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One-Dimensional Many Boson System. III

—— Unitary Transformation and Exactly Dressed Bose Particle ——

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A unitary operator, which transforms free states into the exact eigenstates for a one-dimensional many boson system with repulsive delta-function potential, is explicitly constructed. By the unitary operator, original bare creation and annihilation operators are transformed into new operators which are shown to create or annihilate the exactly dressed bosons with the interaction cloud. The total Hamiltonian, total number and total momentum in the system are expressed in the diagonalized form in terms of the dressed operators. It is shown that the exact ground state is the condensed state of all exactly dressed bosons with zero momentum.

§ 1. Introduction

In a previous paper[1),2)] (hereafter referred to as I and II) the eigenvalue problem in a one-dimensional system composed of n bosons interacting via repulsive delta-function potential has been solved exactly from field theoretical point of view.

The system is described by the Hamiltonian

$$H=\sum_{p}(p^2/2m)a_p{}^* a_p+\sum_{p,q,r}(g/2L)a^*_{p+r}a^*_{q-r}a_q a_p\,. \tag{1·1}$$

For arbitrary quantum numbers $q_1 \leqq q_2 \leqq \cdots \leqq q_n$ ($q_i=(2\pi\hbar/L)\times$integer), the eigenstate of (1·1) for n-boson system is given as

$$|\Psi_{q_1,q_2\cdots,q_n}\rangle=\beta_{q_1,\cdots q_n}\sum_{\{p_{i,j}\,;\,1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n}d(p_{i,j}\,;\,k_{i,j})\prod_{i=1}^{n}a^*{}_{\sum_{j=1,\,j\neq i}^{n}p_{i,j}+q_i}|0\rangle\,, \tag{1·2}$$

where $\beta_{q_1,\cdots,q_n}$ indicates a normalization constant, and

$$d(p_{i,j}\,;\,k_{i,j})=(-k_{i,j})/(p_{i,j}-k_{i,j}),\quad p_{j,i}=-p_{i,j}\,,\quad k_{j,i}=-k_{i,j}\,,\quad (1\leqq i<j\leqq n) \tag{1·3}$$

in which $k_{i,j}$ have been proved to be uniquely determined by (2·26) in I. By using $k_{i,j}$, the eigenenergy is obtained by

$$E_{q_1,\cdots,q_n}=\sum_{i=1}^{n}k_i{}^2/2m=\sum_{i=1}^{n}(1/2m)[\sum_{j=1,\,j\neq i}^{n}k_{i,j}+q_i]^2\,. \tag{1·4}$$

It has been verified in § 3 of II that the ground state is given by the state $|\Psi_{0,0,\cdots,0}\rangle$ where all momenta $q_1, q_2, \cdots, q_n$ in (1·2) are taken as zero. This means that the ground state is a condensed state of dressed bosons with zero momentum. The main purpose of this paper is to make clear the nature of the dressed bosons. In § 2 we will first construct a unitary transformation U_n which transforms the eigenstates for noninteracting n-boson system into the exact eigenstates (1·2). On the basis of the unitary operator U_n, we will construct, in § 3, a unitary operator U which works on the system composed of arbitrary number of bosons. We will show in § 4 that the total Hamiltonian, total momentum and total number operator can be expressed in diagonalized forms by making use of a new creation operator A_p^* and an annihilation operator A_p produced by the unitary transformation U. This fact indicates that the operators A_p^* and A_p are the creation and annihilation operators of an exactly dressed particle which involves all effects of the interactions among bosons. The ground state $|\Psi_{0,0,\cdots,0}\rangle$ is represented by $(1/\sqrt{n!})(A_0^*)^n|0\rangle$, which evidently shows that the ground state is the condensed state of n-dressed bosons with zero momentum. In § 5 we will investigate the limiting properties of U in $g\to 0$.

§ 2. Unitary transformation for n-boson system

The purpose of this section is to construct a unitary transformation U_n (in a system with the fixed total number n) which transforms the free states in a noninteracting n-boson system to the exact eigenstates (1·2).

2.1. *Unitary transformation U_n*

Any eigenstate for the noninteracting n-boson system is expressed as

$$|q_1, q_2, \cdots, q_n\rangle = \alpha_{q_1,q_2,\cdots,q_n}\prod_{i=1}^{n} a_{q_i}^*|0\rangle\ , \tag{2·1}$$

under the condition $q_1 \leqq q_2 \cdots \leqq q_n$ which is required to guarantee that a set of all states given by (2·1) is a complete orthonormal set. The factor $\alpha_{q_1,\cdots,q_n}$ indicates a normalization constant,

$$\alpha_{q_1,\cdots,q_n} = \prod_{j=1}^{l}(1/\sqrt{n_j!})\,;$$

$$q_1 = \cdots = q_{n_1} < q_{n_1+1} = \cdots = q_{n_1+n_2} < \cdots < q_{n_1+\cdots+n_{l-1}+1} = \cdots = q_{n_1+\cdots+n_l},\ \sum_{j=1}^{n} n_j = n\ . \tag{2·2}$$

The transformation operators U_n and U_n^* defined by

$$U_n = \sum_{q_1\leqq q_2\cdots\leqq q_n}\ \sum_{\{p_{i,j}\,;\,1\leqq i<j\leqq n\}} \beta_{q_1,q_2,\cdots,q_n}\alpha_{q_1,\cdots,q_n} \prod_{1\leqq i<j\leqq n} d(p_{i,j}\,;\,k_{i,j}) \prod_{i=1}^{n} a^*_{\sum_{j=1,\,j\neq i}^{n} p_{i,j}+q_i} \prod_{i=1}^{n} a_{q_i} \tag{2·3a}$$

and

$$U_n^* = \sum_{q_1 \leq q_2 \cdots \leq q_n} \sum_{\{p_{i,j}\,;\,1\leq i<j\leq n\}} \beta^*_{q_1,q_2,\cdots,q_n} \alpha^*_{q_1,\cdots,q_n} \prod_{1\leq i<j\leq n} d(p_{i,j}\,;\,k_{i,j}) \prod_{i=1}^{n} a^*_{q_i} \prod_{i=1}^{n} a_{\sum_{j=1, j\neq i}^{n} p_{i,j}+q_i} \tag{2·3b}$$

satisfy the equations

$$|\Psi_{q_1,q_2,\cdots,q_n}\rangle = U_n|q_1, q_2, \cdots, q_n\rangle\,, \quad \langle \Psi_{q_1,q_2,\cdots,q_n}| = \langle q_1, q_2, \cdots, q_n|U_n^*\,. \tag{2·4}$$

If one can verify that

$$U_n^* U_n|q_1, q_2, \cdots, q_n\rangle = |q_1, q_2, \cdots, q_n\rangle\,, \tag{2·5a}$$

$$U_n U_n^*|q_1, q_2, \cdots, q_n\rangle = |q_1, q_2, \cdots, q_n\rangle \tag{2·5b}$$

for any free eigenstate in noninteracting n-boson system, then the operator U_n is a unitary one, since all free eigenstates (2·1) form a complete orthonormal set for n-boson system.

2.2. *Unitarity of U_n in the case of infinitely large g*

Now we begin to verify (2·5a) and (2·5b) in the case of infinitely large g.

(a) Proof of (2·5a)

From (2·30) of I in the case $g\to\infty$, $k_{i,j}$ are given by

$$k_{i,j} = (-\pi\hbar/L)\varepsilon_{i,j}\,; \quad \varepsilon_{i,j} = (j-i)/|j-i|. \quad (i\neq j)$$

In this case the normalization factor $\beta_{q_1,\cdots,q_n}$ is $(\pi/2)^{-n(n-1)/2}$ as will be seen later. The left-hand side of (2·5a), then, becomes

$$\begin{aligned} U_n^* U_n|q_1, \cdots q_n\rangle = &\sum_{q_1' \leq q_2' \cdots \leq q_n'} (\pi/2)^{-n(n-1)} \alpha_{q_1',\cdots,q_n'} \sum_{\{p'_{i,j}, p_{i,j}\,;\,1\leq i<j\leq n\}} \\ &\times \prod_{1\leq i<j\leq n} d(p'_{i,j}\,;(-\pi\hbar/L)\varepsilon_{i,j}) d(p_{i,j}\,;(-\pi\hbar/L)\varepsilon_{i,j}) \\ &\times \sum_{\mu} \prod_{i=1}^{n} \delta\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} p'_{\mu_i,\mu_j} + q'_{\mu_i},\ \sum_{\substack{j=1\\ j\neq i}}^{n} p_{i,j} + q_i\Big) \prod_{i=1}^{n} a^*_{q_i'}|0\rangle\,, \end{aligned} \tag{2·6}$$

where the symbol $\sum_\mu$ indicates the summation over all permutations $\mu = \binom{1,\,2,\,\cdots,\,n}{\mu_1,\,\mu_2,\,\cdots,\,\mu_n}$, and the function $\delta(p', p)$ denotes Kronecker's deltafunction $\delta_{p',p}$.

Noting the relation $d(p'_{i,j}\,;(-\pi\hbar/L)\varepsilon_{i,j}) = d(p'_{j,i}\,;(-\pi\hbar/L)\varepsilon_{j,i})$ which can be seen from the definitions (1·3), we have

$$\prod_{1\leq i<j\leq n} d(p'_{i,j}\,;(-\pi\hbar/L)\varepsilon_{i,j}) = \prod_{1\leq i<j\leq n} d(p'_{\mu_i,\mu_j}\,;(-\pi\hbar/L)\varepsilon_{\mu_i,\mu_j}) \tag{2·7}$$

for any permutation μ. Then the expression (2·6) can be rewritten as

$$U_n{}^* U_n|q_1, q_2, \cdots, q_n\rangle = \sum_{q_1'\leq q_2'\cdots\leq q_n'} (\pi/2)^{-n(n-1)} a_{q_1',\cdots,q_n'}$$

$$\times \sum_{\mu} \sum_{\{p'_{\mu_i,\mu_j}, p_{i,j}; 1\leq i<j\leq n\}} \prod_{1\leq i<j\leq n} d(p'_{\mu_i,\mu_j}; (-\pi h/L)\varepsilon_{\mu_i,\mu_j})$$

$$\times d(p_{i,j}; (-\pi h/L)\varepsilon_{i,j}) \prod_{i=1}^{n} \delta(\sum_{\substack{j=1\\ j\neq i}}^{n} p'_{\mu_i,\mu_j}$$

$$+ q'_{\mu_i}, \sum_{\substack{j=1\\ j\neq i}}^{n} p_{i,j} + q_i) \prod_{i=1}^{n} a^*_{q_i'}|0\rangle . \tag{2·8}$$

Let us here introduce such abbreviated notations as

$$p''_{i,j}(\mu) = p'_{\mu_i,\mu_j}, \quad q''_i(\mu) = q'_{\mu_i} \quad \text{and} \quad \varepsilon''_{i,j}(\mu) = \varepsilon_{\mu_i,\mu_j} \tag{2·9}$$

for a fixed permutation μ. Then one gets

$$U_n{}^* U_n|q_1, q_2, \cdots, q_n\rangle = \sum_{q_1'\leq q_2'\cdots\leq q_n'} (\pi/2)^{-n(n-1)} a_{q_1',\cdots,q_n'} \sum_{\mu} \mathrm{I}(q''(\mu), q) \prod_{i=1}^{n} a^*_{q_i'}|0\rangle , \tag{2·10}$$

where

$$\mathrm{I}(q''(\mu), q) = \sum_{\{p''_{i,j}(\mu), p_{i,j}; 1\leq i<j\leq n\}} \prod_{1\leq i<j\leq n} d(p''_{i,j}(\mu); (-\pi h/L)\varepsilon''_{i,j}(\mu))$$

$$\times d(p_{i,j}; (-\pi h/L)\varepsilon_{i,j}) \prod_{i=1}^{n} \delta(\sum_{\substack{j=1\\ j\neq i}}^{n} p''_{i,j}(\mu) + q''_i(\mu), \sum_{\substack{j=1\\ j\neq i}}^{n} p_{i,j} + q_i). \tag{2·11}$$

The introduction of new variables $r_{i,j}$ defined by

$$r_{i,j} = p''_{i,j}(\mu) - p_{i,j} \quad (1\leq i<j\leq n) \tag{2·12}$$

yields

$$\mathrm{I}(q''(\mu), q) = \sum_{\{r_{i,j}, p_{i,j}; 1\leq i<j\leq n\}} \prod_{1\leq i<j\leq n} d(p_{i,j} + r_{i,j}; (-\pi h/L)\varepsilon''_{i,j}(\mu))$$

$$\times d(p_{i,j}; (-\pi h/L)\varepsilon_{i,j}) \prod_{i=1}^{n} \delta_{\sum_{\substack{j=1\\ j\neq i}}^{n} r_{i,j} + q_i''(\mu), q_i} . \tag{2·13}$$

The multiple summations over $p_{i,j}$ in (2·13) can be carried out by using the formula

$$\sum_{p_{i,j}} d(p_{i,j} + r_{i,j}; (-\pi h/L)\varepsilon''_{i,j}) d(p_{i,j}; (-\pi h/L)\varepsilon_{i,j}) = (\pi/2)^2 \varepsilon''_{i,j}\varepsilon_{i,j}\delta_{r_{i,j}, (-\pi h/L)(\varepsilon''_{i,j} - \varepsilon_{i,j})} , \tag{2·14}$$

since

$$
(\text{l.h.s.})=\begin{cases}\dfrac{\varepsilon''_{i,j}\varepsilon_{i,j}}{4}\dfrac{1}{\dfrac{L}{2\pi h}r_{i,j}+\dfrac{1}{2}(\varepsilon''_{i,j}-\varepsilon_{i,j})}\sum_{l=-\infty}^{+\infty}\left\{\dfrac{1}{l+\dfrac{1}{2}\varepsilon_{i,j}}-\dfrac{1}{l+\dfrac{L}{2\pi h}r_{i,j}+\dfrac{1}{2}\varepsilon''_{i,j}}\right\}=0\,, \\ \qquad\qquad\qquad\qquad \text{for } r_{i,j}+\dfrac{\pi h}{L}(\varepsilon''_{i,j}-\varepsilon_{i,j})\neq 0\,, \\ \dfrac{\varepsilon''_{i,j}\varepsilon_{i,j}}{4}\sum_{l=-\infty}^{+\infty}\dfrac{1}{(l+(1/2)\varepsilon_{i,j})^2}=\dfrac{\pi^2}{4}\varepsilon''_{i,j}\varepsilon_{i,j}\,, \quad \text{for } r_{i,j}+\dfrac{\pi h}{L}(\varepsilon''_{i,j}-\varepsilon_{i,j})=0\,. \end{cases}
$$

Taking the summations over $p_{i,j}$ and next over $r_{i,j}$ in (2·13), one has

$$
\mathrm{I}(q''(\mu),q)=(\pi/2)^{n(n-1)}\prod_{1\leq i<j\leq n}\varepsilon''_{i,j}(\mu)\varepsilon_{i,j}\prod_{i=1}^{n}\delta_{q_i''(\mu)+\sum_{j=1, j\neq i}^{n}(\pi h/L)(\varepsilon_{i,j}-\varepsilon''_{i,j}(\mu)),q_i}\,. \tag{2·15}
$$

Substitution of (2·15) in (2·10) produces

$$
\begin{aligned}
U_n{}^*U_n|q_1,q_2,\cdots,q_n\rangle=&\sum_{q_1'\leq q_2'\cdots\leq q_n'}a_{q_1',q_2',\cdots,q_n'}\sum_{\mu}\Big\{\prod_{1\leq i<j\leq n}\varepsilon''_{i,j}(\mu)\varepsilon_{i,j}\\
&\times\prod_{i=1}^{n}\delta_{q_i''(\mu),q_i+\sum_{j=1, j\neq i}^{n}(\pi h/L)(\varepsilon''_{i,j}(\mu)-\varepsilon_{i,j})}\Big\}\prod_{i=1}^{n}a^*_{q'_i}|0\rangle\,.
\end{aligned} \tag{2·16}
$$

Now let us show that there is no contribution in (2·16) from any nonidentical permutation. For a nonidentical permutation μ, there necessarily exists an integer I for which

$$
\mu_I>\mu_{I+1}\,. \tag{2·17}
$$

Under this permutation, the restriction for the summations in (2·16), namely, $q_1'\leqq q_2'\leqq\cdots\leqq q_n'$ gives

$$
q_I''(\mu)=q'_{\mu_I}\geqq q'_{\mu_{I+1}}=q''_{I+1}(\mu). \tag{2·18}
$$

On the other hand, one can derive the incompatible inequality with (2·18) in the following way. From Kronecker's symbols in (2·16) one obtains

$$
\begin{aligned}
q''_{I+1}(\mu)-q''_I(\mu)=&\,q_{I+1}-q_I-(2\pi h/L)(\varepsilon''_{I,I+1}(\mu)-1)\\
&+\sum_{l=I+2}^{n}(\pi h/L)(\varepsilon''_{I+1,l}(\mu)-\varepsilon''_{I,l}(\mu))\\
&+\sum_{l=1}^{I-1}(\pi h/L)(\varepsilon''_{l,I}(\mu)-\varepsilon''_{l,I+1}(\mu)),
\end{aligned} \tag{2·19}
$$

where one has used $\varepsilon_{l,m}=1$ $(1\leqq l<m\leqq n)$. By using the following relations:

$$
\varepsilon''_{I+1,l}(\mu)-\varepsilon''_{I,l}(\mu)=\frac{\mu_l-\mu_{I+1}}{|\mu_l-\mu_{I+1}|}-\frac{\mu_l-\mu_I}{|\mu_l-\mu_I|}
$$

$$=\begin{cases} 0 & (\mu_l<\mu_{I+1}<\mu_I) \\ 2 & (\mu_{I+1}<\mu_l<\mu_I) \\ 0 & (\mu_{I+1}<\mu_I<\mu_l) \end{cases}$$

$$\varepsilon''_{l,I}(\mu)-\varepsilon''_{l,I+1}(\mu)=\frac{\mu_I-\mu_l}{|\mu_I-\mu_l|}-\frac{\mu_{I+1}-\mu_l}{|\mu_{I+1}-\mu_l|}$$

$$=\begin{cases} 0 & (\mu_l<\mu_{I+1}<\mu_I) \\ 2 & (\mu_{I+1}<\mu_l<\mu_1) \\ 0 & (\mu_{I+1}<\mu_I<\mu_l) \end{cases}$$

(2·19) becomes

$$q''_{I+1}(\mu)-q''_I(\mu)\geqq q_{I+1}-q_I-(2\pi\hbar/L)(\varepsilon''_{I,I+1}(\mu)-1)\geqq 4\pi\hbar/L\,, \tag{2·20}$$

where we have made use of $\varepsilon''_{I,I+1}(\mu)=-1$ and $q_{I+1}\geqq q_I$. The inequality (2·20) is evidently incompatible with (2·18). Thus there are no contributions from nonidentical permutations in (2·16). In the case of identical permutation μ_0, n Kronecker's symbols in (2·16) give

$$q_i'=q_i''(\mu_0)=q_i\,,\qquad (i=1,2,\cdots,n) \tag{2·21}$$

because $\varepsilon''_{l,m}(\mu_0)=1$ $(1\leqq l<m\leqq n)$. Hence we have

$$U_n{}^*U_n|q_1,q_2,\cdots,q_n\rangle=\alpha_{q_1,\cdots,q_n}\prod_{i=1}^n a^*_{q_i}|0\rangle=|q_1,q_2,\cdots,q_n\rangle\,. \tag{2·22}$$

In this way, (2·5a) has been established in the case $g\to\infty$.

(b) Proof of (2·5b)

From (2·3a, b) one has

$$\begin{aligned} &U_nU_n^*|s_1,s_2,\cdots,s_n\rangle \\ &=\sum_{q_1'\leqq q_2'\cdots\leqq q_n'}\;\sum_{q_1\leqq q_2\cdots\leqq q_n}(\pi/2)^{-n(n-1)}\alpha_{q_1',\cdots,q_n'}\alpha_{q_1,\cdots,q_n}\sum_{\{p'_{i,j},p_{i,j};1\leqq i<j\leqq n\}} \\ &\quad\times\prod_{1\leqq i<j\leqq n}d(p'_{i,j};(-\pi\hbar/L)\varepsilon_{i,j})\,d(p_{i,j};(-\pi\hbar/L)\varepsilon_{i,j}) \\ &\quad\times\prod_{i=1}^n a^*_{\sum_{j=1,j\neq i}^n p'_{i,j}+q_i'}\prod_{i=1}^n a_{q_i'}\prod_{i=1}^n a^*_{q_i}\prod_{i=1}^n a_{\sum_{j=1,j\neq i}^n p_{i,j}+q_i}|s_1,s_2,\cdots s_n\rangle\,. \end{aligned} \tag{2·23}$$

Noting that

$$\alpha_{q_1',\cdots,q_n'}\alpha_{q_1,\cdots,q_n}\prod_{i=1}^n a_{q_i'}\prod_{i=1}^n a^*_{q_i}|0\rangle=\prod_{i=1}^n\delta_{q_i',q_i}|0\rangle\,,$$

which is valid under the restriction for the summations in (2·23), one has

$$a_{q_1',\cdots,q_n'}a_{q_1,\cdots,q_n}\prod_{i=1}^{n}a_{q_i'}\prod_{i=1}^{n}a^*_{q_i}\prod_{i=1}^{n}a_{p_i}|s_1,\cdots,s_n\rangle=\prod_{i=1}^{n}\delta_{q_i',q_i}\prod_{i=1}^{n}a_{p_i}|s_1,\cdots,s_n\rangle\,, \tag{2·24}$$

for $q_1'\leqq q_2'\leqq\cdots\leqq q_n'$ and $q_1\leqq q_2\leqq\cdots\leqq q_n$. Use of (2·24) in (2·23) gives

$$\begin{aligned}U_nU_n^*|s_1,\cdots,s_n\rangle=&\sum_{q_1\leqq q_2\cdots\leqq q_n}(\pi/2)^{-n(n-1)}\sum_{\{p'_{i,j},p_{i,j};1\leqq i<j\leqq n\}}\\&\times\prod_{1\leqq i<j\leqq n}d(p'_{i,j};(-\pi\hbar/L)\varepsilon_{i,j})d(p_{i,j};(-\pi\hbar/L)\varepsilon_{i,j})\\&\times\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p'_{i,j}+q_i}\prod_{i=1}^{n}a_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}|s_1,\cdots,s_n\rangle\,.\end{aligned} \tag{2·25}$$

According to the formula A in the Appendix the expression (2·25) can be rewritten as

$$\begin{aligned}U_nU_n^*|s_1,\cdots,s_n\rangle=&(\pi/2)^{-n(n-1)}(1/n!)\sum_{q_1,\cdots,q_n}\sum_{\{p'_{i,j},p_{i,j};1\leqq i<j\leqq n\}}\\&\times\prod_{1\leqq i<i\leqq n}\{d(p'_{i,j};(-\pi\hbar/L)\varepsilon_{i,j})d(p_{i,j};(-\pi\hbar/L)\varepsilon_{i,j})\}\\&\times\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p'_{i,j}+q_i}\prod_{i=1}^{n}a_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}|s_1,\cdots,s_n\rangle\,.\end{aligned} \tag{2·26}$$

Now let us change the summation variables in (2·26) as

$$q_i+\sum_{\substack{j=1\\j\neq i}}^{n}p_{i,j}\to t_i\,,\qquad (i=1,2,\cdots,n) \tag{2·27}$$

and introduce the new variables given by

$$r_{i,j}=p'_{i,j}-p_{i,j}\,.\qquad (1\leqq i<j\leqq n) \tag{2·28}$$

Then, the expression of (2·26) is rewritten as

$$\begin{aligned}U_nU_n^*|s_1,\cdots s_n\rangle=&(\pi/2)^{-n(n-1)}(1/n!)\sum_{t_1,\cdots,t_n}\sum_{\{p_{i,j},r_{i,j};1\leqq i<j\leqq n\}}\\&\times\prod_{1\leqq i<j\leqq n}\{d(p_{i,j}+r_{i,j};(-\pi\hbar/L))d(p_{i,j};(-\pi\hbar/L))\}\\&\times\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}r_{i,j}+t_i}\prod_{i=1}^{n}a_{t_i}|s_1,s_2,\cdots,s_n\rangle\,,\end{aligned} \tag{2·29}$$

where we have used $\varepsilon_{i,j}=1$ $(1\leqq i<j\leqq n)$. Carrying out the sums over the variables $p_{i,j}$ $(1\leqq i<j\leqq n)$ in (2·29) with the aid of the formula (2·14), we have

$$\begin{aligned}U_nU_n^*|s_1,s_2,\cdots,s_n\rangle=&(1/n!)\sum_{t_1,\cdots,t_n}\sum_{\{r_{i,j};1\leqq i<j\leqq n\}}\\&\times\prod_{1\leqq i<j\leqq n}\delta_{r_{i,j},0}\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}r_{i,j}+t_i}\prod_{i=1}^{n}a_{t_i}|s_1,s_2,\cdots,s_n\rangle\end{aligned}$$

$$=(1/n!)\sum_{t_1,\cdots,t_n}\prod_{i=1}^{n}a_{t_i}^{*}\prod_{i=1}^{n}a_{t_i}|s_1, s_2, \cdots, s_n\rangle .$$

Here noting the formula B in the Appendix for $l=0$, one gets

$$U_n U_n^{*}|s_1, s_2, \cdots, s_n\rangle=|s_1, s_2, \cdots, s_n\rangle . \qquad (2\cdot30)$$

Thus the equality (2·5b) has been justified for infinitely large g.

In this way, we can conclude that the transformation operator (2·3a, b) for n-boson system is unitary operator in the case $g\to\infty$, and that the normalization factor $\beta_{q_1,q_2,\cdots,q_n}$ is $(\pi/2)^{-n(n-1)/2}$ in this case.

2.3. *Unitarity of U_n in the case of finite coupling constant*

In § 2 of II, we have established the orthogonality of the exact eigenstates in the case of finite coupling constant g. From this result, one has

$$\langle \Psi_{q_1',q_2',\cdots,q_n'}|\Psi_{q_1,q_2,\cdots,q_n}\rangle=\prod_{i=1}^{n}\delta_{q_i',q_i} . \qquad (2\cdot31)$$

Substitution of (2·4) in (2·31) gives

$$\langle q_1', q_2', \cdots, q_n'|U_n^{*} U_n|q_1, q_2, \cdots, q_n\rangle=\prod_{i=1}^{n}\delta_{q_i',q_i} \qquad (2\cdot32)$$

for any two free eigenstates $|q_1', q_2', \cdots, q_n'\rangle$ and $|q_1, q_2, \cdots, q_n\rangle$. Thus the equality (2·5a) holds in the case of finite coupling constant g.

On the other hand, the condition (2·5b) for completeness of the set of the exact eigenstates (1·2) unfortunately cannot be verified for finite coupling constant g. If one notes that all eigenstates (1·2) for finite g approach continuously their limiting ones for $g\to\infty$, it may be plausible to conclude that the completeness condition

$$\langle q_1', q_2', \cdots, q_n'|U_n U_n^{*}|q_1, q_2, \cdots, q_n\rangle=\prod_{i=1}^{n}\delta_{q_i',q_i} , \qquad (2\cdot33)$$

is valid even for the case of finite g.

From (2·32) and the plausible conjecture (2·33), we may regard the operator U_n for any positive g to be unitary operator in n-boson system.

§ 3. Unitary transformation for the system with arbitrary number of bosons

The purpose of this section is to construct the unitary operator U for the system composed of arbitrary number of bosons, on the basis of the unitary operator U_n for n-boson system, given by (2·3a, b). The conditions which should be imposed on this unitary operator U are given as follows:

$$U|s_1, s_2, \cdots, s_n\rangle = U_n|s_1, s_2, \cdots, s_n\rangle , \quad (n \geqq 2) \tag{3·1a}$$

$$U|0\rangle = |0\rangle , \qquad Ua_p^*|0\rangle = a_p^*|0\rangle , \tag{3·1b}$$

where the conditions (3·1b) come from the facts that the free states $|0\rangle$ and $a_p^*|0\rangle$ themselves are the eigenstates of the total Hamiltonian (1·1).

Now we assume that the unitary operator U can be expressed in the following form:

$$U = 1 + \sum_{n=2}^{\infty} X_n \tag{3·2}$$

and

$$X_n = U_n + \sum_{m=2}^{n-1} h_n(m) \sum_{p_1, p_2, \cdots, p_{n-m}} a_{p_1}^* \cdots a_{p_{n-m}}^* U_m a_{p_1} \cdots a_{p_{n-m}}$$

$$+ h_n(0) \sum_{p_1, \cdots, p_n} a_{p_1}^* \cdots a_{p_n}^* a_{p_1} \cdots a_{p_n} . \tag{3·3}$$

Then it can be readily seen that the operator U satisfies the conditions (3·1b). The expansion coefficients $h_n(m)$ can be determined by the condition (3·1a) in the following way. First observe that $X_l|s_1, \cdots, s_n\rangle = 0$ for $l \geqq n+1$. Then, using (3·2) in (3·1a), one gets the relation

$$X_n|s_1, \cdots, s_n\rangle = (U_n - 1 - \sum_{l=2}^{n-1} X_l)|s_1, \cdots, s_n\rangle . \tag{3·4}$$

Substitution of (3·3) in the left-hand side of (3·4) gives

$$X_n|s_1, \cdots, s_n\rangle = \{U_n + \sum_{m=2}^{n-1} h_n(m) \sum_{p_1, \cdots, p_{n-m}} a_{p_1}^* \cdots a_{p_{n-m}}^* U_m a_{p_1} \cdots a_{p_{n-m}}$$

$$+ h_n(0) \sum_{p_1, \cdots, p_n} a_{p_1}^* \cdots a_{p_n}^* a_{p_1} \cdots a_{p_n}\}|s_1, \cdots, s_n\rangle \tag{3·5a}$$

and introduction of (3·3) into the right-hand side of (3·4) yields

$$(U_n - 1 - \sum_{l=2}^{n-1} X_l)|s_1, \cdots, s_n\rangle$$

$$= \{U_n - 1 - \sum_{l=2}^{n-1} [U_l + \sum_{m=2}^{l-1} h_l(m) \sum_{p_1, \cdots, p_{l-m}} a_{p_1}^* \cdots a_{p_{l-m}}^* U_m a_{p_1} \cdots a_{p_{l-m}}$$

$$+ h_l(0) \sum_{p_1, \cdots, p_l} a_{p_1}^* \cdots a_{p_l}^* a_{p_1} \cdots a_{p_l}]\}|s_1, \cdots, s_n\rangle$$

$$= \{U_n - (1/n!) \sum_{p_1, \cdots, p_n} a_{p_1}^* \cdots a_{p_n}^* a_{p_1} \cdots a_{p_n}$$

$$- \sum_{l=2}^{n-1} (1/(n-l)!) \sum_{p_1, \cdots, p_{n-l}} a_{p_1}^* \cdots a_{p_n}^* a_{p_1} U_l a_{p_1} \cdots a_{p_{n-l}}$$

$$- \sum_{l=2}^{n-1} \sum_{m=2}^{l-1} h_l(m) (1/(n-l)!) \sum_{p_1, \cdots, p_{n-m}} a_{p_1}^* \cdots a_{p_{n-m}}^* U_m a_{p_1} \cdots a_{p_{n-m}}$$

$$-\sum_{l=2}^{n-1} h_l(0)(1/(n-l)!)\sum_{p_1,\cdots,p_n} a^*_{p_1}\cdots a^*_{p_n}a_{p_1}\cdots a_{p_n}\}|s_1,\cdots,s_n\rangle\,, \tag{3·5b}$$

where the second equality is due to the Formula B,

$$\begin{aligned}&a_{q_1}a_{q_2}\cdots a_{q_l}|s_1,\cdots,s_n\rangle\\ &\quad=(1/(n-l)!)\sum_{p_1,\cdots,p_{n-l}} a^*_{p_1}\cdots a^*_{p_{n-l}}a_{q_1}\cdots a_{q_l}a_{p_1}\cdots a_{p_{n-l}}|s_1,\cdots,s_n\rangle\,,\end{aligned} \tag{3·6}$$

which is verified in the Appendix. Comparison of (3·5a) and (3·5b) gives the relations:

$$\begin{aligned}&h_n(n-1)=-(1/1\,!),\\ &h_n(m)=-(1/(n-m)!)-\sum_{l=m+1}^{n-1}(1/(n-l)!)h_l(m)\quad\text{for } 2\leqq m\leqq n-2\,,\\ &h_n(0)=-(1/n\,!)-\sum_{l=2}^{n-1}(1/(n-l)!)h_l(0).\end{aligned} \tag{3·7}$$

The solutions of Eqs. (3·7) are given by

$$h_n(m)=(-1)^{n-m}/(n-m)!\qquad\text{for } 2\leqq m\leqq n-1\,, \tag{3·8a}$$

$$h_n(0)=\{(-1)^{n-1}/(n-1)!\}+(-1)^n/n\,!\,. \tag{3·8b}$$

The fact, that the solutions (3·8a) and (3·8b) satisfy Eq. (3·7) certainly, can be confirmed in the following way:

$$\begin{aligned}&\{\sum_{l=m+1}^{n} h_l(m)/(n-l)!\}+1/(n-m)!=\sum_{l=m}^{n}[(-1)^{l-m}/\{(n-l)!(l-m)!\}]\\ &\quad=(1-1)^{n-m}/(n-m)!=0\,,\quad(2\leqq m\leqq n-1)\end{aligned} \tag{3·9a}$$

$$\begin{aligned}&\{\sum_{l=2}^{n} h_l(0)/(n-l)!\}+1/n\,!=\sum_{l=1}^{n}[(-1)^{l-1}/\{(n-l)!(l-1)!\}]\\ &\qquad\qquad+\sum_{l=0}^{n}[(-1)^{l}/\{(n-l)!\;l\,!\}]=0\,,\end{aligned} \tag{3·9b}$$

where $0\,!$ denotes 1. In this way the operator U is completely determined as

$$\begin{aligned}U=1+\sum_{n=2}^{\infty}\Big\{&U_n+\sum_{l=2}^{n-1}\frac{(-1)^{n-l}}{(n-l)!}\sum_{p_1,\cdots,p_{n-l}} a^*_{p_1}\cdots a^*_{p_{n-l}}U_l a_{p_1}\cdots a_{p_{n-l}}\\ &-\frac{(-1)^n(n-1)}{n\,!}\sum_{p_1,\cdots,p_n} a^*_{p_1}\cdots a^*_{p_n}a_{p_1}\cdots a_{p_n}\Big\}.\end{aligned} \tag{3·10a}$$

The hermite conjugate operator U^* is

$$U^*=1+\sum_{n=2}^{\infty}\Big\{U_n{}^*+\sum_{l=2}^{n-1}\frac{(-1)^{n-l}}{(n-l)!}\sum_{p_1,\cdots,p_{n-l}} a^*_{p_1}\cdots a^*_{p_{n-l}}U_l{}^* a_{p_1}\cdots a_{p_{n-l}}$$

$$-\frac{(-1)^n(n-1)}{n\,!}\sum_{p_1,\cdots,p_n} a_{p_1}^*\cdots a_{p_n}^* a_{p_1}\cdots a_{p_n}\Big\}. \tag{3·10b}$$

Unitarity of the operator U is verified in the following way. First note that

$$U^*|0\rangle=|0\rangle\,,\qquad U^*a_p{}^*|0\rangle=a_p{}^*|0\rangle\,, \tag{3·11}$$

$$U^*|s_1,\cdots,s_n\rangle=U_n{}^*|s_1,\cdots,s_n\rangle\,, \tag{3·12}$$

which are readily confirmed by using (3·6), (3·8a, b) and (3·9a, b). Hence, the combinations of (3·1b) and (3·11) and that of (3·1a) and (3·12) give

$$U^*U|0\rangle=|0\rangle\,,\qquad UU^*|0\rangle=|0\rangle\,, \tag{3·13a}$$

$$U^*Ua_p{}^*|0\rangle=a_p{}^*|0\rangle\,,\qquad UU^*a_p{}^*|0\rangle=a_p{}^*|0\rangle\,, \tag{3·13b}$$

$$U^*U|s_1,\cdots,s_n\rangle=U_n{}^*U_n|s_1,\cdots,s_n\rangle=|s_1,\cdots,s_n\rangle\,,\quad(n\geqq2)$$

$$UU^*|s_1,\cdots,s_n\rangle=U_nU_n{}^*|s_1,\cdots,s_n\rangle=|s_1,\cdots,s_n\rangle\,,\quad(n\geqq2) \tag{3·13c}$$

respectively where unitarity of the operator U_n for n-boson system has been used in (3·13c). The relations (3·13a, b, c) clearly mean that

$$U^*U=1\quad\text{and}\quad UU^*=1\,, \tag{3·14}$$

since all of the states $|0\rangle$, $a_p{}^*|0\rangle$ and $|s_1,\cdots,s_n\rangle$ $(n\geqq2)$ form a complete set for the states of the system with arbitrary number of bosons.

In closing this section, we should point out that the operator U satisfies the following commutation relations:

$$[U,N]=0\,,\qquad [U,P_{\rm tot}]=0\,, \tag{3·15}$$

where N and $P_{\rm tot}$ indicate the total number operator and the total momentum operator defined, respectively, by

$$N=\sum_p a_p{}^*a_p\,,\qquad P_{\rm tot}=\sum_p pa_p{}^*a_p\,. \tag{3·16}$$

The relations (3·15) can be easily confirmed by using (2·3a), (3·10a) and (3·16).

§ 4. Exactly dressed particles and condensation

In § 3 the unitary operator U has been successfully constructed to be given by (3·10a). The unitary transformation produced by the operator U leads to the introduction of the creation and annihilation operators $A_p{}^*$ and A_p defined by

$$A_p^*=Ua_p^*U^*\,,\qquad A_p=Ua_pU^*\,. \tag{4·1}$$

These operators A_p^* and A_p obey evidently the canonical commutation relations:

$$[A_p, A_p^*] = U[a_p, a_p^*]U^* = \delta_{p,q}\ ,$$
$$[A_p, A_q] = 0\ , \qquad [A_p^*, A_q^*] = 0\ . \tag{4·2}$$

Furthermore, the following commutation relations are easily derived from (3·15)

$$[N, A_q^*] = U[N, a_q^*]U^* = A_q^*\ , \quad [N, A_q] = -A_q\ , \tag{4·3}$$

$$[P_{\rm tot}, A_q^*] = U[P_{\rm tot}, a_q^*]U^* = qA_q^*\ , \quad [P_{\rm tot}, A_q] = -qA_q\ . \tag{4·4}$$

Now let us consider the vacuum state $\|0\gg$ defined by

$$A_q\|0\gg = 0 \qquad \text{for all } q\ . \tag{4·5}$$

Then, it is easily seen that

$$\|0\gg = |0\rangle\ , \tag{4·6}$$

because the bare vacuum state $|0\rangle$ satisfies (4·5) from (4·1) and (3·11). Multiplying the state $|0\rangle$ by the operators A_q^*, one can construct a complete orthonormal set as

$$\{|0\rangle,\ A_q^*|0\rangle;\ \alpha_{q_1,\cdots,q_n}A_{q_1}^*\cdots A_{q_n}^*|0\rangle\ , \quad (n\geqq 2,\ q_1\leqq q_2\leqq\cdots\leqq q_n)\}, \tag{4·7}$$

which are readily seen to be the exact eigenstates of the Hamiltonian (1·1). This is due to the following equations:

$$\begin{aligned}|\Psi_{q_1,\cdots,q_n}\rangle &= U\alpha_{q_1,\cdots,q_n}a_{q_1}^*\cdots a_{q_n}^*|0\rangle \\ &= \alpha_{q_1,\cdots,q_n}A_{q_1}^*\cdots A_{q_n}^*|0\rangle\ .\end{aligned} \tag{4·8}$$

If we operate A_q^* to any eigenstate (4·8), we have again the eigenstate of the total Hamiltonian (1·1) in which the particle number is increased by one and the total momentum is increased by q as readily seen from (4·3) and (4·4). From this fact, one can give the following interpretation for the new operators A_q^* and A_q. "*The operators A_q^* and A_q are the creation and annihilation ones of a bose particle* (*with momentum q*) *which has been dressed exactly with the interaction cloud*." Hence the bose particles produced by the operator A_p^* can be called "*the exactly dressed bosons*." In this way it can be seen that the quantum numbers $q_1, \cdots, q_n$ introduced to specify the eigenstate $|\Psi_{q_1,\cdots,q_n}\rangle$ in I indicate the momenta of the exactly dressed bosons, and they are in this sense certainly appropriate quantum numbers.

Here let us reexpress the total number operator N, the total momentum operator $P_{\rm tot}$ and the total Hamiltonian (1·1) in terms of the new operators A_q^* and A_q. By virtue of (3·15), the operators N and $P_{\rm tot}$ are expressed as

$$N = \sum_p a_p^* a_p = U\sum_p a_p^* a_p U^* = \sum_p A_p^* A_p\ , \tag{4·9}$$

$$P_{\rm tot}=\sum_p pa_p^* a_p = U\sum_p pa_p^* a_p U^* = \sum_p pA_p^* A_p\,. \tag{4·10}$$

Next we intend to rewrite the original total Hamiltonian (1·1) in terms of the operators A_q^* and A_q. By denoting the exact eigenstate $|\Psi_{q_1,\cdots,q_n}\rangle$ as

$$\|q_1, q_2, \cdots, q_n\gg = \alpha_{q_1,\cdots,q_n}\prod_{i=1}^{n} A_{q_i}^*|0\rangle = |\Psi_{q_1,\cdots,q_n}\rangle\,, \quad (q_1\leqq\cdots\leqq q_n) \tag{4·11}$$

the eigenequation becomes

$$\hat{H}(A^*, A)\|q_1, \cdots, q_n\gg = E_{q_1,\cdots,q_n}\|q_1, \cdots, q_n\gg\,. \tag{4·12}$$

in which

$$\hat{H}(A^*, A) = H = U^*UH(a^*, a)U^*U = U^*H(A^*, A)U\,, \tag{4·13}$$

where $H(a^*, a)$ indicates the functional form of the total Hamiltonian (1·1) with respect to the operators $a_q{}^*$ and a_q. The functional form $\hat{H}(A^*, A)$ of the total Hamiltonian can be easily obtained by the same method as that for the construction of the unitary operator U in § 3. For this purpose, let us introduce such operators $H_n(n\geqq 1)$ that

$$H_1=\sum_p(p^2/2m)A_p{}^*A_p\,,\quad H_n=(1/n\,!)\sum_{p_1,\cdots,p_n}\hat{E}_{p_1,\cdots,p_n}A_{p_1}^*\cdots A_{p_n}^*A_{p_1}\cdots A_{p_n}\,,\quad (n\geqq 2) \tag{4·14}$$

in which $\hat{E}_{p_1,\cdots,p_n}$ is defined by making use of the eigenenergy $E_{q_1,\cdots,q_n}$ of (1·4) as

$$\hat{E}_{p_1,p_2,\cdots,p_n} = E_{p_{\mu_1},p_{\mu_2},\cdots,p_{\mu_n}}\,, \tag{4·15}$$

where the permutation $\mu=\binom{1\,2\cdots\,n}{\mu_1\,\mu_2\cdots\,\mu_n}$ indicates the rearrangement of the momenta $p_1, p_2, \cdots, p_n$ to the nondecreasing order of magnitude of the momenta, namely, $p_{\mu_1}\leqq p_{\mu_2}\leqq\cdots\leqq p_{\mu_n}$.

Now we assume that the total Hamiltonian $\hat{H}(A^*, A)$ is expanded in the following form:

$$\hat{H}(A^*, A)=\sum_{n=1}^{\infty} Y_n\,, \tag{4·16a}$$

$$Y_n = H_n + \sum_{l=1}^{n-1}\mathcal{Y}_n(l)\sum_{p_1,p_2,\cdots,p_{n-l}} A_{p_1}^*\cdots A_{p_{n-l}}^* H_l A_{p_1}\cdots A_{p_{n-l}}\,. \tag{4·16b}$$

The expansion coefficients $\mathcal{Y}_n(l)$ are determined by the conditions,

$$\hat{H}(A^*, A)\|q_1, \cdots, q_n\gg = H_n\|q_1, \cdots, q_n\gg\,,\quad (n\geqq 1) \tag{4·17}$$

and the results are given by

$$\mathcal{Y}_n(l)=(-1)^{n-l}/(n-l)!\,. \tag{4·18}$$

From (4·18) and (4·16a, b) Eq. (4·17) is derived as follows:

$$\hat{H}(A^*, A)\|q_1, \cdots, q_n\gg=\sum_{l=1}^{n} Y_l\|q_1, \cdots, q_n\gg$$

$$=\{H_n+\sum_{l=1}^{n-1} H_l+\sum_{l=2}^{n}\sum_{m=1}^{l-1}((-1)^{l-m}/(l-m)!)$$

$$\times\sum_{p_1,\cdots,p_{l-m}} A_{p_1}^*\cdots A_{p_{l-m}}^* H_m A_{p_1}\cdots A_{p_{l-m}}\}\|q_1, \cdots, q_n\gg$$

$$=[H_n+\sum_{m=1}^{n-1}\left\{\frac{1}{(n-m)!}+\sum_{l=m+1}^{n}\frac{(-1)^{l-m}}{(l-m)!(n-l)!}\right\}$$

$$\times\sum_{p_1\cdots p_{n-m}} A_{p_1}^*\cdots A_{p_{n-m}}^* H_m A_{p_1}\cdots A_{p_{n-m}}]\|q_1, \cdots, q_n\gg$$

$$=H_n\|q_1, \cdots, q_n\gg=E_{q_1,\cdots,q_n}\|q_1, \cdots, q_n\gg\ , \tag{4·19}$$

where we have used (3·9a) and the formula (3·6) in which the operators $a_p{}^*$ and a_p are replaced by the operators $A_p{}^*$ and A_p, respectively. In this way, the explicit form of the Hamiltonian $\hat{H}(A^*, A)$ expressed in terms of the creation and annihilation operators $A_p{}^*$ and A_p of the exactly dressed bose particles is written as

$$\hat{H}(A^*, A)=\sum_p \frac{p^2}{2m}A_p{}^*A_p+\sum_{n=2}^{\infty}\sum_{p_1,\cdots,p_n}\sum_{l=1}^{n}\frac{(-1)^{n-l}}{(n-l)!\,l\,!}$$

$$\times\hat{E}_{p_1,\cdots,p_l}A_{p_1}^*\cdots A_{p_n}^*A_{p_1}\cdots A_{p_n}\,. \tag{4·20}$$

It should be noted that the Hamiltonian $\hat{H}(A^*, A)$ has the diagonalized form which is constructed by the terms of the products of the number operator $A_p{}^*A_p$.

As has been shown in detail in § 3 of II, the ground state in our interacting n-boson system has been given by $|\Psi_{0,0,\cdots,0}\rangle$, and then the expression of the ground state indicates the existence of zero-momentum condensation. Now this can be exhibited explicitly as

$$|\Psi_{0,0,\cdots,0}\rangle=(1/\sqrt{n\,!})(A_0{}^*)^n|0\rangle\ , \tag{4·21}$$

thanks to the successfull introduction of the operators $A_p{}^*$ of the exactly dressed bose particles. Thus we can conclude that the ground state in our interacting n-boson system is the condensed state of n exactly dressed bose particles with zero momentum.

§ 5. Limiting properties for $g\to 0$

In this section we first investigate the limiting property of $k_{i,j}$ for $g\to 0$. From the limiting property we will find that the unitary operator U approaches the identity operator in the limit $g\to 0$. Secondly we will prove that $k_{i,j}$ cannot be expanded in the power series concerning g when $q_i=q_j$.

We start to verify the limiting properties

$$\lim_{g\to 0} k_{i,j}=0\,, \qquad (1\leqq i<j\leqq n) \tag{5·1}$$

for any set $\{q_i\}$ of the momenta, where $k_{i,j}$ are given by (3·7) in I.

Proof

It has been proved in § 3 of I that the $(n-1)$ simultaneous equations ((3·12) in I) determine uniquely the $(n-1)$ parameters $\varDelta_i$, and then the $n(n-1)/2$ quantities $k_{i,j}$ are obtained from the parameters $\varDelta_i$. In order to prove (5·1), hence, we need to clarify the limiting behaviors of $\varDelta_i$ for $g\to 0$. This is accomplished as follows. First note the inequalities

$$\begin{aligned} F_i(\alpha,\alpha,\cdots,\alpha)&=\alpha-(2\hbar/L)[\cot^{-1}((\hbar/mg)i\alpha)+\cot^{-1}\{(\hbar/mg)(n-i)\alpha\}] \\ &\leqq \alpha-(4\hbar/L)\cot^{-1}\{(\hbar/mg)n\alpha\}, \quad (\text{for } \alpha\geqq 0,\ i=1,2,\cdots,n-1) \end{aligned} \tag{5·2}$$

where the first equality has been derived from substitution of $\alpha(\geqq 0)$ for all $\varDelta_i$ in the function $F_i\,(\varDelta_1,\varDelta_2,\cdots,\varDelta_{n-1})$ (see (3·11) in I). Here noting that the inequality

$$\cot^{-1}x>(x+1)^{-1}\,, \qquad (\text{for } x\geqq 0) \tag{5·3}$$

holds due to the restriction for the region of the function $\cot^{-1}x$ given in (3·3) of I, we have

$$F_i(\alpha,\alpha,\cdots,\alpha)<\alpha-(4\hbar/L)\{(\hbar/mg)n\alpha+1\}^{-1}\,. \tag{5·4}$$

Since the right-hand side of (5·4) becomes zero when

$$\alpha=\alpha_1\equiv(mg/2n\hbar)\{\sqrt{1+(16n\hbar^2/mgL)}-1\}, \tag{5·5}$$

one gets

$$F_i(\alpha_1,\alpha_1,\cdots,\alpha_1)<0\,. \qquad (i=1,2,\cdots,n-1) \tag{5·6}$$

Here remember the method by which it has been proved that there exist the solutions $\varDelta_i(s_1,s_2,\cdots,s_{n-1})$ in the simultaneous equations (3·12) in I, and change the Step 1 in the method into the following statement.

(Step 1′) Let $\varDelta_i^{(1)}$ $(i=1,2,\cdots,n-1)$ be α_1.

The other steps can be accomplished in the same way as in I owing to the inequalities (5·6). The monotonically increasing sequences $\{\varDelta_i^{(1)},\varDelta_i^{(2)},\cdots\}$ are, therefore, obtained and converge to the limiting values $\varDelta_i(s_1,s_2,\cdots,s_{n-1})$ which prove to be the roots of the simultaneous equations. The monotonically increasing properties of the sequences $\{\varDelta_i^{(1)},\varDelta_i^{(2)},\cdots\}$ produce the inequalities

$$\Delta_i > \Delta_i^{(1)} = (mg/2n\hbar)\{\sqrt{1+(16n\hbar^2/mgL)}-1\}. \tag{5·7}$$

Consequently we have

$$\lim_{g\to 0}(\Delta_i/g) \geqq \lim_{g\to 0}(m/2n\hbar)\{\sqrt{1+(16n\hbar^2/mgL)}-1\} = \infty\,, \tag{5·8}$$

which yields the limiting properties (5·1):

$$\lim_{g\to 0} k_{i,j} = (-2\hbar/L)\lim_{g\to 0}\cot^{-1}\{(\hbar/mg)\sum_{l=i}^{j-1}\Delta_l\} = 0\,, \quad (1 \leqq i < j \leqq n)$$

where we have used (3·7) in I. Thus the proof of (5·1) has been established.

Now taking account of the limiting properties (5·1) in the functions $d(p_{i,j}; k_{i,j})$ in (1·3), one has

$$\lim_{g\to 0} d(p_{i,j}; k_{i,j}) = \delta_{p_{i,j},0}\,, \tag{5·9}$$

since the momentum $p_{i,j}$ has discontinuous values $((2\pi\hbar/L)\times \text{integer})$. Substitution of (5·9) in (2·3a) gives

$$\lim_{g\to 0} U_n = \sum_{q_1 \leqq q_2 \leqq \cdots \leqq q_n} (a_{q_1,\cdots,q_n})^2 \prod_{i=1}^{n} a_{q_i}^* \prod_{i=1}^{n} a_{q_i} = (1/n\,!) \sum_{q_1,\cdots,q_n} \prod_{i=1}^{n} a_{q_i}^* \prod_{i=1}^{n} a_{q_i}\,, \tag{5·10}$$

which yields such a limiting property of X_n in (3·3) as

$$\lim_{g\to 0} X_n = \sum_{m=0}^{n}\{(-1)^{n-m}/(n-m)!\ m\,!\} \sum_{p_1,p_2,\cdots,p_n} \prod_{i=1}^{n} a_{p_i}^* \prod_{i=1}^{n} a_{p_i} = 0\,. \quad (n \geqq 2) \tag{5·11}$$

Thus we find from (3·2)

$$\lim_{g\to 0} U = 1\,. \tag{5·12}$$

This means that the exactly dressed operators A_p^* and A_p do approach the bare operators a_p^* and a_p, respectively, in the limit $g \to 0$:

$$\lim_{g\to 0} A_p^* = \lim_{g\to 0}(U a_p^* U^*) = a_p^*\,, \quad \lim_{g\to 0} A_p = a_p\,. \tag{5·13}$$

We next show that $k_{i,j}$ cannot be expanded in the power series concerning g when $q_i = q_j$. From (2·26) in I,

$$\cot(Lk_{i,j}/2\hbar) = (\hbar/mg)\{2k_{i,j} + \sum_{\substack{l=1\\ l\neq i,j}}^{n} (k_{i,l} - k_{j,l})\}, \tag{5·14}$$

where use of $q_i = q_j$ has been made. Let us, now, assume that $k_{s,t}$ can be expanded in the following form:

$$k_{s,t}=\sum_{l=0}^{\infty}c_{s,t}^{(l)}g^l\,, \qquad (1\leqq s<t\leqq n) \tag{5·15}$$

where $c_{s,t}^{(l)}$ indicate the expansion coefficients. From the limiting property (5·1) in $g\to 0$, one can see that $c_{s,t}^{(0)}=0$. Hence substitution of (5·15) in the right-hand side of (5·14) gives the convergent result as

$$\lim_{g\to 0}(\text{r.h.s. of }(5\cdot 14))=(\hbar/m)\{2c_{i,j}^{(1)}+\sum_{\substack{l=1\\ l\neq i,j}}^{n}(c_{i,l}^{(1)}-c_{j,l}^{(1)})\}. \tag{5·16}$$

On the other hand, the left-hand side of (5·14) is divergent due to (5·1) as

$$\lim_{g\to 0}\cot\left(\frac{Lk_{i,j}}{2\hbar}\right)=\begin{cases}+\infty\,, & (i>j)\\ -\infty\,. & (i<j)\end{cases} \tag{5·17}$$

These facts (5·16) and (5·17) contradict with each other, and therefore $k_{i,j}$ can be never expanded in the form of the power series of g when $q_i=q_j$.

From the above result, it seems that the unitary operator U cannot be expanded in the power series concerning g, because the summations with respect to q_i and q_j in the expression (2·3a) of U_n include the case $q_i=q_j$. Therefore one does not reach the concept of the exactly dressed operators $A_p{}^*$ and A_p by the perturbation method.

The above fact can be explained by the following physical consideration. Let the plural momenta $q_{i_1}, q_{i_2}, \cdots, q_{i_l}$ among the momenta $q_1, q_2, \cdots, q_n$ of n bare bose particles take the same value $q_{i_1}(=q_{i_2}=\cdots=q_{i_l})$. Then, the kinetic energies of these plural bare particles at the coordinate system with the velocity q_{i_1}/m become zero, and hence the interaction energies among them cannot be neglected even if the coupling constant g is negligibly small. This implies the impossibility of the expansion of U_n concerning g.

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Appendix

[Formula A]

$$\sum_{q_1\leqq\cdots\leqq q_n}\ \sum_{\{p'_{i,j},p_{i,j};1\leqq i<j\leqq n\}}\ \prod_{1\leqq i<j\leqq n}\{d(p'_{i,j};-\pi\hbar/L)d(p_{i,j};-\pi\hbar/L)\}$$

$$\times\prod_{i=1}^{n}a^*_{\sum_{\substack{j=1\\ j\neq i}}^{n}p'_{i,j}+q_i}\prod_{i=1}^{n}a_{\sum_{\substack{j=1\\ j\neq i}}^{n}p'_{i,j}+q_i}$$

$$=(1/n\,!)\sum_{q_1,\cdots,q_n}\sum_{\{p'_{i,j},p_{i,j};1\leq i<j\leq n\}}\prod_{1\leq i<j\leq n}\{d(p'_{i,j};-\pi\hbar/L)d(p_{i,j};-\pi\hbar/L)\}$$

$$\times\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p'_{i,j}+q_i}\prod_{i=1}^{n}a_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}\,. \tag{A·1}$$

Prior to the verification of (A·1), let us prove the following Lemma.

Lemma

When q_l is equal to $q_{l+1}+2\pi\hbar/L$,

$$\sum_{\{p'_{i,j},p_{i,j};1\leq i<j\leq n\}}\prod_{1\leq i<j\leq n}\{d(p'_{i,j};-\pi\hbar/L)d(p_{i,j};-\pi\hbar/L)\}$$

$$\times\prod_{i=1}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p'_{i,j}+q_i}\prod_{i=1}^{n}a_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}=0\,. \tag{A·2}$$

Consider such a permutation μ as

$$\mu=\begin{pmatrix}1,2,\cdots,n\\ \mu_1,\mu_2,\cdots,\mu_n\end{pmatrix},\qquad \begin{array}{l}\mu_i=i\quad\text{for } i\neq l\,,\quad l+1\\ \mu_l=l+1\,,\quad \mu_{l+1}=l\,,\end{array} \tag{A·3}$$

and denote the transformed variables p'_{μ_i,μ_j} by

$$p''_{i,j}(\mu)=p'_{\mu_i,\mu_j}\,.\quad(\text{for } 1\leq i\leq n,\, 1\leq j\leq n) \tag{A·4}$$

Note that

$$\prod_{1\leq i<j\leq n}d(p'_{i,j};-\pi\hbar/L)=\prod_{\substack{1\leq i<j\leq n\\(i,j)\neq(l,l+1)}}d(p''_{i,j}(\mu);-\pi\hbar/L)d(-p''_{l,l+1}(\mu);-\pi\hbar/L), \tag{A·5}$$

then the left-hand side of (A·2), referred to as B, is expressed as.

$$B=\sum_{\{p''_{i,j}(\mu),p_{i,j};1\leq i<j\leq n\}}\prod_{1\leq i<j\leq n}d(p_{i,j};-\pi\hbar/L)\prod_{\substack{1\leq i<j\leq n\\(i,j)\neq(l,l+1)}}d(p''_{i,j}(\mu);-\pi\hbar/L)$$

$$\times\{-d(p''_{l,l+1}(\mu)-2\pi\hbar/L;-\pi\hbar/L)\}$$

$$\times\prod_{\substack{i=1\\i\neq l,l+1}}^{n}a^*_{\sum_{j=1,j\neq i}^{n}p''_{i,j}(\mu)+q_i}\,a^*_{\sum_{j=1,j\neq l,l+1}^{n}p''_{l+1,j}(\mu)-p''_{l,l+1}(\mu)+q_l}$$

$$\times a^*_{\sum_{j=1,j\neq l,l+1}^{n}p''_{l,j}(\mu)+p''_{l,l+1}(\mu)+q_{l+1}}\prod_{i=1}^{n}a_{\sum_{j=1,j\neq i}^{n}p_{i,j}+q_i}\,, \tag{A·6}$$

in terms of the variables $p''_{i,j}(\mu)$, where we have used

$$d\Bigl(-p''_{l,l+1}(\mu);-\frac{\pi\hbar}{L}\Bigr)=\frac{\pi\hbar/L}{-p''_{l,l+1}(\mu)+\pi\hbar/L}$$

$$=-d\Bigl(p''_{l,l+1}(\mu)-\frac{2\pi\hbar}{L};-\frac{\pi\hbar}{L}\Bigr). \tag{A·7}$$

The relation $q_l=q_{l+1}+2\pi\hbar/L$ and the replacement of $p''_{l,l+1}(\mu)$ by

$p''_{i,l+1}(\mu)+2\pi\hbar/L$ give

$$B=-\sum_{\{p''_{i,j}(\mu),p_{i,j};1\leqq i<j\leqq n\}}\prod_{1\leqq i<j\leqq n}\{d(p''_{i,j}(\mu);-\pi\hbar/L)d(p_{i,j};-\pi\hbar/L)\}$$

$$\times\prod_{i=1}^{n}a^{*}{}_{\sum_{\substack{j=1\\j\neq i}}^{n}p''_{i,j}(\mu)+q_i}\prod_{i=1}^{n}a_{\sum_{\substack{j=1\\j\neq i}}^{n}p_{i,j}+q_i}\,. \tag{A·8}$$

Rewriting here the summation variables $p''_{i,j}(\mu)$ by $p'_{i,j}$, we can find the Lemma (A·2).

Now we turn to the verification of (A·1) on the basis of the Lemma (A·2). The Lemma (A·2) leads to

$$A=\sum_{q_1\leqq q_2\leqq\cdots\leqq q_{l-1}\leqq q_{l+1}+2\pi\hbar/L\leqq q_{l+2}+2\pi\hbar/L\leqq\cdots\leqq q_n+2\pi\hbar/L}$$

$$\times\sum_{\substack{q_l\\q_{l-1}\leqq q_l\leqq q_{l+1}+2\pi k/L}}\sum_{\{p'_{i,j},p_{i,j};1\leqq i<j\leqq n\}}$$

$$\times\prod_{1\leqq i<j\leqq n}\{d(p'_{i,j};-\pi\hbar/L)d(p_{i,j};-\pi\hbar/L)\}\prod_{i=1}^{n}a^{*}{}_{\sum_{\substack{j=1\\j\neq i}}^{n}p'_{i,j}+q_i}\prod_{i=1}^{n}a_{\sum_{\substack{j=1\\j\neq i}}^{n}p_{i,j}+q_i}\,, \tag{A·9}$$

where A indicates the left-hand side of (A·1). Consider here a permutation μ given by

$$\mu=\begin{pmatrix}1,2,\cdots,l-1, & l, & l+1,\cdots,n-1, & n\\ 1,2,\cdots,l-1, & l+1, & l+2,\cdots, \quad n, & l\end{pmatrix}, \tag{A·10}$$

and denote the momenta p'_{μ_i,μ_j}, p_{μ_i,μ_j} and q_{μ_i} by

$$p'''_{i,j}=p'_{\mu_i,\mu_j}\,,\quad p''_{i,j}=p_{\mu_i,\mu_j}\,,\quad q''_i=q_{\mu_i}\,.\quad(1\leqq i\leqq n,1\leqq j\leqq n) \tag{A·11}$$

Here note that

$$\begin{aligned}
&p'_{i,j}=p'''_{i,j} && \text{and}\quad p_{i,j}=p''_{i,j}\,, && (1\leqq i<j\leqq l-1)\\
&p'_{i,j}=p'''_{i,j-1} && \text{and}\quad p_{i,j}=p''_{i,j-1}\,, && (1\leqq i\leqq l-1,\ l+1\leqq j\leqq n)\\
&p'_{i,j}=p'''_{i-1,j-1} && \text{and}\quad p_{i,j}=p''_{i-1,j-1}\,, && (l+1\leqq i<j\leqq n)\\
&p'_{i,l}=p'''_{i,n} && \text{and}\quad p_{i,l}=p''_{i,n}\,, && (1\leqq i\leqq l-1)\\
&p'_{l,j}=-p'''_{j-1,n} && \text{and}\quad p_{l,j}=-p''_{j-1,n}\,. && (l+1\leqq j\leqq n)
\end{aligned} \tag{A·12}$$

Rewriting (A·9) in terms of $p'''_{i,j}$, $p''_{i,j}$ and q''_i by using (A·12), one has

$$A=\sum_{q_1''\leqq q_2''\leqq\cdots\leqq q''_{l-1}\leqq q_l''+2\pi\hbar/L\leqq\cdots\leqq q''_{n-1}+2\pi\hbar/L}\ \sum_{\substack{q_n''\\q''_{l-1}\leqq q''_n\leqq q_l''+2\pi\hbar/L}}\ \sum_{\{p'''_{i,j},p''_{i,j};1\leqq i<j\leqq n\}}$$

$$\times\prod_{1\leqq i<j\leqq n-1}\{d(p'''_{i,j};-\pi\hbar/L)d(p''_{i,j};-\pi\hbar/L)\}$$

$$\times\prod_{i=1}^{l-1}\{d(p'''_{i,n};-\pi\hbar/L)d(p''_{i,n};-\pi\hbar/L)\}\times$$

$$\times \prod_{j=l}^{n-1} \{d(p'''_{j,n}-2\pi\hbar/L\,;\,-\pi\hbar/L)d(p''_{j,n}-2\pi\hbar/L\,;\,-\pi\hbar/L)\}$$

$$\times \prod_{i=1}^{n} a^{*}{}_{\sum_{j=1, j\neq i}^{n} p'''_{i,j}+q_i''} \prod_{i=1}^{n} a_{\sum_{j=1, j\neq i}^{n} p''_{i,j}+q_i''}\,, \tag{A·13}$$

where the same relation as (A·7) has been used. The replacement of $p'''_{j,n}$ and $p''_{j,n}$ $(j=l,\ l+1,\ \cdots n-1)$ by $p'''_{j,n}+2\pi\hbar/L$ and $p''_{j,n}+2\pi\hbar/L$ $(j=l,\ l+1,\ \cdots n-1)$, respectively, gives

$$A=\sum_{q_1''\leq q_2''\leq\cdots\leq q_{l-1}''\leq q_l''+2\pi\hbar/L\leq\cdots\leq q_{n-1}''+2\pi\hbar/L}\ \sum_{q_{l-1}''\leq q_n''\leq q_l''+2\pi\hbar/L}\ \sum_{\{p'''_{i,j},p''_{i,j};1\leq i<j\leq n\}}$$

$$\times \prod_{1\leq i<j\leq n}\{d(p'''_{i,j}\,;\,-\pi\hbar/L)d(p''_{i,j}\,;\,-\pi\hbar/L)\}\prod_{i=1}^{l-1} a^{*}{}_{\sum_{j=1, j\neq i}^{n} p'''_{i,j}+q_i''}$$

$$\times \prod_{i=l}^{n-1} a^{*}{}_{\sum_{j=1, j\neq i}^{n} p'''_{i,j}+q_i''+2\pi\hbar/L}\quad a^{*}{}_{\sum_{j=1, j\neq n}^{n} p'''_{n,j}+q_n''-(2\pi\hbar/L)(n-1)}$$

$$\times \prod_{i=1}^{l-1} a_{\sum_{j=1, j\neq i}^{n} p''_{i,j}+q_i''}\prod_{i=l}^{n-1} a_{\sum_{j=1, j\neq i}^{n} p''_{i,j}+q_i''+2\pi\hbar/L}\quad a_{\sum_{j=1, j\neq n}^{n} p''_{n,j}+q_n''-(2\pi\hbar/L)(n-l)}\,. \tag{A·14}$$

The change of summation variables in (A·14) given by

$$p'''_{i,j}\rightarrow p'_{i,j}\,,\qquad p''_{i,j}\rightarrow p_{i,j}\,,\quad (1\leqq i<j\leqq n)$$

$$q_i''\rightarrow q_i\ (\text{for } 1\leqq i\leqq l-1),\quad q_i''+2\pi\hbar/L\rightarrow q_i\ (\text{for } l\leqq i\leqq n-1)$$

$$q_n''-(2\pi\hbar/L)(n-l)\rightarrow q_n \tag{A·15}$$

leads to

$$A=\sum_{q_1\leq q_2\leq\cdots\leq q_{n-1}}\ \sum_{q_{l-1}(2\pi\hbar/L)(n-l)\leq q_n\leq q_l-(2\pi\hbar/L)(n-l)}\ \sum_{\{p'_{i,j},p_{i,j};1\leq i<j\leq n\}}$$

$$\times \prod_{1\leq i<j\leq n}\{d(p'_{i,j}\,;\,-\pi\hbar/L)d(p_{i,j}\,;\,-\pi\hbar/L)\}\prod_{i=1}^{n} a^{*}{}_{\sum_{j=1, j\neq i}^{n} p'_{i,j}+q_i}\prod_{i=1}^{n} a_{\sum_{j=1, j\neq i}^{n} p_{i,j}+q_i}\,. \tag{A·16}$$

The above result (A·16) means that the original region (which is $q_{n-1}-2\pi\hbar/L<q_n$) of the variable q_n in the left-hand side of (A·1), has been transformed into

$$[q_{l-1}-(2\pi\hbar/L)\{n-(l-1)\}]<q_n\leqq q_l-(2\pi\hbar/L)(n-l), \tag{A·17}$$

where we have made use of the fact that $q_n=(2\pi\hbar/L)\times$ integer. Adding all regions of (A·17) for $l=1, 2, \cdots, n-1$ to the original region produces

$$-\infty\leqq q_n\leqq\infty\,. \tag{A·18}$$

Hence we obtain

$$A=(1/n)\sum_{q_1\leq q_2\leq\cdots\leq q_{n-1}}\ \sum_{q_n\ \text{all}}\ \sum_{\{p'_{i,j},p_{i,j};1\leq i<j\leq n\}}\ \prod_{1\leq i<j\leq n}\{d(p'_{i,j}\,;\,-\pi\hbar/L)\times$$

$$\times d(p_{i,j}\,;\,-\pi h/L)\}\prod_{i=1}^{n} a^*_{\sum_{j=1, j\neq i}^{n} p'_{i,j}+q_i}\prod_{i=1}^{n} a_{\sum_{j=1, j\neq i}^{n} p_{i,j}+q_i}\,. \qquad (\mathrm{A}\cdot 19)$$

A similar extension of the region for all variables q_i leads to the establishment of the formula A.

[Formula B]

For any state $|s_1, s_2, \cdots, s_n\rangle$ in noninteracting n-boson system

$$a_{q_1}\cdots a_{q_l}|s_1, s_2, \cdots, s_n\rangle$$
$$=[(n-l)!]^{-1}\sum_{p_1,\cdots,p_{n-l}} a^*_{p_1}\cdots a^*_{p_{n-l}}a_{q_1}\cdots a_{q_l}a_{p_1}\cdots a_{p_{n-l}}|s_1,\cdots,s_n\rangle\,, \quad (l<n) \qquad (\mathrm{B}\cdot 1)$$

holds.

proof

The state $|s_1, s_2, \cdots, s_n\rangle$ can be written as

$$|s_1, s_2, \cdots, s_n\rangle=\alpha_{s_1,s_2,\cdots,s_n}\prod_{i=1}^{\nu}(a^*_{t_i})^{M_i}|0\rangle\,, \quad (\nu\leqq n)$$
$$M_1+M_2+\cdots+M_\nu=n \quad \text{and} \quad t_1<t_2<\cdots<t_\nu\,, \qquad (\mathrm{B}\cdot 2)$$

where the momenta t_i indicate different ones to each other among $s_1, s_2, \cdots, s_n$. If any one of the momenta $q_1, q_2, \cdots, q_l$ is not equal to all of the momenta $t_1, t_2, \cdots, t_\nu$, then, both sides of (B·1) vanish, namely, the equality of (B·1) holds in this case. In other case,

$$a_{q_1}\cdots a_{q_l}=\prod_{i=1}^{\nu}(a_{t_i})^{L_i}$$

holds. Then the left-hand side of (B·1) becomes

$$[\text{l.h.s. of (B}\cdot 1)]=\alpha_{s_1,\cdots,s_n}\prod_{i=1}^{\nu}\{M_i\,!/(M_i-L_i)!\}(a^*_{t_i})^{M_i-L_i}|0\rangle\,, \qquad (\mathrm{B}\cdot 3)$$

and the right-hand side is

$$[\text{r.h.s. of (B}\cdot 1)]=\alpha_{s_1,\cdots,s_n}[(n-l)!]^{-1}$$
$$\times\sum_{p_1,\cdots,p_{n-l}} a^*_{p_1}\cdots a^*_{p_{n-l}}a_{p_1}\cdots a_{p_{n-l}}\prod_{i=1}^{n}\frac{M_i\,!}{(M_i-L_i)!}(a^*_{t_i})^{M_i-L_i}|0\rangle\,. \qquad (\mathrm{B}\cdot 4)$$

In the summations over $p_1, p_2, \cdots, p_{n-l}$ on (B·4), the nonvanishing contributions come from the following cases, where

$$a_{p_1}\cdots a_{p_{n-l}}=\prod_{i=1}^{\nu}(a_{t_i})^{M_i-L_i} \qquad (\mathrm{B}\cdot 5)$$

holds, and the number of the cases is

$$(n-l)!\prod_{i=1}^{\nu}[(M_i-L_i)!]^{-1}\,. \qquad (\mathrm{B}\cdot 6)$$

Substituting (B·5) and (B·6) in (B·4), we have

$$[\text{r.h.s. of (B·1)}] = \alpha_{s_1,\cdots,s_n}[(n-l)!]^{-1}(n-l)! \prod_{i=1}^{\nu}[(M_i-L_i)!]^{-1}\prod_{i=1}^{\nu}(a_{t_i}^*)^{M_i-L_i}$$

$$\times \prod_{i=1}^{\nu}(a_{t_i})^{M_i-L_i}\prod_{i=1}^{\nu}[M_i\,!\{(M_i-L_i)!\}^{-1}(a_{t_i}^*)^{M_i-L_i}]|0\rangle$$

$$= \alpha_{s_1,\cdots,s_n}\prod_{i=1}^{\nu}[(M_i-L_i)!]^{-1}\prod_{i=1}^{\nu}(a_{t_i}^*)^{M_i-L_i}\prod_{i=1}^{\nu}M_i\,!|0\rangle\,. \qquad \text{(B·7)}$$

This result (B·7) is equal to (B·3), namely, the left-hand side of (B·1). Thus Formula B has been established.

References

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