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One-Dimensional Many Boson System. II

——Orthogonality of Eigenstates and Their Level Structure——

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The analysis for the Lieb and Liniger model is continued on the basis of the result in the preceding paper. It is verified that two eigenstates specified with different sets of quantum numbers are orthogonal to each other. The ground state is shown to be specified by quantum numbers with zero values. The energy levels are investigated systematically.

§ 1. Introduction

In a preceding paper[1] (hereafter referred to as I), the Lieb and Liniger model[2),3)] has been exactly solved from a field theoretical viewpoint. The eigenstates and eigen-energies have been determined uniquely.

In this paper, we continue the analysis on the eigenstates and eigen-energies given in I. Section 2 is devoted to verification of orthogonality of the eigenstates. It is clarified in § 3 that the ground state with minimum energy is specified by all quantum numbers with zero values. This implies that the ground state is the condensed state of bose-particles dressed with interaction cloud which will be introduced in a subsequent paper.[4] Furthermore, the energy levels are classified systematically in the case of an infinitely large coupling constant, and this classification is compared with Lieb's one.[5]

§ 2. Orthogonality of eigenstates

The purpose of this section is to establish orthogonality of two exact eigenstates specified by different sets of quantum numbers. From (2·19) in I, the exact eigenstate for the n-boson system specified by quantum numbers q_1, q_2, $\cdots$, q_n is given as

$$|\Psi_{q_1,q_2,\cdots,q_n}\rangle=\beta_{q_1,q_2,\cdots,q_n}\sum_{\{p_{i,j};1\leq i<j\leq n\}}\prod_{1\leq i<j\leq n}d(p_{i,j};k_{i,j})\prod_{i=1}^{n}a^*_{\sum_{j=1, j\neq i}^{n}p_{ij}+q_i}|0\rangle, \qquad (2\cdot1)$$

by following the same notations and definitions as in I. The scalar product of the states $|\Psi_{q_1,q_2,\cdots,q_n}\rangle$ and $|\Psi_{q'_1,q'_2,\cdots,q'_n}\rangle$ is written as

$$\langle \Psi_{q'_1,q'_2,\cdots,q'_n} | \Psi_{q_1,q_2,\cdots,q_n} \rangle$$
$$= \beta^*_{q'_1,q'_2,\cdots,q'_n} \beta_{q_1,q_2,\cdots,q_n} \sum_{\{p'_{i,j},p_{i,j};1\leq i<j\leq n\}} \prod_{1\leq i<j\leq n} \{ d(p'_{i,j};\, k'_{i,j}) d(p_{i,j};\, k_{i,j}) \}$$
$$\times \sum_{\mu} \prod_{i=1}^{n} \delta\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} p'_{\mu_i,\mu_j} + q'_{\mu_i},\, \sum_{\substack{j=1\\ j\neq i}}^{n} p_{i,j} + q_i \Big), \tag{2·2}$$

where μ indicates a permutation $\binom{1,\,2,\,\cdots,\,n}{\mu_1,\,\mu_2,\,\cdots,\,\mu_n}$,the symbol $\sum_\mu$ means the summation over all $n!$ permutations and $\delta(x, y)$ is the Kronecker delta $\delta_{x,y}$. It is more convenient for the accomplishment of multiple sums in (2·2) to rewrite the summation variables $p'_{i,j} (1\leq i<j\leq n)$ by $p'_{\mu i,\mu j}$ since the Kronecker deltas have the variables p'_{μ_i,μ_j} as their arguments. Hence we, hereafter, use new notations defined as

$$p''_{i,j}(\mu) \equiv p'_{\mu_i,\mu_j}\,, \qquad k''_{i,j}(\mu) \equiv k'_{\mu_i,\mu_j}\,, \qquad (1\leq i<j\leq n) \tag{2·3}$$

and

$$p_i''(\mu) \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} p'_{\mu_i,\mu_j} + q'_{\mu_i}\,, \qquad q''_i(\mu) \equiv q'_{\mu_i}\,, \qquad (i=1, 2, \cdots, n)$$

$$p_i \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} p_{i,j} + q_i\,. \qquad (i=1, 2, \cdots, n) \tag{2·4}$$

Here note from the definitions (2·20) of $d(p_{i,j};\, k_{i,j})$ in I that

$$d(p_{i,j};\, k_{i,j}) = d(p_{j,i};\, k_{j,i}), \tag{2·5a}$$

and

$$d(p'_{i,j};\, k'_{i,j}) = d(p'_{j,i};\, k'_{j,i}). \tag{2·5b}$$

This fact (2·5b) gives

$$\prod_{1\leq i<j\leq n} d(p'_{i,j};\, k'_{i,j}) = \prod_{1\leq i<j\leq n} d(p'_{\mu_i,\mu_j};\, k'_{\mu_i,\mu_j}) = \prod_{1\leq i<j\leq n} d(p''_{i,j}(\mu);\, k''_{i,j}(\mu)) \tag{2·5c}$$

for a permutation μ. Then the expression (2·2) is rewritten as

$$\langle \Psi_{q_1',\cdots,q_n'} | \Psi_{q_1,\cdots,q_n} \rangle = \beta^*_{q_1',\cdots,q_n'} \beta_{q_1,\cdots,q_n} \sum_{\mu} \sum_{\left\{\substack{p''_{i,j}(\mu),p_{i,j}\\ 1\leq i<j\leq n}\right\}} \prod_{1\leq i<j\leq n} d''_{i,j}(\mu) d_{i,j} \prod_{l=1}^{n} \delta_{p''_l(\mu),p_l}\,, \tag{2·6}$$

where

$$d''_{i,j}(\mu) = d(p''_{i,j}(\mu);\, k''_{i,j}(\mu))$$

and

$$d_{i,j} = d(p_{i,j};\, k_{i,j}). \tag{2·7}$$

For convenience of later discussion, let us introduce the quantities $k''_i(\mu)$ defined by

$$k_i''(\mu)=\sum_{\substack{j=1\\ j\neq i}}^{n} k_{i,j}''(\mu)+q_i''(\mu). \qquad (1\leqq i\leqq n) \tag{2·8}$$

It should be noted that the number of summation variables in (2·6) is $n(n-1)$, whereas the number of the Kronecker symbols is $(n-1)$ except one which gives total momentum conservation. Thus, at first sight, it seems difficult to carry out the multiple summations in (2·6). However this difficulty can be overcome in the following way.

From the assumption that two sets of momenta $(q_1', q_2', \cdots, q_n')$ and $(q_1, q_2, \cdots, q_n)$ are different from each other, we can find, at least, one quantity k_{i_1} among $(k_1, k_2, \cdots, k_n)$ to satisfy the relation

$$k_{i_1}\neq k_1', k_2', \cdots, k_n'. \tag{2·9a}$$

This is reasoned as follows. If we cannot find such a quantity k_{i_1}, namely if all $k_i(i=1, 2, \cdots, n)$ are equal to any ones among $(k_1', k_2', \cdots, k_n')$ respectively, then we have $k_i=k_i'(i=1, 2, \cdots, n)$ according to the inequalities $k_l<k_{l+1}$ and $k_l'<k_{l+1}'(l=1, 2, \cdots, n-1)$ (see (3·25) in I). This means $\sum_{l=1}^n q_l=\sum_{l=1}^n q_l'$ and $s_l=s_l'(l=1, 2, \cdots, n-1)$ from (3·4), (3·12) and (3·13) in I, that is, $q_l=q_l'(l=1, 2, \cdots, n)$. This is incompatible with the initial assumption. Hence the above relations (2·9a) have been established. Then from (2·9a) we have

$$k_{i_1}\neq k_{i_1}''(\mu) \qquad \text{for all permutation.} \tag{2·9b}$$

Now let us carry out the multiple summations over $p_{l,m}$ and $p_{l,m}''(\mu)$ in (2·6) step by step in the following way.

[Step 1] The identities

$$\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n} k_{i_1,i_2}d_{i_1,i_2}^{-1}=-p_{i_1}+k_{i_1}$$

and

$$\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n} k_{i_1,i_2}''(\mu)d_{i_1,i_2}''^{-1}(\mu)=-p_{i_1}''(\mu)+k_{i_1}''(\mu)$$

lead to

$$\prod_{l=1}^{n}\delta_{p_l''(\mu),p_l}=\frac{1}{k_{i_1}-k_{i_1}''(\mu)}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\{k_{i_1,i_2}\mathrm{d}_{i_1,i_2}^{-1}-k_{i_1,i_2}''(\mu)\mathrm{d}_{i_1,i_2}''^{-1}(\mu)\}\prod_{l=1}^{n}\delta_{p_l''(\mu),p_l}, \tag{2·10}$$

where we have made use of (2·9b). Substitution of (2·10) in (2·6) produces

$$\langle\Psi_{q_1',\cdots,q_n'}|\Psi_{q_1,\cdots,q_n}\rangle=\beta_{q_1',\cdots,q_n'}^*\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\{p_{i,j}''(\mu),p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n}\{d_{i,j}''(\mu)d_{i,j}\}$$

$$\times\frac{1}{k_{i_1}-k''_{i_1}(\mu)}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\{k_{i_1,i_2}d^{-1}_{i_1,i_2}-k''_{i_1,i_2}(\mu)d''^{-1}_{i_1,i_2}(\mu)\}\prod_{l=1}^{n}\delta_{p''_l(\mu),p_l}\,. \tag{2·11}$$

Here note that

$$\prod_{l=1}^{n}\delta_{p''_l(\mu),p_l}=\prod_{\substack{l=1\\ l\neq i_1}}^{n}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}\,, \tag{2·12}$$

where

$$Q=\sum_{l=1}^{n}q_l \qquad \text{and} \qquad Q'=\sum_{l=1}^{n}q'_l\,. \tag{2·13}$$

The use of (2·12) in (2·11) and the separation of the terms having the summation variables $p''_{i_1,i_2}(\mu)$ or p_{i_1,i_2} in (2·11) give

$$\begin{aligned}\langle\Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle&=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\\ &\cdot\Bigg[\sum_{\{p''_{i,j}(\mu),p_{i,j};1\leq i<j\leq n,(i,j)\neq[i_1,i_2]\}}\ \prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2]}}\{d_{i,j}d''_{i,j}(\mu)\}\\ &\times\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}\sum_{p_{i_1,i_2},p''_{i_1,i_2}(\mu)}\frac{1}{k_{i_1}-k''_{i_1}(\mu)}\\ &\times\{k_{i_1,i_2}d^{-1}_{i_1,i_2}-k''_{i_1,i_2}(\mu)d''^{-1}_{i_1,i_2}(\mu)\}d_{i_1,i_2}d''_{i_1,i_2}(\mu)\delta(p''_{i_2}(\mu),p_{i_2})\Bigg],\end{aligned}$$

where use has been made of (2·5a, b), the symbol $[i_1, i_2]$ is defined by

$$[i_1,i_2]=\begin{cases}(i_1,i_2) & \text{for}\quad i_1<i_2\,,\\ (i_2,i_1) & \text{for}\quad i_2<i_1\,,\end{cases} \tag{2·14}$$

and, therefore, $(i,j)\neq[i_1,i_2]$ implies that $i\neq i_1$, or $j\neq i_2$; for $i_1<i_2$) and ($i\neq i_2$ or $j\neq i_1$; for $i_1>i_2$). Hereafter we abbreviate the notation of the summation as

$$\sum_{\{p''_{i,j}(\mu),p_{i,j};1\leq i<j\leq n,(i,j)\neq[i_1,i_2]\}}\longrightarrow\sum_{\substack{\{p''_{i,j}(\mu),p_{i,j}\}\\ (i,j)\neq[i_1,i_2]}}.$$

Next observe that

$$\sum_{p_{i_1,i_2},p''_{i_1,i_2}(\mu)}d_{i_1,i_2}\delta_{(p''_{i_2}(\mu),p_{i_2})}=\frac{Lk_{i_1,i_2}}{2h}\cot\left(\frac{Lk_{i_1,i_2}}{2h}\right)=\frac{Lk_{i_1,i_2}}{2mg}(k_{i_1}-k_{i_2}) \tag{2·15a}$$

and

$$\sum_{p_{i_1,i_2},p''_{i_1,i_2}(\mu)}d''_{i_1,i_2}(\mu)\delta_{(p''_{i_2}(\mu),p_{i_2})}=\frac{Lk''_{i_1,i_2}(\mu)}{2mg}(k''_{i_1}(\mu)-k''_{i_2}(\mu)), \tag{2·15b}$$

where we have summed over firstly $p''_{i_1,i_2}(\mu)$ and secondly p_{i_1,i_2} in (2·15a) by using (2·25) and (2·26) in I, and reversely in (2·15b). Then one has

$$\langle \Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2]}}d_{i,j}d''_{i,j}(\mu)$$

$$\times\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}\left(\frac{-L}{2mg}\right)k''_{i_1,i_2}(\mu)k_{i_1,i_2}$$

$$\times\left\{2-\frac{k_{i_1}+k_{i_2}-k''_{i_1}(\mu)-k''_{i_2}(\mu)}{k_{i_1}-k''_{i_1}(\mu)}\right\}. \tag{2·16}$$

Here let us show that the contribution (we denote it by Z) from the first term in the curly bracket on the right-hand side of (2·16) vanishes out. If we consider such a permutation $\mu'=\begin{pmatrix}1,2,\cdots,n\\ \mu_1',\mu_2',\cdots,\mu_n'\end{pmatrix}$ as

$$\mu'_l=\mu_l,\quad (l\neq i_1,\ i_2)\quad \mu'_{i_1}=\mu_{i_2},\quad \mu'_{i_2}=\mu_{i_1}\ , \tag{2·17}$$

then, we obtain the following relations:

$$k''_{i_1,i_2}(\mu)=k'_{\mu_{i_1},\mu_{i_2}}=-k''_{i_1,i_2}(\mu') \tag{2·18a}$$

and

$$\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2]}}d_{i,j}d''_{i,j}(\mu)\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}$$

$$=\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu')\}\\ (i,j)\neq[i_1,i_2]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2]}}d_{i,j}d''_{i,j}(\mu')\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p''_l(\mu'),p_l}\delta_{Q',Q}\ . \tag{2·18b}$$

Using the relations (2·18a, b) in Z yields

$$Z=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu')\}\\ (i,j)\neq[i_1,i_2]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2]}}d_{i,j}d''_{i,j}(\mu')$$

$$\times\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p''_l(\mu'),p_l}\delta_{Q',Q}(-L/2mg)(-k''_{i_1,i_2}(\mu'))k_{i_1,i_2}\times 2\ . \tag{2·19a}$$

As the sum concerning μ in (2·19a) is taken over all permutations, the summation over μ can be changed to the one over μ'. From this fact, one obtains

$$Z=(-1)\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu'}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu')\}\\ (i,j)\neq[i_1,i_2]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2]}}d_{i,j}d''_{i,j}(\mu')$$

$$\times\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p''_l(\mu'),p_l}\delta_{Q',Q}(-L/2mg)k''_{i_1,i_2}(\mu')k_{i_1,i_2}\times 2=-Z\ . \tag{2·19b}$$

Thus Z is equal to zero. Therefore we see that

$$\langle \Psi_{q_1'\cdots q_n'} | \Psi_{q_1\cdots q_n}\rangle = \beta^*_{q_1',\cdots,q_n'}\beta_{q_1\cdots q_n}\sum_\mu \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n} \sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2]}} \prod_{\substack{1\le i<j\le n\\ (i,j)\neq[i_1,i_2]}} d_{i,j}d''_{i,j}(\mu)$$
$$\times \prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n} \delta_{p_l''(\mu),p_l}\delta_{Q',Q}(L/2mg)k''_{i_1,i_2}(\mu)k_{i_1,i_2}$$
$$\times(k_{i_1}+k_{i_2}-k''_{i_1}(\mu)-k''_{i_2}(\mu))/(k_{i_1}-k''_{i_1}(\mu)). \tag{2·20}$$

[Step 2] In the same way as the derivation of (2·10), we can obtain

$$(k_{i_1}+k_{i_2}-k''_{i_1}(\mu)-k''_{i_2}(\mu))\prod_{l\neq i_1,i_2}\delta_{p_l''(\mu),p_l}\delta_{Q',Q}$$
$$=\sum_{\substack{i_3=1\\ i_3\neq i_1,i_2}}^{n}\{k_{i_1,i_3}d^{-1}_{i_1,i_3}+k_{i_2,i_3}d^{-1}_{i_2,i_3}-k''_{i_1,i_3}(\mu)d''^{-1}_{i_1,i_3}(\mu)$$
$$-k''_{i_2,i_3}(\mu)d''^{-1}_{i_2,i_3}(\mu)\}\prod_{\substack{l=1\\ l\neq i_1,i_2}}^{n}\delta_{p_l''(\mu),p_l}\delta_{Q',Q}\,. \tag{2·21}$$

Substitution of (2·21) in (2·20) yields

$$\langle \Psi_{q_1',\cdots,q_n'} | \Psi_{q_1,\cdots,q_n}\rangle = \beta^*_{q_1',\cdots,q_n'}\beta_{q_1,\cdots,q_n}\sum_\mu \sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}} \sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}} \prod_{\substack{1\le i<j\le n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)$$
$$\times \prod_{\substack{l=1\\ l\neq i_1,i_2,i_3}}^{n} \delta_{p_l''(\mu),p_l}\delta_{Q',Q}(L/2mg)k''_{i_1,i_2}(\mu)k_{i_1,i_2}$$
$$\times\sum_{p_{i_1,i_3},p_{i_2,i_3},p''_{i_1,i_3}(\mu),p''_{i_2,i_3}(\mu)}\{-k''_{i_1,i_3}(\mu)d''^{-1}_{i_1,i_3}(\mu)+k_{i_1,i_3}d^{-1}_{i_1,i_3}-k''_{i_2,i_3}(\mu)d''^{-1}_{i_2,i_3}(\mu)$$
$$+k_{i_2,i_3}d^{-1}_{i_2,i_3}\}d_{i_1,i_3}d''_{i_1,i_3}(\mu)d_{i_2,i_3}d''_{i_2,i_3}(\mu)\delta(p''_{i_3}(\mu),p_{i_3})$$
$$/(k_{i_1}-k''_{i_1}(\mu)), \tag{2·22}$$

where the symbol $(i,j)\neq[i_1,i_2,i_3]$ implies that $(i,j)\neq[i_1,i_2]$, $[i_1,i_3]$ and $[i_2,i_3]$. If one takes the sums over p_{i_1,i_3}, $p''_{i_1,i_3}(\mu)$, p_{i_2,i_3} and $p''_{i_2,i_3}(\mu)$ by the same procedures as the ones mentioned below (2·15a, b) one has

$$\langle \Psi_{q_1',\cdots,q_n'} | \Psi_{q_1,\cdots,q_n}\rangle = \beta^*_{q_1',\cdots,q_n'}\beta_{q_1,\cdots,q_n}\sum_\mu \sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}} \sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}} \prod_{\substack{1\le i<j\le n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)$$
$$\times\prod_{l\neq i_1,i_2,i_3}\delta_{p_l''(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$
$$\times\prod_{1\le l<m\le 3}k''_{i_l,i_m}(\mu)k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$
$$\times\frac{1}{(k_{i_1}-k''_{i_1}(\mu))(k_{i_1}-k_{i_2})(k''_{i_1}(\mu)-k''_{i_2}(\mu))}$$
$$\times\left\{\left(\frac{1}{k''_{i_1}(\mu)-k''_{i_3}(\mu)}-\frac{1}{k_{i_1}-k_{i_3}}\right)\right.$$
$$\left.+\left(\frac{1}{k''_{i_2}(\mu)-k''_{i_3}(\mu)}-\frac{1}{k_{i_2}-k_{i_3}}\right)\right\}, \tag{2·23}$$

where it should be noted that all denominators in (2·23) do not vanish from (3·25) in I. Now let us show that in the curly bracket of (2·23) the terms $(1/(k''_{i_2}(\mu)-k''_{i_3}(\mu))-1/(k_{i_2}-k_{i_3}))$ can be replaced by $(1/2)(1/(k''_{i_1}(\mu)-k''_{i_3}(\mu))-1/(k_{i_1}-k_{i_3}))$ in the following way. For this purpose we denote the contribution from the first term $(1/(k''_{i_2}(\mu)-k''_{i_3}(\mu)))$ by A,

$$A=\beta^*_{q'_1\cdots q'_n}\beta_{q_1\cdots q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}}d_{i,j}d''_{i,j}(\mu)$$

$$\times\prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$

$$\times\prod_{1\leq l<m\leq 3}k''_{i_l,i_m}(\mu)k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times\frac{1}{(k_{i_1}-k''_{i_1}(\mu))(k_{i_1}-k_{i_2})(k''_{i_1}(\mu)-k''_{i_2}(\mu))(k''_{i_2}(\mu)-k''_{i_3}(\mu))}\,. \tag{2·24a}$$

Rewriting the summation over the permutation μ by that over $\mu'=\binom{1,\,2\cdots n}{\mu'_1\,\mu'_2\cdots\mu'_n}$,

$$\mu'_l=\mu_l\,,\quad(l\neq i_2,\,i_3);\quad \mu'_{i_2}=\mu_{i_3};\quad \mu'_{i_3}=\mu_{i_2}\,, \tag{2·25}$$

we have

$$A=\beta^*_{q'_1\cdots q'_n}\beta_{q_1\cdots q_n}\sum_{\mu'}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu')\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}}d_{i,j}d''_{i,j}(\mu')$$

$$\times\prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu'),p_l}(-1)(L/2mg)^4$$

$$\times\prod_{1\leq l<m\leq 3}k''_{i_l,i_m}(\mu')k_{i_l,i_m}(k''_{i_l}(\mu')-k''_{i_m}(\mu'))(k_{i_l}-k_{i_m})$$

$$\times\frac{1}{(k_{i_1}-k''_{i_1}(\mu'))(k_{i_1}-k_{i_2})(k''_{i_1}(\mu')-k''_{i_3}(\mu'))(k''_{i_3}(\mu')-k''_{i_2}(\mu'))}\,, \tag{2·24b}$$

since

$$\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}}d''_{i,j}(\mu')=\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}}d''_{i,j}(\mu),$$

$$\prod_{1\leq l<m\leq 3}k''_{i_l,i_m}(\mu')(k''_{i_l}(\mu')-k''_{i_m}(\mu'))=\prod_{1\leq l<m\leq 3}k''_{i_l,i_m}(\mu)(k''_{i_l}(\mu)-k''_{i_m}(\mu)), \tag{2·26}$$

and the summation over all μ' is equivalent to the one over all μ. Taking the sum of (2·24a) and (2·24b) and dividing it by 2, one has

$$A=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}}d_{i,j}d''_{i,j}(\mu)$$

$$\times\prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$

$$\times \prod_{1\leq l<m\leq 3} k''_{i_l,i_m}(\mu) k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times \frac{1/2}{(k_{i_1}-k''_{i_1}(\mu))(k_{i_1}-k_{i_2})(k''_{i_1}(\mu)-k''_{i_2}(\mu))(k''_{i_1}(\mu)-k''_{i_3}(\mu))} . \qquad (2\cdot 27\mathrm{a})$$

Next let us consider the contribution (we denote it by B) from the second term $(-1/(k_{i_2}-k_{i_3}))$. Write down its expression by firstly exchanging i_2 with i_3 and next rewrite the summation over μ by one over μ' in (2·25). The sum of this expression and the original one, multiplied by one half yields

$$B=\beta^*_{q'_1\cdots q'_n}\beta_{q_1\cdots q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\sum_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)$$

$$\times \prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$

$$\times \prod_{1\leq l<m\leq 3} k''_{i_l,i_m}(\mu) k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times \frac{-1/2}{(k_{i_1}-k''_{i_1}(\mu))(k_{i_1}-k_{i_2})(k''_{i_1}(\mu)-k''_{i_2}(\mu))(k_{i_1}-k_{i_3})} . \qquad (2\cdot 27\mathrm{b})$$

Substituting A and B in (2·23) instead of the second term in the curly bracket of (2·23), we have

$$\langle \Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)$$

$$\times \prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$

$$\times \prod_{1\leq l<m\leq 3} k''_{i_l,i_m}(\mu) k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times \frac{(1+1/2)[k_{i_1}-k''_{i_1}(\mu)-(k_{i_3}-k''_{i_3}(\mu)]}{(k_{i_1}-k''_{i_1}(\mu))(k_{i_1}-k_{i_2})(k''_{i_1}(\mu)-k''_{i_2}(\mu))(k_{i_1}-k_{i_3})(k''_{i_1}(\mu)-k''_{i_3}(\mu))} . \qquad (2\cdot 28)$$

Now let us construct another expression by exchanging i_2 with i_3 in (2·28). Then the sum of the result and the original expression (2·28), multiplied by one half, becomes

$$\langle \Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)$$

$$\times \prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$

$$\times \prod_{1\leq l<m\leq 3} k''_{i_l,i_m}(\mu) k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times \frac{(1+1/2)(1/2)[3(k_{i_1}-k''_{i_1}(\mu))-\sum_{l=1}^{3}(k_{i_l}-k''_{i_l}(\mu))]}{(k_{i_1}-k''_{i_1}(\mu))\prod_{l=2}^{3}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))} . \qquad (2\cdot 29)$$

Next we will intend to show that the contribution from the first term, $3(k_{i_1}-k''_{i_1}(\mu))$, in the bracket of (2·29) vanishes out. For this purpose let us consider the following two permutations μ' and μ'':

$$\mu'_l=\mu''_l=\mu_l\,, \qquad (l\neq i_1,\ i_2,\ i_3)$$

$$\mu'_{i_1}=\mu''_{i_2}=\mu_{i_3}\,, \qquad \mu'_{i_2}=\mu''_{i_3}=\mu_{i_1}\,, \qquad \mu'_{i_3}=\mu''_{i_1}=\mu_{i_2}\,. \tag{2·30}$$

In the same way as the derivation of (2·27a) from (2·24a), the contribution from the first term $3(k_{i_1}-k''_{i_1}(\mu))$, can be expressed as

$$\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)\prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(-1)(L/2mg)^4$$
$$\times\prod_{1\leq l<m\leq 3}k''_{i_l,i_m}(\mu)k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})\{D(\mu)+D(\mu')+D(\mu'')\}/3\,, \tag{2·31}$$

where

$$D(\mu)=\frac{(1/2)(1+1/2)\times 3}{(k_{i_1}-k_{i_2})(k_{i_1}-k_{i_3})(k''_{i_1}(\mu)-k''_{i_2}(\mu))(k''_{i_1}(\mu)-k''_{i_3}(\mu))}\,.$$

We can easily observe that

$$\frac{1}{3}(D(\mu)+D(\mu')+D(\mu''))=\frac{3}{4}\frac{1}{(k_{i_1}-k_{i_2})(k_{i_1}-k_{i_3})}\prod_{1\leq l<m\leq 3}\frac{1}{(k''_{i_l}(\mu)-k''_{i_m}(\mu))}$$
$$\times\{k''_{i_2}(\mu)-k''_{i_3}(\mu)+k''_{i_1}(\mu)-k''_{i_2}(\mu)+k''_{i_3}(\mu)-k''_{i_1}(\mu)\}=0\,.$$

This fact leads to

$$\langle\Psi_{q'_1\cdots q'_n}|\Psi_{q_1\cdots q_n}\rangle=\beta^*_{q'_1\cdots q'_n}\beta_{q_1\cdots q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,i_3]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,i_3]}} d_{i,j}d''_{i,j}(\mu)$$
$$\times\prod_{l\neq i_1,i_2,i_3}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}(L/2mg)^4$$
$$\times\prod_{1\leq l<m\leq 3}k''_{i_l,i_m}(\mu)k_{i_l,i_m}(k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$
$$\times\frac{(1+1/2)(1/2)\sum_{l=1}^{3}(k_{i_l}-k''_{i_l}(\mu))}{(k_{i_1}-k''_{i_1}(\mu))\prod_{l=2}^{3}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))}\,. \tag{2·32}$$

This is the result in Step 2. The result (2·40) in a general Step α will be established by mathematical induction in the following way.

[Step (α-1)] Let us assume that the expression of the scalar product in Step (α-1) is given by

$$\langle\Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2\\ \cdots\cdots\\ i_\alpha\neq i_1,i_2,\cdots,i_{\alpha-1}}}\frac{\alpha}{2^{\alpha-1}}\left(\frac{L}{2mg}\right)^{(\alpha-1)^2}$$

$$\times \sum_{\substack{\{p_{i,j}, p''_{i,j}(\mu)\} \\ (i,j)\neq[i_1,i_2,\cdots,i_\alpha]}} \prod_{\substack{1\leq i<j\leq n \\ (i,j)\neq[i_1,i_2,\cdots,i_\alpha]}} d_{i,j} d''_{i,j}(\mu) \prod_{l\neq i_1,i_2,\cdots,i_\alpha} \delta_{p'_l(\mu),p_l} \delta_{Q',Q}$$

$$\times \prod_{1\leq l<m\leq\alpha} k''_{i_l,i_m}(\mu) k_{i_l,i_m} (k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times \frac{\sum_{l=1}^{\alpha}(k_{i_l}-k''_{i_l}(\mu))}{(k_{i_1}-k''_{i_1}(\mu))\prod_{l=2}^{\alpha}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))} . \tag{2·33}$$

[Step α] First note the Formula A

$$\sum_{l=1}^{\alpha}(k_{i_l}-k''_{i_l}(\mu)) \prod_{\substack{l=1 \\ l\neq i_1,\cdots,i_\alpha}}^{n} \delta_{p'_l(\mu),p_l}\delta_{Q',Q}$$
$$= \sum_{\substack{i_{\alpha+1}=1 \\ i_{\alpha+1}\neq i_1\cdots i_\alpha}}^{n} \{\sum_{l=1}^{\alpha}(k_{i_l,i_{\alpha+1}} d^{-1}_{i_l,i_{\alpha+1}} - k_{i_l,i''_{\alpha+1}}(\mu) d''^{-1}_{i_l,i_{\alpha+1}}(\mu))\} \prod_{\substack{l=1 \\ l\neq i_1\cdots i_\alpha}}^{n} \delta_{p'_l(\mu),p_l}\delta_{Q',Q} , \tag{2·34}$$

of which the proof is given in the Appendix. Substitution of (2·34) in (2·33) and the summations over $p_{i_l,i_{\alpha+1}}$ and $p''_{i_l,i_{\alpha+1}}(\mu)$, $(l=1, 2, \cdots, \alpha)$ yield

$$\langle \Psi_{q'_1,\cdots,q'_n} | \Psi_{q_1,\cdots,q_n} \rangle = \beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n} \sum_{\mu} \sum_{\substack{i_2\neq i_1 \\ i_3\neq i_1,i_2 \\ \cdots\cdots \\ i_{\alpha+1}\neq i_1,i_2,\cdots,i_\alpha}} \sum_{\substack{\{p_{i,j}, p''_{i,j}(\mu)\} \\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}} \prod_{\substack{1\leq i<j\leq n \\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}} d_{i,j} d''_{i,j}(\mu)$$

$$\times \prod_{l\neq i_1,i_2,\cdots,i_{\alpha+1}} \delta_{p'_l(\mu),p_l}\delta_{Q',Q} \frac{\alpha}{2^{\alpha-1}} \left(\frac{L}{2mg}\right)^{(\alpha-1)^2}$$

$$\times \prod_{1\leq l<m\leq\alpha} k''_{i_l,i_m}(\mu) k_{i_l,i_m} (k''_{i_l}(\mu)-k''_{i_m}(\mu))(k_{i_l}-k_{i_m})$$

$$\times \frac{1}{k_{i_1}-k''_{i_1}(\mu)} \frac{1}{\prod_{l=2}^{\alpha}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))} \left(\frac{L}{2mg}\right)^{2\alpha-1}$$

$$\times \prod_{l=1}^{\alpha} k''_{i_l,i_{\alpha+1}}(\mu) k_{i_l,i_{\alpha+1}} (k''_{i_l}(\mu)-k''_{i_{\alpha+1}}(\mu))(k_{i_l}-k_{i_{\alpha+1}})$$

$$\times \sum_{l=1}^{\alpha}\left\{\frac{1}{k_{i_l}-k_{i_{\alpha+1}}} - \frac{1}{k''_{i_l}(\mu)-k''_{i_{\alpha+1}}(\mu)}\right\}$$

$$= \beta^*_{q'_1\cdots q'_n}\beta_{q_1\cdots q_n} \sum_{\mu} \sum_{\substack{i_2\neq i_1 \\ i_3\neq i_1,i_2 \\ \cdots\cdots \\ i_{\alpha-1}\neq i_1,i_2,\cdots,i_\alpha}} \Bigg[\sum_{\substack{\{p_{i,j}, p''_{i,j}(\mu)\} \\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}} \prod_{\substack{1\leq i<j\leq n \\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}} d_{i,j} d''_{i,j}(\mu)$$

$$\times \prod_{l\neq i_1,i_2,\cdots,i_{\alpha-1}} \delta_{p'_l(\mu),p_l}\delta_{Q',Q} \frac{\alpha}{2^{\alpha-1}} \left(\frac{L}{2mg}\right)^{\alpha^2}$$

$$\times \prod_{1\leq l<m\leq\alpha+1} k_{i_l,i_m} k''_{i_l,i_m}(\mu)(k_{i_l}-k_{i_m})(k''_{i_l}(\mu)-k''_{i_m}(\mu))$$

$$\times \frac{1}{(k_{i_1}-k''_{i_1}(\mu))\prod_{l=2}^{\alpha+1}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))} \Bigg]$$

$$\times \left\{ (k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu) - k_{i_1} + k_{i_{\alpha+1}}) + \sum_{l=2}^{\alpha} \left(\frac{(k_{i_1} - k_{i_{\alpha+1}})(k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu))}{(k_{i_l} - k_{i_{\alpha+1}})} - \frac{(k_{i_1} - k_{i_{\alpha+1}})(k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu))}{(k''_{i_l}(\mu) - k''_{i_{\alpha+1}}(\mu))} \right) \right\}. \qquad (2\cdot35)$$

In the same way as the transformation of (2·23) into (2·28), it is shown that each term below second term in the curly bracket of (2·35) can be transformed into half of the first term, $(1/2)(k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu) - k_{i_1} + k_{i_{\alpha+1}})$. In order to verify this fact, let us consider the permutation $\mu' = \binom{1,2\cdots n}{\mu_1'\mu_2'\cdots\mu_n'}$ defined as

$$\mu'_m = \mu_m\,, \qquad (m \neq i_l,\ i_{\alpha+1})$$

$$\mu'_{i_l} = \mu_{i_{\alpha+1}}\,, \qquad \mu'_{i_{\alpha+1}} = \mu_{i_l}\,,$$

where l indicates an any integer among $2, 3, \cdots, \alpha$. Now replace the sum over the permutation μ in the contribution from the second part $X(\mu) = -((k_{i_1} - k_{i_{\alpha+1}}) \times (k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu))/(k''_{i_l}(\mu) - k''_{i_{\alpha+1}}(\mu)))$ of the l-th term in the curly bracket of (2·35) by the one over the permutation μ'. Since all factors in the bracket of (2·35) remain unchanged for this replacement, the contribution from $X(\mu)$ is equal to the one obtained by the replacement of $X(\mu)$ by $(X(\mu) + X(\mu'))/2$, which is

$$(X(\mu) + X(\mu'))/2 = -(k_{i_1} - k_{i_{\alpha+1}})/2\,.$$

In the contribution from the first part $Y(\mu) = (k_{i_1} - k_{i_{\alpha+1}})(k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu))/(k_{i_l} - k_{i_{\alpha+1}})$ of the l-th term, on the other hand, first exchange i_l with $i_{\alpha+1}$, and second replace the sum over μ by the one over μ'. Then, it is permitted that the term $Y(\mu)$ in (2·35) is replaced by $(k''_{i_1}(\mu) - k''_{i_{\alpha+1}}(\mu))/2$. These results give

$$\langle \Psi_{q'_1,\cdots,q'_n} | \Psi_{q_1,\cdots,q_n} \rangle = \beta^*_{q'_1,\cdots,q'_n} \beta_{q_1,\cdots,q_n} \sum_{\mu} \sum_{\substack{i_2 \neq i_1 \\ i_3 \neq i_1, i_2 \\ \cdots\cdots \\ i_{\alpha+1} \neq i_1, i_2 \cdots i_\alpha}} \sum_{\substack{\{p_{i,j}, p''_{i,j}(\mu)\} \\ (i,j) \neq [i_1, i_2, \cdots, i_{\alpha+1}]}}$$

$$\times \Bigg[\prod_{\substack{1 \leq i < j \leq n \\ (i,j) \neq [i_1 \cdots i_{\alpha+1}]}} d_{i,j} d''_{i,j}(\mu) \prod_{l \neq i_1, \cdots, i_{\alpha+1}} \delta_{p''_l(\mu), p_l} \delta_{Q',Q}$$

$$\times \frac{\alpha(\alpha+1)}{2^\alpha} \left(\frac{L}{2mg} \right)^{\alpha^2} \frac{\prod_{1 \leq l < m \leq \alpha+1} k_{i_l, i_m} k''_{i_l, i_m}(\mu)(k_{i_l} - k_{i_m})(k''_{i_l}(\mu) - k''_{i_m}(\mu))}{(k_{i_1} - k''_{i_1}(\mu)) \prod_{l=2}^{\alpha+1} (k_{i_1} - k_{i_l})(k''_{i_1}(\mu) - k''_{i_l}(\mu))} \Bigg]$$

$$\times \{ (k''_{i_1}(\mu) - k_{i_1}) + (k_{i_{\alpha+1}} - k''_{i_{\alpha+1}}(\mu)) \}. \qquad (2\cdot36)$$

Next let us consider the symmetrization of the term $(k_{i_{\alpha+1}} - k''_{i_{\alpha+1}}(\mu))$ with respect to the suffixes $i_2, i_3, \cdots, i_{\alpha+1}$, as follows. Note that all factors in the bracket of (2·36) remain unchanged under the exchange of $i_l(2\leq l\leq\alpha)$ with $i_{\alpha+1}$. Summing the $(\alpha-1)$ expressions obtained by those exchanges and the original one (2·36), and dividing the sum by α, we have

$$\langle \Psi_{q'_1\cdots q'_n}|\Psi_{q_1\cdots q_n}\rangle$$

$$=\beta^*_{q'_1\cdots q'_n}\beta_{q_1\cdots q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2\\ \cdots\cdots\\ i_{\alpha+1}\neq i_1,i_2,\cdots,i_\alpha}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}}d_{i,j}d''_{i,j}(\mu)$$

$$\times\prod_{l\neq i_1,i_2,\cdots,i_{\alpha+1}}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}\frac{\alpha+1}{2^\alpha}\left(\frac{L}{2mg}\right)^{\alpha^2}$$

$$\times\frac{\prod_{1\leq l<m\leq\alpha+1}k_{i_l,i_m}k''_{i_l,i_m}(\mu)(k_{i_l}-k_{i_m})(k''_{i_l}(\mu)-k''_{i_m}(\mu))}{\prod_{l=2}^{\alpha+1}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_2}(\mu))}$$

$$\times\frac{\{(\alpha+1)(k''_{i_1}(\mu)-k_{i_1})+\sum_{l=1}^{\alpha+1}(k_{i_l}-k''_{i_l}(\mu))\}}{(k_{i_1}-k''_{i_1}(\mu))}. \qquad (2\cdot37)$$

Here we turn our attention to the contribution from the first term in the curly bracket of (2·37). In the same way as the derivation of (2·32) from (2·29), let us consider α permutations $\mu^{(2)}, \mu^{(3)}, \cdots, \mu^{(\alpha+1)}$ defined as

$$\mu^{(s)}_m=\mu_m\,,\quad (m\neq i_1,\ i_s);\quad \mu^{(s)}_{i_1}=\mu_{i_s};\quad \mu^{(s)}_{i_s}=\mu_{i_1}\,.$$

Then the same consideration as in the case of (2·31) leads to

$$\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2\\ \cdots\cdots\\ i_{\alpha+1}\neq i_1,i_2,\cdots,i_\alpha}}\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,\cdots,i_{\alpha+1}]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,\cdots,i_{\alpha+1}]}}d_{i,j}d''_{i,j}(\mu)$$

$$\times\prod_{l\neq i_1,\cdots,i_{\alpha+1}}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}\frac{\alpha+1}{2^\alpha}\left(\frac{L}{2mg}\right)^{\alpha^2}$$

$$\times\frac{\prod_{1\leq l<m\leq\alpha+1}k_{i_l,i_m}k''_{i_l,i_m}(\mu)(k_{i_l}-k_{i_m})(k''_{i_l}(\mu)-k''_{i_m}(\mu))}{\prod_{l=2}^{\alpha+1}(k_{i_1}-k_{i_l})}$$

$$\times\left[\sum_{s=1}^{\alpha+1}\frac{-1}{\prod_{\substack{l=1\\ l\neq s}}^{\alpha+1}(k''_{i_s}(\mu)-k''_{i_l}(\mu))}\right] \qquad (2\cdot38)$$

for the contribution from the first term in the curly bracket of (2·37). We can observe here that

$$\sum_{s=1}^{\alpha+1}\frac{1}{\prod_{\substack{l=1\\ l\neq s}}^{\alpha+1}(k''_{i_s}(\mu)-k''_{i_l}(\mu))}=0\,,$$

by taking the limit of infinitely large x for the identity

$$\frac{x}{\prod_{s=1}^{\alpha+1}(x-k''_{i_s}(\mu))}=\sum_{s=1}^{\alpha+1}\frac{x}{x-k''_{i_s}(\mu)}\times\frac{1}{\prod_{\substack{l=1\\ l\neq s}}^{\alpha+1}(k''_{i_s}(\mu)-k''_{i_l}(\mu))}\,. \quad (2\cdot39)$$

Thus the contribution from the first term in the curly bracket of (2·37) becomes zero, and hence we have

$$\begin{aligned}\langle\Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle&=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2\\ \cdots\cdots\\ i_{\alpha+1}\neq i_1,\cdots,i_\alpha}}\frac{\alpha+1}{2^\alpha}\left(\frac{L}{2mg}\right)^{\alpha^2}\\&\times\sum_{\substack{\{p_{i,j},p''_{i,j}(\mu)\}\\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}}\prod_{\substack{1\leq i<j\leq n\\ (i,j)\neq[i_1,i_2,\cdots,i_{\alpha+1}]}}d_{i,j}d''_{i,j}(\mu)\prod_{l\neq i_1,\cdots,i_{\alpha+1}}\delta_{p''_l(\mu),p_l}\delta_{Q',Q}\\&\times\prod_{1\leq l<m\leq\alpha+1}k_{i_l,i_m}k''_{i_l,i_m}(\mu)(k_{i_l}-k_{i_m})(k''_{i_l}(\mu)-k''_{i_m}(\mu))\\&\times\frac{\sum_{l=1}^{\alpha+1}(k_{i_l}-k''_{i_l}(\mu))}{(k_{i_1}-k''_{i_1}(\mu))\prod_{l=2}^{\alpha+1}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))}\,.\end{aligned} \quad (2\cdot40)$$

In this way the expression (2·40) in Step α has been established.

By mathematical induction the result in final Step $(n-1)$ is given by

$$\begin{aligned}&\langle\Psi_{q'_1,\cdots,q'_n}|\Psi_{q_1,\cdots,q_n}\rangle\\&=\beta^*_{q'_1,\cdots,q'_n}\beta_{q_1,\cdots,q_n}\sum_{\mu}\sum_{\substack{i_2\neq i_1\\ i_3\neq i_1,i_2\\ \cdots\cdots\\ i_n\neq i_1,i_2,\cdots,i_{n-1}}}\frac{n}{2^{n-1}}\left(\frac{L}{2mg}\right)^{(n-1)^2}\delta_{Q',Q}\\&\times\prod_{1\leq l<m\leq n}k_{i_l,i_m}k''_{i_l,i_m}(\mu)(k_{i_l}-k_{i_m})(k''_{i_l}(\mu)-k''_{i_m}(\mu))\\&\times\frac{\sum_{l=1}^{n}(k_{i_l}-k''_{i_l}(\mu))}{(k_{i_1}-k''_{i_1}(\mu))\prod_{l=2}^{n}(k_{i_1}-k_{i_l})(k''_{i_1}(\mu)-k''_{i_l}(\mu))}\\&=0\,,\end{aligned} \quad (2\cdot41)$$

where the final equality is due to the fact that $\delta_{Q',Q}(\sum_{l=1}^{n}(k_{i_l}-k''_{i_l}(\mu)))=\delta_{Q',Q}(Q-Q')=0$.

Thus we can conclude that two states with different sets of momenta $(q'_1,\cdots,q'_n)$ and $(q_1,\cdots,q_n)$ are orthogonal to each other. It should be noted, on the other hand, that the normalization factor is not determined*) by the discussion mentioned above, and this problem remains an open question.

§ 3. Ground state and low-lying excitations

In this section we will show that the state with minimum energy is given by $|\Psi_{0,0,\cdots,0}\rangle$ and will investigate the low-lying excitations in a simple case of infinitely large g.

3.1. *Ground state*

The energy eigenvalue (2·27) in I can be expressed in terms of the $(n-1)$ parameters $\Delta_l(s_1, s_2, \cdots, s_{n-1})$ introduced in (3·4) of I. In order to show this fact, let us first note that the parameters $k_i(i=1, 2, \cdots, n)$ are given as

$$k_i=\frac{Q}{n}+\frac{1}{n}\sum_{l=1}^{n}(k_i-k_l)=\frac{Q}{n}+\frac{1}{n}\sum_{l=1}^{n-1}\chi_l{}^i\Delta_l(s_1, s_2, \cdots, s_{n-1}), \tag{3·1}$$

and

$$\chi_l{}^i=\begin{cases} l & \text{for} \quad l\leqq i-1\,, \\ -(n-l) & \text{for} \quad l\geqq i\,, \end{cases} \tag{3·2}$$

where Q indicates the total momentum $(q_1+q_2+\cdots+q_n)$ and equals the sum $(k_1+k_2+\cdots+k_n)$ from (3·13) of I. Then the expression (2·27) in I for the energy eigenvalue is rewritten as

$$E_{q_1,q_2,\cdots,q_n}=\frac{1}{2mn^2}\{nQ^2$$
$$+\sum_{i=1}^{n}\sum_{l,j=1}^{n-1}\chi_l{}^i\chi_j{}^i\Delta_l(s_1, \cdots, s_{n-1})\Delta_j(s_1, \cdots, s_{n-1})\}, \tag{3·3}$$

where use has been made of the fact that $\sum_{i=1}^{n}\sum_{l=1}^{n-1}\chi_l{}^i\Delta_l=\sum_{i=1}^{n}\sum_{l=1}^{n}(k_i-k_l)=0$. The definition (3·2) for $\chi_l{}^i$ yields

$$\sum_{i=1}^{n}\chi_l{}^i\chi_j{}^i=\begin{cases} n(n-j)l & \text{for} \quad l\leqq j\,, \\ n(n-l)j & \text{for} \quad l\geqq j\,. \end{cases} \tag{3·4}$$

From (3·4), the energy eigenvalue expressed in terms of the parameters $\Delta_l(s_1, s_2, \cdots, s_{n-1})$ is given by

$$E_{q_1,q_2,\cdots,q_n}=\frac{Q^2}{2mn}+\varepsilon(s_1, s_2, \cdots, s_{n-1}) \tag{3·5}$$

*) In the case $g\to\infty$, the normalization factor can be determined, as will be shown in a subsequent paper.[4)]

and

$$\varepsilon(s_1, s_2, \cdots, s_{n-1})=\frac{1}{2mn}\{\sum_{l=1}^{n-1}(n-l)l\Delta_l^2(s_1, \cdots, s_{n-1})$$
$$+\sum_{1\leq l<j\leq n-1}2(n-j)l\Delta_l(s_1, \cdots, s_{n-1})\Delta_j(s_1, \cdots, s_{n-1})\}, \tag{3·6}$$

where the first term in (3·5) evidently indicates the kinetic energy of the center of mass, and the second one implies the energy of the system about the center of mass. The energy eigenvalue (3·5) is completely specified by the total momentum Q and the $(n-1)$ quantities $s_i=q_{i+1}-q_i$ as readily seen from (3·5) and (3·6). In this sense, the $(n-1)$ quantities s_i can be regarded as internal quantum numbers.

In (3·6), all coefficients in front of the product of two parameters Δ_i are positive integers, and further the parameters Δ_i are positive and have the minimum values $\Delta_i(0, 0, \cdots, 0)$ at all $s_i=0$ from (3·33) in I. Hence we have

$$\varepsilon(s_1, s_2, \cdots, s_{n-1})\geqq\varepsilon(0, 0, \cdots, 0), \tag{3·7}$$

where the equality holds only for the case when all $s_i=0$, namely only when $q_1=q_2=\cdots=q_n$. Thus it can be readily seen from (3·5) and (3·7) that the total energy (3·5) takes minimum value $\varepsilon(0, 0, \cdots, 0)$ only when $Q=0$ and $s_1=s_2=\cdots=s_{n-1}=0$, that is, $q_1=q_2=\cdots=q_n=0$. Hence the ground state of our interacting system is given by a single state $|\Psi_{0,0,\cdots,0}\rangle$. It will be shown in a subsequent paper[4] that the ground state is a condensed state of bose particles with zero momentum which are exactly dressed with interaction cloud.

3.2. *Low-lying excitations*

The investigation of excitation energy can be achieved in principle by solving the simultaneous equations (3·12) in I for various sets of quantum numbers $(q_1, q_2, \cdots, q_n)$. They, however, are transcendental equations, and hence we are forced to solve them numerically. In order to avoid this unfortunate situation, we restrict ourselves to the simple case of infinitely large g. For this case, we will investigate in detail the excitation energy. In this investigation, we intend to discuss the Type II excitation which has been pointed out by Lieb,[5] and further to clarify the appropriateness of the quantum numbers $(q_1, q_2, \cdots, q_n)$ for the description of energy levels.

In the case of infinitely large g, the quantities k_i are given by (2·30) in I. For given quantum numbers,

$$q_1\leqq q_2\leqq\cdots\leqq q_n\ , \tag{3·8}$$

the eigenenergy is written as

$$E_{q_1,q_2,\cdots,q_n}=E_G+\frac{1}{2m}\left\{\sum_{i=1}^{n}\frac{2\pi\hbar}{L}(2i-n-1)q_i+\sum_{i=1}^{n}q_i{}^2\right\}, \tag{3·9}$$

from (2·27) and (2·30) in I, where E_G indicates the ground state energy

$$E_G=n\frac{\hbar^2}{2m}\frac{\pi^2}{3}\left\{\left(\frac{n}{L}\right)^2-\frac{1}{L^2}\right\}. \tag{3·10}$$

i) Single excitation

The restriction (3·8) for quantum numbers leads to two cases of single excitation, that is, ($q_1=0$, $q_2=0$, $\cdots$, $q_{n-1}=0$, $q_n=p$) or ($q_1=-p$, $q_2=0$, $q_3=0$, $\cdots q_n=0$) for $p>0$. From (3·9), the energies for these cases become

$$E_{0,0,\cdots,0,p}=E_G+\varepsilon_p \tag{3·11a}$$

and

$$E_{-p,0,\cdots,0}=E_G+\varepsilon_{-p}\ , \tag{3·11b}$$

respectively, where

$$\varepsilon_p=\frac{\pi\hbar}{m}\frac{(n-1)}{L}|p|+\frac{p^2}{2m}\ , \tag{3·12a}$$

which indicates the energy of single excitation. In the limit of $n\to\infty$ and $L\to\infty$ for a fixed $\rho=n/L$, one has

$$\varepsilon_p=\frac{\pi\hbar}{m}\rho|p|+\frac{p^2}{2m}\ . \tag{3·12b}$$

The spectrum (3·12b) has evidently phonon character in the region of small momentum p, and the phonon velocity c is given by

$$c=\frac{\pi\hbar}{m}\rho\ . \tag{3·13}$$

This result that there appears phonon spectrum in low-lying excitation has been pointed out by Girardeau[3] and Lieb[2] by analogy with the excitation of a fermion from the surface of Fermi-sphere in non-interacting many-fermion system.

ii) Double excitation

The restriction (3·8) for quantum numbers gives three cases for double excitation, and the energy and total momentum in each cases are given as follows:
(Cese a) ($q_1=q_2=\cdots=q_{n-2}=0$, $q_{n-1}=p_1$, $q_n=p_2$; $0<p_1\leqq p_2$)

$$E_{0,0,\cdots 0,p_1,p_2}=E_G+\varepsilon_{p_1}+\varepsilon_{p_2}-\frac{\hbar}{2m}\frac{4\pi}{L}p_1 \tag{3·14a}$$

and

$$Q = p_1 + p_2 .$$

(Case b) $(q_1 = -p_2,\ q_2 = -p_1,\ q_3 = q_4 = \cdots = q_n = 0;\ 0 < p_1 \leqq p_2)$

$$E_{-p_2, -p_1, 0, 0, \cdots, 0} = E_G + \varepsilon_{p_1} + \varepsilon_{p_2} - \frac{\hbar}{2m}\frac{4\pi}{L} p_1 \tag{3·14b}$$

and

$$Q = -(p_1 + p_2).$$

(Case c) $(q_1 = -p_1,\ q_2 = q_3 = \cdots = q_{n-1} = 0,\ q_n = p_2;\ p_1 > 0,\ p_2 > 0)$

$$E_{-p_1, 0, 0, \cdots, 0, p_2} = E_G + \varepsilon_{p_1} + \varepsilon_{p_2} \tag{3·14c}$$

and

$$Q = p_2 - p_1 .$$

As readily seen from (3·14a, b) the excitation energy for double excitation involves the term $-2\pi\hbar p_1/mL$ which comes from the effect of interaction. Since this term vanishes out in the limit of $n \to \infty$ and $L \to \infty$ for a fixed $\rho = n/L$, the excitation energy becomes the sum of two single excitation energies, $\varepsilon_{p_1} + \varepsilon_{p_2}$.

iii) Multiple excitation

The multiple excitation, in which l quantum numbers have nonzero values, has the following three cases:

(Case a) $(q_1 = q_2 = \cdots = q_{n-1} = 0,\ q_{n-l+1} = p_1,\ q_{n-l+2} = p_2, \cdots, q_n = p_l;\ 0 < p_1 \leqq p_2 \cdots \leqq p_l)$

$$E_{0,0,\cdots,p_1,p_2,\cdots,p_l} = E_G + \frac{1}{2m}\left\{\sum_{i=1}^{l} \frac{2\pi\hbar}{L}(n - 2l + 2i - 1) p_i + \sum_{i=1}^{l} p_i^2\right\},$$

$$Q = \sum_{i=1}^{l} p_i . \tag{3·15a}$$

(Case b) $(q_1 = p_1, \cdots, q_l = p_l,\ q_{l+1} = \cdots = q_n = 0;\ p_1 \leqq \cdots \leqq p_l < 0)$

$$E_{p_1,p_2,\cdots,p_l,0,\cdots,0} = E_G + \frac{1}{2m}\left\{\sum_{i=1}^{l} \frac{2\pi\hbar}{L}(2i - n - 1) p_i + \sum_{i=1}^{l} p_i^2\right\},$$

$$Q = \sum_{i=1}^{l} p_i . \tag{3·15b}$$

(Case c) $(q_1 = p_1, \cdots, q_j = p_j,\ q_{j+1} = \cdots = q_{n-l+j} = 0,\ q_{n-l+j+1} = p_{j+1}, \cdots, q_n = p_l;\ p_1 \leqq p_2 \leqq \cdots \leqq p_j < 0 < p_{j+1} \leqq \cdots \leqq p_l,\ j < l)$

$$E_{p_1,p_2,\cdots,p_j,0,\cdots,0,p_{j+1},\cdots,p_l} = E_G + \frac{1}{2m}\left\{\sum_{i=1}^{j} \frac{2\pi\hbar}{L}(2i - n - 1) p_i \right.$$

$$\left. + \sum_{i=j+1}^{l} \frac{2\pi\hbar}{L}(n - 2l + 2i - 1) p_i + \sum_{i=1}^{l} p_i^2\right\},$$

$$Q=\sum_{i=1}^{l} p_i \,. \tag{3·15c}$$

When l/n can be ignored in the limit of $n\to\infty$ and $L\to\infty$ for a fixed density $\rho = n/L$, the total energy in each case becomes

$$(\text{Total Energy})=E_G+\sum_{i=1}^{l}\varepsilon_{p_i}\,, \tag{3·16}$$

which indicates that the interaction energies among multiple excitations can be neglected in the case $l/n\to 0$. For small momenta p_i, the total energy (3·16) is reduced to

$$(\text{Total Energy})=E_G+\sum_{i=1}^{l}c|p_i|. \tag{3·17}$$

The expression (3·17) means evidently that a multiple excitation is composed of multiple phonons. When l/n cannot be ignored in the above limit, on the other hand, the interaction energies among multiple excitations become so important that they give rise to a drastic change in the excitation energy. This will be shown in a forthcoming paper.[6)]

By analogy with the excitations of Fermi gas, Lieb has proposed type I, type II and umklapp excitations as the elementary excitations in the interacting many boson system. As has been stated by himself,*) however, there are some difficulties.**) These difficulties come from the unnatural viewpoint that the excitations in the bose system has been analyzed by analogy with the ones of Fermi gas in spite of bose system (although the authors think that this viewpoint is obliged to be taken by a historical situation).

If this unfortunate viewpoint is changed to our standpoint, there does not appear those difficulties. This can be seen as follows. It will be shown in our subsequent paper that new creation $A_p{}^*$ and annihilation A_p operators can be introduced by using a unitary transformation U,

$$A_p{}^*=Ua_p{}^*U^*, \qquad A_p=Ua_pU^*$$

and of course satisfy the usual Bose commutation relation. Then, the exact excited state with l excitations with momenta $p_1, p_2, \cdots, p_l$ is expressed as

$$|\text{exact excited state}\rangle=A_{p_1}^*A_{p_2}^*\cdots A_{p_l}^*(A_0{}^*)^{n-l}|0\rangle, \tag{3·18}$$

*) See page 1618 in Lieb's paper.[5)]

**) Another difficulties can be pointed out as follows. In Lieb's classification scheme, the excited state with two excitations, which is constructed by bringing about a excitation of momentum p after exciting one with q is different from the excited state constructed by the opposite order. Furthermore, after a type I excitation with momentum $p(>0)$ has been excited, it is forbidden to excite one more type I excitation with $p+2\pi\hbar/L$ in the case $g\to\infty$. This means that Lieb's excitations have no Bose nature.

in terms of the creation operators A_p^*. Thus, by virtue of Bose nature of A_p^*, the difficulty mentioned in the footnote*) has been overcome. Furthermore it is evident that there do not**) appear such other difficulties as stated by Lieb.

As readily seen from (3·18) the quantum numbers q_i introduced in our paper imply the momenta of the exactly dressed bose particles created by new operators A_p^*, and, therefore, they are appropriate quantum numbers for description of the many boson system.***)

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Appendix

Formula A: Let the set $\{i_\beta; \beta=1, 2, \cdots, \alpha\}$ to be any subset of the integers 1, 2, $\cdots$, n. Then, the following relation holds:

$$\sum_{\beta=1}^{\alpha}(k_{i_\beta}-k''_{i_\beta}(\mu)) \prod_{\substack{l=1\\ l\neq i_1,\cdots,i_\alpha}}^{n} \delta_{p''_l(\mu),p_l}\delta_{Q',Q}$$

$$= \sum_{\substack{j=1\\ j\neq i_1,\cdots,i_\alpha}}^{n} \sum_{\beta=1}^{\alpha}\{k_{i_\beta,j}d^{-1}_{i_\beta,j}-k''_{i_\beta,j}(\mu)d''^{-1}_{i_\beta,j}(\mu)\} \prod_{\substack{l=1\\ l\neq i_1,\cdots,i_\alpha}}^{n} \delta_{p''_l(\mu),p_l}\delta_{Q',Q}\,, \tag{A·1}$$

where $Q'=\sum_{i=1}^{n}q'_i$ and $Q=\sum_{i=1}^{n}q_i$.

This is justified as follows. First note that

$$0=[Q'-Q-\sum_{\substack{l=1\\ l\neq i_1,\cdots,i_\alpha}}^{n}(p''_l(\mu)-p_l)] \prod_{\substack{l=1\\ l\neq i_1,\cdots,i_\alpha}}^{n} \delta_{p''_l(\mu),p_l}\delta_{Q',Q}\,. \tag{A·2}$$

By using the facts that $Q'=\sum_{l=1}^{n}p''_l(\mu)$ and $Q=\sum_{n=1}^{n}p_l$, the right-hand side of (A·2) can be rewritten as follows:

$$0=\sum_{\beta=1}^{\alpha}[(\sum_{\substack{j=1\\ j\neq i_\beta}}^{n} p''_{i_\beta,j}(\mu)+q''_{i_\beta}(\mu))-(\sum_{\substack{j=1\\ j\neq i_\beta}}^{n} p_{i_\beta,j}+q_{i_\beta})] \prod_{\substack{l=1\\ l\neq i_1,\cdots,i_\alpha}}^{n} \delta_{p''_l(\mu),p_l}\delta_{Q',Q}$$

---

*) See the footnote on p. 1234.

**) As readily seen from (3·18), our each excitation is specified by only one momentum p of the operators A_p^*, and the momentum p has no provisos in contrast with Lieb's classification scheme (see page 1618 in Lieb's paper[5]).

***) On the other hand, Lieb and Liniger's quantum numbers n_i are related to our quantum numbers as

$$n_i=\frac{L}{2\pi h}(q_{i+1}-q_i)+1\,,$$

and they have no physical meaning.

$$
= \sum_{\beta=1}^{\alpha} \Big[\sum_{\substack{j=1 \\ j \neq i_\beta}}^{n} \{ p''_{i_\beta,j}(\mu) - k''_{i_\beta,j}(\mu)) - (p_{i_\beta,j} - k_{i_\beta,j}) \}
$$

$$
+ (k''_{i_\beta}(\mu) - k_{i_\beta}) \Big] \prod_{\substack{l=1 \\ l \neq i_1, \cdots, i_\alpha}}^{n} \delta_{p''_l(\mu), p_l} \delta_{Q',Q}
$$

$$
= \sum_{\beta=1}^{\alpha} \Big[\sum_{\substack{j=1 \\ j \neq i_\beta}}^{n} \{ k_{i_\beta,j} d_{i_\beta,j}^{-1} - k''_{i_\beta,j}(\mu) d''^{-1}_{i_\beta,j}(\mu) \} + (k''_{i_\beta}(\mu) - k_{i_\beta}) \Big] \prod_{\substack{l=1 \\ l \neq i_1, \cdots, i_\alpha}}^{n} \delta_{p''_l(\mu), p_l} \delta_{Q',Q} \,, \tag{A·3}
$$

where we have used the definitions (2·20) and (2·22) in I for $d_{i,j} \equiv d(p_{i,j};\, k_{i,j})$ and k_i, respectively. We can observe that

$$
\sum_{\beta=1}^{\alpha} \sum_{\substack{j=1 \\ j \neq i_\beta}}^{n} \{ k_{i_\beta,j} d_{i_\beta,j}^{-1} - k''_{i_\beta,j}(\mu) d''^{-1}_{i_\beta,j}(\mu) \}
$$

$$
= \sum_{\substack{j=1 \\ j \neq i_1, i_2, \cdots, i_\alpha}}^{n} \sum_{\beta=1}^{\alpha} \{ k_{i_\beta,j} d_{i_\beta,j}^{-1} - k''_{i_\beta,j}(\mu) d''^{-1}_{i_\beta,j}(\mu) \}, \tag{A·4}
$$

holds according to the antisymmetricity of $k_{i,j}$ and $k''_{i,j}(\mu)$, and the symmetric property of $d_{i,j}$ and $d''_{i,j}(\mu)$ for exchange of their suffixes. Substitution of (A·4) in (A·3) gives Formula A.

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