our scale.

More ambitious particle physicists attempt to construct the unified theory in which all the four interactions— gravitational, weak, electromagnetic and strong interactions— are described by a single entity. To do so gravitational interaction needs to be quantized because the other three interactions have already been quantized.

Because quantum gravity is essentially the physics at the Planck scale which is far beyond our experience, it is no wonder that we have not yet acquired a consensus on what the theory of quantum gravity would look like. In consequence there are many approaches to quantum gravity [50]. Among them, the following three approaches are vigorously investigated:

i) superstring theory. In the mid seventies perturbative quantization of general relativity turned out to be non-renormalizable. This result lead particle physicists to consider that general relativity is the low energy effective theory of a more fundamental theory which is presumably renormalizable (or finite). One such candidate is the theory of superstrings, in which the gravity is incorporated as a massless excitation mode of the closed string. Recently it has been clarified that the strong coupling phase of a superstring theory is described by the weak coupling phase of another superstring theory[48][87]. A series of these recent discoveries is expected to accelerate developments in nonperturbative aspects of superstring theory;

ii) discretized quantum gravity. Another natural reaction to the non-renormalizability of general relativity is to consider the nonperturbative quantization of general relativity. There are two main approaches to implement this, discretized quantum gravity and canonical quantum gravity. In discretized quantum gravity, we first approximate the spacetime manifold and the Einstein-Hilbert action by a simplicial complex (a connected set of four-simplices) and by the Regge action respectively[75]. Quantization is performed by numerical path integral methods, namely by taking a summation of various numerical configurations. There are several choices of independent variables, the two most typical ones among which are seen in Regge calculus and in dynamical triangulations[2]. In Regge calculus we fix a triangulation and use the edge lengths of four-simplices as dynamical variables, whereas in dynamical triangulations we set the edge lengths to unity and sum up all the possible ways of triangulations. While we have not yet extracted physically meaningful information on four dimensional gravity, this approach is expected to yield some intuitive picture on quantum gravity in the near future;

iii) canonical quantization of general relativity. Some general relativitists consider that splitting the metric into background and quantum fluctuation parts is the origin of the failure in the perturbative quantization of general relativity. They therefore anticipate that it may be possible to quantize general relativity if we deal with the metric as a whole. Inspired by this anticipation canonical quantization for general relativity has been investigated for about thirty years. While the metric formulation seems to be too complicated to solve, since Ashtekar discovered new variables which simplifies the canonical formulation[4], Ashtekar's formalism has been actively investigated by many people [82]. This Ashtekar's formulation for general relativity is now considered to be one of the promising approaches to quantum gravity.

In the following we will explain more precisely this canonical approach.

## 1.2 Issues on canonical quantum gravity

Canonical quantization in the metric formulation starts from the (3+1)-decomposition of the spacetime metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{1.1}$$

By plugging this into the Einstein-Hilbert action $I_{EH} = \int_M d^4x\sqrt{-g}R$, we obtain the ADM action[3]

$$I_{ADM} = \int dt \int_{M^{(3)}} d^3x[\tilde{\pi}^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a], \tag{1.2}$$

where $\mathcal{H}^a$ and $\mathcal{H}$ are respectively the momentum constraint and the Hamiltonian constraint. If we perform Dirac's quantization, these two first class constraints yield the constraint equations imposed on the wavefunction $\Psi[q_{ab}]$. The momentum constraint $\hat{\mathcal{H}}^a$ requires $\Psi[q_{ab}]$ to be invariant under spatial diffeomorphisms. The Hamiltonian constraint $\hat{\mathcal{H}}$ yields the notorious Wheeler-De Witt (WD) equation[32]

$$\left\{\sqrt{q}^{-1}(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd})\frac{\delta}{\delta q_{ab}}\frac{\delta}{\delta q_{cd}} + \sqrt{q}^{(3)}R(q)\right\}\Psi[q_{ab}] = 0, \tag{1.3}$$

where ${}^{(3)}R(q)$ is the 3-dimensional scalar curvature. This WD equation is considered as the dynamical equation in canonical quantum gravity.

Canonical quantum gravity is known to possess intrinsically the following conceptual problems:

i) Because the WD equation is the Klein-Gordon type equation with signature

$$(-,+,+,+,+,+) \times \infty^3,$$

we cannot construct any positive semi-definite inner-product which is conserved by the WD equation. Thus we do not know how to interpret the wave function $\Psi[q_{ab}]$.

ii) The "issue of time" in quantum gravity[58]. The Hamiltonian of canonical quantum gravity is given by a linear combination of the first class constraints $(\mathcal{H}, \mathcal{H}^a)$. As a result the evolution of the physical wavefunction $\Psi[q_{ab}]$ (or of physical observables) w.r.t. the coordinate time $t$ is trivial. This naturally follows from general covariance of the theory. Several remedies for this problem have been considered. For example, to identify a gravitational degree of freedom or a matter degree of freedom with a physical time. The former is called the "intrinsic time" and the latter is called the "extrinsic time".

These conceptual problems originally stems from lack of our knowledge on the quantization of diffeomorphism invariant field theories. In order to resolve these problems, we therefore have to investigate more extensively these diffeomorphism invariant quantum field theories such as quantum gravity. When we try to carry out the program of canonical quantum gravity, however, we always run into the serious technical problem. Namely, the Wheeler-De Witt equation (1.3) is so complicated that we have not found even one solution to it.[1] This severe fact had confronted the researchers of canonical quantum gravity until 1986 when the new canonical variables were discovered by Ashtekar [4].

## 1.3 Ashtekar's formalism and solutions to the Wheeler-De Witt equation

As we will see in §2, the Wheeler-De Witt equation takes a simple form if written in terms of Ashtekar's new variables. We therefore expect that the WD equation is solved in Ashtekar's formalism. In fact several types of solutions have been found. Roughly speaking they are classified into two classes.

One is the class of "Wilson loop solutions" [52][49][36][37] whose fundamental ingredient is the parallel propagator of the Ashtekar connection $A_a^i$ along a curve $\alpha : [0,1] \to M^{(3)}$:

$$h_\alpha[0,1] = \mathcal{P}\exp\{\int_0^1 ds\dot{\alpha}^a(s)A_a^i(\alpha(s))J_i\}.$$

Recently there have been remarkable developments in the application of spin network states (the extended objects of Wilson loops, see §3) to Ashtekar's formalism [8] [9] [10] [12] [14] [78] [79]. Inspired by these developments, progress in the area of constructing solutions from parallel propagators is expected to be made in the near future.

While finding Wilson loop solutions is a promising program, as yet we have not reached the point where we can extract physically interesting results. The other class of solutions, however, are considered to have definite significance in quantum gravity. They are the topological solutions[55][20] which are regarded as "vacuum states" in quantum gravity [83][66]. These solutions therefore guarantees at least that the efforts made in canonical quantum gravity will not completely be in vain.

## 1.4 An outline of this paper

In this paper we will make a survey of developments and issues on the solutions for canonical quantum gravity.

§2 and §3 are devoted to an extensive review of the bases necessary to read this paper.

[1] Quite recently a class of solutions to the WD equation (1.3) have been found[57]. Their physical significance has not been clarified as yet.

In §2 we explain classical and quantum theory of Ashtekar's formalism[5]. We followed the derivation by way of the chiral Lagrangian[24]. In §3 the results on the spin network states are reviewed. The notion of spin network states is a kind of generalization of link-functionals in lattice gauge theory[56]. It was introduced into quantum general relativity in order to construct a mathematically rigorous formulation of the loop representation[77], which is expected to describe neatly the nonperturbative nature of canonical quantum gravity. While recent progress on the spin network states is striking[8][9][10] [12][14][78][79], their underlying ideas have been obscured because of the necessity for mathematical rigor. This makes this area of "applied" spin network states inaccessible to the usual physicists. So in that section we will make a maximal attempt to provide the results in the forms which are familiar to the physicists[2]. In particular we give several explicit formulae for the measures defined on spin network states. They are useful in proving the consistency of the measures and, in particular, in showing completely that the set of spin network states forms an orthonormal basis w.r.t. these measures (the induced Haar measure and the induced heat-kernel measure).

Most of the subject in §4~§6 is based on the author's recent works[36][37][38][39].

In §4 we present Wilson loop solutions. We first compute the action of "renormalized" scalar constraint on the spin network states. Then we provide simple solutions which we call "combinatorial solutions" to the renormalized WD equation. After that we explain why we cannot extract physically interesting results from these solutions. As an attempt to construct physically interesting solutions, we will introduce in §5 the lattice approach to Ashtekar's formalism. There a set of nontrivial solutions—"multi-plaquette solutions"—to a discretized WD equation is given. Some properties of these solutions indicate a direction of future developments in this area. Topological solutions are investigated in §6. We will focus on the semiclassical interpretation of these solutions and show that they semiclassically represent the spacetimes which are vacuum solutions to the Einstein equation. In particular in the case of a vanishing cosmological constant, each topological solution corresponds to a family of (3+1)-dimensional Lorentzian structures with a fixed projection to the Lorentz group. In that section we will also show that Ashtekar's formulation for $N = 1$ supergravity also possesses topological solutions by unraveling the relation between $GSU(2)$ BF theory and $N = 1$ Ashtekar's formalism. In §7 a brief comment is made on solutions[31], including the Jones polynomial [22][23][41][42] in the loop representation[77], which are not properly contained in the above two categories.

Here we will explain the notation used in this paper:

1. $\alpha, \beta, \cdots$ denote $SO(3,1)$ local Lorentz indices;

2. $\mu, \nu, \cdots$ stand for spacetime indices;

[2] To a reader who is interested in the mathematically rigorous formulation, we recommend to read [6] and references therein.

3. $a, b, \cdots$ are used for spatial indices;

4. $A, B, \cdots$ represent left-handed $SL(2, \mathbf{C})$ spinor indices;

5. $i, j, \cdots$ denote indices for the adjoint representation of (the left-handed part of) $SL(2, \mathbf{C})$. $J_i$ stands for the $SL(2, \mathbf{C})$ generator subject to the commutation relation $[J_i, J_j] = \epsilon_{ijk} J_k$;

6. $\tilde{\epsilon}^{abc}$( $\underset{\sim}{\epsilon}_{abc}$) is the Levi-Civita alternating tensor density of weight $+1$ $(-1)$ with $\tilde{\epsilon}^{123} = \underset{\sim}{\epsilon}_{123} = 1$;

7. $\epsilon^{ijk}$ is the antisymmetric (pseudo-)tensor with $\epsilon^{123} = 1$;

8. $\epsilon^{AB}$ $(\epsilon_{AB})$ is the antisymmetric spinor with $\epsilon^{12} = \epsilon_{12} = 1$;[3]

9. relation between a symmetric rank-2 spinor $\phi^{AB}$ and its equivalent vector $\phi^i$ in the adjoint representation is given by $\phi^{AB} = \phi^i (J_i)^{AB}$, where $(J_i)^A{}_B$ is the $SL(2, \mathbf{C})$ generator in the spinor representation subject to $(J_i)^A{}_C (J_j)^C{}_B = -\frac{1}{4}\delta^{ij}\delta^A_B + \frac{1}{2}\epsilon^{ijk}(J_k)^A{}_B$ ;

10. $D = dx^\mu D_\mu$ denotes the covariant exterior derivative with respect to the $SL(2, \mathbf{C})$ connection $A = A^i J_i$ and ;

11. indices located between ( and ) ([ and ]) are regarded as symmetrized (antisymmetrized).

For analytical simplicity, we will consider in this paper that the spacetime $M$ has the topology $\mathbf{R} \times M^{(3)}$ with $M^{(3)}$ being a compact, oriented, 3 dimensional manifold without boundary.

# 2 Ashtekar's formalism

## 2.1 Classical theory

In this section we provide a review of Ashtekar's formulation for canonical general relativity. Ashtekar's formalism were originally derived from the first order ADM formalism by exploiting the complex canonical transformation [4][5]. In order to derive it from the outset, however, it is more convenient to use the chiral Lagrangian formalism[24]. So we will take the latter route.

[3]These antisymmetric spinors are used to raise and lower the spinor index: $\varphi^A = \epsilon^{AB}\varphi_B$, $\varphi_A = \varphi^B \epsilon_{BA}$.

First we consider the usual first-order formalism of general relativity, whose action is called the Einstein-Palatini action with a cosmological constant $\Lambda$:

$$I_{EP} = -\frac{1}{2}\int_M \epsilon_{\alpha\beta\gamma\delta} e^\alpha \wedge e^\beta \wedge (R^{\gamma\delta} - \frac{\Lambda}{6} e^\gamma \wedge e^\delta), \tag{2.1}$$

where $e^\alpha$ is the vierbein and $R^{\alpha\beta} = d\omega^{\alpha\beta} + \omega^\alpha{}_\gamma \wedge \omega^{\gamma\beta}$ denotes the curvature of the spin-connection $\omega^{\alpha\beta}$. Equations of motion are derived from the variational principle w.r.t. the vierbein $e^\alpha$ and the spin-connection $\omega^{\alpha\beta}$. The results are respectively given by

$$\begin{aligned} 0 &= \epsilon_{\alpha\beta\gamma\delta} e^\beta \wedge (R^{\gamma\delta} - \frac{\Lambda}{3} e^\gamma \wedge e^\delta) \\ 0 &= \epsilon_{\alpha\beta\gamma\delta} e^\gamma \wedge (de^\delta + \omega^\delta{}_\epsilon \wedge e^\epsilon). \end{aligned} \tag{2.2}$$

If we assume that the vierbein $e^\alpha$ is non-degenerate, the second equation is equivalent to the torsion-free condition

$$de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0. \tag{2.3}$$

The result of plugging its solution into $I_{EP}$ yields the Einstein-Hilbert action

$$I_{EH} = \int_M d^4x \sqrt{g}(R - 2\Lambda), \tag{2.4}$$

where $R$ denotes the scalar curvature. This implies that the two equations in (2.2) are equivalent to the Einstein equation provided that the vierbein is non-degenerate.

The (3+1)-decomposition of $I_{EP}$ yields the first-order ADM formalism which is essentially equivalent to the usual ADM formalism[3]. The resulting Wheeler-De Witt (WD) equation is as complicated as that in the ADM formalism. This Einstein-Palatini action is therefore not suitable for canonical quantization of general relativity.

In deriving Ashtekar's formalism from the outset, we need another action which is classically equivalent to the Einstein-Hilbert action. This is the complex chiral action (or the Plebanski action [73]):[4]

$$\begin{aligned} I_{CC} &= -i\int_M P^{(-)\alpha\beta}{}_{\gamma\delta}\, e_\alpha \wedge e_\beta \wedge (R^{\gamma\delta} - \frac{\Lambda}{6} e^\gamma \wedge e^\delta) \\ &= \frac{1}{2} I_{EP} - \frac{i}{2}\int_M e_\alpha \wedge e_\beta \wedge R^{\alpha\beta}. \end{aligned} \tag{2.5}$$

We can easily see that this action is equivalent to $\frac{1}{2} I_{EH}$ under the torsion-free condition $de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0$, which is the equation of motion derived from the real part of $I_{CC}$. As long as we regard $(e^\alpha, \omega^{\beta\gamma})$ to be real-valued, we can deal with the real and imaginary parts of $I_{CC}$ separately in deriving equations of motion. The complex chiral action $I_{CC}$ is therefore classically equivalent to the Einstein-Hilbert action.[5]

[4] The definitions and properties of $P^{(-)\alpha\beta}{}_{\gamma\delta}$ and $P^{(-)i}_{\alpha\beta}$ are listed in Appendix A.

[5] The difference of the overall factor by 2 is not important because we can always change the overall factor by taking different unit of length.

Using eq.(A.4) this complex chiral action can be cast into the BF action:

$$I_{CC} = i\int_M (\Sigma^i \wedge F^i + \frac{\Lambda}{6}\Sigma^i \wedge \Sigma^i), \tag{2.6}$$

where $F^i = dA^i + \frac{1}{2}\epsilon^{ijk}A^j \wedge A^k$ is the curvature of the $SL(2,\mathbf{C})$ connection $A^i$ and $\Sigma^i$ is an $SL(2,\mathbf{C})$ Lie algebra-valued two-form. The new variables $(A^i, \Sigma^i)$ are related with the old ones $(e^\alpha, \omega^{\alpha\beta})$ by:

$$\begin{aligned} A^i &\equiv -iP^{(-)i}_{\alpha\beta}\omega^{\alpha\beta} = -\frac{1}{2}\epsilon^{ijk}\omega^{jk} - i\omega^{0i} && (2.7) \\ \Sigma^i &\equiv iP^{(-)i}_{\alpha\beta}e^\alpha \wedge e^\beta = \frac{1}{2}\epsilon^{ijk}e^j \wedge e^k + ie^0 \wedge e^k. && (2.8) \end{aligned}$$

Namely, $A^i$ is the anti-self-dual part of the spin-connection and $\Sigma^i$ is the anti-self-dual two-form constructed from the vierbein.

In order to rewrite the action in the canonical form, we have to perform a (3+1)-decomposition. We consider the spacetime $M$ to be homeomorphic to $\mathbf{R} \times M^{(3)}$ and use $t$ and $(x^a)$ as coordinates for $\mathbf{R}$ and the spatial hypersurface $M^{(3)}$ respectively. The result is

$$I_{CC} = i\int dt \int_{M^{(3)}} (\tilde{\pi}^{ai}\dot{A}^i_a + A^i_t G^i + \Sigma^i_{ta}\Phi^{ai}), \tag{2.9}$$

where $\tilde{\pi}^{ai} \equiv \frac{1}{2}\tilde{\epsilon}^{abc}\Sigma^i_{bc}$ plays the role of the momentum conjugate to $A^i_a$, and $G^i$ and $\Phi^{ai}$ are the first class constraints *in BF theory*:

$$\begin{aligned} G^i &= D_a\tilde{\pi}^{ai} \equiv \partial_a\tilde{\pi}^{ai} + \epsilon^{ijk}A^j_a\tilde{\pi}^{ak} && (2.10) \\ \Phi^{ai} &= \frac{1}{2}\tilde{\epsilon}^{abc}F^i_{bc} + \frac{\Lambda}{3}\tilde{\pi}^{ai}. && (2.11) \end{aligned}$$

In order to obtain the action for Ashtekar's formalism we further have to express $\Sigma^i_{ta}$ in terms of $\tilde{\pi}^{ai}$ by solving $\Sigma^i = iP^{(-)i}_{\alpha\beta}e^\alpha \wedge e^\beta$. Let us first fix the Lorentz boost part of $SL(2,\mathbf{C})$ gauge transformation by choosing the spatial gauge:

$$e^0 = -Ndt, \quad e^i = e^i_a(dx^a + N^a dt), \tag{2.12}$$

where $N$ and $N^a$ are the lapse function and the shift vector, and $e^i_a dx^a$ yields the induced dreibein on $M^{(3)}$. Plugging this expression into eq.(2.8) and performing a straightforward calculation, we find

$$\begin{aligned} \tilde{\pi}^{ai} &= \det(e^j_b)e^a_i \equiv \tilde{e}^{ai} && (2.13) \\ \Sigma^i_{ta} &= N^b \underset{\sim}{\epsilon}_{bac}\tilde{\pi}^{ci} + \frac{i}{2}\underset{\sim}{N}\epsilon^{ijk}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bj}\tilde{\pi}^{ck}, && (2.14) \end{aligned}$$

where $e^a_i$ denotes the co-dreibein $e^a_i e^i_b = \delta^a_b$ and $\underset{\sim}{N} \equiv N\{\det(\tilde{\pi}^{ai})\}^{-\frac{1}{2}}$ is an $SO(3,\mathbf{C})$ invariant scalar density of weight $-1$ which plays the role of a new Lagrange multiplier.

Here we should note that eq.(2.14) is covariant under $SO(3,\mathbf{C})$ gauge transformations. Eq.(2.14) therefore holds under arbitrary gauge, as long as the vierbein $e^\alpha$ can be written in the form of eq.(2.12) in a particular gauge. This is always possible if the dreibein $(e^i_a)$ is non-degenerate. A more detailed investigation[67] shows that this is possible if there is a spacelike hypersurface at each point in the spacetime.

Now the desired action is obtained by substituting eq.(2.14) into the BF action (2.9):

$$I_{CC} = i\int dt \int_{M^{(3)}} d^3x(\tilde{\pi}^{ai}\dot{A}^i_a + A^i_t G^i + N^a\mathcal{V}_a + \frac{i}{2}\underset{\sim}{N}\mathcal{S}). \tag{2.15}$$

From this action we see that there are three kinds of first class constraints in Ashtekar's formalism, namely, Gauss' law constraint (2.10), the vector constraint $\mathcal{V}_a$, and the scalar constraint $\mathcal{S}$.[6] The latter two constraints are of the following form

$$\begin{aligned} \mathcal{V}_a &\equiv \underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{ci}\Phi^{bi} = -\tilde{\pi}^{bi}F^i_{ab} & (2.16)\\ \mathcal{S} &\equiv \epsilon^{ijk}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{ai}\tilde{\pi}^{bj}\Phi^{ck} = \epsilon^{ijk}\tilde{\pi}^{ai}\tilde{\pi}^{bj}(F^k_{ab} + \frac{\Lambda}{3}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{ck}) & (2.17) \end{aligned}$$

Next we look into the constraint algebra. For this aim it is convenient to use smeared constraints:

$$\begin{aligned} G(\theta) &= i\int_{M^{(3)}} d^3x\theta^i G^i = -i\int_{M^{(3)}} d^3x D_a\theta^i\tilde{\pi}^{ai} \\ \mathcal{D}(\vec{N}) &= -i\int_{M^{(3)}} d^3x N^a(\mathcal{V}_a + A^i_a G^i) = i\int_{M^{(3)}} d^3x\tilde{\pi}^{ai}\mathcal{L}_{\vec{N}}A^i_a \\ \mathcal{S}(\underset{\sim}{N}) &= \frac{1}{2}\int_{M^{(3)}} d^3x\underset{\sim}{N}\mathcal{S}, \end{aligned} \tag{2.18}$$

where $\theta^i$ is an $SO(3,\mathbf{C})$ Lie algebra-valued scalar, $\vec{N} = (N^a)$ is a vector on $M^{(3)}$ and $\underset{\sim}{N}$ is a scalar density of weight $-1$. We will refer to $\mathcal{D}(\vec{N})$ as the 'diffeomorphism constraint'.

Under the Poisson bracket

$$\{A^i_a(x), \tilde{\pi}^{bj}(y)\}_{PB} = -i\delta^{ij}\delta^b_a\delta^3(x,y), \tag{2.19}$$

these smeared constraints generate gauge transformations in a broad sense. Gauss' law constraint and the diffeomorphism constraint respectively generate small $SL(2,\mathbf{C})$ gauge transformations and small spatial diffeomorphisms:

$$\begin{aligned} \{(A^i_a, \tilde{\pi}^{ai}), G(\theta)\}_{PB} &= (-D_a\theta^i, (\theta\times\tilde{\pi}^a)^i) \\ \{(A^i_a, \tilde{\pi}^{ai}), \mathcal{D}(\vec{N})\}_{PB} &= (\mathcal{L}_{\vec{N}}A^i_a, \mathcal{L}_{\vec{N}}\tilde{\pi}^{ai}), \end{aligned} \tag{2.20}$$

[6] We should note that $G^i$, $\mathcal{V}_a$ and $\mathcal{S}$ are respectively of density weight $+1$, $+1$ and $+2$, while they are not explicitly shown by the tilde.

where we have used the notation $(\theta \times \tilde{\pi}^a)^i \equiv \epsilon^{ijk}\theta^j\tilde{\pi}^{ak}$. The scalar constraint generates (infinitesimal) many-fingered time evolutions

$$\begin{aligned}
\{A_a^i, S(\underset{\sim}{N})\}_{PB} &= -i\underset{\sim}{N}\epsilon^{ijk}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bj}\Phi^{ck} + \frac{\Lambda}{3}\varphi_a^i \\
\{\tilde{\pi}^{ai}, S(\underset{\sim}{N})\}_{PB} &= -\tilde{\epsilon}^{abc}D_b\varphi_c^i,
\end{aligned} \tag{2.21}$$

where $\Phi^{ai}$ is the constraint (2.11) in BF theory and $\varphi_a^i \equiv -\frac{i}{2}\underset{\sim}{N}\epsilon^{ijk}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bj}\tilde{\pi}^{ck}$ is an $SO(3,\mathbf{C})$ Lie algebra-valued one-form.

Using these results we can now easily compute the Poisson brackets between smeared constraints. We find

$$\begin{aligned}
\{G(\theta'), G(\theta)\}_{PB} &= -G(\theta \times \theta') \\
\{G(\theta), \mathcal{D}(\vec{N})\}_{PB} &= -G(\mathcal{L}_{\vec{N}}\theta) \\
\{\mathcal{D}(\vec{M}), \mathcal{D}(\vec{N})\}_{PB} &= -\mathcal{D}([\vec{N}, \vec{M}]_{LB}) \\
\{S(\underset{\sim}{N}), G(\theta)\}_{PB} &= 0 \\
\{S(\underset{\sim}{M}), \mathcal{D}(\vec{N})\}_{PB} &= -S(\mathcal{L}_{\vec{N}}\underset{\sim}{M}) \\
\{S(\underset{\sim}{N}), S(\underset{\sim}{M})\}_{PB} &= -i\int_{M^{(3)}} d^3x K^a \mathcal{V}_a = \mathcal{D}(\vec{K}) + G(K^a A_a),
\end{aligned} \tag{2.22}$$

where $[\vec{N}, \vec{M}]_{LB} = (N^b\partial_b M^a - M^b\partial_b N^a)$ stands for the Lie bracket and $K^a \equiv (\underset{\sim}{N}\partial_b\underset{\sim}{M} - \underset{\sim}{M}\partial_b\underset{\sim}{N})\tilde{\pi}^{bj}\tilde{\pi}^{aj}$ is a vector on $M^{(3)}$. The first three equations are the manifestation of the fact that $SL(2,\mathbf{C})$ gauge transformations and spatial diffeomorphisms form a semi-direct product group. The fourth and the fifth equations mean that the scalar constraint $S$ transforms trivially under $SL(2,\mathbf{C})$ gauge transformations and as a scalar density of weight $+2$ under spatial diffeomorphisms. It is only the last equation which is somewhat difficult to derive. It involves non-trivial structure functionals and is expected to cause one of the most formidable obstacles to quantum Ashtekar's formalism.

Next we will explain the reality conditions[4][5]. In demonstrating the classical equivalence of the complex chiral action with the Einstein-Hilbert action, it has been indispensable that the vierbein $e^\alpha$ and the spin-connection $\omega^{\alpha\beta}$ are real-valued. This tells us that, in order to extract full information on general relativity from Ashtekar's formalism, we have to impose some particular conditions called "reality conditions" on the canonical variables $(A_a^i, \tilde{\pi}^{ai})$. According to the treatment of the $SL(2,\mathbf{C})$ gauge degrees of freedom, there are two alternative ways of imposing reality conditions:

1. Fix the Lorentz boost degrees of freedom by choosing the spatial gauge (2.12). As a consequence the gauge group reduces to $SU(2)$. The classical reality conditions in this case take the following form (the bar denotes the complex conjugation)

$$\overline{\tilde{\pi}^{ai}} = \tilde{\pi}^{ai}, \quad A_a^i + \overline{A_a^i} = -\epsilon^{ijk}\omega_a^{jk}(e), \tag{2.23}$$

where ${}^{(3)}\omega^{ij} \equiv \omega_a^{ij}(e)dx^a$ is the spin-connection on $M^{(3)}$ which satisfies the torsion-free condition w.r.t. the dreibein ${}^{(3)}e^i \equiv e_a^i dx^a$

$$^{(3)}d^{(3)}e^i +^{(3)} \omega^{ij} \wedge^{(3)} e^j = 0.$$

2. Do not fix the gauge and keep the full $SL(2,\mathbf{C})$ gauge degrees of freedom. The reality conditions in this case are merely pullbacks of eqs.(2.7)(2.8) to $M^{(3)}$, namely

$$\begin{aligned} A_a^i &= -\frac{1}{2}\epsilon^{ijk}\omega_a^{jk} - i\omega_a^{0i} \\ \tilde{\pi}^{ai} &= \frac{1}{2}\tilde{\epsilon}^{abc}(\epsilon^{ijk}e_b^j e_c^k + 2ie_b^0 e_c^i) \end{aligned} \tag{2.24}$$

must hold for real $e_a^\alpha$ and real $\omega_a^{\alpha\beta}$.

We cannot say which is absolutely better than the other. It is clever to make a relevant choice according to each problem to work with.

Before concluding this subsection we will comment on the Euclidean case. The Euclidean counterpart of the Lorentz group $SO(3,1)^\uparrow \cong SL(2,\mathbf{C})/\{\pm 1\}$ is the four-dimensional rotation group $SO(4) \cong (SU(2) \times SU(2))/\{\pm 1\}$. As a result the anti-self-dual part of a rank two tensor takes a real value, for example,

$$\begin{aligned} A_E^i &= -\frac{1}{2}\epsilon^{ijk}\omega^{jk} + \omega^{0i}, \\ \Sigma_E^i &= -\frac{1}{2}\epsilon^{ijk}e^j \wedge e^k + e^0 \wedge e^i. \end{aligned} \tag{2.25}$$

Using this the Euclidean counterpart of the complex chiral action are given by

$$\begin{aligned} (I_{CC})^E &= \int_M (\Sigma_E^i \wedge F_E^i - \frac{\Lambda}{6}\Sigma_E^i \wedge \Sigma_E^i) \\ &= \frac{1}{2}(I_{EP})^E - \int_M e^\alpha \wedge e^\beta \wedge R^{\alpha\beta}, \end{aligned} \tag{2.26}$$

where $(I_{EP})^E$ is the Euclidean version of the Einstein-Palatini action.

If we deal with this action in the first order formalism, namely if we regard $A_E^i$ and $e^\alpha$ as independent variables, $(I_{CC})^E$ does not coincide with the Euclidean Einstein-Hilbert action $(I_{EH})^E$ *even on-shell.* This is because information on the self-dual part of the torsion-free condition is lost. However, if we impose in advance the torsion-free condition $de^\alpha + \omega^{\alpha\beta} \wedge e^\beta = 0$, $(I_{CC})^E$ coincides with $(I_{EH})^E$ owing to the first Bianchi identity. In other words, in order to reproduce the result of Euclidean general relativity from that of Euclidean Ashtekar's formalism, we have to impose the torsion-free condition by hand. In the canonical treatment, this amounts to the real canonical transformation from $\omega_a^{0i}$ to $(A_E^i)_a$ which is generated by $\frac{1}{2}\int_{M^{(3)}} \epsilon^{ijk}\omega_a^{ij}\tilde{\pi}^{ak}$.

## 2.2 Quantization

Next we consider the quantization of Ashtekar's formalism. There are two well-known quantization methods for systems involving first order constraints such as Ashtekar's formalism, namely the reduced phase space quantization and the Dirac quantization.

In the reduced phase space quantization[7], we first construct the reduced phase space which consists only of the physical degrees of freedom and then perform the canonical quantization on this reduced phase space. The reduced phase space is constructed by first solving the first class constraints completely and then removing all the gauge degrees of freedom which are generated by the first class constraints. In general relativity this amounts to finding all the diffeomorphism equivalence classes of the solutions to the Einstein equation. It is in practice impossible to carry out this. So we usually adopt Dirac's quantization procedure[33] when we canonically quantize gravity.

In Dirac's quantization, we first promote canonical variables in the unconstrained phase space to quantum operators by replacing $i$ times the Poisson brackets with the corresponding quantum commutation relations. The first class constraints become the operator equations imposed on physical wave functions. When the first class constraints are at most linear in momenta, this prescription is known to yield the same result as that in the reduced phase space method, up to a minor subtlety (see Appendix D of [5]). Because the scalar constraint is at least quadratic in momenta, we expect that these two methods lead to different results for quantum Ashtekar's formalism.

Let us now perform Dirac's quantization. The canonical variables $(A_a^i, \tilde{\pi}^{ai})$ are promoted to the fundamental quantum operators $(\hat{A}_a^i, \hat{\tilde{\pi}}^{ai})$ subject to the commutation relations

$$[\hat{A}_a^i(x), \hat{\tilde{\pi}}^{bj}(y)] = \delta^{ij}\delta_a^b\delta^3(x,y). \tag{2.27}$$

In this paper we will take the connection representation (or holomorphic representation) in which the $SL(2,\mathbf{C})$ connection $\hat{A}_a^i$ is diagonalized. Wave functions are thus given by holomorphic functionals $\Psi[A] = < A|\Psi >$ of the $SL(2,\mathbf{C})$ connection. The action of $\hat{A}_a^i$ and $\hat{\tilde{\pi}}^{ai}$ on these wavefunctions are respectively represented by multiplication by $A_a^i$ and by functional derivative w.r.t. $A_a^i$:

$$\begin{aligned} \hat{A}_a^i(x)\Psi[A] &= A_a^i \cdot \Psi[A] \\ \hat{\tilde{\pi}}^{ai}(x)\Psi[A] &= -\frac{\delta}{\delta A_a^i(x)}\Psi[A]. \end{aligned} \tag{2.28}$$

Next we impose the constraint equations:

$$\hat{G}^i\Psi[A] = 0 \, . \tag{2.29}$$

[7]For more detailed explanations of the reduced phase space quantization, we refer the reader to [88] or to Appendix D of [5].

$$\hat{\mathcal{D}}_a\Psi[A] = 0 \tag{2.30}$$
$$\hat{\mathcal{S}}\Psi[A] = 0. \tag{2.31}$$

Gauss' law constraint (2.29) and the diffeomorphism constraint (2.30) respectively require the physical wavefunctions to be invariant under small $SL(2,\mathbf{C})$ gauge transformations and small spatial diffeomorphisms[8]

$$\Psi[A^g] = \Psi[A] \tag{2.32}$$
$$\Psi[\phi^*A] = \Psi[A]. \tag{2.33}$$

The scalar constraint (2.31) yields Ashtekar's version of the Wheeler-De Witt equation. Because this equation involves at least second order functional derivative, its rigorous treatment requires some regularization. In association with this there is an issue on the operator ordering in the constraints. At the formal level there are two plausible candidates:

1. Putting $\hat{\tilde{\pi}}^{ai}$ to the right [52][19]. A virtue of this ordering is that $\hat{G}^i$ and $\hat{\mathcal{D}}_a$ correctly generate $SL(2,\mathbf{C})$ gauge transformations and spatial diffeomorphisms. In this ordering, however, commutator $[\hat{\mathcal{S}},\hat{\mathcal{S}}]$ fails to vanish weakly because the structure functionals appear to the right of the constraints.

2. Putting $\hat{\tilde{\pi}}^{ai}$ to the left [4][55]. This ordering has a merit that the commutator algebra of the constraints $\hat{G}^i$, $\hat{\mathcal{V}}_a$ (not $\hat{\mathcal{D}}_a$) and $\hat{\mathcal{S}}$ formally closes. A demerit of this ordering is that $\hat{\mathcal{D}}_a$ (or $\hat{\mathcal{V}}_a$) does not generate diffeomorphisms correctly.[9]

We should note that the above discussion is formal in the sense that we deal with non-regularized constraints.

In order to extract physical information from the physical wave functions, we also need to construct physical observables and to specify a physical inner product. Physical observables are self-adjoint operators which commutes with the constraints at least weakly. Finding the physical inner product has been a longstanding problem in quantum Ashtekar's formalism because it is intimately related with the reality conditions. As we have seen before, Ashtekar's canonical variables have to satisfy the nontrivial reality conditions (2.23) or (2.24), which have to be implemented in the quantum theory as nontrivial adjointness conditions. Because the adjointness conditions can be attributed to the problem of inner product, the issue of reality conditions has not been taken so seriously as yet. Quite recently, however, a promising candidate for the physical inner product has been proposed[84]. If it turns out that this inner product is genuinely physical and useful, we can say that the program of quantizing Ashtekar's formalism has made great progress.

[8] $A^g_a = (A^g)^i_a J_i = gA^i_a J_i g^{-1} + g\partial_a g^{-1}$ and $\phi^*A^i_a(x) = \partial_a\phi^b(x)A^i_b(\phi(x))$ respectively denote the image of the connection $A^i_a(x)$ under the small $SL(2,\mathbf{C})$ gauge transformation $g(x)$ and the pullback of $A^i_a(x)$ by the spatial diffeomorphism $\phi : M^{(3)} \to M^{(3)}$.

[9] In the loop representation, however, $\hat{\mathcal{V}}_a$ generates diffeomorphism correctly by virtue of the relation (7.3). For a detailed analysis, see e.g. ref.[21].

# 3 Spin network states

Recently there have been a large number of remarkable developments in the program of applying spin network states [71] to quantum gravity in terms of Ashtekar's new variables [14] [8] [9] [10] [78] [79]. As one of the promising approaches to canonical quantum gravity, this program is expected to make further progress in the near future. Unfortunately, however, most of the results are given in the form of mathematical theorems and propositions which are unfamiliar to most of the physicists. This makes it obscure that these notions have originary been brought into quantum general relativity in order to concretize the simple physical ideas. In this section we make a maximal attempt to explain some of these results on the spin network states in the language of physics and to clarify the underlying ideas. In particular, we provide explicit formulae for two types of measures defined on spin network states[8][9]. Because we are only interested in its application to quantum gravity, we will restrict our attention to the case where the gauge group is $SU(2)$ or $SL(2,\mathbf{C})$. Our discussion is, however, applicable to any compact gauge group $G$ and its complexification $G^{\mathbf{C}}$ if some necessary modifications are made. For simplicity we will only deal with the unitary representations of $SU(2)$.

## 3.1 Backgrounds and the definition

As is seen in §2, we can regard Ashtekar's formalism as a kind of $SL(2,\mathbf{C})$ gauge theory. A common feature of gauge theories is that, by virtue of Gauss' law constraint (2.29), the wavefunction in the connection representation is given by a functional of the connection which is invariant under small gauge transformations. We will henceforth restrict our attention to the wavefunctions which are invariant under all the gauge transformations including large gauge transformations.

In order to construct gauge invariant functionals of the connection $A^i$, it is convenient to use the parallel propagator $h_\alpha[0,1]$ of $A^i$ evaluated along the curve $\alpha : [0,1] \to M^{(3)}$

$$h_\alpha[0,1] = \mathcal{P}\exp\{\int_0^1 ds\dot{\alpha}^a(s)A^i_a(\alpha(s))J_i\}, \tag{3.1}$$

where $\mathcal{P}$ stands for the path ordering with smaller $s$ to the left. In spite of the fact that the gauge transformation of the connection is inhomogenous, the parallel propagator transforms homogeneously under the gauge transformation $g(x)$:

$$\begin{aligned} A_a(x) &\to g(x)A_a g^{-1}(x) - \partial_a g(x) g^{-1}(x) \\ h_\alpha[0,1] &\to g(\alpha(0))h_\alpha[0,1]g^{-1}(\alpha(1)), \end{aligned} \tag{3.2}$$

where we have set $A_a \equiv A^i_a J_i$.

Thus it is considerably straightforward to construct gauge invariant functionals from these parallel propagators $h_\alpha[0,1]$. The simplest example is the Wilson loop along a loop

$\gamma : [0,1] \to M^{(3)}$ $(\gamma(0) = \gamma(1))$

$$W(\gamma, \pi) \equiv \mathrm{Tr}\pi(h_\gamma[0,1]). \tag{3.3}$$

Its gauge invariance follows immediately from eq.(3.2). Considerably many early works on Ashtekar's formalism were based on the use of these Wilson loops. In fact the "reconstruction theorem" proposed by Giles[43] guarantees that, in the case where the gauge group is compact, Wilson loops suffice to extract all the gauge invariant information on the connection.[10]

However, the framework based on the Wilson loop has several drawbacks in order to provide efficient tools. Among them, particularly serious ones are the following:

i) (Overcompleteness) it is known that the set of Wilson loops form an overcomplete basis in $\mathcal{A}/\mathcal{G}$. In order to extract necessary information on the connection, we have to impose algebraic constraints such as Mandelstam identities[64] on Wilson loops.

ii) (Non-locality) The action of local operators on Wilson loops frequently induces the change of global properties of the loops on which the Wilson loops are defined. For example, it often happens that orientation of some loops has to be inverted in order to make the resulting composite loop to be consistent.

These drawbacks become the sources of extra intricacy when we carry out calculations by means of Wilson loops. In this sense the Wilson loops are not so suitable for practical calculations. Naturally a question arises as to whether or not there exist more convenient tools composed of parallel propagators which give rise to complete (but not overcomplete) basis in $\mathcal{A}/\mathcal{G}$ and on which the action of local operators is described by purely local manipulation. The answer is yes. Spin network states satisfy these requirements.

A spin network state is defined on a 'graph'. A graph $\Gamma = (\{e\}, \{v\})$ consists of a set $\{v\}$ of 'vertices' and a set $\{e\}$ of 'edges'. Each vertex $v \in M^{(3)}$ is a point on $M^{(3)}$ and each edge $e : [0,1] \to M^{(3)}$ is a curve on $M^{(3)}$ which connects two vertices (or a loop on $M^{(3)}$ which is based at a vertex). An example of a graph is shown in figure1.

In order to construct a unique spin network state, we have only to determine a 'colored graph' $(\Gamma, \{\pi_e\}, \{i_v\})$ as follows: i) evaluate along each edge $e$ a parallel propagator of the connection $A_a dx^a$ in a representation $\pi_e$ of $SL(2,\mathbf{C})$; and ii) equip each vertex $v$ with an intertwining operator[11]

$$i_v : \left(\bigotimes_{e(1)=v} \pi_e^*\right) \otimes \left(\bigotimes_{e(0)=v} \pi_e\right) \longrightarrow 1 \text{ (trivial)}, \tag{3.4}$$

[10]In the $SL(2,\mathbf{C})$ case it was shown that the Wilson loop is sufficient to separate all the separable points of the space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes of connections [7].In other words the Wilson loop misses some information on the null rotation part of the holonomies (parallel propagators along the loops).

[11]We can regard each intertwining operator as an invariant tensor in the corresponding tensor product representation.

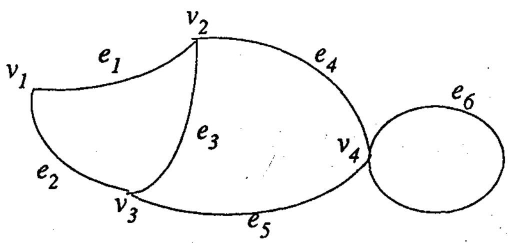



Figure 1: An example of a graph.

where $\pi^*$ denotes the conjugate representation of $\pi$ defined by $\pi^*(g) \equiv^T \pi(g^{-1})$ ($^T$ represents taking the transpose). The corresponding spin network state is defined as

$$< A|\Gamma, \{\pi_e\}, \{i_v\} > \equiv \prod_{v \in \{v\}} i_v \cdot \left( \bigotimes_{e \in \{e\}} \pi_e(h_e[0,1]) \right). \tag{3.5}$$

Gauge invariance of the spin network states thus defined is manifest from the transformation law(3.2) of the parallel propagators and from the defining property (3.4) of the intertwining operators.

It is obvious that the spin network states form a complete basis because the Wilson loops are regarded as particular spin network states. Moreover, it turns out that they are not overcomplete, namely, we do not have to impose Mandelstam identities. We will see in the next subsections that they actually yield an orthonormal basis w.r.t. some inner products. Another merit of spin network states is that the action of local operators is described in terms of purely local operations on the graph. Thus we can concentrate only on the local structures in the practical calculation. While we do not show this generically, an example is demonstrated in the next section.

Here we should explain the consistency of the spin network states. In practical calculation it is sometimes convenient to consider the graph $\Gamma$ as a subgraph $\Gamma < \Gamma'$ of a larger graph $\Gamma'$, where $\Gamma < \Gamma'$ means that all the edges and the vertices in $\Gamma$ is contained in $\Gamma'$, namely, $\{e\}, \{v\} \in \Gamma'$. The spin network state $< A|\Gamma, \{\pi_e\}, \{i_v\} >$ is then regarded as the spin network state $< A|\Gamma', \{\pi'_{e'}\}, \{i'_{v'}\} >$ which is determined as follows.

A graph $\Gamma'$ which is larger than $\Gamma$ is obtained from $\Gamma$ by a sequence of the following four moves[14]:

i) adding a vertex $v' \notin \Gamma$;

ii) adding an edge $e' \notin \Gamma$;
iii) subdividing an edge $\Gamma \in e = e_1 \cdot e_2$ with $e_1(1) = e_2(0) = v' \notin \Gamma$; and
iv) reverting the orientation of an edge $\Gamma \ni e \to e' = e^{-1} \in \Gamma'$.

In order to obtain the spin network state $< A|\Gamma', \{\pi'_{e'}\}, \{i'_{v'}\} >$ which is equal to $< A|\Gamma, \{\pi_e\}, \{i_v\} >$, we have to associate with each move the following operation:
for i) we equip the vertex $v'$ with the unity $i'_{v'} = 1$;
for ii) we provide the edge $e'$ with the trivial representation $\pi'_{e'}(h_{e'}[0,1]) = 1$;
for iii) we fix the representation and the intertwining operator as $\pi'_{e_1} = \pi'_{e_2} = \pi_e$ and as $i'_{v'_*} = \delta^J_I$ respectively, where $\delta^J_I$ denotes the Cronecker delta in the representation $\pi_e$; and for iv) the corresponding representation for $e'$ is given by the conjugate representation $\pi'_{e'} = (\pi_e)^*$.

Action of local operators on spin network states is necessarily independent of the choice of the graph, because the operators do not perceive the difference of the graphs on which the identical spin network state is defined. However this consistency gives rise to a criterion for defining well-defined measures on $\overline{\mathcal{A}/\mathcal{G}}$,[12] namely, a well-defined measure should be invariant under the above four moves.

Because we only deal with the case where the gauge group is $SL(2, \mathrm{C})$ or $SU(2)$, some simplifications are possible. In particular, because the conjugate representation $\pi^*$ in $SL(2, \mathrm{C})$ is unitary equivalent to the original representation $\pi$,[13] we do not have to worry about the orientation of the edges. We can therefore neglect the consistency under the fourth move.

## 3.2 The induced Haar measure on $SU(2)$ gauge theories

As is seen in §2, Ashtekar's formalism for Euclidean gravity is embedded in the $SU(2)$ gauge theory. Besides, an Ashtekar-like formulation exists also in Lorentzian gravity which deals with the $SU(2)$ connection [15]. For these formulations the induced Haar measure is used for a natural measure of the theory.

The induced Haar measure on $\overline{\mathcal{A}/\mathcal{G}}$ is defined as follows. First we define a 'generalized connection' $A_\Gamma$ on the graph $\Gamma \subset M^{(3)}$ as a map

$$A_\Gamma : \{e\} = (e_1, \ldots, e_{n_\Gamma}) \to (SU(2))^{n_\Gamma}.$$

The space $\overline{\mathcal{A}_\Gamma}$ is the set of all the generalized connections on $\Gamma$. Of course the usual connection $A_a$ gives rise to a generalized connection by the relation $A_\Gamma(e) = h_e[0,1]$. The

[12] The space $\overline{\mathcal{A}/\mathcal{G}}$ of equivalence classes of generalized connections modulo generalized gauge transformations is a completion of the space $\mathcal{A}/\mathcal{G}$. Roughly speaking it is obtained by allowing the parallel propagators $h_e[0,1]$ which cannot be obtained by integrating the well-defined connection $A_a$ on $M^{(3)}$. The cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$ are obtained from the superposition of spin network states on a graph $\Gamma$ in the limit of making $\Gamma$ larger and larger[8].

[13] In fact if we use the convention in which $J_2$ in each representation is a real skew-symmetric matrix, then the representation $\pi$ and its conjugate $\pi^*$ are related by the unitary matrix $\pi(\exp(\pi J_2))$.

space $\overline{(\mathcal{A}/\mathcal{G})}_\Gamma$ is the quotient space of $\overline{\mathcal{A}}_\Gamma$ modulo generalized gauge transformations:

$$\begin{aligned} &g : \{v\} = (v_1, \ldots, v_{N_\Gamma}) \to (SU(2))^{N_\Gamma}, \\ &A_\Gamma(e) \to g(e(0))A_\Gamma(e)g^{-1}(e(1)). \end{aligned}$$

Next we consider a function $f_\Gamma(A_\Gamma)$ on $\overline{(\mathcal{A}/\mathcal{G})}_\Gamma$. We will assume that the pull-back $(p_\Gamma)^* f_\Gamma$ of $f_\Gamma$ becomes a cylindrical function on $\overline{\mathcal{A}/\mathcal{G}}$, where $p_\Gamma$ denotes the projection map

$$p_\Gamma : \overline{\mathcal{A}} \to \overline{\mathcal{A}}_\Gamma.$$

It is obvious that $(p_\Gamma)^* f_\Gamma$ can be expressed by a linear combination of spin network states defined on $\Gamma$. For $(p_\Gamma)^* f_\Gamma$ the induced Haar measure $d\mu_H$ on $\overline{\mathcal{A}/\mathcal{G}}$ is defined as

$$\int d\mu_H(A)(p_\Gamma)^* f_\Gamma(A) \equiv \int d\mu_\Gamma(A_\Gamma) f_\Gamma(A_\Gamma) \equiv \int_{\overline{\mathcal{A}}_\Gamma} \left( \prod_{e \in \{e\}} d\mu(A_\Gamma(e)) \right) f_\Gamma(A_\Gamma), \tag{3.6}$$

where $d\mu$ denotes the Haar measure on $SU(2)$. The consistency condition of this measure is provided by the following equation

$$\int d\mu_\Gamma(A_\Gamma) f_\Gamma(A_\Gamma) = \int d\mu_{\Gamma'}(A_{\Gamma'})(p_{\Gamma\Gamma'})^* f_\Gamma(A_{\Gamma'}), \tag{3.7}$$

where $\Gamma' > \Gamma$, and

$$p_{\Gamma\Gamma'} : \overline{\mathcal{A}}_{\Gamma'} \to \overline{\mathcal{A}}_\Gamma$$

is the projection from $\overline{\mathcal{A}}_{\Gamma'}$ onto $\overline{\mathcal{A}}_\Gamma$. It was shown that this consistency condition indeed holds [8]. We will presently see this by a concrete calculation. Before doing so we provide the explicit formula for the induced Haar measure on the spin network states.

Let us first calculate the inner product between two spin network states $< A|\Gamma, \{\pi_{p_e}\}, \{i_v\} >$ and $< A|\Gamma, \{\pi_{q_e}\}, \{i'_v\} >$ which are defined on the same graph $\Gamma$:[14]

$$\begin{aligned} &< \Gamma, \{\pi_{p_e}\}, \{i_v\} | \Gamma, \{\pi_{q_e}\}, \{i'_v\} >_{|d\mu_H} \\ &\equiv \int d\mu_H(A) \overline{< A|\Gamma, \{\pi_{p_e}\}, \{i_v\} >} < A|\Gamma, \{\pi_{q_e}\}, \{i'_v\} > \\ &= \overline{(\prod_v i_v)} \cdot (\prod_v i'_v) \cdot \prod_e \left( \int d\mu(A_\Gamma(e)) \overline{\pi_{p_e}(A_\Gamma(e))_{I_e}{}^{J_e}} \pi_{q_e}(A_\Gamma(e))_{I'_e}{}^{J'_e} \right). \end{aligned} \tag{3.8}$$

In obtaining the last expression we have used eq.(3.5). By plugging eq.(B.3) into this equation we find

$$\begin{aligned} < \Gamma, \{\pi_{p_e}\}, \{i_v\} | \Gamma, \{\pi_{q_e}\}, \{i'_v\} >_{|d\mu_H} &= \overline{(\prod_v i_v)} \cdot (\prod_v i'_v) \cdot \left( \prod_e \frac{\delta_{p_e, q_e}}{p_e + 1} \delta_{I'_e}^{I_e} \delta_{J_e}^{J'_e} \right) \\ &= \prod_e \frac{1}{p_e + 1} \delta_{p_e, q_e} \prod_v < i_v | i'_v >, \end{aligned} \tag{3.9}$$

[14] As in Appendix B $\pi_p$ denotes the spin-$\frac{p}{2}$ representation of $SL(2, \mathbf{C})$.

where we have used the symbol $< i_v|i'_v >$ to mean the complete contraction of two intertwining operators

$$< i_v|i'_v >\equiv \overline{i_v}\cdot i'_v\cdot\left(\prod_{e:e(0)=v}\delta^{I_e}_{I'_e}\prod_{e:e(1)=v}\delta^{J'_e}_{J_e}\right). \tag{3.10}$$

We are now in a position to prove the consistency (3.7). Consistency under the moves i) and ii) are trivial and thus we have only to show consistency under the move iii). This immediately follows from the first equality in eq.(3.9) if we substitute expressions like $\pi_{p_e}(A_\Gamma(e))_I{}^J = \pi_{p_e}(A_\Gamma(e_1))_I{}^K\pi_{p_e}(A_\Gamma(e_2))_K{}^J$ into the induced Haar measure defined on $\Gamma'$. An only possibility of the change stems from the subdivided edge $e = e_1\cdot e_2$ which yields the following contribution to the measure

$$\frac{\delta_{p_e,q_e}}{p_e+1}\delta^I_{I'}\delta^{K'}_K\times\frac{\delta_{p_e,q_e}}{p_e+1}\delta^K_{K'}\delta^{J'}_J.$$

By taking care not to contract two $(\delta_{p_e,q_e})$'s, we get the same result as that from the measure defined on $\Gamma$:

$$\frac{\delta_{p_e,q_e}}{p_e+1}\delta^I_{I'}\delta^{J'}_J.$$

Thus we have demonstrated concretely that the consistency holds for the induced Haar measure on $SU(2)$ gauge theories.

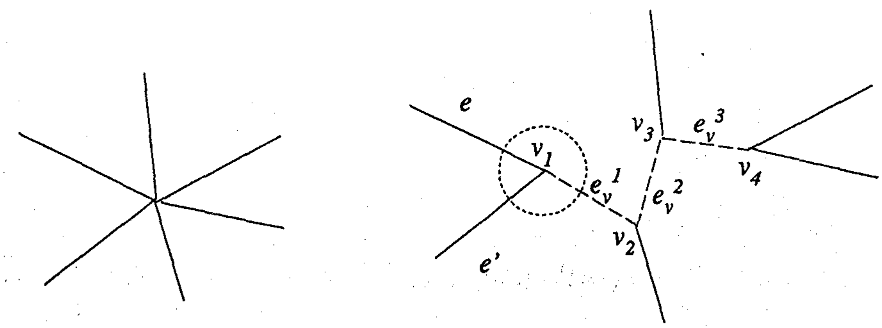
Eq.(3.9) also tells us that two spin network states are orthogonal with each other if the representations $\pi_{p_e}$ and $\pi_{q_e}$ do not coincide on one or more edges $e\in\Gamma$. As a corollary of this result and the consistency it follows that the inner product of two spin network states $< A|\Gamma,\{\pi_e\},\{i_v\}>$ and $< A|\Gamma',\{\pi'_{e'}\},\{i'_{v'}\}>$ vanishes if the former is not equivalent to a spin network state on $\Gamma'$ and also if the latter is not equivalent to that on $\Gamma$.

In order to show completely that the spin network states form an orthonormal basis, the contraction $< i_v|i'_v >$ of intertwining operators remains to be calculated. For this goal we first have to provide a complete set of intertwining operators. A convenient choice is given as follows.

We know that the intertwining operator for a trivalent vertex is, up to an overall constant, uniquely fixed by the Clebsch-Gordan coefficient

$$< p,J;q,K|r,L >: \pi_p\otimes\pi_q\otimes\pi^*_r\to 1 \text{ (trivial)}.$$

Thus a complete set of intertwining operators for an $n$-valent vertex $v$ is given by decomposing the vertex into $n-2$ trivalent vertices $(v_1,v_2,\ldots,v_{n-2})$ and by assigning to the resulting $n-3$ virtual edges $(e^1_V,\ldots,e^{n-3}_V)$ $n-3$ irreducible representations $(\pi_{r_1},\ldots,\pi_{r_{n-3}})$ which satisfy triangular inequations [56][79]. In this way each basic intertwining operator for an $n$-valent vertex is specified by an non-negative integer-valued $n-3$ dimensional vector $(r_1,\ldots,r_{n-3})$. The way of decomposing an $n$-valent vertex into $n-2$ trivalent




(a) (b)

Figure 2: An example of decomposing a six-valent vertex (a) into four trivalent vertices (b). Dashed lines denote the virtual edges.

vertices is not unique and thus we have to choose one. We can obtain the complete set starting from an arbitrary choice. An example for the $n = 6$ case is shown in figure2.

Let us now show that any such choice of the basis for intertwining operators yields an orthonormal basis for $< i_v|i'_v >$ (eq.(3.10)). Because there exist at least one trivalent vertex connecting two true edges, like the one which is enclosed by the circle in figure2, we will start from one such edge, say $v_1$. We assume that the two true edges $e$ and $e'$ emanating from $v_1$ are respectively equipped with the spin-$\frac{p}{2}$ and the spin-$\frac{q}{2}$ representations. The two non-negative integer-valued vectors which specify $i_v$ and $i'_v$ are respectively supposed to have components $r_1$ and $r'_1$ which correspond to the virtual edge $e^1_V$ with an end point $v_1$. By paying attention only to the vertex $v_1$, we find

$$\begin{aligned} < i_v|i'_v > &\propto \sum_{J,J'=1}^{p+1} \sum_{K,K'=1}^{q+1} \overline{< p, J; q, K|r_1, M >} < p, J'; q, K'|r'_1, M' > \delta^{J'}_J \delta^{K'}_K \\ &= \delta_{r_1, r'_1} \delta^M_{M'}. \end{aligned} \tag{3.11}$$

In deriving the last equation we have used the fact that the Clebsch-Gordan Coefficients $< p, J; q, K|r, M >$ yield the unitary transformation:

$$\pi_p \otimes \pi_q \to \pi_{|p-q|} \oplus \pi_{|p-q|+2} \oplus \cdots \oplus \pi_{p+q}.$$

Eq.(3.11) shows that the trivalent vertex $v_1$ turns the Cronecker delta in the tensor product representation $\pi_p \otimes \pi_q$ into that in the irreducible representation $\pi_{r_1}$ $(= \pi_{r'_1})$. This holds also at the other trivalent vertices. Thus we can successively perform the calculation like eq.(3.11). The final result is

$$< i_v = (r_1, \ldots, r_{n-3})|i'_v = (r'_1, \ldots, r'_{n-3}) >= C(v) \prod_{k=1}^{n-3} \delta_{r_k, r'_k}, \tag{3.12}$$

where $C(v)$ is an overall constant factor which depends on the way of decomposing the $n$-valent vertex $v$.

Eq.(3.9) together with eq.(3.12) shows that the spin network states form an orthonormal basis w.r.t. the induced Haar measure (3.6) if they are multiplied by appropriate overall constant factors.

A remarkable fact is that this measure is invariant under spatial diffeomorphisms. This is obvious because it does not depend on any background structure. The induced Haar measure is therefore expected to be a 'prototype' whose appropriate modification yields the physical measure in canonical quantum gravity.

## 3.3 The coherent state transform and the induced heat-kernel measure

In the last subsection we have derived the explicit formula for the inner product of the spin network states w.r.t. the induced Haar measure on $SU(2)$ gauge theories. However, because the gauge group of Ashtekar's formulation for Lorentzian gravity is $SL(2,\mathbf{C})$ which is noncompact, we cannot use the induced Haar measure as it is. Nevertheless we can define some types of measures on $SL(2,\mathbf{C})$ gauge theories, one of which is the induced heat-kernel measure[9]. This measure is closely related to the 'coherent state transform' which maps a functional of the $SU(2)$ connection to a holomorphic functional of the $SL(2,\mathbf{C})$ connection. We expect that, using this kind of transform, many of the machineries obtained in the framework of $SU(2)$ spin network states can be brought into that of $SL(2,\mathbf{C})$ ones. In this subsection we distinguish the objects in $SL(2,\mathbf{C})$ gauge theories from those in $SU(2)$ by attaching the superscript $^{\mathbf{C}}$ to the former.

In order to define the coherent state transform and the induced heat-kernel measure we first furnish each edge $e \in \Gamma$ with a length function $l(e) > 0$. The coherent state transform

$$C_t^l : L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H) \to L^2(\overline{\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}}}, d\nu_t^l) \cap \mathcal{H}(\overline{\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}}}),$$

from the space of square integrable functionals w.r.t. $d\mu_H$ on $\overline{\mathcal{A}/\mathcal{G}}$ to the space of holomorphic functionals which are square integrable w.r.t. the induced heat-kernel measure $d\nu_t^l$ on $\overline{\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}}}$ is given by a consistent family $\{C^l_{t,\Gamma}\}_{|\Gamma}$ of transforms

$$C^l_{t,\Gamma} : L^2(\overline{(\mathcal{A}/\mathcal{G})}_\Gamma, d\mu_\Gamma) \to L^2(\overline{(\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}})}_\Gamma, d\nu^l_{t,\Gamma}) \cap \mathcal{H}(\overline{(\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}})}_\Gamma).$$

For a function $f_\Gamma(A_\Gamma)$ on $\overline{(\mathcal{A}/\mathcal{G})}_\Gamma$ the transform $C^l_{t,\Gamma}$ is defined as

$$C^l_{t,\Gamma}[f_\Gamma](A^{\mathbf{C}}_\Gamma) \equiv \int_{\overline{\mathcal{A}_\Gamma}} \left( \prod_e d\mu(A_\Gamma(e)) \rho_{l(e)t}(A_\Gamma(e)^{-1} A^{\mathbf{C}}_\Gamma(e)) \right) f_\Gamma(A_\Gamma(e)), \tag{3.13}$$

where, as in Appendix B, $\rho_t(g)$ is the heat-kernel for the Casimir operator $\Delta = J_i J_i$ of $SU(2)$. For $(p_\Gamma)^* f_\Gamma(A) \in L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$ the coherent state transform is given by

$$C_t^l[(p_\Gamma)^* f_\Gamma](A^{\mathbf{C}}) = C_{t,\Gamma}^l[f_\Gamma](p_\Gamma(A^{\mathbf{C}})). \tag{3.14}$$

In particular, by using eq.(B.9), we find

$$C_t^l[< A|\Gamma, \{\pi_{p_e}\}, \{i_v\} >](A^{\mathbf{C}}) = \exp(-\frac{t}{2}\sum_e l(e)\frac{p_e(p_e+2)}{4}) < A^{\mathbf{C}}|\Gamma, \{\pi_{p_e}\}, \{i_v\} > . \tag{3.15}$$

From this equation we immediately see that $C_t^l$ satisfies the consistency condition iff $l(e \cdot e') = l(e) + l(e')$ and $l(e^{-1}) = l(e)$ hold.

The induced heat-kernel measure $d\nu_t^l$ for the functional $(p_\Gamma)^* f_\Gamma$ on $\overline{(\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}})}$ is defined as:

$$\begin{aligned} \int_{\overline{\mathcal{A}^{\mathbf{C}}}} d\nu_t^l(A^{\mathbf{C}})(p_\Gamma)^* f_\Gamma(A^{\mathbf{C}}) &\equiv \int_{\overline{\mathcal{A}^{\mathbf{C}}}_\Gamma} d\nu_{t,\Gamma}^l(A_\Gamma^{\mathbf{C}}) f_\Gamma(A_\Gamma^{\mathbf{C}}) \\ &\equiv \int_{\overline{\mathcal{A}^{\mathbf{C}}}_\Gamma} \left(\prod_e d\nu_{l(e)t}(A_\Gamma^{\mathbf{C}}(e))\right) f_\Gamma(A_\Gamma^{\mathbf{C}}), \end{aligned} \tag{3.16}$$

where $d\nu_t(g^{\mathbf{C}})$ is Hall's averaged heat-kernel measure [46].

Now, from Hall's theorem which is explained in Appendix B, it follows that the coherent state transform $C_t^l$ is an isometric isomorphism of $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$ onto $L^2(\overline{\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}}}, d\nu_t^l) \cap \mathcal{H}(\overline{\mathcal{A}^{\mathbf{C}}/\mathcal{G}^{\mathbf{C}}})$ [9]. Thus we see that the spin network states form an orthonormal basis w.r.t. the induced heat-kernel measure:

$$\begin{aligned} &< \Gamma, \{\pi_{p_e}\}, \{i_v\}|\Gamma, \{\pi_{q_e}\}, \{i'_v\} >_{|d\nu_t^l} \\ &\equiv \int_{\overline{\mathcal{A}^{\mathbf{C}}}} d\nu_t^l(A^{\mathbf{C}}) \overline{< A^{\mathbf{C}}|\Gamma, \{\pi_{p_e}\}, \{i_v\} >} < A^{\mathbf{C}}|\Gamma, \{\pi_{q_e}\}, \{i'_v\} > \\ &= \exp(t\sum_e l(e)\frac{p_e(p_e+2)}{4}) < \Gamma, \{\pi_{p_e}\}, \{i_v\}|\Gamma, \{\pi_{q_e}\}, \{i'_v\} >_{|d\mu_H} \\ &= \exp(t\sum_e l(e)\frac{p_e(p_e+2)}{4}) \prod_e \frac{\delta_{p_e,q_e}}{p_e+1} \prod_v < i_v|i'_v > . \end{aligned} \tag{3.17}$$

Similarly to the coherent state transform, the consistency of this measure is satisfied iff $l(e \cdot e') = l(e) + l(e')$ and $l(e^{-1}) = l(e)$ hold.

Here we make a few remarks. The coherent state transform $C_t^l$ and the induced heat-kernel measure $d\nu_t^l$ are not invariant under diffeomorphisms, because their definitions require to introduce a length function $l(e)$ which is not diffeomorphism invariant. Thus we can use this measure only in the context which do not respect the diffeomorphism invariance. In the lattice formulation explained in §5, however, we do not have to worry so much about the diffeomorphism invariance because it is manifestly violated by the introduction of a lattice. So it is possible that $d\nu_t^l$ play an important role in the lattice formulation.

The invariance of the induced heat-kernel measure $d\nu_t^l(A^{\mathbf{C}})$ under generalized $SU(2)$ gauge transformations follows from the bi-$SU(2)$ invariance of the averaged heat-kernel measure $d\nu_t(g^{\mathbf{C}})$. However, because $d\nu_t(g^{\mathbf{C}})$ is not bi-$SL(2,\mathbf{C})$ invariant, $d\nu_t^l(A^{\mathbf{C}})$ is not invariant under full $SL(2,\mathbf{C})$ gauge transformations. This measure is therefore useful when we fix the Lorentz boost gauge degrees of freedom.

We should note that the induced heat-kernel measure does not yield a physical measure for quantum gravity because it does not implement the correct reality conditions. This measure, however, may be useful for examining some property of the wavefunctions.

## 3.4 The Wick rotation transform

Finding the physical inner product for quantum Ashtekar's formalism in the Lorentzian signature is one of the most formidable problem because the physical inner product must implement non-trivial reality conditions. To the author's knowledge no works on the inner product which have been made so far have reached a decisive conclusion. Quite recently, however, a promising candidate has appeared for the construction scheme of the physical inner product. Leaving the detail to the original literature[84], here we will briefly explain the underlying idea.

As is mentioned before, there is a formulation for Lorentzian gravity[15] whose configuration variable is a real $SU(2)$ connection

$$(A')^i_a = -\frac{1}{2}\epsilon^{ijk}\omega^{jk}_a + \omega^{0i}_a. \tag{3.18}$$

While the Wheeler-De Witt equation in this formulation is as complicated as that in the ADM formalism, this formulation has a merit that the reality conditions are given by trivial self-adjointness conditions. A physical measure in this formalism is therefore given by the induced Haar measure explained in §§3.2. The main idea in ref.[84] is to obtain Ashtekar's $SL(2,\mathbf{C})$ connection

$$A^i_a = -\frac{1}{2}\epsilon^{ijk}\omega^{jk}_a - i\omega^{0i}_a$$

from the real $SU(2)$ connection (3.18), by way of the "Wick rotation transform":

$$(A')^i_a \to A^i_a = e^{\hat{C}}(A')^i_a e^{-\hat{C}}, \tag{3.19}$$

where $\hat{C}$ is the operator version of the "Wick rotator"

$$C \equiv \frac{\pi}{2}\int_{M^{(3)}} d^3x\,\omega^{0i}_a\tilde{e}^{ai}.$$

We expect that the "Wick rotated measure", the measure which is induced from the induced Haar measure by this Wick rotation, implements the correct reality conditions. However, to find a fully regularized operator version of this transform seems to be considerably difficult, because $C$ involves an expression which is non-polynomial in the canonical variables $(A_a^i, \tilde{e}^{ai})$ (or $((A')_a^i, \tilde{e}^{ai})$). If we find a well-defined Wick rotation transform, we can assert that we have made a large step forward toward completing the program of canonical quantum gravity in terms of Ashtekar's new variables.

## 3.5 Diffeomorphism invariant states

As we have seen in the previous subsections spin network states provide us with a complete basis in the space $\overline{\mathcal{A}/\mathcal{G}}$ of equivalence classes of (generalized) connections up to gauge transformations. Thus, as long as we work with the spin network states, Gauss' law constraint (2.29) is automatically solved. As it is, however, spin network states are not invariant under spatial diffeomorphisms. A further device is therefore required in order to construct diffeomorphism invariant states, namely to solve the integrated diffeomorphism constraint (2.33).

We know that an integrated version $\hat{U}(\phi)$ of the diffeomorphism constraint operator $\hat{\mathcal{D}}(\vec{N}) = -i\int_{M^{(3)}} d^3x \mathcal{L}_{\vec{N}} A_a^i \delta/\delta A_a^i$ generates a small spatial diffeomorphism

$$\hat{U}(\phi)A_a^i(x)\hat{U}(\phi^{-1}) = (\phi^{-1})^* A_a^i(x) \equiv \partial_a(\phi^{-1}(x))^b A_b^i(\phi^{-1}(x)).$$

This affects the transformation of the parallel propagator $h_\alpha[0,1]$ as a distortion of the curve $\alpha$

$$\hat{U}(\phi)h_\alpha[0,1]\hat{U}(\phi^{-1}) = h_{\phi^{-1}\circ\alpha}[0,1]. \tag{3.20}$$

Thus we see that the integrated diffeomorphism operator $\hat{U}(\phi)$ induces a diffeomorphism of the graph $\Gamma$ on which the spin network states are defined:

$$\begin{aligned}\hat{U}(\phi) < A|\Gamma, \{\pi_e\}, \{i_v\} > &= < A|\phi^{-1}\circ\Gamma, \{\pi'_{\phi^{-1}\circ e} = \pi_e\}, \{i'_{\phi^{-1}(v)} = i_v\} > \\ &\equiv < A|\phi^{-1}\circ\Gamma, \{\pi_e\}, \{i_v\} >,\end{aligned} \tag{3.21}$$

where $\phi\circ\Gamma \equiv (\{\phi\circ e\}, \{\phi(v)\})$.

Naively considering, taking the following average yields the basis of diffeomorphism invariant states:

$$\{Vol(\mathrm{Diff}_0(M^{(3)}))\}^{-1}\int_{\mathrm{Diff}_0(M^{(3)})}[\mathcal{D}\phi] < A|\phi\circ\Gamma, \{\pi_e\}, \{i_v\} >, \tag{3.22}$$

where $[\mathcal{D}\phi]$ is an 'invariant measure' on the space $\mathrm{Diff}_0(M^{(3)})$ of small diffeomorphisms on $M^{(3)}$. This average is formal in the sense that it is indefinite because it involves the ratio of two divergent expressions, namely, $Vol(\mathrm{Diff}_0(M^{(3)}))$ and $\int_{\mathrm{Diff}_0(M^{(3)})}[\mathcal{D}\phi]$.

A prescription to make this formal average be mathematically rigorous was given in ref.[10]. A rough outline of the strategy used there is the following. We first consider the space $\Phi \equiv Cyl^{\infty}(\overline{\mathcal{A}/\mathcal{G}})$ of smooth cylindrical functions each of which is given by a linear combination of the spin network states defined on a sufficiently large graph. We can thus perform an orthogonal decomposition of $\varphi \in \Phi$ as:

$$\varphi[A] = \sum_{\Gamma'} \sum_{\{\pi_{e'}\}} \varphi_{\Gamma',\{\pi_{e'}\}}[A],$$

where $\varphi_{\Gamma',\{\pi_{e'}\}}$ denotes the projection of $\varphi$ onto the space of spin network states defined on a graph $\Gamma'$ with fixed representations $\{\pi_{e'}\}$ assigned to the edges $\{e'\}$. Now we can define the average of $< A|\Gamma, \{\pi_e\}, \{i_v\} >$ over the group $\mathrm{Diff}_0(M^{(3)})$ of small diffeomorphisms:

$$\Psi^{DI}_{([\Gamma],\{\pi_e\},\{i_v\})}[\varphi] \equiv \sum_{\Gamma_2 \in [\Gamma]} < \Gamma_2, \{\pi_e\}, \{i_v\} | \varphi >, \tag{3.23}$$

where $[\Gamma]$ stands for the orbit of the graph $\Gamma$ under $\mathrm{Diff}_0(M^{(3)})$, and $< \mid >$ represents some inner product on $\overline{\mathcal{A}/\mathcal{G}}$ which is invariant under diffeomorphisms. Eq.(3.23) is well-defined as a distributional wavefunction which belongs to the topological dual $\Phi'$ of $\Phi$. This is because, if we use for example the induced Haar measure as the inner product $< \mid >$, nonvanishing contributions to eq.(3.23) are only from the inner products in which the spin network states appearing in the bra $< \mid$ and in the ket $\mid >$ correspond to the identical graph $\Gamma$ with coincident colored edges $\{(e, \pi_e)\}$. Hence, from the spin network states $< A|\Gamma, \{\pi_e\}, \{i_v\} >$, we can construct a basis $\{\Psi^{DI}_{([\Gamma],\{\pi_e\},\{i_v\})}\}$ of diffeomorphism (and gauge) invariant states.[15]

Here a remark should be made. In the defining equation (3.23) we have assumed the existence of a diffeomorphism invariant inner product $< \mid >$. Because the induced Haar measure enjoys this property, the prescription sketched here applies to Ashtekar's formulation for Euclidean gravity. In the Lorentzian signature, however, we do not yet know whether we can define diffeomorphism invariant states through eq.(3.23). This is because the induced heat-kernel measure is not invariant under diffeomorphisms. While we can construct a diffeomorphism invariant measure [9] by way of the Baez measure[12], it does not seem to be suitable for the present purpose because it is not faithful. However, it is noteworthy that, if appropriately regularized, the Wick rotated measure explained in the last subsection may provide a desired inner product owing to the diffeomorphism invariance of the Wick rotator $C$. A hard task remaining there is to find a convenient orthonormal basis w.r.t. this Wick rotated measure such as the spin network state basis w.r.t. the induced Haar measure.

[15] The notion of a discretized measure on loops which leads to diffeo invariant measures was first introduced into quantum gravity by ref.[74]. This kind of diffeo invariant measures were first constructd by Gel'fand and collaborators using various point sets.

## 3.6 Area and volume operators

In order to extract physical information from the wavefunctions we need physical observables each of which is a self-adjoint operator which commutes with the constraint operators at least weakly. Necessarily physical observables have to be invariant under $SL(2, \mathrm{C})$ gauge transformations and spatial diffeomorphisms.

While we do not know any physical observables in pure gravity, we can construct a large set of gauge and diffeomorphism invariant operators, namely, area operators and volume operators [78]. In matter-coupled theories these operators are expected to become physical observables.

First we explain the area operators. Classically the area $A(D)$ of a two-dimensional region $D \in M^{(3)}$ is given by

$$A(D) = \int_D d^2\sigma \left(\underset{\sim}{n}_a \underset{\sim}{n}_b \tilde{\pi}^{ai} \tilde{\pi}^{bi}\right)^{\frac{1}{2}}, \tag{3.24}$$

where $\underset{\sim}{n}_a \equiv \underset{\sim}{\epsilon}_{abc} \partial_{\sigma^1} x^b \partial_{\sigma^2} x^c$ is the "normal vector" to $D$. Because this expression contains second order functional derivative, some regularization is necessary in passing to the quantum theory. A clever regularization was provided in [78]. We first divide the region $D$ into subregions $D = \sum_I D_I$ with the coordinate area of each subregion $D_I$ assumed to be of order $\delta^2$. Then the operator version of $A(D)$ is defined as

$$\begin{aligned} \hat{A}(D) &= \lim_{\delta \to 0} \sum_I \hat{A}(D_I) \\ \{\hat{A}(D_I)\}^2 &\equiv \int_{D_I} d^2\sigma \underset{\sim}{n}_a(x(\sigma)) \int_{D_I} d^2\sigma' \underset{\sim}{n}_b(x(\sigma')) \hat{\tilde{\pi}}^{ai}(x(\sigma)) \hat{\tilde{\pi}}^{bi}(x(\sigma')). \end{aligned} \tag{3.25}$$

We can easily compute the action of this area operator on the parallel propagator $\pi(h_\alpha[0,1])$ in the representation $\pi$ evaluated along a smooth curve $\alpha$ which does not intersect with itself. Because the action of $\hat{\tilde{\pi}}^{ai} = -\delta/\delta A_a^i$ on the parallel propagator is given by

$$\hat{\tilde{\pi}}^{ai}(x) h_\alpha[0,1] = -\int_0^1 ds \delta^3(x, \alpha(s)) \dot{\alpha}^a(s) h_\alpha[0,s] J_i h_\alpha[s,1],$$

the action of $\{\hat{A}(D_I)\}^2$ on the parallel propagator is computed as

$$\{\hat{A}(D_I)\}^2 \cdot \pi(h_\alpha[0,1]) = \begin{cases} \pi(\Delta)\pi(h_\alpha[0,1]) & \text{if} \quad \alpha \cap D_I = x_0 \in M^{(3)} \\ 0 & \text{if} \quad \alpha \cap D_I = \emptyset, \end{cases} \tag{3.26}$$

where $\Delta \equiv J_i J_i$ is the Casimir operator of $SL(2, \mathbf{C})$. We assumed that $D_I$ is so small and $\alpha$ is so well-behaved that they intersect with each other at most once.

Here we should note that, if $\pi$ is the spin-$\frac{p}{2}$ representation, the Casimir $\pi(\Delta) = -\frac{p(p+2)}{4}$ is negative. Thus $\pi(h_\alpha[0,1])$ is an eigenfunction of the squared-area operator $\{\hat{A}(D_I)\}^2$ *with a negative eigenvalue.* If we take the minus sign seriously, it seems to follow that the spin network state defined on a piecewise smooth graph with each edge colored by a finite dimensional representation does not correspond to any classical spacetimes with correct Lorentzian signature. This may suggest that we should work with the spin network states i) which is defined on graphs which are not piecewise analytic and/or ii) each of whose edges are colored by an infinite dimensional representation. At present we do not know the correct solution to this "issue of negative squared-area", or even do not know whether or not we should confront this issue seriously. However, if we take an optimistic attitude and avoid this issue by taking only the absolute value of the Casimir, we obtain the result derived in [78]:

$$|\hat{A}(D)| \cdot \pi(h_\alpha[0,1]) = n(D,\alpha)\sqrt{|\pi(\Delta)|}\pi(h_\alpha[0,1]), \tag{3.27}$$

where $n(D,\alpha)$ is the number of intersections between the two-dimensional region $D$ and the curve $\alpha$. The area operators therefore exhibit a discrete spectrum. It was shown that the volume operators also have discrete eigenvalues[78]. This "discreteness of area and volume" is considered to suggest that the spacetime reveals some microstructure in the Planck regime.

Next we consider the volume operators. A naive expression of the operator $\hat{V}(\mathcal{R})$ which measures the volume of a three-dimensional region $\mathcal{R}$ is given by

$$\hat{V}(\mathcal{R}) = \int_{\mathcal{R}} d^3x \left(\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}\hat{\tilde{\pi}}^{ai}\hat{\tilde{\pi}}^{bj}\hat{\tilde{\pi}}^{ck}\right)^{\frac{1}{2}}. \tag{3.28}$$

Because this operator involves the third order functional derivative, we need some regularization in order to obtain a well-defined result. Regularized versions of this operator are proposed both in the continuum case[78] and in the discretized case [62]. For detail we refer the reader to the references and here we briefly explain the essence.

We see from eq.(3.28) that the nonvanishing contributions of its action on the spin network states arise only from vertices. Thus we can concentrate only on vertices. The results in [78][62] tell us that the action of $\hat{V}$ vanishes on the bi- or trivalent vertex. In order to have nonvanishing volume, we have to consider spin network states which are defined on graphs with at least four-valent vertices.

## 3.7 Spin network states in terms of spinor propagators

In the previous subsections we have developed the framework of spin network states by using arbitrary finite-dimensional representations. Because the spin-$\frac{p}{2}$ representation is expressed as a symmetrized tensor-product of $p$ copies of spinor representations

$$\begin{aligned} \pi_p(h_\alpha[0,1]) &\cong h_\alpha[0,1]^{A_1}{}_{(B_1}\cdots h_\alpha[0,1]^{A_p}{}_{B_p)} \\ h_\alpha[0,1]^A{}_B &\equiv \pi_1(h_\alpha[0,1])^A{}_B, \end{aligned} \tag{3.29}$$

we can in principle do our analysis in terms only of spinor propagators. While it is desirable that the ultimate results should be described in terms of arbitrary irreducible representations, it is sometimes more convenient to calculate purely by means of the spinor representation. Actually the analysis in the next section is based only on the spinor representation. Thus it would be proper to develop here the framework in terms of spinor propagators.

As a result of the defining equation of $SL(2,\mathbf{C})$, the spinor propagator is subject to the following identity:

$$\epsilon^{AD}\epsilon_{BC}h_\alpha[0,1]^C{}_D = (h_\alpha[0,1]^{-1})^A{}_B = h_{\alpha^{-1}}[0,1]^A{}_B. \tag{3.30}$$

This identity is useful because it enables us to choose relevant orientations of the propagators according to a particular problem.

There are three invariant tensors in the spinor representation, i.e. the invariant spinors $\epsilon_B{}^A \equiv \delta^A_B$, $\epsilon^{AB}$ and $\epsilon_{AB}$. Because all the intertwining operators are constructed from these invariant spinors, the total rank of an intertwining operator is even. Thus the sum of the numbers of spinor propagators ending and starting at a vertex is necessarily even. Using the identity (3.30) and the equation $\epsilon_{AC}\epsilon^{BC} = \delta^B_A$, we can equalize at each vertex the number of in-coming propagators with that of out-going propagators. As a consequence we can regard any intertwining operator to be a linear combination of the products of $\delta^B_A$.

In addition to eq.(3.30), there are three useful identities in the spinor representation. Two of them are the two-spinor identity

$$\delta^B_A\delta^D_C - \delta^D_A\delta^B_C = \epsilon_{AC}\epsilon^{BD} \quad (\text{ or } \phi^A\epsilon^{BC} + \phi^B\epsilon^{CA} + \phi^C\epsilon^{AB} = 0) \tag{3.31}$$

and the Fiertz identity

$$(J_i)^A{}_B(J_i)^C{}_D = -\frac{1}{2}(\delta^A_D\delta^C_B - \frac{1}{2}\delta^A_B\delta^C_D). \tag{3.32}$$

The third identity

$$\epsilon_{ijk}(J_j)^A{}_B(J_k)^C{}_D = \frac{1}{2}\{(J_i)^A{}_D\delta^C_B - (J_i)^C{}_B\delta^A_D\} \tag{3.33}$$

is obtained by combining the Fiertz identity and the defining equation:

$$(J_i)^A{}_C(J_j)^C{}_B = -\frac{1}{4}\delta_{ij}\delta^A_B + \frac{1}{2}\epsilon_{ijk}(J_k)^A{}_B. \tag{3.34}$$

The identity (3.33) plays an important role when we calculate the action of the scalar constraint.

The two-spinor identity tells us that the antisymmetrized tensor product of the two identical spinor propagators gives the trivial representation:

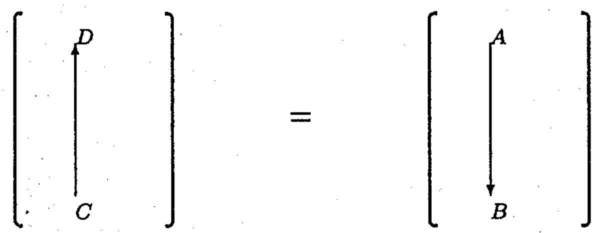
$$h_\alpha[0,1]^A{}_{[B}h_\alpha[0,1]^C{}_{D]} = \frac{1}{2}\epsilon^{AC}\epsilon_{BD} \tag{3.35}$$

and that, at an intersection $\alpha(s_0) = \beta(t_0)$ of two curves $\alpha$ and $\beta$, the following identity holds:

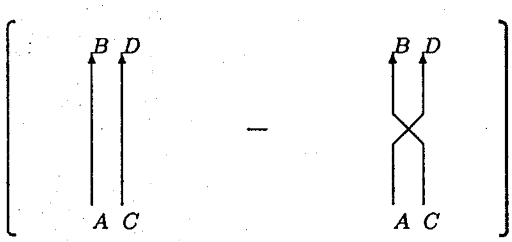
$$
\begin{aligned}
&h_\alpha[0,1]^A{}_B h_\beta[0,1]^C{}_D - (h_\alpha[0,s_0]h_\beta[t_0,1])^A{}_D (h_\beta[0,t_0]h_\alpha[s_0,1])^C{}_B \\
&= (h_\alpha[0,s_0]h_{\beta^{-1}}[1-t_0,1])^A{}_E \epsilon^{EC} (h_{\beta^{-1}}[0,1-t_0]h_\alpha[s_0,1])^F{}_B \epsilon_{FD}. \qquad (3.36)
\end{aligned}
$$

Owing to eq.(3.35) we do not have to take account of antisymmetrizing the identical propagators.

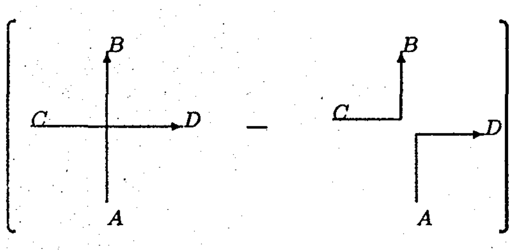
In practical calculation it is frequently convenient to introduce the graphical representation. We will denote a spinor propagator $h_\alpha[0,1]^A{}_B$ by an arrow from $\alpha(0)$ to $\alpha(1)$ with its tail (tip) being equipped with the spinor index $A$ ($B$). Identities (3.30),(3.35) and (3.36) are then expressed as follows:

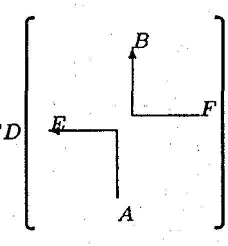
$\epsilon^{AD}\epsilon_{BC}$ 

, (3.30)'



$= \epsilon^{AC}\epsilon_{BD}$ , (3.35)'



$= \epsilon^{EC}\epsilon_{FD}$ 

. (3.36)'

# 4 Combinatorial solutions to the Wheeler-De Witt equation

In the last section we have prepared mathematical tools to deal with spin network states. A virtue of using spin network states is that the action of local operators can be described purely in terms of local manipulations on the colored graphs under consideration, while we have to allow for changes of global properties when we deal with Wilson loops. In this section we will explicitly see this in the context of extending the solution found by Jacobson and Smolin[52], which is a linear combination of spinor Wilson loops, to arbitrary finite dimensional representations[36].

For analytical simplicity we will restrict the type of vertices $\{v\}$ appearing in the graph $\Gamma$ to those at which two smooth curves intersect. We further assume that the tangent vectors of the two curves are linearly independent. We will henceforth call those vertices "regular four-valent vertices". In this and the next sections we consider the case where the cosmological constant vanishes, i.e. $\Lambda = 0$.

## 4.1 Action of the scalar constraint on spin network states

In order to construct solutions to the Wheeler-De Witt equation (2.31) from spin network states, we have to evaluate the action of the scalar constraint operator [16]

$$\hat{S}(\underset{\sim}{N}) = \int_{M^{(3)}} d^3x \frac{\underset{\sim}{N}(x)}{2} \epsilon^{ijk} F^i_{ab}(x) \frac{\delta}{\delta A^j_a(x)} \frac{\delta}{\delta A^k_b(x)}$$

on the spin network states. Because this operator contains second order functional derivative, some regularization have to be prescribed. Here we will use the point-splitting regularization [77][20][19][21]:

$$\begin{aligned}\hat{S}^\epsilon(\underset{\sim}{N}) \equiv& \int_{M^{(3)}} d^3x \int_{M^{(3)}} d^3x \frac{\underset{\sim}{N}(x)}{2} \tilde{f}_\epsilon(x,y) \underset{\sim}{\epsilon}_{abc} \mathrm{Tr}(-4h_{yx}[0,1]\tilde{B}^c(x) J_i h_{xy}[0,1] J_j) \\ &\times \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(y)},\end{aligned} \tag{4.1}$$

where $h_{xy}[0,1]$ is the parallel propagator along a curve from $x$ to $y$ which shrinks to $x$ in the limit $y \to x$, $\tilde{B}^c \equiv \tilde{B}^{ci} J_i \equiv \frac{1}{2}\tilde{\epsilon}^{abc} F^i_{ab} J_i$ is the magnetic field of the $SL(2,\mathbf{C})$ connection and Tr in this section stands for the trace in the spinor representation. The regulator $\tilde{f}_\epsilon(x,y)$ is subject to the condition

$$\tilde{f}_\epsilon(x,y) \stackrel{\epsilon\to 0}{\longrightarrow} \delta^3(x,y).$$

In order to define a particular regulator we must fix a local coordinate frame. Because a usual choice of the coordinate frame breaks diffeomorphism covariance, the resulting regularized scalar constraint are usually not covariant under diffeomorphisms. Here we will use a somewhat tricky choice of the local coordinate frame which is similar to that used in the (2+1)-dimensional version of Ashtekar's formalism[36].

[16] As is explained in §§2.2, there is an issue on the choice of operator ordering which is intimately related to the problem on the closure of the constraint algebra [55][19][21]. We will not discuss on these issues and simply choose the ordering in which the momenta are placed to the right of the $SL(2,\mathbf{C})$ connections.

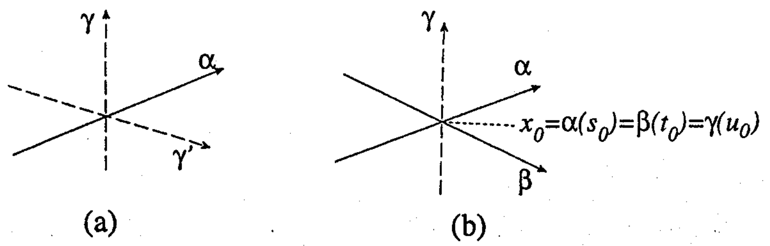



Figure 3: Local curvilinear coordinate frames at an analytic part (a) and at a regular four-valent vertex (b). The solid and the dashed lines respectively denote the true curves and the virtual ones in the graph.

It is obvious that the action of $\hat{\mathcal{S}}^\epsilon(\underset{\sim}{N})$ on the spin network states is nonvanishing only on the curves belonging to the graph in question. Eq.(3.20) tells us that these curves are subject to the action of the diffeomorphism constraint. We therefore expect that, if we introduce a local curvilinear coordinate frame in which these curves play the role of coordinate curves, we can define a regularization of the scalar constraint which preserves diffeomorphism covariance. In this subsection we will execute such a regularization procedure.

In order to fix a curvilinear coordinate frame completely we need three linearly independent curves. In the case under our consideration, we have at most two curves intersecting at a point. So we have to introduce one or two "virtual curves" as is shown in figure 3.

Because the scalar constraint involves two functional derivatives, it is convenient to separate its action as follows:

$$\hat{\mathcal{S}}(\underset{\sim}{N}) = \hat{\mathcal{S}}_1(\underset{\sim}{N}) + \hat{\mathcal{S}}_2(\underset{\sim}{N}) \qquad (\text{or} \quad \hat{\mathcal{S}}^\epsilon(\underset{\sim}{N}) = \hat{\mathcal{S}}_1^\epsilon(\underset{\sim}{N}) + \hat{\mathcal{S}}_2^\epsilon(\underset{\sim}{N})), \tag{4.2}$$

where $\hat{\mathcal{S}}_1(\underset{\sim}{N})$ and $\hat{\mathcal{S}}_2(\underset{\sim}{N})$ stand for, respectively, the action of $\hat{\mathcal{S}}(\underset{\sim}{N})$ on the single spinor propagators and that on the pairs of spinor propagators. This separation simplifies considerably the evaluation of the action of the Hamiltonian constraint on spin-network states.

Now we demonstrate a few examples of the calculation.

First we consider the action of the regularized Hamiltonian (4.1) on a single spinor propagator $h_\alpha[0,1]^A{}_B$ along a smooth curve $\alpha$. We can easily see that this vanishes. A functional derivative on a propagator (in an arbitrary representation)

$$\frac{\delta}{\delta A_a^i(x)} h_\alpha[0,1] = \int_0^1 ds \delta^3(x,\alpha(s))\dot{\alpha}^a(s) h_\alpha[0,s] J_i h_\alpha[s,1] \tag{4.3}$$

necessarily picks a distributional factor $\int_0^1 ds\delta^3(x,\alpha(s))\dot{\alpha}^a(s)$ which involves the tangent vector. By carrying out the other functional derivative and the integrations in eq.(4.1), we find that $\hat{\mathcal{S}}^\epsilon h_\alpha[0,1] = \hat{\mathcal{S}}_1^\epsilon h_\alpha[0,1]$ involves the expression

$$\int_0^1 ds \int_0^1 dt \tilde{f}_\epsilon(\alpha(s),\alpha(t)) \underset{\sim}{\epsilon}_{abc}\dot{\alpha}^a(s)\dot{\alpha}^b(t). \tag{4.4}$$

In our coordinate choice this expression vanishes because $\dot{\alpha}^a(s) \propto \dot{\alpha}^a(t)$ even for $s \neq t$.

By a similar reasoning we see that the action of the regularized scalar constraint $\hat{\mathcal{S}}^\epsilon(\underset{\sim}{N})$ on a pair of spinor propagators along a smooth curve $\alpha$ vanishes. Summarizing these two results we find:

$$\hat{\mathcal{S}}_1^\epsilon(\underset{\sim}{N})h_\alpha[0,1]^A{}_B = \hat{\mathcal{S}}_2^\epsilon(\underset{\sim}{N})(h_\alpha[0,1]^A{}_B h_\alpha[0,1]^C{}_D) = 0. \tag{4.5}$$

In consequence we have only to pay attention to points on the graph at which the analyticity of the curve breaks, namely to vertices and kinks.

Next we calculate the action on a single spinor propagator

$$h_{\alpha_1\cdot\beta_2}[0,1]^A{}_B \equiv (h_\alpha[0,s_0]h_\beta[t_0,1])^A{}_B$$

along a curve $\alpha_1\cdot\beta_2$ with a kink[17]:

$$\begin{aligned}
&\hat{\mathcal{S}}_1^\epsilon(\underset{\sim}{N})h_{\alpha_1\cdot\beta_2}[0,1]^A{}_B \\
&= \frac{1}{2}\int_0^{s_0} ds \int_{t_0}^1 dt \underset{\sim}{\epsilon}_{abc}\dot{\alpha}^a(s)\dot{\beta}^b(t)\tilde{f}_\epsilon(\alpha(s),\beta(t)) \\
&\times \Big\{\underset{\sim}{N}(\alpha(s))\mathrm{Tr}(-4h_{ts}\tilde{B}^c(\alpha(s))J_j h_{st}J_k)h_\alpha[0,s]J_j h_\alpha[s,s_0]h_\beta[t_0,t]J_k h_\beta[t,1] \\
&- \underset{\sim}{N}(\beta(t))\mathrm{Tr}(-4h_{st}\tilde{B}^c(\beta(t))J_j h_{ts}J_k)h_\alpha[0,s]J_k h_\alpha[s,s_0]h_\beta[t_0,t]J_j h_\beta[t,1]\Big\}^A{}_B,
\end{aligned} \tag{4.6}$$

where we have abbreviated $h_{\alpha(s)\beta(t)}[0,1]$ as $h_{st}$. In order to go further we have to determine explicitly the regulator $\tilde{f}_\epsilon$. By introducing the curvilinear coordinate and by exploiting the fact that $\delta^3(x,y)$ transforms as a scalar density of weight $+1$, we define $\tilde{f}_\epsilon$ as

$$\tilde{f}_\epsilon(\alpha(s),\beta(t)) \equiv |\underset{\sim}{\epsilon}_{abc}\dot{\alpha}^a(s_0)\dot{\beta}^b(t_0)\dot{\gamma}^c(u_0)|^{-1}\frac{1}{(2\epsilon)^3}\theta(\epsilon-|s-s_0|)\theta(\epsilon-|t-t_0|)\theta(\epsilon-|u-u_0|). \tag{4.7}$$

By plugging this definition into eq.(4.6) and by using $(h_{st})^A{}_B = \delta_B^A + O(\epsilon)$ and similar approximations, we find

$$\begin{aligned}
&\hat{\mathcal{S}}_1^\epsilon(\underset{\sim}{N})h_{\alpha_1\cdot\beta_2}[0,1]^A{}_B \\
&= \frac{1}{16\epsilon}\{2\underset{\sim}{n}\cdot\tilde{B}^i(x_0)\epsilon_{ijk}h_\alpha[0,s_0]J_j J_k h_\beta[t_0,1] + O(\epsilon)\}^A{}_B,
\end{aligned} \tag{4.8}$$

[17] In this section we assume that the curves $\alpha$ and $\beta$ intersect with each other at $\alpha(s_0)=\beta(t_0)=x_0$. $\alpha_1 \subset \alpha$ is the curve from $\alpha(0)$ to $\alpha(s_0)$ and $\beta_2 \subset \beta$ is the curve from $\beta(t_0)$ to $\beta(1)$. We also assume that the virtual curve $\gamma$ intersects with $\alpha$ and $\beta$ at $\gamma(u_0)=x_0$.

where we have used the notations $\underset{\sim}{n} \cdot \tilde{B}^i(x_0) \equiv \underset{\sim}{n}_c(x_0)\tilde{B}^{ci}(x_0)$ and

$$\underset{\sim}{n}_c(x_0) \equiv \underset{\sim}{N}(x_0)\underset{\sim}{\epsilon}_{cab}\dot{\alpha}^a(s_0)\dot{\beta}^b(t_0)|\underset{\sim}{\epsilon}_{abc}\dot{\alpha}^a(s_0)\dot{\beta}^b(t_0)\dot{\gamma}^c(u_0)|^{-1}.$$

One might claim that the $O(\epsilon)$ part between the braces in eq.(4.8) is in fact crucial because this part yields the $O(1)$ contribution if we naively take the limit of sending $\epsilon$ to zero. However, this part does not survive if we consider the "flux-tube regularization"[52] in which each curve is replaced by some extended object such as a ribbon or a tube. This is expected to hold also when we consider the "extended loop" regularization[16] in which the distributional expression $\int ds\delta^3(x, \alpha(s))\dot{\alpha}^a(s)$ is replaced by a smooth vector field $X^a(x)$ subject to some defining equations. Taking these into account it seems to be appropriate to consider the "renormalized" scalar constraint [77][20][19]:

$$\hat{S}^{ren}(\underset{\sim}{N}) \equiv \lim_{\epsilon\to 0} 16\epsilon\hat{S}^\epsilon(\underset{\sim}{N}) \qquad \left(\hat{S}^{ren}_I(\underset{\sim}{N}) \equiv \lim_{\epsilon\to 0} 16\epsilon\hat{S}^\epsilon_I(\underset{\sim}{N}) \quad (I=1,2)\right). \tag{4.9}$$

Now by using the identity (3.33), we can further simplify eq.(4.6) and obtain the final result

$$\begin{aligned} \hat{S}^{ren}_1(\underset{\sim}{N})h_{\alpha_1\cdot\beta_2}[0,1]^A{}_B &= \{h_\alpha[0,s_0]\underset{\sim}{n}\cdot\tilde{B}(x_0)h_\beta[t_0,1]\}^A{}_B \\ &= \underset{\sim}{n}_c(x_0)\frac{1}{2}\tilde{\epsilon}^{abc}\Delta_{ab}(\alpha_1\cdot\beta_2,x_0)h_{\alpha_1\cdot\beta_2}[0,1]^A{}_B. \end{aligned} \tag{4.10}$$

$\Delta_{ab}$ here stands for the "area derivative" [64] acting on the functional of graphs which is defined by[18]

$$\sigma^{ab}\Delta_{ab}(\alpha,x_0)\Psi[\alpha,\cdots] \equiv \Psi[\alpha_1\cdot\gamma_{x_0}\cdot\alpha_2,\cdots] - \Psi[\alpha,\cdots] + O((\sigma^{ab}\sigma^{ab})^{\frac{3}{4}}). \tag{4.11}$$

for arbitrary small loop $\gamma_{x_0}$ based at the point $x_0$.

Action on the other configurations can be calculated similarly. We list the action of $\hat{S}^{ren}(\underset{\sim}{N})$ on all the basic configurations in Appendix C. There we make use of the graphical representation.

Since we have at hand the basic action of the scalar constraint, it is not difficult to calculate its action on general spin network states. The idea is the following. Appendix C tells us that the action of $\hat{S}^{ren}(\underset{\sim}{N})$ has nonvanishing contributions only at vertices. Because the action of $\hat{S}^{ren}(\underset{\sim}{N})$ is local, we can separate its action on the individual vertices, i.e.

$$\hat{S}(\underset{\sim}{N})\Psi_\Gamma(A) = \sum_{v\in\{v\}\subset\Gamma}\left(\hat{S}^{ren}(\underset{\sim}{N})\Psi_\Gamma(A)\right)|_v, \tag{4.12}$$

where $\Psi_\Gamma(A)$ is the wave function on $\overline{\mathcal{A}/\mathcal{G}}$ which can be expressed by a superposition of spin network states defined on a graph $\Gamma$.

[18] We regard $\alpha$ as a part of the graph under consideration. $\sigma^{ab} \equiv \frac{1}{2}\int_{\gamma_{x_0}} x^a dx^b$ is the coordinate area element of the small loop $\gamma_{x_0}$.

If we use the separation (4.2), we can easily evaluate the action on each vertex $v$. First we add up the contributions from all the kinks and thus obtain the action of $\hat{\mathcal{S}}_1^{ren}(\underset{\sim}{N})$ on the vertex $v$. We then compute the sum of the action of $\hat{\mathcal{S}}_2^{ren}(\underset{\sim}{N})$ on all the pairs of parallel propagators. The total sum of these contributions yields the action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ at the vertex $v$. Symbolically we can write:

$$\begin{aligned}\left(\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})\Psi_\Gamma(A)\right)|_v &= \sum_{k\in K_v}\begin{pmatrix}\text{the action of } \hat{\mathcal{S}}_1^{ren}(\underset{\sim}{N}) \\ \text{on a kink } k\end{pmatrix} \\ &\quad + \sum_{p\in P_v}\begin{pmatrix}\text{the action of } \hat{\mathcal{S}}_2^{ren}(\underset{\sim}{N}) \\ \text{on a pair } p \text{ of propagators}\end{pmatrix}, \end{aligned} \tag{4.13}$$

where $K_v$ ($P_v$) is the set of all the kinks (all the pairs of parallel propagators) at the vertex $v$. Using eqs.(4.12), (4.13) and the equations in Appendix C, we can therefore reduce the problem of evaluating the action of the Hamiltonian constraint on the spin network states to that of combinatorics. This applies also to more general operators which involve the product of a finite number of momenta $\tilde{\pi}_a^i$.

We see that the action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ defined above is covariant under spatial diffeomorphisms *assuming that the virtual curves are also subject to the distortion $\gamma \to \gamma \circ \phi^{-1}$ under the action of the integrated diffeomorphism constraint (3.21).* This regularization is, however, explicitly depends on the artificial structure, namely on the choice of the virtual curve and on its parametrization, through $\underset{\sim}{n}_a$. In this sense our analysis given here is only heuristic. In order to obtain more physically solid results, a more justifiable regularization method is longed for which does not depends on any artificial structure and therefore which respects diffeomorphism covariance of the scalar constraint.

## 4.2 Topological solutions

Now that we have obtained the basic action of the renormalized scalar constraint $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$, let us construct solutions to the renormalized Wheeler-De Witt equation (WD equation)

$$\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})\Psi[A] = 0. \tag{4.14}$$

As we have seen in the last subsection and in Appendix C, the nonvanishing action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ on spin network states necessarily involves the area derivative $\Delta_{ab}$ at the vertex $v$. Hence the action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ vanishes if the result of the area derivative vanishes everywhere, namely if the $SL(2,\mathbf{C})$ connection is flat:

$$\tilde{B}^{ci}(x) \equiv \frac{1}{2}\tilde{\epsilon}^{abc}F_{ab}^i(x) = 0. \tag{4.15}$$

We therefore find the following 'distributional' solution called "topological solution" [20]:

$$\Psi_{topo}[A] \equiv \psi[A]\prod_{x\in M^{(3)}}\prod_{a,i}\delta(\tilde{B}^{ai}(x)), \tag{4.16}$$

where $\psi[A]$ is an arbitrary gauge-invariant functional of the $SL(2,\mathbf{C})$ connection $A_a^i$. Because it has a support only on flat connections, the topological solution represented in terms of spin network states depends only on homotopy classes of the colored graphs. As a corollary, it follows that the topological solution is invariant under diffeomorphisms and so it is a solution to all the constraint. A more detailed analysis on these topological solutions will be made in §6.

## 4.3 Analytic loop solutions

One of the important results obtained in §§4.1 is that the nonvanishing contributions to the action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ are only from vertices, i.e. the points where the analyticity of the curves breaks down. From this result we realize that, if we consider the spin network states consisting only of Wilson loops evaluated along smooth loops $\{\alpha_i\}$ $(i=1,\cdots,I)$ without any intersection:

$$\Psi_{\{(\alpha_i,\pi_i)\}}[A]=\prod_{i=1}^{I}W(\alpha_i,\pi_i), \tag{4.17}$$

then these states solve the renormalized WD equation[52]. These states are related with the solutions to the Hamiltonian constraint which have been found in the loop representation[77].

## 4.4 Combinatorial solutions

If we look at eqs.(C.2), (C.3) and (C.8) carefully, we find that the following linear combination of spinor propagators has a vanishing action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$:

$$h_\alpha[0,1]^A{}_B h_\beta[0,1]^C{}_D-2h_{\alpha_1\cdot\beta_2}[0,1]^A{}_D h_{\beta_1\cdot\alpha_2}[0,1]^C{}_B. \tag{4.18}$$

This is the solution found by Jacobson and Smolin[52]. We therefore expect that, even if the graph has vertices, some appropriate linear combinations of spin network states defined on a graph may solve the Hamiltonian constraint. Here we will find such "combinatorial solutions" which yield a set of extended versions of eq.(4.18) to arbitrary finite dimensional representations. We have seen in eq.(4.12) that $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ acts on each vertices independently. In order to be a solution to the renormalized WD equation, the spin network state must solve this equation *at every vertex.* We can construct a solution by looking for intertwining operators which give the vanishing action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$ *at each vertex,* and by gluing these solutions at adjacent (but separate) vertices using an adequate parallel propagator as a glue. Thus it is sufficient to concentrate only on one vertex, say, at $x_0$.

Let us now demonstrate explicitly that there exists a set of solutions to the renormalized WD equation which are extensions of Jacobson and Smolin's solution to spin network states defined on a regular four-valent graph[36].

In order to do so we first define the following two sets of functionals:

$$C^r(m,n)^{A_1\cdots A_m;C_1\cdots C_n}_{B_1\cdots B_m;D_1\cdots D_n} \equiv \prod_{i=1}^{m}(h_\alpha[0,s_0]^{A_i}{}_{E_i} h_\alpha[s_0,1]^{F_i}{}_{B_i}) \prod_{j=1}^{n}(h_\beta[0,t_0]^{C_j}{}_{G_j} h_\beta[t_0,1]^{H_j}{}_{D_j})$$

$$\times \sum_{\substack{1\le k_1<\cdots<k_r\le m \\ 1\le l_1<\cdots<l_r\le n}} \sum_{\sigma\in P_r} \left( \prod_{i=1}^{r} \delta^{E_{k_i}}_{H_{l_{\sigma_i}}} \delta^{G_{l_{\sigma_i}}}_{F_{k_i}} \prod_{\substack{k'=1 \\ k'\ne k_1,\cdots,k_r}}^{m} \delta^{E_{k'}}_{F_{k'}} \prod_{\substack{l'=1 \\ l'\ne l_1,\cdots,l_r}}^{n} \delta^{G_{l'}}_{H_{l'}} \right) \tag{4.19}$$

$$B^r(m,n)^{A_1\cdots A_m;C_1\cdots C_n}_{B_1\cdots B_m;D_1\cdots D_n} \equiv \prod_{i=1}^{m}(h_\alpha[0,s_0]^{A_i}{}_{E_i} h_\alpha[s_0,1]^{F_i}{}_{B_i}) \prod_{j=1}^{n}(h_\beta[0,t_0]^{C_j}{}_{G_j} h_\beta[t_0,1]^{H_j}{}_{D_j})$$

$$\times \sum_{\substack{1\le k_1<\cdots<k_r\le m \\ 1\le l_1<\cdots<l_r\le n}} \sum_{\sigma\in P_r} \sum_{i=1}^{r} \left\{ \prod_{\substack{j=1 \\ j\ne i}}^{r} \delta^{E_{k_i}}{}_{H_{l_{\sigma_i}}} \delta^{G_{l_{\sigma_i}}}{}_{F_{k_i}} \right.$$

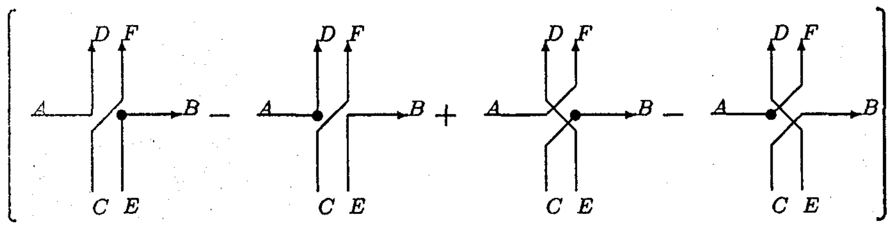
$$\times(\underset{\sim}{n}\cdot\tilde{B}^{E_{k_i}}{}_{H_{l_{\sigma_i}}} \delta^{G_{l_{\sigma_i}}}{}_{F_{k_i}} - \delta^{E_{k_i}}{}_{H_{l_{\sigma_i}}} \underset{\sim}{n}\cdot\tilde{B}^{G_{l_{\sigma_i}}}{}_{F_{k_i}})$$

$$\left. \times \prod_{\substack{k'=1 \\ k'\ne k_1,\cdots,k_r}}^{m} \delta^{E_{k'}}{}_{F_{k'}} \prod_{\substack{l'=1 \\ l'\ne l_1,\cdots,l_r}}^{n} \delta^{G_{l'}}{}_{H_{l'}} \right\}.$$

$P_r$ in the above expression is the group of permutations of $r$ numbers $(l_1,\cdots,l_r)$. We should notice that $B^0(m,n)=B^{\min(m,n)+1}(m,n)=0$. Intuitively, the $C^r(n,m)$-functional is obtained as follows: Prepare $m$ propagators $h_\alpha$ along $\alpha$ and $n$ propagators $h_\beta$ along $\beta$; then choose $r$ pairs of $h_\alpha$ and $h_\beta$; cut and reglue each pair in the orientation preserving fashion; and finally sum up the results obtained from all the choices of $r$ pairs. The $B^r(m,n)$-functional is the result of the action of $\hat{\mathcal{S}}^{ren}_1(\underset{\sim}{N})$ on the $C^r(m,n)$-functional.

We will sometimes omit the 'external' spinor indices each of which the tip or the tail of a parallel propagator $h_\alpha[0,1]$ or $h_\beta[0,1]$ is equipped with, because they only serve as the labels of the propagators and do not play any essential role in the following calculation.

By using eqs.(C.2-8) and the following useful identity



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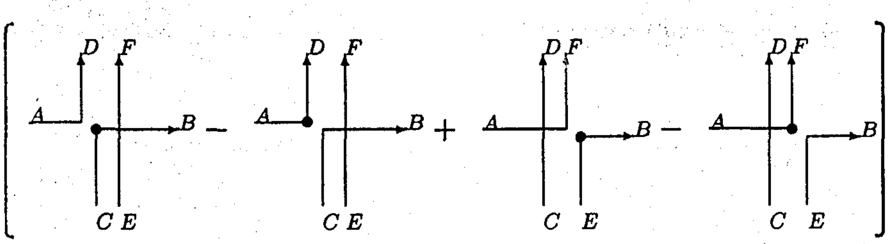

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which is easily proved by means of the two-spinor identity (3.31), and by performing somewhat lengthy calculations on permutations and combinations, we find

$$\hat{S}^{ren}(\underset{\sim}{N})C^r(m,n) = (m+n-2r+1)B^r(m,n) + 2B^{r+1}(m,n). \tag{4.20}$$

Now it is an elementary exercise of the linear algebra to derive from eq.(4.20) the equation

$$\hat{S}^{ren}(\underset{\sim}{N}) \left( \sum_{r=0}^{\min(m,n)} \left( \prod_{i=1}^{r} \frac{-2}{m+n-2i+1} \right) C^r(m,n) \right) = 0. \tag{4.21}$$

Thus we have found a set of solutions to the renormalized WD equation *at one vertex* $x_0 = \alpha(s_0) = \beta(t_0)$:

$$\prod_{i=1}^{m} (h_\alpha[0,s_0]^{A_i}{}_{E_i} h_\alpha[s_0,1]^{F_i}{}_{B_i}) \prod_{j=1}^{n} (h_\beta[0,t_0]^{C_j}{}_{G_j} h_\beta[t_0,1]^{H_j}{}_{D_j}) \times I^\circ(m,n)^{E_1\cdots E_m;G_1\cdots G_n}_{F_1\cdots F_m;H_1\cdots H_n}, \tag{4.22}$$

where $I^\circ(m,n)$ is the relevant intertwining operator:

$$\begin{aligned}
I^\circ(m,n)^{E_1\cdots E_m;G_1\cdots G_n}_{F_1\cdots F_m;H_1\cdots H_n} &\equiv \sum_{r=0}^{\min(m,n)} \left( \prod_{i=1}^{r} \frac{-2}{m+n-2i+1} \right) I^r(m,n)^{E_1\cdots E_m;G_1\cdots G_n}_{F_1\cdots F_m;H_1\cdots H_n}, \\
I^r(m,n)^{E_1\cdots E_m;G_1\cdots G_n}_{F_1\cdots F_m;H_1\cdots H_n} &\equiv \sum_{\substack{1\le k_1<\cdots<k_r\le m \\ 1\le l_1<\cdots<l_r\le n}} \sum_{\sigma\in P_r} \\
&\times \left( \prod_{i=1}^{r} \delta^{E_{k_i}}_{H_{l_{\sigma_i}}} \delta^{G_{l_{\sigma_i}}}_{F_{k_i}} \prod_{\substack{k'=1 \\ k'\neq k_1,\cdots,k_r}}^{m} \delta^{E_{k'}}_{F_{k'}} \prod_{\substack{l'=1 \\ l'\neq l_1,\cdots,l_r}}^{n} \delta^{G_{l'}}_{H_{l'}} \right).
\end{aligned} \tag{4.23}$$

We should note that eq.(4.22) remains to be the solution even if we permutate the external indices. Moreover, from eq.(3.35), we see that the antisymmetrized tensor product of two spinor propagators yields the invariant spinor. As a result we have only to consider symmetrized tensor products of the spinor propagators, say

$$h_\alpha[0,s_0]^{A_1}{}_{(E_1}\cdots h_\alpha[0,s_0]^{A_m}{}_{E_m)}.$$

Taking these into account we can provide the procedure to construct a combinatorial solution defined on a regular four-valent graph $\Gamma^{reg}$:
1) Extract from $\Gamma^{reg}$ all the smooth loops $\alpha_i : [0,1] \to M^{(3)}$ $(i = 1,2,\cdots,N; \alpha_i(0) = \alpha_i(1))$.
2) Equip each loop $\alpha_i$ with a Wilson loop in the spin-$\frac{p_i}{2}$ representation, which is given by the symmetrized trace of $p_i$ spinor propagators:

$$h_{\alpha_i}[0,1]^{A_1}{}_{(A_1}\cdots h_{\alpha_i}[0,1]^{A_{p_i}}{}_{A_{p_i})}.$$

3) At each point where two loops, say, $\alpha_i$ and $\alpha_j$ intersect, cut the spinor Wilson loops and rejoin them by using as a glue the intertwining operator $I^\circ(p_i,p_j)$ defined by eq.(4.23).

We will denote the combinatorial solution constructed by this procedure as $\Psi^{\Gamma^{reg}}_{\{p_i\}}[A]$. If we fix a regular four-valent graph $\Gamma^{reg}$ which consists of loops $\{\alpha_i\}$ $(i = 1,\cdots,N)$, we can construct $(\mathbf{Z}_+)^N$ combinatorial solutions where $\mathbf{Z}_+$ is the set of non-negative integers. The analytic loop solutions described in the last subsection are contained in the combinatorial solutions $\Psi^{\Gamma^{reg}}_{\{p_i\}}[A]$ as particular cases in which all the loops $\alpha_i$ in the graph $\Gamma^{reg}$ do not intersect.

Extension of Jacobson and Smolin's solutions to the case where three or more smooth curves intersect at a point was explored in ref.[49] and several extended solutions were found in the spinor representation. While we will not explicitly examine here, it is in principle possible to generalize these extended solutions to arbitrary finite dimensional representations. In order to carry out this, we need the action of the renormalized scalar constraint on vertices at which three or more smooth curves intersect.

## 4.5 Issues on the combinatorial solutions

In this section, we have constructed the simplest type of solutions which are composed of spin network states, namely the combinatorial solutions $\{\Psi^{\Gamma^{reg}}_{\{p_i\}}[A]\}$ each of which are completely determined by fixing a regular four-valent graph $\Gamma^{reg}$ and a set of non-negative integers $\{p_i\}$ assigned to each smooth loops in $\Gamma^{reg}$. It turns out, however, that the classical counterparts of these combinatorial solutions are spacetimes with degenerate metric $\det(q_{ab}) = 0$. We can easily see this by the following discussion.

First we are aware that the scalar constraint can be expressed by the product of the magnetic field $\tilde{B}^{ai}$ and the 'densitized co-dreibein' $\tilde{e}^i_a \equiv \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\tilde{\pi}^{bj}\tilde{\pi}$ :

$$\hat{\mathcal{S}} = \epsilon_{ijk}F^i_{ab}\hat{\tilde{\pi}}^{aj}\hat{\tilde{\pi}}^{bk} = 2\tilde{B}^{ai}\hat{\tilde{e}}^i_a. \tag{4.24}$$

From the way of their construction, we see that the combinatorial solutions satisfy the following equation

$$(\hat{\tilde{e}}^i_a)^{ren}\Psi^{\Gamma^{reg}}_{\{p_i\}}[A] = 0, \tag{4.25}$$

where $(\hat{\tilde{e}}^i_a)^{ren}$ is the renormalized densitized co-dreibein

$$(\hat{\tilde{e}}^i_a)^{ren}(x) \equiv \lim_{\epsilon\to 0} 2\epsilon \int_{M^{(3)}} d^3y \tilde{f}_\epsilon(x,y)\frac{1}{2}\underset{\sim}{\epsilon}_{abc}\epsilon^{ijk}\frac{\delta}{\delta A^j_b(x)}\frac{\delta}{\delta A^k_c(y)}.$$

On the other hand, the volume operator (3.28) can be rewritten by using $\hat{\tilde{e}}^i_a$

$$\hat{V}(\mathcal{R}) = \int_{\mathcal{R}} d^3x \left(\frac{1}{3}\hat{\tilde{\pi}}^{ai}\hat{\tilde{e}}^i_a\right)^{\frac{1}{2}}. \tag{4.26}$$

By comparing eqs.(4.25) and (4.26) we conclude that, if there exists a regularization which can deal with $\hat{\mathcal{S}}$ and $\hat{V}$ consistently, the combinatorial solutions correspond to spatial geometries whose volume element vanishes everywhere in $M^{(3)}$. While at present we do not know such a regularization in the continuum theory, this situation actually happens for some lattice version of these operators.

Besides this undesirable feature, there is an argument[65] that the combinatorial solutions are 'spurious solutions' which are not genuine solutions to the Wheeler-De Witt equation $\hat{\mathcal{H}}\Psi = 0$ in quantum general relativity. The reasoning is the following. Classically the scalar constraint $\mathcal{S}$ in Ashtekar's formalism is essentially the volume element $(\det(\tilde{e}^{ai}))^{\frac{1}{2}}$ times the Hamiltonian constraint $\mathcal{H}$ in the ADM formalism. In the present operator ordering, $\Psi^{\Gamma^{reg}}_{\{p_i\}}[A]$ is eliminated by $(\widehat{\det(\tilde{e}^{ai})})^{\frac{1}{2}}$ and not by $\hat{\mathcal{H}}$. Thus we can consider that the combinatorial solutions are 'spurious solutions' which do not realize correctly the quantum version of spacetime diffeomorphism invariance.

The above arguments indicate that a correct solution $\Psi^{corr}[A]$ to the WD equation must satisfy the conditions

$$\hat{\mathcal{S}}\Psi^{corr}[A] = 0 \quad \text{and} \quad \hat{\tilde{e}}^i_a\Psi^{corr}[A] \neq 0. \tag{4.27}$$

In search for these correct solutions in the framework of spin network states, contributions from the magnetic field $\tilde{B}^{ai}$, or equivalently the area derivative $\Delta_{ab}$ will play an essential role. Unfortunately, we do not yet know how to define the area derivative unambiguously in the continuum theory. Under this situation, it seems to be useful to investigate as a heuristic model the lattice discretized formulation, because it allows us to fix a definition of the area derivative. So let us explore in the next section a lattice version of Ashtekar's formalism.

# 5 Discretized Ashtekar's formalism

A lattice discretized version of Ashtekar's formalism was first proposed by Renteln and Smolin[76] and later developed by Loll[61][62]. This formulation is based on the idea of the Hamiltonian lattice gauge theory of Kogut and Susskind[56].

We will first explain the setup. This model is defined on a 3 dimensional cubic lattice $\Gamma^N$ of size N. We will label lattice sites by $n$ and three positive directions of links by $\hat{a}$. We take the lattice spacing to be $a$. The $SL(2,\mathbf{C})$ connection $A^i_a(x)$ is replaced by the link variables $V(n,\hat{a})^A{}_B \in SL(2,\mathbf{C})$, which is regarded as a parallel propagator along a link:

$$\begin{aligned} V(n,\hat{a})^A{}_B &= \mathcal{P}\exp[a\int_0^1 ds A^i_{\hat{a}}(n+s\hat{a})J_i]^A{}_B \\ &= [\mathbf{1}+\frac{a}{2}(A^i_{\hat{a}}(n)+A^i_{\hat{a}}(n+\hat{a}))J_i+O(a^2)]^A{}_B. \end{aligned} \tag{5.1}$$

Next we recall that the conjugate momentum $\tilde{\hat{\pi}}^{ai}$ acts on the parallel propagator $h_\alpha[0,1]$ as

$$\tilde{\hat{\pi}}^{ai}(x)h_\alpha[0,1] = -\int_0^1 ds\delta^3(x,\alpha(s))\dot{\alpha}^a(s)h_\alpha[0,s]J_ih_\alpha[s,1].$$

From this we see that a lattice analog of the conjugate momentum $\tilde{\hat{\pi}}^{ia}$ is given by

$$\pi^i(n,\hat{a}) = p^i_L(n,\hat{a}) + p^i_R(n,\hat{a}), \tag{5.2}$$

where $p^i_L$ and $p^i_R$ respectively stand for the left translation and the right translation operators:

$$\begin{aligned} p^i_L(n,\hat{a})V(n,\hat{b})^A{}_B &= \delta_{\hat{a},\hat{b}}(J_iV(n,\hat{b}))^A{}_B \\ p^i_R(n,\hat{a})V(n-\hat{b},\hat{b})^A{}_B &= \delta_{\hat{a},\hat{b}}(V(n-\hat{b},\hat{b})J_i)^A{}_B. \end{aligned} \tag{5.3}$$

The relation between momenta in the lattice and continuum theories is given by

$$\pi^i(n,\hat{a}) = -2a^2\tilde{\hat{\pi}}^{\hat{a}i}(n) + O(a^3). \tag{5.4}$$

While redundant for the completeness of the variables, it is sometimes convenient to introduce link variables in the negative direction $V(n,-\hat{a}) \equiv V(n-\hat{a},\hat{a})^{-1}$. Then by using the relation

$$p^i_R(n,\hat{a}) = -p^i_L(n,-\hat{a}), \tag{5.5}$$

the lattice momenta are expressed in terms only of the left translation

$$\pi^i(n,\hat{a}) = p^i_L(n,\hat{a}) - p^i_L(n,-\hat{a}). \tag{5.6}$$

The lattice counterpart of the curvature $F^i_{ab}$ is provided by plaquette loop variables:

$$\begin{aligned} V(n,P_{\hat{a}\hat{b}})^A{}_B &\equiv [V(n,\hat{a})V(n+\hat{a},\hat{b})V(n+\hat{a}+\hat{b},-\hat{a})V(n+\hat{b},-\hat{b})]^A{}_B \\ &= [\mathbf{1}+a^2F^i_{\hat{a}\hat{b}}(n)J_i+O(a^3)]^A{}_B. \end{aligned} \tag{5.7}$$

In order to formulate a lattice version of Ashtekar's formalism, we have to translate the constraint operators $\hat{G}^i$, $\hat{\mathcal{D}}_a$ and $\hat{\mathcal{S}}$ in terms of lattice variables. Among them, Gauss' law constraint $\hat{G}^i$ is solved by considering only gauge-invariant functionals of link variables, namely, spin network states defined on the lattice $\Gamma^N$. While it is difficult in general to impose the diffeomorphism constraint (2.30) in lattice formulations, we formally solve this constraint by regarding our lattice to be a purely topological object[61]. This is based on the idea of diffeomorphism invariant states explained in §§3.4. Thus we are left only with the scalar constraint $\hat{\mathcal{S}}$.

A plausible candidate for the discretized version of the scalar constraint which respects the symmetry under $\frac{\pi}{2}$ rotations around the $\hat{a}$-axis is given by [37]

$$\begin{aligned} \hat{\mathcal{S}}_I(n) &\equiv \sum_{\hat{a}<\hat{b}}\epsilon^{ijk}\mathrm{Tr}(-\widetilde{V}(n,\hat{a}\hat{b})J_k)\pi^i(n,\hat{a})\pi^j(n,\hat{b}) \\ &= a^6\hat{\mathcal{S}}(n)+O(a^7), \end{aligned} \tag{5.8}$$

where $\widetilde{V}(n,\hat{a}\hat{b})\equiv\frac{1}{4}(V(n,P_{\hat{a}\hat{b}})+V(n,P_{\hat{b},-\hat{a}})+V(n,P_{-\hat{a},-\hat{b}})+V(n,P_{-\hat{b},\hat{a}}))$ is the 'averaged' plaquette loop variable. The action of $\hat{\mathcal{S}}_I$ on lattice spin network states can be described in the form which is analogous to that of $\hat{\mathcal{S}}^{ren}$ in the last section. The only difference is that the insertion of $\underset{\sim}{n}\cdot\tilde{B}$ in the latter is replaced by the insertion of $\frac{1}{4}(\widetilde{V}-\widetilde{V^{-1}})$. The discretized WD equation using $\hat{\mathcal{S}}_I$ therefore has as its solutions the lattice version of combinatorial solutions $\Psi^{\Gamma^N}_{\{p_i\}}[V]$ in which the smooth loops are replaced by straight Polyakov loops [61].

However we are interested only in finding a clue to construct 'correct' solutions which become solutions only after the action of the area derivative is taken into account. For this purpose $\hat{\mathcal{S}}_I$ does not seem to be suitable because it contains $16\times 3$ terms at a vertex. So we will henceforth use in our analysis the 'truncated' scalar constraint

$$\hat{\mathcal{S}}_{II}(n)=\sum_{\hat{a}<\hat{b}}\sum_{\eta_1,\eta_2=\pm}\epsilon^{ijk}\mathrm{Tr}(-V(n,P_{\eta_1\hat{a},\eta_2\hat{b}})J_k)p^i_L(n,\eta_1\hat{a})p^j_L(n,\eta_2\hat{b}). \tag{5.9}$$

The number of terms appearing in $\hat{\mathcal{S}}_{II}$ is a quarter of that in $\hat{\mathcal{S}}_I$. This enables us to simplify the analysis.

We do not think that our discretized WD equation yield solutions which become in the continuum limit genuine solutions to the continuum WD equation. From our analysis, however, we expect to extract some instructive information on the correct solutions.

We can define measures in the lattice formulation in quite an analogous manner to that in which the measures for spin network states are defined. For example, the induced

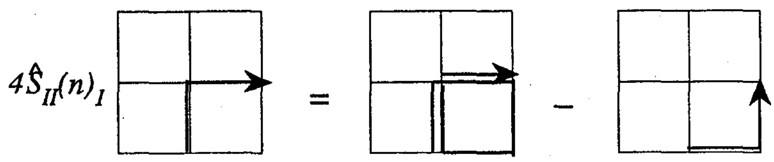


Figure 4: The action of the truncated scalar constraint on a single kink.

Haar measure is defined by eq.(3.6) with the edges $\{e\}$in the graph $\Gamma$ being replaced by the links $\{\ell\}$ in the lattice $\Gamma^N$. The induced heat-kernel measure is also defined in a similar way by using eq.(3.17). In this case, however, we need to fix a length function $l(\ell)$. A convenient and reasonable choice is to set $l(\ell) = a$. In the following we will adopt this choice.

## 5.1 Multi-plaquette solutions

Let us now search for non-trivial solutions to the truncated WD equation[37]

$$\hat{\mathcal{S}}_{II}(n)\Psi[V] = 0 \quad \text{for} \quad {}^\forall n \in \Gamma^N. \tag{5.10}$$

For this aim we have to evaluate the action of $\hat{\mathcal{S}}_{II}(n)$ on lattice spin network states. This is executed in essentially the same way as that of calculating the action of $\hat{\mathcal{S}}^{ren}(\underset{\sim}{N})$. In particular, we can separate its action as

$$\hat{\mathcal{S}}_{II}(n) = \hat{\mathcal{S}}_{II}(n)_1 + \hat{\mathcal{S}}_{II}(n)_2, \tag{5.11}$$

where $\hat{\mathcal{S}}_{II}(n)_1$ and $\hat{\mathcal{S}}_{II}(n)_2$ denote respectively the action on the single link-chains and that on the pairs of link-chains.

For example, the action on kinks $V(n-\hat{a},\hat{a})V(n,\hat{b})$ $(\hat{a} \neq \hat{b})$ is computed as follows

$$\begin{aligned}
\hat{\mathcal{S}}_{II}(n)_1 \cdot \{V(n-\hat{a},\hat{a})V(n,\hat{b})\}^A{}_B &= \frac{1}{4}\{V(n-\hat{a},\hat{a})(V(n,P_{\hat{b},-\hat{a}}) - V(n,P_{-\hat{a},\hat{b}}))V(n,\hat{b})\}^A{}_B, \\
\hat{\mathcal{S}}_{II}(n)_2 \cdot \left(\{V(n-\hat{a},\hat{a})V(n,\hat{b})\}^A{}_B \{V(n-\hat{a},\hat{a})V(n,\hat{b})\}^C{}_D\right) &= 0.
\end{aligned} \tag{5.12}$$

Topologically, the former action can be interpreted as taking the difference of the results of inserting plaquettes with the opposite orientations (figure 4). The action on the other types of vertices can also be interpreted in terms of combinatorial topology of the lattice graphs.

Now we are in a position to construct the simplest "nontrivial solutions" on which the action of the area derivative essentially contributes to the vanishing action of $\hat{S}_{II}$.

We first consider the action on the trace of the $p$-th power of a plaquette. This is calculated by using eq.(5.12)

$$\hat{S}_{II}(n)\cdot \mathrm{Tr}(V(n,P_{\hat{a}\hat{b}})^p) = \frac{p}{4}\left(\mathrm{Tr}(V(n,P_{\hat{a}\hat{b}})^{p+1}) - \mathrm{Tr}(V(n,P_{\hat{a}\hat{b}})^{p-1})\right). \tag{5.13}$$

This equation is reinterpreted as

$$\hat{S}_{II}(n)\cdot \mathrm{Tr}F(V(n,P_{\hat{a}\hat{b}})) = \mathrm{Tr}\left[\frac{V(n,P_{\hat{a}\hat{b}})^2-1}{4}\frac{d}{dV}F(V(n,P_{\hat{a}\hat{b}}))\right], \tag{5.14}$$

where $F$ denotes an arbitrary polynomial. We can readily extend this equation to the case where $F$ is a function which can be expressed by a Laurent series. Thus we find

$$\hat{S}_{II}(n)\cdot \mathrm{Tr}\log\left(\frac{1-V(n,P_{\hat{a}\hat{b}})}{1+V(n,P_{\hat{a}\hat{b}})}\right) = 1. \tag{5.15}$$

As for the action of $\hat{S}_{II}(m)$ with $m \neq n$, the following can be said. When $m$ coincides with one of the vertices of the plaquette $P_{\hat{a}\hat{b}}$, the result is identical to eq.(5.15) owing to the symmetry of $\hat{S}_{II}$. When $m$ does not coincide, on the other hand, the action necessarily vanishes. These results are summarized as

$$\hat{S}_{II}(m)\cdot \mathrm{Tr}\log\left(\frac{1-V(n,P_{\hat{a}\hat{b}})}{1+V(n,P_{\hat{a}\hat{b}})}\right) = \begin{cases} 2 & \text{for } m = n, n+\hat{a}, n+\hat{b}, n+\hat{a}+\hat{b}, \\ 0 & \text{for } m \neq n, n+\hat{a}, n+\hat{b}, n+\hat{a}+\hat{b}. \end{cases} \tag{5.16}$$

Now we can provide the prescription for constructing "multi-plaquette solutions" on which the action of the area derivative is essential[37]: i) prepare a connected set of plaquettes $\{P\}$in which each vertex belongs to at least two plaquettes; ii) assign to each plaquette $P$ a weight factor $w(P)$ so that the sum of weight factors of the plaquettes which meet at each vertex vanishes; iii) the following expression yields a solution to eq.(5.10)

$$< V|\{w(P)\} >\equiv \sum_{P\in\{P\}} w(P)\mathrm{Tr}\log\left(\frac{1-V(P)}{1+V(P)}\right). \tag{5.17}$$

Now we will investigate whether or not the multi-plaquette solutions are normalizable w.r.t. some measures on the lattice gauge theory. It is convenient to rewrite the result by means of the Clebsch-Gordan decomposition. We see that $\log(\frac{1-V}{1+V})$ is proportional to the "one-plaquette state"[37]

$$< V|\log(\frac{1-P}{1+P}) >\equiv \sum_{p=0}^{\infty}\frac{1}{(2p+3)(2p+1)}\mathrm{Tr}S(V(P)^{2p+1}), \tag{5.18}$$

where $\mathrm{Tr}S(V^m)$ is the symmetrized trace

$$\mathrm{Tr}S(V^m) \equiv V^{A_1}{}_{(A_1} \cdots V^{A_m}{}_{A_m)} = \mathrm{Tr}\pi_m(V).$$

First we examine the normalizability w.r.t. the induced Haar measure $d\mu_H(V)$. Owing to the consistency of the measure and because one-plaquette states defined on different plaquettes are orthogonal with each other, it is sufficient to investigate the norm of a one-plaquette state

$$\| < V| \log(\frac{1-P}{1+P}) > \|_{d\mu_H} = \int d\mu_H(V(P)) | < A| \log(\frac{1-P}{1+P}) > |^2.$$

Using the formula (B.5) this is easily evaluated as

$$\| < V| \log(\frac{1-P}{1+P}) > \|_{d\mu_H} = \sum_{p=0}^{\infty} \frac{1}{(2p+3)^2(2p+1)^2} < \infty, \tag{5.19}$$

namely the multi-plaquette solutions are normalizable w.r.t. the induced Haar measure.

Next we consider the induced heat-kernel measure $d\nu_t^l(V)$. The above reduction of the problem holds also in this case. By taking $l(P) = 4a$ into account and by using eq.(B.10) we find

$$\begin{aligned} \| < V| \log(\frac{1-P}{1+P}) > \|_{d\nu_t^l} &\equiv \int d\nu_t^l(V(P)) | < V| \log(\frac{1-P}{1+P}) > |^2 \\ &= \sum_{p=0}^{\infty} \frac{e^{at(2p+3)(2p+1)}}{(2p+3)^2(2p+1)^2} \to \infty. \end{aligned} \tag{5.20}$$

The multi-plaquette solutions therefore turn out to be non-normalizable w.r.t. the induced heat-kernel measure.

## 5.2 Lessons to the continuum theory

In this section we have investigated a lattice version of the WD equation and obtained a set of nontrivial solutions to the truncated WD equation (5.10), namely, multi-plaquette solutions (5.17).

We should note that, while these solutions are nontrivial in the sense that they have non-vanishing action of the densitized co-triad operator, they still correspond to three dimensional manifolds with degenerate metric. This is because the bi- and trivalent vertices have zero eigenvalue of the volume operator [62]. In this sense we have achieved only a half of our goal. In order to find solutions with non-degenerate metric we have to deal with lattice spin network states containing at least four-valent vertices. This seems to be a quite hard task and is left to the future investigation.

We expect, however, that these multi-plaquette solutions already provide us with some lessons which are helpful in searching for correct solutions to the continuum WD equation.

We have seen in the last subsection that the multi-plaquette solutions are not normalizable w.r.t. the induced heat-kernel measure. This can be traced back to the fact that each multi-plaquette solution is expressed by a sum of infinitely many spin network states. This seems to be indispensable when we want to have a non-trivial mechanism of cancelling the action of the scalar constraint. Thus we conjecture that the correct solutions are, if any, not normalizable w.r.t. the induced heat-kernel measure on $SL(2, \mathbf{C})$ while they may be normalizable w.r.t. the induced Haar measure on $SU(2)$. But this should not be taken so seriously, because the induced heat-kernel measure at most serves as a kinematical measure as is explained in §§3.3. Or we may rather use this non-normalizability w.r.t. the induced heat-kernel measure as a criterion for determining whether an obtained solution is physical or not.

We can expect that a 'correct' physical state in the lattice formulation consists of an infinite number of lattice spin network states each of which have their 'partners'. The partners are different from the original term by addition or removal of two plaquettes. If we try to reproduce this situation in the continuum theory, we have to extend our framework of the spin network states so that we can deal with graphs which are not piecewise analytic [19]. This seems to be a difficult but essential subject left to us.

# 6 Topological solutions

In the last two sections we have seen that any solutions which have constructed so far by means of spin network states do not correspond to spacetimes with non-degenerate metric. From this one might suspect that canonical quantization of general relativity do not yield any physically meaningful results and thus it is not worthwhile investigating Ashtekar's formulation for quantum gravity much further. It is the existence of topological solutions which sweeps off this suspicion. In this section we will look into these topological solutions focusing on their spacetime geometrical aspects.

## 6.1 $SL(2, \mathbf{C})$ BF theory and topological solutions

In order to see the existence of topological solutions, it is the easiest to show the relation between Ashtekar's formalism and $SL(2, \mathbf{C})$ BF theory[18]. As we have seen in §§2.1, Ashtekar's formalism is obtained from $SL(2, \mathbf{C})$ BF theory

$$-iI_{BF} = \int_M (\Sigma^i \wedge F^i + \frac{\Lambda}{6}\Sigma^i \wedge \Sigma^i) \tag{6.1}$$

[19] A piecewise analytic graph is a graph which can be composed of a finite number of smooth edges without intersection. While most of the results on the spin network states are verified on the piecewise analytic graphs, it is desirable that they should be extended to the piecewise smooth graphs or more general contexts. A piecewise smooth graph may contain smooth curves with infinite number of intersections in a finite interval.

by imposing the algebraic constraint (2.8) which is equivalent to

$$\Sigma^i \wedge \Sigma^j = \frac{1}{3}\delta^{ij}\Sigma^k \wedge \Sigma^k. \tag{6.2}$$

The first class constraints $(G^i, \mathcal{V}_a, \mathcal{S})$ in Ashtekar's formalism can therefore be expressed as linear combinations of those $(G^i, \Phi^{ai})$ in $SL(2,\mathbf{C})$ BF theory:

$$\begin{aligned} G^i &= G^i \\ \mathcal{V}_a &= C^i_{ab}\Phi^{bi} \\ \mathcal{S} &= \widetilde{C}_{ai}\Phi^{ai}, \end{aligned} \tag{6.3}$$

where $C^i_{ab}$ and $\widetilde{C}_{ai}$ are functionals only of the momenta $\tilde{\pi}^{ai}$.

In consequence, if we take the operator ordering with momenta to the left, all the solutions to the quantum $SL(2,\mathbf{C})$ BF constraints

$$\begin{aligned} \hat{G}^i\Psi[A] &= -D_a\left(\frac{\delta}{\delta A^i_a}\Psi[A]\right) = 0 \\ \hat{\Phi}^{ai}\Psi[A] &= \left(\frac{1}{2}\tilde{\epsilon}^{abc}F^i_{bc} - \frac{\Lambda}{3}\frac{\delta}{\delta A^i_a}\right)\Psi[A] = 0 \end{aligned} \tag{6.4}$$

are contained in the solution space of quantum Ashtekar's formalism. They are the topological solutions, which take different forms according to whether the cosmological constant $\Lambda$ vanishes or not.

For $\Lambda \neq 0$ we have "Chern-Simons solutions" [55][27]

$$\Psi^{CS}_I[A] = I[A]\exp\{\frac{3}{2\Lambda}S_{CS}[A]\}, \tag{6.5}$$

where $I[A]$ is some topological invariant of the $SL(2,\mathbf{C})$ connection and $S_{CS}[A]$ is the Chern-Simons invariant

$$S_{CS}[A] \equiv \int_{M^{(3)}}\left(A^i dA^i + \frac{1}{3}\epsilon^{ijk}A^i \wedge A^j \wedge A^k\right). \tag{6.6}$$

We see from eq.(6.5) that the number of linearly independent Chern-Simons solutions is equal to that of principal $SL(2,\mathbf{C})$ bundles over $M^{(3)}$. This is consistent with the classical result because the reduced phase space for $SL(2,\mathbf{C})$ BF theory with $\Lambda \neq 0$ is a set of discrete points each of which represents a principal $SL(2,\mathbf{C})$ bundle over $M^{(3)}$[18].

For $\Lambda = 0$, all the gauge-invariant wavefunctions with their support only on flat connections become solutions to eq.(6.4). This was first pointed out by Brencowe[20]. Each of these solutions is formally described as

$$\Psi_{topo}[A] = \psi[A]\prod_{x\in M^{(3)}}\left(\prod_{i,a}\delta(\tilde{\epsilon}^{abc}F^i_{bc}(x))\right), \tag{6.7}$$

where $\psi[A]$ is a gauge-invariant functional of the $SL(2,\mathbf{C})$ connection. Because the defining region of $\psi[A]$ is reduced to the space of flat connections, we can regard $\psi[A]$ as a function on the moduli space $\mathcal{N}$ of flat $SL(2,\mathbf{C})$ connections on $M^{(3)}$ modulo small gauge transformations. This allows us to give an alternative expression for $\Psi_{topo}[A]$

$$\Psi_{topo}[A] = \int_{\mathcal{N}} dn\psi(n)\Psi_n[A] \tag{6.8}$$

$$\Psi_n[A] = \int[dg(x)]\prod_x\prod_{a,i}\delta(A_a^i(x) = g[A_{0n}]_a^i(x)), \tag{6.9}$$

where $A_{0n}$ is a flat $SL(2,\mathbf{C})$ connection which represent a point $n$ on the moduli space $\mathcal{N}$ and

$$g[A]_a^i(x)J_i \equiv g(x)A_a^i(x)J_i g^{-1}(x) + g(x)\partial_a g^{-1}(x)$$

denotes the gauge-transformed $SL(2,\mathbf{C})$ connection. This expression of the topological solutions is essentially a kind of Fourier transform of the expression given in ref.[66]. It was discussed in [66] that it is sufficient to take the integration region in eq.(6.9) to be the space of small $SU(2)$ gauge transformations.

Because the topological solutions are solutions to $SL(2,\mathbf{C})$ BF theory, most of their properties can be clarified by investigating the quantization of this theory. Some of the works of this kind can be seen, for example, in [27][28][60][29][13] and several interesting results have been found. However, if we investigate the pure BF theory we cannot extract any information on the spacetime geometry in which the general relativitist are the most interested. In order to investigate spacetime geometrical feature of the solutions to quantum gravity, however, we need a physical inner product and physical observables which measure quantities necessary to reconstruct spacetimes. This is left to the future investigation because we do not know any physical inner product and any such physical observables. We can nevertheless provide particular types of solutions with spacetime geometrical interpretations *semiclassically*, and this is the case with the topological solutions.

In the following we will give semiclassical interpretations for these topological solutions. We first review the $\Lambda \neq 0$ case and then look into the $\Lambda = 0$ case.

## 6.2 WKB orbits of the Chern-Simons solutions

It is a well known fact that the Chern-Simons solution takes the form of a WKB wavefunction

$$\Psi_I^{CS}[A] = I[A]e^{iW[A]}, \tag{6.10}$$

where $W[A] = -i\frac{3}{2\Lambda}S_{CS}[A]$ is regarded as a Hamilton principal functional subject to the Hamilton-Jacobi equation:

$$\frac{\partial}{\partial t}W[A] + H(A_a^i, \tilde{\pi}^{ai} = -i\frac{\delta}{\delta A_a^i}W[A]) = 0. \tag{6.11}$$

Hamiltonian in eq.(6.11) is given by

$$\begin{aligned} H &= -G(A_t - N^a A_a) + \mathcal{D}(\vec{N}) + S(\underset{\sim}{N}) \\ &= i\int_{M^{(3)}} d^3x (D_a A_t^i \tilde{\pi}^{ai} - \Sigma_{ta}^i(\tilde{\pi}^{bj})\Phi^{ai}), \end{aligned} \tag{6.12}$$

where $\Sigma_{ta}^i(\tilde{\pi}^{bj})$ is the solution to eq.(2.8) whose explicit form is given by eq.(2.14). Now we can semiclassically interpret the WKB wavefunction (6.10) by investigating a family of corresponding classical solutions (WKB orbits). This was first investigated by Kodama[55] in the minisuperspace model. Here we will review his result in terms of generic spacetimes. WKB orbits are determined by the following equations

$$\tilde{\pi}^{ai} = -i\frac{\delta}{\delta A_a^i} W = -\frac{3}{\Lambda}\tilde{B}^{ai} \tag{6.13}$$

$$\begin{aligned} \frac{\partial}{\partial t} A_a^i &= \{A_a^i, H\}_{PB} \\ &= D_a A_t^i - \frac{\Lambda}{3}\Sigma_{ta}^i(\tilde{\pi}^{bj}) - i\int_{M^{(3)}} d^3x \{A_a^i, \Sigma_{ta}^i(\tilde{\pi}^{bj})\}_{PB}\Phi^{ai}. \end{aligned} \tag{6.14}$$

Eq.(6.13) is called the Ashtekar-Renteln ansatz[11]. Because the last term in eq.(6.14) vanishes on account of eq.(6.13), these equations can be summarized into a covariant form

$$F_{\mu\nu}^i = -\frac{\Lambda}{3}\Sigma_{\mu\nu}^i. \tag{6.15}$$

If we stick to the region of Lorentzian signature, the WKB orbits are subject to further restrictions. Namely, by imposing the (covariant) reality conditions (2.7)(2.8), we find

$$R^{\alpha\beta} = \frac{\Lambda}{3} e^\alpha \wedge e^\beta. \tag{6.16}$$

This tells us that the Chern-Simons solutions correspond to a family of spacetimes which are locally de Sitter[20]:

$$ds_{DS}^2 = \frac{3}{\Lambda}[-d\xi^2 + \cosh^2\xi (d\Omega_{(3)})^2], \tag{6.17}$$

where $(d\Omega_{(3)})^2$ denotes the standard line element on $S^3$.

However, this is not the whole story. From the Ashtekar-Renteln ansatz (6.13), we can read off the 3-metric $q_{ab}$ which is given by

$$(q^{-1})^{ab} = -\frac{\Lambda}{3}(\det(\tilde{B}^{ck}))^{-1}\tilde{B}^{ai}\tilde{B}^{bi}. \tag{6.18}$$

Sticking to the Lorentzian signature thus amounts to restricting ourselves to the region $\det(\tilde{B}^{ck}) < 0$. However, because $\Psi_I^{CS}[A]$ is well-defined on the whole space of $SL(2, \mathbf{C})$

[20] For simplicity we have assumed that the cosmological constant $\Lambda$ is positive.

connections, there seems to be no natural reason to rule out the region with $\det(\tilde{B}^{ck}) > 0$ (i.e. with $\det(q_{ab}) < 0$)[55]. Spacetimes in such a region corresponds to Euclidean manifolds with signature $(-,-,-,-)$ which represent spacetimes fluctuating about $H^4$ (the 4 dimensional hyperbolic hypersurface):

$$ds^2_{H^4} = -\frac{3}{\Lambda}[d\xi^2 + \sinh^2\xi(d\Omega_{(3)})^2]. \tag{6.19}$$

Moreover, if we consider the pure imaginary time $\xi' = i\xi$ ($\xi$: real) also, we obtain the Euclidean spacetimes with signature $(+,+,+,+)$ fluctuating about the four-sphere $S^4$:

$$ds^2_{S^4} = \frac{3}{\Lambda}[d\xi^2 + \sin^2\xi(d\Omega_{(3)})^2]. \tag{6.20}$$

By appropriately assembling these results, the Chern-Simons solution $\Psi_I^{CS}[A]$ is considered to correspond with spacetimes fluctuating about the sequence

$$H^4 \to S^4 \to dS^4.$$

This gives the picture of "creation of the Lorentzian spacetimes ($dS^4$) from a mother Euclidean space ($H^4$) by way of a temporary bridge ($S^4$)"[55].

Because the deSitter space is a vacuum solution to the Einstein equation in the case of a positive cosmological constant, we expect that the Chern-Simons solutions are vacuum states of quantum gravity with $\Lambda > 0$. This expectation was confirmed by investigating stability of the classical background (6.13) w.r.t. small perturbations [83].

Here we will make a remark. In this section the reality conditions are imposed *classically* in order to extract information on the real quantities $(e^\alpha, \omega^{\alpha\beta})$. In the actual quantum theory, however, reality conditions are imposed *quantum mechanically* through the inner product. As a result the self-dual part $C^{(+)\rho}{}_{\sigma\mu\nu}$ of the conformal curvature may fluctuate[83], while its anti-self dual part $C^{(-)\rho}{}_{\sigma\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$ are fixed almost completely by eq.(6.15) to be

$$C^{(-)\rho}{}_{\sigma\mu\nu} = 0, \quad R_{\mu\nu} = \Lambda g_{\mu\nu}.$$

## 6.3 Semiclassical interpretation of $\Psi_{topo}[A]$

Let us now look into the case of a vanishing cosmological constant. Because $\Psi_{topo}[A]$ does not possess a WKB structure, one may consider that we cannot interpret $\Psi_{topo}[A]$ semiclassically. We can nevertheless give $\Psi_{topo}[A]$ a semiclassical interpretation.

To see this, let us first consider the simplest example, namely the harmonic oscillator with the hamiltonian $h = \frac{1}{2}p^2 + \frac{1}{2}q^2$. If we introduce the complex variable $z \equiv q + ip$, the ground state $< q|0 >= \psi_0(q)$ of this system satisfies

$$\hat{z} \cdot \psi_0(q) = (q + \hbar\frac{d}{dq})\psi_0(q) = 0. \tag{6.21}$$

The normalized ground state is therefore given by

$$\psi_0(q) = (\pi\hbar)^{-1/4} e^{-\frac{1}{2\hbar}q^2}. \tag{6.22}$$

Thus the probability density that the coordinate takes the value $q$ and the one that the momentum takes the value $p$ are, respectively, given by

$$\begin{aligned} | < q|0 > |^2 &= \frac{1}{\sqrt{\pi\hbar}} e^{\frac{1}{\hbar}q^2} \xrightarrow{\hbar\to 0} \delta(q), \\ | < p|0 > |^2 &= \frac{1}{\sqrt{\pi\hbar}} e^{\frac{1}{\hbar}p^2} \xrightarrow{\hbar\to 0} \delta(p). \end{aligned} \tag{6.23}$$

We can conclude that the ground state $|0>$ semiclassically corresponds to the origin $(q,p) = (0,0)$ of the phase space. Of course this is not the case in the quantum sense because $\hat{\bar{z}} \cdot \psi_0(q)$ does not vanish. The essential thing is that the quantum commutator $[\hat{z}, \hat{\bar{z}}] = 2\hbar$ is negligible in the semiclassical region and thus we can 'simultaneously diaginalize' $\hat{z}$ and $\hat{\bar{z}}$ (or equivalently $\hat{q}$ and $\hat{p}$) under the semiclassical approximation.

This 'semiclassical interpretation' is expected to apply also to the topological solutions for Ashtekar's formalism. From its definition, it is obvious that the topological solutions (6.8) correspond to classical solutions with

$$F^i_{ab} = 0. \tag{6.24}$$

Because the temporal components $A^i_t$ only play the role of gauge parameters, they can be arbitrary chosen to yield

$$F^i_{\mu\nu} = 0. \tag{6.25}$$

Now, by classically imposing the reality conditions (2.7), we see that $\Psi_{topo}[A]$ correspond to spacetimes which are flat[21]

$$R^{\alpha\beta} = 0. \tag{6.26}$$

This suggests that the topological solutions represent vacuum states of quantum gravity in the $\Lambda = 0$ case. Some affirmative evidence for this suggestion was provided in ref.[66], which appeals to the analogy of $\Psi_{topo}[A]$ to the typical vacuum states of quantum field theories. For example, that $\Psi_{topo}[A]$ carry the topological degrees of freedom (the moduli $n \in \mathcal{N}$ of flat $SL(2, \mathbf{C})$ connections) is essentially the same as that the vacuum states in a quantum field theory carry the similar ones (e.g. the soliton numbers).

Unlike in the case with $\Lambda \neq 0$ we can exploit from the topological solutions for $\Lambda = 0$ more detailed information on the spacetime. As is immediately seen in eq.(6.9), we can at least in principle distinguish different moduli by means of the 'characteristic' topological

[21] As in the case of $\Psi^{CS}_I[A]$, self-dual part of the conformal curvature still survives slightly in the full quantum theory because the quantum reality conditions cannot completely fix this part.

solutions $\Psi_n[A]$. Giving semiclassical interpretation to $\Psi_n[A]$ thus amounts to finding a family of flat spacetimes corresponding to the moduli $n$.

This is the situation similar to that which we have encountered in (2+1)-dimensional gravity[85]. There the reduced phase space is the moduli space of flat connections in a non-compact gauge theory and a point on the moduli space is related to a (2+1)-dimensional spacetime through a geometric structure [25].[22] This relation was explicitly established in the cases of simple topologies with $\Lambda = 0$[26][63], $\Lambda > 0$[34] and with $\Lambda < 0$[35].

Difference between the problem at hand and that in (2+1)-dimensions is that, in the former, we need only to work out the relation of the space $\mathcal{N}$ of canonical coordinates to the spacetimes, while in the latter we have established the relation between the phase space and the set of flat (or maximally symmetric) spacetimes. We can therefore expect easily that the relation between $\mathcal{N}$ and the space of flat spacetimes is one to infinity (not countable).[23]

Let us investigate in practice what spacetimes correspond to a point $n$ on the moduli space $\mathcal{N}$[39]. For this purpose we will first provide a parametrization of $\mathcal{N}$.

We should note that the moduli space $\mathcal{N}$ in general consists of disconnected sectors which are related with one another by large gauge transformations[18], namely by gauge transformations which cannot be continuously connected to the identity. Because we are now interested in a family of spacetime metrics which corresponds to a point $n \in \mathcal{N}$, it is sufficient to investigate only one of these sectors. For simplicity, we will choose the sector $\mathcal{N}_0$ which is connected to the trivial connection $A_a^i = 0$.

In order to parametrize $\mathcal{N}_0$, it is convenient to use holonomies along non-contractible loops:

$$H(\alpha) \equiv h_\alpha[0,1] \equiv \mathcal{P}\exp\left(\int_0^1 ds\dot{\alpha}^a(s)A_a\right), \tag{6.27}$$

where $\alpha : [0,1] \to M^{(3)}$ ($\alpha(0) = \alpha(1) = x_0$) is a loop on $M^{(3)}$. Because the connection $A_a$ is flat the holonomy depends only on the homotopy class $[\alpha]$ of the loop $\alpha$:

$$H(\alpha) = H(\alpha') \equiv H[\alpha] \quad \text{if} \quad [\alpha] = [\alpha']. \tag{6.28}$$

The (large or small) gauge transformation $A_a(x) \to g(x)A_a(x)g^{-1}(x) + g(x)\partial_a g^{-1}(x)$ on the connection $A_a(x)$ is cast into the conjugation by $g(x_0)$ on the holonomy

$$H[\alpha] \to g(x_0)H[\alpha]g^{-1}(x_0). \tag{6.29}$$

The moduli space $\mathcal{N}_0$ is therefore identical to the space of equivalence classes of homomorphisms from the fundamental group $\pi_1(M^{(3)})$ to the group $SL(2,\mathrm{C})$ modulo conjugations:

$$\mathcal{N}_0 = \mathrm{Hom}(\pi_1(M^{(3)}), SL(2,\mathrm{C}))/\sim . \tag{6.30}$$

[22] For a detailed explanation of the geometric structure, see for example, refs.[44][69].

[23] The relation between the reduced phase space of $SL(2,\mathrm{C})$ BF theory and the space of flat spacetimes are much more complicated [39]. In this paper we will not go into its detail.

Once we have parametrized the moduli space $\mathcal{N}_0$ by using holonomies, we can work out the explicit relation between the moduli $n$ and flat spacetimes. Because we know from eq.(6.25) that the $SL(2,\mathbf{C})$ connection $A^i_a$ is flat, it can be written in the form of a pure gauge on the universal covering $\widetilde{M} \approx \mathbf{R} \times \widetilde{M^{(3)}}$ [24]

$$A^i(\tilde{x}^\mu)J_i = \Lambda^{-1}(\tilde{x}^\mu)d\Lambda(\tilde{x}^\mu), \tag{6.31}$$

where $\Lambda(\tilde{x}^\mu) \in SL(2,\mathbf{C})$ is the integrated connection which is subject to the periodicity condition

$$\Lambda(\gamma + \tilde{x}^\mu) = H[\gamma]\Lambda(\tilde{x}^\mu). \tag{6.32}$$

Next by imposing the reality conditions (2.7) classically, we can (up to gauge degrees of freedom) completely fix a spin connection $\omega^\alpha{}_\beta(\tilde{x}^\mu)$ on $\widetilde{M}$ as

$$\omega^\alpha{}_\beta(\tilde{x}^\mu) = (\Lambda^{-1}(\tilde{x}^\mu))^\alpha{}_\gamma d\Lambda(\tilde{x}^\mu)^\gamma{}_\beta, \tag{6.33}$$

where $\Lambda(\tilde{x}^\mu)^\alpha{}_\beta \equiv (\mathcal{P}\exp\int_0^{\tilde{x}^\mu}\omega(\tilde{y}^\nu))^\alpha{}_\beta$ is the integrated spin connection which is related to $\Lambda(\tilde{x}^\mu)$ in eq.(6.31) by the last two equations in Appendix A.

In order to construct spacetime metrics which are related to this spin connection, we have to search for the vierbein $e^\alpha$ which are single-valued on $M$ and which satisfy the torsion-free condition

$$de^\alpha(\tilde{x}^\mu) + \omega^\alpha{}_\beta(\tilde{x}^\mu) \wedge e^\beta(\tilde{x}^\mu) = 0.$$

Using eq.(6.33), the torsion-free condition is cast into the closedness condition

$$d[\Lambda(\tilde{x}^\mu)^\alpha{}_\beta e^\beta(\tilde{x}^\mu)] = 0.$$

Because $\widetilde{M}$ is simply-connected by definition, this equation is completely solved by

$$e^\alpha(\tilde{x}^\mu) = (\Lambda^{-1}(\tilde{x}^\mu))^\alpha{}_\beta dX^\beta(\tilde{x}^\mu), \tag{6.34}$$

where the set $\{X^\alpha(\tilde{x}^\mu)\}$ is considered to be the embedding of $\widetilde{M}$ into (the universal covering of) an adequate subspace of the (3+1)-dimensional Minkowski space $M^{3+1}$. In order for the vierbein $e^\alpha(\tilde{x}^\mu)$ to be single-valued on $M$, it must satisfy the 'periodicity condition'

$$e^\alpha(\gamma + \tilde{x}^\mu) = e^\alpha(\tilde{x}^\mu) \quad \text{for} \quad {}^\forall[\gamma] \in \pi_1(M), \tag{6.35}$$

(plus some conditions necessary when $\widetilde{M}$ is not contractible to a point). By substituting eq.(6.34) into eq.(6.35) we find

$$dX^\alpha(\gamma + \tilde{x}^\mu) = d(H[\gamma]^\alpha{}_\beta X^\beta(\tilde{x}^\mu)), \tag{6.36}$$

[24] The universal covering $\widetilde{M^{(3)}}$ of $M^{(3)}$ is the space of all the homotopy classes of curves in $M^{(3)}$ starting from, say, the origin $x = 0$. We will decompose the point $\tilde{x}$ on $\widetilde{M^{(3)}}$ as $\tilde{x} = \gamma + x$, which means the curve which first passes along the loop $\gamma$ beginning at the origin $x = 0$ and then goes from the origin to the point $x$ on $M^{(3)}$ by way of the shortest path measured by some positive-definite background metric on $M^{(3)}$. We will denote by $\tilde{x}^\mu = (t, \tilde{x})$ a point on $\widetilde{M}$.

where $H[\gamma]^{\alpha}{}_{\beta}$ stands for the holonomy of the spin connection $\omega^{\alpha}{}_{\beta}$ evaluated along the loop $\gamma$. The relation between $H[\gamma]^{\alpha}{}_{\beta}$ and $H[\gamma]$ is also given by the last two equations in Appendix A. By integrating eq.(6.36) we obtain the most important result:

$$X^{\alpha}(\gamma+\tilde{x}^{\mu}) = H[\gamma]^{\alpha}{}_{\beta}X^{\beta}(\tilde{x}^{\mu}) + V[\gamma]^{\alpha}, \tag{6.37}$$

which states that the periodicity condition of the embedding functions $\{X^{\alpha}(x^{\mu})\}$ is given by Poincaré transformations which are isometries of the Minkowski space. Consistency among the periodicity conditions imposed on all the loops in $\pi_1(M) \cong \pi_1(M^{(3)})$ requires that the set of Poincaré transformations $\{(H[\gamma]^{\alpha}{}_{\beta}, V[\gamma]^{\alpha})| \quad \gamma \in \pi_1(M^{(3)})\}$ should give a homomorphism from $\pi_1(M^{(3)})$ to the (3+1)-dimensional Poincaré group. This is precisely the Lorentzian structure on the manifold $M \approx \mathbf{R} \times M^{(3)}$ [44]. Thus we can say that the moduli of flat $SL(2,\mathbf{C})$ connections on a spacetime manifold $M \approx \mathbf{R} \times M^{(3)}$ specifies a Lorentz transformation part of Lorentzian structures each of which belongs to

$$\mathrm{Hom}(\pi_1(M^{(3)}), (\mathcal{P}^{3+1})^{\uparrow}_{+})/\sim,$$

where $(\mathcal{P}^{3+1})^{\uparrow}_{+}$ denotes the proper orthochronous Poincaré group in (3+1)-dimensions and $\sim$ stands for the equivalence under the conjugation by (proper orthochronous) Poincaré transformations.

Here we should note that a physically permissible Lorentzian structure must be of rank 3 [25] in order to avoid untamable spacetime singularities, and that its action must be spacelike in order to render the spacetime subject to the strong causality condition[47]. We will see in the next subsection that these conditions restrict severely the semiclassically permissible region of the moduli space $\mathcal{N}$, by examining the simple case in which the spatial 3-manifold is homeomorphic to a 3-torus $T^3$.

## 6.4 The case of $M^{(3)} \approx T^3$

Let us now consider the $M^{(3)} \approx T^3$ case[39]. We first construct the moduli space $\mathcal{N}_0$.

Because $\pi_1(T^3) \cong \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$ is an abelian group, the holonomy group on $T^3$ is generated by three commuting elements $(H[\alpha], H[\beta], H[\gamma])$ of $SL(2,\mathbf{C})$. By taking appropriate conjugations we find that the moduli space $\mathcal{N}_0$ consists of several 'disconnected' sectors [26]:

$$\mathcal{N}_0 = \mathcal{N}_S \oplus \left( \bigoplus_{n_1,n_2,n_3 \in \{0,1\}} \mathcal{N}_F^{n_1,n_2,n_3} \right). \tag{6.38}$$

[25] The rank here is meant to be the maximal number of generators in the subgroup.

[26] As in (2+1)-dimensional cases [63][34] [35], these sectors are in fact connected in a non-Hausdorff manner. However, we will not go into this problem.

The standard sector $\mathcal{N}_S$ is characterized by

$$\begin{aligned} H[\alpha] &= \exp((u+ia)J_1) \\ H[\beta] &= \exp((v+ib)J_1) \\ H[\gamma] &= \exp((w+ic)J_1), \end{aligned} \tag{6.39}$$

where $u, v, w$ are real numbers defined modulo $4\pi$ and $a, b, c \in \mathbf{R}$. Gauge equivalence under $g(x_0) = \exp(\pi J_2)$ further imposes the following equivalence condition

$$(a, b, c; u, v, w) \sim -(a, b, c; u, v, w). \tag{6.40}$$

As a consequence the standard sector $\mathcal{N}_S$ has the topology $(T^3 \times \mathbf{R}^3)/\mathbf{Z}_2$. A simple connection which represent the point (6.39) on $\mathcal{N}_S$ is obtained by making an appropriate gauge choice and an adequate choice of periodic coordinates $(x, y, z)$ with period 1. We find

$$A_a dx^a = [(u+ia)dx + (v+ib)dy + (w+ic)dz]J_1 \equiv (dW_1 + idW_2)J_1 = dWJ_1. \tag{6.41}$$

The flat sectors $\mathcal{N}_F^{n_1,n_2,n_3}$ are parametrized by the following holonomies:

$$\begin{aligned} H[\alpha] &= (-1)^{n_1}\exp(\xi(J_2+iJ_1)) \\ H[\beta] &= (-1)^{n_2}\exp(\eta(J_2+iJ_1)) \\ H[\gamma] &= (-1)^{n_3}\exp(\zeta(J_2+iJ_1)), \end{aligned} \tag{6.42}$$

where $\xi, \eta, \zeta$ are complex numbers which do not vanish simultaneously. Gauge equivalence under $g(x_0) = \exp(-i\kappa J_3)$ with $\kappa \in \mathbf{C}$ tells us that $(\xi, \eta, \zeta)$ provide the homogeneous coordinates on $\mathbf{CP}^2$:

$$(\xi, \eta, \zeta) \sim e^{\kappa}(\xi, \eta, \zeta). \tag{6.43}$$

Thus we find $\mathcal{N}_F^{n_1,n_2,n_3} \approx \mathbf{CP}^2$. Because two flat sectors are related with each other by a large local Lorentz transformation like $g(x) = \exp(2\pi(\delta n_1 x + \delta n_2 y + \delta n_3 z)J_3)$, they are considered as corresponding to the same set of spacetimes. Thus in the following we will consider only one flat sector $\mathcal{N}_F \equiv \mathcal{N}_F^{0,0,0}$. As in $\mathcal{N}_S$ we can find a connection which represent a point (6.42) in $\mathcal{N}_F$. The result is

$$A_a dx^a = (\xi dx + \eta dy + \zeta dz)(J_2+iJ_1) \equiv d\Xi(J_2+iJ_1) = (d\Xi_1 + id\Xi_2)(J_2+iJ_1). \tag{6.44}$$

Now we can discuss what Lorentzian structures correspond to each point on $\mathcal{N}_0' \equiv \mathcal{N}_S \oplus \mathcal{N}_F$. We should bear in mind that, because $\pi_1(T^3) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, the Lorentzian structure in question is generated by three Poincaré transformations which mutually commute.

First we consider the standard sector $\mathcal{N}_S$. Because the parameters $(u, v, w)$ and $(a, b, c)$ respectively give rise to rotations in the $(X^2, X^3)$-plane and Lorentz boosts in the $(X^0, X^1)$-plane, we have to separately consider the four cases.

I) For $(u,v,w) \neq \vec{0} \neq (a,b,c)$, the Lorentzian structures are necessarily embedded into the rank 2 subgroup $\{R_{23}, L_1\}$ of the Poincaré group[27]. We therefore expect that the spacetime given by one of these Lorentzian structures suffers from singularities. This is indeed the case. Such spacetime inevitably has singularities at which the metric degenerates and through which the orientation of the local Lorentz frame defined by $\{e^\alpha\}$ is inverted[39]. Thus these spacetimes cannot be considered to be physical, at least classically.

II) For $(a,b,c) = \vec{0} \neq (u,v,w)$, we can embed the Lorentzian structure into a rank 3 subgroup of the Poincaré group $\{R_{23}, T^0, T^1\}$, where $T^\alpha$ denotes the group of translations in the $X^\alpha$-direction. This subgroup, however, includes time-translations. The corresponding spacetimes thus involve timelike-tori and so they are not considered to be physical. Of course we could use the Lorentzian structure which is embedded in the rank 2 subgroup $\{R_{23}, T^1\}$. But in this case singularities similar to those appeared in case I) always exist in the resulting spacetime. This case therefore corresponds to a set of spacetimes not allowed in general relativity.

III) For $(u,v,w) = \vec{0} \neq (a,b,c)$, the situation drastically changes. The Lorentzian structure in this case can be embedded into the rank 3 subgroup $\{L_1, T^2, T^3\}$. Fortunately the action of this subgroup on the region $\{(X^0)^2 - (X^1)^2 > 0\} \in M^{3+1}$ is spacelike. This indicates that this case corresponds to a set of well-behaved spacetimes.

More precisely, by plugging into eq.(6.34) the integrated spin connection constructed from $A = idW_2 J_1$, we find the equation

$$\begin{aligned}
\cosh W_2 e^0 - \sinh W_2 e^1 &= dX^0 \\
-\sinh W_2 e^0 + \cosh W_2 e^1 &= dX^1 \\
e^2 &= dX^2 \\
e^3 &= dX^3.
\end{aligned} \tag{6.45}$$

A choice of embedding functions which yields well-behaved spacetimes is:

$$(X^\alpha) = (\tau \cosh(W_2 + \alpha), -\tau \sinh(W_2 + \alpha), \vec{\beta} \cdot \vec{x} + \psi_2, \vec{\gamma} \cdot \vec{x} + \psi_3),$$

where $\vec{\beta}$ and $\vec{\gamma}$ are constant vectors in $\mathbf{R}^3$. $\tau$, $\alpha$, $\psi_2$ and $\psi_3$ are single-valued functions on $M$. Substituting this into eq.(6.45), we find

$$\begin{aligned}
e^0 &= d\tau \cosh \alpha + \tau \sinh \alpha d(W_2 + \alpha) \\
e^1 &= -d\tau \sinh \alpha - \tau \cosh \alpha d(W_2 + \alpha) \\
e^2 &= dX^2 = \vec{\beta} \cdot d\vec{x} + d\psi_2 \\
e^3 &= dX^3 = \vec{\gamma} \cdot d\vec{x} + d\psi_3.
\end{aligned} \tag{6.46}$$

[27] $R_{ij}$ and $L_k$ respectively stand for the group of rotations on the $(X^i, X^j)$-plane and that of Lorentz boosts in the $(X^0, X^k)$-plane

This vierbein gives a physically permissible spacetime whose only pathology is the initial singularity at $\tau = 0$:

$$ds^2 = -d\tau^2 + \tau^2 d(W_2 + \alpha)^2 + (dX^2)^2 + (dX^3)^2. \tag{6.47}$$

IV) For $(a, b, c) = (u, v, w) = \vec{0}$. In this case also we can embed the geometric structure into the rank 3 subgroup $\{T^1, T^2, T^3\}$, whose action on the whole Minkowski space is spacelike. We expect this case to correspond to spacetimes without any singularity. Indeed we see that the spacetimes which correspond to this case take the following form

$$ds^2 = -dT^2 + dX^i dX^i, \tag{6.48}$$

where $X^i \equiv \vec{a}^i \cdot \vec{x}$ and each $\vec{a}^i$ is a constant vector in $\mathbf{R}^3$. These spacetimes in general have no singularity.

Next we consider the flat sector $\mathcal{N}_F$. We deal with two cases separately.

I)' For $\mathrm{Re}(\vec{\xi})$ and $\mathrm{Im}(\vec{\xi})$ being linear independent, with $\vec{\xi} \equiv (\xi, \eta, \gamma)$, the Lorentzian structure in question is embedded into the subgroup $\{N^{1+}, N^{2+}, T^+\}$, where $T^+$ is the translation in the $X^+ (\equiv X^0 + X^3)$- direction and $N^{\hat{i}+}$ $(\hat{i} = 1, 2)$ is the null-rotation which stabilizes $X^+$ and $X^{\hat{i}}$. Because this subgroup contains translation in the null-direction, there is a possibility that closed null curves appear. An inspection [39] shows that, while the corresponding spacetimes[28]

$$ds^2 = -dt^2 - e^t dt dZ_+ + e^{2t} d\Xi d\bar{\Xi} \tag{6.49}$$

may or may not have closed causal curves[29], they certainly violate the strong causality condition[47]. These spacetimes are therefore not so desirable in general relativity.

II)' For $\mathrm{Im}(\vec{\xi}) = 0 \neq \mathrm{Re}(\vec{\xi})$ (or equivalently for $\vec{\xi} = \frac{1}{\cos\phi} e^{i\phi} \mathrm{Im}(\vec{\xi})$, $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$), the Lorentzian structure is embedded into the rank 3 subgroup $\{N^{2+}, T^+, T^2\}$. Also in this case the resulting spacetimes spoil the strong causality condition in the same way that the spacetime (6.49) does.

To summarize, each point on $\mathcal{N}_0' = \mathcal{N}_S \oplus \mathcal{N}_F$ yields a family of Lorentzian structures whose projection onto the Lorentz group is given by the corresponding $SL(2, \mathbf{C})$ holonomy group. While there is a subspace $\{(u, v, w) = 0\} \in \mathcal{N}_S$ each point on which yield a family of physically permissible spacetimes, most of the points on $\mathcal{N}_0'$ give rise to a family of spacetimes which suffer from singularities or which ruin the strong causality condition. Most of the characteristic topological solutions $\Psi_n[A]$ therefore correspond to a family of spacetimes which are not permissible from the viewpoint of general relativity.

How to resolve this issue? Two methods are conceivable. One is to impose a selection rule which extract only the subspace $\{(u, v, w) = 0\} \in \mathcal{N}_S$. However, this seems to be

[28] $Z_+$ is defined as $Z_+ \equiv \vec{a}_+ \cdot \vec{x}$ with $\vec{a}_+$ being a constant vector in $\mathbf{R}^3$. $\bar{\Xi}$ denotes the complex conjugate of $\Xi$.

[29] In fact this metric has at worst closed null curves.

artificial and so not desirable compared to the situation of the Chern-Simons solutions. The other is to consider the singularities to be annihilated owing to the uncertainty principle in quantum gravity. This is possible because any topological solutions $\Psi_{topo}[A]$ cannot fix a spacetime (or a Lorentzian structure) and, in particular, $\Psi_n[A]$ corresponds to a family of spacetimes. As for the flat sector $\mathcal{N}_F$ which violates the strong causality condition, taking account of the fact that this sector 'almost degenerates' with the origin of $\mathcal{N}_S$ in a non-Hausdorff manner, corresponding spacetimes are in fact some 'superposition" of (6.48) and (6.49) which may restore the strong causality.

In the actual quantum gravity, we have to take account also of the "right-handed graviton mode" $C^{(+)\rho}{}_{\sigma\mu\nu}$. To elucidate what actually happens to the topological solutions is thus postponed until its full treatment in quantum gravity will be made possible.

## 6.5 The Euclidean case

So far we have investigated what family of spacetimes specify the topological solution $\Psi_I^{CS}[A]$ or $\Psi_n[A]$ putting emphasis on the case of Lorentzian signature. For completeness it would be proper to mention the Euclidean signature case. The case with signature $(-,-,-,-)$ has not yet been studied so much. The case with signature $(+,+,+,+)$ has been investigated in ref.[1]. We will briefly review the results.

We first note that, in the Euclidean case, we cannot exploit reality conditions to extract the full information on the spin connection $\omega^{\alpha\beta}$ from that on the anti-self-dual connection

$$A_E^i = -\frac{1}{2}\epsilon^{ijk}\omega^{jk} + \omega^{0i}.$$

Some of the degrees of freedom concerning the self-dual connection thus remains intact even if we impose the torsion-free condition $de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0$.

For $\Lambda \neq 0$, the Chern-Simons solutions

$$\Psi_I^{CS}[A_E] = I[A_E]e^{-\frac{3}{2\Lambda}S_{CS}[A_E]} \tag{6.50}$$

are the topological solutions also for the Euclidean case and the equation for the WKB orbits

$$F_E^i = \frac{\Lambda}{3}\Sigma_E^i = \frac{\Lambda}{3}(-\frac{1}{2}\epsilon^{ijk}e^j \wedge e^k + e^0 \wedge e^i) \tag{6.51}$$

yields the conformally self-dual Euclidean manifolds with a fixed Ricci curvature $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

Thus the family of WKB orbits corresponding to the Chern-Simons solutions are equal to the moduli space of conformally self-dual Riemannian manifolds, which was shown by Hitchin[17] to yield either $S^4$ with the standard metric or $\mathbf{CP}^2$ with the Fubini-Study metric in the case where the four dimensional manifold $M$ is compact and $\Lambda$ is positive.

For $\Lambda = 0$, the topological solutions $\Psi_{topo}[A_E]$ have their support only on flat anti-self-dual connections

$$F_E^i = 0,$$

namely on self-dual four dimensional manifolds $M$. If $M$ is compact, then by the theorem of Hitchin[17] the topology of $M$ is limited to that of $T^4$, $K3$ or their quotients by some discrete groups. His theorem further informs us that $\Psi_{topo}[A_E]$ is characterized by the moduli of Einstein-Kählerian manifolds with vanishing real first Chern class.

As a result of the flatness of the connection $A_E^i$, the equations which determine the two-form $\Sigma_E^i$ can be written as

$$\begin{aligned} \Sigma_E^i \wedge \Sigma_E^j &= \frac{1}{3}\delta^{ij}\Sigma^k \wedge \Sigma^k \\ d\Sigma_E^i &= 0. \end{aligned} \tag{6.52}$$

In general the second equation holds only locally. If the canonical bundle over $M$ is trivial, however, these equations hold also globally. In this case the corresponding manifolds are called hyper-Kählerian. On the other hand, manifolds whose canonical bundles are non-trivial are called locally hyper-Kählerian.

## 6.6 Topological solutions for $N = 1$ supergravity

Up to here in this paper we have explored several aspects of Ashtekar's formulation for pure gravity. As a matter of fact Ashtekar's formulation can be applied also to supergravities with $N = 1$[51][24] and with $N = 2$[59]. These Ashtekar's formulation for supergravities can also be cast into (graded) BF theories with the two-forms subject to the algebraic constraints like eq.(6.2)[38]. This relation elegantly explains the existence of topological solutions also for $N = 1$ supergravity [80][66] and for $N = 2$ supergravity [80][38]. Here we will briefly see this in the $N = 1$ case only. The similar arguments hold also in the $N = 2$ case. For a detail, we refer the reader to ref.[38].

Ashtekar's formulation for $N = 1$ supergravity was first derived by Jacobson[51]. Later a more convenient derivation by way of the chiral action was discovered in ref.[24]. We will follow the latter case.[30]

Our starting point is the following chiral action with a cosmological constant $\Lambda = g^2$ (possibly with $g = 0$)[31]

$$-iI_{BF}^{N=1} = \int \left( \begin{array}{r} \Sigma^i \wedge (F^i + \lambda g(\frac{\sigma^i}{2i})_{AB}\psi^A \wedge \psi^B) + 2\chi^A \wedge D\psi_A \\ -\frac{g^2}{6}\Sigma^i \wedge \Sigma^i - \frac{g}{3\lambda}\chi_A \wedge \chi^A \end{array} \right), \tag{6.53}$$

[30] We will follow the convention used in refs.[80][81].

[31] To avoid the confusion we will denote in this subsection the $SL(2, \mathbf{C})$ generator in the spinor representation as $(\frac{\sigma_i}{2i})^A{}_B$, with $\sigma_i$ being the Pauli matrix.

where $A^i$ and $\Sigma^i$ denote as in the bosonic case the self-dual part of the spin connection and the self-dual two-form respectively, and $\psi^A$ and $\chi^A$ respectively stand for the left-handed gravitino and the left-handed two-form which corresponds to the right-handed gravitino[32]. In consequence $\Sigma^{AB} \equiv \Sigma^i(\frac{\sigma_i}{2i})^{AB}$ and $\chi^A$ are subject to the algebraic constraints

$$\begin{aligned} \Sigma^{ABCD} &\equiv \Sigma^{(AB} \wedge \Sigma^{CD)} = 0, \\ \Xi^{ABC} &\equiv \Sigma^{(AB} \wedge \chi^{C)} = 0. \end{aligned} \qquad (6.54)$$

We can easily show that eq.(6.53) actually coincides with the BF action with the gauge group being the graded group $GSU(2)$:

$$-iI_{BF}^{N=1} = \int \mathrm{STr}(\mathcal{B} \wedge \mathcal{F} - \frac{g^2}{6}\mathcal{B} \wedge \mathcal{B}), \qquad (6.55)$$

where $\mathcal{B} = \Sigma^i J_i - \frac{1}{\lambda g}\chi^A J_A$ is a $GSU(2)$-valued two-form and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature two form of the $GSU(2)$ connection $\mathcal{A} = A^i J_i + \psi^A J_A$. $(J_i, J_A)$ are the generators of the graded Lie algebra $GSU(2)$[70]:

$$[J_i, J_j] = \epsilon_{ijk}J_k, \quad [J_i, J_A] = (\frac{\sigma^i}{2i})_A{}^B J_B, \quad \{J_A, J_B\} = -2\lambda g(\frac{\sigma^i}{2i})_{AB}J_i, \qquad (6.56)$$

where $\{\ ,\ \}$ denotes the anti-commutation relation. STr stands for the $GSU(2)$ invariant bilinear form which is unique up to an overall constant factor

$$\begin{aligned} &\mathrm{STr}(J_iJ_j) = \delta_{ij}, \quad \mathrm{STr}(J_AJ_B) = -2\lambda g\epsilon_{AB}, \\ &\mathrm{STr}(J_AJ_i) = \mathrm{STr}(J_iJ_A) = 0. \end{aligned} \qquad (6.57)$$

Similarly to the bosonic case Ashtekar's formalism is attainable by performing (3+1)-decomposition of the action (6.53). We first obtain

$$\begin{aligned} -iI_{BF}^{N=1} &= \int dt \int_{M^{(3)}} \mathrm{STr}(\tilde{\Pi}^a \dot{\mathcal{A}}_a + \mathcal{A}_t \mathbf{G} + \mathcal{B}_{ta}\Phi^a) \\ &= \int dt \int_{M^{(3)}} (\tilde{\pi}^{ai}\dot{A}^i_a + 2\tilde{\pi}^A\dot{\psi}_{aA} + A^i_t G^i - 2\psi_{tA}L^A + \Sigma^i_{ta}\Phi^{ai} - 2\chi_{taA}\Phi^{aA}) \end{aligned} \qquad (6.58)$$

where we have set $\tilde{\Pi}^a = \frac{1}{2}\tilde{\epsilon}^{abc}\mathcal{B}_{bc} = \tilde{\pi}^{ai}J_i - \frac{1}{\lambda g}\tilde{\pi}^{aA}J_A$. From this we can read two types of constraints in $GSU(2)$ BF theory, one is Gauss' law constraint which generates the $GSU(2)$ gauge transformations

$$\begin{aligned} \mathbf{G} &= \mathcal{D}_a\tilde{\Pi}^a \equiv \partial_a\tilde{\Pi}^a + [\mathcal{A}_a, \tilde{\Pi}^a] = G^iJ_i - \frac{1}{\lambda g}L^AJ_A, \\ G^i &= D_a\tilde{\pi}^{ai} - 2(\frac{\sigma^i}{2i})_{AB}\psi^A_a\tilde{\pi}^{aB} \\ L^A &= D_a\tilde{\pi}^{aA} + \lambda g\tilde{\pi}^{ai}(\frac{\sigma^i}{2i})^A{}_B\psi^B_a. \end{aligned} \qquad (6.59)$$

[32]Note that $\chi^A$ and $\psi^A$ are Grassmann odd fields. Whether an object is Grassmann even or odd can be determined by whether the number of its Lorentz spinor indices is even or odd.

And the other is the constraint which generates the extended Kalb-Ramond symmetry[53]

$$\begin{aligned}
\Phi^a &= \frac{1}{2}\tilde{\epsilon}^{abc}\mathcal{F}_{bc} - \frac{g^2}{3}\tilde{\Pi}^a \\
&= \Phi^{ai}J_i + \Phi^{aA}J_A, \\
\Phi^{ai} &= \frac{1}{2}\tilde{\epsilon}^{abc}(F^i_{bc} + 2\lambda g(\frac{\sigma^i}{2i})_{AB}\psi^A_b\psi^B_c) - \frac{g^2}{3}\tilde{\pi}^{ai} \\
\Phi^{aA} &= \tilde{\epsilon}^{abc}D_b\psi^A_c + \frac{g}{3\lambda}\tilde{\pi}^{aA}.
\end{aligned} \tag{6.60}$$

Next we solve the algebraic constraints (6.54) for the lagrange multiplier $(\Sigma^i_{ta}, \chi^A_{ta})$. By substituting the results into eq.(6.58) we find that Gauss' law constraint remains intact while the other constraint survives only partially, namely, what survive are only the following linear combinations

$$\begin{aligned}
R^A &= \frac{1}{2}\underset{\sim}{\epsilon}_{abc}\epsilon^{ijk}\tilde{\pi}^{bj}\tilde{\pi}^{ck}(\frac{\sigma^i}{2i})^{AB}\Phi^a_B, \\
\mathcal{S} &= \frac{1}{4}\underset{\sim}{\epsilon}_{abc}\epsilon^{ijk}\tilde{\pi}^{bj}\tilde{\pi}^{ck}\Phi^{ai} - 2\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bi}(\frac{\sigma^i}{2i})_{AB}\tilde{\pi}^{cB}\Phi^{aA}, \\
\mathcal{V}_a &= \frac{1}{2}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bi}\Phi^{ci} + \underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bA}\Phi^c_A.
\end{aligned} \tag{6.61}$$

Physically, $R^A$ generates right-supersymmetry transformations, $\mathcal{S}$ generates many-fingered time evolutions, and $\mathcal{V}_a$ generates spatial diffeomorphisms. In passing, among Gauss' law constraint, $G^i$ generates local Lorentz transformations for left-handed fields and $L^A$ generates left-supersymmetry transformations.

Quantization also proceeds as in the bosonic case by replacing $i$ times the Poisson brackets with the quantum commutators. If we use the representation in which $A^i_a$ and $\psi^A_a$ are diagonalized, the action of the operators $\hat{\tilde{\pi}}^{ai}$ and $\hat{\tilde{\pi}}^{aA}$ on the wavefunction $\Psi[\mathcal{A}_a]$ is represented by functional derivatives:

$$\hat{\tilde{\pi}}^{ai}(x)\cdot\Psi[\mathcal{A}_a] = -\frac{\delta}{\delta A^i_a(x)}\Psi[\mathcal{A}], \quad \hat{\tilde{\pi}}^{aA}(x)\cdot\Psi[\mathcal{A}_a] = \frac{1}{2}\frac{\delta}{\delta\psi_{aA}(x)}\Psi[\mathcal{A}_a]. \tag{6.62}$$

The physical wave functions in quantum $N=1$ supergravity have to satisfy the first class constraint equations:

$$\hat{G}^i\Psi[\mathcal{A}_a] = \hat{L}^A\Psi[\mathcal{A}_a] = 0 \tag{6.63}$$

$$\hat{R}^A\Psi[\mathcal{A}_a] = \hat{\mathcal{S}}\Psi[\mathcal{A}_a] = \hat{\mathcal{V}}_a\Psi[\mathcal{A}_a] = 0. \tag{6.64}$$

Gauss' law constraint (6.63) requires the wavefunctions to be invariant under the $GSU(2)$ gauge transformations. When we solve the constraints $(R^A, \mathcal{S}, \mathcal{V}_a)$, we should note that these constraints are linear combinations of the constraints $(\Phi^{ai}, \Phi^{aA})$ in BF theory, with the coefficients being polynomials in the momenta $(\tilde{\pi}^{ai}, \tilde{\pi}^{aA})$. Thus, as in the bosonic case, if we take the ordering with the momenta to the left, solutions to $GSU(2)$ BF theory are involved in the solution space for quantum $N=1$ supergravity in the Ashtekar form. They are the topological solutions for $N=1$ supergravity.

For $g \neq 0$, we have the $N=1$ Chern-Simons solutions [80][81][40]

$$\Psi_I^{CS}[\mathcal{A}_a] = I[\mathcal{A}_a] e^{-\frac{3}{2g^2} S_{CS}^{N=1}[\mathcal{A}_a]}, \tag{6.65}$$

where $I[\mathcal{A}_a]$ is some topological invariant and $S_{CS}^{N=1}[\mathcal{A}_a]$ is the Chern-Simons functional for the $GSU(2)$ connection $\mathcal{A}$

$$\begin{aligned} S_{CS}^{N=1} &= \int_{M^{(3)}} \mathrm{STr}(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \\ &= \int_{M^{(3)}} (A^i dA^i + \frac{1}{3}\epsilon^{ijk} A^i \wedge A^j \wedge A^k - 2\lambda g \psi^A \wedge D\psi_A). \end{aligned} \tag{6.66}$$

For $g = 0$, we have the following formal expression for the topological solutions

$$\Psi_{topo}[\mathcal{A}_a] = F[\mathcal{A}_a] \prod_{x\in M^{(3)}} \left( \prod_{a,i} \delta(\tilde{\epsilon}^{abc} F_{bc}^i(x)) \prod_{a,A} \delta(\tilde{\epsilon}^{abc} D_b \psi_c^A(x)) \right), \tag{6.67}$$

where $F[\mathcal{A}_a]$ is an arbitrary $GSU(2)$-invariant functional of the connection $\mathcal{A}_a$. Owing to the delta functions $F[\mathcal{A}_a]$ reduces to the function on the moduli space $\mathcal{GN}$ of flat $GSU(2)$ connections. Alternatively, we can write $\Psi_{topo}[\mathcal{A}_a]$ formally as

$$\begin{aligned} \Psi_{topo}[\mathcal{A}_a] &= \int_{gn\in\mathcal{GN}} d(gn) F(gn) \Psi_{gn}[\mathcal{A}_a] \\ \Psi_{gn}[\mathcal{A}_a] &\equiv \int [d\mathcal{G}(x)] \prod_{x\in M^{(3)}} \prod_{a,(i,A)} \delta(\mathcal{A}_a(x) = \mathcal{G}[\mathcal{A}_{0,gn}]_a(x)), \end{aligned} \tag{6.68}$$

where $\mathcal{A}_{0,gn}$ is a flat $GSU(2)$ connection which represent a point $gn$ on the moduli space $\mathcal{GN}$ and

$$\mathcal{G}[\mathcal{A}]_a(x) \equiv \mathcal{G}(x)\mathcal{A}_a(x)\mathcal{G}^{-1}(x) + \mathcal{G}(x)\partial_a\mathcal{G}^{-1}(x)$$

denotes the gauge-transformed $GSU(2)$ connection. The integration region in eq.(6.68) is taken to be the space of small $GSU(2)$ gauge transformations. This expression of topological solutions is essentially a kind of Fourier transform of the $N=1$ topological solutions given in ref.[66].

The solutions (6.65) and (6.67) are considered to be vacuum states of $N=1$ supergravity for the same reason as in the bosinic case[66]. It is expected, however, that the spacetimes corresponding to these topological solutions are considerably different from those in the bosonic case[40], because in supergravity the existence of the gravitino influences the nonvanishing value of the torsion through the equation:

$$de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = -\frac{i}{2}\sigma^\alpha_{A\bar{A}}\overline{\psi}^{\bar{A}} \wedge \psi^A, \tag{6.69}$$

where $\overline{\psi}^{\bar{A}}$ is the right-handed gravitino and $\sigma^{\alpha}_{A\bar{A}}$ stands for the "soldering form" in the Minkowski space $M^{3+1}$ [72]. This was explicitly established in the minisuperspace model for $g \neq 0$[81]. There it was demonstrated that, due to the effect of the gravitino, behavior of the WKB orbits in general deviates from that of the deSitter space.

## 6.7 Summary and discussion on §6

In this section we have looked into the topological solutions. Semiclassically these solutions correspond to a family of vacuum solutions to general relativity i.e. spacetimes with no graviton excitation, and thus they are expected to be vacuum states of quantum gravity. This expectation is confirmed by the fact that the topological solutions are also physical states in a topological field theory, $SL(2, \mathrm{C})$ BF theory, and that they naturally carry the topological degrees of freedom such as the moduli of flat $SL(2, \mathrm{C})$ connections which yield a family of Lorentzian structures on the flat spacetime. These topological solutions as vacuum states of quantum gravity naturally extends to $N = 1, 2$ supergravities. In supergravities, however, corresponding spacetimes are nontrivial because of the existence of the gravitino.

The existence of the topological solutions is intimately related with the fact that Ashtekar's formulation for canonical gravity can be cast into the form of $SL(2, \mathrm{C})$ BF theory with the two-form field $\Sigma^i$ subject to the algebraic constraint. In association with this there are several attempts to interpret Einstein gravity as a kind of "unbroken phase" of some topological field theories [68][54]. This idea is interesting because it may provide us with some information on a more mathematically tractable formulation on the quantum gravity as in the case of (2+1)-dimensional gravity[85]. Now from the result in §§6.6 we expect that these idea can be extended to supergravities. Probably this deserves studying because the existence of the supersymmetry is believed by most of the high energy phenomenologists and from such a viewpoint supergravities seem to be more realistic than pure gravity.

# 7 Other solutions and remaining issues

In this paper we have given an outline of two types of solutions, namely the Wilson loop solutions and the topological solutions, for quantum Ashtekar's formalism. As far as the author knows there are further two types of solutions for canonical quantum gravity. Here we will make a brief introduction of these solutions. After that we will discuss remaining issues on Ashtekar's formulation for canonical quantum gravity.

## 7.1 Loop representation and the Jones polynomial

In this paper we have worked solely with the connection representation in which the wavefunctions are represented by holomorphic functionals $\Psi[A_a^i]$ of the $SL(2, \mathbf{C})$ connection. As a matter of fact, there is another representation which is extensively explored in Ashtekar's formulation for quantum gravity. It is the loop representation [77] in which the loop functionals $\Psi[\gamma]$ play the role of wavefunctions. Relation between the connection representation and the loop representation is given by the formal "loop transform"[77]

$$\Psi[\gamma] = \int [dA_a^i(x)] \overline{W[\gamma]} \Psi[A_a^i], \tag{7.1}$$

where $W[\gamma] \equiv W[\gamma, \pi_1]$ is the Wilson loop in the spinor representation and $[dA_a^i(x)]$ is some gauge- and diffeomorphism invariant measure on the space of connections. If we use spin network states in stead of Wilson loops, this formal transformation can be made rigorous[79] at least in the Euclidean case if we use as an invariant measure $[dA_a^i(x)]$ the induced Haar measure $d\mu_H(A)$ explained in §§3.2:

$$\Psi[\Gamma, \{\pi_e\}, \{i_v\}] = \int_{\mathcal{A}/\mathcal{G}} d\mu_H \overline{< A_a^i | \Gamma, \{\pi_e\}, \{i_v\} >} \Psi[A_a^i]. \tag{7.2}$$

To find out a mathematically rigorous transformation in the Lorentzian case is left to the future investigation because we do not know a well-defined invariant measure in this case.

By means of the transformation (7.1), the operator $\hat{O}_C$ in the connection representation is inherited to the loop representation as an operator $\hat{O}_L$ which is subject to the relation:

$$\hat{O}_L \overline{W[\gamma]} = \overline{\hat{O}_C^\dagger W[\gamma]}, \tag{7.3}$$

where $\dagger$ denotes the adjoint operator w.r.t. the measure $[dA_a^i(x)]$. We can exploit this relation when we define the constraint operators in the loop representation. Owing to the existence of $\dagger$, operator ordering in the loop representation is the opposite to that in the connection representation. A merit of the loop representation is that we can easily implement the diffeomorphism constraint (2.30) by dealing only with the functionals of diffeomorphism equivalence classes of loops (or of colored graphs). So we have only to concentrate on the scalar constraint. For simplicity we will consider the Euclidean case in which the reality conditions for $(\hat{A}_E)_a^i$ and $(\hat{\tilde{\pi}}_E)^{ai}$ are given by the usual self-adjointness condition.

Owing to the relation (7.3) loop transform of the solutions in the connection representation all yields the solutions in the loop representation. For example, the Wilson loop solutions tranforms to the characteristic functions of particular colored graphs[77].

Let us now consider the loop transform of the Chern-Simons solution (6.50) with $I[A_E] = 1$[41][22]. According to ref.[86], the loop transform of $\Psi_1^{CS}[A_E]$ coincides with the Kauffman bracket $\mathrm{KB}_\Lambda[\gamma]$[33]

$$\begin{aligned}\Psi_1^{CS}[\gamma] &= \int[dA_E]\overline{W[\gamma]}e^{-\frac{3}{2\Lambda}S_{CS}[A_E]} \\ &= \mathrm{KB}_\Lambda[\gamma] \\ &= e^{\Lambda \mathrm{Gauss}[\gamma]}\mathrm{JP}_\Lambda[\gamma],\end{aligned} \tag{7.4}$$

where $\mathrm{Gauss}[\gamma]$ and $\mathrm{JP}_\Lambda[\gamma]$ denote respectively the Gauss self-linking number and the Jones polynomial[34]. The Kauffman bracket $\mathrm{KB}_\Lambda[\gamma]$ is thus considered to be a solution to the WD equation in the case where the cosmological constant $\Lambda$ is nonvanishing. It was also shown [41] that $\exp(\Lambda \mathrm{Gauss}[\gamma])$ is also annihilated by the scalar constraint $\hat{\mathcal{S}}$ with $\Lambda \neq 0$. From this we can easily show that the second coefficient of the Jones Polynomial is a solution to the WD equation with $\Lambda = 0$[23][42]. From this fact a conjecture arose that the coefficients of the Jones polynomial are solutions to the WD equation with vanishing cosmological constant[41]. It was shown, however, that the third coefficient does not solve the WD equation[45].

This approach of "finding solutions to the WD equation out of knot invariants" is expected to develop as a method which complements the approach of constructing solutions from the spin network states. The extended loop representation[16] is considered to provide us with a systematic mathematical tool for pursuing this avenue.

In this approach it is indispensable to find an invariant measure on the space of $SL(2, \mathbf{C})$ connections which makes the transform (7.1) well-defined. Otherwise the loop representation cannot be qualified as a representation of Lorentzian quantum gravity.

## 7.2 The wormhole solution in $N = 1$ supergravity

As we have seen so far, finding solutions to the constraint equations in Ashtekar's formalism has made a steady progress. What about other formalisms of canonical quantum gravity? In supergravity the Hamiltonian constraint and the momentum constraint are expressed as appropriate linear combinations of quantum commutators between the left-SUSY generator $\hat{L}_A$ and the right-SUSY generator $\hat{\overline{L}}_{\bar{A}}$. It is therefore possible to easily solve the constraints without recourse to Ashtekar's new variables. In fact, starting from the ansatz

$$\Psi[e_a^i] = \prod_{x \in M^{(3)}} \hat{L}^A(x)\hat{L}_A(x)g(e_a^\alpha), \tag{7.5}$$

[33]In quantum gravity we consider the knot invariants to be extended to the knots with kinks and/or intersections.

[34]In deriving the second equation we have used the property of $SU(2)$ Wilson loops: $\overline{W[\gamma]} = W[\gamma^{-1}] = W[\gamma]$.

where $g(e_a^\alpha)$ is a local Lorentz invariant functional of the vierbein $e_a^\alpha$, we can get a solution[31] which is somewhat similar to the topological solutions discussed in §§6.6:

$$\Psi_{WH}[e_a^i] = \int [d\epsilon^A(x)] \int [d\theta^i(x)] \exp(\int_{M^{(3)}} d^3x \epsilon^A \hat{L}_A) \cdot \exp(-\frac{1}{2} \int_{M^{(3)}} d^3x \tilde{\epsilon}^{abc} (e^\theta)_a^\alpha \partial_b (e^\theta)_{c\alpha}), \tag{7.6}$$

where $[d\epsilon^A(x)]$ and $[d\theta^i(x)]$ are $SU(2)$ invariant measures and $(e^\theta)_a^\alpha$ denotes the vierbein which is obtained from $e_a^\alpha$ by the left-handed local Lorentz transformation generated by $\theta^i(x)$. By investigating its reduction to the spatially homogeneous model, it turns out that this solution should be interpreted as a worm-whole state[31].

## 7.3 Remaining issues

In this paper we have looked into the developments in the program of solving the Wheeler-De Witt equation. We have seen that, up to now, we have found several types of solutions to the WD equation. In order to complete the program of canonical quantum gravity in terms of Ashtekar's new variables, however, there still remain several problems aside from finding complete set of solutions. We will conclude this paper by listing a few of these problems:

1) Regularization and operator ordering. The solutions obtained so far are formal in the sense that we have worked with constraint equations which are not, or which are at most incompletely, regularized. In order to obtain mathematically rigorous solutions, the constraint operators must be regularized correctly. The problem of operator ordering considered in §§2.2 is in fact meaningful only when we regularize the constraints. It is desirable that the regularized constraint operators should form a closed commutator algebra and, in particular, that they should respect the covariance under spatial diffeomorphisms.[35] It is likely that the regularization which satisfies this condition, if any, should be independent of artificial background structures. At present, the extended loop representation[16] seems to be the only candidate which enjoys this property. However, it is possible that anomalies inevitably appear in the quantum commutator algebra. Because Dirac's quantization cannot apply in this case, a modified treatment of the constraint is required[66] as that of the Virasoro constraints in string theory. Anyway, a remarkable statement was made in ref.[66] that the topological solutions still remains physical states even in the presence of anomalies, if an appropriate modification of the quantization procedure is made.

[35] In Dirac's quantization we usually require the quantum commutator algebra of constraints to close in order to ensure the solvability of constraint equations. One might consider this closedness condition to be redundant in quantum gravity because we have already found solutions at least heuristically. The author thinks, however, that the existence of the Wilson-loop solutions is not sufficient to guarantee the existence of *phisically intriguing solutions*. While there may have not been any signals of anomalous terms in the analysis made so far on the spin network states, this is possibly due to the simple structure of the space of spin network states defined on *piecewise-analytic graphs*.

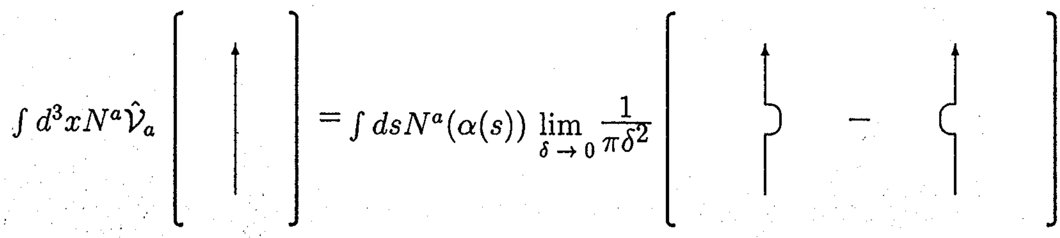


Figure 5: Action of the vector constraint on a parallel propagator.

2) How to deal with the diffeomorphism constraint? In §4 and §5 we have solved only the integrated diffeomorphism constraint (3.21). In order to assert that we have solved the diffeomorphism constraint (2.30), however, this is not sufficient. The action of $\hat{\mathcal{V}}_a$ on the parallel propagator is calculated as

$$\int d^3xN^a(x)\hat{\mathcal{V}}_a(x)h_\alpha[0,1] = \int_0^1 dsN^a(\alpha(s))\dot{\alpha}^b(s)\Delta_{ab}(\alpha,\alpha(s))h_\alpha[0,1]. \tag{7.7}$$

This is visualized for example in figure 5 as the difference between the results of inserting infinitesimal loops which are sitting in the direction of $\dot{\alpha}^a(s)$ and which are stretching toward the opposite directions. Thus in a microscopic viewpoint this may change the differential topology of curves. When we deal with piecewise analytic graphs, no problem arises because this operation is not well-defined (appendix C of [10]). However, because it is almost definite that in quantum gravity we should work with graphs which are not piecewise analytic, it is possible that different treatments yield different results. Roughly speaking there are two options: i) consider only the integrated version (3.21) as relevant; ii) solve the entire equation (2.30) by appropriately regularizing the expression like (7.7). It is then possible that only combinatorial topology of the graph is relevant and that some solutions are given by approriate extensions of the combinatorial solutions to non-piecewise analytic graphs. While in §4 and §5 we have followed the attitude i), the discussion in ref.[42] seems to based on ii). Anyway, what is the correct treatment of $\hat{\mathcal{V}}_a$ will be finally determined by some physical requirement.

3) The physical inner product. This problem is a central problem in Ashtekar's formalism because it is closely related with that of implementing the reality conditions. As is briefly explained in §§3.4 a plausible candidate for the physical inner product is the Wick rotated measure [84]. The problem is whether it is appropriately regularized because this measure is induced by the Wick rotator which involves a non-polynomial functional of conjugate momenta.

4) Finding physical observables. Not only the physical inner product but also physical

observables are necessary to extract quantum gravitational information from solutions to the constraints. We know however little about them because the diffeomorphism invariance severely restricts the possible form of observables. In the matter-coupled gravity, area and volume operators are expected to provide a large set of physical observables. In the topological sector, namely the sector which is described purely by $SL(2, \mathbf{C})$ BF theory, a set of nonlocal observables can be constructed by using the BRST descent sequences[28]. However, what we are interested in is a set of physical observables which measure e.g. graviton excitations efficiently. Unfortunately, as far as the author knows, such a set has not yet been discovered.

It is not likely that these problems are resolved separately, because they are intimately related with each other. While it seems to take great efforts to solve these formidable issues, we have a flash of hope. The existence of the topological solutions guarantees that our labors toward this direction will not be wasted. We expect that a true theory of quantum gravity will appear as an extension of this program of nonperturbative canonical quantum gravity.

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# Appendix A The projector $P^{(-)i}_{\alpha\beta}$

Here we provide the definition and some properties of the projector $P^{(-)i}_{\alpha\beta}$. First we define the projection operator $P^{(-)\alpha\beta}{}_{\gamma\delta}$ into the space of anti-self-dual Lorentz bi-vectors:

$$P^{(-)\alpha\beta}{}_{\gamma\delta} = \frac{1}{4}(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma - i\epsilon^{\alpha\beta}{}_{\gamma\delta}), \tag{A.1}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the totally anti-symmetric pseudo tensor with $\epsilon^{0123} = \epsilon^{123} = 1$. We use the metric $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \mathrm{diag}(-1,1,1,1)$ to raise or lower the Lorentz indices. This projection operator possesses the following properties

$$P^{(-)\alpha\beta}{}_{\gamma\delta} = -\frac{i}{2}\epsilon^{\alpha\beta}{}_{\alpha'\beta'}P^{(-)\alpha'\beta'}{}_{\gamma\delta} = -\frac{i}{2}P^{(-)\alpha\beta}{}_{\gamma'\delta'}\epsilon^{\gamma'\delta'}{}_{\gamma\delta} = P^{(-)\alpha\beta}{}_{\alpha'\beta'}P^{(-)\alpha'\beta'}{}_{\gamma\delta}. \tag{A.2}$$

The projector $P^{(-)i}{}_{\alpha\beta}$ is defined as

$$\begin{aligned} P^{(-)i}{}_{\alpha\beta} &\equiv \frac{1}{2}(\delta^0_\alpha\delta^i_\beta - \delta^0_\beta\delta^i_\alpha - i\epsilon^{0i}{}_{\alpha\beta}) \\ &= 2P^{(-)0i}{}_{\alpha\beta} = -i\epsilon^{ijk}P^{(-)jk}{}_{\alpha\beta}. \end{aligned} \tag{A.3}$$

This projector satisfies the following identities

$$P^{(-)i}{}_{\gamma\delta}P^{(-)i\alpha\beta} = -P^{(-)\alpha\beta}{}_{\gamma\delta} \tag{A.4}$$

$$\eta^{\beta\delta}P^{(-)i}{}_{\alpha\beta}P^{(-)j}{}_{\delta\gamma} = \frac{i}{2}\epsilon^{ijk}P^{(-)k}{}_{\alpha\gamma} + \frac{1}{4}\delta^{ij}\eta_{\alpha\gamma}. \tag{A.5}$$

Using this projector we can give the relation between $SO(3,1)$ representation $\Lambda^\alpha{}_\beta$ and $SO(3,\mathbf{C})$ representation $\Lambda^{ij}$ of the (proper orthochronous) Lorentz group:

$$\Lambda^{ij} = -P^{(-)i}{}_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta P^{(-)j\gamma\delta}. \tag{A.6}$$

This $SO(3,\mathbf{C})$ representation is obtained as the adjoint representation of $SL(2,\mathbf{C})$:

$$(e^{\theta^k J_k})^{ij}\Phi^j J_i = e^{\theta^k J_k}\Phi^j J_j e^{-\theta^k J_k}, \tag{A.7}$$

where $(J_k)^{ij} = \epsilon^{ikj}$ is the $SL(2,\mathbf{C})$ generator in the adjoint representation.

# Appendix B Invariant measures on $SU(2)$ and $SL(2,\mathbf{C})$

Here we give useful formulae for the invariant measures on $SU(2)$ and $SL(2,\mathbf{C})$, namely, the Haar measure and the (averaged) heat-kernel measure. As for detailed discussion on measures on general compact Lie groups and on their complexifications, we refer the reader to refs.[30] and [46].

For simplicity we will consider only finite dimensional representations, namely unitary representations in $SU(2)$. Because any unitary representation is equivalent to the direct sum of a finite number of irreducible representations, it is sufficient to deal only with irreducible representations, i.e. spin-$\frac{p}{2}$ representations $\pi_p$ ($p \in \mathbf{Z}$).

## B.4 The Haar measure on $SU(2)$

The Haar measure $d\mu(g)$ on $SU(2)$ ($g \in SU(2)$)is completely determined by the following two conditions[30]:

$$\text{normalization} \quad : \quad \int d\mu(g) = 1 \tag{B.1}$$

$$\text{bi-}SU(2) \text{ invariance} \quad : \quad d\mu(h_1 g h_2) = d\mu(g), \tag{B.2}$$

where $h_1$ and $h_2$ are some fixed elements of $SU(2)$. By exploiting the unitarity $\overline{\pi_p(g)_I{}^J} = \pi_p(g^{-1})_J{}^I$ $(I, J = 1, \ldots, p+1)$, we immediately find

$$\int d\mu(g) \overline{\pi_p(g)_I{}^J} \pi_q(g)_{I'}{}^{J'} = \frac{1}{p+1} \delta_{p,q} \delta^I_{I'} \delta^{J'}_J . \tag{B.3}$$

This formula plays a fundamental role when we consider the measure on $SU(2)$ gauge theories.

While eq.(B.3) suffices for any computation of inner products or expectation values in the framework of spin network states, it is sometimes convenient if we know formulae in terms only of the spinor representation $V^A{}_B \equiv \pi_1(V)^A{}_B$. Their computations are straightforward if we use the property $\epsilon_{AC} V^A{}_B V^C{}_D = \epsilon_{BD}$. We provide only two main results. One is

$$\int d\mu(V) V^{A_1}{}_{B_1} V^{A_2}{}_{V_2} \cdots V^{A_{2m}}{}_{B_{2m}} = \frac{1}{(m+1)! m! 2^m} \sum_{\sigma \in P_{2m}} \prod_{k=1}^{m} \epsilon^{A_{\sigma_{2k-1}} A_{\sigma_{2k}}} \epsilon_{B_{\sigma_{2k-1}} B_{\sigma_{2k}}} . \tag{B.4}$$

where $m$ is a non-negative integer and $P_n$ denotes the group of permutations of $n$ entries. The integration vanishes if the integrand is the product of an odd number of copies of spinor representation. The other is

$$\int d\mu(V) \overline{V^{A_1}{}_{(B_1} \cdots V^{A_n}{}_{B_n)}} V^{C_1}{}_{(D_1} \cdots V^{C_m}{}_{D_m)} = \frac{1}{n+1} \delta_{n,m} \delta^{C_1}_{(A_1} \cdots \delta^{C_n}_{A_n)} \delta^{(B_1}_{D_1} \cdots \delta^{B_n)}_{D_n} . \tag{B.5}$$

In deriving the latter equation we have used the unitarity

$$\overline{V^B{}_A} = (V^{-1})^A{}_B = \epsilon_{BC} \epsilon^{AD} V^C{}_D .$$

We should notice that eq.(B.3) and eq.(B.5) are two different realizations of the same formula.

## B.5 The heat-kernel measure on $SL(2, \mathbf{C})$

While the Haar measure provides a useful basis for the inner product on compact gauge theories, its analogue does not exist for complexified gauge groups such as $SL(2, \mathbf{C})$. However, there is a candidate for the well-defined measure on complexified gauge groups.

This was discovered by Hall[46] in the context of extending the coherent state transform to compact Lie groups.

The key ingredient is the coherent state transform

$$C_t : L^2(SU(2), d\mu) \to L^2(SL(2,\mathbf{C}), d\nu_t) \cap \mathcal{H}(SL(2,\mathbf{C}))$$

($t > 0$) from the space of functions on $SU(2)$ which are square-integrable w.r.t. the Haar measure $d\mu$ to the space of holomorphic functions on $SL(2,\mathbf{C})$ which are square-integrable w.r.t the "averaged" heat-kernel measure $d\nu_t$. Explicitly $C_t$ is given by

$$C_t[F](g) = \int_{SU(2)} d\mu(x) F(x) \rho_t(x^{-1}g), \tag{B.6}$$

where $F(x) \in L^2(SU(2), d\mu)$ and $g \in SL(2,\mathbf{C})$. $\rho_t(g)$ is the analytic continuation to $SL(2,\mathbf{C})$ of the heat kernel for the Casimir operator $\Delta = X_i X_i$ on $SU(2)$:[36]

$$\begin{aligned} \frac{d}{dt}\rho_t(x) &= \frac{1}{2}\Delta\rho_t(x) \\ \lim_{t\to 0}\rho_t(x) &= \delta(x, \mathbf{1}) \end{aligned} \tag{B.7}$$

For our purpose the following theorem is the most important:

**Theorem2 (Hall [46])**: *The coherent state transform $C_t$ is an isometric isomorphism of $L^2(SU(2), d\mu)$ onto $L^2(SL(2,\mathbf{C}), d\nu_t) \cap \mathcal{H}(SL(2,\mathbf{C}))$.*

We can express this theorem by

$$\int_{SL(2,\mathbf{C})/SU(2)} d\nu_t(g) \overline{C_t[F](g)} C_t[F'](g) = \int_{SU(2)} d\mu(x) \overline{F(x)} F'(x). \tag{B.8}$$

In order to obtain the explicit formula we further need eq.(30) of ref.[46]

$$C_t[\pi(x)](g) = e^{\frac{t}{2}\pi(\Delta)}\pi(g). \tag{B.9}$$

Using these results and eq.(B.3) we find the desired formula:

$$\int d\nu_t(g) \overline{\pi_p(g)_I{}^J} \pi_q(g)_{I'}{}^{J'} = \frac{\delta_{p,q}}{p+1} e^{t\frac{p(p+2)}{4}} \delta^I_{I'} \delta^{J'}_J. \tag{B.10}$$

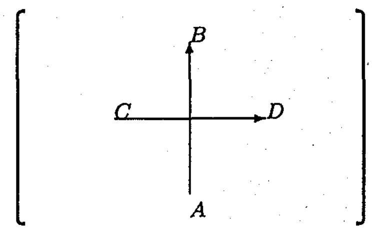
For the explicit expression of $d\nu_t$, we refer the reader to [46]. Here we only say that $d\nu_t$ is essentially the heat-kernel measure on $SL(2,\mathbf{C})/SU(2)$ which is bi-$SU(2)$ invariant and left-$SL(2,\mathbf{C})$ invariant, and that they are defined by using the Laplace-Beltrami operator on $SL(2,\mathbf{C})$.

[36] $X_i$ stands for the left-invariant vector field corresponding to the $SU(2)$ generator $J_i$. $\Delta$ can be regarded as the Laplace-Beltrami operator on $SU(2)$

# Appendix C Basic action of the renormalised scalar constraint

Here we list the action of the renormalized scalar constraint $\hat{S}^{ren}(\underset{\sim}{N})$ on the basic configurations in terms of the graphical notation. We will denote the parallel propagators along the curves $\alpha$ and $\beta$ by an upward vertical arrow and a horizontal arrow oriented to the right respectively. For example, we write the tensor product of the propagators along $\alpha$ and $\beta$ as follows [37]:

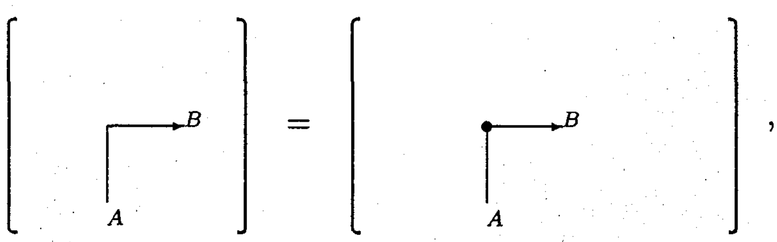
$$h_\alpha[0,1]_A{}^B h_\beta[0,1]_C{}^D \quad = $$




(C.1)

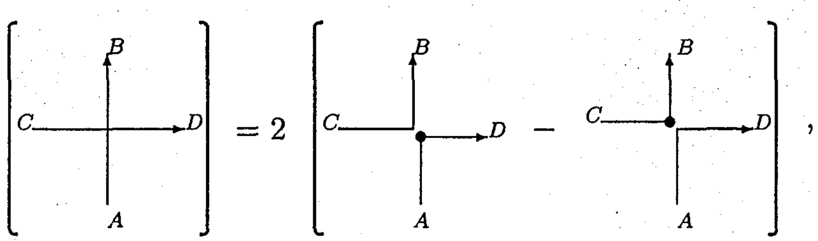
As is seen in §§4.1, we have only to concentrate on the action at vertices. In the present case the only vertex is at $x_0$. In the following we provide some of the basic actions on the regular four-valent vertex $x_0$ in the graphical notation. The dot in the diagram denotes the the location in which the magnetic field $\underset{\sim}{n} \cdot \tilde{B}$ is inserted. Using equations (C.2-8) and identities(3.30)(3.35) (3.36), we can write out all the basic action, namely the action of the renormalized scalar constraint $\hat{S}^{ren}(\underset{\sim}{N})$ on any single propagator and any pair of propagators, in the regular four-valent graph.

$\hat{S}_1^{ren}(\underset{\sim}{N})$



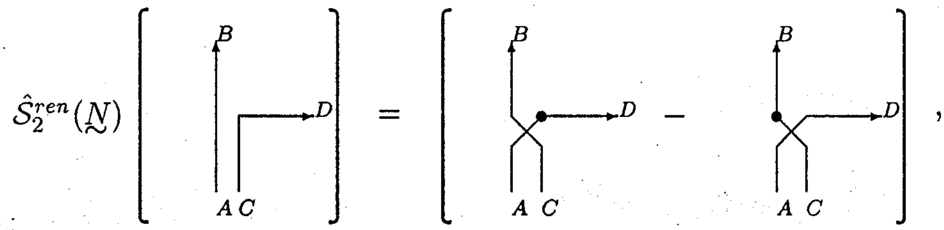

(C.2)

$\hat{S}_2^{ren}(\underset{\sim}{N})$

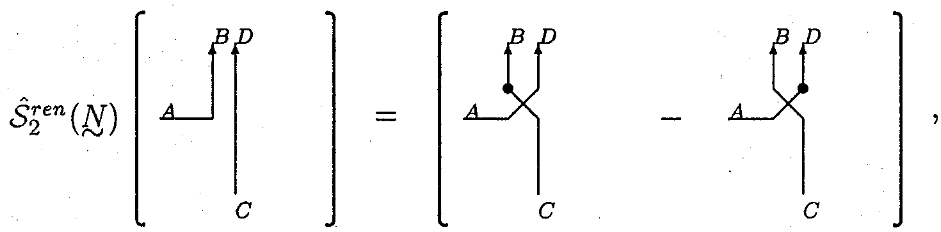



(C.3)

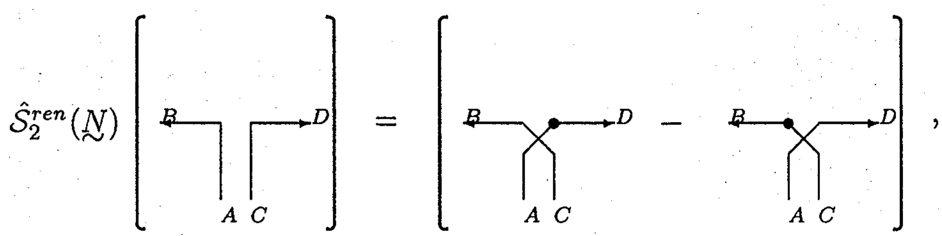
[37] We will assume that $\alpha$ and $\beta$ intersects once at $x_0 = \alpha(s_0) = \beta(t_0)$.



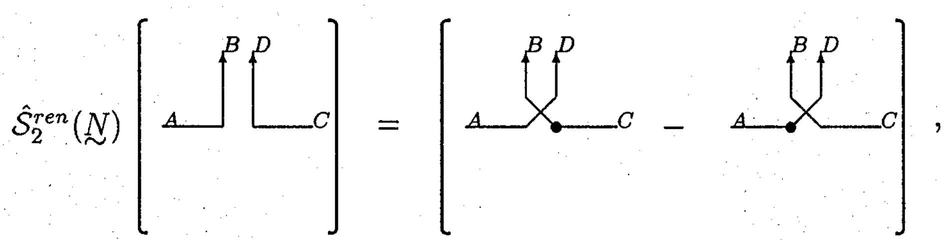
(C.4)



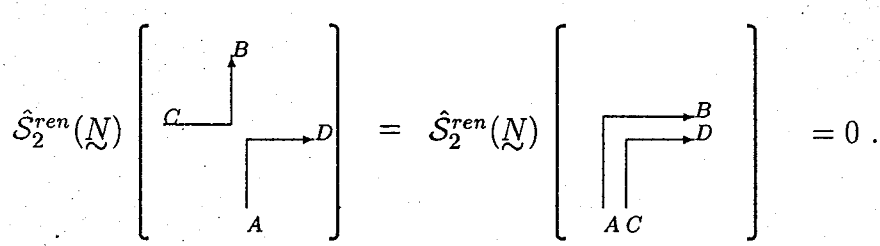
(C.5)



(C.6)



(C.7)



(C.8)

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