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One-Dimensional Many Boson System. IV

—— *Condensation and Excitation Energy* ——

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The analysis for a one-dimensional many boson system with a repulsive delta-function potential is continued. It is shown that the expectation value of the number $a_0^*a_0$ of bare bosons with zero momentum for the exact ground state $|\Psi_{0,0,\cdots,0}\rangle$ is negligibly small compared with the total number n in the limit of an infinitely large coupling constant g; i.e., $\langle\Psi_{0,\cdots,0}|a_0^*a_0|\Psi_{0,\cdots,0}\rangle/n\to 0$(for $n\to\infty$, $g\to\infty$). This result shows a striking contrast to the result $\langle\Psi_{0,\cdots,0}|A_0^*A_0|\Psi_{0,\cdots,0}\rangle/n=1$ for an arbitrary g, where $A_0^*A_0$ indicates the number operator of exactly dressed bosons with zero momentum. It is clarified that the energy of a dressed boson strongly depends upon the number distribution $\{n_q\}$ of the other dressed bosons in the system. The excitation energy of a dressed boson has phonon character when $n_0/n\neq 0$, where n_0 denotes the number of dressed bosons with zero momentum. *When n_0/n tends to zero, the phonon character disappears in a drastic way.* Some characteristic phenomena in liquid helium II are discussed on the basis of the results of our previous papers and the present paper.

§1. Introduction

In a one-dimensional many boson system described by the Hamiltonian,

$$H=\sum_p\frac{p^2}{2m}a_p^*a_p+\sum_{p,q,r}\frac{g}{2L}a_{p+r}^*a_{q-r}^*a_qa_p\,,\quad (g>0)\tag{1·1}$$

the eigenvalue problem has been solved exactly in previous papers[1),2)] (hereafter referred to as I, II). For arbitrary quantum numbers $q_1\leqq q_2\leqq\cdots\leqq q_n$, the eigenstate and eigenenergy are given by

$$|\Psi_{q_1,q_2,\cdots,q_n}\rangle=\beta_{q_1,\cdots,q_n}\sum_{\{p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n}d(p_{i,j};k_{i,j})\prod_{i=1}^n a^*_{\sum_{j=1,j\neq i}^n p_{i,j}+q_i}|0\rangle\,,\tag{1·2}$$

$$E_{q_1,q_2,\cdots,q_n}=\sum_{i=1}^n(1/2m)k_i^2\,,\quad k_i=\sum_{\substack{j=1\\j\neq i}}^n k_{i,j}+q_i\,,\quad (i=1,\cdots,n)\tag{1·3}$$

where $d(p_{i,j};k_{i,j})=-k_{i,j}/(p_{i,j}-k_{i,j})$, $p_{j,i}=-p_{i,j}$, $k_{j,i}=-k_{i,j}$ $(1\leqq i<j\leqq n)$, $k_{i,j}$ are uniquely determined by (2·26) in I, and $\beta_{q_1,\cdots,q_n}$ denotes the normalization constant. In the limit $g\to\infty$, $k_{i,j}$ and $\beta_{q_1,\cdots,q_n}$ reduce to

$$k_{i,j}=-(\pi\hbar/L)\varepsilon_{i,j}\,,\qquad \varepsilon_{i,j}=(j-i)/|j-i|,\quad (i\neq j)\tag{1·4}$$

$$\beta_{q_1,\cdots,q_n}=(\pi/2)^{-n(n-1)/2}\,.\tag{1·5}$$

In our third paper[3] (hereafter referred to as III), a unitary operator U, which transforms free eigenstates for a noninteracting system into the exact eigenstates (1·2), has been successfully constructed (see (3·10a), (2·3a), (2·2) in III) in the form as

$$U=1+\sum_{n=2}^{\infty}\left\{U_n+\sum_{l=2}^{n-1}\frac{(-1)^{n-l}}{(n-l)!}\sum_{p_1,\cdots,p_{n-l}}\prod_{i=1}^{n-l}a_{p_i}^*U_l\prod_{i=1}^{n-l}a_{p_i}\right.$$
$$\left.-\frac{(-1)^n(n-1)}{n!}\sum_{p_1,\cdots,p_n}\prod_{i=1}^{n}a_{p_i}^*\prod_{i=1}^{n}a_{p_i}\right\}, \tag{1·6}$$

where

$$U_n=\sum_{q_1\leq q_2\leq\cdots\leq q_n}\sum_{\{p_{i,j};1\leq i<j\leq n\}}\beta_{q_1,\cdots,q_n}\alpha_{q_1,\cdots,q_n}\prod_{1\leq i<j\leq n}d(p_{i,j};k_{i,j})\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}\prod_{i=1}^{n}a_{q_i}. \tag{1·7}$$

The unitary transformation leads to the introduction of the creation and annihilation operators of an exactly drossed particle with momentum p, as

$$A_p^*=Ua_p^*U^*\,,\qquad A_p=Ua_pU^*\,. \tag{1·8}$$

Then, the eigenstate (1·2) is simply expressed in terms of the dressed operators A_q^* by (see (4·11) in III)

$$|\Psi_{q_1,q_2,\cdots,q_n}\rangle=\alpha_{q_1,q_2,\cdots,q_n}\prod_{i=1}^{n}A_{q_i}^*|0\rangle\equiv\|q_1,\,q_2,\,\cdots,\,q_n\gg\,. \tag{1·9}$$

From (4·21) in III, the exact ground state $|\Psi_{0,0,\cdots,0}\rangle$ is expressed by $\sqrt{1/n\,!}\,A_0^{*n}|0\rangle$, which indicates evidently the condensed state of n *exactly dressed bosons* with zero momentum, that is,

$$\langle\Psi_{0,\cdots,0}|A_0^*A_0|\Psi_{0,\cdots,0}\rangle/n=1\,. \tag{1·10}$$

On the other hand, Lenard et al.[4),5)] have shown that, in the limit $g\to\infty$, the zero momentum condensation for *bare bosons* does not exist, i.e.,

$$\lim_{g\to\infty}\langle\Psi_{0,\cdots,0}|a_0^*a_0|\Psi_{0,\cdots,0}\rangle/n=0\,,\quad(L\to\infty\text{ for fixed }n/L) \tag{1·11}$$

although their calculations for (1·11) have been carried out in the first quantized form.[6)] In § 2, the result (1·11) is confirmed from the second quantized form. Section 3 is devoted to showing that the unitary operator U is invariant under any Galilei-transformation. Owing to this fact, the dressed operator A_p^* has the same transformation property, $G(v)A_p^*G^*(v)=A_{p+mv}^*$, as that of the bare operator a_p^*, $G(v)a_p^*G^*(v)=a_{p+mv}^*$, where $G(v)$ indicates a Galilei-transformation operator with a relative velocity v.

The main purpose of this paper is to investigate the properties of the excitation energy in detail. As will be shown in § 4, the phonon character of the excitation energy is lost with disappearance of the condensate of the dressed bosons. In this case we can see that the functional form of the excitation energy versus momentum depends sensitively upon the distribution of dressed bosons. In the final section, we will discuss some characteristic properties of liquid helium on the basis of the present theory.

§ 2. Number distribution of bare bose particles

In this section, we will calculate the expectation value of $a_0^* a_0$ for the ground state $|\Psi_{0,0,\cdots,0}\rangle$ in the case $g\to\infty$. The expectation value is shown to be expressed in the form of Toeplitz determinant.[7] By using Lenard's arguments,[4] this determinant becomes zero in the limit $L\to\infty$ for fixed n/L. This indicates that there does not exist zero-momentum condensation of bare bosons.

The expectation value of the operator $a_p^* a_p$ sandwiched by the exact ground state $|\Psi_{0,\cdots,0}\rangle$ in the case $g\to\infty$ is given by

$$\bar{n}_{\text{bare}}(p)=\langle\Psi_{0,\cdots,0}|a_p^* a_p|\Psi_{0,\cdots,0}\rangle$$

$$=(\pi/2)^{-n(n-1)}\sum_{\{p'_{s,t},p_{s,t};1\leqq s<t\leqq n\}}\prod_{1\leqq s<t\leqq n}\{d(p'_{s,t};(-\pi h/L)\varepsilon_{s,t})d(p_{s,t};(-\pi h/L)\varepsilon_{s,t})\}$$

$$\times\sum_{l=1}^{n}\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p_{l,j},\ p)\sum_{\mu}\prod_{l=1}^{n}\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p'_{\mu_l,\mu_j},\ \sum_{\substack{j=1\\j\neq l}}^{n}p_{l,j}),\tag{2·1}$$

where use of (1·4) and (1·5) has been made, the symbol $\sum_\mu$ indicates the summation over all permutations $\mu=\binom{1,\ 2,\ \cdots,\ n}{\mu_1,\ \mu_2,\ \cdots,\ \mu_n}$ and $\delta(p,\ q)$ implies the Kronecker delta-function $\delta_{p,q}$. Noting the relation

$$\prod_{1\leqq s<t\leqq n}d(p'_{s,t};(-\pi h/L)\varepsilon_{s,t})=\prod_{1\leqq s<t\leqq n}d(p'_{\mu_s,\mu_t};(-\pi h/L)\varepsilon_{\mu_s,\mu_t}),\tag{2·2}$$

which is derived from the definitions below (1·3), one has

$$\bar{n}_{\text{bare}}(p)=(\pi/2)^{-n(n-1)}\sum_{l=1}^{n}\sum_{\mu}\sum_{\{p'_{\mu_s,\mu_t},p_{s,t};1\leqq s<t\leqq n\}}$$

$$\times\prod_{1\leqq s<t\leqq n}d(p'_{\mu_s,\mu_t};(-\pi h/L)\varepsilon_{\mu_s,\mu_t})d(p_{s,t};(-\pi h/L)\varepsilon_{s,t})$$

$$\times\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p_{l,j},\ p)\prod_{l=1}^{n}\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p'_{\mu_l,\mu_j},\ p_{l,j}).\tag{2·3}$$

For brevity, let us introduce the following notations:

$$p''_{s,t}(\mu)=p'_{\mu_s,\mu_t}\,,\qquad \varepsilon''_{s,t}(\mu)=\varepsilon_{\mu_s,\mu_t}=(\mu_t-\mu_s)/|\mu_t-\mu_s|.\tag{2·4}$$

Then, we have

$$\bar{n}_{\rm bare}(p)=(\pi/2)^{-n(n-1)}\sum_{I=1}^{n}\sum_{\mu}I_I(\mu;\,p), \tag{2·5}$$

$$I_I(\mu;\,p)=\sum_{\{p''_{s,t}(\mu),p_{s,t};1\leq s<t\leq n\}}\prod_{1\leq s<t\leq n}$$
$$\times d(p''_{s,t}(\mu);\,(-\pi\hbar/L)\varepsilon''_{s,t}(\mu))d(p_{s,t};\,(-\pi\hbar/L)\varepsilon_{s,t}) \tag{2·6}$$
$$\times\delta(\sum_{\substack{j=1\\j\neq I}}^{n}p_{I,j},\,p)\prod_{l=1}^{n}\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p''_{l,j}(\mu),\,\sum_{\substack{j=1\\j\neq l}}^{n}p_{l,j}).$$

Here the product of n kronecker symbols in (2·6) is reduced by the antisymmetricity of $p''_{s,t}(\mu)$ and $p_{s,t}$ as

$$\prod_{l=1}^{n}\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p''_{l,j}(\mu),\,\sum_{\substack{j=1\\j\neq l}}^{n}p_{l,j})=\prod_{\substack{l=1\\l\neq I}}^{n}\delta(\sum_{\substack{j=1\\j\neq l}}^{n}p''_{l,j}(\mu),\,\sum_{\substack{j=1\\j\neq l}}^{n}p_{l,j}). \tag{2·7a}$$

These $(n-1)$ kronecker symbols give

$$p''_{l,I}(\mu)=p_{l,I}+\sum_{\substack{j=1\\j\neq l,I}}^{n}(p_{l,j}-p''_{l,j}(\mu)).\quad(1\leq l\leq n,\ l\neq I) \tag{2·7b}$$

Use of (2·7a, b) in (2·6) yields

$$I_I(\mu;\,p)=\sum_{\left\{p''_{s,t}(\mu),p_{s,t};\substack{1\leq s<t\leq n\\s\neq I,t\neq I}\right\}}\prod_{\substack{1\leq s<t\leq n\\s\neq I,t\neq I}}$$
$$\times d(p''_{s,t}(\mu);\,(-\pi\hbar/L)\varepsilon''_{s,t}(\mu))d(p_{s,t};\,(-\pi\hbar/L)\varepsilon_{s,t})$$
$$\times\sum_{\left\{p_{l,I};\substack{1\leq l\leq n\\l\neq I}\right\}}\prod_{\substack{l=1\\l\neq I}}^{n}\{d(p_{l,I}+\sum_{\substack{j=1\\j\neq l,I}}^{n}(p_{l,j}-p''_{l,j}(\mu));\,(-\pi\hbar/L)\varepsilon''_{l,I}(\mu))$$
$$\times d(p_{l,I};\,(-\pi\hbar/L)\varepsilon_{l,I})\}\delta(\sum_{\substack{j=1\\j\neq I}}^{n}p_{I,j},\,p). \tag{2·8}$$

By changing the summation variables $p''_{s,t}(\mu)$ $(s\neq I,\ t\neq I)$ in (2·8) as

$$r_{s,t}=p''_{s,t}(\mu)-p_{s,t}\,,\qquad(s\neq I,\ t\neq I) \tag{2·9}$$

the expression (2·8) is rewritten by

$$I_I(\mu;\,p)=\sum_{\left\{r_{s,t},p_{s,t};\substack{1\leq s<t\leq n\\s\neq I,t\neq I}\right\}}\prod_{\substack{1\leq s<t\leq n\\s\neq I,t\neq I}}$$
$$\times d(p_{s,t}+r_{s,t};\,(-\pi\hbar/L)\varepsilon''_{s,t}(\mu))d(p_{s,t};\,(-\pi\hbar/L)\varepsilon_{s,t})$$
$$\times\sum_{\{p_{l,I};1\leq l\leq n,l\neq I\}}\prod_{\substack{l=1\\l\neq I}}^{n}\{d(p_{l,I}-\sum_{\substack{j=1\\j\neq l,I}}^{n}r_{l,j};\,(-\pi\hbar/L)\varepsilon''_{l,I}(\mu))$$
$$\times d(p_{l,I};\,(-\pi\hbar/L)\varepsilon_{l,I})\}\delta(-\sum_{\substack{j=1\\j\neq I}}^{n}p_{j,I},\,p). \tag{2·10}$$

The summations over $p_{s,t}(s\neq I,\ t\neq I)$ for the fixed values of $r_{s,t}$ produce

$$I_I(\mu;\,p)=\sum_{\{r_{s,t};1\leq s<t\leq n \atop s\neq I,t\neq I\}}(\pi/2)^{(n-1)(n-2)}\prod_{1\leq s<t\leq n \atop s\neq I,t\neq I}\varepsilon''_{s,t}(\mu)$$
$$\times\delta(r_{s,t},\,(\pi h/L)(\varepsilon_{s,t}-\varepsilon''_{s,t}(\mu)))$$
$$\times\sum_{\{p_{l,I};1\leq l\leq n,l\neq I\}}\prod_{l=1 \atop l\neq I}^{n}\{d(p_{l,I}-\sum_{j=1 \atop j\neq l,I}r_{l,j};\,(-\pi h/L)\varepsilon''_{l,I}(\mu))$$
$$\times d(p_{l,I};\,(-\pi h/L)\varepsilon_{l,I})\}\delta(-\sum_{j=1 \atop j\neq I}^{n}p_{j,I},\,p), \tag{2·11}$$

where we have made use of (1·4) (note $\varepsilon_{s,t}=1$ for $s<t$) and the formula (2·14) of III, i.e.,

$$\sum_{p_{s,t}}d(r_{s,t}+p_{s,t};\,(-\pi h/L)\varepsilon''_{s,t}(\mu))d(p_{s,t};\,(-\pi h/L)\varepsilon_{s,t})$$
$$=(\pi/2)^2\varepsilon''_{s,t}(\mu)\delta(r_{s,t};\,(\pi h/L)(\varepsilon_{s,t}-\varepsilon''_{s,t}(\mu))). \tag{2·12}$$

Next, carrying out the summations over $r_{s,t}$, we have

$$I_I(\mu;\,p)=(\pi/2)^{(n-1)(n-2)}\prod_{1\leq s<t\leq n \atop s\neq I,t\neq I}\varepsilon''_{s,t}(\mu)\sum_{\{p_{l,I};1\leq l\leq n \atop l\neq I\}}\delta(-\sum_{l=1 \atop l\neq I}^{n}p_{l,I},\,p)$$
$$\times\prod_{l=1 \atop l\neq I}^{n}\{d(p_{l,I}+\sum_{j=1 \atop j\neq l,I}^{n}(\pi h/L)(\varepsilon''_{l,j}(\mu)-\varepsilon_{l,j});(-\pi h/L)\varepsilon''_{l,I}(\mu))$$
$$\times d(p_{l,I};\,(-\pi h/L)\varepsilon_{l,I})\}. \tag{2·13}$$

We can observe here that

$$d(p_{l,I}+\sum_{j=1 \atop j\neq l,I}\frac{\pi h}{L}(\varepsilon''_{l,j}(\mu)-\varepsilon_{l,j});\,\frac{-\pi h}{L}\varepsilon''_{l,I}(\mu))$$
$$=\frac{(\pi h/L)\varepsilon''_{l,I}(\mu)}{p_{l,I}+(\pi h/L)\sum_{j=1 \atop j\neq l}^{n}(\varepsilon''_{l,j}(\mu)-\varepsilon_{l,j})+(\pi h/L)\varepsilon_{l,I}} \tag{2·14}$$

and

$$\sum_{j=1 \atop j\neq l}^{n}(\varepsilon''_{l,j}(\mu)-\varepsilon_{l,j})=\{(n-\mu_l)-(\mu_l-1)\}-\{(n-l)-(l-1)\}=2(l-\mu_l). \tag{2·15}$$

By inserting (2·14) and (2·15) into (2·13), and by using new notations as $p_l=-p_{l,I}(1\leq l\leq n,\ l\neq I)$, we get

$$I_I(\mu;\,p)=\left(\frac{\pi}{2}\right)^{(n-1)(n-2)}\prod_{1\leq s<t\leq n}\varepsilon''_{s,t}(\mu)\sum_{\{p_l;1\leq l\leq n \atop l\neq I\}}\delta(\sum_{l=1 \atop l\neq I}^{n}p_l,\,p)$$

$$\times\prod_{\substack{l=1\\l\neq I}}^{n}\left\{\frac{(\pi h/L)\varepsilon_{l,I}}{-p_l+(2\pi h/L)(l-\mu_l)+(\pi h/L)\varepsilon_{l,I}}\times\frac{(\pi h/L)\varepsilon_{l,I}}{-p_l+(\pi h/L)\varepsilon_{l,I}}\right\}, \tag{2·16}$$

where we have used the relations

$$\varepsilon''_{l,I}(\mu)=\varepsilon''_{l,I}(\mu)\varepsilon_{l,I}\ (l<I);\quad \varepsilon''_{l,I}(\mu)=\varepsilon''_{I,l}(\mu)\varepsilon_{l,I}\ (l>I). \tag{2·17}$$

Substitution of (2·16) in (2·5) yields

$$\bar{n}_{\mathrm{bare}}(p)=\left(\frac{\pi}{2}\right)^{-2(n-1)}\sum_{I=1}^{n}\sum_{\mu}\varepsilon(\mu)\frac{1}{L}\int_0^L dxe^{-ipx/h}$$
$$\times\prod_{l=1}^{I-1}\sum_{p_l}\frac{(\pi h/L)^2e^{ip_lx/h}}{(p_l-\pi h/L)(p_l+(2\pi h/L)(\mu_l-l)-\pi h/L)}$$
$$\times\prod_{l=I+1}^{n}\sum_{p_l}\frac{(\pi h/L)^2e^{ip_lx/h}}{(p_l+\pi h/L)(p_l+(2\pi h/L)(\mu_l-l)+\pi h/L)}, \tag{2·18}$$

where we have made use of

$$\delta(\sum_{\substack{l=1\\l\neq I}}^{n}p_l,\ p)=(1/L)\int_0^L dx\ \exp(i(\sum_{\substack{l=1\\l\neq I}}^{n}p_l-p)x/h),$$

and $\varepsilon(\mu)$ indicates $\prod_{1\leq l<m\leq n}\varepsilon''_{l,m}(\mu)$ which equals $+1$ or -1 according to whether μ is even or odd permutation, as has been proved in Formula A of the Appendix.

Now let us change the summation variables in (2·18) as

$$p_l\rightarrow p_l+(2\pi h/L)l\,,\qquad (1\leq l\leq I-1)$$
$$p_l\rightarrow p_l+(2\pi h/L)(l-1)\qquad (I+1\leq l\leq n)$$

Then the expression (2·18) is rewritten by

$$\bar{n}_{\mathrm{bare}}(p)=(\pi/2)^{-2(n-1)}(1/L)$$
$$\times\int_0^L dxe^{-ip\cdot x/h}e^{i(2\pi/L)n(n-1)x/2}\sum_{I=1}^{n}\det(a_{l,m}^{(I)}(x)), \tag{2·19}$$

where $\det(a_{l,m}^{I}(x))$ indicates a determinant of the $n\times n$ matrix $(a_{l,m}^{(I)}(x))$,

$$a_{l,m}^{(I)}(x)=\sum_{q}\frac{(\pi h/L)^2e^{iqx/h}}{(q+(2\pi h/L)l-\pi h/L)(q+(2\pi h/L)m-\pi h/L)},\quad \begin{Bmatrix}l\neq I,\ 1\leq l\leq n\\ 1\leq m\leq n\end{Bmatrix},$$

$$a_{I,m}^{(I)}(x)=1\,.\qquad (1\leq m\leq n) \tag{2·20}$$

According to Formula B in the Appendix, the sum of the determinants can be modified as

$$\sum_{I=1}^{n}\det(a_{l,m}^{(I)}(x))=(\pi^2/4)^{(n-1)}e^{-i(\pi/L)n(n-1)x}\det(C_{s-t}(\alpha)), \tag{2·21}$$

where $\alpha=2\pi x/L$, and $\det(C_{s-t}(\alpha))$ denotes a determinant of the $(n-1)\times(n-1)$ Toeplitz matrix, of which elements are given by

$$C_l(\alpha)=2\delta_{l,0}\cos\frac{\alpha}{2}-\delta_{l,1}-\delta_{l,-1}+\frac{2}{\pi}$$
$$\times\left\{\frac{\sin((l+1)|\alpha|/2)}{l+1}+\frac{\sin((l-1)|\alpha|/2)}{l-1}-2\frac{\cos(\alpha/2)\sin(l|\alpha|/2)}{l}\right\}. \tag{2·22}$$

Then, we have

$$\bar{n}_{\mathrm{bare}}(p)=(1/2\pi)\int_0^{2\pi}d\alpha\, e^{-i(L/2\pi\hbar)p\alpha}\det(C_{s-t}(\alpha)). \tag{2·23}$$

The Toeplitz determinant[7] in (2·23) agrees with the starting expression (20) in Lenard's paper. He has shown that

$$\det(C_{s-t}(\alpha))<\left|\frac{en}{\sin(\alpha/2)}\right|^{1/2}, \qquad \text{and hence } \bar{n}_{\mathrm{bare}}(0)\leqq O(n^{1/2}).$$

This result indicates that there does not exist the zero momentum condensation of bare bose particles, i.e.,

$$\lim_{n\to\infty}\langle\Psi_{0,0,\cdots,0}|a_0{}^*a_0|\Psi_{0,0,\cdots,0}\rangle/n=\lim_{n\to\infty}(\bar{n}_{\mathrm{bare}}(0)/n)=0\,. \tag{2·24}$$

However, one should note that

$$\langle\Psi_{0,0,\cdots,0}|A_0{}^*A_0|\Psi_{0,0,\cdots,0}\rangle/n=1\,, \quad (\text{for all } n=1,2,\cdots) \tag{2·25}$$

for dressed bosons, and thus it should be emphasized that the ground state is not the condensed state of the *bare bosons* but that of the *dressed bosons* with zero momentum in one-dimensional many boson system. This property will also be kept presumably even in the three-dimensional system.

§3. Galilei-invariance of the unitary operator

As has been shown in III, the unitary operator U has been constructed in a coordinate system of inertia S. It is not trivial whether the unitary operator U' in another coordinate system S' is identical with U. In this section we will show that the operator U is invariant under a Galilei-transformation $G(v)$ from S to S', that is, $G(v)UG(v)^*=U$, where v is the relative velocity between S and S'.

As is shown in Formula C of the Appendix, the Galilei-transformation is given by

$$G(v)=1+\sum_{n=1}^{\infty}\Big[G_n(v)+\sum_{l=1}^{n-1}\frac{(-1)^{n-l}}{(n-l)!}\sum_{p_1,\cdots,p_{n-l}}a_{p_1}^*\cdots a_{p_{n-l}}^*G_l(v)a_{p_1}\cdots a_{p_{n-l}}$$

$$+\frac{(-1)^n}{n!}\sum_{p_1,\cdots,p_n} a^*_{p_1}\cdots a^*_{p_n} a_{p_1}\cdots a_{p_n}\Bigg], \tag{3·1}$$

where the velocity $v=(2\pi h/mL)\times$integer, and

$$G_n(v)=\sum_{p_1\leqq p_2\leqq\cdots\leqq p_n} \alpha_{p_1+mv,p_2+mv,\cdots,p_n+mv}\alpha_{p_1,p_2,\cdots,p_n}\prod_{i=1}^{n} a^*_{p_i+mv}\prod_{i=1}^{n} a_{p_i}\,, \tag{3·2}$$

in which $\alpha_{q_1,q_2,\cdots,q_n}$ denotes the normalization factor defined by (2·2) in III. By using the formula C, the Galilei-transformation for the operator U_n of (1·7) becomes

$$\begin{aligned}G(v)U_nG(v)^* &= \sum_{q_1\leqq\cdots\leqq q_n}\alpha_{q_1,\cdots,q_n}\beta_{q_1,\cdots,q_n}\sum_{\{p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n} d(p_{i,j};k_{i,j})\\ &\qquad\times\prod_{i=1}^{n} a^*_{\sum_{j=1,\,j\neq i}^{n} p_{i,j}+q_i+mv}\prod_{i=1}^{n} a_{q_i+mv}\\ &=\sum_{q'_1\leqq\cdots\leqq q'_n}\alpha_{q'_1-mv,\cdots,q'_n-mv}\beta_{q'_1-mv,\cdots,q'_n-mv}\\ &\qquad\times\sum_{\{p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n} d(p_{i,j};k_{i,j})\prod_{i=1}^{n} a^*_{\sum_{j=1,\,j\neq i}^{n} p_{i,j}+q_i'}\prod_{i=1}^{n} a_{q_i'}\,,\end{aligned} \tag{3·3}$$

where $q_i'=q_i+mv(i=1,\cdots,n)$. Since the parameters $k_{i,j}$ have been uniquely determined as the functions of $s_i=q_{i+1}-q_i(i=1,\cdots,n-1)$ in § 3 of I, we obtain

$$k_{i,j}\equiv k_{i,j}(s_1,\cdots,s_n)=k_{i,j}(s_i',\cdots,s_n')\equiv k'_{i,j}\,, \tag{3·4}$$

where $s_i'=q'_{i+1}-q_i'$. Noting that

$$\alpha_{q_1'-mv,\cdots,q_n'-mv}=\alpha_{q_1',\cdots,q_n'}\,,\quad \beta_{q_1'-mv,\cdots,q_n'-mv}=\beta_{q_1',\cdots,q_n'}\,, \tag{3·5}$$

and substituting (3·4) and (3·5) in (3·3), one gets

$$\begin{aligned}G(v)U_nG(v)^* &= \sum_{q_1'\leqq\cdots\leqq q_n'}\alpha_{q_1',\cdots,q_n'}\beta_{q_1',\cdots,q_n'}\\ &\qquad\times\sum_{\{p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n} d(p_{i,j};k'_{i,j})\prod_{i=1}^{n} a^*_{\sum_{j=1,\,j\neq i}^{n} p_{i,j}+q_i'}\prod_{i=1}^{n} a_{q_i'}\,,\end{aligned} \tag{3·6}$$

which is identical with (1·7), that is,

$$G(v)U_nG(v)^*=U_n\,. \tag{3·7}$$

As readily seen from the definition (1·6) of U and (3·7), the Galilei-transformation for U is

$$G(v)UG(v)^*=U\,, \tag{3·8}$$

which indicates that the operator U is invariant under Galilei-transformation.

By virtue of the Galilei-invariance (3·8) for U, the dressed operators A_p^* and A_p are transformed as

$$G(v)A_p^*G(v)^*=A_{p+mv}^*\,,\quad G(v)A_pG(v)^*=A_{p+mv}\,, \tag{3·9}$$

since $G(v)A_p^*G(v)^*=G(v)Ua_p^*U^*G(v)^*=UG(v)a_p^*G(v)^*U^*=Ua_{p+mv}^*U^*=A_{p+mv}^*$ from (3·8) and (1·8). From (3·9) one can readily see for the exact eigenstate (1·9) that

$$G(v)\|q_1,\cdots,q_n\gg=\|q_1+mv,\cdots,q_n+mv\gg. \tag{3·10}$$

Hence, the ground state $(1/\sqrt{n!})(A_0^*)^n|0\rangle$ is transformed as

$$G(v)(1/\sqrt{n!})(A_0^*)^n|0\rangle=(1/\sqrt{n!})(A_{mv}^*)^n|0\rangle\,. \tag{3·11}$$

This means clearly that the ground state in a moving n-boson system with velocity v is the condensed state of the dressed bosons with momentum mv.

In closing this section, let us show that the Galilei-transformation for the Hamiltonian (1·1) is obtained by using Formula C as

$$G(v)H(a^*,a)G^*(v)=H(a^*,a)-v\sum_p pa_p^*a_p+(mv^2/2)\sum_p a_p^*a_p\,. \tag{3·12}$$

From (3·12) and the Galilei-invariance (3·8) of U, the Hamiltonian $\hat{H}(A^*,A)$ expressed in terms of the dressed operators A_p^* and A_p in (4·20) of III is shown to be transformed as

$$\begin{aligned}G(v)\hat{H}(A^*,A)G(v)^*&=G(v)U^*UHU^*UG(v)^*\\&=\hat{H}(A^*,A)-v\sum_p pA_p^*A_p+(mv^2/2)\sum_p A_p^*A_p\,.\end{aligned} \tag{3·13}$$

§ 4. Disappearance of phonon character

We first investigate the energy of an exactly dressed boson with momentum p, which is defined by the increased energy of the system when the single dressed boson is added to the system specified by the state $\|q_1,\cdots,q_n\gg$. The first purpose of this section is to show that the energy of a dressed boson depends not only on its momentum p but also on the quantum numbers $q_1,\cdots,q_n$, namely, the number distribution $\{n_q\}$ of exactly dressed bosons of the state. On the other hand, the single excitation energy of the system, which is different from the energy mentioned above, is defined by the increased energy of the system when a dressed boson with zero momentum is excited to have momentum p. The second purpose of this section is to point out that the phonon character of the single excitation energy is lost drastically with disappearance of the zero momentum condensate of dressed bosons.

For the first purpose, we consider $[\hat{H}(A^*, A), A_p{}^*]\|q_1, \cdots, q_n \gg$ which becomes

$$[\hat{H}(A^*, A), A_p{}^*]\|q_1, \cdots, q_n\gg = \omega_p(q_1, \cdots, q_n) A_p{}^* \|q_1, \cdots, q_n\gg , \tag{4·1}$$

$$\omega_p(q_1, \cdots, q_n) = \hat{E}_{p,q_1,\cdots,q_n} - \hat{E}_{q_1,\cdots,q_n} , \tag{4·2}$$

where $\hat{H}(A^*, A)$ is the total Hamiltonian rewritten in terms of the dressed operators as (4·20) of III, and $\hat{E}_{p,q_1,\cdots,q_n}$ is defined by (4·15) of III. The energy difference $\omega_p(q_1, \cdots, q_n)$ defined above indicates the energy of an exactly dressed boson. As readily seen from (4·2), the energy of a dressed boson is dependent on its momentum p and the quantum numbers $q_1, q_2, \cdots, q_n$ of the eigenstate to which one dressed boson with momentum p is added. One should note here that there are interaction energies between the added dressed particle and the original n dressed particles.

In various approximations in many body problems, one has been tried to find such operators $\alpha_p{}^*$ that satisfy approximately the equation

$$[H, \alpha_p{}^*] = \nu_p \alpha_p{}^* . \tag{4·3}$$

At a glance, Eq. (4·3) resembles (4·1) in their forms. However, if one calculates the commutator $[\hat{H}(A^*, A), A_p{}^*]$ directly, one can get

$$\begin{aligned}
&[\hat{H}(A^*, A), A_p{}^*] \\
&\quad = \frac{p^2}{2m} A_p{}^* + \sum_{n=2}^{\infty} \sum_{p_1,p_2,\cdots,p_{n-1}} \sum_{l=1}^{n} \frac{(-1)^{n-l}}{(n-l)!(l-1)!} \hat{E}_{p_1,p_2,\cdots,p_{l-1},p} A_p{}^* \prod_{i=1}^{n-1} A^*_{p_i} \prod_{i=1}^{n-1} A_{p_i} \\
&\quad\quad + \sum_{n=2}^{\infty} \sum_{p_1,p_2,\cdots,p_{n-1}} \sum_{l=1}^{n-1} \frac{(-1)^{n-l}}{(n-l-1)!\, l!} \hat{E}_{p_1,\cdots,p_l} A_p{}^* \prod_{i=1}^{n-1} A^*_{p_i} \prod_{i=1}^{n-1} A_{p_i} ,
\end{aligned} \tag{4·4}$$

where use has been made of (4·2) and (4·20) in III. It should be noted that the right-hand side of (4·4) has a nonlinear form with respect to $A_q{}^*$ and A_q, and has a quite different form from the linear form on the right-hand side of (4·3). This nonlinearity yields the q_i-dependence of $\omega_p(q_1, \cdots, q_n)$ in (4·2). If we introduce operators $\alpha_p{}^*$ satisfying (4·3) in solving the many body problem, the energy ν_p cannot have such state-dependence as $\omega_p(q_1, \cdots, q_n)$ in (4·2). In other words, there does not exist any interaction between the additional quasi-particle described by $\alpha_p{}^*$ and the original n quasi-particles.

Let us now investigate the case $g \to \infty$, since the q_i-dependence of $\omega_p(q_1, \cdots, q_n)$ can be clarified explicitly. First we rewrite the eigenstate $\|q_1, \cdots, q_n\gg$ and the energy $E_{q_1,q_2,\cdots,q_n}$ in terms of the number distributions $\{n_q\}$ of dressed bosons,

$$\|q_1, \cdots, q_n\gg = \prod_q (1/\sqrt{n_q!})(A_q{}^*)^{n_q}|0\rangle , \tag{4·5}$$

$$E_{q_1,\cdots,q_n} \equiv E(\{n_q\})$$

$$=(1/2m)\sum_q \sum_{l_q=1}^{n_q}\{q+(2\pi h/L)(\sum_{r<q} n_r+l_q)-(\pi h/L)(n+1)\}^2 , \tag{4·6}$$

where use of (2·27) and (2·30) in I has been made. From (4·2), the energy $\omega_p(\{n_q\})$ of a dressed boson with momentum p can be rewritten in terms of $\{n_q\}$ by

$$\omega_p(\{n_q\})=E(\{n_q'\})-E(\{n_q\}), \tag{4·7}$$

$$n_q'=n_q(q\neq p), \qquad n_p'=n_p+1 . \tag{4·8}$$

Substitution of (4·6) in (4·7) gives

$$\begin{aligned}\omega_p(\{n_q\})=&\frac{1}{2m}\Big[\Big\{p+\frac{2\pi h}{L}(\sum_{r<p} n_r+n_p+1)-\frac{\pi h}{L}(n+2)\Big\}^2\\&+\sum_{q\leqq p}\sum_{l_q=1}^{n_q}\frac{-2\pi h}{L}\Big\{q+\frac{2\pi h}{L}(\sum_{r<q} n_r+l_q)-\frac{\pi h}{L}(n+1)\Big\}\\&+\sum_{q>p}\sum_{l_q=1}^{n_q}\frac{2\pi h}{L}\Big\{q+\frac{2\pi h}{L}(\sum_{r<q} n_r+l_q)-\frac{\pi h}{L}(n+1)\Big\}+\sum_q\Big(\frac{\pi h}{L}\Big)^2 n_q\Big]\\=&\frac{1}{2m}\Big[\Big\{p+\frac{\pi h}{L}(\sum_{r\leqq p} n_r-\sum_{r>p} n_r)\Big\}^2-\frac{2\pi h}{L}\sum_{q\leqq p}\Big\{q+\frac{\pi h}{L}(\sum_{r<q} n_r-\sum_{r>q} n_r)\Big\}n_q\\&+\frac{2\pi h}{L}\sum_{q>p}\Big\{q+\frac{\pi h}{L}(\sum_{r<q} n_r-\sum_{r>q} n_r)\Big\}n_q\Big]+\frac{1}{2m}\sum_q\Big(\frac{\pi h}{L}\Big)^2 n_q\\=&\frac{1}{2m}\Big[\Big\{p+\frac{\pi h}{L}(\sum_{r<p} n_r-\sum_{r>p} n_r)\Big\}^2\\&+\Big(\frac{\pi h}{L}\Big)^2 n_p^2+\frac{2\pi h}{L}\sum_q \varepsilon(q-p)\Big\{q+\frac{\pi h}{L}(\sum_{r<q} n_r-\sum_{r>q} n_r)\Big\}n_q\Big]\\&+\frac{1}{2m}\sum_q\Big(\frac{\pi h}{L}\Big)^2 n_q ,\end{aligned} \tag{4·9}$$

where we have used $\sum_q n_q=n$ and the function $\varepsilon(q-p)$ is defined as

$$\varepsilon(q-p)=\begin{cases}1 \quad , & (q>p)\\ 0 \quad , & (q=p)\\ -1 . & (q<p)\end{cases} \tag{4·10}$$

The last term in (4·9) can be neglected in the limit $n\to\infty$ and $L\to\infty$ for a fixed number density $\rho=n/L$. Separating out the number n_0 of dressed bosons with zero momentum in (4·9), we have

$$\begin{aligned}\omega_p(\{n_q\})=&\frac{1}{2m}\Big[p^2+2p\frac{\pi h}{L}\Big\{n_0+(\sum_{r<0} n_r-\sum_{r>0} n_r)+2\sum_{0<r<p} n_r\Big\}\\&-\frac{4\pi h}{L}\sum_{0<q<p} n_q\Big\{q+\frac{\pi h}{L}(n_q+2\sum_{0<r<q} n_r)\Big\}+\Big(\frac{2\pi h}{L}\sum_{0<r<p} n_r\Big)^2\Big]+\omega_0(\{n_q\})\end{aligned} \tag{4·11a}$$

for $p>0$,

$$\omega_p(\{n_q\})=\frac{1}{2m}\Big[p^2-2p\frac{\pi\hbar}{L}\{n_0-(\sum_{r<0}n_r-\sum_{r>0}n_r)+2\sum_{p<r<0}n_r\}$$
$$+\frac{4\pi\hbar}{L}\sum_{p<q<0}n_q\Big\{q-\frac{\pi\hbar}{L}(n_q+2\sum_{q<r<0}n_r)\Big\}+\Big(\frac{2\pi\hbar}{L}\sum_{p<r<0}n_r\Big)^2\Big]$$
$$+\omega_0(\{n_q\}) \tag{4·11b}$$

for $p<0$, and

$$\omega_0(\{n_q\})=\frac{1}{2m}\Big[\Big(\frac{\pi\hbar}{L}\Big)^2\{n_0^2+2n_0(\sum_{r<0}n_r-\sum_{r>0}n_r)-(\sum_{r<0}n_r-\sum_{r>0}n_r)^2\}$$
$$-\frac{2\pi\hbar}{L}\sum_{q<0}n_q\Big\{q-\frac{\pi\hbar}{L}(n_q+2\sum_{q<r<0}n_r)\Big\}$$
$$+\frac{2\pi\hbar}{L}\sum_{q>0}n_q\Big\{q+\frac{\pi\hbar}{L}(n_q+2\sum_{0<r<q}n_r)\Big\}\Big] \tag{4·11c}$$

for $p=0$, where the relations

$$\sum_{r<q}n_r-\sum_{r>q}n_r=n_0+(\sum_{r<0}n_r-\sum_{r>0}n_r)+n_q+2\sum_{0<r<q}n_r\,,\quad (q>0) \tag{4·12a}$$

$$\sum_{r<q}n_r-\sum_{r>q}n_r=-n_0+(\sum_{r<0}n_r-\sum_{r>0}n_r)-n_q-2\sum_{q<r<0}n_r\quad (q<0) \tag{4·12b}$$

have been inserted in (4·9). Next we can observe that

$$(\sum_{0<r<p}n_r)^2=\sum_{\substack{0<q<p\\0<r<p}}n_qn_r=\sum_{0<q<p}n_q^2+2\sum_{0<q<p}n_q\sum_{0<r<q}n_r$$
$$=\sum_{0<q<p}n_q(n_q+2\sum_{0<r<q}n_r),\quad (p>0)$$
$$(\sum_{p<r<0}n_r)^2=\sum_{p<q<0}n_q(n_q+2\sum_{q<r<0}n_r).\quad (p<0)$$

Use of the above relations in (4·11a, b, c) gives

$$\omega_p(\{n_q\})=\frac{1}{2m}\Big[p^2+2p\frac{\pi\hbar}{L}\{n_0+(\sum_{r<0}n_r-\sum_{r>0}n_r)$$
$$+2\sum_{0<r<p}n_r\}-\frac{4\pi\hbar}{L}\sum_{0<q<p}qn_q\Big]+\omega_0(\{n_q\}) \tag{4·13a}$$

for $p>0$,

$$\omega_p(\{n_q\})=\frac{1}{2m}\Big[p^2-2p\frac{\pi\hbar}{L}\{n_0-(\sum_{r<0}n_r-\sum_{r>0}n_r)$$
$$+2\sum_{p<r<0}n_r\}+\frac{4\pi\hbar}{L}\sum_{p<q<0}qn_q\Big]+\omega_0(\{n_q\}) \tag{4·13b}$$

for $p<0$ and

$$\omega_0(\{n_q\})=\frac{1}{2m}\left[\left(\frac{\pi h}{L}n\right)^2-\frac{2\pi h}{L}\sum_{q<0}qn_q+\frac{2\pi h}{L}\sum_{q>0}qn_q\right] \tag{4·13c}$$

for $p=0$. In this way, we have confirmed that the energy $\omega_p(\{n_q\})$ of an added dressed particle to the state $\| q_1,\cdots,q_n\gg$ with momentum p has strong correlation with the number distribution $\{n_q\}$ assigned to the eigenstate. It should be emphasized again that the energy $\omega_p(\{n_q\})$ cannot be determined without the knowledge of the number distribution $\{n_q\}$ of the original state.

Next we intend to discuss the single excitation energy which is defined by

$$\epsilon_p(q_1,\cdots,q_{n-1})=\hat{E}_{p,q_1,\cdots,q_{n-1}}-\hat{E}_{0,q_1,\cdots,q_{n-1}}\,. \tag{4·14a}$$

This energy indicates the increased energy of the system, when the momentum of one exactly dressed boson in the sea composed of $n-1$ dressed bosons is changed from zero to p. In terms of the number distribution, the excitation energy can be written as

$$\epsilon_p(\{n_q\})=\omega_p(\{n_q\})-\omega_0(\{n_q\}), \tag{4·14b}$$

where it should be noted that $\Sigma_q n_q=n$ in the case of (4·13a, b, c), but

$$\sum_q n_q=n-1 \tag{4·15}$$

in the above case (4·14b). Changing n by $n-1$ in (4·13a, b, c) and introducing them in (4·14b), we obtain

$$\epsilon_p(\{n_q\})=\frac{1}{2m}\left[p^2+2p\frac{\pi h}{L}\{n_0+(\sum_{r<0}n_r-\sum_{r>0}n_r)+2\sum_{0<r<p}n_r\}-\frac{4\pi h}{L}\sum_{0<q<p}qn_q\right] \tag{4·16a}$$

for $p\geqq 0$, and

$$\epsilon_p(\{n_q\})=\frac{1}{2m}\left[p^2-2p\frac{\pi h}{L}\{n_0-(\sum_{r<0}n_r-\sum_{r>0}n_r)+2\sum_{p<r<0}n_r\}+\frac{4\pi h}{L}\sum_{p<q<0}qn_q\right] \tag{4·16b}$$

for $p<0$.

We now investigate in detail the properties of the excitation energy expressed by (4·16a, b).

(Case 1) Let us first consider the case when the number of the excited dressed bosons in the system is negligibly small compared with the total number n,

$$\sum_{q\neq 0} n_q/n \approx 0\,, \quad \text{that is,} \quad n_0/n \approx 1\,, \tag{4·17}$$

in the limit $n\to\infty$ and $L\to\infty$ for a fixed number density $\rho=n/L$. In this case, the excitation energy $\varepsilon_p(\{n_q\})$ (4·16) is reduced to

$$\epsilon_p(\{n_q\})=\frac{\pi\hbar}{m}\rho|p|+\frac{p^2}{2m}\,. \tag{4·18}$$

The result (4·18) indicates that the effect of the interactions among dressed bosons can be neglected in this case. When the momentum p is small, the excitation energy (4·18) has phonon character with the phonon velocity c_0,

$$c_0=(\pi\hbar/m)\rho\,. \tag{4·19}$$

(Case 2) Next let us consider the case when the number of the excited dressed bosons cannot be neglected and the distribution is an even function of q, that is,

$$n_q=n_{-q}\,. \tag{4·20}$$

In this case, the excitation energy (4·16a, b) becomes

$$\epsilon_p(\{n_q\})=\frac{1}{2m}\left[p^2+2\pi\hbar\rho_0|p|+\frac{4\pi\hbar}{L}|p|\sum_{0<q<|p|}n_q-\frac{4\pi\hbar}{L}\sum_{0<q<|p|}qn_q\right], \tag{4·21}$$

$$\rho_0=n_0/L\,. \tag{4·22}$$

If the distribution n_q is a regular function of q except at $q=0$ and $n_q \xrightarrow[q\to 0]{}$ finite, the contributions from the third and fourth terms in (4·21) are estimated to be the order of p^2 for small p. In this case, the phonon velocity c is determined by

$$c=(\pi\hbar/m)\rho_0\,. \tag{4·23}$$

The phonon velocity c decreases as the number n_0 of dressed bosons with zero momentum decreases. When $\rho_0=(n_0/L)$ tends to zero, the phonon character of the excitation energy is lost drastically.

Let us investigate in more detail the behavior of the excitation energy (4·21) by expanding the distribution function n_q in the power series of $|q|$ as

$$n_q=|q|^\alpha(\xi_0+\xi_1|q|+\cdots\cdots) \quad \text{for small } q \text{ except at } q=0\,. \tag{4·24}$$

For the convergence of the summation in (4·21), we assume

$$\alpha>-1\,. \tag{4·25}$$

Then the behavior of the excitation energy (4·21) for small $|p|$ is expressed*) as

*) The summations in (4·21) have been changed to integration, for example, as

$$\frac{2\pi\hbar}{L}\sum_{0<q<|p|}n_q \xrightarrow[\text{fixed } n/L]{L\to\infty} \int_0^{|p|}n_q dq=\sum_{l=0}^{\infty}\frac{\xi_l}{\alpha+l+1}|p|^{\alpha+l+1}$$

for small $|p|$.

$$\epsilon_p(\{n_q\})=\frac{1}{2m}\Big[2\pi\hbar\rho_0|p|$$

$$+\frac{2\xi_0}{(1+\alpha)(2+\alpha)}|p|^{2+\alpha}+p^2+\frac{2\xi_1}{(2+\alpha)(3+\alpha)}|p|^{3+\alpha}+\cdots\cdots\Big]. \quad (4\cdot26)$$

Note that the phonon velocity c is given by the same formula as (4·23), because $2+\alpha>1$ due to (4·25). Furthermore it should be pointed out that the power $(2+\alpha)$ of the second term in (4·26) can be smaller than 2 for the distribution n_q with $-1<\alpha<0$. In this case, the excitation energy versus momentum for small $|p|$ does not have the spectrum of a free particle in the limit $\rho_0\to 0$. In this way, it has been clarified that the excitation energy of the exactly dressed boson has the strong dependence on the distribution of dressed bosons.

The drastic disappearance of phonon character in the limit $\rho_0\to 0$ in the one-dimensional system will play an important role even in the analysis of the phenomena in liquid helium II. This opinion will be discussed in the next section.

§ 5. Discussion

We discuss some characteristic phenomena of liquid He-II on the basis of the results obtained in I, II, III and the present paper. For this purpose, let us summarize our results: The one-dimensional many-boson system has been described completely with the concept of *the exactly dressed boson.* The dressed boson has the following properties.

i) They obey bose-statistics (see (4·2) in III).

ii) Their total number is conserved (see (4·9) and (4·20) in III).

iii) The ground state is the condensed state of all dressed bosons with zero momentum (see (4·21) in III).

iv) The energy of a dressed boson depends strongly upon the number distribution of the other dressed bosons (see (4·13a, b, c)).

v) The phonon velocity decreases as ρ_0 becomes small, and in the limit $n_0/n\to 0$ the excitation of the exactly dressed boson loses phonon character (see (4·21) and (4·26)).

London[8] has pointed out that in the system of noninteracting many bose particles there does exist a phase transition which arises due to the Bose-Einstein condensation of the bare bose particles. On the basis of the model, he gave the microscopic foundation for two-fluid model in Liquid He-II qualitatively, but could not explain its superfluidity. Landau,[9] on the other hand, has succeeded in the interpretation of the superfluidity at zero temperature and the behavior of the specific heat at temperature lower than the λ-point by assuming the existence of the phonon-roton excitation spectrum. In his theory, however, the microscopic foundation for the super component is not clarified, and the logarithmic divergence of the specific heat at the λ-point is not explained.

As easily seen from the matters mentioned above, both theories do not bear the overall explanation of the phenomena in Liquid He-II. Observing the properties of the exactly dressed bosons, on the other hand, we can see that the present theory holds the point of advantage in both theories. If one can introduce the dressed bosons*),**) with interaction cloud in a three-dimensional many boson system, one will be able to explain the characteristic properties of Liquid He-II qualitatively as follows. According to the bose nature of the dressed bosons and the conservation of their total number there does necessarily occur the Bose-Einstein condensation of the dressed bosons with zero momentum. Then we can regard the condensed assembly and the remaining one (this indicates the assembly of the dressed bosons with nonzero momentum) as the super component and the normal component in Liquid He-II, respectively. Thus we can give the microscopic foundation for the two-fluid model. When a dressed boson with zero momentum is excited, the excitation energy will presumably display the phonon-roton character due to the effect of the interaction cloud. Hence, the dressed bosons with zero momentum have no friction with the wall owing to the Landau criterion. It should be noted here that the superfluidity***) can be observed not only at zero temperature but also at non-zero temperature below the λ-point, since the macroscopic number of the dressed bosons is condensed at zero momentum owing to the Bose-Einstein condensation. Even in a three-dimensional system, we may also expect that there exists a similar drastic change of the excitation energy (4·21) when the condensate vanishes. It seems to the present authors that this drastic change may serve as the origin of the logarithmic divergence of the specific heat at the λ-point. Thus, Landau's idea can be harmonized with London's idea under the concept of the dressed bosons.

If our viewpoint mentioned above is valid in liquid helium II, the super component is composed of the condensate of dressed bosons (not of bare helium atoms). Therefore, the authors hope that the quantity $\langle \Psi_{0,\cdots,0}|A_0{}^*A_0|\Psi_{0,\cdots,0}\rangle$ rather than the quantity****) $\langle \Psi_{0,\cdots,0}|a_0{}^*a_0|\Psi_{0,\cdots,0}\rangle/n$ will be measured experimentally.

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*) An approximate introduction of the dressed bosons has been proposed by Sasaki and Matsuda.[10]

**) The introduction of the dressed bosons seems to be possible approximately, but the term "approximately" does not imply the perturbation method as readily seen from the discussion in § 5 in III.

***) In Landau's theory, it is not clear why there exists the super component at non-zero temperature below the λ-point.

****) This quantity has been measured by the neutron scattering experiments.[11]

Appendix

[Formula A]

For a permutation $\mu=\binom{1,\ 2,\ \cdots,\ n}{\mu_1,\ \mu_2,\ \cdots,\ \mu_n}$ the quantity $\varepsilon(\mu)$ defined by

$$\varepsilon(\mu)=\prod_{1\leq l<m\leq n}\frac{\mu_m-\mu_l}{|\mu_m-\mu_l|}\,, \tag{A·1}$$

equals $+1$ or -1 according to whether μ is an even or odd permutation.

This formula is proved as follows. For the identical permutation, the formula (A·1) holds apparently. Next for any permutation ν

$$\nu=\begin{pmatrix}1, & 2, & \cdots, & n\\ \nu_1, & \nu_2, & \cdots, & \nu_n\end{pmatrix}, \tag{A·2}$$

let us consider a transposition μ, which exchanges ν_l and $\nu_m (l<m)$,

$$\mu=\begin{pmatrix}\nu_1, & \cdots, & \nu_l, & \cdots, & \nu_m, & \cdots, & \nu_n\\ \nu_1, & \cdots, & \nu_m, & \cdots, & \nu_l, & \cdots, & \nu_n\end{pmatrix}. \tag{A·3}$$

Then, denoting the product of ν and μ by ν',

$$\nu'=\begin{pmatrix}1, & 2, & \cdots, & n\\ \nu_1', & \nu_2', & \cdots, & \nu_n'\end{pmatrix};\quad \nu_s'=\nu_s(s\neq l,m),\quad \nu_m'=\nu_l\,,\quad \nu_l'=\nu_m\,, \tag{A·4}$$

and noting the inequality $l<m$, we have

$$\begin{aligned}\varepsilon(\nu')&=\prod_{1\leq s<t\leq n}\frac{\nu_t'-\nu_s'}{|\nu_t'-\nu_s'|}=\prod_{\substack{1\leq s<t\leq n\\ s\neq l,t\neq m}}\frac{\nu_t'-\nu_s'}{|\nu_t'-\nu_s'|}\\ &\times\left(\prod_{s=1}^{l-1}\frac{\nu_m'-\nu_s'}{|\nu_m'-\nu_s'|}\right)\frac{\nu_m'-\nu_l'}{|\nu_m'-\nu_l'|}\left(\prod_{s=l+1}^{m-1}\frac{\nu_m'-\nu_s'}{|\nu_m'-\nu_s'|}\right)\left(\prod_{t=m+1}^{n}\frac{\nu_t'-\nu_m'}{|\nu_t'-\nu_m'|}\right)\\ &\times\left(\prod_{s=1}^{l-1}\frac{\nu_l'-\nu_s'}{|\nu_l'-\nu_s'|}\right)\left(\prod_{t=l+1}^{m-1}\frac{\nu_t'-\nu_l'}{|\nu_t'-\nu_l'|}\right)\left(\prod_{t=m+1}^{n}\frac{\nu_t'-\nu_l'}{|\nu_t'-\nu_l'|}\right)\\ &=\prod_{1\leq s<t\leq n}\frac{\nu_t-\nu_s}{|\nu_t-\nu_s|}(-1)(-1)^{m-l-1}(-1)^{m-l-1}=-\varepsilon(\nu).\end{aligned} \tag{A·5}$$

In general, an arbitrary permutation ν can be expressed by a product of some transpositions $\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(I)}$ as

$$\nu=\mu^{(1)}\cdot\mu^{(2)}\cdot\cdots\cdot\mu^{(I)}\,. \tag{A·6}$$

Then, the repeated use of the above result (A·5) gives

$$\varepsilon(\nu)=(-1)^I\,. \tag{A·7}$$

Thus the formula A has been established.

[Formula B]

$$\sum_{I=1}^{n} \det(a_{l,m}^{(I)}(x)) = (\pi^2/4)^{n-1} e^{-i(\pi/L)n(n-1)x} \det(C_{s-t}(\alpha)), \tag{B·1}$$

where $\alpha = 2\pi x/L$, and

$$a_{l,m}^{(I)}(x) = \sum_q \frac{\left(\dfrac{\pi h}{L}\right)^2 e^{iq\cdot x/h}}{\left(q+\dfrac{2\pi h}{L}l-\dfrac{\pi h}{L}\right)\left(q+\dfrac{2\pi h}{L}m-\dfrac{\pi h}{L}\right)} \quad (1 \leqq l \leqq n,\ 1 \leqq m \leqq n,\ l \neq I),$$

$$a_{I,m}^{(I)}(x) = 1 \quad (1 \leqq m \leqq n). \tag{B·2}$$

The $\det(C_{s-t}(\alpha))$ stands for a determinant of $(n-1)\times(n-1)$ matrix of which an element with $(j+l)$-th row and j-th column is given by

$$C_l(\alpha) = 2\delta_{l,0}\cos\left(\frac{\alpha}{2}\right) - \delta_{l,1} - \delta_{l,-1}$$
$$+\frac{2}{\pi}\left\{\frac{\sin((l+1)\alpha/2)}{l+1} + \frac{\sin((l-1)\alpha/2)}{l-1} - 2\frac{\cos(\alpha/2)\sin(l\alpha/2)}{l}\right\}. \tag{B·3}$$

Proof

Subtracting the second column of the determinant of $a_{l,m}^{(I)}$ from the first column and performing a similar subtraction of the m-th column from the $(m-1)$-th column step by step $(m=2, 3, \cdots, n)$, we get

$$\det(a_{l,m}^{(I)}(x)) = \begin{vmatrix} b_{1,1}^{(I)}, & \cdots\cdots & b_{1,n-1}^{(I)}, & a_{1,n}^{(I)}(x) \\ \cdots & \cdots & \cdots & \cdots \\ b_{I-1,1}^{(I)}, & \cdots\cdots & b_{I-1,n-1}^{(I)}, & a_{I-1,n}^{(I)}(x) \\ 0 & \cdots\cdots & 0 & 1 \\ b_{I+1,1}^{(I)} & \cdots\cdots & b_{I+1,n-1}^{(I)}, & a_{I+1,n}^{(I)}(x) \\ \cdots & \cdots & \cdots & \cdots \\ b_{n,1}^{(I)}, & \cdots\cdots & b_{n,n-1}^{(I)} & a_{n,n}^{(I)}(x) \end{vmatrix}$$

$$= (-1)^{I+n} \begin{vmatrix} b_{1,1}^{(I)} & \cdots\cdots & b_{1,n-1}^{(I)} \\ \cdots & \cdots & \cdots \\ b_{I-1,1}^{(I)} & \cdots\cdots & b_{I-1,n-1}^{(I)} \\ b_{I+1,1}^{(I)} & \cdots\cdots & b_{I+1,n-1}^{(I)} \\ \cdots & \cdots & \cdots \\ b_{n,1}^{(I)} & \cdots\cdots & b_{n,n-1}^{(I)} \end{vmatrix}, \tag{B·4}$$

where

$$b_{l,m}^{(I)}=a_{l,m}^{(I)}(x)-a_{l,m+1}^{(I)}(x)=\sum_q \frac{2\left(\frac{\pi h}{L}\right)^3 e^{iq\cdot x/h}}{\left(q+\frac{2\pi h}{L}l-\frac{\pi h}{L}\right)\left[\left(q+\frac{2\pi h}{L}m\right)^2-\left(\frac{\pi h}{L}\right)^2\right]}\,.$$

$$\begin{pmatrix}1\leqq m\leqq n-1\\ 1\leqq l\leqq n,\ l\neq I\end{pmatrix} \tag{B·5}$$

Then the summation of $\det(a_{l,m}^{(I)}(x))$ over $I=1, 2, \cdots, n$ gives

$$\sum_{I=1}^{n}\det(a_{l,m}^{(I)}(x))=\begin{vmatrix} b_{1,1} & \cdots & b_{1,n-1} & 1\\ \cdots & \cdots & \cdots & \cdots \\ & & & \\ b_{n,1} & \cdots & b_{n,n-1} & 1\end{vmatrix}, \tag{B·6}$$

where

$$b_{l,m}=\sum_q \frac{2\left(\frac{\pi h}{L}\right)^3 e^{iq\cdot x/h}}{\left(q+\frac{2\pi h}{L}l-\frac{\pi h}{L}\right)\left[\left(q+\frac{2\pi h}{L}m\right)^2-\left(\frac{\pi h}{L}\right)^2\right]}\,.$$

$$(1\leqq l\leqq n,\ 1\leqq m\leqq n-1) \tag{B·7}$$

Now, by subtracting the second row of the determinant on the right-hand side of (B·6) from the first row and by carrying out similar subtractions of the $l+1$-th row from the l-th row step by step, Eq. (B·6) becomes

$$\sum_{I=1}^{n}\det(a_{l,m}^{(I)}(x))=\begin{vmatrix} g_{1,1}\, g_{1,2} & \cdots\cdots & g_{1,n-1} & 0\\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ g_{n-1,1} & \cdots\cdots & g_{n-1,n-1} & 0\\ b_{n,1} & \cdots\cdots & b_{n,n-1} & 1\end{vmatrix}=\begin{vmatrix} g_{1,1} & \cdots\cdots & g_{1,n-1}\\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ g_{n-1,1} & \cdots\cdots & g_{n-1,n-1}\end{vmatrix}, \tag{B·8}$$

where

$$g_{l,m}=b_{l,m}-b_{l+1,m}=\sum_q \frac{4\left(\frac{\pi h}{L}\right)^4 e^{iq\cdot x/h}}{\left[\left(q+\frac{2\pi h}{L}l\right)^2-\left(\frac{\pi h}{L}\right)^2\right]\left[\left(q+\frac{2\pi h}{L}m\right)^2-\left(\frac{\pi h}{L}\right)^2\right]}$$

$$=\frac{\pi^2}{4}e^{-i((l+m)/2)(2\pi/L)x}\Bigg[\ 2\cos\left(\frac{\pi x}{L}\right)\delta_{l,m}-\delta_{l,m+1}-\delta_{l+1,m}$$

$$+\frac{2}{\pi}\left\{\frac{\sin\left(\frac{\pi}{L}(l-m+1)x\right)}{l-m+1}+\frac{\sin\left(\frac{\pi}{L}(l-m-1)x\right)}{l-m-1}\right.$$

$$-2\frac{\sin\left(\frac{\pi}{L}(l-m)x\right)}{l-m}\cos\left(\frac{\pi}{L}x\right)\Bigg\}\Bigg] . \tag{B·9}$$

Here, when the denominators in the curly bracket of (B·9) are equal to zero, its term denotes the limiting value such as

$$\frac{\sin\left(\frac{\pi}{L}(l-m)x\right)}{l-m}=\frac{\pi}{L}x \qquad \text{for } l=m \text{ and so on .} \tag{B·10}$$

By taking out the factors $(\pi^2/4)e^{-i(l/2)(2\pi/L)x}$ and $e^{-i(m/2)(2\pi/L)x}$ from the l-th row and the m-th column, respectively, we have

$$\sum_{l=1}^{n}\det(a_{l,m}^{(l)}(x))=e^{-i(n(n-1)/2)(2\pi/L)x}\left(\frac{\pi^2}{4}\right)^{n-1}\det(C_{s-t}(\alpha)), \tag{B·11}$$

where $\alpha=2\pi x/L$, and

$$C_l(\alpha)=2\delta_{l,0}\cos(\alpha/2)-\delta_{l,1}-\delta_{l,-1}$$
$$+\frac{2}{\pi}\left\{\frac{\sin((l+1)\alpha/2)}{l+1}+\frac{\sin((l-1)\alpha/2)}{l-1}-2\frac{\cos(\alpha/2)\sin(l\alpha/2)}{l}\right\}. \tag{B·12}$$

Thus, the proof of Formula B is completed.

[Formula C]

The operator $G(v)$ given by (3·1) is a unitary operator and yields the following Galilei-transformation:

$$G(v)a_p{}^*G(v)^*=a_{p+mv}^* , \quad G(v)a_pG(v)^*=a_{p+mv} . \tag{C·1}$$

Proof

The operator $G_n(v)$ in (3·2) can be readily seen to transform the free eigenstate in (2·1) of III as follows:

$$G_n(v)|q_1,\cdots,q_n\rangle=\sum_{p_1\leqq p_2\leqq\cdots\leqq p_n}a_{p_1+mv,\cdots,p_n+mv}a_{p_1,\cdots,p_n}$$
$$\times\prod_{i=1}^{n}a_{p_i+mv}^*\prod_{i=1}^{n}a_{p_i}a_{q_1,\cdots,q_n}\prod_{i=1}^{n}a_{q_i}^*|0\rangle$$
$$=\sum_{p_1\leqq p_2\leqq\cdots\leqq p_n}a_{p_1+mv,\cdots,p_n+mv}\prod_{i=1}^{n}\delta_{p_i,q_i}\prod_{i=1}^{n}a_{p_i+mv}^*|0\rangle$$
$$=|q_1+mv,\cdots,q_n+mv\rangle . \tag{C·2}$$

From (C·2) and $G_n{}^*(v)=G_n(-v)$ which is due to the definition (3·2), we have

$$G_n{}^*(v)G_n(v)|q_1,\cdots,q_n\rangle=|q_1,\cdots,q_n\rangle ,$$

$$G_n(v)G_n(v)^*|q_1, \cdots, q_n\rangle = |q_1, \cdots, q_n\rangle . \qquad (n \geqq 1) \tag{C·3}$$

Now, in the same way as the derivation of the unitary operator U (see (3·1) ~(3·12) in III), the operator $G(v)$ satisfies

$$G(v)|0\rangle = |0\rangle , \quad G(v)|q_1, \cdots, q_n\rangle = G_n(v)|q_1, \cdots, q_n\rangle , \quad (n \geqq 1)$$

$$G^*(v)|0\rangle = |0\rangle , \quad G^*(v)|q_1, \cdots, q_n\rangle = G_n{}^*(v)|q_1, \cdots, q_n\rangle . \quad (n \geqq 1) \tag{C·4}$$

From (C·4) and (C·3), we can see that

$$G^*(v)G(v)|0\rangle = |0\rangle , \quad G(v)G^*(v)|0\rangle = |0\rangle ,$$

$$G^*(v)G(v)|q_1, \cdots, q_n\rangle = |q_1, \cdots, q_n\rangle ,$$

$$G(v)G^*(v)|q_1, \cdots, q_n\rangle = |q_1, \cdots, q_n\rangle . \quad (n \geqq 1) \tag{C·5}$$

Since the free states $\{|0\rangle, |q_1, \cdots, q_n\rangle (n \geqq 1)\}$ form a complete set for a system with arbitrary number of bosons, we can conclude that

$$G(v)G(v)^* = 1 , \qquad G(v)^* G(v) = 1 . \tag{C·6}$$

By this unitary operator $G(v)$, the operators $a_p{}^*$ and a_p are proved to be transformed as (C·1), because

$$\begin{aligned} G(v)a_p{}^*G(v)^*|q_1, \cdots, q_n\rangle &= G(v)a_p{}^*G_n{}^*(v)|q_1, \cdots, q_n\rangle \\ &= G(v)a_p{}^*|q_1 - mv, \cdots, q_n - mv\rangle \\ &= G_{n+1}(v)a_p{}^*|q_1 - mv, \cdots, q_n - mv\rangle = a_{p+mv}^*|q_1, \cdots, q_n\rangle . \quad (\text{for all } n) \end{aligned} \tag{C·7}$$

Thus, the formula C has been verified.

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