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Energy Spectrum of One-Dimensional Many Boson System

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An energy spectrum is exactly examined for a one-dimensional many boson system where bosons are interacting to each other through a delta-functional repulsive potential. After a unitary transformation, the Hamiltonian is diagonalized to be the following compact form $\sum_p(p^2/2m)n_p+(\pi\hbar/2mL)\times\sum_{p,q}|p-q|n_pn_q+(1/6m)(\pi\hbar/L)^2[(\sum_p n_p)^3-\sum_p n_p]$ at the infinitely large limit of the coupling constant. This form is non-linear with respect to the numbers n_p of quasi-particles. The non-linear terms are Galilean-invariant and produce a phonon-like spectrum. In a case of a finite coupling constant g, the total energy is expanded to power series of $(1/g)$.

§ 1. Introduction

A one-dimensional system of interacting many bosons was exactly solved by Giradau,[1] and Lieb and Liniger[2] on the basis of the first quantization formalism. The orthogonal and complete set of eigenstates was obtained by Sasaki and Kebukawa[4)~6)]on the basis of the field theory formalism. They also found a unitary transformation which transforms the total Hamiltonian to the diagonal form.

The energy levels of the system have been investigated by Lieb[3] in terms of the fermion momenta (not boson). The use of the fermion momenta originates in the fact that the energy of the interacting bose system agrees with the one of the non-interacting fermion system[1] in the limit of the infinitely large coupling constant.

However his classification scheme includes the following three difficulties as mentioned by himself;[3] first every excitation is described in terms of two parameters (two fermion momenta before and after excitation) instead of only one as physicists normally hope to do for a bose system. Second, there are too many provisos between these momenta by the exclusion principle which is needless in the original bose system. Third, his scheme includes two types of supplementary umklapp excitations besides two types of normal excitations.

These difficulties have been overcome by Sasaki and Kebukawa[4)~6)] in the second quantized form through a unitary transformation U diagonalizing the total Hamiltonian completely. They have introduced the dressed boson operators by transforming the original bare boson operators through the unitary transformation U. Then, the ground state has been expressed by the product of the creation-operators of dressed bosons with zero momentum. Each excitation state has been produced by changing zero momentum of one dressed boson to non-zero momentum p. Thus, each excitation is distinguished by only one momentum p which can take arbitrary value in contrast to Lieb's scheme where there are many provisos. Accordingly, the difficulties mentioned above have been removed.

However, the expression of the eigenenergies in the previous paper is rather complicated. In this paper, we rewrite the total energy by using the number of the dressed bosons and succeed in obtaining the compact form as

$$E=\sum_p \frac{p^2}{2m} n_p + \frac{1}{2m}\frac{\pi\hbar}{L}\sum_{p,q}|p-q|n_p n_q + \frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2 \frac{N(N^2-1)}{3} \tag{1·1}$$

in an infinitely large limit of the coupling constant g, where n_p denotes the number of the dressed bosons with momentum p, N is the total number of the bosons and L is the length of the one-dimensional space. Moreover, in the case of a finite value of the coupling constant, we obtain expansion of the energy to the power series of $(1/g)$. The calculations in this paper are not a mere reexpression of the energy, but give a physically meaningful form as follows: The right-hand side of Eq. (1·1) is constructed by two parts: One term is the kinetic energy of the dressed bosons which is Galilean-covariant and the residual terms Galilean-invariant terms. The latter terms have non-linear forms with respect to the dressed boson number. This nonlinearity derives the essential property of the interacting many boson system. Namely, these non-linear terms produce the drastic change in the excitation spectrum, that is, the phonon-like spectrum appears or disappears according to the change of the momentum distribution of the dressed bosons. In the final section, we will discuss possibility of an extension of this theory to a three-dimensional many boson system.

§ 2. Energy spectrum in the infinitely large coupling constant

Let us investigate the following interacting many boson system of which a total Hamiltonian is

$$H=\sum_p \frac{p^2}{2m} a_p{}^* a_p + \frac{g}{2L}\sum_{p,q,r} a_{p+r}^* a_{q-r}^* a_p a_q\,, \tag{2·1}$$

where m indicates the mass of one boson, g the coupling constant, and p, q or r the momentum which takes one of such values as $(2\pi\hbar/L)\times$ integer owing to the periodic boundary condition of the length L. The operators a_p and $a_p{}^*$ are the annihilation and creation operators which obey the commutation relations

$$[a_p, a_p{}^*]=\delta_{p,q}$$

and

$$[a_p, a_q]=[a_p{}^*, a_q{}^*]=0\,, \tag{2·2}$$

where $\delta_{p,q}$ is the Kronecker-delta function. Here, we shall briefly summarize the results of the previous papers.[4)~6)] The Hamiltonian (2·1) is diagonalized by the unitary operator U:[5)]

$$U=1+\sum_{N=2}^{\infty}\left\{U_N+\sum_{l=2}^{N-1}\frac{(-1)^N}{(N-l)!}\sum_{p_1p_2,\cdots p_{N-l}} a_{p_1}^*\cdots a_{p_{N-l}}^* U_l a_{p_1}\cdots a_{p_{N-l}}\right.$$
$$\left. -\frac{(-1)^N(N-1)}{N!}\sum_{p_1,\cdots p_N} a_{p_1}^*\cdots a_{p_N}^* a_{p_1}\cdots a_{p_N}\right\}, \tag{2·3}$$

where the operator U_N denotes the unitary transformation that produces the exact eigenstates of the fixed total number N and is written as

$$U_N=\sum_{q_1\leqq q_2\leqq\cdots\leqq q_N}\sum_{\{p_{ij}\}}\beta_{q_1\cdots q_N}\alpha_{q_1\cdots q_N}\prod_{1\leqq i<j\leqq N}\frac{k_{ij}}{k_{ij}-p_{ij}}\prod_{i=1}^{N}a^*_{q_i+\Sigma_j p_{ij}}\prod_{i=1}^{N}a_{q_i}\,. \tag{2·4}$$

In this equation, $\alpha_{q_1,\cdots,q_N}$ and $\beta_{q_1,\cdots,q_N}$ are the normalization constants and are given by

$$\alpha_{q_1,\cdots,q_N}=\prod_p\frac{1}{\sqrt{n_p!}} \tag{2·5a}$$

and

$$\beta_{q_1,\cdots,q_N}=\left(\sqrt{\frac{2}{\pi}}\right)^{N(N-1)},\quad (\text{in the limit } g\to\infty) \tag{2·5b}$$

where n_p is the number of the momentum p in $q_1, q_2, \cdots, q_N$. The running parameters p_{ij} indicate the transfer momenta from the i-th boson to the j-th boson. The constant values k_{ij} are uniquely determined by the following coupled equations:

$$\cot\left(\frac{L}{2\hbar}k_{ij}\right)=\frac{\hbar}{mg}\{q_i-q_j+2k_{ij}+\sum_{l\neq i,j}(k_{il}-k_{jl})\}\,, \tag{2·6}$$

where

$$k_{ij}=-k_{ji}\,,\quad -\pi\hbar/L\leqq k_{ij}<0\,.\quad (1\leqq i<j\leqq N) \tag{2·7}$$

The exact eigenstate of the Hamiltonian (2·1) is described by

$$|\psi_{q_1\cdots q_N}\rangle=U_N\alpha_{q_1\cdots q_N}a^*_{q_1}\cdots a^*_{q_N}|0\rangle\,, \tag{2·8}$$

where $|0\rangle$ is a vacuum state. Then, the eigenvalue $E_{q_1\cdots q_N}$ of the total energy becomes

$$E_{q_1\cdots q_N}=\frac{1}{2m}\sum_i[q_i+\sum_j k_{ij}]^2\,. \tag{2·9}$$

These results have been obtained in the previous paper.

Now, in this section, let us derive a reexpression of the total energy in the limit $g\to\infty$. In this limit, the value k_{ij} is easily determined from Eqs. (2·6) and (2·7) as

$$k_{ij}=\frac{\pi\hbar}{L}\eta_{ij}\,, \tag{2·10}$$

where

$$\eta_{ij}=\begin{cases}-1 & \text{for}\quad i<j\,,\\ 0 & \text{for}\quad i=j\,,\\ 1 & \text{for}\quad i>j\,.\end{cases} \tag{2·11}$$

Using the unitary oprator U of Eq. (2·3), we define the creation and annihilation operators of one dressed boson in the interaction cloud as

$$A_p{}^*=Ua_p{}^*U^*\,,\qquad A_p=Ua_pU^*\,. \tag{2·12}$$

The creation operator $A_p{}^*$ of the dressed boson produces any eigenstates of the total Hamiltonian:

$$|\psi_{q_1\cdots q_N}\rangle = a_{q_1\cdots q_N} A^*_{q_1}\cdots A^*_{q_N}|0\rangle\,. \tag{2·13}$$

Then, the number operator of the dressed boson with momentum p is given by

$$n_p = A_p{}^* A_p = U a_p{}^* a_p U^*\,. \tag{2·14}$$

We can reexpress the total energy in terms of these number operators n_p as will be mentioned below. This reexpression starts with substitution of Eqs. (2·10) and (2·11) into Eq. (2·9). Then, Eq. (2·9) becomes

$$\begin{aligned} E_{q_1\cdots q_N} &= \frac{1}{2m}\sum_{i=1}^{N}\Big[q_i + \frac{\pi\hbar}{L}\sum_{j=1}^{N}\eta_{ij}\Big]^2 \\ &= \frac{1}{2m}\sum_{i=1}^{N} q_i{}^2 + \frac{1}{m}\frac{\pi\hbar}{L}\sum_{i=1}^{N}\Big(q_i\sum_{l=1}^{N}\eta_{il}\Big) + \frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2 \sum_{ijl}\eta_{ij}\eta_{il}\,. \end{aligned} \tag{2·15}$$

The antisymmetric property of the suffices of η_{il} gives

$$\sum_{i=1}^{N}\Big(q_i\sum_{l=1}^{N}\eta_{il}\Big) = \frac{1}{2}\sum_{i,l}(q_i - q_l)\eta_{il}\,, \tag{2·16}$$

and the right-hand side of Eq. (2·16) becomes

$$(q_i - q_l)\eta_{il} = \begin{cases} -(q_i - q_l) & \text{for} \quad i<l\,, \\ (q_i - q_l) & \text{for} \quad i>l\,. \end{cases} \tag{2·17}$$

If we remember the restriction among the values of the momenta $q_1, q_2, \cdots, q_N$ in the sum of Eq. (2·4), then we can rewrite Eq. (2·17) into the absolute value of the momentum difference, that is,

$$(q_i - q_l)\eta_{il} = |q_i - q_l|\,. \tag{2·18}$$

By combining Eq. (2·18) with Eq. (2·16), we obtain

$$\sum_{i,l} q_i\eta_{il} = \frac{1}{2}\sum_{i,l}|q_i - q_l|\,. \tag{2·19}$$

In order to calculate the third term on the right-hand side of Eq. (2·15), we use the following property:

$$\sum_{l=1}^{N}\eta_{il} = -(i-1) + N - i = N + 1 - 2i\,, \tag{2·20}$$

which yields

$$\sum_{i,j,l}\eta_{ij}\eta_{il} = \sum_{i=1}^{N}[N+1-2i]^2 = \frac{1}{3}N(N^2-1)\,. \tag{2·21}$$

By substituting Eqs. (2·19) and (2·21) into Eq. (2·15), we have

$$E_{q_1,\cdots,q_N} = \frac{1}{2m}\sum_i q_i{}^2 + \frac{1}{2m}\frac{\pi\hbar}{L}\sum_{i,l}|q_i - q_l| + \frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2\frac{N(N^2-1)}{3}\,. \tag{2·22}$$

Let us rewrite the expression (2·22) in terms of the number operator which is defined by Eq. (2·14). We easily obtain it as follows:

$$E=\frac{1}{2m}\sum_p p^2 n_p+\frac{1}{2m}\frac{\pi\hbar}{L}\sum_{p,q}|p-q|n_p n_q+\frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2\frac{N(N^2-1)}{3}, \qquad (2\cdot 23)$$

which is equivalent to

$$H=\frac{1}{2m}\sum_p p^2 A_p{}^*A_p+\frac{1}{2m}\frac{\pi\hbar}{L}\sum_{p,q}|p-q|A_p{}^*A_pA_q{}^*A_q$$
$$+\frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2\frac{1}{3}[(\sum_p A_p{}^*A_p)^3-\sum_p A_p{}^*A_p].$$

Consequently we have succeeded in rewriting the total energy into the simple form (2·23) in the case $g\to\infty$. This result immediately leads that the total Hamiltonian is diagonalized as follows:

$$U^*HU=\frac{1}{2m}\sum_p p^2 a_p{}^*a_p+\frac{1}{2m}\frac{\pi\hbar}{L}\sum_{p,q}|p-q|a_p{}^*a_pa_q{}^*a_q$$
$$+\frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2\frac{1}{3}[(\sum_p a_p{}^*a_p)^3-\sum_p a_p{}^*a_p], \qquad (2\cdot 24)$$

where we make use of the inverse transformation

$$U^*A_p{}^*U=a_p{}^*, \qquad U^*A_pU=a_p. \qquad (2\cdot 25)$$

Next, let us investigate the property of the energy spectrum by using this compact form. We define the excitation energy ε_p as an energy-increase of the system when the dressed boson in the interaction cloud is excited from the zero momentum state to the state with momentum p. This excitation is described by the change of the number distribution from $\{n_q\}$ to $\{n_q'\}$, which are related to each other,

$$n_0'=n_0-1,$$
$$n_p'=n_p+1,$$
$$n_q'=n_q \quad \text{for} \quad q\neq 0 \quad \text{and} \quad q\neq p. \qquad (2\cdot 26)$$

Then, the excitation energy ε_p is easily obtaind from Eq. (2·23) as

$$\varepsilon_p=E(\{n_q'\})-E(\{n_q\})$$
$$=\frac{p^2}{2m}+\frac{1}{m}\frac{\pi\hbar}{L}\sum_q(|p-q|-|q|)n_q-\frac{1}{m}\frac{\pi\hbar}{L}|p|. \qquad (2\cdot 27)$$

When the value n_0/L is non-zero in the limit $L\to\infty$, the excitation energy becomes

$$\varepsilon_p \xrightarrow[\text{fixed } n_0/L]{L\to\infty} \frac{\pi\hbar}{m}\frac{n_0}{L}|p|+\frac{p^2}{2m}+\frac{\pi\hbar}{mL}\sum_{q\neq 0}(|p-q|-|q|)n_q. \qquad (2\cdot 28)$$

This indicates that ε_p is proportional to the absolute value of p in the region of small value of $|p|$. This means that the phonon velocity is equal to $\pi\hbar n_0/mL$.

Next, let us investigate Galilean covariancy of the total energy. The first term on the right-hand side of Eq. (2·23) is decomposed into the kinetic energy of the center of mass and the remaining term,

$$\frac{1}{2m}\sum_p p^2 n_p=\frac{1}{2mN}(\sum_p pn_p)^2+\frac{1}{4mN}\sum_{p,q}(p-q)^2 n_pn_q. \qquad (2\cdot 29)$$

Then, the total energy is expressed as

$$E=\frac{1}{2mN}(\sum_{p}pn_p)^2+\Big[\frac{1}{4mN}\sum_{p,q}(p-q)^2n_pn_q+\frac{\pi\hbar}{2mL}\sum_{p,q}|p-q|n_pn_q$$
$$+\left(\frac{\pi\hbar}{L}\right)^2\frac{N(N^2-1)}{6m}\Big]. \qquad (2\cdot30)$$

The terms in the square parenthesis on the right-hand side of Eq. (2·30) are easily seen to be invariant under Galilean transformation, because its functional forms are made of momentum differences only. Thus, the total energy is decomposed into the kinetic energy of center of mass (the first term in Eq. (2·30)) and the Galilean-invariant part (the residual terms in Eq. (2·30)). Namely, the expression (2·30) of the total energy shows to be Galilean-covariant.

§ 3. Expansion

In this section, we expand k_{ij} into the power series of $(1/g)$. We put k_{ij} as

$$k_{ij}=\sum_{n=0}^{\infty}\left(\frac{1}{g}\right)^n k_{ij}^{(n)}, \qquad (3\cdot1)$$

where $k_{ij}^{(n)}$ is a coefficient of n-th order in the expansion. In the limit $g\to\infty$, k_{ij} is expressed by Eq. (2·10) and therefore the zeroth order coefficient becomes

$$k_{ij}^{(0)}=\frac{\pi\hbar}{L}\eta_{ij} \qquad (3\cdot2)$$

which yields the coupled equation determining the coefficients $\{k_{ij}^{(n)}\}$ as

$$\cot\left[\frac{\pi}{2}\eta_{ij}+\frac{L}{2\hbar}\sum_{n=1}^{\infty}\left(\frac{1}{g}\right)^n k_{ij}^{(n)}\right]$$
$$=\frac{\hbar}{mg}\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_{l=1}^{N}(\eta_{il}-\eta_{jl})+\sum_{n=1}^{\infty}\sum_{l=1}^{N}\left(\frac{1}{g}\right)^n(k_{il}^{(n)}-k_{jl}^{(n)})\right], \qquad (3\cdot3)$$

where we use Eqs. (2·6), (2·7), (3·1) and (3·2). Since $\pi\eta_{ij}/2$ is equal to $\pi/2$ or $-\pi/2$, the coupled equation (3·3) is rewritten as

$$\frac{L}{2\hbar}\sum_{n=1}^{\infty}\left(\frac{1}{g}\right)^n k_{ij}^{(n)}=-\mathrm{Arctan}\ \frac{\hbar}{mg}\Big[q_i-q_j+\frac{\pi\hbar}{L}\sum_{l}(\eta_{il}-\eta_{jl})$$
$$+\sum_{n=1}^{\infty}\sum_{l}\left(\frac{1}{g}\right)^n(k_{il}^{(n)}-k_{jl}^{(n)})\Big], \qquad (3\cdot4)$$

where the value of the function of Arctan is chosen in the range between $-\pi/2$ and $\pi/2$ owing to the restriction (2·7).

Now, Eq. (3·4) has the following property: The right-hand side of Eq. (3·4) can be determined up to the $(s+1)$-th order of $(1/g)$ only by using the coefficients $k_{il}^{(n)}$ up to the s-th order. Therefore, we can determine the coefficient $k_{ij}^{(s+1)}$ by using Eq. (3·4) by substituting the lower order coefficients $k_{ij}^{(n)}$ $(0\leq n\leq s)$ into the right-hand side of Eq. (3·4).

Let us determine the coefficients $k_{ij}^{(n)}$ up to the third order by making use of the expansion of Arctan x:

$$\mathrm{Arctan}\ x=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\cdots. \tag{3·5}$$

Then, Eq. (3·4) becomes

$$\begin{aligned}&\frac{L}{2\hbar}\left(\frac{1}{g}k_{ij}^{(1)}+\frac{1}{g^2}k_{ij}^{(2)}+\frac{1}{g^3}k_{ij}^{(3)}+\cdots\right)\\&=-\frac{\hbar}{mg}\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_l(\eta_{il}-\eta_{jl})\right]\\&\quad-\frac{\hbar}{m}\sum_l\left[\frac{1}{g^2}(k_{il}^{(1)}-k_{jl}^{(1)})+\frac{1}{g^3}(k_{il}^{(2)}-k_{jl}^{(2)})\right]\\&\quad+\frac{1}{3}\left(\frac{\hbar}{mg}\right)^3\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_l(\eta_{il}-\eta_{jl})\right]^3+\mathrm{Order}\left(\frac{1}{g^4}\right).\end{aligned} \tag{3·6}$$

If we pick up the terms with the same power of $(1/g)$ on both sides of Eq. (3·6), we obtain the coefficient $k_{ij}^{(n)}$from the first order to the third order successively.
(first order part)

$$k_{ij}^{(1)}=-\frac{2\hbar^2}{Lm}\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_l(\eta_{il}-\eta_{jl})\right], \tag{3·7}$$

(second order part)

$$\frac{L}{2\hbar}k_{ij}^{(2)}=-\frac{\hbar}{m}\sum_l(k_{il}^{(1)}-k_{jl}^{(1)}), \tag{3·8}$$

(third order part)

$$\frac{L}{2\hbar}k_{ij}^{(3)}=-\frac{\hbar}{m}\sum_l(k_{il}^{(2)}-k_{jl}^{(2)})+\frac{1}{3}\left(\frac{\hbar}{m}\right)^3\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_l(\eta_{il}-\eta_{jl})\right]^3. \tag{3·9}$$

By substituting Eq. (3·7) into Eq. (3·8), the second order coefficient $k_{ij}^{(2)}$ becomes

$$\begin{aligned}k_{ij}^{(2)}&=\left(\frac{2\hbar^2}{Lm}\right)^2\sum_l\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right]\\&=\left(\frac{2\hbar^2}{Lm}\right)^2N\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right].\end{aligned} \tag{3·10}$$

Similarly, we put Eq. (3·10) into Eq. (3·9), and then obtain

$$\begin{aligned}k_{ij}^{(3)}&=-\frac{2\hbar^2}{Lm}\sum_l\left(\frac{2\hbar^2}{Lm}\right)^2N\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right]\\&\quad+\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{jl})\right]^3\\&=-\left(\frac{2\hbar^2}{Lm}\right)^3N^2\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right]\end{aligned}$$

$$+\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\Big[(q_i-q_j)^3+3(q_i-q_j)^2\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})$$

$$+3(q_i-q_j)\left(\frac{\pi\hbar}{L}\right)^2\sum_{st}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})$$

$$+\left(\frac{\pi\hbar}{L}\right)^3\sum_{stu}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})\Big]. \tag{3·11}$$

Thus, we have determined the value of k_{ij} up to the third order of the power of $(1/g)$.

§ 4. $1/g$ power expansion of the energy

In this section, we expand the total energy to the $1/g$ power series and obtain the coefficients up to the third order. The total energy is expanded as follows:

$$E=E_0+\frac{1}{g}E_1+\frac{1}{g^2}E_2+\frac{1}{g^3}E_3+\cdots. \tag{4·1}$$

In order to obtain the coefficients E_n, we substitute $k_{ij}^{(n)}$ obtained in the previous section into the representation (2·9):

$$E=\frac{1}{2m}\sum_i\Big[q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}+\sum_j\Big(\frac{1}{g}k_{ij}^{(1)}+\frac{1}{g^2}k_{ij}^{(2)}+\frac{1}{g^3}k_{ij}^{(3)}\Big)\Big]^2+\text{Order}\left(\frac{1}{g^4}\right)$$

$$=\frac{1}{2m}\sum_i\left(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\right)^2$$

$$+\frac{1}{2m}\frac{1}{g}\sum_i\Big[2\Big(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\Big)\sum_j k_{ij}^{(1)}\Big]$$

$$+\frac{1}{2m}\frac{1}{g^2}\sum_i\Big[2\Big(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\Big)\sum_j k_{ij}^{(2)}+\sum_j k_{ij}^{(1)}\sum_l k_{il}^{(1)}\Big]$$

$$+\frac{1}{2m}\frac{1}{g^3}\sum_i\Big[2\Big(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\Big)\sum_j k_{ij}^{(3)}+2\sum_j k_{ij}^{(1)}\sum_l k_{il}^{(2)}\Big]+\text{Order}\left(\frac{1}{g^4}\right). \tag{4·2}$$

By comparing Eq. (4·2) with Eq. (4·1), we obtain each coefficient as follows:

$$E_0=\frac{1}{2m}\sum_i\left(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\right)^2, \tag{4·3}$$

$$E_1=\frac{1}{2m}\sum_i\Big[2\Big(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\Big)\sum_j k_{ij}^{(1)}\Big], \tag{4·4}$$

$$E_2=\frac{1}{2m}\sum_i\Big[2\Big(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\Big)\sum_j k_{ij}^{(2)}+\sum_j k_{ij}^{(1)}\sum_l k_{il}^{(1)}\Big], \tag{4·5}$$

$$E_3=\frac{1}{2m}\sum_i\Big[2\Big(q_i+\frac{\pi\hbar}{L}\sum_s\eta_{is}\Big)\sum_j k_{ij}^{(3)}+2\sum_j k_{ij}^{(1)}\sum_l k_{il}^{(2)}\Big]. \tag{4·6}$$

The coefficient E_0 agrees with the total energy (2·22) of the limit $g\to\infty$:

$$E_0=\frac{1}{2m}\sum_i q_i^2+\frac{1}{2m}\frac{\pi\hbar}{L}\sum_{i,j}|q_i-q_j|+\frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2\frac{N(N^2-1)}{3}. \tag{4·7}$$

Here, we write down two identities useful in the higher order calculation:

$$\sum_i(q_i\sum_j k_{ij}^{(n)})=\frac{1}{2}\sum_{ij}(q_i-q_j)k_{ij}^{(n)},$$

$$\sum_{ijs}\eta_{is}k_{ij}^{(n)}=\frac{1}{2}\sum_{ijs}(\eta_{is}-\eta_{js})k_{ij}^{(n)}, \tag{4·8}$$

which are easily derived from the antisymmetric property $k_{ij}^{(n)}=-k_{ji}^{(n)}$.

Then, Eq. (4·4) is rewritten by using Eq. (3·7) as

$$\begin{aligned}E_1&=-\frac{1}{2m}\frac{2\hbar^2}{Lm}\sum_{ij}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{jt})\right\}\\&=-\frac{1}{2m}\frac{2\hbar^2}{Lm}\sum_{ij}\left[(q_i-q_j)^2+\frac{2\pi\hbar}{L}\sum_s(q_i-q_j)(\eta_{is}-\eta_{js})\right.\\&\qquad\left.+\left(\frac{\pi\hbar}{L}\right)^2\sum_{st}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})\right].\end{aligned} \tag{4·9}$$

The second and third terms in the square parenthesis are rewritten by Eq. (2·19)

$$\sum_{ijs}(q_i-q_j)(\eta_{is}-\eta_{js})=\sum_{ijs}q_i\eta_{is}+\sum_{ijs}q_j\eta_{js}=N\sum_{is}|q_i-q_s|, \tag{4·10}$$

$$\sum_{ijst}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})=\sum_{ijst}(\eta_{is}\eta_{it}+\eta_{js}\eta_{jt})=2N\sum_{ist}\eta_{is}\eta_{it}, \tag{4·11}$$

where we use the antisymmetric property for the suffices. Substituting Eqs. (4·10) and (4·11) into Eq. (4·9), we have

$$E_1=-\frac{1}{2m}\frac{2\hbar^2}{Lm}\left[\sum_{ij}(q_i-q_j)^2+\frac{2\pi\hbar}{L}N\sum_{is}|q_i-q_s|+\left(\frac{\pi\hbar}{L}\right)^2\frac{2N^2(N^2-1)}{3}\right], \tag{4·12}$$

where Eq. (2·21) is employed. Similiarly, by making use of Eqs. (4·5), (4·8) and (3·10), E_2 becomes

$$\begin{aligned}E_2&=\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^2N\sum_{ij}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{jt})\right\}\\&\quad+\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^2\sum_{ijl}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\left\{q_i-q_l+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{lt})\right\}\\&=\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^2\frac{3}{2}N\sum_{ij}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\\&\qquad\times\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{jt})\right\}.\end{aligned} \tag{4·13}$$

By the calculation similiar to that of E_1, we can easily derive

$$E_2=\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^2\frac{3N}{2}\left[\sum_{ij}(q_i-q_j)^2+\frac{2\pi\hbar}{L}N\sum_{is}|q_i-q_s|+\left(\frac{\pi\hbar}{L}\right)^2\frac{2N^2(N^2-1)}{3}\right]. \tag{4·14}$$

Consequently, we obtain the total energy up to the second order of the power of $(1/g)$ as follows:

$$E=\frac{1}{2m}\sum_i \dot{q}_i^2+\frac{1}{2m}\left\{-\frac{2\hbar^2}{Lmg}+\left(\frac{2\hbar^2}{Lmg}\right)^2\frac{3N}{2}\right\}\sum_{ij}(q_i-q_j)^2$$
$$+\frac{1}{2m}\left\{1-\frac{2\hbar^2}{Lmg}2N+\left(\frac{2\hbar^2}{Lmg}\right)^2 3N^2\right\}\left\{\frac{\pi\hbar}{L}\sum_{is}|q_i-q_s|+\left(\frac{\pi\hbar}{L}\right)^2\frac{N(N^2-1)}{3}\right\}$$
$$+\text{Order}\left(\frac{1}{g^3}\right). \tag{4·15}$$

Rewriting this expression in terms of the number distribtion $\{n_p\}$ of the dressed bosons, we get

$$E=\frac{1}{2m}\sum_p p^2 n_p+\frac{1}{2m}\left\{-\frac{2\hbar^2}{Lmg}+\left(\frac{2\hbar^2}{Lmg}\right)^2\frac{3N}{2}\right\}\sum_{p,q}(p-q)^2 n_p n_q$$
$$+\frac{1}{2m}\left\{1-\frac{2\hbar^2}{Lmg}2N+\left(\frac{2\hbar^2}{Lmg}\right)^2 3N^2\right\}\left\{\frac{\pi\hbar}{L}\sum_{p,q}|p-q|n_p n_q+\left(\frac{\pi\hbar}{L}\right)^2\frac{N(N^2-1)}{3}\right\}$$
$$+\text{Order}\left(\frac{1}{g^3}\right). \tag{4·16}$$

If we decompose the first term on the right-hand side of Eq. (4·16) into the sum of the kinetic energy of the center of mass and the Galilean invariant terms, we obtain

$$E=\frac{1}{2mN}\left(\sum_p pn_p\right)^2+\frac{1}{4mN}\left\{1-2\left(\frac{2\hbar^2 N}{mgL}\right)+3\left(\frac{2\hbar^2 N}{mgL}\right)^2\right\}$$
$$\times\left[\sum_{p,q}\left\{(p-q)^2+\frac{2\pi\hbar N}{L}|p-q|\right\}n_p n_q+2\left(\frac{\pi\hbar N}{L}\right)^2\frac{N^2-1}{3}\right]+\text{Order}\left(\frac{1}{g^3}\right). \tag{4·17}$$

Thus, we have accomplished to rewrite the total energy to the compact form as Eqs. (4·15)~(4·17). Next, let us calculate the third order coefficient E_3. By substituting the representation (3·7), (3·10) and (3·11) of $k_{ij}^{(n)}$ into Eq. (4·6) and by employing Eq. (4·8), we obtain

$$E_3=\frac{1}{2m}\sum_{ij}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\left\{-\left(\frac{2\hbar^2}{Lm}\right)^3 N^2\left[q_i-q_j+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{jt})\right]\right.$$
$$+\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\left[(q_i-q_j)^3+3(q_i-q_j)^2\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{jt})\right.$$
$$+3(q_i-q_j)\left(\frac{\pi\hbar}{L}\right)^2\sum_{tu}(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})$$
$$\left.\left.+\left(\frac{\pi\hbar}{L}\right)^3\sum_{tuv}(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})(\eta_{iv}-\eta_{jv})\right]\right\}$$
$$-\frac{2}{2m}\left(\frac{2\hbar^2}{Lm}\right)^3 N\sum_{ijl}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\left\{q_i-q_l+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{lt})\right\}$$
$$=-\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^3 2N^2\sum_{ij}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})\right\}\left\{q_i-q_j+\frac{\pi\hbar}{L}\sum_t(\eta_{it}-\eta_{jt})\right\}$$

$$+\frac{1}{2m}\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\sum_{ij}\Big[(q_i-q_j)^4+(1+3)(q_i-q_j)^3\frac{\pi\hbar}{L}\sum_s(\eta_{is}-\eta_{js})$$

$$+(3+3)(q_i-q_j)^2\left(\frac{\pi\hbar}{L}\right)^2\sum_{s,t}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})$$

$$+(3+1)(q_i-q_j)\left(\frac{\pi\hbar}{L}\right)^3\sum_{stu}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})$$

$$+\left(\frac{\pi\hbar}{L}\right)^4\sum_{stuv}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})(\eta_{iv}-\eta_{jv})\Big]. \tag{4·18}$$

Through a slightly long calculation given in the Appendix, Eq. (4·18) becomes

$$E_3=-\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^3 2N^2\Big[\sum_{ij}(q_i-q_j)^2+\frac{2\pi\hbar}{L}N\sum_{is}|q_i-q_s|$$

$$+\left(\frac{\pi\hbar}{L}\right)^2\frac{2N^2(N^2-1)}{3}\Big]+\frac{1}{2m}\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\Big[\sum_{ij}(q_i-q_j)^4$$

$$+4\frac{\pi\hbar}{L}\sum_{ijs}|q_i-q_s|\{(q_i-q_j)^2+(q_i-q_j)(q_s-q_j)+(q_s-q_j)^2\}$$

$$+6\left(\frac{\pi\hbar}{L}\right)^2\sum_{ijst}\{2(q_i-q_j)^2\eta_{is}\eta_{it}+|q_i-q_s||q_j-q_t|\}$$

$$+4\left(\frac{\pi\hbar}{L}\right)^3\{2N\sum_{istu}q_i\eta_{is}\eta_{it}\eta_{iu}+N(N^2-1)\sum_{is}|q_i-q_s|\}$$

$$+\left(\frac{\pi\hbar}{L}\right)^4\frac{8}{15}N^2(2N^4-5N^2+3)\Big]. \tag{4·19}$$

Thus, we have accomplished the power expansion of the total energy up to the third order.

§ 5. Phonon velocity and ground state energy

In this section, we calculate the phonon velocity up to the third order of the power of $(1/g)$ and examine the relation between the phonon velocity and the ground state energy.

The state of the single excitation is characterized by the number distribution function $\{n_q''\}$ as

$$n_0''=N-1,$$

$$n_p''=1,$$

$$n_q''=0 \qquad \text{for all } q. \quad (q\neq 0,\ q\neq p) \tag{5·1}$$

Therefore, the single excitation energy $\varepsilon_p{}^0$ is defined by

$$\varepsilon_p{}^0=E(\{n_q''\})-E_G, \tag{5·2}$$

where E_G denotes the ground state energy. Substituting the expression (4·16) into Eq. (5·2), we can calculate the single excitation energy $\varepsilon_p{}^0$ up to the second order of the power of $(1/g)$ as

$$\varepsilon_p{}^0=\frac{1}{2m}p^2+\frac{1}{2m}\left\{-\frac{2\hbar^2}{Lmg}+\left(\frac{2\hbar^2}{Lmg}\right)^2\frac{3N}{2}\right\}2(N-1)p^2$$

$$+\frac{1}{2m}\left\{1-\frac{2\hbar^2}{Lmg}2N+\left(\frac{2\hbar^2}{Lmg}\right)^2 3N^2\right\}\frac{\pi\hbar}{L}2N|p|+\text{Order }\left(\frac{1}{g^3}\right)$$

$$=\frac{1}{2m}\left\{1-\frac{2\hbar^2}{Lmg}2N+\left(\frac{2\hbar^2}{Lmg}\right)^2 3N^2\right\}\left(p^2+\frac{2\pi\hbar}{L}N|p|\right)$$

$$+\text{Order }\left(\frac{1}{g^3}\text{ or }\frac{1}{L}\right). \tag{5·3}$$

Now, the phonon velocity c is defined by

$$c=\lim_{|p|\to 0}(\varepsilon_p{}^0/|p|) \tag{5·4}$$

which is expanded into the power series as

$$c=c_0+\frac{1}{g}c_1+\frac{1}{g^2}c_2+\frac{1}{g^3}c_3+\cdots. \tag{5·5}$$

Therefore, the coefficients c_i up to the second order are derived from Eqs. (5·3) and (5·4):

$$c_0=\frac{\pi\hbar}{m}\frac{N}{L}, \tag{5·6}$$

$$c_1=-2\frac{\pi\hbar}{m}\frac{2\hbar^2}{m}\left(\frac{N}{L}\right)^2, \tag{5·7}$$

$$c_2=3\frac{\pi\hbar}{m}\left(\frac{2\hbar^2}{m}\right)^2\left(\frac{N}{L}\right)^3, \tag{5·8}$$

where we take a limit $L\to\infty$ under a fixed value of N/L.

We calculate the third order coefficient c_3 by making use of Eq. (4·19), and obtain

$$c_3=-4\frac{\pi\hbar}{m}\left(\frac{2\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^4+\frac{1}{2m}\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3 4\left(\frac{\pi\hbar}{L}\right)^3\{2N^4+2N^4\}$$

$$=\left\{-4\frac{\pi\hbar}{m}\left(\frac{2\hbar^2}{m}\right)^3+\frac{16}{3}\frac{\hbar}{m}\left(\frac{\pi\hbar^2}{m}\right)^3\right\}\left(\frac{N}{L}\right)^4. \tag{5·9}$$

Consequently, the phonon velocity c is given by

$$c=\frac{\pi\hbar}{m}\left[\frac{N}{L}-\frac{2}{g}\frac{2\hbar^2}{m}\left(\frac{N}{L}\right)^2+\frac{3}{g^2}\left(\frac{2\hbar^2}{m}\right)^2\left(\frac{N}{L}\right)^3-\frac{4}{g^3}\left(\frac{2\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^4\right]$$

$$+\frac{16}{3g^3}\frac{\hbar}{m}\left(\frac{\pi\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^4+\text{Order }\left(\frac{1}{g^4}\right). \tag{5·10}$$

Next, let us expand the ground state energy E_G to the power series as follows:

$$E_G=E_{G0}+\frac{1}{g}E_{G1}+\frac{1}{g^2}E_{G2}+\frac{1}{g^3}E_{G3}+\cdots. \tag{5·11}$$

By putting the number distribution function $\{n_q\}$ as

$$n_0=N, \qquad n_q=0 \qquad \text{for all} \quad q, \quad (q\neq 0) \tag{5·12}$$

in Eqs. (4·16) and (4·19), we obtain the coefficients $E_{G0}\sim E_{G3}$ in the expansion (5·11) of the ground state energy,

$$E_{G0}=\frac{1}{3}\frac{\pi^2\hbar^2}{2m}\left(\frac{N}{L}\right)^2 N,$$

$$E_{G1}=-\frac{2}{3}\frac{\pi^2\hbar^2}{2m}\frac{2\hbar^2}{m}\left(\frac{N}{L}\right)^3 N,$$

$$E_{G2}=\frac{3}{3}\frac{\pi^2\hbar^2}{2m}\left(\frac{2\hbar^2}{m}\right)^2\left(\frac{N}{L}\right)^4 N,$$

$$E_{G3}=-\frac{4}{3}\frac{\pi^2\hbar^2}{2m}\left(\frac{2\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^5 N+\frac{1}{2m}\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\left(\frac{\pi\hbar}{L}\right)^4\frac{16}{15}N^6,$$

$$=-\frac{4}{3}\frac{\pi^2\hbar^2}{2m}\left(\frac{2\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^5 N+\frac{16}{45}\left(\frac{\pi\hbar^2}{m}\right)^4\left(\frac{N}{L}\right)^5 N, \tag{5·13}$$

where we take a limit $L\to\infty$ under a fixed value of N/L.

As is well known, a sound velocity v of liquid or gas is related to its ground state energy E_G from a macroscopic argument. The relation is

$$v^2=\frac{L^2}{mN}\frac{\partial^2 E_G}{\partial L^2}. \tag{5·14}$$

Let us show that the relation (5·14) also holds for the microscopical phonon velocity. First, we expand the square of the phonon velocity into the power of $(1/g)$,

$$c^2=c_0^2+\frac{1}{g}2c_0c_1+\frac{1}{g^2}(2c_0c_2+c_1^2)+\frac{1}{g^3}(2c_0c_3+2c_1c_2)+\cdots. \tag{5·15}$$

Each coefficient of the same power of $(1/g)$ in Eq. (5·15) becomes the middle sides of Eqs. (5·16)~(5·19) by using Eqs. (5·6)~(5·9):

$$c_0^2=\left(\frac{\pi\hbar}{m}\right)^2\left(\frac{N}{L}\right)^2=\frac{L^2}{mN}\frac{\partial^2 E_{G0}}{\partial L^2}, \tag{5·16}$$

$$2c_0c_1=-4\left(\frac{\pi\hbar}{m}\right)^2\frac{2\hbar^2}{m}\left(\frac{N}{L}\right)^3=\frac{L^2}{mN}\frac{\partial^2 E_{G1}}{\partial L^2}, \tag{5·17}$$

$$2c_0c_2+c_1^2=10\left(\frac{\pi\hbar}{m}\right)^2\left(\frac{2\hbar^2}{m}\right)^2\left(\frac{N}{L}\right)^4=\frac{L^2}{mN}\frac{\partial^2 E_{G2}}{\partial L^2}, \tag{5·18}$$

$$2c_0c_3+2c_1c_2=-20\left(\frac{\pi\hbar}{m}\right)^2\left(\frac{2\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^5+\frac{32}{3}\frac{\pi\hbar^2}{m^2}\left(\frac{\pi\hbar^2}{m}\right)^3\left(\frac{N}{L}\right)^5$$

$$=\frac{L^2}{mN}\frac{\partial^2 E_{G3}}{\partial L^2}. \tag{5·19}$$

On the other hand, if we take the second partial derivatives of Eqs. (5·13), we obtain the last equalities in Eqs. (5·16)~(5·19). Consequently we have verified Eq. (5·14) in an explicit form up to the third order.

§ 6. Conclusion and discussion

In the previous sections, we have investigated the $(1/g)$ power expansion of the energy spectrum for the interacting many boson system in one-dimensional space. This scheme is accomplished by using the number of the dressed boson, and the result gives a physically meaningful expression. Moreover, we have obtained the relation between the phonon velocity and the ground state energy explicitly. Taking consideration of the simplification of the energy-expression, we may also reexpress the unitary transformation U to a more compact form, but do not succeed in doing it.

In conclusion, let us discuss a utilization of the present results in three-dimensional system. Some of the fundamental properties clarified in this paper may hold in the three-dimensional system, also. Especially the Galilean-invariant term in the total energy may be extended to the three-dimensional system where its total energy may have a similiar functional form $|\boldsymbol{p}-\boldsymbol{q}|n_p n_q$ ($\boldsymbol{p}$ and $\boldsymbol{q}$ are three-dimensional vectors). This idea gives a new viewpoint on a microscopic theory of Liquid He II as shown in Refs. 7)~9).

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Appendix

Let us rewrite each term of Eq. (4·18) as follows:

$$\begin{aligned}\sum_{ijs}(q_i-q_j)^3(\eta_{is}-\eta_{js}) &= 2\sum_{ijs}(q_i-q_j)^3\eta_{is}\\ &= \sum_{ijs}[(q_i-q_j)^3-(q_s-q_j)^3]\eta_{is}\\ &= \sum_{ijs}(q_i-q_s)[(q_i-q_j)^2+(q_i-q_j)(q_s-q_j)+(q_s-q_j)^2]\eta_{is}\\ &= \sum_{ijs}|q_i-q_s|[(q_i-q_j)^2+(q_i-q_j)(q_s-q_j)+(q_s-q_j)^2]\,, \qquad (\text{A}\cdot 1)\end{aligned}$$

where the final equality is derived from Eq. (2·18). Next the third term in the square parenthesis of Eq. (4·18) is reexpressed as

$$\begin{aligned}&\sum_{i,j,s,t}(q_i-q_j)^2(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})\\ &=\sum_{i,j,s,t}[2(q_i-q_j)^2\eta_{is}\eta_{it}-2(q_i-q_j)^2\eta_{is}\eta_{jt}]\\ &=\sum_{i,j,s,t}[2(q_i-q_j)^2\eta_{is}\eta_{it}+4q_iq_j\eta_{is}\eta_{jt}]\end{aligned}$$

$$=\sum_{i,j,s,t}[2(q_i-q_j)^2\eta_{is}\eta_{it}+|q_i-q_s||q_j-q_t|]\,, \tag{A·2}$$

where we use the asymmetric property for the suffices of η_{is} and Eq. (2·19). The fourth term in the square parenthesis of Eq. (4·18) becomes

$$\sum_{ijstu}(q_i-q_j)(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})$$

$$=2\sum_{ijstu}q_i(\eta_{is}\eta_{it}\eta_{iu}-3\eta_{is}\eta_{it}\eta_{ju}+3\eta_{is}\eta_{jt}\eta_{ju}-\eta_{js}\eta_{jt}\eta_{ju})\,. \tag{A·3}$$

In the summation on the right-hand side of Eq. (A·3), we can use the following properties:

$$\sum_{ju}\eta_{ju}=0\,,$$

$$\sum_{jstu}\eta_{js}\eta_{jt}\eta_{ju}=\sum_j(2j-N-1)^3=0\,. \tag{A·4}$$

Then, Eq. (A·3) is rewritten as

$$\sum_{ijstu}(q_i-q_j)(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})$$

$$=2\sum_{ijstu}[q_i\eta_{is}\eta_{it}\eta_{iu}+3q_i\eta_{is}\eta_{jt}\eta_{ju}]$$

$$=2N\sum_{istu}q_i\eta_{is}\eta_{it}\eta_{iu}+N(N^2-1)\sum_{is}|q_i-q_s|\,, \tag{A·5}$$

where we use Eq. (2·21).

Finally, we calculate the fifth term in the square parenthesis of Eq. (4·18) as follows:

$$\sum_{ijstuv}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})(\eta_{iv}-\eta_{jv})$$

$$=\sum_{ijstuv}[2\eta_{is}\eta_{it}\eta_{iu}\eta_{iv}-8\eta_{is}\eta_{it}\eta_{iu}\eta_{jv}+6\eta_{is}\eta_{it}\eta_{ju}\eta_{jv}]\,. \tag{A·6}$$

Now, we calculate the first term on the right-hand side of Eq. (A·6):

$$\sum_{ijstuv}\eta_{is}\eta_{it}\eta_{iu}\eta_{iv}=N\sum_{i=1}^{N}(2i-N-1)^4=\frac{N^2}{15}[3N^4-10N^2+7]\,. \tag{A·7}$$

If we put Eqs. (A·7) and (2·21) into (A·6), we obtain

$$\sum_{ijstuv}(\eta_{is}-\eta_{js})(\eta_{it}-\eta_{jt})(\eta_{iu}-\eta_{ju})(\eta_{iv}-\eta_{jv})$$

$$=\frac{2N^2}{15}[3N^4-10N^2+7]+\frac{6N^2(N^2-1)^2}{9}$$

$$=\frac{8}{15}N^2[2N^4-5N^2+3]\,. \tag{A·8}$$

By substituting Eqs. (A·1), (A·2), (A·5) and (A·8) into Eq. (4·18), we get the final result:

$$
\begin{aligned}
E_3=&-\frac{1}{2m}\left(\frac{2\hbar^2}{Lm}\right)^3 2N^2\left[\sum_{ij}(q_i-q_j)^2+\frac{2\pi\hbar}{L}N\sum_{i,s}|q_i-q_s|\right.\\
&\left.+\left(\frac{\pi\hbar}{L}\right)^2\frac{2N^2(N^2-1)}{3}\right]+\frac{1}{2m}\frac{1}{3}\frac{2\hbar}{L}\left(\frac{\hbar}{m}\right)^3\left[\sum_{ij}(q_i-q_j)^4\right.\\
&+4\frac{\pi\hbar}{L}\sum_{ijs}|q_i-q_s|\{(q_i-q_j)^2+(q_i-q_j)(q_s-q_j)+(q_s-q_j)^2\}\\
&+6\left(\frac{\pi\hbar}{L}\right)^2\sum_{ijst}\{2(q_i-q_j)^2\eta_{is}\eta_{it}+|q_i-q_s||q_j-q_t|\}\\
&+4\left(\frac{\pi\hbar}{L}\right)^3\{2N\sum_{istu}q_i\eta_{is}\eta_{it}\eta_{iu}+N(N^2-1)\sum_{is}|q_i-q_s|\}\\
&\left.+\left(\frac{\pi\hbar}{L}\right)^4\frac{8}{15}N^2(2N^4-5N^2+3)\right].
\end{aligned}
\tag{A$\cdot$9}
$$

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