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One-Dimensional Many Boson System. I

——*Exact Solution in Field Theoretical Form*——

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The eigenvalue problem in a one-dimensional many boson system with a repulsive delta-function potential has been solved exactly from the standpoint of field theory. All eigenstates are expressed in a simple and compact form by introducing appropriate quantum numbers for their specification. The energy eigenvalue and eigenstate are determined by solutions of simultaneous equations. It is verified that the simultaneous equations have unique solutions.

§ 1. Introduction

Several microscopic theories[1)~4)] for an interacting many boson system have recently succeeded in deriving the definite form of the interactions among elementary excitations and the observed elementary excitation spectrum of He-II.[5)] These developments encourage us to intend to make clear, for example, the unknown mechanism of the λ-transition from a microscopic point of view. For this achievement it is necessary to clarify at least qualitatively the whole structure of energy levels involving not only the low-lying excitation but also the multiple excitation, since the level structure of the multiple excitation is indispensable for understanding the mechanism of the λ-transition. The whole structure, unfortunately, has not been clarified on account of the fact that the interactions among elementary excitations are obliged to be treated by the perturbation method. Here is an interesting model, proposed by Girardeau,[6)] Lieb and Liniger,[7)] in the following sense: Their model has a merit that the eigenvalue problem can be solved exactly, and the solvability suggests that it is possible to make clear the whole structure of energy levels. As being one-dimensional, however, this model is oversimplified concerning dimensionality. Nevertheless it can be expected that clarification of whole structure of energy levels in their model may give a clue for the analysis of whole structure in a three-dimensional many boson system. This expectation leads us to the further investigation of the above model in the present and forthcoming papers.[8)~10)]

The Lieb and Liniger model is a one-dimensional system of many bosons interacting via a repulsive delta-function potential with strength g which involves the Girardeau model as the case $g\to\infty$. They have shown that the exact eigenfunction and eigen-energy are determined by the solutions of simultaneous equations, and have precisely investigated the behavior of ground state energy as a function of g. In their work, however, there are some remaining problems: (i) How are determined concretely the solutions of their simultaneous equations? (ii) Does the state called the ground state by them have certainly a minimum energy? (iii) Do all of their eigen-functions form a complete orthonormal set? (iv) How is it possible to investigate systematically the whole structure of energy levels? Particularly, it is a serious obstacle to solving the above third problem that their eigen-functions are composed of tremendously many terms, namely, $(n!)^2$ terms (n indicates the total number of particles).

In this paper, the eigenvalue problem for the Hamiltonian (see (2·1)) in Lieb and Liniger's model[7] is solved exactly from the standpoint of the field theory (this standpoint is different from the one of the *first quantization formalism* in the works of Girardeau, Lieb and Liniger). Consequently, every eigenstate is expressed in a simple and compact form by introducing appropriate quantum numbers for its specification. This dissolves the obstacle mentioned above, and we can give the answers about the above remaining problems.

In § 2, the eigenvalue problem is solved exactly, and it is shown that the eigenstates and energy eigenvalues are determined by the solutions of simultaneous transcendental equations. Section 3 is devoted to the solution of the problem (i), namely, we will give the successive method to obtain approximate solutions of simultaneous equations that approach closely the exact solutions as much as one wishes. In a subsequent paper,[8] other problems (ii) and (iii) will be discussed on the basis of the results obtained in the present paper.

The following further development will be given in forthcoming papers.[9),10] We can construct explicitly such a unitary transformation U that transforms the free states to the exact eigenstates in the interacting system. On the basis of the unitary transformation U, new operators A_p^* and A_p are introduced, and it is shown that these indicate the creation and annihilation operators of new bose particles dressed exactly with interaction cloud. When being expressed in terms of the new operators, the ground state $|G\rangle$ proves to be the condensed state of all exactly dressed particles with zero momentum. This gives a remarkable result $\langle G|A_0{}^*A_0|G\rangle/n=1$ in contrast to the result $\langle G|a_0{}^*a_0|G\rangle/n=0$ obtained by Lenard et al.[11),12] in the case of infinitely large g. We can investigate the whole structure of the energy levels in terms of the number distribution of the exactly dressed particles, especially the excitation energy ε_p which is defined as the energy increase of the system when the momentum of an exactly dressed particle is changed from zero to p in the sea composed of the other many dressed particles with the number distribution $\{n_q\}$. Then, we have such a remarkable result that the excitation energy ε_p does depend upon the number distribution $\{n_q\}$, especially strongly upon the number n_0 of exactly dressed particles with zero momentum.

In other words the excitation energy ε_p does "*lose the phonon character drastically*" when the ratio n_0/n (n indicates the total number of particles) tends to zero. On the basis of the properties of the exactly dressed particles we will discuss some characteristic properties in Liquid He-II in forthcoming papers.[10] Such a fact should be noted here that the unitary transformation U cannot be expanded in a power series of the coupling constant g, although it approaches the identical transformation in the limit $g\to 0$. This means that we cannot reach the above results by the perturbation method.

§ 2. Eigenvalue problem

We consider a system of many bosons of mass m interacting via a repulsive delta-function potential with a positive strength g in a one-dimensional region of length L. This system is described by the Hamiltonian

$$H=\sum_p \frac{p^2}{2m}a_p{}^* a_p+\sum_{p,q,r}\frac{g}{2L}a_{p+r}^* a_{q-r}^* a_q a_p\,, \tag{2·1}$$

where $a_p{}^*$ and a_p indicate creation and annihilation operators of a bose particle with momentum $p=2\pi hl/L$(l; integer), and are subjected to usual commutation relations

$$[a_p,\, a_q{}^*]=\delta_{p,q}\,,\quad [a_p,\, a_q]=0\,,\quad [a_p{}^*,\, a_q{}^*]=0\,.$$

2a. *Two-boson system*

In order to obtain the clue for searching the exact eigenstates of many boson system, we will investigate a system of interacting two bosons. The eigenstate for the two-boson system is expressed by the following form:

$$|\Psi_Q\rangle=\sum_p c(-p,\, p+Q)a_{-p}^* a_{p+Q}^*|0\rangle, \tag{2·2}$$

where $|0\rangle$ indicates the vacuum state, that is, $a_p|0\rangle=0$ for all p, and Q denotes the total momentum of the system. The expansion coefficient $c(-p,\, p+Q)$ is assumed to have the symmetric property

$$c(-p,\, p+Q)=c(p+Q,\, -p), \tag{2·3}$$

due to bose nature of the system. The eigen-energy E and the coefficient $c(-p,\, p+Q)$ are determined by the equation $(H-E)|\Psi_Q\rangle=0$, and one has

$$\left\{E-\frac{1}{2m}(p^2+(p+Q)^2)\right\}c(-p,\, p+Q)=\frac{g}{L}\sum_q c(-q,\, q+Q), \tag{2·4}$$

by making use of (2·3). From this equation, one gets

$$c(-p,\, p+Q)=a(Q)/\{E-(p^2+(p+Q)^2)/2m\}, \tag{2·5}$$

where $\alpha(Q)$ indicates a normalization constant. Then the equation to determine the eigenenergy E is given by

$$\frac{g}{L}\sum_q \frac{1}{E-(q^2+(q+Q)^2)/2m}=1\,. \tag{2·6}$$

By expressing the momenta q and Q as $2\pi\hbar j/L$ and $2\pi\hbar M/L$ (j, M; integer), respectively, Eq. (2·6) is rewritten as

$$\sum_{j=-\infty}^{+\infty}\frac{1}{((L/2\pi\hbar)^2 4mE-M^2)-(2j+M)^2}=\frac{L}{g}\frac{1}{4m}\left(\frac{2\pi\hbar}{L}\right)^2. \tag{2·7}$$

Since the right-hand side of (2·7) is always positive for the positive coupling constant g, the quantity $\{(L/2\pi\hbar)^2 4mE-M^2\}$ must be positive to ensure the existence of solutions of Eq. (2·7), and hence can be denoted by ξ^2,

$$\xi^2=(L/2\pi\hbar)^2 4mE-M^2\,. \qquad (\xi\neq 0) \tag{2·8}$$

The summation over j on the left-hand side of (2·7) gives

$$\cot(\pi\xi/2)=2\pi\hbar^2\xi/mgL \qquad \text{for even integer } M \tag{2·9a}$$

and

$$\cot(\pi(\xi+1)/2)=2\pi\hbar^2\xi/mgL \qquad \text{for odd integer } M\,, \tag{2·9b}$$

where we have made use of the formula

$$\sum_{j=-\infty}^{+\infty}\frac{1}{x^2-(2j)^2}=\frac{\pi}{2x}\cot\left(\frac{\pi x}{2}\right),\quad \sum_{j=-\infty}^{+\infty}\frac{1}{x^2-(2j+1)^2}=\frac{\pi}{2x}\cot\left(\frac{\pi(x+1)}{2}\right).$$

Equations (2·9a, b) can be solved by the graphical method. As readily seen from the intersecting points in Fig. 1, the eigenvalues of ξ are given by two sequences: The one denotes positive eigenvalues of ξ, and the other negative ones.

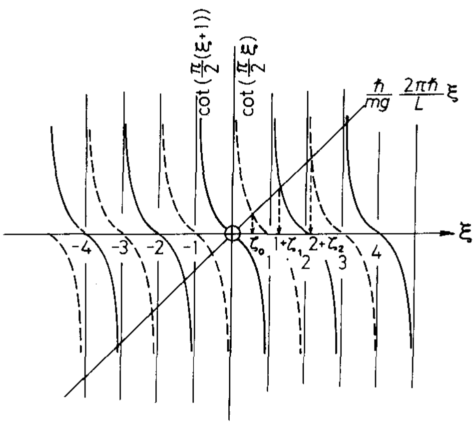

Fig. 1. The solutions of Eqs. (2·9a, b) by the graphical method. The schematic behaviors of the functions $\cot(\pi\xi/2)$ and $\cot(\pi(\xi+)/2)$ are drawn by the dotted and solid curved lines, respectively. The straight line indicates $2\pi\hbar^2\xi/mgL$. The solutions of Eqs. (2·9a, b) are determined from the intersecting points between the straight line and the curved lines, where the point $\xi=0$ is excluded due to $\xi^2>0$.

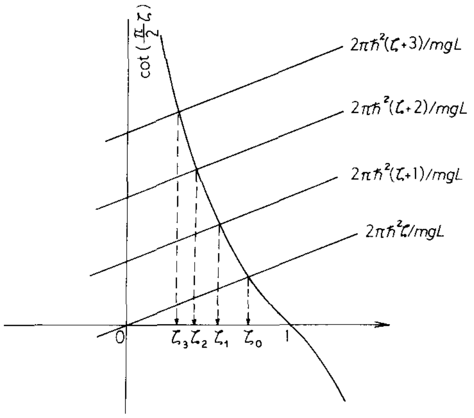

Fig. 2. The graphical solutions of Eq. (2·11). The behaviors of the function $2\pi h^2(l+\zeta_l)/mgL$ are drawn by the straight lines. The curved line indicates $\cot(\pi\zeta_l/2)$. The intersecting points between the curved line and the straight ones determine the solutions $(\zeta_0, \zeta_1, \zeta_2, \cdots)$ of Eq. (2·11).

Both sequences give the same result for the energy eigenvalues from (2·8) and for eigenstates from (2·2) and (2·5). In the following, therefore, we adopt the sequence of positive eigenvalues of ξ for avoiding the double specification of the energy eigenvalues and eigenstates. Then the eigenvalues of ξ are given by

$$\xi_l = l + \zeta_l\,, \qquad 0 < \zeta_l \leqq 1 \tag{2·10a}$$

and

$$\begin{aligned} l &= 2j, \quad (j=0,1,2\cdots) && \text{for even integer } M\,, \\ l &= 2j+1\,, (j=0,1,2\cdots) && \text{for odd integer } M\,, \end{aligned} \tag{2·10b}$$

where the quantity ζ_l indicates the deviation of the eigenvalue ξ_l from an integer $l(\geqq 0)$, and takes unity in the case of infinitely large g. For a given integer $l(\geqq 0)$ the deviation ζ_l is determined as a root of the equation

$$\cot(\pi\zeta_l/2) = 2\pi h^2(l+\zeta_l)/mgL\,, \tag{2·11}$$

in the region $0 < \zeta_l \leqq 1$. The graphical solution of (2·11) is shown in Fig. 2. As readily seen from Fig. 2, for a given integer $l(\geqq 0)$ Eq. (2·11) has a unique solution within $0 < \zeta_1 \leqq 1$. From (2·8), the energy eigenvalues are given by

$$E_{l,Q} = (\pi h/L)^2\{(M-l-\zeta_l)^2 + (M+l+\zeta_l)^2\}/2m\,, \quad (l=0,1,2,\cdots) \tag{2·12}$$

and the eigenstates are expressed as

$$| \Psi_{l,Q}\rangle = \sum_p \frac{a(Q)}{(1/2m)[(\pi h/L)^2\{(M-l-\zeta_l)^2+(M+l+\zeta_l)^2\} - p^2 - (p+Q)^2]} a^*_{-p} a^*_{p+Q}|0\rangle\,, \quad (l=0,1,2,\cdots) \tag{2·13}$$

from (2·2), (2·5) and (2·12). In this way the eigenvalue problem for a two-boson system has been solved.

At first sight it seems impossible to extend the form of the eigenstate (2·13) in a two-boson system to the case of an n-boson system. As will be seen later, however, this difficulty can be overcome by rearranging the expression (2·13) to a simple form with the help of introduction of new quantum numbers. This is due to the fact that an appropriate choice of the variables is essential for the treatment of many body system. Noting that the quantities $(M\pm l)/2$ are integers from (2·10a, b), let us introduce

$$q_1=2\pi h(M-l)/2L \text{ and } q_2=2\pi h(M+l)/2L\,, \tag{2·14a}$$

as new quantum numbers instead of l and $Q=2\pi hM/L$. These new quantum numbers q_1 and q_2 are subjected to

$$q_1\leqq q_2 \tag{2·14b}$$

due to $l\geqq 0$. Furthermore, instead of ζ_l, we define the quantity k_{12}

$$k_{12}=-\pi h\zeta_l/L\,. \tag{2·15}$$

If one makes use of the new variables q_1, q_2 and k_{12}, Eq. (2·11) can be rewritten as

$$\cot(Lk_{12}/2h)=h(2k_{12}+q_1-q_2)/mg \tag{2·16a}$$

within the region

$$-(\pi h/L)\leqq k_{12}<0\,. \tag{2·16b}$$

Then the energy eigenvalue (2·12) is expressed in terms of the root k_{12} of (2·16a) as

$$E_{q_1,q_2}=(1/2m)\{(k_{12}+q_1)^2+(-k_{12}+q_2)^2\} \tag{2·17}$$

for given quantum numbers q_1 and q_2. On the other hand, the eigenstate (2·13) is rearranged as follows:

$$\begin{aligned}
|\Psi_{q_1,q_2}\rangle &=\sum_p \frac{2ma(Q)}{(k_{12}+q_1)^2+(-k_{12}+q_2)^2-(p+q_1+q_2)^2-p^2}a^*_{p+q_1+q_2}a^*_{-p}|0\rangle \\
&=\sum_p \frac{2ma(Q)}{(k_{12}+q_1)^2+(-k_{12}+q_2)^2-(p+q_1)^2-(-p+q_2)^2}a^*_{p+q_1}a^*_{-p+q_2}|0\rangle \\
&=\sum_p \frac{ma(Q)}{-(p-k_{12})(p+k_{12}+q_1-q_2)}a^*_{p+q_1}a^*_{-p+q_2}|0\rangle \\
&=\sum_p \frac{-ma(Q)}{(2k_{12}+q_1-q_2)}\left\{\frac{1}{p-k_{12}}-\frac{1}{p+k_{12}+q_1-q_2}\right\}a^*_{p+q_1}a^*_{-p+q_2}|0\rangle
\end{aligned}$$

$$=\beta_{q_1,q_2}\sum_{p_{12}}\frac{-k_{12}}{p_{12}-k_{12}}a^*_{p_{12}+q_1}a^*_{-p_{12}+q_2}|0\rangle, \tag{2·18}$$

where β_{q_1,q_2} indicates a new normalization constant, and we have replaced the variable p of the second term in the curly bracket of the fifth expression by $-p-q_1+q_2$. Thus the eigenstate (2·13) has been rewritten in such a simple form as (2·18).

The coefficient in (2·13) has the symmetric property for exchange of the momenta $-p$ and $p+Q$. On the other hand, the coefficient $d(p_{12};k_{12})=(-k_{12})/(p_{12}-k_{12})$ in (2·18) has no such a symmetric property (although this sacrifice is recovered by taking the sum over p_{12} and by using bose nature of the operators in (2·18)). In compensation for the sacrifice of the symmetricity of the coefficient $d(p_{12};k_{12})$, we have been able to obtain the simple expression (2·18) of the eigenstate. In the rearranged state (2·18), the new quantum numbers q_1 and q_2 mean the momenta attached to two bosons, respectively, and the momentum p_{12} denotes a transferred momentum between them. This significance of the new variables and the simple form of the expansion coefficient $d(p_{12};k_{12})$ in a two-boson system give the clue for searching the exact eigenstates of many boson system.

2b. *n-boson system*

By the aid of the clue mentioned above, we *assume* that the eigenstate for n-boson system is given by the following form,

$$|\Psi_{q_1,q_2,\cdots,q_n}\rangle=\beta_{q_1,q_2,\cdots,q_n}\sum_{\{p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n}d(p_{i,j};k_{i,j})\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}|0\rangle, \tag{2·19}$$

where the factors $d(p_{i,j};k_{i,j})$ are defined as

$$d(p_{i,j};k_{i,j})=(-k_{i,j})/(p_{i,j}-k_{i,j}), \tag{2·20}$$

and $\beta_{q_1,q_2,\cdots,q_n}$ indicates the normalization constant. The n momenta $q_i=(\text{integer})\times 2\pi h/L$ $(i=1,2,\cdots,n)$ indicate the quantum numbers corresponding to the momenta q_1 and q_2 in (2·18). Similarly $p_{i,j}=(\text{integer})\times 2\pi h/L$ and $k_{i,j}(1\leqq i<j\leqq n)$ have been introduced as the generalized ones from p_{12} and k_{12} in (2·18), where $k_{i,j}$ $(1\leqq i<j\leqq n)$ are the quantities to be determined by the eigen-equation for the state (2·19). For convenience of notations, $-p_{i,j}$ and $-k_{i,j}$ $(1\leqq i<j\leqq n)$ have been expressed as $p_{j,i}$ and $k_{j,i}$ in the suffixes of the creation operators in (2·19), respectively,

$$p_{j,i}=-p_{i,j}\,,\qquad k_{j,i}=-k_{i,j}\,.\quad(1\leqq i<j\leqq n)$$

In order to show that the state (2·19) is really the exact eigenstate of (2·1), we operate the Hamiltonian (2·1) to the state (2·19), and have

$$H|\Psi_{q_1,q_2,\cdots,q_n}\rangle=\beta_{q_1,q_2,\cdots,q_n}\sum_{\{p_{i,j};1\leq i<j\leq n\}}\left\{\sum_{\alpha=1}^{n}\frac{1}{2m}(\sum_{\beta\neq\alpha}p_{\alpha,\beta}+q_\alpha)^2\prod_{1\leq i<j\leq n}d(p_{i,j};k_{i,j})\right.$$

$$\left.+\sum_{1\leq\alpha<\beta\leq n}\frac{g}{L}\sum_{r}d(r;k_{\alpha,\beta})\prod_{\substack{1\leq i<j\leq n\\(i,j)\neq(\alpha,\beta)}}d(p_{i,j};k_{i,j})\right\}\prod_{i=1}^{n}a^*_{\sum_{j=1,\,j\neq i}^{n}p_{i,j}+q_i}|0\rangle,\tag{2·21}$$

where the symbol $(i,j)\neq(\alpha,\beta)$ indicates that the factor $d(p_{\alpha,\beta};k_{\alpha,\beta})$ should be excluded. For brevity let us here introduce the following variables defined by

$$p_i=\sum_{\substack{j=1\\j\neq i}}^{n}p_{i,j}+q_i\,;\qquad k_i=\sum_{\substack{j=1\\j\neq i}}^{n}k_{i,j}+q_i\,.\quad(i=1,2,\cdots,n)\tag{2·22}$$

Then the first term in the curly bracket of (2·21) is rewritten as

$$\begin{aligned}\sum_{\alpha=1}^{n}(\sum_{\beta\neq\alpha}p_{\alpha,\beta}+q_\alpha)^2/2m&=\sum_{\alpha=1}^{n}(p_\alpha^2-k_\alpha^2+k_\alpha^2)/2m\\&=\sum_{\alpha=1}^{n}k_\alpha^2/2m+\sum_{\alpha=1}^{n}(p_\alpha+k_\alpha)\sum_{\beta\neq\alpha}(p_{\alpha,\beta}-k_{\alpha,\beta})/2m\\&=\sum_{\alpha=1}^{n}k_\alpha^2/2m+\sum_{1\leq\alpha<\beta\leq n}(p_{\alpha,\beta}-k_{\alpha,\beta})(p_\alpha-p_\beta+k_\alpha-k_\beta)/2m\,.\end{aligned}\tag{2·23}$$

Substitution of this final form in (2·21) gives

$$\begin{aligned}&(H-\sum_{\alpha=1}^{n}k_\alpha^2/2m)|\Psi_{q_1,q_2,\cdots,q_n}\rangle\\&=\beta_{q_1,q_2,\cdots,q_n}\sum_{\{p_{i,j};1\leq i<j\leq n\}}\sum_{1\leq\alpha<\beta\leq n}[(-k_{\alpha,\beta}/2m)\{(p_\alpha-p_\beta)\\&\quad+(k_\alpha-k_\beta)-(mg/\hbar)\cot(Lk_{\alpha,\beta}/2\hbar)\}\\&\quad\times\prod_{\substack{1\leq i<j\leq n\\(i,j)\neq(\alpha,\beta)}}d(p_{i,j};k_{i,j})]\prod_{i=1}^{n}a^*_{\sum_{j=1,\,j\neq i}^{n}p_{i,j}+q_i}|0\rangle,\end{aligned}\tag{2·24}$$

where we have made use of the formula

$$\sum_{l=-\infty}^{+\infty}\frac{1}{l-x}=-\pi\cot(\pi x).\tag{2·25}$$

The first term in the curly bracket of (2·24) vanishes. This can be seen in the following way. Replacement of the summation variables $p_{\alpha,\beta}$ by $[-p_{\alpha,\beta}-q_\alpha+q_\beta-\sum_{j\neq\alpha,\beta}(p_{\alpha,j}-p_{\beta,j})]$ gives

$$\sum_{\{p_{i,j};1\leq i<j\leq n\}}(p_\alpha-p_\beta)\prod_{\substack{1\leq i<j\leq n\\(i,j)\neq(\alpha,\beta)}}d(p_{i,j};k_{i,j})\prod_{\substack{i=1\\j\neq\alpha,\beta}}^{n}a^*_{p_i}a^*_{\sum_{j=1,\,j\neq\alpha,\beta}^{n}p_{\alpha,j}+p_{\alpha,\beta}+q_\alpha}\,a^*_{\sum_{j=1,\,j\neq\alpha,\beta}^{n}p_{\beta,j}-p_{\alpha,\beta}+q_\beta}|0\rangle$$

$$= \sum_{\{p_{i,j};1\leqq i<j\leqq n\}} -(p_\alpha - p_\beta) \prod_{\substack{1\leqq i<j\leqq n \\ (i,j)\neq(\alpha,\beta)}} d(p_{i,j};\, k_{i,j})$$

$$\times \prod_{\substack{i=1 \\ i\neq\alpha,\beta}}^{n} a_{p_i}^* a^*_{\sum_{j=1, j\neq\alpha,\beta}^{n} p_{\beta,j} - p_{\alpha,\beta} + q_\beta}\, a^*_{\sum_{j=1, j\neq\alpha,\beta}^{n} p_{\alpha,j} + p_{\alpha,\beta} + q_\alpha} |0\rangle$$

$$= 0\,,$$

where the last equality is due to the bose nature of the operators. Then, if one demands that the unknown $k_{i,j}$ are determined as the roots of the simultaneous equations

$$\cot\left(\frac{Lk_{i,j}}{2h}\right) = \frac{h}{mg}(k_i - k_j)$$

$$= \frac{h}{mg}\{2k_{i,j} + \sum_{l\neq i,j}(k_{i,l} - k_{j,l}) + q_i - q_j\}, \quad (1\leqq i<j\leqq n) \tag{2·26}$$

one can see that (2·19) becomes really the eigenstate for n-boson system and the energy eigenvalue is given by

$$E_{q_1,q_2,\cdots,q_n} = \sum_{i=1}^{n} k_i^2/2m\,. \tag{2·27}$$

The use of all roots of the simultaneous equations (2·26) in (2·19) and (2·27) yields redundant eigenstates and eigen-energies. Similar situation has appeared in the case of the two-boson system and the redundant solution for (2·16a) has been excluded by the restrictions (2·14b) and (2·16b). In analogy with these restrictions, we adopt*) the restrictions

$$q_1 \leqq q_2 \leqq \cdots \leqq q_n\,, \tag{2·28}$$

$$-\pi h/L \leqq k_{i,j} < 0\,, \quad (1\leqq i<j\leqq n) \tag{2·29}$$

for eliminating the redundant eigenstates and eigenenergies in our many boson system.

The case of infinitely large g is so simple that the equations (2·26) can be easily solved under the conditions (2·28) and (2·29), and the results are given by

$$k_{i,j} = -\frac{\pi h}{L}, \quad (1\leqq i<j\leqq n); \quad k_i = \frac{2\pi h}{L}\left(i - \frac{n+1}{2}\right) + q_i\,. \quad (i=1, 2, \cdots, n) \tag{2·30}$$

In the general case, however, the simultaneous equations (2·26) cannot be solved analytically, since they are transcendental equations. In spite of this difficulty, C. N. Yang and C. P. Yang[13] have verified that the equations (2·26) have unique

*) In fact it can be shown that the eigenstates corresponding to the roots outside the restricted regions (2·28) and (2·29) are necessarily involved in the eigenstates in the regions (2·28) and (2·29).

solutions, but the procedure to obtain concretely the solutions has not been given in their work. In the next section, therefore, we will show a successive method by which the solutions can be obtained numerically.

§ 3. Successive method to solve simultaneous equations

In this section we will give a successive method to obtain approximate solutions of Eqs. (2·26) which approach closely the exact solutions as much as one wishes. On the basis of this new method, we will prove that the equations (2·26) have unique solutions $k_{i,j}$ under the restrictions (2·28) and (2·29), where the method of this verification is different from C. N. Yang and C. P. Yang's one. Here the term "unique solutions" means that for any set of quantum numbers (q_1, $q_2, \cdots, q_n$) satisfying (2·28), there necessarily exists only one set of the solutions $k_{i,j}$ of the equations within the regions (2·29).

For convenience of later discussion, let us regard the definitions of k_i in (2·22) and the conditions (2·26) for $k_{i,j}$ as the coupled equations

$$k_{i,j} = -\frac{2h}{L}\cot^{-1}\left[\frac{h}{mg}(k_j - k_i)\right], \qquad (1 \leqq i < j \leqq n) \tag{3·1}$$

$$k_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{i,j} + q_i\,, \qquad (i = 1, 2, \cdots, n) \tag{3·2}$$

to determine k_i and $k_{i,j}$ as the functions concerning given quantum numbers q_1, q_2, $\cdots q_n$, where the region of the function $\cot^{-1}(x)$ for $x \geqq 0$ has been taken as

$$0 < \cot^{-1}(x) \leqq \pi/2\,, \qquad (x \geqq 0) \tag{3·3}$$

due to the restrictions (2·29). Equations (3·1) indicate that $n(n-1)/2$ unknown $k_{i,j}$ are determined from unknown differences $(k_j - k_i)$. From this fact it becomes easy to treat the coupled equations (3·1) and (3·2) if one can obtain the closed equations for the differences $(k_j - k_i)$. This aim is accomplished by introducing $(n-1)$ parameters $\varDelta_i$ in the following way,

$$\varDelta_i = k_{i+1} - k_i = \sum_{l \neq i+1} k_{i+1,l} - \sum_{l \neq i} k_{i,l} + q_{i+1} - q_i\,, \tag{3·4}$$

which are nonnegative from (3·1) and (2·29) as

$$\varDelta_i = -\frac{mg}{h}\cot\left(\frac{Lk_{i,i+1}}{2h}\right) \geqq 0\,. \tag{3·5}$$

By noting that a difference $(k_j - k_i)$ can be expressed by

$$k_j - k_i = \varDelta_{j-1} + \varDelta_{j-2} + \cdots + \varDelta_i\,, \qquad (1 \leqq i < j \leqq n) \tag{3·6}$$

the right-hand side of Eqs. (3·1) can be rewritten as

$$k_{i,j} = -\frac{2\hbar}{L}\cot^{-1}\left[\frac{\hbar}{mg}\sum_{l=i}^{j-1}\Delta_l\right], \tag{3·7}$$

in terms of the parameters Δ_l. Substitution of (3·7) in (3·4) gives the equations for the parameters Δ_l,

$$\begin{aligned}\Delta_i &= -2k_{i,i+1} - \sum_{l=1}^{i-1}(k_{l,i+1}-k_{l,i}) + \sum_{l=i+2}^{n}(k_{i+1,l}-k_{i,l}) + q_{i+1} - q_i \\ &= \frac{4\hbar}{L}\cot^{-1}\left(\frac{\hbar}{mg}\Delta_i\right) + \frac{2\hbar}{L}\sum_{l=1}^{i-1}\left[\cot^{-1}\left\{\frac{\hbar}{mg}(\Delta_i + X_{i,l})\right\} - \cot^{-1}\left(\frac{\hbar}{mg}X_{i,l}\right)\right] \\ &\quad + \frac{2\hbar}{L}\sum_{l=i+2}^{n}\left[\cot^{-1}\left\{\frac{\hbar}{mg}(\Delta_i + X_{l,i+1})\right\} - \cot^{-1}\left(\frac{\hbar}{mg}X_{l,i+1}\right)\right] + s_i\,, \\ &\qquad (i=1, 2, \cdots, n-1)\end{aligned} \tag{3·8}$$

where

$$\begin{aligned}X_{i,l} &= \Delta_l + \Delta_{l+1} + \cdots + \Delta_{i-1}\,, \qquad (1 \leqq l < i \leqq n) \\ X_{l,i+1} &= \Delta_{i+1} + \Delta_{i+2} + \cdots + \Delta_{l-1}\,, \qquad (1 \leqq i+1 < l \leqq n)\end{aligned} \tag{3·9}$$

and

$$s_i = q_{i+1} - q_i \geqq 0\,. \qquad (i=1, 2, \cdots, n-1) \tag{3·10}$$

The last inequality is due to (2·28). If one defines the functions $F_i(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ by

$$\begin{aligned}F_i(\Delta_1, \cdots, \Delta_i, \cdots, \Delta_{n-1}) = \Delta_i - \frac{2\hbar}{L}\Bigg\{&2\cot^{-1}\left(\frac{\hbar}{mg}\Delta_i\right) \\ &+ \sum_{l=1}^{i-1}\left[\cot^{-1}\left\{\frac{\hbar}{mg}(\Delta_i + X_{i,l})\right\} - \cot^{-1}\left(\frac{\hbar}{mg}X_{i,l}\right)\right] \\ &+ \sum_{l=i+2}^{n}\left[\cot^{-1}\left\{\frac{\hbar}{mg}(\Delta_i + X_{l,i+1})\right\} - \cot^{-1}\left(\frac{\hbar}{mg}X_{l,i+1}\right)\right], \\ &(i=1, 2, \cdots, n-1)\end{aligned} \tag{3·11}$$

Eqs. (3·8) are rewritten as

$$F_i(\Delta_1, \Delta_2, \cdots, \Delta_i, \cdots, \Delta_{n-1}) = s_i\,. \qquad (i=1, 2, \cdots, n-1) \tag{3·12}$$

Thus the coupled equations (3·1) and (3·2) are decomposed into $n(n-1)/2$ equations (3·7) to determine $k_{i,j}$ and $(n-1)$ equations (3·12) for parameters Δ_i under the restrictions (3·5) and (3·10). Hence if one can obtain the solutions Δ_i of Eqs. (3·12), the parameters $k_{i,j}$ are determined from (3·7) by making use of the solutions Δ_i. As the solutions Δ_i give the differences $(k_{i+1}-k_i)$, the magnitudes of k_i themselves seem at a glance to remain undetermined. However, if one makes use of the relation

$$\sum_{i=1}^{n} k_i = \sum_{\substack{i,j\\ i\neq j}} k_{i,j} + \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} q_i \tag{3·13}$$

which is due to antisymmetricity of $k_{i,j}$, all k_i themselves can be determined.

As readily seen from the above argument, our purpose is to prove that the simultaneous equations (3·12) have unique solutions under the conditions (3·5) for given $s_1, s_2, \cdots s_n$ satisfying (3·10). This will be accomplished in the following way: First the existence of one set of the solutions Δ_i will be verified, and next the uniqueness of the solutions will be established.

Let us first note that the function $\cot^{-1}x$ is a monotonically decreasing one in x, and the function $(\cot^{-1}(x+y)-\cot^{-1}y)$ is a monotonically increasing one in y for $x\geqq 0$ and $y\geqq 0$. On the basis of these facts, the schematical behaviors of the functions $F_i(\Delta_1, \Delta_2, \cdots, \Delta_i, \cdots, \Delta_{n-1})$ are shown in Fig. 3. As readily seen from this figure, the function $F_i(\Delta_1, \cdots, \Delta_i, \cdots, \Delta_{n-1})$ has two characteristic properties within the regions $\Delta_l \geqq 0$:

(i) The first is that the function $F_i(\Delta_1, \cdots, \Delta_i, \cdots, \Delta_{n-1})$ is a monotonically increasing one of the variable Δ_i.

(ii) The second is that the function $F_i(\Delta_1, \cdots \Delta_i, \cdots, \Delta_{n-1})$ is a monotonically decreasing one for other variables $\Delta_k (k\neq i)$ than Δ_i.

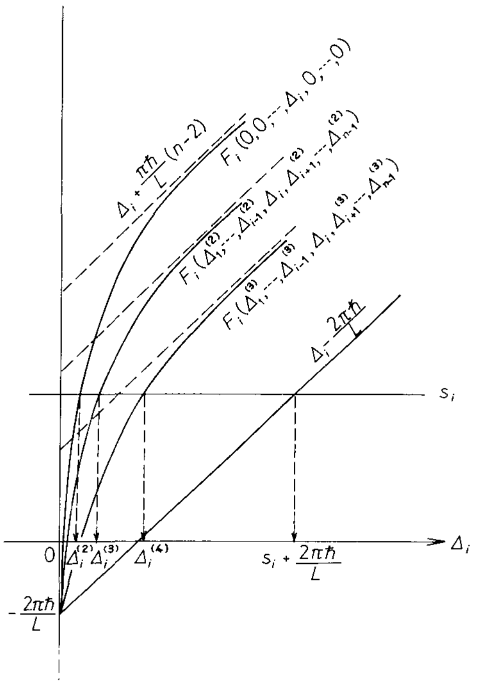

Fig. 3. The schematical behaviors of the function $F_i(\Delta_1, \cdots, \Delta_i, \cdots, \Delta_{n-1})$ concerning the variable Δ_i are drawn by curved lines for fixed values $\Delta_l = \Delta_l^{(\alpha-1)} (l\neq i,\ l=1, 2, \cdots, n-1)$ in the case of Step α ($\alpha=1, 2, \cdots$). The dotted lines indicate the asymptotic ones of the function $F_i(\Delta_1^{(\alpha-1)}, \cdots, \Delta_{i-1}^{(\alpha-1)}, \Delta_i, \Delta_{i+1}^{(\alpha-1)}, \cdots, \Delta_{n-1}^{(\alpha-1)})$ in the limit $\Delta_i\to\infty$. The function $F_i(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ is a monotonically increasing function in Δ_i and a monotonically decreasing one in $\Delta_l (l\neq i)$. Hence the lower bound of the function $F_i(\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_{n-1})$ is given as

$$F_i(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}) \geqq \lim_{\Delta_1\to\infty}\cdots \lim_{\Delta_{i-1}\to\infty} \lim_{\Delta_{i+1}\to\infty}\cdots \lim_{\Delta_{n-1}\to\infty} F_i(\Delta_1, \cdots, \Delta_{n-1}) = \Delta_i - \frac{4\hbar}{L}\cot^{-1}\left(\frac{\hbar}{mg}\Delta_i\right) \geqq \Delta_i - \frac{2\pi\hbar}{L},$$

and is denoted by the straight line. The solution $\Delta_i^{(\alpha)}$ of Eq. (3·16) is determined from the intersecting point between the flat line and the curved line $F_i(\Delta_1^{(\alpha-1)}, \cdots, \Delta_{i-1}^{(\alpha-1)}, \Delta_i, \Delta_{i+1}^{(\alpha-1)}, \cdots, \Delta_{n-1}^{(\alpha-1)})$.

In virtue of these characteristic properties (i) and (ii), we can show the existence of the solutions $\varDelta_i(s_1, s_2, \cdots, s_{n-1})$ of the simultaneous equations (3·12) within the regions (3·5) and (3·10) by the following successive procedure.

(Step 1) Firstly let $\varDelta_i(i=1, 2, \cdots, n-1)$ to be zero, and write them by $\varDelta_i^{(1)}(=0)$.

(Step 2) Secondly write the solutions of the equations

$$F_i(0, 0, \cdots, 0, \varDelta_i^{(2)}, 0, \cdots, 0)=s_i\,, \quad (i=1, 2, \cdots, n-1) \tag{3·14}$$

as $\varDelta_i^{(2)}$.

(Step 3) Similarly the quantities $\varDelta_i^{(3)}(i=1, 2, \cdots, n-1)$ denote the solutions of the equations

$$F_i(\varDelta_1^{(2)}, \cdots, \varDelta_{i-1}^{(2)}, \varDelta_i^{(3)}, \varDelta_{i+1}^{(2)}, \cdots, \varDelta_{n-1}^{(2)})=s_i\,. \quad (i=1, 2, \cdots, n-1) \tag{3·15}$$

(Step α) Generally we denote the solutions of the following equations by $\varDelta_i^{(\alpha)}$ $(i=1, 2, \cdots, n-1)$:

$$F_i(\varDelta_1^{(\alpha-1)}, \varDelta_2^{(\alpha-1)}, \cdots, \varDelta_{i-1}^{(\alpha-1)}, \varDelta_i^{(\alpha)}, \varDelta_{i+1}^{(\alpha-1)}, \cdots, \varDelta_{n-1}^{(\alpha-1)})=s_i\,. \quad (i=1, 2, \cdots, n-1) \tag{3·16}$$

Repeating the above procedure infinitely, we have the $(n-1)$ infinite sequences $\{\varDelta_i^{(1)}, \varDelta_i^{(2)}, \varDelta_i^{(3)}, \cdots\}(i=1, 2, 3, \cdots, n-1)$.

Now note that the solutions $\varDelta_i^{(2)}$ of Eqs. (3·14) are given by the intersecting point of the curved lines $F_i(0, 0, \cdots, 0, \varDelta_i, 0, \cdots, 0)$ and the flat line of $s_i \geqq 0$ in Fig. 3. This intersecting points give positive values for the solutions $\varDelta_i^{(2)}$ due to the property (i). Hence one gets

$$\varDelta_i^{(2)}>\varDelta_i^{(1)}=0\,. \quad (i=1, 2, \cdots, n-1) \tag{3·17}$$

According to the property (ii), these inequalities yield

$$F_i(\varDelta_1^{(2)}, \varDelta_2^{(2)}, \cdots, \varDelta_{i-1}^{(2)}, \varDelta_i, \varDelta_{i+1}^{(2)}, \cdots, \varDelta_{n-1}^{(2)})<F_i(0, 0, \cdots, 0, \varDelta_i, 0, \cdots, 0) \tag{3·18}$$

for any positive value $\varDelta_i$. Then, from the property (i) and (3·18), the intersecting point between the curved line $F_i(\varDelta_1^{(2)}, \varDelta_2^{(2)}, \cdots, \varDelta_{i-1}^{(2)}, \varDelta_i, \varDelta_{i+1}^{(2)}, \cdots, \varDelta_{n-1}^{(2)})$ and the flat line of $s_i \geqq 0$ gives the inequality

$$\varDelta_i^{(3)}>\varDelta_i^{(2)}\,. \quad (i=1, 2, \cdots, n-1) \tag{3·19}$$

Generally if one gets

$$\varDelta_i^{(\alpha-1)}>\varDelta_i^{(\alpha-2)}\,, \quad (i=1, 2, \cdots, n-1) \tag{3·20}$$

in Step $(\alpha-1)$, the inequality

$$\begin{aligned} &F_i(\varDelta_1^{(\alpha-1)}, \cdots, \varDelta_{i-1}^{(\alpha-1)}, \varDelta_i, \varDelta_{i+1}^{(\alpha-1)}, \cdots, \varDelta_{n-1}^{(\alpha-1)}) \\ &\qquad <F_i(\varDelta_1^{(\alpha-2)}, \cdots, \varDelta_{i-1}^{(\alpha-2)}, \varDelta_i, \varDelta_{i+1}^{(\alpha-2)}, \cdots, \varDelta_{n-1}^{(\alpha-2)}), \end{aligned} \tag{3·21}$$

holds for positive value $\varDelta_i$ due to the property (ii). In the same way as in the derivation of (3·19), one can find

$$\varDelta_i^{(\alpha)} > \varDelta_i^{(\alpha-1)} . \qquad (i=1, 2, \cdots, n-1) \tag{3·22}$$

Thus, by mathematical induction, we can conclude that the infinite sequences $(\varDelta_i^{(1)}, \varDelta_i^{(2)}, \varDelta_i^{(3)}, \cdots)(i=1, 2, \cdots, n-1)$ are monotonically increasing ones. Meanwhile the infinite sequences have upper bounds given by

$$\varDelta_i^{(\alpha)} \leqq \frac{2\pi h}{L} + s_i , \qquad (i=1, 2, \cdots, n-1) \tag{3·23}$$

as readily seen from Fig. 3. Hence, from a well-known theorem in mathematics, the infinite sequences converge and have definite limiting values, that is,

$$\lim_{\alpha\to\infty} \varDelta_i^{(\alpha)}(s_1, s_2, \cdots, s_{n-1}) = \varDelta_i(s_1, s_2, \cdots, s_{n-1}) > 0 . \; (i=1, 2, \cdots, n-1) \tag{3·24}$$

In (3·24), the positive nature of $\varDelta_i$ is due to (3·17), namely, the limiting values satisfy the restrictions (3·5). This means that

$$k_1 < k_2 < k_3 < \cdots < k_n , \tag{3·25}$$

hold from the definitions (3·4).

Now do the limiting values $\varDelta_i(s_1, s_2, \cdots, s_{n-1})$ obtained above satisfy the simultaneous equations (3·12)? This is verified as follows. According to the procedure to obtain $\varDelta_i^{(\alpha)}$, Eqs. (3·16) hold for any integer $\alpha(\geqq 2)$. Note here that the functions $F_i(\varDelta_1, \varDelta_2, \cdots, \varDelta_i, \cdots, \varDelta_{n-1})$ are continuous ones concerning all $\varDelta_l$ in the regions $\varDelta_l \geqq 0$. Then taking the limit of $\alpha \to \infty$ in Eqs. (3·16) gives

$$F_i(\varDelta_1(s_1, \cdots, s_{n-1}), \cdots, \varDelta_i(s_1, \cdots, s_{n-1}), \cdots, \varDelta_{n-1}(s_1, s_2, \cdots, s_{n-1})) = s_i .$$
$$(i=1, 2, \cdots, n-1) \tag{3·26}$$

Thus, the limiting values $\varDelta_i(s_1, s_2, \cdots, s_{n-1})$ are certainly the solutions of the simultaneous equations (3·12) under the restrictions (3·5) and (3·10).

Here let us consider the case in the finite Step α mentioned above. From (3·16), the $\varDelta_i^{(\alpha)}$ can be determined numerically and they are seen to be the approximate solutions of (3·12). Moreover, when α is increased to be large, the approximate solutions $\varDelta_i^{(\alpha)}$ approach closely the exact solutions of (3·12) as much as one wishes (the evaluation of $\varDelta_i^{(\alpha)}$, however, is very tedious). In this way we have founded the successive method to solve the simultaneous equation (2·26) numerically.

Next let us verify the uniqueness of the solutions $\varDelta_i(s_1, s_2, \cdots, s_{n-1})$. If there exist different two sets of the solutions, $(\varDelta_1(s_1, \cdots, s_{n-1}), \varDelta_2(s_1, \cdots, s_{n-1})\cdots, \varDelta_{n-1}(s_1, \cdots, s_{n-1}))$ and $(\varDelta_1'(s_1, \cdots, s_{n-1}), \varDelta_2'(s_1, \cdots, s_{n-1}), \cdots, \varDelta_{n-1}'(s_1, \cdots, s_{n-1}))$ which satisfy the simultaneous equations (3·12), the relations between magnitudes of the solutions in the two sets are given either by

$$\text{(Case 1)}\quad \Delta_{l_j}(s_1, s_2, \cdots, s_{n-1}) < \Delta'_{l_j}(s_1, s_2, \cdots, s_{n-1}), \quad (j=1, 2, \cdots, k;\ k \leqq n-1) \tag{3·27a}$$

and

$$\Delta_l(s_1, s_2, \cdots, s_{n-1}) \geqq \Delta'_l(s_1, s_2, \cdots, s_{n-1}), \quad (l \neq l_1, l_2, \cdots, l_k) \tag{3·27b}$$

or by

$$\text{(Case 2)}\quad \Delta_i(s_1, s_2, \cdots, s_{n-1}) \geqq \Delta'_i(s_1, s_2, \cdots, s_{n-1}). \quad (i=1, 2, \cdots, n-1) \tag{3·28}$$

In Case 1 let us consider the function $G_{l_1,l_2,\cdots,l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ defined by

$$G_{l_1,l_2,\cdots,l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}) = \sum_{j=1}^{k} F_{l_j}(\Delta_1, \Delta_2, \cdots, \Delta_{l_j}, \cdots, \Delta_{n-1}). \tag{3·29}$$

According to Lemma A in the Appendix, the function $G_{l_1,l_2,\cdots,l_k}(\Delta_1, \cdots, \Delta_{n-1})$ is the increasing function of $\Delta_{l_j}(j=1, 2, \cdots, k)$ and the decreasing one of $\Delta_l(l \neq l_1, l_2, \cdots, l_k)$. Using these properties of the function $G_{l_1,l_2,\cdots,l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$, we have the inequality

$$\begin{aligned} G_{l_1,l_2,\cdots,l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}) &< G_{l_1,l_2,\cdots,l_k}(\Delta_1, \cdots, \Delta'_{l_1}, \cdots, \Delta'_{l_k}, \cdots, \Delta_{n-1}) \\ &\leqq G_{l_1,l_2,\cdots,l_k}(\Delta'_1, \cdots, \Delta'_{n-1}), \end{aligned}$$

where for brevity, the solutions $\Delta_i(s_1, s_2, \cdots, s_{n-1})$ and $\Delta_i'(s_1, s_2, \cdots, s_{n-1})$ are denoted by Δ_i and Δ'_i, respectively. This inequality is incompatible with the fact that $G_{l_1,l_2,\cdots,l_k}(\Delta_1, \cdots, \Delta_{n-1})$ and $G_{l_1,l_2,\cdots,l_k}(\Delta'_1, \Delta'_2, \cdots, \Delta'_{n-1})$ must have the same value $s_{l_1} + s_{l_2} + \cdots + s_{l_k}$ from the equations (3·12). Hence in Case 1 one can conclude that there do not exist such two sets of the solutions. In Case 2 one has

$$G_{1,2,\cdots,n-1}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}) \geqq G_{1,2,\cdots,n-1}(\Delta'_1, \Delta'_2, \cdots, \Delta'_{n-1}), \tag{3·30}$$

where $G_{1,2,\cdots,n-1}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}) = \sum_{i=1}^{n} F_i(\Delta_1, \cdots, \Delta_i, \cdots, \Delta_{n-1})$. The equality in (3·30) holds only in the case $\Delta_i = \Delta'_i$ for all $i=1, 2, \cdots, n-1$. The inequality in (3·30) is inconsistent with the fact that both $G_{1,2,\cdots,n-1}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ and $G_{1,2,\cdots,n-1}(\Delta'_1, \Delta'_2, \cdots, \Delta'_{n-1})$ take the same value $s_1 + s_2 + \cdots + s_{n-1}$ from (3·12). Thus it has been verified that there do not exist different two sets of the solutions for the simultaneous equations (3·12). Namely, the uniqueness of the solutions in (3·12) has been established.

In this way it has been proved that the simultaneous equations (3·12) for the $(n-1)$ parameters Δ_i under the restrictions (3·5) have unique solutions for arbitrarily given nonnegative momenta s_i. From this result we can conclude that the simultaneous equations (2·26) under the restrictions (2·29) have unique solutions $k_{i,j}$ for any set of quantum numbers $(q_1, q_2, \cdots, q_n)$ satisfying the conditions (2·28), because the solutions $k_{i,j}$ are obtained by substituting the solutions

$\Delta_i(s_1, s_2, \cdots, s_{n-1})$ obtained above in Eqs. (3·7).

In closing this section let us point out the following Lemma B of which proof is given in the Appendix.

Lemma B

$$\Delta_i(s_1, s_2, \cdots, s_{n-1}) > \Delta_i(s_1', s_2', \cdots, s_{n-1}'), \quad (i=1, 2, \cdots, n-1) \tag{3·31}$$

where $\Delta_i(s_1, s_2, \cdots, s_{n-1})$ and $\Delta_i(s_1', s_2', \cdots, s_{n-1}')$ indicate the solutions of the simultaneous equations (3·12) for such two sets of nonnegative momenta $\{s_l\}$ and $\{s_l'\}$, respectively, as

$$s_l = s_l', \qquad (l \neq k,\ l=1, 2, \cdots, n-1)$$

and (3·32)

$$s_k > 0, \qquad s_k' = 0.$$

On the basis of Lemma B, we can readily see that

$$\Delta_i(s_1, s_2, \cdots, s_{n-1}) > \Delta_i(0, 0, \cdots, 0), \quad (i=1, 2, \cdots, n-1) \tag{3·33}$$

for $(s_1, s_2, \cdots, s_{n-1}) \neq (0, 0, \cdots, 0)$. This inequality means evidently that the quantity $\Delta_i(s_1, s_2, \cdots, s_{n-1})$ takes the minimum value only for the case $s_i=0$ $(i=1, 2, \cdots, n-1)$. This important fact will be used in the discussion about the ground state energy in § 3 of a subsequent paper.

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Appendix

Lemma A: Let the function $G_{l_1, l_2, \cdots, l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ to be given by

$$G_{l_1, l_2, \cdots, l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}) = \sum_{j=1}^{k} F_{l_j}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1}), \tag{A·1}$$

where $l_j (j=1, 2, \cdots, k)$ take integers among $1, 2, 3, \cdots, n-1$, and are ordered as $l_1 < l_2 < \cdots < l_k$. Then, the function $G_{l_1, l_2, \cdots, l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ is a monotonically increasing function of $\Delta_{l_j} (j=1, 2, \cdots, k)$ and a monotonically decreasing one of $\Delta_l (l \neq l_1, l_2, \cdots, l_k)$.

This can be proved as follows. Let any one among the variables $\Delta_{l_j} (j=1, 2, \cdots, k)$ to be Δ_{l_i}. Then, picking out the terms involving the variables Δ_{l_i} in the function $G_{l_1,l_2,\cdots,l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$, we have

$$
\begin{aligned}
&\Delta_{l_i}-\frac{2\hbar}{L}\Big[2\cot^{-1}\Big(\frac{\hbar}{mg}\Delta_{l_i}\Big)+\sum_{l=1}^{l_i-1}\cot^{-1}\Big\{\frac{\hbar}{mg}(\Delta_{l_i}+X_{l_i,l})\Big\} \\
&\quad+\sum_{l=l_i+2}^{n}\cot^{-1}\Big\{\frac{\hbar}{mg}(\Delta_{l_i}+X_{l,l_i+1})\Big\}\Big] \\
&\quad-\frac{2\hbar}{L}\sum_{j=i+1}^{k}\sum_{l=1}^{l_i}\Big[\cot^{-1}\Big\{\frac{\hbar}{mg}(\Delta_{l_j}+X_{l_j,l})\Big\}-\cot^{-1}\Big\{\frac{\hbar}{mg}X_{l_j,l}\Big\}\Big] \\
&\quad-\frac{2\hbar}{L}\sum_{j=1}^{i-1}\sum_{l=l_i+1}^{n}\Big[\cot^{-1}\Big\{\frac{\hbar}{mg}(\Delta_{l_j}+X_{l,l_j+1})\Big\}-\cot^{-1}\Big\{\frac{\hbar}{mg}X_{l,l_j+1}\Big\}\Big] \\
&=\Delta_{l_i}-\frac{2\hbar}{L}\sum_{l=1}^{l_i}\Big[\cot^{-1}\Big(\frac{\hbar}{mg}X_{l_i+1,l}\Big) \\
&\qquad+\sum_{j=i+1}^{k}\Big\{\cot^{-1}\Big(\frac{\hbar}{mg}X_{l_j+1,l}\Big)-\cot^{-1}\Big(\frac{\hbar}{mg}X_{l_j,l}\Big)\Big\}\Big] \\
&\qquad-\frac{2\hbar}{L}\sum_{l=l_i+1}^{n}\Big[\cot^{-1}\Big(\frac{\hbar}{mg}X_{l,l_i}\Big) \\
&\qquad+\sum_{j=1}^{i-1}\Big\{\cot^{-1}\Big(\frac{\hbar}{mg}X_{l,l_j}\Big)-\cot^{-1}\Big(\frac{\hbar}{mg}X_{l,l_j+1}\Big)\Big\}\Big] \\
&=\Delta_{l_i}-\frac{2\hbar}{L}\sum_{l=1}^{l_i}\Big[\cot^{-1}\Big(\frac{\hbar}{mg}X_{l_k+1,l}\Big) \\
&\qquad+\sum_{j=i+1}^{k}\Big\{\cot^{-1}\Big(\frac{\hbar}{mg}X_{l_{(j-1)}+1,l}\Big)-\cot^{-1}\Big(\frac{\hbar}{mg}X_{l_j,l}\Big)\Big\}\Big] \\
&\qquad-\frac{2\hbar}{L}\sum_{l=l_i+1}^{n}\Big[\cot^{-1}\Big(\frac{\hbar}{mg}X_{l,l_1}\Big) \\
&\qquad+\sum_{j=1}^{i-1}\Big\{\cot^{-1}\Big(\frac{\hbar}{mg}X_{l,l_{(j+1)}}\Big)-\cot^{-1}\Big(\frac{\hbar}{mg}X_{l,l_j+1}\Big)\Big\}\Big],
\end{aligned}
\tag{A·2}
$$

where we have used the following facts,

$$\Delta_{l_i}=X_{l_i+1,l_i}\,,$$

$$\Delta_{l_i}+X_{l_i,l}=X_{l_i+1,l}$$

and

$$\Delta_{l_i}+X_{l,l_i+1}=X_{l,l_i}$$

to derive the first equality. Noting here that

$$l_{(j-1)}+1\leqq l_j$$

and

$$l_{(j+1)} \geqq l_j + 1 ,$$

we can see that all terms in the curly brackets in the final expression of (A·2) are monotonically decreasing functions of Δ_{l_i} or zero. Hence the function $G_{l_1,l_2,\cdots,l_k}(\Delta_1,\Delta_2,\cdots,\Delta_{n-1})$ is a monotonically increasing function of $\Delta_{l_i}(i=1,2,\cdots,k)$. On the other hand, the function $G_{l_1,l_2,\cdots,l_k}(\Delta_1, \Delta_2, \cdots, \Delta_{n-1})$ is a monotonically decreasing one of $\Delta_l(l \neq l_1, l_2, \cdots, l_k)$, since all functions $F_{l_j}(j=1, 2, \cdots, k)$ are decreasing ones concerning the variables $\Delta_l(l \neq l_j)$. Thus Lemma A has been established.

Lemma B mentioned at the end of § 3 can be verified as follows. The solutions in the simultaneous equations (3·12) for sets of momenta, $\{s_l\}$ and $\{s_l'\}$ of which the relation is given by (3·32), are denoted by $\Delta_i(\{s_l\})$ and $\Delta_i(\{s_l'\})$, respectively. Now let us find the solutions $\Delta_i(\{s_l\})$ of (3·12) for the set of momenta $\{s_l\}$ in the following method.

(Step 1) Let $\Delta_i^{(1)}(\{s_l\})$ to be $\Delta_i(\{s_l'\})$ for $i=1, 2, \cdots, n-1$.

(Step J) In a general Step $J(J \geqq 2)$, take $\Delta_i^{(J)}(\{s_l\})$ to be the solutions of the equations

$$F_i(\Delta_1^{(J-1)}(\{s_l\}), \cdots, \Delta_{i-1}^{(J-1)}(\{s_l\}), \Delta_i^{(J)}(\{s_l\}), \Delta_{i+1}^{(J-1)}(\{s_l\}), \cdots, \Delta_{n-1}^{(J-1)}(\{s_l\}))=s_i . \tag{B·1}$$

By this method, we have the $(n-1)$ infinite sequences $(\Delta_i^{(1)}(\{s_l\}), \Delta_i^{(2)}(\{s_l\}), \cdots)$. Using a similar discussion to the one in verifying the existence of solutions of Eqs. (3·12), in Step 2, one gets

$$\Delta_i^{(2)}(\{s_l\})=\Delta_i(\{s_l'\}) \qquad (i \neq k) \tag{B·2a}$$

and

$$\Delta_k^{(2)}(\{s_l\})>\Delta_k(\{s_l'\}). \tag{B·2b}$$

In Step 3, the following relations hold:

$$\Delta_i^{(3)}(\{s_l\})>\Delta_i^{(2)}(\{s_l\})=\Delta_i(\{s_l'\}) \qquad (i \neq k) \tag{B·3a}$$

and

$$\Delta_k^{(3)}(\{s_l\})=\Delta_k^{(2)}(\{s_l\})>\Delta_k(\{s_l'\}), \tag{B·3b}$$

where (B·3a) is due to the inequality (B·2b) and the property (ii) of the function $F_i(\Delta_1, \cdots, \Delta_i, \cdots, \Delta_{n-1})$. Generally in Step $J(J \geqq 4)$ one can verify that

$$\Delta_i^{(J)}(\{s_l\})>\Delta_i^{(J-1)}(\{s_l\})>\Delta_i(\{s_l'\}). \quad (i=1, 2, \cdots, n-1) \tag{B·4}$$

As readily seen from the discussion in § 3, there exist the limiting values $\Delta_i(\{s_l\})$ in the $(n-1)$ infinite sequences $\{\Delta_i^{(1)}(\{s_l\}), \Delta_i^{(2)}(\{s_l\}), \Delta_i^{(3)}(\{s_l\}), \cdots\}$:

$$\lim_{J\to\infty} \Delta_i^{(J)}(\{s_l\}) = \Delta_i(\{s_l\}), \qquad (i=1, 2, \cdots, n-1) \tag{B$\cdot$5}$$

which are the unique solutions of the simultaneous equations (3·12). The combination of (B·4) and (B·5) yields

$$\Delta_i(\{s_l\}) > \Delta_i(\{s_l'\}). \qquad (i=1, 2, \cdots, n-1) \tag{B$\cdot$6}$$

Thus Lemma B has been established.

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