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規則右辺に照合外変数を含む条件付き項書換え系における階層合流性のモジュラ性

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あらまし 条件付き項書換え系は, 条件付き書換え規則における, 規則左辺に出現しない変数 (extra variable, 照合外変数と訳した) の分布の状況によって, いくつかのクラスに分類される. 本論文では, 「規則右辺の変数は, 規則左辺に出現する変数か, 条件に出現する照合外変数でなければならない」というクラス (3-CTRSs) に対し, 階層合流性 (level-confluence) を満たす CTRS の直和 (disjoint union) は階層合流性を満たすことを示す.

キーワード 条件付き項書換え系, 照合外変数, 階層合流性, モジュラ性

Modularity of Level-Confluence for Conditional Term Rewriting Systems with Extra Variables in Right-Hand Sides

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Abstract It has been proved that every disjoint union of level-confluent conditional term rewriting systems (CTRSs) with extra variables in the conditions of the rewrite rules is also level-confluent. In this paper, we show that this statement is also valid for CTRSs with extra variables in the right-hand sides of the rewrite rules.

key words conditional term rewriting system, extra variable, level-confluence, modularity

1 Introduction

It is well-known that conditional narrowing is complete for level-confluent and terminating conditional term rewriting systems with extra-variables in the right-hand sides of the rewrite rules (3-CTRSs)[6]. However, whether level-confluence is a modular property of 3-CTRSs has been unknown until now. Here, a property of CTRSs is said to be modular, if the property is conserved under merging systems with no common function symbols and under dividing into systems with no common function symbols.

We have proved that every disjoint union of level-confluent 3-CTRSs is also level-confluent[7]. That is, we have proved the important half of the modularity of level-confluence for 3-CTRSs. Our proof is based on the proof of the modularity of level-confluence for CTRSs with extra variables in the conditions of the rewrite rules (2-CTRSs)[5]. In this paper, for space limitation, we omit to describe the part of our proof, which are essentially the same as the proof for 2-CTRSs. We present discussions in our proof, which a character of 3-CTRSs require.

Reduction relation in a CTRS R is defined by the TRSs $\cup_{k\geq 0} R_k$ which are constructed inductively by R. In the proof of the modularity of level-confluence for CTRSs, for every $k \geq 0$, the union of two TRSs R_k and S_k, which are in the TRSs to define reduction relations in R and S respectively, is considered. If R and S are 3-CTRSs, then R_k and S_k are not TRSs in general: they may be TRSs with extra variables in the right-hand sides of the rewrite rules, what we call by eTRSs. This fact causes some problems in the proof of the modularity of level-confluence. The two of them are very serious. The first one is as follows: we want to show confluence of the union $R_k \oplus S_k$ for confluent R_k and S_k, but we cannot apply the Toyama's theorem which states that confluence is a modular property of TRSs [1, 2]. Fortunately, it has been proved that confluence is a modular property of eTRSs[3].

We also have proved this fact by another way[7]. The paper [3] has attempted to prove the Toyama's theorem for TRSs by using category theory and has obtained the result which is valid also for eTRSs. The method in [3] seems to be very strong so that it will be used to solve other problems, but it is in fact a difficult method since it employs category theory. To the contrary, our proof does not require any mathematical tools except those used in usual TRS papers.

The most difficult problem in proving the modularity of confluence for eTRSs is how to deal with increase of the rank of a term in a reduction. For example, suppose that two disjoint eTRSs are given and one eTRS contains a rule $A(x) \to B(x,y)$. Let us consider the union of them. In the reduction $A(s) \to B(s,t)$ for terms s and t, the rank of $B(s,t)$ may become larger than that of $A(s)$ due to the arbitrariness of t. By using this example, we say shortly the way to treat this problem: we consider another reduction $A(s) \to B(s,t')$, where $B(s,t')$ is obtained from $B(s,t)$ by replacing its all principal subterms appearing in t by variables. In $A(s) \to B(s,t')$, the rank does not increase.

The second problem, yielded by the fact that R_k and S_k are eTRSs, is that collapsing reduction is not strongly terminating in the union $R_k \oplus S_k$. Of course, weak termination of collapsing reduction in eTRSs is sufficient for the proof of the modularity of level-confluence for 3-CTRSs. However, we do not prove weak termination of collapsing reduction. We take the following alternative: we introduce a restricted collapsing reduction and prove strong termination of this reduction. Furthermore, this reduction is shown to have the same ability, as ordinary collapsing reduction, to prove the modularity of level-confluence. That is, every normal form w.r.t. reduction which we introduce is a preserved term. Here, we remark that some of the notions used in solving the second problem are due to our argument about the first problem.

The rest of this paper is organized as follows. In Section 2, we will explain some notions and notations for TRSs and modularity. In Section 3, we will show that every disjoint union of two confluent eTRSs is confluent. The solution for the second problem described above will be shown in Section 4.

2 Preliminaries

For space limitation, we write here only notions and notations which seem peculiar to this paper. Please refer to for instance [4] and [7] for details.

A term rewriting system (or a conditional term rewriting system) is a pair (F, R) of a set F of function symbols and a set R of rewrite rules. The root occurrence of every term is denoted by λ. The symbol at an occurrence o of a term s is denoted by $sym(s, o)$. Contexts such as $C[,\cdots,]$ or $C[\ ]$ are sometimes denoted by C, simply.

Let (F, R) and (G, S) be disjoint conditional term rewriting systems (i.e. $F \cup G = \emptyset$). We denote the elements in F by capital letters and the elements in G by small letters. We assume that every element of F has the black "color" and every element of G has the white "color". For notions defined for F or R and symmetrically for G or S, we describe only the definition for F or R. Let $T_F = T(F, V)$ and

$T_G = T(G, V)$, respectively. A term in T_F is called a black term. We call an element of $T(F \cup \{\square\}, V)$ by a black context or a black layer. Let $T_\oplus = T(F \cup G, V)$. A term in $T_\oplus$ whose root symbol is in F is called a top black term. For two symbols a and b in $F \cup G \cup V$, we denote $a \sim b$, if a and b are function symbols belonging to the same set, or at least one of them is a variable. Otherwise, we denote $a \not\sim b$. For a term $t \in T_\oplus$, $T_B(t)$ is defined as follows. If t is a black term or a variable, then $T_B(t) \equiv t$. If t is a top black term and $t \equiv C[\![t_1, \cdots, t_n]\!]$, then $T_B(t) \equiv C[, \cdots,]$. If t is a top white term, then $T_B(t) \equiv \square$.

3 Confluence for disjoint unions of confluent TRSs with extra variables (eTRSs)

In this section, we will show confluence of the disjoint union of two confluent TRSs with extra variables in the right-hand sides of the rewrite rules.

In the rest of this paper, we call an unconditional rewrite rule $l \to r$ satisfying $l \notin V$ and $Var(l) \supseteq Var(r)$, by an *ordinary rewrite rule*.

Definition 3.1 TRS with extra variables (eTRS)
If a pair of terms denoted by $l \to r$ satisfies $l \notin V$ and $Var(l) \not\supseteq Var(r)$, then we call it by a *rewrite rule with extra variables*, or an *e-rule*. For this rule, every element of the set $Var(r) - Var(l)$ is called an *extra variable*, or an *e-variable* of the e-rule $l \to r$. A set which consists of ordinary rewrite rules and e-rules is called a *term rewriting system with extra variables*, or an *eTRS*. □

Reduction relation in a eTRS is defined similarly to that in a TRS.

In this section and the next section, let (F, R) and (G, S) be disjoint eTRSs and suppose that they are confluent on $T_F = T(F, V)$ and $T_G = T(G, V)$, respectively. If no further comment, then we assume that a reduction is on $T_\oplus = T(F \cup G, V)$. We use $\to$ for the abbreviation of $\to_{R \oplus S}$. The word "rewrite rule" or "rule" means either ordinary rewrite rule or e-rule.

Suppose that a reduction sequence $\alpha : t \xrightarrow{*} t'$ is given and a reduction step $s \to s'$ in α is done by an e-rule. Consider the subterms of s' which are the terms substituted for the e-variables of the e-rule. We give a "mark" for every occurrence of s', which is at such subterms. We let every marked symbol in s' have also the mark through $s' \xrightarrow{*} t'$, if the symbol remains.

Definition 3.2 e-occurrence
Suppose that a reduction sequence $\alpha : t \xrightarrow{*} t'$ is given. Let s be a term in α and o be an occurrence of s. Then, we give a value 0 or 1, called by the *e-value* of o of s *in* α, denoted by $e_\alpha(s, o)$, inductively as follows.

First (for the basis), if $s \equiv t$, then we let $e_\alpha(s, o)$ be 0 for every occurrence o of s. Next (for the inductive step), suppose that $u \to s$ is a reduction step in α and every occurrence of u has an e-value. Assume that $u|_p$ is rewritten by a rule $l \to r$. Let O_r be the set of the occurrences of r, which are not at the variables of r, O_v be the set of the occurrences at the variables of r, which are not the e-variables, and O_e be the set of the occurrences at the e-variables of r, respectively. Then, for every occurrence o of s, we define the e-value $e_\alpha(s, o)$ as follows.

1. If $o \not\geq p$, then $e_\alpha(s, o) = e_\alpha(u, o)$.
2. If $o = p \cdot q$ for an occurrence $q \in O_r$, then $e_\alpha(s, o) = e_\alpha(u, p)$.
3. Suppose the case that $o \geq p \cdot q$ for an occurrence $q \in O_v$. By introducing o', we can write $o = p \cdot q \cdot o'$. Assume that the variable at the occurrence q of r is at the occurrence q' of l. Then, $e_\alpha(s, o) = e_\alpha(u, p \cdot q' \cdot o')$.
4. If $o \geq p \cdot q$ for an occurrence $q \in O_e$, then $e_\alpha(s, o) = 1$.

If $e_\alpha(s, o) = 1$, then we say that o is an *e-occurrence* of s *in* α. □

Definition 3.3 ep-subterm
Suppose that a reduction sequence $\alpha : t \xrightarrow{*} t'$ is given and s is a subterm of a term in α, whose root occurrence is not an e-occurrence in α. A subterm $s|_o$ at an occurrence o of s is an *ep-subterm* of s *in* α, if the following two conditions hold.

- $e_\alpha(s, o) = 1$.
- Let p be the maximal occurrence which satisfies $p < o$ and $e_\alpha(s, p) = 0$. Then, $sym(s, q) \not\sim sym(s, o)$ for every occurrence q such that $p \leq q < o$.

If the ep-subterms of s in α are $s_1, \cdots, s_n$, then we denote $s \equiv C\lceil s_1, \cdots, s_n \rceil_\alpha$. □

Here, we remark that there always exists the occurrence p in the second condition of Definition 3.3, for every e-occurrence o of every term s in α, provided each eTRS of a union is confluent. We can prove it by the fact that reduction sequences such as $A(x) \to B(x, y) \xrightarrow{*} y$ is impossible. This impossibility is due to confluence of each eTRS.

Proposition 3.1 For every term s in $\alpha : t \xrightarrow{*} t'$, $e_\alpha(s, \lambda) = 0$. □

(Proof) Assume that there exists a term s in α which satisfies $e_\alpha(s, \lambda) = 1$. Suppose that $u \to u'$ is a reduction step in α, and that this is done by an e-rule $l \to r$. Let w be a term substituted for an e-variable in r, which has s as its subterm. Without loss of generality, let $l \to r \in R$. Considering the fact that $u' \xrightarrow{*} s$, the following reduction sequence is possible: $C[s_1, \cdots, s_k, \cdots, s_n] \xrightarrow{*}_R s_k$ on $T_\oplus$, where $C[s_1, \cdots, s_n]$ is a subterm of u', s_k is a subterm of w and $C[, \cdots,]$ is a black layer which is the redex (containing r) of the reduction. Furthermore, there exists a black layer $u'' \equiv D[, \cdots,]$ in u, such that $D[, \cdots,] \to_R C[, \cdots,]$ by $l \to r$. We notice that $u'' \to_R C[, \cdots, x, \cdots,]$ is possible, where $C[, \cdots, x, \cdots,]$ is obtained from $C[, \cdots,]$ by replacing its kth □ from the leftmost by x. Since $C[, \cdots,]$ is the redex in the reduction $C[s_1, \cdots, s_k, \cdots, s_n] \xrightarrow{*}_R s_k$, $C[, \cdots, x, \cdots,] \xrightarrow{*}_R x$ is also possible. Hence, we have $u'' \xrightarrow{*}_R x$. For a variable y such that $x \not\equiv y$, $u'' \xrightarrow{*}_R y$ is also possible. However, the fact $x \xleftarrow{*}_R u'' \xrightarrow{*}_R y$ contradicts to confluence of R. □

Hence, we can determine whether a term s in α has ep-subterms.

Here, we define notions which are similar to the notions of outer reduction and inner reduction.

Definition 3.4 e-outer reduction, e-inner reduction
Suppose that a reduction sequence $\alpha : t \xrightarrow{*} t'$ is given. A reduction step $s \to s'$ in α is distinguished as follows.

- If s has no ep-subterm in α, then $s \to s'$ is an *e-outer reduction step in* α.
- Suppose that $s \equiv C\lceil s_1, \cdots, s_n \rceil_\alpha \to s'$.
  - If a redex in $C[\,]$ is rewritten in $s \to s'$, then $s \to s'$ is an *e-outer reduction step in* α.
  - If a redex in one of $s_1, \cdots, s_n$ is rewritten, then $s \to s'$ is an *e-inner reduction step in* α.

A reduction sequence $u \xrightarrow{*} u'$ in α is *an e-outer reduction sequence in* α, if every reduction step in $u \xrightarrow{*} u'$ is an e-outer reduction step in α. If α is an e-outer reduction sequence in α, we say that α is an *e-outer reduction sequence.*

A conversion $\beta : t_1 \xleftrightarrow{*} t_2$ is an *e-outer conversion*, if every reduction sequence in β is an e-outer reduction sequence. □

The next proposition is similar to Proposition 1.10 of [2] for outer reductions in TRSs. We omit the proof since it is also similar to the proof of that proposition.

Proposition 3.2 Suppose that an e-outer reduction $s \equiv C\lceil s_1, \cdots, s_n \rceil_\alpha \to C'\lceil u_1, \cdots, u_m \rceil_\alpha \equiv s'$ in $\alpha : t \xrightarrow{*} t'$ is given. If $\langle s_1, \cdots, s_n \rangle \propto \langle p_1, \cdots, p_n \rangle$, then we can have another reduction $p \equiv C[p_1, \cdots, p_n] \to C'[q_1, \cdots, q_m] \equiv p'$. In this case, if $\{u_1, \cdots, u_m\} - \{s_1, \cdots, s_n\} = \{u_{i_1}, \cdots, u_{i_l}\}$, then we can let each of $q_{i_1}, \cdots, q_{i_l}$ be arbitrary terms. □

Here, we define e-mono reduction and show its confluence.

Definition 3.5 e-mono reduction
For a reduction step $s \to s'$, if s' has no ep-subterm in $s \to s'$, then $s \to s'$ is called an *e-mono reduction step*.

If every reduction step in $\alpha : t \xrightarrow{*} t'$ is an e-mono reduction step, then α is an *e-mono reduction sequence.*

A conversion $\beta : t_1 \xleftrightarrow{*} t_2$ is an *e-mono conversion*, if every reduction sequence in β is an e-mono reduction sequence. □

The below proposition states that every e-mono reduction step has the same property as the reduction steps in every disjoint union of TRSs.

Proposition 3.3 Consider an e-mono reduction step $s \to s'$. Let u be the minimal special subterm of s, which is rewritten in this reduction. We can write $s \equiv C[u] \to C[u'] \equiv s'$, for a context $C[\ ]$ and a term u'. Assume that $u \equiv D[\![u_1, \cdots, u_n]\!]$. Then $u' \equiv D'\langle\!\langle u_{i_1}, \cdots, u_{i_m} \rangle\!\rangle$ for a set of indices $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, m\}$. □

(Proof) Suppose that $u' \equiv D'\langle\!\langle t_1, \cdots, t_N \rangle\!\rangle$ and that some $t_i (1 \leq i \leq N)$ has no identical term in $u_1, \cdots, u_n$. The root occurrence of u' is not an e-occurrence in $s \to s'$, and the root occurrence of t_i is an e-occurrence in $s \to s'$. Combining this with the fact that t_i is a principal subterm of u', t_i is an ep-subterm of s' in $s \to s'$. This fact contradicts to the assumption that $s \to s'$ is an e-mono reduction step. Hence every t_i has an identical term in $u_1, \cdots, u_n$, and we have this proposition. □

By the consequence of the above proposition, we have the following.

Proposition 3.4 For every e-mono conversion $t' \xleftarrow{*} t \xrightarrow{*} t''$, we have an e-mono conversion $t' \downarrow t''$. □

(Proof) By the consequence of Proposition 3.3, we can construct completely the same proof as that of confluence for disjoint unions of confluent TRSs[2], by using only e-mono reduction steps to obtain a conversion $t' \downarrow t''$. □

Concerning with e-mono reduction, we append the following two symmetrical propositions. We need them to prove confluence of e-outer reduction.

Proposition 3.5 If every term in $\alpha : t \xrightarrow{*} t'$ has no ep-subterm in α, then α is an e-mono reduction sequence. □

(Proof) Take a reduction step $\beta : s \rightarrow s'$ in α. By the assumption, s' has no ep-subterm in α. Considering the definition of e-occurrence, we notice that every e-occurrence of s' in β is an e-occurrence of s' in α. Combining this with the fact that s' has no ep-subterm in α, s' has no ep-subterm also in β. Hence, β is an e-mono reduction step. □

Proposition 3.6 If $\alpha : t \xrightarrow{*} t'$ is an e-mono reduction sequence, then every term in α has no ep-subterm in α. □

(Proof) See [7]. □

Here, we show confluence of e-outer reduction. The proof is essentially the same as that of confluence of "monochrome outer reduction" in TRSs (Proposition 3.1 in [2]).

Proposition 3.7 For every e-outer conversion $t' \xleftarrow{*} t \xrightarrow{*} t''$, we have an e-outer conversion $t' \downarrow t''$. □

(Proof) Let α_1 and α_2 be $t \xrightarrow{*} t'$ and $t \xrightarrow{*} t''$, respectively. Note that they are e-outer reduction sequences. Here, we consider the ep-subterms defined in α_1 and α_2. Let $W = \{w_1, \cdots, w_n\}$ be the set of the ep-subterms appearing in α. We prepare fresh variables $x_1, \cdots, x_n$ which satisfy $x_i \equiv x_j$ iff $w_i \equiv w_j$ for every $1 \leq i < j \leq n$, and we let $X = \{x_1, \cdots, x_n\}$.

Take a reduction step $u \rightarrow u'$ in α_1. Let v and v' be the terms obtained from u and u' respectively, by replacing their every ep-subterm $w_i \in W$ by $x_i \in X$. Since $u \rightarrow u'$ is an e-outer reduction step in α_1, we have $v \rightarrow v'$ by the consequence of Proposition 3.2. Repeating the corresponding replacement and the application of Proposition 3.2 for every reduction step in α_1, we have a reduction sequence $\alpha'_1 : s \xrightarrow{*} s'$ for α_1. Since every term in α'_1 has no ep-subterm in α'_1, α'_1 is an e-mono reduction sequence, by Proposition 3.5. Similarly, $\alpha'_2 : s \xrightarrow{*} s''$ which is obtained from α_2 by the similar way is an e-mono reduction sequence.

Hence, for the conversion $s' \xleftarrow{*} s \xrightarrow{*} s''$ obtained by combining α'_1 and α'_2, we have two e-mono reduction sequences $\beta'_1 : s' \xrightarrow{*} s^\star$ and $\beta'_2 : s'' \xrightarrow{*} s^\star$ for a term $s^\star$, by Proposition 3.4.

Take a reduction step $p \rightarrow p'$ in β'_1. Let q and q' be the terms obtained from p and p' respectively, by replacing their every variable $x_i \in X$ by $w_i \in W$. It is obvious that $q \rightarrow q'$ is possible. By repeating this for every reduction step in β'_1, we have a reduction sequence β_1. Let β_2 be the reduction sequence obtained from β'_2 by the similar way. Note that β_1 and β_2 are represented by $t' \xrightarrow{*} t^\star$ and $t'' \xrightarrow{*} t^\star$ for a term $t^\star$, respectively. We notice that β'_1 is an e-mono reduction sequence, by Proposition 3.6. Consequently, even if a term in β_1 has an ep-subterm w in β_1, w is not rewritten through β_1. Hence, β_1 is an e-outer reduction sequence. Similarly, β_2 is an e-outer reduction sequence. Here, we have obtained an e-outer conversion $\beta : t' \xrightarrow{*} t^\star \xleftarrow{*} t''$ for α. □

In the next proposition, we use a notation such as (i, j) instead of $t_{i,j}$.

Proposition 3.8 Suppose that the following two reduction sequences are given: $(1,1) \xrightarrow{*} (2,1) \xrightarrow{*} \cdots \xrightarrow{*} (m,1)$, where $(i,1) \xrightarrow{*} (i+1,1)$ is an e-outer reduction sequence for every $i \in \{1, \cdots, m-1\}$, and $(1,1) \xrightarrow{*} (1,2) \xrightarrow{*} \cdots \xrightarrow{*} (1,n)$, where $(1,j) \xrightarrow{*} (1,j+1)$ is an e-outer reduction sequence for every $j \in \{1, \cdots, n-1\}$.

Then, we have the following two reduction sequences: $(1,n) \xrightarrow{*} (2,n) \xrightarrow{*} \cdots \xrightarrow{*} (m,n)$, where $(i,n) \xrightarrow{*} (i+1,n)$ is an e-outer reduction sequence for every $i \in \{1, \cdots, m-1\}$, and $(m,1) \xrightarrow{*} (m,2) \xrightarrow{*} \cdots \xrightarrow{*} (m,n)$, where $(m,j) \xrightarrow{*} (m,j+1)$ is an e-outer reduction sequence for every $j \in \{1, \cdots, n-1\}$. □

(Proof) Apply Proposition 3.7 for $(m-1)(n-1)$ times, as the diagram in Figure 3. Every $\xrightarrow{*}$ appearing in the diagram is an e-outer reduction sequence. □

It is easy to notice that the following fact: *every one step reduction is an e-outer reduction sequence.*

Hence, for a conversion $\alpha : t' \xleftarrow{*} t \xrightarrow{*} t''$ by a union of eTRSs, by corresponding every reduction step in α to a suitable e-outer reduction sequence in the premise of the above proposition, we can have a common reduct of t' and t''.

Theorem 3.1 Let (F, R) and (G, S) be disjoint eTRSs. If R and S are confluent on T_F and T_G respectively, then $R \oplus S$ is confluent on $T_\oplus$. □

$$\begin{array}{ccccccc}
(1,1) & \xrightarrow{*} & (2,1) & \xrightarrow{*}\cdots\xrightarrow{*} & (m-1,1) & \xrightarrow{*} & (m,1)\\
\downarrow * & & \downarrow * & & \downarrow * & & \downarrow *\\
(1,2) & \xrightarrow{*} & (2,2) & \xrightarrow{*}\cdots\xrightarrow{*} & (m-1,2) & \xrightarrow{*} & (m,2)\\
\downarrow * & & \downarrow * & & \downarrow * & & \downarrow *\\
\vdots & & \vdots & & \vdots & & \vdots\\
\downarrow * & & \downarrow * & & \downarrow * & & \downarrow *\\
(1,n-1) & \xrightarrow{*} & (2,n-1) & \xrightarrow{*}\cdots\xrightarrow{*} & (m-1,n-1) & \xrightarrow{*} & (m,n-1)\\
\downarrow * & & \downarrow * & & \downarrow * & & \downarrow *\\
(1,n) & \xrightarrow{*} & (2,n) & \xrightarrow{*}\cdots\xrightarrow{*} & (m-1,n) & \xrightarrow{*} & (m,n)
\end{array}$$

Figure 1: The diagram for the proof of Proposition 3.8

(Proof) Suppose that a conversion $t' \overset{*}{\leftarrow} t \overset{*}{\rightarrow} t''$ by $R \oplus S$, which consists of terms in $T_\oplus$ is given. Assume that $t \overset{*}{\rightarrow} t'$ and $t \overset{*}{\rightarrow} t''$ have m and n reduction steps, respectively. By corresponding the ith ($1 \leq i \leq m$) reduction step of $t \overset{*}{\rightarrow} t'$ to $(i,1) \overset{*}{\rightarrow} (i+1,1)$ of Proposition 3.8 and the jth ($1 \leq j \leq n$) reduction step of $t \overset{*}{\rightarrow} t''$ to $(1,j) \overset{*}{\rightarrow} (1,j+1)$ of Proposition 3.8 respectively, we have $t' \overset{*}{\rightarrow} t^* \overset{*}{\leftarrow} t''$ for a term $t^* \equiv (m,n)$ of Proposition 3.8. □

4 Collapsing reduction in eTRSs

The proof of the modularity of level-confluence for 2-CTRSs requires termination of collapsing reduction in TRSs [5]. Similarly, the proof of it for 3-CTRSs requires termination of collapsing reduction in eTRSs. Unfortunately, collapsing reduction is not strongly terminating in eTRSs. Of course, weak termination of collapsing reduction in eTRSs is sufficient for the proof for 3-CTRSs. However, we do not prove weak termination of collapsing reduction, as we have mentioned in Introduction. We have said that we introduce a restricted collapsing reduction: we restrict collapsing reduction to e-mono reduction. Such a reduction, called e-mono collapsing reduction, is shown to be strongly terminating. In this section, we also show that a normal form w.r.t. e-mono collapsing reduction has the same property as that w.r.t. collapsing reduction. That is, a normal form w.r.t. e-mono collapsing reduction is a preserved term.

Definition 4.1 e-mono collapsing reduction
A collapsing reduction $s \overset{*}{\rightarrow} t$ is called an *e-mono collapsing reduction*, if $s \overset{*}{\rightarrow} t$ is an e-mono reduction sequence. □

Proposition 4.1 E-mono collapsing reduction is strongly terminating. □

(Proof) By Proposition 3.3, we can construct completely the same proof as termination of collapsing reduction for TRSs (Proposition 2.5 in [2]). □

Definition 4.2 non-e principal subterm, rank without ep-subterms
Suppose that a term s in $\alpha : t \overset{*}{\rightarrow} t'$ is denoted by $s \equiv C[\![s_1, \cdots, s_n]\!]$. Let $s_{i_1}, \cdots, s_{i_m}$ be the terms in $s_1, \cdots, s_n$, which are not ep-subterms of s in α. We call them by the *non-e principal subterms* of s *in* α. In this case, we denote $s \equiv D\lfloor s_{i_1}, \cdots, s_{i_m} \rfloor_\alpha$, where $D[, \cdots,]$ is the context obtained from s by replacing $s_{i_1}, \cdots, s_{i_m}$ by holes.
The *rank without ep-subterms* $\underline{rank}_\alpha(u)$ of a term u *in* α is defined as follows.

- If u has no non-e principal subterm in α, then $\underline{rank}_\alpha(u) = 1$.
- If $u \equiv E\lfloor u_1, \cdots, u_k \rfloor_\alpha$, then $\underline{rank}_\alpha(u) = 1 + \max\{\underline{rank}_\alpha(u_h) | 1 \leq h \leq k\}$.

□

Definition 4.3 First, a normal form of u w.r.t. e-mono collapsing reduction by rules in R is denoted by $\psi_R(u)$. Similarly, a normal form of u w.r.t. e-mono collapsing reduction by rules in S is denoted by $\psi_S(u)$.
Next, suppose that s is a term in a reduction sequence $\alpha : t \overset{*}{\rightarrow} t'$. $\chi_\alpha(s)$ is a term obtained from s by replacing its every top black ep-subterm s_b in α by $\psi_R(s_b)$ and its every top white ep-subterm s_w in α by $\psi_S(s_w)$, respectively. Note that $s \overset{*}{\rightarrow} \chi_\alpha(s)$. □

Definition 4.4 Suppose that w is a term in a reduction sequence $\alpha' : u \overset{*}{\rightarrow} u'$ and $w \equiv C\lceil w_1, \cdots, w_n \rceil_{\alpha'}$. Then, we say that $C[, \cdots,]$ is the *non-ep part* of w *in* α' and denote it by $N_{\alpha'}(w)$. □

Definition 4.5 Suppose that s is a term in a sequence $\alpha : t \overset{*}{\rightarrow} t'$. Let α' be $t \overset{*}{\rightarrow} s \overset{*}{\rightarrow} \chi_\alpha(s)$. Then, we denote $N_{\alpha'}(\chi_\alpha(s))$ by $N\chi_\alpha(s)$, for short. □

The notions of non-e principal subterm and rank without ep-subterms, and the notations χ, N and $N\chi$ are for terms in a given reduction sequence. Note that they can be extended the respective notions and notations for a subterm of a term in a given reduction sequence, whose root occurrence is not an e-occurrence in the sequence.

Proposition 4.2 Suppose that a top black term s in $\alpha : t \xrightarrow{*} t'$ has no non-e principal subterm and let $s \equiv D\lceil s_1, \cdots, s_n \rceil_\alpha$. Then, $N\chi_\alpha(s) \equiv D[T_B(\psi_S(s_1)), \cdots, T_B(\psi_S(s_n))]$. □

(Proof) Since s has no non-e principal subterm, $s_1, \cdots, s_n$ are the principal subterms of s. Hence, they are top white and we have $\chi_\alpha(s) \equiv D[\psi_S(s_1), \cdots, \psi_S(s_n)]$. Let α' be $t \xrightarrow{*} s \xrightarrow{*} \chi_\alpha(s)$. For a $s_i (1 \leq i \leq m)$, $\psi_S(s_i)$ may be a top black term. Since D is a black layer, the non-ep part of $\chi_\alpha(s)$ in α' absorbs the topmost black layer of $\psi_S(s_i)$. Hence, we have $N\chi_\alpha(s) \equiv D[T_B(\psi_S(s_1)), \cdots, T_B(\psi_S(s_n))]$. □

Proposition 4.3 Suppose that s is a term in $\alpha : t \xrightarrow{*} t'$. If $s \equiv C\lfloor s_1, \cdots, s_n \rfloor_\alpha$, then $N\chi_\alpha(s) \equiv N\chi_\alpha(C)\lfloor N\chi_\alpha(s_1), \cdots, N\chi_\alpha(s_n) \rfloor_\alpha$. □

(Proof) In this proof, we omit to write the suffix α in expressions. For every $i (1 \leq i \leq n)$, since the root occurrence of s_i is not an e-occurrence in α, each ep-subterm of s which is a subterm of s_i is an ep-subterm of s_i. Hence, $\chi(s) \equiv \chi(C)\lfloor \chi(s_1), \cdots, \chi(s_n) \rfloor$. Let α' be $t \xrightarrow{*} s \xrightarrow{*} \chi(s)$. Since χ rewrites the ep-subterms of s in α, the root occurrence of $\chi(s_i)$ is not an e-occurrence in α', for every $i (1 \leq i \leq n)$. Hence, $N\chi(t) \equiv N\chi(C)\lfloor N\chi(s_1), \cdots, N\chi(s_n) \rfloor$. □

In the below argument, we use the above two propositions without mentioning explicitly.

Definition 4.6 If $s \to s'$ is an e-mono reduction step, then we denote it by $s \to_m s'$. □

Proposition 4.4 Suppose that $s \to s'$ is a reduction step in a reduction sequence $\alpha : t \xrightarrow{*} t'$. Then we have $N\chi_\alpha(s) \to_m N\chi_\alpha(s')$ or $N\chi_\alpha(s) \equiv N\chi_\alpha(s')$. □

(Proof) See [7]. □

Proposition 4.5 Every destructive reduction $s \to s'$ is an e-mono collapsing reduction. □

(Proof) First, every destructive reduction is a collapsing reduction. Next, we show that this is an e-mono reduction step. Since this is a destructive reduction, the form of the rule used in the reduction must be such as $l \to x$. If we let $x \notin Var(l)$, then we have two reductions $l \to x$ and $l \to y$ for two different variables x and y. This contradicts to confluence of the eTRS to which the rule $l \to x$ belongs. Hence, we have $x \in Var(l)$, that is, the rule used in $s \to s'$ is not an e-rule. Thus, $s \to s'$ is an e-mono reduction step. □

Proposition 4.6 Suppose that s is a normal form w.r.t. e-mono collapsing reduction. Then, every reduction $s \to s'$ is not a destructive reduction. □

(Proof) Assume that $s \to s'$ is a destructive reduction. By Proposition 4.5, this is an e-mono collapsing reduction. However, this contradicts to the assumption that t is a normal form w.r.t. e-mono collapsing reduction. □

Proposition 4.7 Suppose that a top black term s in $\alpha : t \xrightarrow{*} t'$ is denoted by $s \equiv C\lfloor s_1, \cdots, s_n \rfloor_\alpha$, and let $C \equiv D\lceil u_1, \cdots, u_m \rceil_\alpha$. Then, there exists a black layer (may be □) w_i for every $u_i (1 \leq i \leq m)$, and $T_B(N\chi_\alpha(s)) \equiv T_B(s)[w_1, \cdots, w_m]$. □

(Proof) In this proof, we omit to write the suffix α in expressions. Note that $D[, \cdots,]$ is a black layer and $s_1, \cdots, s_n, u_1, \cdots, u_m$ are top white terms. First, $T_B(s) \equiv T_B(C) \equiv D$. Next, consider $T_B(N\chi(s))$. Since $s_1, \cdots, s_n$ are the non-e principal subterms of s and χ rewrites the ep-subterms of s, the topmost white layer of $s_1, \cdots, s_n$ is not rewritten by χ. Hence, $\chi(s_1), \cdots, \chi(s_n)$ are still top white terms and $N\chi(s_1), \cdots, N\chi(s_n)$ are also top white terms. We use this fact to derive the third expression of the following from the second expression.

$$\begin{aligned} T_B(N\chi(s)) &\equiv T_B(N\chi(C)\lfloor N\chi(s_1), \cdots, N\chi(s_n) \rfloor) \\ &\equiv T_B(N\chi(C)) \\ &\equiv T_B(D[T_B(\psi_S(u_1)), \cdots, T_B(\psi_S(u_m))]) \\ &\equiv D[T_B(\psi_S(u_1)), \cdots, T_B(\psi_S(u_m))] \\ &\equiv T_B(s)[T_B(\psi_S(u_1)), \cdots, T_B(\psi_S(u_m))] \end{aligned}$$

The last expression is derived by $T_B(s) \equiv D$ which is the first thing we have proved in the proof. Here, $T_B(\psi_S(u_i))$ is a black layer (may be □) for every $i (1 \leq i \leq m)$. Hence, we have this proposition. □

The following is the result of this section.

Proposition 4.8 Every normal form w.r.t. e-mono collapsing reduction is a preserved term. □

(Proof) Suppose that w is a normal form w.r.t. e-mono collapsing reduction. Every special subterm of w is also a normal form w.r.t. e-mono collapsing reduction. We will show that all of them are root preserved terms. Take a special subterm t of w. Without loss of generality, assume that $root(t) \in F$.

Suppose that a reduction sequence $t \xrightarrow{*} t'$ is given. We will show that $root(s) \in F$, for every term s in $t \xrightarrow{*} t'$. We use induction on the length N of $t \xrightarrow{*} t'$. If $N = 0$, then it is obvious. Consider the case that $N > 0$. Suppose that $t \xrightarrow{*} t'' \to t'$ and assume that $root(t^{\star}) \in F$ for every term $t^{\star}$ in $\alpha : t \xrightarrow{*} t''$. In the below, we omit to write the suffix α in expressions. Repeated application of Proposition 4.4 to each step in $t \xrightarrow{*} t''$ yields an e-mono reduction sequence $N\chi(t) \xrightarrow{*} N\chi(t'')$. Combining this with the fact that $N\chi(t) \equiv t$ is a normal form w.r.t. e-mono collapsing reduction, $N\chi(t'')$ is also a normal form w.r.t. e-mono collapsing reduction. Hence, by Proposition 4.6, $T_B(N\chi(t''))$ is not disappear whichever $N\chi(t'')$ is reduced to. By Proposition 4.7, $T_B(N\chi(t'')) \equiv T_B(t'')[w_1, \cdots, w_m]$ for black layers (may be $\Box$) $w_i (1 \leq i \leq m)$. Hence, $T_B(t'')$ is also not disappear whichever t'' is reduced to. Thus, we have $root(t') \in F$. Here, we have proved that $root(s) \in F$ for every term s in $t \xrightarrow{*} t'$. $\Box$

5 Result

By using results obtained in the preceding sections, we have proved the following theorem[7]. The rest of the proof has been obtained from the corresponding part of the proof for 2-CTRSs[5], by modifying it slightly.

Theorem 5.1 Every disjoint union of two level-confluent join 3-CTRSs is level-confluent. $\Box$

Some future research concerning with this result is also described in [7].

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