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A Microscopic Theory of the Interacting Bose System

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A new theory of the interacting bose system is proposed. In contrast with the Bogoliubov theory, the condensate in the present theory is composed of the interacting dressed atoms and the total number of the dressed atoms is conserved. The second-order corrections to the ground-state energy and the excitation energy are evaluated to be found that the results in the present theory coincide with those obtained by Sunakawa et al. On the basis of this situation, it is shown that the present formalism is essentially equivalent to the theory of Sunakawa et al.

§ 1. Introduction

Many microscopic theories of an interacting Boson system have so far been proposed to account for peculiar properties of He II. The theories may be classified into two groups: The first group is the Bogoliubov theory[1] and its modified ones.[2),3] The theories belonging to the second group are those which make use of the density-fluctuation and its canonical variables as collective variables.[4]

On the basis of London's original idea,[5] Bogoliubov assumed the existence of macroscopic occupation of the condensate and set the zero momentum single atom operators a_0 and $a_0{}^*$ equal to a c-number $\sqrt{n_0}$. With the aid of this assumption, he succeeded in deriving the phonon character of the excitation spectrum in the lowest approximation. In this type of theories, however, the effects of the phonon-phonon interaction have so far not been taken into consideration definitely. On the other hand, the theories in the second group are the microscopic theoretical version of the Landau phenomenological theory.[6] As a representative of such theories, we shall adopt hereafter the theory developed by Sunakawa, Yamasaki and Kebukawa[4),7] (hereafter referred to as the S.Y.K. theory). In this theory, they introduce the density-fluctuation operator and the velocity operator as the collective variables. In the recent articles,[7] they have obtained a definite expression for the proper phonon-phonon interactions, in which every vertex-function is convergent and vanishes in the limit when the interaction among atoms disappears. In this respect, the S.Y.K. theory is in advance of the present Bogoliubov formalism.

The first purpose of the present paper is to propose a new theory which gives an improvement of the original Bogoliubov theory. In the Bogoliubov theory, the condensate is assumed to be composed of free atoms and the number of the Bogoliubov quasi-particles does not conserve. From physical point of view, however, the condensate in the realistic system should be composed of interacting atoms with zero-momentum. In order to consider such a condensate and to restore London's original idea that the λ-transition occurs when the condensate disappears, the new variables describing the interacting dressed atoms are introduced so as to conserve the total number of the dressed atoms, and we can obtain a definite expression for the phonon-phonon interactions, which make us calculate the higher-order corrections for physical quantities.

The second purpose of this paper is to clarify the relation between the present theory which belongs to the first group and the S.Y.K. theory belonging to the second group.

In § 2 the new operators $A_{\boldsymbol{p}}$ and $A_{\boldsymbol{p}}^*$ are introduced which satisfy the condition that the number of the dressed atoms is conserved. When the original system is described in terms of these new operators in the form of power series concerning $N^{-1/2}$, it will be shown that the ground-state energy and the excitation energy in the lowest-order coincide with those in the S.Y.K. theory, where N is the total number of the atoms. By virtue of the expansion concerning $N^{-1/2}$, the form of the interactions among the dressed atoms is definitely determined, and the corrections to the ground-state energy and the excitation energy due to the interactions are evaluated in § 3 up to order N^{-1}. It is shown that the phonon character of the excitation spectrum in the lowest-order is restored even in the higher-order correction by combining the contributions from various terms of the interactions. In § 4 the results for the ground-state energy and the excitation energy obtained in § 3 are modified to be found that they coincide precisely with those obtained in the S.Y.K. theory. The fact suggests that the present formalism has some relation with the S.Y.K. theory. After a unitary transformation, it has been proved that the present theory which belongs to the first group is essentially equivalent to the S.Y.K. theory in spite of the fact that two theories are developed on the quite different bases.

§ 2. Formulation

We consider a system of N interacting Bose particles of mass m enclosed in a cubic box of volume Ω. In a second quantized formalism, the Hamiltonian of the system is expressed by

$$H=\sum_{\boldsymbol{p}}\frac{\hbar^2\boldsymbol{p}^2}{2m}a_{\boldsymbol{p}}^*a_{\boldsymbol{p}}+\frac{1}{2\Omega}\sum_{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}}V(\boldsymbol{k})\,a_{\boldsymbol{p}+\boldsymbol{k}}^*a_{\boldsymbol{q}-\boldsymbol{k}}^*a_{\boldsymbol{q}}a_{\boldsymbol{p}}\,, \tag{2·1}$$

where $V(\boldsymbol{k})$ is the Fourier transform of the effective interparticle potential $\mathcal{V}(\boldsymbol{x})$, i.e.,

$$V(\boldsymbol{k}) = \int d^3x\,\mathcal{V}(\boldsymbol{x})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{2·2}$$

The operators $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^*$ denote the annihilation and creation operators which satisfy the commutation relations

$$[a_p, a_q^*] = \delta_{p,q}, \quad [a_p, a_q] = [a_p^*, a_q^*] = 0. \qquad (2\cdot3)$$

Let us now introduce a set of new operators $A_{\boldsymbol{p}}$ and $A_{\boldsymbol{p}}^*$ which are defined from the original set of operators $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^*$ by the following unitary transformation:

$$A_p = e^{-F(a,a^*)} a_p e^{F(a,a^*)} \quad \text{and} \quad A_p^* = e^{-F(a,a^*)} a_p^* e^{F(a,a^*)}, \qquad (2\cdot4)$$

where F is given by

$$F(a, a^*) = \sum_{p \neq 0} f(\boldsymbol{p}) (a_p^* a_{-p}^* \chi_0 \chi_0 - \chi_0^* \chi_0^* a_p a_{-p}). \qquad (2\cdot5)$$

In $(2\cdot5)$, the operators χ_0 and χ_0^* are expressed as

$$\chi_0 = (a_0^* a_0 + 1)^{-1/2} a_0 \quad \text{and} \quad \chi_0^* = a_0^* (a_0^* a_0 + 1)^{-1/2}, \qquad (2\cdot6)$$

respectively. The function $f(\boldsymbol{p})$ is a real function to be determined later. The unitary transformation $(2\cdot4)$ is a generalization of the well-known Bogoliubov transformation, in which the operators χ_0 and χ_0^* are replaced by a c-number. A similar transformation has already been utilized by Iwamoto and others.[3] However, the transformed operators used by them do not change the number of atoms and describe the collective modes of excitations in the system. On the other hand, the operators A_p and A_p^* in $(2\cdot4)$ change the number of atoms by one and thus they represent a dressed atom interacting with other atoms. Furthermore, the operators A_p and A_p^* are defined by $(2\cdot4)$ not only for the case $\boldsymbol{p} \neq 0$ but also for $\boldsymbol{p} = 0$ in contrast with the theories of Iwamoto and others. In these respects, the present formalism is different from their theories.

Owing to the unitary character of the transformation $(2\cdot4)$, one can readily see that

$$[A_p, A_q^*] = \delta_{p,q}, \quad [A_p, A_q] = [A_p^*, A_q^*] = 0, \qquad (2\cdot7)$$

for all $\boldsymbol{p}$ and $\boldsymbol{q}$. Since the operator F in $(2\cdot5)$ conserves the number of the atoms, the number operator $\tilde{N} = \sum_{\text{all } p} a_p^* a_p$ commutes with F, and we see that

$$\tilde{N} = \sum_{\text{all } p} a_p^* a_p = \sum_{\text{all } p} A_p^* A_p. \qquad (2\cdot8)$$

Therefore, the total number of the bare atoms is equal to that of the dressed atoms expressed by the operators A_p and A_p^*. This situation indicates the remarkable fact that it becomes possible for us to consider the Bose-Einstein condensation of the dressed atoms instead of condensation of bare atoms which has been assumed in the Bogoliubov theory. In the case of the theories which utilize the collective modes, one cannot consider the condensate since the number of the modes does not conserve.

In order to express the Hamiltonian $(2\cdot1)$ in terms of the new operators

A_p and A_p^*, we need to make use of the inverse transformation of (2·4). The inverse transformation can be expressed as follows:

$$a_p = e^{F(a,a^*)} A_p e^{-F(a,a^*)} = e^{F(A,A^*)} A_p e^{-F(A,A^*)}, \tag{2·9}$$

where

$$F(A, A^*) = \sum_{p\neq 0} f(p)\,(A_p^* A_{-p}^* \xi_0 \xi_0 - \xi_0^* \xi_0^* A_p A_{-p}), \tag{2·10}$$

$$\xi_0 = e^{-F(a,a^*)} \chi_0 e^{F(a,a^*)} = \frac{1}{\sqrt{A_0^* A_0 + 1}} A_0 . \tag{2·11}$$

Operating ξ_0 and ξ_0^* in (2·11) on the states defined by

$$A_0 | N_0\rangle = \sqrt{N_0}\,|N_0 - 1\rangle \quad \text{and} \quad A_0^* | N_0\rangle = \sqrt{N_0+1}\,|N_0+1\rangle ,$$

one has

$$\xi_0^* | N_0\rangle = |N_0 + 1\rangle ,$$
$$\xi_0 | N_0\rangle = |N_0 - 1\rangle \quad \text{for } N_0 \geqq 1 \quad \text{and} \quad \xi_0 | N_0 = 0\rangle = 0 . \tag{2·12}$$

From (2·12), one can readily see that

$$\xi_0 \xi_0^* = 1 , \tag{2·13a}$$

$$\xi_0^* \xi_0 = 1 - | N_0 = 0\rangle\langle N_0 = 0| , \tag{2·13b}$$

and has the commutation relation

$$[\xi_0, \xi_0^*] = | N_0 = 0\rangle\langle N_0 = 0| . \tag{2·13c}$$

As a good approximation in a many-boson system, we may replace (2·13) by

$$\xi_0 \xi_0^* = 1 , \quad \xi_0^* \xi_0 = 1$$

and

$$[\xi_0, \xi_0^*] = 0 , \tag{2·14}$$

because we can assume that the physically significant states in the many-boson system are always occupied by the zero-momentum physical particles described by the operators A_0 and A_0^*. The detailed discussion of this argument will be found in the Appendix.

With the aid of the assumption (2·14), the transformation (2·9) can be rewritten as

$$a_p = M_p(A, A^*) = \cosh(2f(p))\,A_p - \sinh(2f(p))\,A_{-p}^* \xi_0 \xi_0$$

and

$$a_p^* = M_p^*(A, A^*) = \cosh(2f(p))\,A_p^* - \sinh(2f(p))\,\xi_0^* \xi_0^* A_{-p} \tag{2·15}$$

for $p \neq 0$. For the case $p = 0$, we have

$$a_0 = e^{F(A,A^*)} A_0 e^{-F(A,A^*)} = e^{F(A,A^*)} \sqrt{A_0^* A_0 + 1}\xi_0 e^{-F(A,A^*)}$$
$$= e^{F(A,A^*)} \xi_0 \sqrt{A_0^* A_0} e^{-F(A,A^*)} = e^{F(A,A^*)} \xi_0 \sqrt{\tilde{N} - \sum_p{}' A_p^* A_p} e^{-F(A,A^*)}$$
$$= \xi_0 \sqrt{\tilde{N} - \sum_p{}' M_p^*(A, A^*) M_p(A, A^*)} \qquad (2\cdot16\text{a})$$

and

$$a_0^* = \sqrt{\tilde{N} - \sum_p{}' M_p^*(A, A^*) M_p(A, A^*)}\, \xi_0^*, \qquad (2\cdot16\text{b})$$

which give the expressions for the operators a_0 and a_0^* represented in terms of the variables A_p and A_p^*. In the last equality of (2·16a), we have used Eqs. (2·10) and (2·14). The prime in the sum over $\boldsymbol{p}$ implies that the terms with $\boldsymbol{p}=0$ should be excluded.

Since we have employed the approximate relations (2·14) in deriving (2·15) and (2·16), one needs to verify whether the operators a_p and a_p^* given by (2·15) and (2·16) still satisfy the commutation relations (2·3) or not, on assuming the commutator (2·7) for A_p and A_p^*. For the case $\boldsymbol{p}\neq 0$ and $\boldsymbol{q}\neq 0$, it can be easily seen that

$$[a_p, a_q^*] = (\cosh^2(2f(\boldsymbol{p})) - \sinh^2(2f(\boldsymbol{p}))\xi_0^2 \xi_0^{*2})\delta_{\boldsymbol{p},\boldsymbol{q}} = \delta_{\boldsymbol{p},\boldsymbol{q}}, \qquad (2\cdot17)$$

where the first equality is due to the relations (2·14) and the second equality owes to (2·13a). In the same way, we have

$$[a_p, a_q] = [a_p^*, a_q^*] = 0. \qquad (2\cdot18)$$

By virtue of the commutativity of M_p with ξ_0^*, the commutator $[a_p, a_0^*]$ for $\boldsymbol{p}\neq 0$ becomes

$$[a_p, a_0^*] = [M_p, \sqrt{\tilde{N} - \sum_q{}' M_q^* M_q}]\xi_0^* = 0. \qquad (2\cdot19)$$

In a quite similar way, we have

$$[a_p^*, a_0] = [a_p, a_0] = [a_p^*, a_0^*] = 0. \qquad (2\cdot20)$$

For the case of the commutator $[a_0, a_0^*]$, we have

$$[a_0, a_0^*] = \xi_0(\tilde{N} - \sum_p{}' M_p^* M_p)\xi_0^* - \sqrt{\tilde{N} - \sum_p{}' M_p^* M_p}\,\xi_0^* \xi_0 \sqrt{\tilde{N} - \sum_q{}' M_q^* M_q}$$
$$= \xi_0 \xi_0^* (\tilde{N} + 1 - \sum_p{}' M_p^* M_p) - (\tilde{N} - \sum_p{}' M_p^* M_p) = 1. \qquad (2\cdot21)$$

The result (2·21) comes from the relation $[\tilde{N}, \xi_0^*] = \xi_0^*$. If the operators ξ_0 and ξ_0^* on the right-hand side of (2·16) are replaced by a c-number, a_0 and a_0^* becomes commutable and the present theory reduces essentially to the original Bogoliubov formalism. From the results (2·17)~(2·21), we see that the commutation relations (2·3) are kept to be valid for the operators a_p and a_p^* expressed by the right-hand sides of (2·15) and (2·16).

In the original Bogoliubov theory, the operators a_0 and $a_0{}^*$ are replaced directly by a c-number $\sqrt{n_0}$, where n_0 is the number of zero-momentum bare atoms. This assumption gives the excitation energy

$$E_{\boldsymbol{p}}=\hbar^2 p(2m)^{-1}(\boldsymbol{p}^2+4mn_0V(\boldsymbol{p})/\hbar^2\Omega)^{1/2},$$

which leads to the vanishing phonon velocity at the λ-point. Since this result contradicts with experiments, the number of the condensate n_0 in $E_{\boldsymbol{p}}$ should be replaced by the total number of atoms N. This can be readily achieved by replacing the number operator $\tilde{N}$ in (2·16) by a c-number N and by expanding (2·16) in powers of N^{-1} in the following way:

$$a_0=\xi_0 N^{1/2}[1-(2N)^{-1}\sum_{\boldsymbol{p}}{}' M_{\boldsymbol{p}}{}^* M_{\boldsymbol{p}}+\cdots]$$

and

$$a_0{}^*=N^{1/2}[1-(2N)^{-1}\sum_{\boldsymbol{p}}{}' M_{\boldsymbol{p}}{}^* M_{\boldsymbol{p}}+\cdots]\xi_0{}^*. \tag{2·22}$$

With the aid of (2·15) and (2·22), we can rewrite the Hamiltonian (2·1) in terms of the operators $A_{\boldsymbol{p}}$ and $A_{\boldsymbol{p}}{}^*$ as a power-series expansion concerning $N^{-1/2}$. Firstly, we extract out the zero-momentum operators a_0 and $a_0{}^*$ from the sum in (2·1) and have

$$H=U_0+H_1+H_2+H_3,$$

$$U_0=\rho V(0)(N-1)/2,$$

$$H_1=\sum_{\boldsymbol{p}}{}'\frac{\hbar^2\boldsymbol{p}^2}{2m}a_{\boldsymbol{p}}{}^*a_{\boldsymbol{p}}+\frac{\rho}{2N}\sum_{\boldsymbol{p}}{}' V(\boldsymbol{p})(a_{\boldsymbol{p}}{}^*a_{-\boldsymbol{p}}^*a_0{}^2+a_0{}^{*2}a_{\boldsymbol{p}}a_{-\boldsymbol{p}}+2a_{\boldsymbol{p}}{}^*a_{\boldsymbol{p}}a_0{}^*a_0),$$

$$H_2=\frac{\rho}{N}\sum_{\substack{\boldsymbol{p},\boldsymbol{k}\\ \boldsymbol{p}+\boldsymbol{k}\neq 0}}{}' V(\boldsymbol{k})(a_{\boldsymbol{p}}{}^*a_{\boldsymbol{k}}{}^*a_{\boldsymbol{p}+\boldsymbol{k}}a_0+a_0{}^*a_{\boldsymbol{p}+\boldsymbol{k}}^*a_{\boldsymbol{p}}a_{\boldsymbol{k}})$$

and

$$H_3=\frac{\rho}{2N}\sum_{\substack{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}\\ \boldsymbol{p}+\boldsymbol{k}\neq 0\\ \boldsymbol{q}-\boldsymbol{k}\neq 0}}{}' V(\boldsymbol{k})a_{\boldsymbol{p}+\boldsymbol{k}}^*a_{\boldsymbol{q}-\boldsymbol{k}}^*a_{\boldsymbol{q}}a_{\boldsymbol{p}}, \tag{2·23}$$

where $\rho=N/\Omega$. Inserting (2·15) and (2·22) into (2·23) and choosing the function $f(\boldsymbol{p})$ as

$$\cosh(2f(\boldsymbol{p}))=(4\lambda_{\boldsymbol{p}})^{-1/2}(1+\lambda_{\boldsymbol{p}}),\qquad \sinh(2f(\boldsymbol{p}))=(4\lambda_{\boldsymbol{p}})^{-1/2}(1-\lambda_{\boldsymbol{p}})$$

and

$$\lambda_{\boldsymbol{p}}=p(\boldsymbol{p}^2+4mNV(\boldsymbol{p})/\hbar^2\Omega)^{-1/2}, \tag{2·24}$$

we can obtain the Hamiltonian in terms of the operators $A_{\boldsymbol{p}}$ and $A_{\boldsymbol{p}}{}^*$ as the power-series concerning $N^{-1/2}$ in the following form:

$$H=H_0{}^{(A)}+H_{\rm I}{}^{(A)}+H_{\rm II}{}^{(A)}+\cdots, \tag{2·25}$$

$$H_0{}^{(A)}=E_G{}^B+\sum_{\boldsymbol{p}}{}' E_{\boldsymbol{p}}{}^B A_{\boldsymbol{p}}{}^* A_{\boldsymbol{p}}\,, \tag{2·26}$$

$$H_{\rm I}{}^{(A)}=\frac{1}{\sqrt{N}}\sum_{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}}{}' \delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0} R_{\rm I}{}^{(2,1)}(\boldsymbol{p},\boldsymbol{k};-\boldsymbol{q})\,(A_{\boldsymbol{p}}{}^*A_{\boldsymbol{k}}{}^*A_{-\boldsymbol{q}}\xi_0+\xi_0{}^*A_{-\boldsymbol{q}}^*A_{\boldsymbol{k}}A_{\boldsymbol{p}})$$
$$+\frac{1}{\sqrt{N}}\sum_{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}}{}' \delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0} R_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})\,(A_{\boldsymbol{p}}{}^*A_{\boldsymbol{k}}{}^*A_{\boldsymbol{q}}{}^*\xi_0{}^3+\xi_0{}^{*3}A_{\boldsymbol{q}}A_{\boldsymbol{k}}A_{\boldsymbol{p}}) \tag{2·27}$$

and

$$H_{\rm II}{}^{(A)}=\frac{1}{N}R_{\rm II}{}^{(0,0)}+\frac{1}{N}\sum_{\boldsymbol{p}}{}' R_{\rm II}{}^{(1,1)}(\boldsymbol{p})\,A_{\boldsymbol{p}}{}^*A_{\boldsymbol{p}}$$
$$+\frac{1}{N}\sum_{\boldsymbol{p}}{}' R_{\rm II}{}^{(2,0)}(\boldsymbol{p})\,(A_{\boldsymbol{p}}{}^*A_{-\boldsymbol{p}}^*\xi_0{}^2+\xi_0{}^{*2}A_{\boldsymbol{p}}A_{-\boldsymbol{p}})$$
$$+\frac{1}{N}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s},0} R_{\rm II}{}^{(2,2)}(\boldsymbol{p},\boldsymbol{q};-\boldsymbol{r},-\boldsymbol{s})\,A_{\boldsymbol{p}}{}^*A_{\boldsymbol{q}}{}^*A_{-\boldsymbol{r}}A_{-\boldsymbol{s}}$$
$$+\frac{1}{N}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s},0} R_{\rm II}{}^{(3,1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};-\boldsymbol{s})$$
$$\times(A_{\boldsymbol{p}}{}^*A_{\boldsymbol{q}}{}^*A_{\boldsymbol{r}}{}^*A_{-\boldsymbol{s}}\xi_0{}^2+\xi_0{}^{*2}A_{-\boldsymbol{s}}^*A_{\boldsymbol{r}}A_{\boldsymbol{q}}A_{\boldsymbol{p}})$$
$$+\frac{1}{N}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s},0} R_{\rm II}{}^{(4,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s})$$
$$\times(A_{\boldsymbol{p}}{}^*A_{\boldsymbol{q}}{}^*A_{\boldsymbol{r}}{}^*A_{\boldsymbol{s}}{}^*\xi_0{}^4+\xi_0{}^{*4}A_{\boldsymbol{s}}A_{\boldsymbol{r}}A_{\boldsymbol{q}}A_{\boldsymbol{p}})\,. \tag{2·28}$$

It should be noticed here that the lowest-order term $H_0{}^{(A)}$ has been diagonalized by virtue of the choice (2·24) of the function $f(\boldsymbol{p})$. The ground-state energy $E_G{}^B$ and the excitation energy $E_{\boldsymbol{p}}{}^B$ in the lowest approximation are given by

$$E_G{}^B=N\left(\frac{\rho}{2}V(0)-\frac{1}{N}\sum_{\boldsymbol{k}}{}'\frac{\hbar^2\boldsymbol{k}^2}{8m\lambda_{\boldsymbol{k}}{}^2}(\lambda_{\boldsymbol{k}}-1)^2\right)$$

and

$$E_{\boldsymbol{p}}{}^B=\hbar^2\boldsymbol{p}^2/2m\lambda_{\boldsymbol{p}}\,, \tag{2·29}$$

respectively. One note here that the number of the condensate n_0 is replaced by the total number N in the excitation energy $E_{\boldsymbol{p}}{}^B$ of (2·29). The vertex-functions in (2·27) and (2·28) are defined by

$$R_{\rm I}{}^{(2,1)}(\boldsymbol{p},\boldsymbol{k};-\boldsymbol{q})=\frac{\hbar^2}{16m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}}}\left[\frac{\boldsymbol{p}^2}{\lambda_{\boldsymbol{p}}}(1-\lambda_{\boldsymbol{p}}{}^2)(\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}+1)\right.$$
$$\left.+\frac{\boldsymbol{k}^2}{\lambda_{\boldsymbol{k}}}(1-\lambda_{\boldsymbol{k}}{}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}+1)+\frac{\boldsymbol{q}^2}{\lambda_{\boldsymbol{q}}}(1-\lambda_{\boldsymbol{q}}{}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}-1)\right],$$

$$R_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})=\frac{\hbar^2}{48m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}}}\left[\frac{\boldsymbol{p}^2}{\lambda_{\boldsymbol{p}}}(1-\lambda_{\boldsymbol{p}}{}^2)(\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}-1)\right.$$

$$+\frac{\boldsymbol{k}^2}{\lambda_{\boldsymbol{k}}}(1-\lambda_{\boldsymbol{k}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}-1)+\frac{\boldsymbol{q}^2}{\lambda_{\boldsymbol{q}}}(1-\lambda_{\boldsymbol{q}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}-1)\Big],$$

$$R_{\text{II}}{}^{(0,0)}=\frac{\hbar^2}{64m}\sum_{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{\boldsymbol{k}^2(1-\lambda_{\boldsymbol{k}}^2)}{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}{}^2\lambda_{\boldsymbol{q}}}[(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}+1)(\lambda_{\boldsymbol{p}}-1)(\lambda_{\boldsymbol{q}}-1)$$
$$-2\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{p}}-1)^2(\lambda_{\boldsymbol{k}}-1)],$$

$$R_{\text{II}}{}^{(1,1)}(\boldsymbol{p})=-\frac{\hbar^2}{16m}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{1}{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}}\Big[\boldsymbol{p}^2(1-\lambda_{\boldsymbol{p}}^2)\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{k}}-1)^2$$
$$+\frac{\boldsymbol{k}^2}{\lambda_{\boldsymbol{k}}}(1-\lambda_{\boldsymbol{k}}^2)\{\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{p}}{}^2+1)(\lambda_{\boldsymbol{k}}-1)-(\lambda_{\boldsymbol{q}}-1)(\lambda_{\boldsymbol{p}}{}^2\lambda_{\boldsymbol{q}}-1)\}\Big],$$

$$R_{\text{II}}{}^{(2,0)}(\boldsymbol{p})=-\frac{\hbar^2}{32m}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{1}{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}}\Big[\boldsymbol{p}^2(1-\lambda_{\boldsymbol{p}}^2)\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{k}}-1)^2$$
$$+\frac{\boldsymbol{k}^2}{\lambda_{\boldsymbol{k}}}(1-\lambda_{\boldsymbol{k}}^2)\{\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{p}}{}^2-1)(\lambda_{\boldsymbol{k}}-1)-(\lambda_{\boldsymbol{q}}-1)(\lambda_{\boldsymbol{p}}{}^2\lambda_{\boldsymbol{q}}+1)\}\Big],$$

$$R_{\text{II}}{}^{(2,2)}(\boldsymbol{p},\boldsymbol{q};-\boldsymbol{r},-\boldsymbol{s})$$
$$=\frac{\hbar^2}{64\times 2m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}}\Big[\Big\{\frac{(\boldsymbol{p}+\boldsymbol{q})^2}{\lambda_{\boldsymbol{p}+\boldsymbol{q}}^2}(1-\lambda_{\boldsymbol{p}+\boldsymbol{q}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}-1)(\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}-1)$$
$$+\frac{(\boldsymbol{p}+\boldsymbol{r})^2}{\lambda_{\boldsymbol{p}+\boldsymbol{r}}^2}(1-\lambda_{\boldsymbol{p}+\boldsymbol{r}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{r}}+1)(\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{s}}+1)$$
$$+\frac{(\boldsymbol{p}+\boldsymbol{s})^2}{\lambda_{\boldsymbol{p}+\boldsymbol{s}}^2}(1-\lambda_{\boldsymbol{p}+\boldsymbol{s}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{s}}+1)(\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}+1)\Big\}$$
$$+\{\text{the terms obtained by exchanging variables as } (\boldsymbol{p}\leftrightarrow\boldsymbol{q}) \text{ in the previous three terms}\}\Big],$$

$$R_{\text{II}}{}^{(3,1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};-\boldsymbol{s})$$
$$=\frac{\hbar^2}{32\times 3m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}}\Big[\frac{(\boldsymbol{p}+\boldsymbol{s})^2}{\lambda_{\boldsymbol{p}+\boldsymbol{s}}^2}(1-\lambda_{\boldsymbol{p}+\boldsymbol{s}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{s}}+1)(\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}-1)$$
$$+(\text{two terms obtained by exchanging variables as } (\boldsymbol{p}\leftrightarrow\boldsymbol{q}) \text{ and } (\boldsymbol{p}\leftrightarrow\boldsymbol{r}))\Big]$$

and

$$R_{\text{II}}{}^{(4,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s})$$
$$=\frac{\hbar^2}{128\times 6m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}}\Big[\frac{(\boldsymbol{p}+\boldsymbol{q})^2}{\lambda_{\boldsymbol{p}+\boldsymbol{q}}^2}(1-\lambda_{\boldsymbol{p}+\boldsymbol{q}}^2)(\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}-1)(\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}-1)$$
$$+\Big(\text{five terms obtained by exchanging variables as } (\boldsymbol{p}\leftrightarrow\boldsymbol{r}),\ (\boldsymbol{p}\leftrightarrow\boldsymbol{s}),\ (\boldsymbol{q}\leftrightarrow\boldsymbol{r}),\ (\boldsymbol{q}\leftrightarrow\boldsymbol{s}) \text{ and } \begin{pmatrix}\boldsymbol{p}\leftrightarrow\boldsymbol{r}\\ \boldsymbol{q}\leftrightarrow\boldsymbol{s}\end{pmatrix}\Big)\Big], \qquad (2\cdot 30)$$

where the vertex-functions have been symmetrized with each momentum variable such as

$$R_{\rm I}^{(2,1)}(\boldsymbol{p},\boldsymbol{k};-\boldsymbol{q})=R_{\rm I}^{(2,1)}(\boldsymbol{k},\boldsymbol{p};-\boldsymbol{q}),$$
$$R_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})=R_{\rm I}^{(3,0)}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{p})=R_{\rm I}^{(3,0)}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{k}),\quad \text{etc.}$$

In (2·28), we have dropped the terms which are higher in the power of N^{-1} compared with those written in (2·28). As an example, the constant term on the right-hand side of (2·28) is given by direct evaluation as

$$R_{\rm II}^{(0,0)}+\frac{\hbar^2}{32m}\sum_{\boldsymbol{k}}{}'\frac{\boldsymbol{k}^2}{\lambda_{\boldsymbol{k}}^3}(\lambda_{\boldsymbol{k}}^2-1)\{(\lambda_{\boldsymbol{k}}^2-1)(2\lambda_{\boldsymbol{k}}-1)+(\lambda_{\boldsymbol{k}}-1)^3\}. \qquad (2\cdot31)$$

As will be seen by replacing summations in (2·31) with integrals, the extra-term in (2·31) is higher-order in N^{-1} than the first term and thus it can be dropped. Furthermore, one should note here that the interaction Hamiltonians in (2·25) conserve the number of dressed atoms in contrast with the theory described by collective modes.

The vertex-functions in (2·30) seems at a glance to be singular at low momentum limits. For example, the integrand in $R_{\rm II}^{(0,0)}$ is singular at $\boldsymbol{p}=0$ or $\boldsymbol{q}=0$. The function $R_{\rm I}^{(2,1)}(\boldsymbol{p},\boldsymbol{k};-\boldsymbol{q})$ is also divergent in the limit $\boldsymbol{q}=0$. Although the excitation energy $E_{\boldsymbol{p}}^B$ in the lowest-order (2·29) assures the existence of the phonon, the higher-order effects from the interactions between the Bogoliubov phonons have a possibility to distroy the phonon character of the excitations in the system on account of the singular properties of the vertex-functions. By taking correctly account of all contributions in the same order in N^{-1}, however, it will be shown in § 3 that the singularities disappear to have the phonon excitation in order N^{-1}.

Before closing this section, we should give a comment concerning the method to determine the function $f(\boldsymbol{p})$. In the theories of Giradeau-Arnowitt, Takano and Iwamoto,[3] the function corresponding to $f(\boldsymbol{p})$ has been decided so as to diagonalize all terms which have the forms like $A_{\boldsymbol{p}}{}^*A_{\boldsymbol{p}}$, $A_{\boldsymbol{p}}{}^*A^*_{-\boldsymbol{p}}$ and $A_{\boldsymbol{p}}A_{-\boldsymbol{p}}$ irrespective of order $N^{-1/2}$, and thus they have obtained the excitation spectrum with an energy gap at zero momenta in their lowest approximation. In the present theory, on the other hand, the Hamiltonian (2·25) is expressed in the power-series concerning $N^{-1/2}$, and only the terms in the lowest-order has been diagonalized. Because of this prescription, the energy gap has not been produced in the lowest-order excitation spectrum.

§ 3. Second-order corrections in the perturbation theory

In this section, we calculate the ground-state energy and the excitation energy*)

*) It is a good approximation only for the condition $N_0/N\sim1$ that we calculate the physical quantities by the perturbational method up to order N^{-1}, although the Hamiltonian (2·25) given by the sum of the terms of all orders concerning $N^{-1/2}$ is correct even for the case $N_0/N\sim0$ according to the discussion in the Appendix.

up to order N^{-1}, and it is shown in the next section that the results coincide with those obtained by the S.Y.K. theory which makes use of the velocity operator and the density-fluctuation operator as the collective variables.

The ground-state for the Hamiltonian (2·25) composed of N dressed atoms is given in the lowest-order by

$$|G\rangle = \frac{1}{\sqrt{N!}}(A_0{}^*)^N|0\rangle\,, \tag{3·1}$$

where the state $|0\rangle$ is defined by

$$A_{\boldsymbol{p}}|0\rangle = 0 \qquad \text{for all } \boldsymbol{p}\,.$$

The one phonon state in the lowest approximation is represented as

$$|\boldsymbol{p}\rangle = A_{\boldsymbol{p}}{}^*\xi_0|G\rangle\,. \tag{3·2}$$

One should note here that the definition of the state $|\boldsymbol{p}\rangle$ is slightly different from the traditional definition of the one particle state which is expressed as $|\boldsymbol{p}\rangle = A_{\boldsymbol{p}}{}^*|G\rangle$, and that the number of the dressed atoms is conserved by virtue of the existence of the operator ξ_0 in (3·2).

(i) *Ground-state energy*

On the basis of (2·25) and (3·1), the ground-state energy up to order N^{-1} can readily be obtained from the perturbation theory in the following form:

$$E_G = E_G{}^B + \Delta E_G \tag{3·3}$$

and

$$\Delta E_G = \frac{1}{N}R_{\mathrm{II}}{}^{(0,0)} - \frac{3!}{N}\sum_{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}}{}' \delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{|R_{\mathrm{I}}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_{\boldsymbol{p}}{}^B + E_{\boldsymbol{k}}{}^B + E_{\boldsymbol{q}}{}^B}\,. \tag{3·4}$$

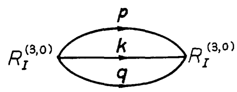

Fig. 1. The diagrammatic representation of the contribution to the ground-state energy in order N^{-1}.

As will be easily seen, the integral in $R_{\mathrm{II}}{}^{(0,0)}$ is absolutely convergent, although the integrand in $R_{\mathrm{II}}{}^{(0,0)}$ is singular at $\boldsymbol{p}=0$ (or $\boldsymbol{q}=0$). The vertex-function $R_{\mathrm{I}}{}^{(3,0)}$ itself is singular at zero momenta, but the integrals in the second term on the right-hand side of (3·4) are also absolutely convergent for small momenta owing to the existence of the phase factors in the integrals. For large momenta, it is also absolutely convergent if we assume a soft-repulsive core as the interatomic potential $\mathcal{V}(\boldsymbol{x})$.

(ii) *Excitation energy*

With the aid of (2·25) and (3·2), we calculate the excitation energy up to order N^{-1} to yield the result

$$E(\boldsymbol{p})=E_{\boldsymbol{p}}{}^{B}+\varDelta E_{\boldsymbol{p}}\,, \tag{3·5}$$

where

$$\varDelta E_{\boldsymbol{p}}=\frac{1}{N}R_{\mathrm{II}}{}^{(1,1)}(\boldsymbol{p})-\frac{2}{N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{|R_{\mathrm{I}}{}^{(2,1)}(-\boldsymbol{k},-\boldsymbol{q};\boldsymbol{p})|^2}{E_{\boldsymbol{k}}{}^{B}+E_{\boldsymbol{q}}{}^{B}-E_{\boldsymbol{p}}{}^{B}}$$
$$-\frac{18}{N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{|R_{\mathrm{I}}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_{\boldsymbol{p}}{}^{B}+E_{\boldsymbol{k}}{}^{B}+E_{\boldsymbol{q}}{}^{B}}\,. \tag{3·6}$$

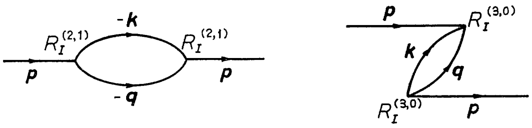

Fig. 2. The diagrammatic representation of the excitation energy in order N^{-1}.

In a similar way to the case of (3·4), the integrals in (3·6) are absolutely convergent for small and large $\boldsymbol{k}$ (and $\boldsymbol{q}$), when we assume the soft-repulsive core as the interatomic potential. In the limit $\boldsymbol{p}\to 0$, all integrals on the right-hand side of (3·6) can be expanded as

$$\frac{1}{N}R_{\mathrm{II}}{}^{(1,1)}(\boldsymbol{p})=\frac{cK_1}{p}+L_1+\text{(order of }p\text{)},$$
$$\frac{2}{N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{|R_{\mathrm{I}}{}^{(2,1)}(-\boldsymbol{k},-\boldsymbol{q};\boldsymbol{p})|^2}{E_{\boldsymbol{k}}{}^{B}+E_{\boldsymbol{q}}{}^{B}-E_{\boldsymbol{p}}{}^{B}}=\frac{cK_2}{p}+L_2+\text{(order of }p\text{)},$$
$$\frac{18}{N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\frac{|R_{\mathrm{I}}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_{\boldsymbol{p}}{}^{B}+E_{\boldsymbol{k}}{}^{B}+E_{\boldsymbol{q}}{}^{B}}=\frac{cK_3}{p}+L_3+\text{(order of }p\text{)}, \tag{3·7}$$

where $c=\sqrt{4mNV(0)/\hbar^2\Omega}$. Thus the phonon character of the excitation energy seems at a glance not to be assured. However, the detailed investigation of (3·7) shows that constants K_1, K_2 and K_3 are given by

$$K_1=2K_2=2K_3=\frac{\hbar^2}{16mN}\sum_{\boldsymbol{k}}{}'\frac{\boldsymbol{k}^2(1-\lambda_{\boldsymbol{k}}{}^2)^2}{\lambda_{\boldsymbol{k}}{}^3}\,, \tag{3·8}$$

and thus we see that $K_1-K_2-K_3=0$. The constants L_1, L_2 and L_3 in (3·7) are found to be $L_1=L_2=L_3=0$. Namely, this situation shows that the combination of the contributions from the matrix-elements in order N^{-1} gives a vanishing value for the coefficients which are propotional to p^{-1}. Although the contribution from each vertex-function does not give the phonon character of the excitation spectrum, the combination of the matrix-elements in order N^{-1} yields the phonon spectrum.

By employing a variational method based on the pair theory, Giradeau and Arnowitt[3] have shown that there appears an energy gap at $\boldsymbol{p}=0$ in the excitation spectrum. However, non-existence of the energy gap in the many boson system has been proved by Hugenholtz and Pines[8] in a general way. In consideration of the corrections from residual interactions which have been neglected in the pair theory, Takano[3] has evaluated the corrections to have shown that the energy gap really disappears. In the present theory, on the other hand, the excitation energy $E_p{}^B$ in the lowest-order approximation has no energy gap in contrast with the Giradeau and Arnowitt theory and moreover the corrections due to the residual interaction (2·25) also show the phonon behavior in a correct way.

§ 4. Relation between the S.Y.K. theory and the present formalism

As mentioned in the Introduction, one of the main purpose of the present paper is to clarify the relation between the Bogoliubov-type theories and the theories which make use of the density-fluctuation as the collective variable.

In this section, we clarify the relation between the present theory which belongs to the line of the Bogoliubov formalism and the S.Y.K. theory which utilizes the density-fluctuation operator and the velocity operator. In the recent papers in the S.Y.K. theory,[7] the system is described by the operators b_p and $b_p{}^*$ and various physical quantities are calculated up to order N^{-1}. Their results are absolutely convergent and the phonon character of the excitation energy is ensured. On the other hand, the results in the present theory exhibited in the last section are also absolutely convergent and the phonon character of the excitation energy is assumed. Therefore it will be natural for us to consider that there may exist some relation between two theories.

In order to obtain a clue to find the relation, we rewrite the ground-state energy (3·3) in the following way. The vertex-function $R_{\rm I}{}^{(3,0)}$ in (3·4) can be rewritten as

$$R_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})=(E_{\boldsymbol{p}}{}^B+E_{\boldsymbol{k}}{}^B+E_{\boldsymbol{q}}{}^B)\,g_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})+\frac{1}{3!}F_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}),\tag{4·1}$$

where

$$g_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})=\frac{1}{24\sqrt{\lambda_p\lambda_k\lambda_q}}(-1+\lambda_p\lambda_k+\lambda_k\lambda_q+\lambda_p\lambda_q-2\lambda_p\lambda_k\lambda_q)\tag{4·2}$$

and

$$F_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})=\frac{\hbar^2}{4m\sqrt{\lambda_p\lambda_k\lambda_q}}[(\boldsymbol{p}\cdot\boldsymbol{k})\lambda_q(\lambda_p-1)((\lambda_k-1)$$
$$+(\boldsymbol{k}\cdot\boldsymbol{q})\lambda_p(\lambda_k-1)(\lambda_q-1)+(\boldsymbol{p}\cdot\boldsymbol{q})\lambda_k(\lambda_p-1)(\lambda_q-1)].\tag{4·3}$$

The vertex-function $F_{\rm I}{}^{(3,0)}$ in (4·3) is that obtained in the S.Y.K. theory. Introduction of (4·1) into (3·4) gives

$$\Delta E_G=\frac{1}{N}R_{\rm II}{}^{(0,0)}-\frac{12}{N}\sum_{\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}T_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})\,g_{\rm I}{}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})$$

$$-\frac{1}{3!N}\sum_{p,k,q}{}'\delta_{p+k+q,0}\frac{|F_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_p{}^B+E_k{}^B+E_q{}^B}, \tag{4·4}$$

where

$$\begin{aligned}T_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}) &=\frac{1}{3!}F_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})+\frac{1}{2}(E_p{}^B+E_k{}^B+E_q{}^B)g_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})\\ &=R^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})-\frac{1}{2}(E_p{}^B+E_k{}^B+E_q{}^B)g_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})\\ &=\frac{\hbar^2}{96m\sqrt{\lambda_p\lambda_k\lambda_q}}\Big[\frac{\boldsymbol{p}^2}{\lambda_p}(-1+\lambda_q\lambda_k-\lambda_p\lambda_k-\lambda_p\lambda_q\\ &\qquad+2\lambda_p{}^2+2\lambda_p\lambda_k\lambda_q-2\lambda_p{}^2\lambda_q\lambda_k)\\ &\qquad+(\text{two terms obtained by exchanging variables as}\\ &\qquad\qquad(\boldsymbol{p}\leftrightarrow\boldsymbol{q})\text{ and }(\boldsymbol{p}\leftrightarrow\boldsymbol{k})\Big].\end{aligned} \tag{4·5}$$

After rearrangement of (4·4), we can readily see that

$$\begin{aligned}\Delta E_G=\frac{1}{N}F_{\rm II}^{(0,0)}-\frac{1}{3!N}\sum_{p,k,q}{}'\delta_{p+k+q,0}\frac{|F_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_p{}^B+E_k{}^B+E_q{}^B}\\ +\frac{\hbar^2}{32mN}\sum_{p,k,q}{}'(\boldsymbol{p}\cdot\boldsymbol{k})\frac{(\lambda_p-1)^2(\lambda_k-1)^2}{\lambda_p\lambda_k},\end{aligned} \tag{4·6}$$

where $F_{\rm II}^{(0,0)}$ is a function derived in the S.Y.K. theory and is given by

$$F_{\rm II}^{(0,0)}=\frac{\hbar^2}{16mN}\sum_{p,k,q}{}'\delta_{p+k+q,0}\boldsymbol{k}^2(1-\lambda_p)(1-\lambda_k)(1-\lambda_q). \tag{4·7}$$

The first and second terms on the right-hand side of (4·6) coincide precisely with the result in the S.Y.K. theory. The extra-term in (4·6) is absolutely convergent and vanishes on account of the odd property of the integrand.

Next, we consider the excitation energy. The vertex-function $R_{\rm I}^{(2,1)}$ in (3·6) is transformed as

$$\begin{aligned}&R_{\rm I}^{(2,1)}(-\boldsymbol{k},-\boldsymbol{q};\boldsymbol{p})\\ &=(E_k{}^B+E_q{}^B-E_p{}^B)g_{\rm I}^{(2,1)}(-\boldsymbol{k},-\boldsymbol{q};\boldsymbol{p})+\frac{1}{2!}F_{\rm I}^{(2,1)}(-\boldsymbol{k},-\boldsymbol{q};\boldsymbol{p}),\end{aligned} \tag{4·8}$$

where

$$\begin{aligned}&g_{\rm I}^{(2,1)}(-\boldsymbol{k},-\boldsymbol{q};\boldsymbol{p})\\ &=\frac{1}{8\sqrt{\lambda_p\lambda_k\lambda_q}}(1+\lambda_p\lambda_k+\lambda_p\lambda_q-\lambda_k\lambda_q-2\lambda_p\lambda_k\lambda_q)\end{aligned} \tag{4·9}$$

and

$$F_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})=\frac{\hbar^2}{4m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}}}[(\boldsymbol{k}\cdot\boldsymbol{q})\lambda_{\boldsymbol{p}}(\lambda_{\boldsymbol{k}}-1)(\lambda_{\boldsymbol{q}}-1)$$
$$+(\boldsymbol{q}\cdot\boldsymbol{p})\lambda_{\boldsymbol{k}}(\lambda_{\boldsymbol{q}}-1)(\lambda_{\boldsymbol{p}}+1)+(\boldsymbol{k}\cdot\boldsymbol{p})\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{k}}-1)(\lambda_{\boldsymbol{p}}+1)]. \quad (4\cdot10)$$

The vertex-function $F_{\mathrm{I}}^{(2,1)}$ is that used in the S.Y.K. theory. Introducing (4·1) and (4·8) into (3·6), we have

$$\Delta E_{\boldsymbol{p}}=\frac{1}{N}R_{\mathrm{II}}^{(1,1)}(\boldsymbol{p})-\frac{1}{N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}(4T_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})$$
$$+36T_{\mathrm{I}}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})g_{\mathrm{I}}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q}))$$
$$-\frac{1}{2N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\left\{\frac{|F_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})|^2}{E_{\boldsymbol{k}}^B+E_{\boldsymbol{q}}^B-E_{\boldsymbol{p}}^B}+\frac{|F_{\mathrm{I}}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_{\boldsymbol{p}}^B+E_{\boldsymbol{k}}^B+E_{\boldsymbol{q}}^B}\right\}, \quad (4\cdot11)$$

where

$$T_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})=\frac{1}{2!}F_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})$$
$$+\frac{1}{2}(E_{\boldsymbol{k}}^B+E_{\boldsymbol{q}}^B-E_{\boldsymbol{p}}^B)g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})$$
$$=R_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})-\frac{1}{2}(E_{\boldsymbol{k}}^B+E_{\boldsymbol{q}}^B-E_{\boldsymbol{p}}^B)g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})$$
$$=\frac{\hbar^2}{32m\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}}}\Big[\frac{\boldsymbol{k}^2}{\lambda_{\boldsymbol{k}}}(1+\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}-\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{p}}+\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}-2\lambda_{\boldsymbol{k}}^2+2\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}-2\lambda_{\boldsymbol{k}}^2\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}})$$
$$+\frac{\boldsymbol{q}^2}{\lambda_{\boldsymbol{q}}}(1+\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}+\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{p}}-\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}-2\lambda_{\boldsymbol{q}}^2+2\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}-2\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}^2\lambda_{\boldsymbol{p}})$$
$$+\frac{\boldsymbol{p}^2}{\lambda_{\boldsymbol{p}}}(-1+\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}+\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{p}}+\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}+2\lambda_{\boldsymbol{p}}^2-2\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}-2\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{p}}^2)\Big]. \quad (4\cdot12)$$

After the straightforward evaluation of the first two terms on the right-hand side of (4·11), one obtains

$$\Delta E_{\boldsymbol{p}}=\frac{1}{N}F_{\mathrm{II}}^{(1,1)}(\boldsymbol{p})-\frac{1}{2N}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}\left\{\frac{|F_{\mathrm{I}}^{(2,1)}(-\boldsymbol{k},\ -\boldsymbol{q};\boldsymbol{p})|^2}{E_{\boldsymbol{k}}^B+E_{\boldsymbol{q}}^B-E_{\boldsymbol{p}}^B}+\frac{|F_{\mathrm{I}}^{(3,0)}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{q})|^2}{E_{\boldsymbol{p}}^B+E_{\boldsymbol{k}}^B+E_{\boldsymbol{q}}^B}\right\}$$
$$-\frac{\hbar^2}{8mN}\frac{(\lambda_{\boldsymbol{p}}^2+1)}{\lambda_{\boldsymbol{p}}}\sum_{\boldsymbol{q}}{}'(\boldsymbol{p}\cdot\boldsymbol{q})\frac{(\lambda_{\boldsymbol{q}}-1)^2}{\lambda_{\boldsymbol{q}}}, \quad (4\cdot13)$$

where

$$F_{\mathrm{II}}^{(1,1)}(\boldsymbol{p})=-\frac{\hbar^2}{8m}\lambda_{\boldsymbol{p}}\sum_{\boldsymbol{k},\boldsymbol{q}}{}'\delta_{\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q},0}(\boldsymbol{p}^2+\boldsymbol{k}^2+\boldsymbol{q}^2)(\lambda_{\boldsymbol{k}}-1)(\lambda_{\boldsymbol{q}}-1). \quad (4\cdot14)$$

The first and second terms in (4·13) coincide with the formula derived in the S.Y.K. theory. The remaining term in (4·13) is absolutely convergent and vanishes owing to the odd property of the integrand. As has already been shown in the S.Y.K. theory, the excitation energy $E_p^B+\Delta E_p$ in (4·13) shows a phonon-like behavior at small $\boldsymbol{p}$ and the result (4·13) is consistent with the argument in (3·7)～(3·9).

The results (4·6) and (4·13) suggest us the existence of a unitary transformation between the operators A_p and A_p^* used in the present theory and those b_p and b_p^* in the S.Y.K. theory. However, the operators A_p and A_p^* change the number of atoms by one and the operators b_p and b_p^* do not change the number of atoms since b_p and b_p^* describe the collective mode in the system. By taking this situation into consideration, we introduce a unitary transformation defined by

$$\alpha_p = e^{-G(A,A^*)}A_p e^{G(A,A^*)} \quad \text{and} \quad \alpha_p^* = e^{-G(A,A^*)}A_p^* e^{G(A,A^*)} \tag{4·15}$$

for all $\boldsymbol{p}$.

The form of the operator $G(A, A^*)$ should be determined to obtain the vertex-functions in the S.Y.K. theory, in which the singularities at small momenta involved in the vertex-functions in (2·30) are eliminated. This program can be performed, for example, by determining the operator G in order $N^{-1/2}$ so as to cancel out the first terms on the right-hand sides of (4·1) and (4·8). In this way, the function $G(A, A^*)$ is expressed as

$$\begin{aligned} G(A,A^*) = {} & \frac{1}{\sqrt{N}}\sum_{r,s,t}{}'\delta_{r+s+t,0}[g_{\mathrm{I}}^{(3,0)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{t})(A_r^*A_s^*A_t^*\xi_0^3-\xi_0^{*3}A_tA_sA_r) \\ & +g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r},\boldsymbol{s};-\boldsymbol{t})(A_r^*A_s^*A_{-t}\xi_0-\xi_0^*A_{-t}^*A_sA_r)] \\ & +\frac{1}{N}\sum_{r}{}' g_{\mathrm{II}}^{(2,0)}(\boldsymbol{r})(A_r^*A_{-r}^*\xi_0^2-\xi_0^{*2}A_{-r}A_r) \\ & +\frac{1}{N}\sum_{r,s,t,u}{}'\delta_{r+s+t+u,0}[g_{\mathrm{II}}^{(4,0)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{u}) \\ & \times(A_r^*A_s^*A_t^*A_u^*\xi_0^4-\xi_0^{*4}A_uA_tA_sA_r) \\ & +g_{\mathrm{II}}^{(3,1)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{t};-\boldsymbol{u})(A_r^*A_s^*A_t^*A_{-u}\xi_0^2-\xi_0^{*2}A_{-u}^*A_tA_sA_r) \\ & +g_{\mathrm{II}}^{(2,2)}(\boldsymbol{r},\boldsymbol{s};-\boldsymbol{t},-\boldsymbol{u})A_r^*A_s^*A_{-t}A_{-u}], \end{aligned} \tag{4·16}$$

where the functions $g_{\mathrm{I}}^{(3,0)}$ and $g_{\mathrm{I}}^{(2,1)}$ are given by (4·2) and (4·9), respectively. The new functions $g_{\mathrm{II}}^{(2,0)}$, $g_{\mathrm{II}}^{(4,0)}$, $g_{\mathrm{II}}^{(3,1)}$ and $g_{\mathrm{II}}^{(2,2)}$ are defined by

$$g_{\mathrm{II}}^{(2,0)}(\boldsymbol{r}) = \frac{1}{2!\,8}\sum_{s,t}{}'\delta_{r+s+t,0}(\lambda_s-1)(\lambda_t-1), \tag{4·17}$$

$$\begin{aligned} g_{\mathrm{II}}^{(4,0)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{u}) = {} & \frac{1}{4!\,16}\frac{1}{\sqrt{\lambda_r\lambda_s\lambda_t\lambda_u}}\{\lambda_r(\lambda_s-1)(\lambda_t-1)(\lambda_u-1) \\ & +\lambda_s(\lambda_r-1)(\lambda_t-1)(\lambda_u-1)+\lambda_t(\lambda_r-1)(\lambda_s-1)(\lambda_u-1) \\ & +\lambda_u(\lambda_r-1)(\lambda_s-1)(\lambda_t-1)\}, \end{aligned} \tag{4·18}$$

$$g_{\mathrm{II}}^{(3,1)}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}; -\boldsymbol{u}) = \frac{1}{3!\,16}\frac{1}{\sqrt{\lambda_r\lambda_s\lambda_t\lambda_u}}\{\lambda_r+\lambda_s+\lambda_t-\lambda_u$$

$$-(2+3\lambda_u)(\lambda_r\lambda_s+\lambda_s\lambda_t+\lambda_t\lambda_r)+3\lambda_r\lambda_s\lambda_t+10\lambda_r\lambda_s\lambda_t\lambda_u$$

$$+2\lambda_{u+r}\lambda_u\lambda_r(\lambda_s+\lambda_t)+2\lambda_{u+s}\lambda_u\lambda_s(\lambda_r+\lambda_t)+2\lambda_{u+t}\lambda_u\lambda_t(\lambda_r+\lambda_s)$$

$$-4\lambda_r\lambda_s\lambda_t\lambda_u(\lambda_{u+r}+\lambda_{u+s}+\lambda_{u+t})\} \qquad (4\cdot19)$$

and

$$g_{\mathrm{II}}^{(2,2)}(\boldsymbol{r}, \boldsymbol{s}; -\boldsymbol{t}, -\boldsymbol{u}) = \frac{1}{(2!)^2\,16}\frac{1}{\sqrt{\lambda_r\lambda_s\lambda_t\lambda_u}}[-\lambda_r-\lambda_s+\lambda_t+\lambda_u+2(\lambda_r\lambda_s-\lambda_t\lambda_u)$$

$$+(3-\lambda_{r+t}-\lambda_{r+u}-\lambda_{s+t}-\lambda_{s+u})\{\lambda_r\lambda_s(\lambda_t+\lambda_u)-\lambda_t\lambda_u(\lambda_r+\lambda_s)\}]. \qquad (4\cdot20)$$

The operators ξ_0 and ξ_0^* are transformed by the unitary transformation (4·15) in the following way:

$$\eta_0 = e^{-G(A, A^*)}\xi_0 e^{G(A, A^*)}. \qquad (4\cdot21)$$

As will be seen later, the operators b_p and b_p^* in the S.Y.K. theory are identical with the operators $\eta_0^*\alpha_p$ and $\alpha_p^*\eta_0$, respectively.

In order to express the Hamiltonian (2·25) in terms of the new operators α_p and α_p^*, we introduce an inverse-transformation to (4·15) in the following way:

$$A_p = e^{G(\alpha, \alpha^*)}\alpha_p e^{-G(\alpha, \alpha^*)}, \qquad (4\cdot22)$$

where

$$G(\alpha, \alpha^*) = \frac{1}{\sqrt{N}}\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}}{}' \delta_{\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{t},0}[g_{\mathrm{I}}^{(3,0)}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})(\alpha_r^*\alpha_s^*\alpha_t^*\eta_0^3-\eta_0^{*3}\alpha_t\alpha_s\alpha_r)$$

$$+g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r}, \boldsymbol{s}; -\boldsymbol{t})(\alpha_r^*\alpha_s^*\alpha_{-t}\eta_0-\eta_0^*\alpha_{-t}^*\alpha_s\alpha_r)]$$

$$+\frac{1}{N}\sum_{\boldsymbol{r}}{}' g_{\mathrm{II}}^{(2,0)}(\boldsymbol{r})(\alpha_r^*\alpha_{-r}^*\eta_0^2-\eta_0^{*2}\alpha_{-r}\alpha_r)$$

$$+\frac{1}{N}\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{u}}{}' \delta_{\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{t}+\boldsymbol{u},0}[g_{\mathrm{II}}^{(4,0)}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u})$$

$$\times(\alpha_r^*\alpha_s^*\alpha_t^*\alpha_u^*\eta_0^4-\eta_0^{*4}\alpha_u\alpha_t\alpha_s\alpha_r)$$

$$+g_{\mathrm{II}}^{(3,1)}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}; -\boldsymbol{u})(\alpha_r^*\alpha_s^*\alpha_t^*\alpha_{-u}\eta_0^2-\eta_0^{*2}\alpha_{-u}^*\alpha_t\alpha_s\alpha_r)$$

$$+g_{\mathrm{II}}^{(2,2)}(\boldsymbol{r}, \boldsymbol{s}; -\boldsymbol{t}, -\boldsymbol{u})\alpha_r^*\alpha_s^*\alpha_{-t}\alpha_{-u}], \qquad (4\cdot23)$$

$$\eta_0 = e^{-G(A, A^*)}\frac{1}{\sqrt{A_0^*A_0+1}}A_0 e^{G(A, A^*)} = \frac{1}{\sqrt{\alpha_0^*\alpha_0+1}}\alpha_0. \qquad (4\cdot24)$$

On assuming the relations

$$\eta_0\eta_0^* = \eta_0^*\eta_0 = 1, \qquad (4\cdot25)$$

which correspond to (2·14), we expand (4·22) in powers of $N^{-1/2}$ and have

$$A_{\boldsymbol{p}}=\alpha_{\boldsymbol{p}}+\frac{1}{\sqrt{N}}\Gamma_{\mathrm{I}}(\boldsymbol{p})+\frac{1}{N}\Gamma_{\mathrm{II}}(\boldsymbol{p})+\cdots$$

and

$$A_{\boldsymbol{p}}^*=\alpha_{\boldsymbol{p}}^*+\frac{1}{\sqrt{N}}\Gamma_{\mathrm{I}}^*(\boldsymbol{p})+\frac{1}{N}\Gamma_{\mathrm{II}}^*(\boldsymbol{p})+\cdots, \tag{4·26}$$

where

$$\Gamma_{\mathrm{I}}^*(\boldsymbol{p})=\sum_{\boldsymbol{r},\boldsymbol{s}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s}-\boldsymbol{p},0}[g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r},\boldsymbol{s};\boldsymbol{p})\alpha_{\boldsymbol{r}}^*\alpha_{\boldsymbol{s}}^*\eta_0-2g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{r},\boldsymbol{p};\boldsymbol{s})\eta_0^*\alpha_{\boldsymbol{s}}^*\alpha_{-\boldsymbol{r}}$$
$$-3g_{\mathrm{I}}^{(3,0)}(\boldsymbol{p},-\boldsymbol{r},-\boldsymbol{s})\eta_0^{*3}\alpha_{-\boldsymbol{r}}\alpha_{-\boldsymbol{s}}] \tag{4·27}$$

and

$$\Gamma_{\mathrm{II}}^*(\boldsymbol{p})=\sum_{\boldsymbol{r},\boldsymbol{s}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s}-\boldsymbol{p},0}[9(g_{\mathrm{I}}^{(3,0)}(\boldsymbol{p},-\boldsymbol{r},-\boldsymbol{s}))^2-(g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r},\boldsymbol{s};\boldsymbol{p}))^2]\alpha_{\boldsymbol{p}}^*$$
$$+\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{u}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s},\boldsymbol{u}}\delta_{\boldsymbol{t}+\boldsymbol{u},\boldsymbol{p}}[\{g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r},\boldsymbol{s};\boldsymbol{u})g_{\mathrm{I}}^{(2,1)}(\boldsymbol{u},\boldsymbol{t};\boldsymbol{p})$$
$$+3g_{\mathrm{I}}^{(3,0)}(\boldsymbol{r},\boldsymbol{s},-\boldsymbol{u})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{u},\boldsymbol{p};\boldsymbol{t})\}\alpha_{\boldsymbol{r}}^*\alpha_{\boldsymbol{s}}^*\alpha_{\boldsymbol{t}}^*\eta_0^2$$
$$+\{-3g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{r},-\boldsymbol{s};-\boldsymbol{u})g_{\mathrm{I}}^{(3,0)}(-\boldsymbol{u},-\boldsymbol{t},\boldsymbol{p})$$
$$+3g_{\mathrm{I}}^{(3,0)}(-\boldsymbol{r},-\boldsymbol{s},\boldsymbol{u})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{t},\boldsymbol{p};\boldsymbol{u})\}\eta_0^{*4}\alpha_{-\boldsymbol{r}}\alpha_{-\boldsymbol{s}}\alpha_{-\boldsymbol{t}}$$
$$+\{9g_{\mathrm{I}}^{(3,0)}(\boldsymbol{r},\boldsymbol{s},-\boldsymbol{u})g_{\mathrm{I}}^{(3,0)}(-\boldsymbol{u},-\boldsymbol{t},\boldsymbol{p})$$
$$-g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r},\boldsymbol{s};\boldsymbol{u})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{t},\boldsymbol{p};\boldsymbol{u})\}\alpha_{\boldsymbol{r}}^*\alpha_{\boldsymbol{s}}^*\alpha_{-\boldsymbol{t}}$$
$$+\{-3g_{\mathrm{I}}^{(3,0)}(-\boldsymbol{r},-\boldsymbol{s},\boldsymbol{u})g_{\mathrm{I}}^{(2,1)}(\boldsymbol{u},\boldsymbol{t};\boldsymbol{p})$$
$$-g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{r},-\boldsymbol{s};-\boldsymbol{u})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{u},\boldsymbol{p};\boldsymbol{t})\}\eta_0^{*2}\alpha_{\boldsymbol{t}}^*\alpha_{-\boldsymbol{r}}\alpha_{-\boldsymbol{s}}]$$
$$+\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{u}}{}'\delta_{\boldsymbol{r}+\boldsymbol{t},\boldsymbol{u}}\delta_{\boldsymbol{s}+\boldsymbol{u},\boldsymbol{p}}[\{2g_{\mathrm{I}}^{(2,1)}(\boldsymbol{r},-\boldsymbol{u};-\boldsymbol{t})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{u},\boldsymbol{p};\boldsymbol{s})$$
$$-2g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{t},\boldsymbol{u};\boldsymbol{r})g_{\mathrm{I}}^{(2,1)}(\boldsymbol{u},\boldsymbol{s};\boldsymbol{p})\}\alpha_{\boldsymbol{r}}^*\alpha_{\boldsymbol{s}}^*\alpha_{-\boldsymbol{t}}$$
$$+\{6g_{\mathrm{I}}^{(2,1)}(\boldsymbol{t},-\boldsymbol{u};-\boldsymbol{r})g_{\mathrm{I}}^{(3,0)}(-\boldsymbol{u},-\boldsymbol{s},\boldsymbol{p})$$
$$+2g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{r},\boldsymbol{u};\boldsymbol{t})g_{\mathrm{I}}^{(2,1)}(-\boldsymbol{s},\boldsymbol{p};\boldsymbol{u})\}\eta_0^{*2}\alpha_{\boldsymbol{t}}^*\alpha_{-\boldsymbol{r}}\alpha_{-\boldsymbol{s}}]$$
$$-2g_{\mathrm{II}}^{(2,0)}(\boldsymbol{p})\alpha_{-\boldsymbol{p}}-\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{t}+\boldsymbol{p},0}4g_{\mathrm{II}}^{(4,0)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{p})\eta_0^{*4}\alpha_{\boldsymbol{r}}\alpha_{\boldsymbol{s}}\alpha_{\boldsymbol{t}}$$
$$+\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{t}-\boldsymbol{p},0}g_{\mathrm{II}}^{(3,1)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{t};\boldsymbol{p})\alpha_{\boldsymbol{r}}^*\alpha_{\boldsymbol{s}}^*\alpha_{\boldsymbol{t}}^*\eta_0^2$$
$$-\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{u}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{u}+\boldsymbol{p},0}3g_{\mathrm{II}}^{(3,1)}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{p};-\boldsymbol{u})\eta_0^{*2}\alpha_{-\boldsymbol{u}}^*\alpha_{\boldsymbol{r}}\alpha_{\boldsymbol{s}}$$
$$+\sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}}{}'\delta_{\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{t}-\boldsymbol{p},0}2g_{\mathrm{II}}^{(2,2)}(\boldsymbol{r},\boldsymbol{s};-\boldsymbol{t},\boldsymbol{p})\alpha_{\boldsymbol{r}}^*\alpha_{\boldsymbol{s}}^*\alpha_{-\boldsymbol{t}}. \tag{4·28}$$

Introducing (4·26) into (2·25), we can represent the Hamiltonian (2·25) in terms of the operators $\alpha_{\boldsymbol{p}}$ and $\alpha_{\boldsymbol{p}}^*$. After the straightforward calculation up to order

N^{-1}, we have

$$H = H_0^{(\alpha)} + H_{\rm I}^{(\alpha)} + H_{\rm II}^{(\alpha)} + \cdots , \tag{4·29}$$

$$H_0^{(\alpha)} = E_G^B + \sum_{\boldsymbol{p}}{}' E_{\boldsymbol{p}}^B \alpha_{\boldsymbol{p}}^* \alpha_{\boldsymbol{p}} , \tag{4·29a}$$

$$H_{\rm I}^{(\alpha)} = \frac{1}{3!\sqrt{N}} \sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r},0} F_{\rm I}^{(3,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) (\alpha_{\boldsymbol{p}}^*\alpha_{\boldsymbol{q}}^*\alpha_{\boldsymbol{r}}^*\eta_0^3 + \eta_0^{*3}\alpha_{\boldsymbol{r}}\alpha_{\boldsymbol{q}}\alpha_{\boldsymbol{p}})$$
$$+ \frac{1}{2!\sqrt{N}} \sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r},0} F_{\rm I}^{(2,1)}(\boldsymbol{p},\boldsymbol{q};-\boldsymbol{r}) (\alpha_{\boldsymbol{p}}^*\alpha_{\boldsymbol{q}}^*\alpha_{-\boldsymbol{r}}\eta_0 + \eta_0^*\alpha_{-\boldsymbol{r}}^*\alpha_{\boldsymbol{q}}\alpha_{\boldsymbol{p}}) \tag{4·29b}$$

and

$$H_{\rm II}^{(\alpha)} = \frac{1}{N} F_{\rm II}^{(0,0)} + \frac{1}{N} \sum_{\boldsymbol{p}}{}' F_{\rm II}^{(1,1)}(\boldsymbol{p}) \alpha_{\boldsymbol{p}}^*\alpha_{\boldsymbol{p}}$$
$$+ \frac{1}{2N} \sum_{\boldsymbol{p}}{}' F_{\rm II}^{(2,0)}(\boldsymbol{p}) (\alpha_{\boldsymbol{p}}^*\alpha_{-\boldsymbol{p}}^*\eta_0^2 + \eta_0^{*2}\alpha_{-\boldsymbol{p}}\alpha_{\boldsymbol{p}})$$
$$+ \frac{1}{3!N} \sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s},0} F_{\rm II}^{(3,1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};-\boldsymbol{s})$$
$$\times (\alpha_{\boldsymbol{p}}^*\alpha_{\boldsymbol{q}}^*\alpha_{\boldsymbol{r}}^*\alpha_{-\boldsymbol{s}}\eta_0^2 + \eta_0^{*2}\alpha_{-\boldsymbol{s}}^*\alpha_{\boldsymbol{r}}\alpha_{\boldsymbol{q}}\alpha_{\boldsymbol{p}})$$
$$+ \frac{1}{(2!)^2 N} \sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}{}' \delta_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s},0} F_{\rm II}^{(2,2)}(\boldsymbol{p},\boldsymbol{q};-\boldsymbol{r},-\boldsymbol{s}) \alpha_{\boldsymbol{p}}^*\alpha_{\boldsymbol{q}}^*\alpha_{-\boldsymbol{r}}\alpha_{-\boldsymbol{s}} , \tag{4·29c}$$

where the vertex-functions $F_{\rm I}^{(3,0)}$, $F_{\rm I}^{(2,1)}$, $F_{\rm II}^{(0,0)}$, $F_{\rm II}^{(1,1)}$ have been given in Eqs. (4·3), (4·10), (4·7), (4·14), respectively, and the remaining functions are expressed as

$$F_{\rm II}^{(2,0)}(\boldsymbol{p}) = F_{\rm II}^{(1,1)}(\boldsymbol{p}),$$

$$F_{\rm II}^{(3,1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};-\boldsymbol{s}) = \frac{\hbar^2}{4m}\frac{1}{\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}} [\{(\boldsymbol{p}\cdot\boldsymbol{q})\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}(\lambda_{\boldsymbol{p}+\boldsymbol{q}}(\lambda_{\boldsymbol{p}}-1)(\lambda_{\boldsymbol{q}}-1)$$
$$+\lambda_{\boldsymbol{q}}(\lambda_{\boldsymbol{p}}-1)(\lambda_{\boldsymbol{p}+\boldsymbol{r}}-1)+\lambda_{\boldsymbol{p}}(\lambda_{\boldsymbol{q}}-1)(\lambda_{\boldsymbol{q}+\boldsymbol{r}}-1))$$
$$+(\boldsymbol{p}\cdot\boldsymbol{s})\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}(\lambda_{\boldsymbol{p}}-1)(\lambda_{\boldsymbol{p}+\boldsymbol{q}}+\lambda_{\boldsymbol{p}+\boldsymbol{r}}-2)\}$$
$$+\{\text{two terms obtained by cyclic exchange of } \boldsymbol{p},\ \boldsymbol{q} \text{ and } \boldsymbol{r}\}]$$

and

$$F_{\rm II}^{(2,2)}(\boldsymbol{p},\boldsymbol{q};-\boldsymbol{r},-\boldsymbol{s}) = \frac{\hbar^2}{4m}\frac{1}{\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}} \Big[(\boldsymbol{p}\cdot\boldsymbol{s})\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{q}}((\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{s}}-1)(\lambda_{\boldsymbol{p}+\boldsymbol{s}}-1)$$
$$+\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{s}}(\lambda_{\boldsymbol{p}+\boldsymbol{r}}+\lambda_{\boldsymbol{q}+\boldsymbol{s}}-2)-(\lambda_{\boldsymbol{p}}-1)(\lambda_{\boldsymbol{s}}-1))$$
$$+\Big\{\text{three terms obtained by exchanging variables as } (\boldsymbol{p}\leftrightarrow\boldsymbol{q}),\ (\boldsymbol{r}\leftrightarrow\boldsymbol{s})$$
$$\text{and } \begin{pmatrix}\boldsymbol{p}\leftrightarrow\boldsymbol{q}\\ \boldsymbol{r}\leftrightarrow\boldsymbol{s}\end{pmatrix}\Big\}\Big]$$

$$+\frac{\hbar^2}{2m}\frac{1}{\sqrt{\lambda_p\lambda_q\lambda_r\lambda_s}}\{(\boldsymbol{p}\cdot\boldsymbol{q})+(\boldsymbol{r}\cdot\boldsymbol{s})\}\lambda_p\lambda_q\lambda_r\lambda_s(\lambda_{p+s}+\lambda_{p+r}-2). \quad (4\cdot30)$$

From the above expressions, we can see that the Hamiltonian (4·29) coincides to that in the S.Y.K. theory, if we carry out the following replacements:

$$b_p=\eta_0{}^*\alpha_p \quad \text{and} \quad b_p{}^*=\alpha_p{}^*\eta_0 \quad (4\cdot31)$$

for $\boldsymbol{p}\neq0$. This fact indicates that the present formalism is essentially equivalent to the S.Y.K. theory in spite of the fact that both formalisms are developed on quite different bases. The total number $\tilde{N}^{(\alpha)}$ of the dressed atoms expressed by the operators α_p and $\alpha_p{}^*$ conserves, since we can show that

$$\tilde{N}^{(\alpha)}=\alpha_0{}^*\alpha_0+\sum_p{}'\alpha_p{}^*\alpha_p=e^{-G(A,A^*)}\sum_{\text{all }p}A_p{}^*A_pe^{G(A,A^*)}$$
$$=\sum_{\text{all }p}A_p{}^*A_p=\sum_{\text{all }p}a_p{}^*a_p, \quad (4\cdot32)$$

where use has been made of Eq. (2·8). However, the number $\sum_p' b_p{}^*b_p$ of the excitation modes in the S.Y.K. theory does not conserve in contrast with the total number $\tilde{N}^{(\alpha)}$ in the present theory. In this respect, the present theory is different from the S.Y.K. theory.

The conservation of the number of the dressed atoms described by the operators α_p and $\alpha_p{}^*$ indicates the existence of the Bose-Einstein condensation of the dressed atoms and thus the occurence of the λ-transition in the bose system can be at least qualitatively realized. It should be emphasized here that the condensate in the present theory is not bare atoms as in the case of the Bogoliubov theory but the dressed atoms expressed by the operators α_0 and $\alpha_0{}^*$. However, the calculation of the temperature T_λ at the λ-point based on the present theory is very difficult because of the interactions among excitations near the λ-point and this problem remains still as an open question.

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Appendix

In (2·13a)∼(2·13c), the projection operator $|N_0=0\rangle\langle N_0=0|$ has been ignored and Eq. (2·13a)∼(2·13c) have been replaced by (2·14). Since we are mainly concerned with the approximation of the commutator in (2·14), we consider the following function instead of F in (2·10), for simplicity:

$$F=x\xi_0{}^2-x^*\xi_0{}^{*2}, \quad (\text{A}\cdot1)$$

where x is an arbitrary c-number. In order to see the condition for the approximation, let us consider the difference between the exact transformation and the approximate transformation. When the operator e^F is expanded in powers of F, the exact transformation becomes as

$$e^F=\sum_{l=0}^{\infty}\frac{1}{l!}(x\xi_0{}^2-x^*\xi_0{}^{*2})^l. \qquad (\mathrm{A\cdot 2})$$

Every term in the power-series involves products of the operators ξ_0 and $\xi_0{}^*$, which can be replaced by normal products with the aid of the exact relation $\xi_0\xi_0{}^*=1$. For example, the product $\xi_0{}^{*2}\xi_0{}^2\xi_0{}^{*2}\xi_0{}^2$ reduces to $\xi_0{}^{*2}\xi_0{}^2$. If the normal product which has the form $\xi_0{}^{*k}\xi_0{}^m$ is operated on the state $|N_0\rangle$, the result is expressed as

$$\xi_0{}^{*k}\xi_0{}^m|N_0\rangle\begin{cases}=0 & \text{for } N_0\leqq m-1\,,\\ =\xi_0{}^{*\,k-m}|N_0\rangle & \text{for } N_0\geqq m \text{ and } k\geqq m\,,\\ =\xi_0{}^{m-k}|N_0\rangle & \text{for } N_0\geqq m \text{ and } k<m\,.\end{cases} \qquad (\mathrm{A\cdot 3})$$

Therefore, the product $\xi_0{}^{*k}\xi_0{}^m$ can be expressed as

$$\xi_0{}^{*k}\xi_0{}^m\begin{cases}=\xi_0{}^{*\,k-m}(1-|m-1\rangle\langle m-1|-\cdots-|N_0=0\rangle\langle N_0=0|) & \text{for } k\geqq m\,,\\ =\xi_0{}^{m-k}(1-|m-1\rangle\langle m-1|-\cdots-|N_0=0\rangle\langle N_0=0|) & \text{for } k<m\,.\end{cases} \qquad (\mathrm{A\cdot 4})$$

The approximation used in (2·14) corresponds to ignoreing the existence of the projection operators on the right-hand side of (A·4), i.e.,

$$\xi_0{}^{*k}\xi_0{}^m\begin{cases}=\xi_0{}^{*\,k-m} & \text{for } k\geqq m\,,\\ =\xi_0{}^{m-k} & \text{for } k<m\,.\end{cases} \qquad (\mathrm{A\cdot 5})$$

Employing the approximation (A·5), we obtain the approximate transformation for (A·2) as

$$e^F\approx\sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{1}{(l-k)!\,k!}x^{l-k}(-x^*)^k\times\begin{cases}\xi_0{}^{*\,4k-2l} & \text{for } 2k\geqq l\,,\\ \xi_0{}^{2l-4k} & \text{for } 2k<l\,.\end{cases} \qquad (\mathrm{A\cdot 6})$$

If we assume that the physically-significant states in many-boson system are always occupied by more than N_0 particles with zero momenta described by the operators A_0 and $A_0{}^*$, the terms on the right-hand side of (A·6) coincide with the terms of (A·2) up to order $l=(N_0/2)+1$ from the operation of both terms on these states. Although the approximation (A·6) is not correct for the case $l\geqq(N_0/2)+2$, it will be seen that the error is negligibly small according to the following consideration. In the case $l=(N_0/2)+2$, the exact form of the l-th order is given by

$$\frac{1}{((N_0/2)+2)!}(x\xi_0{}^2-x^*\xi_0{}^{*2})^{N_0/2+2}=\frac{1}{((N_0/2)+2)!}\times[x^{N_0/2+2}\xi_0{}^{N_0+4}+x^{N_0/2+1}(-x^*)\{\xi_0{}^{*2}\xi_0{}^{N_0+2}+((N_0/2)+1)\xi_0{}^{N_0}\}+\cdots], \qquad (\mathrm{A\cdot 7})$$

and the corresponding term of the approximation (A·6) becomes

$$(l\text{-th term in (A·6)}) = \frac{1}{((N_0/2)+2)!}$$
$$\times [x^{N_0/2+2}\xi_0{}^{N_0+4} + x^{N_0/2+1}(-x^*)((N_0/2)+2)\xi_0{}^{N_0} + \cdots]. \quad \text{(A·8)}$$

It should be noticed that only the second terms on the right-hand sides of (A·7) and (A·8) are different. Operating (A·7) and (A·8) on the state $|N_0\rangle$ and comparing the two results, one can see that the relative error of the approximation (A·8) is $((N_0/2)+1)^{-1}$. If the number N_0 is about 10^6, which satisfies the condition $N_0/N \sim 0$, the relative error is about 10^{-6}. In the same way, the approximation can be seen to be valid for the other cases. Thus it is seen that the assumption (2·14) (or (A·5)) does not demand the existence of finite fraction of condensate in the physical states.

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