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Doctoral Dissertation

Nuclear structure with bare nucleon-nucleon interaction in tensor optimized few-body model

Kaori Horii

Research Center for Nuclear Physics,
Osaka University
Doctoral Dissertation

Nuclear structure with bare nucleon-nucleon interaction in tensor optimized few-body model
(核力が織りなすテンソル最適化少数系モデルによる核構造)

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Osaka University
Abstract

It is important to understand the structure of nuclear many-body systems in terms of the bare nucleon-nucleon interactions. The difficulty of treating the nuclear force is the presence of the strong short-range repulsion and the strong tensor interaction. In addition, it is necessary to include the effects of three-body force in many-body nucleon system. However many nuclear models have been developed to adopt the effective nucleon-nucleon interaction resulting from G-matrix calculation. Since these models cannot treat the properties of the intrinsic nuclear force, we have to develop the foundation of the theoretical framework to treat the nuclear interaction in many-body system.

In this study, we propose a tensor-optimized few-body model (TOFM) in the few body framework. The tensor interaction plays significant roles to determine the nuclear structure. In TOFM, the configurations caused by one-operation of the tensor operator to the S-wave ground state are introduced in the total wave function. The physical concept of TOFM comes from the tensor-optimized shell-model (TOSM), which can be applicable to the study of medium and heavy nuclei. The two methods of TOFM and TOSM can describe the deuteron-like tensor correlation bringing high momentum components in finite nuclei and provide a good reproduction of the binding energy with the bare nucleon-nucleon interaction. We apply the TOFM to the s-shell and p-shell nuclei and investigate the defect of TOSM quantitatively. We suggest the modification of the short-range correlation would improve the results of TOSM. It is also shown that TOFM reproduces more than 90% of the total binding energy and of the tensor contribution of the rigorous few-body calculation for s-shell nuclei. From these results we conclude that TOFM is good enough to use as an approximation in the variational calculation for the total binding energy and the TOFM wave function can treat the property of the $NN$ interaction, in particular the tensor interaction. This study is very encouraging to extend our TOFM framework to the nuclei beyond s-shell. We extend the TOFM to the p-shell region, such as $^5\text{He}$, and show the TOFM results in case of $^5\text{He}$ nucleus. We work out the variational calculation for $^5\text{He}$ and discuss the structure difference between two resonance $3/2^-$ and $1/2^-$ states. In particular, we shed light on the roles of the tensor interaction in two states, such as the $LS$ splitting energy.

We further study the effects of many-body forces in our framework. In this study we treat explicitly $\Delta(1232)$ isobar degrees of freedom in the bare interaction, which can be the origin of the three-body forces via the pion exchange. We adopt the Argonne $\Delta$ model potential and study the explicit role of $\Delta$ in nuclei. It is surprising that the additional $NN$ and $\Delta\Delta$ states generate strong tensor interactions, and change various matrix elements from the results of the only $NN$ space. It turns out that $\Delta$ plays significant roles in nuclear structure.
Acknowledgments

My deepest appreciation goes to Prof. Hiroshi Toki who provided carefully considered feedback for many questions and valuable comments for my thesis. Special thanks to Prof. Atsushi Hosaka and Prof. Kazuyuki Ogata whose opinions and informations have helped me very much throughout the production of this thesis. I am grateful to Prof. Kiyomi Ikeda and Prof. Takayuki Myo who gave me invaluable comments and made enormous contributions to my thesis. I would like to express my sincere thanks to Prof. Tadashi Shimoda, Prof. Nori Aoi, and Prof. Toru Sato for the review of this thesis. I want to thank Prof. Hisashi Horiuchi and Akihiro Tosaki for their warm encouragements. Special thanks go to Dr. Hiroyuki Kamano and all the members of RCNP theory group. I gratefully appreciate the exempt of school expenses from Osaka University and the support from Research Center for Nuclear Physics (RCNP) that made it possible to complete my thesis. Finally, I would also like to express my gratitude to my family for their moral support and warm encouragements.
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Chapter 1

Introduction

1.1 Aim of this study

The aim of this study is to understand the structure of finite nuclei in terms of the microscopic nuclear interaction. One of the most important challenges is to treat the properties of the bare nucleon-nucleon ($NN$) interaction in calculating the nuclear structure. The bare nuclear interaction has complicated structures such as the strong short-range repulsion and the strong tensor interaction and additionally various spin-isospin dependences with many operators. Moreover effects of three-body force should be also take into account to reproduce the experimental data of many-body nuclear system. Many conventional nuclear models have been developed by using effective nuclear interactions due to the difficulty of treating the bare nuclear interaction directly. The effective interactions do not include the intrinsic properties of nuclear interactions manifestly. Hence it is important to describe the nuclear many-body system by using the realistic $NN$ interaction.

At the present time, few-body calculations with ab initio method have been developed and succeeded in the reproduction of the ground and few excited states for light nuclei. However, it is extremely difficult to extend these frameworks to heavier nuclei because of the difficulty of treating large model space exactly. In addition, the treatment of three-body interaction so far has been phenomenological, while there are discussions that non-nucleonic degrees of freedom may play important roles after being truncated in the nucleon space. Therefore, it is desired to develop a microscopic method starting from the bare $NN$ interaction.

1.2 Bare nucleon-nucleon interaction

It is not very straightforward to describe the nuclear interaction from a microscopic theory. We have to consider not only the meson-exchange but also the contribution of quark and gluon degrees of freedom in the nuclear forces. We recall that the nuclear interaction in many-body system has a complicated structure with spin and isospin including multi-nucleon interaction. There are many attempts to describe the bare $NN$ interaction such as Argonne potential [2, 3] and Bonn potential [4, 5] in reproducing the nucleon-nucleon scattering data and the deuteron properties. There are slight differences in these...
phenomenological $NN$ interactions due to their model settings, for example a potential shape, but these interactions tell us some characteristic properties of $NN$ interaction in common. The notable features of the bare $NN$ interaction is the existences of the strong repulsive interaction at short distance caused by quark dynamics, and the strong tensor interaction at intermediate and long-range distance due to the pion exchange between nucleons.

The tensor force is dominantly provided the meson theory, and was originally proposed by Yukawa [1]. The strong tensor force originates in the pion-exchange interaction as,

$$3(\vec{\tau}_1 \cdot \vec{\tau}_2)(\vec{\sigma}_1 \cdot \hat{q})(\vec{\sigma}_2 \cdot \hat{q}) \frac{q^2}{m^2 + q^2} = (\vec{\tau}_1 \cdot \vec{\tau}_2) \left[(\vec{\sigma}_1 \cdot \vec{\sigma}_2) \frac{q^2}{m^2 + q^2} + S_{12} \frac{q^2}{m^2 + q^2}\right]$$ (1.1)

$$S_{12} = 3(\vec{\sigma}_1 \cdot \hat{q})(\vec{\sigma}_2 \cdot \hat{q}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$$

$$= [\gamma_2(\hat{q}) \cdot [\vec{\sigma}_1 \times \vec{\sigma}_2]]_2.$$ (1.2)

The pion exchange interaction is illustrated in Fig. 1.1. The second term in Eq. (1.1) represents the tensor force with the tensor operator $S_{12}$ defined in Eq. (1.2), which plays crucial roles in intermediate and long-range distances due to the small pion mass about $m_\pi = 140$ MeV corresponding to about 1.4 fm in the interaction range. In addition, these tensor forces will be enhanced with the increase of the momentum exchanged between the nucleons. The first term in Eq. (1.1) contributes to the central force with the spin and isospin operators. However there is the strong short-range repulsion in the central force, the contributions of the first term in Eq. 1.1 are suppressed by the strong repulsion. As a results, the first term of the central force gives the small contribution with low momentum component. Hence the treatment of the pion exchange interaction is to treat the strong tensor force in many body systems.

In nuclear structure, it is difficult to handle these two different properties of the $NN$ interaction, the strong short-range repulsion and the tensor force, and so many nuclear models have been developed to adopt the effective nucleon-nucleon interaction resulting from $G$-matrix calculations. The $G$-matrix interaction for the Bruckner Hartree-Fock theory integrate out high momentum components of nucleon wave functions, and the effect of the tensor force is renormalized into central and spin-orbit force without high
momentum components, and also the density-dependent term. Hence the information of the intrinsic NN interaction has been implicit in the conventional nuclear models. It is an important problem to develop treatment of the tensor force explicitly in many-body system and to understand the dynamics brought by the tensor force in finite nuclei.

Now we would like to introduce a traditional model of NN interaction formed by non relativistic framework of two-body potential taking examples as Argonne V8 model (AV8'). We show the radial dependences of the central and tensor interactions in Fig.1.2. We can confirm the existence of the hard core at less than 0.5 fm in the central potential. The triplet- even ($^3E$) part has the deepest pocket at the distance of 0.8 fm. Besides we also have a characteristic tensor interaction. In particular the triplet-even channel ($^3E$) has a very strong attraction at a short- and intermediate- range, which generates the high-momentum components.

The Argonne potential is based on a phenomenological model by fitting the NN scattering data and some deuteron properties. On the other hand, we would like to mention the recent study of $NN$ interaction in the first principle lattice QCD simulation [6]. The construction of the NN interaction from the fundamental QCD theory is one of the aims in the lattice QCD. Ref. [6] gives the calculated results of the $NN$ interactions in lattice QCD. As shown in Fig. 1.3, the lattice QCD calculations present the existence of a strong short-range repulsion in the central interaction and strong attraction of the tensor interaction. These features of the derived $NN$ interaction is similar to the AV8' potential case. However calculations cannot still reach a real pion mass, so that when the light pion mass is used in the calculation, the tensor interaction would become larger. The QCD derived $NN$ interaction would be given from the lattice QCD in the future.
1.3 Effects of the nuclear interaction in deuteron

In this section, we would like to discuss the effects of the bare $NN$ interaction on the deuteron properties. We show the calculated results of the deuteron wave function with the AV8' potential in Fig. 1.4 and various components of the Hamiltonian in Table 1.1. These results are useful to understand the roles of the short-range repulsion and tensor interaction [3,7].

As shown in Fig. 1.4, the calculated results of the deuteron wave functions show that the $S$-wave component is suppressed at the short distance due to the strong short-range repulsion, and the $D$-wave component has a more compact spatial structure than the $S$-wave one. This compact $D$-wave brings a high momentum component which leads to a large (DD) kinetic energy of 8.57 MeV as listed in Table 1.1. Although the $D$-wave probability is about 5%, the dominant attraction is caused by the tensor interaction, which is the source to produce the $D$-wave component. As a result, the small binding energy of 2.24 MeV is obtained by canceling the interaction energy with the large kinetic energy. In the case of the AV8' potential, the tensor contribution amounts to almost 80% of the total attraction as shown in Table 1.1. From these results we learn that the tensor interaction creates the characteristic $D$-wave function in the deuteron. In particular, the large matrix element of the tensor interaction is caused by the transition from the $S$-wave to the $D$-wave component. We call this two-nucleon pair consisting of a proton and a neutron, a deuteron-like state, which plays a significant role in providing a large tensor attraction even for many-body systems. Hence these features of the $NN$ interaction should appear in finite nuclei. In particular it is considered that a proton-neutron ($pn$) pairs is favored to a deuteron-like structure in order to gain the tensor correlation in nuclei. Actually, it is reported experimentally that $pn$ pair forms a strong correlation in nuclei [8,9]. Subedi et al mentioned that a large fraction of $pn$ pair is observed than the
1.4 Ab initio calculations and problems

One of the most efficient methods of treating the bare \( NN \) interaction for finite nuclei is few-body methods [10]. Fortunately we have a powerful few-body technique to describe light nuclei by using the relative or Jacobi coordinates, as illustrated in Fig. 1.6 as examples of three-body and four-body systems. The bench mark calculation of the \(^4\)He nucleus using various few-body methods have shown that it is possible to take into account the tensor interaction and the short-range repulsion exactly in terms of these relative coordinates, where the matrix element of the tensor interaction accounts for more than half of the total attraction [10].

In the few-body framework, the calculated results are extremely good for nuclear ground states and a few excited states by including a phenomenological three-body interaction. Pieper and Wiringa calculated the binding energies of the light mass nuclei using the Green’s function Monte Carlo method up to the mass number \( A \sim 12 \) in Fig 1.5, where about 70 \( \sim \) 80 % of the entire \( NN \) interaction comes from the one-pion exchange.
interaction which is the dominant source of the tensor interaction [11, 12].

However, the necessary antisymmetrization of all the particle coordinates and many configurations limit the application of the few-body method to larger mass nuclei. One of the reasons for the difficulty is the large variational model space in which the whole energy is minimized. This fact makes it extremely difficult to apply the few-body methods to heavier mass nuclei. Moreover the additional nuclear force have adopted the phenomenological three-body interaction which is based on the three-body interaction model with the one delta excitation like Fujita-Miyazawa type. In this model, the treatment of the delta excitation is effective by using only nucleon spaces, therefore, the relation between the delta excitation and the three-body force have not been investigated. It is important to study the delta and the origin of the three-body interaction in terms of the more microscopic nuclear interaction. The pion exchange force plays significant roles in the transformation a nucleon into a delta resonance. Hence we propose

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**Figure 1.5:** The calculated results of the binding energies of various nuclei using the Green’s function Monte Carlo method in Ref [11].

**Figure 1.6:** Relative coordinate for three-body and four-body system.
to treat the explicitly delta degrees of freedom in two-body correlation levels as well as two-nucleon correlation without using phenomenological three-body force. We can describe the additional forces, not only three-body force but also higher-body force, by starting from the two-body interaction.

Hence, we need a new theoretical framework, which can apply to heavier nuclei with the bare $NN$ interaction and treat not only the two-body force but also the additional effect of many-body force. We would like to propose a tensor optimized few-body model (TOFM) in order to discuss the essential features of the nuclear structure, which are induced by the NN interaction, and provide a foundation of the many-body framework in terms of the few-body calculation.

1.5 Plan of this thesis

In Chapter 2 we express the physical concept and formulation for tensor optimized few-body model, and verify the validity of TOFM in the s-shell nuclei. In Chapter 3 we further apply TOFM to the p-shell nuclei and try to focus on the discussion of the relation between the tensor force and the doublet splitting structure in $^5\text{He}$. In Chapter 4 we take into account the delta degrees of freedom in two-body nuclear force. We study the roles of the tensor force induced by the $\Delta(1232)$ isobar resonances so as to consider the effect of the three-body force. Finally in Chapter 5 we summarize this thesis and mention what we have to carry out for various studies stated in the outlook.
Chapter 2

Tensor optimized few-body model for s-shell nuclei

2.1 Physical concept of TOFM and the purpose of this chapter

As we discussed in the previous section, the strong tensor interaction play significance roles to bind finite nuclei and to determine the nuclear structure. In the variational calculation, it would be preferable to obtain the tensor correlation efficiently by taking most important configurations. There have been successful developments to handle the strong tensor correlation in the reasonable shell model space, which we call tensor optimized shell-model (TOSM). The TOSM wave function is described in the shell model basis, which can be applicable to the study of medium and heavy nuclei by using the bare NN interaction. TOSM is a method for treating the strong tensor interaction in the $2p-2h$ model space \[7, 13\]. The strong tensor interaction oughts to excite two nucleons in a spin-saturated shell model state into two-particle states. Hence, it is a minimum requirement to take $2p-2h$ excitations in order to treat the tensor interaction, which provides a large attraction to form nucleus. They use Gaussian basis functions of various ranges (short-range in particular) so that the variational wave function is able to describe the strong tensor correlation. The same concept of treating the strong tensor interaction is adapted in the framework of extended Bruckner Hartree-Fock (EBHF) theory for finite and infinite nuclei \[14, 15\].

As for another important feature of the bare NN interaction, the strong short-range repulsion, it is hard to express the relative motion between two nucleons by using the single-particle coordinates of the shell model base in TOSM. The TOSM framework uses the central correlation part of the unitary correlation operator method (UCOM) by truncating the resulting many-body operators at the two-body level \[20, 21\].

Myo et al. introduced TOSM to describe the spin-orbit splitting effect in $^5$He due to the blocking mechanism of the $p_{1/2}$ orbit caused by the deuteron-like tensor correlation \[16\]. Another important application of TOSM was the formation of the halo structure in $^{11}$Li \[17\]. Again, the tensor correlation is responsible for disfavoring a configuration of two neutrons in the $p_{1/2}$ orbit and makes the contribution of two neutrons in the $s_{1/2}$ orbit large to be the source of the halo structure.
Table 2.1: The calculated result of TOSM for $^4$He with AV8' potential. Units are in MeV for the total and various matrix elements.

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Kinetic</th>
<th>Central</th>
<th>Tensor</th>
<th>LS</th>
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<tr>
<td>FY [10]</td>
<td>-25.94</td>
<td>102.39</td>
<td>-55.26</td>
<td>-68.35</td>
<td>-4.72</td>
</tr>
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</table>

In addition, a theoretical study was performed by Myo et al. to use a bare $NN$ interaction AV8' in TOSM for $^4$He and He and Li isotopes, and the reproductions of the energy levels of He and Li isotopes are very successful [13, 18, 19]. In particular for $^4$He study, the numerical results were compared with those obtained by few-body methods [13] in Table 2.1. Table 2.1 shows the total energy and various matrix elements of the Hamiltonian components for $^4$He compared with the exact calculation using Faddeev-Yakubovsky (FY) equations in Ref. [10]. Since the comparison with the rigorous calculation was quantitatively very successful, it may be a good idea to introduce the spirit of the TOSM approximation in the few-body framework. This is a physical meaning of TOFM, and the physical concept of TOFM comes from the tensor-optimized shell model (TOSM). This idea corresponds to taking the $S$-wave configuration as the basis and adding a $D$-wave state in a relative coordinate, which is directly connected by the tensor interaction with the $S$-wave configuration. We will explain the framework of TOFM in more detail in a later subsection.

On the other hand, the binding energy of $^4$He calculated in TOSM was about 3 MeV smaller, and the tensor matrix element was about 20% smaller than the results of the few-body methods. Hence, it is very important to know from where the difference between TOSM and the rigorous calculations comes from in order to use the concept of TOSM for medium and heavy nuclei. Hence we have two purposes in this chapter in order to extend our framework to larger mass nuclei. One is to develop a tensor-optimized few-body method using the spirit of TOSM and study how good is the approximation of TOFM for the s-shell nuclei by comparing with few-body calculation. For s-shell nuclei, these nuclei have already been solved exactly by some few-body methods. The other is to compare with the results of $^4$He calculated in TOSM and discuss what should be done to improve the present TOSM description of nuclei, and propose the improvement strategy of TOSM.

### 2.2 Tensor optimized shell model

We first discuss TOSM, which was developed for the description of the tensor interaction in the shell model basis [13]. The TOSM wave function for a closed shell nucleus is written as

$$|\Psi\rangle = C_0|0\rangle + \sum_\alpha C_{\alpha}|2p - 2h : \alpha\rangle.$$  \hspace{1cm} (2.1)

Here, $|0\rangle$ denotes a $0p-0h$ spin saturated shell model state, while $2p-2h$ states $|2p - 2h : \alpha\rangle$ are those excited by operation of the tensor operator $S_{12}(\hat{x}) \equiv \{Y_2(\hat{x}) \times [\sigma_1 \times \sigma_2]\}_0$.
on two particles in the 0p-0h state. In the tensor operator, \( Y_2(\hat{x}) \) denotes the spherical harmonics with the angle of relative coordinate \( \hat{x} \), and \( \sigma \)'s are the spin operators of two particles. Hence, the two components are connected by the tensor operator as

\[
\langle 2p - 2h : \alpha | S_{12}(\hat{x}) | 0 \rangle \neq 0 ,
\]

where the quantum number \( \alpha \) denotes all the possible \( 2p-2h \) states connected by the tensor operator. The \( 2p-2h \) states are constructed to treat the tensor correlation by taking various Gaussian range wave functions, particularly those with small Gaussian ranges to optimize the role of the tensor interaction [16]. Hence, the essential approximation of TOSM is the inclusion of the variational states, which are excited by only one operation of the tensor interaction. As for the short-range correlation, they used the central correlation part of the UCOM [20] in the TOSM framework, because it is difficult to express the short-range behavior in the shell model basis [13].

2.3 Tensor optimized few-body model

2.3.1 The formulation of Tensor optimized few-body model

We are able to treat the short-range part by the variational method in TOFM. The few-body method uses relative coordinates and, hence, it is more efficient to treat the tensor and short-range correlations, since the tensor interaction and short-range repulsion are expressed using the relative coordinates. We only have to introduce all the states connected by one operation of the tensor interaction and take enough Gaussian functions to express both the tensor and short-range correlations. Let us start with introducing the following tensor-optimized wave function for a few-body system with \( A = 2 \), 3- and 4-body systems by taking the essential features of rigorous calculations in the few-body methods. Hence, we write a few-body wave function as a linear combination of \( S \)- and \( D \)-wave components,

\[
|\Psi\rangle = |\Psi_S\rangle + |\Psi_D\rangle .
\]

The condition of the \( D \)-wave component should be

\[
\langle \Psi_D | S_{12}(\hat{x}) | \Psi_S \rangle \neq 0 .
\]

On the basis of the TOFM approximation, we truncate the model space drastically as explained in Fig. 2.1 for \( ^4\text{He} \) as an example. As compared with the rigorous calculation, the TOFM wave function includes those states connected directly with the \( S \)-wave state by the tensor operator. This truncation makes the variational space much smaller than the rigorous method. In particular, the most essential \( D \)-wave state is the configuration containing a \( Y_2 \)-function for the Jacobi coordinate \( x_1 = r_1 - r_2 \), because it is most efficient to take the \( S \)- and \( D \)-wave couplings of tensor interaction in the coordinate \( x_1 \). In the TOFM wave function, the superpositions of the \( S \)-wave function and \( Y_2 \) function in the \( D \)-state provide the deuteron-like state as explained in the introduction.
Figure 2.1: The $S$-wave and $D$-wave components in TOFM

\[ |\Psi_S\rangle \quad |\Psi_D\rangle \]

Fig. 2.1: The $S$-wave $|\Psi_S\rangle$ and $D$-wave $|\Psi_D\rangle$ functions for the case of $^{4}\text{He}$ in TOFM are shown in the left-hand side. The full model space of few-body methods is shown in the right-hand side, where the total angular momentum $L$ is constructed by a vector summation of orbital angular momenta, $\ell_1$, $\ell_2$, and $\ell_3$, of each Jacobi coordinate. All the possible combinations for $L$ to make the total spin 0 are considered in the rigorous calculations. We need many configurations with various $\ell_1$, $\ell_2$, and $\ell_3$. On the other hand, the TOFM wave function takes only a single $Y_2$ function for each Jacobi coordinate in the description of the $D$-wave state.

The $S$-wave component for $s$-shell nuclei with $A \leq 4$ is written as

\[ |\Psi_S\rangle = \sum_{i=1}^{N_S} C_{ls}^S|\psi_0(A_S, x)\chi_i(\sigma)\chi_i(\tau)\rangle. \tag{2.5} \]

The spin wave function with total spin $s$ for the $A = 4$ system is written as

\[ \chi_i(\sigma) = [([\chi_1^{1/2}(\sigma_1) \times \chi_1^{1/2}(\sigma_2)]_{s12} \times \chi_1^{1/2}(\sigma_3)]_{s123} \times \chi_1^{1/2}(\sigma_4)]_s. \tag{2.6} \]

The iso-spin wave function is written similarly as

\[ \chi_i(\tau) = [([\chi_1^{1/2}(\tau_1) \times \chi_1^{1/2}(\tau_2)]_{n12} \times \chi_1^{1/2}(\tau_3)]_{n123} \times \chi_1^{1/2}(\tau_4)]_n. \tag{2.7} \]

On the other hand, the spatial wave function is written as [24]

\[ \psi_0(A_S, x) = \exp\left(-\frac{1}{2} \hat{x} A_S \hat{x}\right). \tag{2.8} \]

Here, $x$ represents the Jacobi coordinate vector $x = (x_1, x_2, \ldots)$ and $\hat{x} A_S \hat{x}$ means the short-hand notation of $\sum_{ij}^{N_p-1} A_S^{ij} x_i \cdot x_j$. $A_S$ is a $(N - 1) \times (N - 1)$ symmetric matrix and includes $N(N - 1)/2$ variational parameters. The Jacobi coordinate vectors $x_i$ with $i = 1, \ldots, N_p - 1$ are expressed in terms of the particle coordinates $r_j$ with $j = 1, \ldots, N_p$, where $N_p$ is the number of particles of a nucleus. This variational function is able to express the short-range correlation precisely [25].
As for the D-wave component, we have one $Y_{2M}$ function in the spatial wave function and perform all the necessary permutations so that all the particle pairs profit the use of the strong tensor interaction.

$$\psi_{2M}(\mathbf{A}_D, \mathbf{x}) = \exp \left( -\frac{1}{2} \mathbf{x} A_D \mathbf{x} \right) |\tilde{\mathbf{u}} \mathbf{x}|^2 Y_{2M}(\tilde{\mathbf{u}} \mathbf{x}) .$$  (2.9)

The global vector $\tilde{\mathbf{u}} \mathbf{x}$ is defined as the linear combination of the vector $\mathbf{x}$ as $\tilde{\mathbf{u}} \mathbf{x} = \sum_i u_i \mathbf{x}_i$, where $u_i$ is a variational parameter. Here we fix the parameter of the global vector like $\tilde{\mathbf{u}} \mathbf{x} = \mathbf{x}_1$ or $\tilde{\mathbf{u}} \mathbf{x} = \mathbf{x}_2$ and or $\tilde{\mathbf{u}} \mathbf{x} = \mathbf{x}_3$, because we want to know what configurations play significant roles to take into account the tensor correlations in the few-body calculation efficiently. The Gaussian wave function with the function $\tilde{x} A_D \mathbf{x} = \sum_{ij} A_{ij} x_i \cdot x_j$ is able to optimize the role of the tensor interaction with small-range Gaussian functions in the D-wave part as the case of TOSM. Since the spatial wave function now has a finite angular momentum, we have to couple the spatial wave function with the spin wave function to obtain a desired total spin of $J$.

$$|\Psi_D\rangle = \sum_{i_D}^{N_D} C_{i_D} D_{iD} \sigma |\psi_2(A_D, \{x_i\}) \chi_\sigma(|\{\sigma_i\}|) \chi_\tau(|\{\tau_i\}|) \rangle .$$  (2.10)

We take a sufficient amount of $N_D$ in order to include short-range as well as long-range Gaussian wave functions for the D-wave function to express fully the tensor correlation. In the case of $^4$He nucleus, these correlated Gaussian basis wave functions can include the H type configuration of the four particle coordinate due to the effects of the rearrangement and the antisymmetrization. Therefore we do not take the H type configuration explicitly in the TOF calculation.

We should know the role of the antisymmetrization operator $\mathcal{A}$, which is defined as $\mathcal{A} = \sum_{i=1}^{N} p_i \mathcal{P}_i$. It consists of the particle exchange operator with the phase $p_i$ due to even or odd permutation, where the operator $\mathcal{P}_i$ makes the transformation of the Jacobi coordinate $x_P = \mathcal{T}_ix$, and the exponent of the correlated Gaussian basis function and the generalized coordinate in the spherical harmonics take the form after the antisymmetrization of the particles as $A_P = \mathcal{T}_i A \mathcal{T}_i$ and $\nu_P = \mathcal{T}_i u$.

The spin and isospin wave functions are also subject to the antisymmetrization. Thus, the antisymmetrized wave function contains quite a general expression of wave functions and the manipulation of various matrix elements is highly involved. We introduce the representation of the global vector for the basis wave functions in the same way as the Niigata group [22].

### 2.3.2 The variational calculation with stochastic variational method

We have to obtain the wave function and energy in a few-body framework by minimizing the total energy $\delta \left( \langle \psi_H \mathcal{W} \rangle / \langle \psi \mathcal{W} \rangle \right) = 0$. The ranges of the spatial wave functions $A_S$ in Eq. (2.8) and $A_D$ in Eq. (2.9) are chosen irrationally, and the amplitudes $C_i^S$ and $C_i^D$ are determined by solving the eigen value problem for the Hamiltonian. We perform the variational calculation with stochastic variational method (SVM) [23–25]. In the stochastic variational method, the Gaussian ranges $A_S$ and $A_D$ are generated randomly,
and the most suitable wave functions are chosen successively. Thus, the most suitable parameters are selected as follows: First, some candidate parameters are generated in the description of one base. Next by solving eigen value problem, we can obtain the energy for each candidate parameters and select the parameters of the base which corresponds to the lowest energy. And then the number of base is increased with these processes until the energy will converge. Hence we can get the energy convergence and write the wave function dimension as small little as possible.

We may take also the intermediate spin and isospin quantum numbers in Eqs. (2.6), (2.7) and (2.10) with constraint such that the final spin and isospin are $s$ and $t$, respectively.

### 2.3.3 The many body Hamiltonian and the bare nucleon-nucleon interaction

The many body Hamiltonian of $N$ particle system is given as

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} - T_{cm} + \sum_{i<j}^{N} V_{ij} \ . \quad (2.11)$$

$T_{cm}$ is the center of mass kinetic energy. We take the bare nucleon-nucleon interaction for $V_{ij}$ such as Argonne V8 potential (AV8'). Now we would like to introduce a traditional model of $NN$ interaction formed by nonrelativistic framework of two-body potential taking examples as Argonne V18 potential (AV18) and simplified version AV8'. The Argonne potential model describes the pion exchange in a local approximation and expresses the short- and intermediate-range part with a phenomenological parameters. AV18 is constructed by a sum of an electro magnetic part ($v^E$), a one-pion exchange part at a long-range distance part ($v^\pi$), and a short- and intermediate phenomenological distance parts ($v^R$),

$$v_{ij} = v^E_{ij} + v^\pi_{ij} + v^R_{ij} \ . \quad (2.12)$$

Here we get through the details of charge-dependent structure and explain a general pion-exchange part in Argonne potential model. The pion exchange part is formed by the Yukawa and tensor functions with the exponential cut-off,

$$v^\pi_{ij} = f^2 \frac{1}{3} m_\pi c^2 [Y_\mu(r_{ij}) \tau_i \cdot \tau_j + T_\mu(r_{ij}) S_{ij}] \ , \quad (2.13)$$

using a coupling constant $f^2 = 0.075$ and an average of the pion mass $m_\pi = (m_{\pi^0} + 2m_{\pi^\pm})/3$. The Yukawa and tensor functions are,

$$Y_\mu(r) = \frac{e^{-\mu r}}{\mu r} \left(1 - e^{-\mu r}\right) \quad (2.14)$$

$$T_\mu(r) = \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right) \frac{e^{-\mu r}}{\mu r} \left(1 - e^{-\mu r}\right)^2 \ ,$$

with a cut-off $c = 2.1 fm^{-2}$ and $\mu = mc/\hbar$. 

13
The electromagnetic term is introduced for the charge-independence and charge-symmetry breaking in the strong force, and contains such as one and two-photon exchanges Coulomb interaction. The remaining phenomenological part is a sum of a short- and an intermediate-range potentials. The short-range part expresses the hard core with Wood-Saxon function, and the intermediate-range part is assumed to come from the two-pion exchange processes and is given by the square of tensor function. This phenomenological part is determined by the 40 parameters so as to fit various data of the NN system.

Then, the one-pion exchange and phenomenological parts are described as a sum of 18 operators,

\[ v_j^p + v_k^p = \sum_{p=1,18} v_p(r_{ij}) \hat{\Omega}_j^p, \]  

(2.15)

where a charge independent part has 14 operators component and three additional charge-dependent and one charge-asymmetric operators. The 14 operator consists of the central, tensor, \((L\cdot S), L^2\), \((L\cdot S)^2\) terms. On the other hand the AV8' has only eight operator terms, where the \(L^2\) and \((L\cdot S)^2\) terms in AV18 are projected onto the central, tensor, \((L\cdot S)\) terms to in such a way that the interaction is unchanged in the \(S-\) and \(P-\) wave channels. We shall see the details in the next chapter.

The first 14 operators consists of the central, tensor, \((L\cdot S), L^2\), and \((L\cdot S)^2\),

\[ \hat{\Omega}_{ij}^{p=1,14} = 1, \tau_i \cdot \tau_j, \sigma_i \cdot \sigma_j, (\tau_i \cdot \tau_j)(\sigma_i \cdot \sigma_j), S_{ij}, (\tau_i \cdot \tau_j)S_{ij}, \]

\[ (L\cdot S), (\sigma_i \cdot \sigma_j)(L\cdot S), L^2, (\tau_i \cdot \tau_j)L^2, (\sigma_i \cdot \sigma_j)L^2, \]

\[ (\tau_i \cdot \tau_j)(\sigma_i \cdot \sigma_j)L^2, (L\cdot S)^2, (\tau_i \cdot \tau_j)(L\cdot S)^2, \]

and the remaining 4 operators is three charge-dependent and one charge-asymmetric operator,

\[ \hat{\Omega}_{ij}^{p=15,18} = T_{ij}, (\sigma_i \cdot \sigma_j)T_{ij}, S_{ij}T_{ij}, (\tau_i \cdot \tau_j) \],

(2.16)

(2.17)

where \(T_{ij} = 3 \tau_i \tau_j - \tau_i \cdot \tau_j\). On the basis of the AV18 potential, the AV8' potential reduce the first 14 operators to the eight operators terms,

\[ v_1' = v_1 + \frac{5}{4}v_9 + \frac{3}{4}v_{10} + \frac{3}{4}v_{11} + \frac{9}{4}v_{12} + \frac{3}{4}v_{13} + \frac{3}{4}v_{14}, \]

\[ v_2' = v_2 + \frac{3}{4}v_9 + \frac{1}{4}v_{10} + \frac{1}{4}v_{11} + \frac{3}{4}v_{12} + \frac{1}{4}v_{13} + \frac{1}{4}v_{14}, \]

\[ v_3' = v_3 + \frac{1}{4}v_9 + \frac{3}{4}v_{10} + \frac{3}{4}v_{11} + \frac{1}{4}v_{12} + \frac{3}{4}v_{13} + \frac{1}{4}v_{14}, \]

\[ v_4' = v_4 + \frac{1}{4}v_9 - \frac{1}{4}v_{10} - \frac{1}{4}v_{11} + \frac{5}{12}v_{12} + \frac{1}{12}v_{13} + \frac{1}{12}v_{14}, \]

\[ v_5' = v_5 - \frac{5}{16}v_{13} - \frac{5}{16}v_{14}, \]

\[ v_6' = v_6 - \frac{5}{48}v_{13} - \frac{5}{48}v_{14}, \]

\[ v_7' = v_7 - \frac{1}{2}v_9 + \frac{3}{2}v_{10} - \frac{1}{2}v_{11} + \frac{3}{2}v_{12} - \frac{9}{8}v_{13} + \frac{15}{8}v_{14}, \]

\[ v_8' = v_8 + \frac{1}{2}v_9 - \frac{3}{2}v_{10} + \frac{1}{2}v_{11} + \frac{3}{2}v_{12} + \frac{9}{8}v_{13} - \frac{15}{8}v_{14}, \]
otherwise in a representation of the spin and isospin channel,

\[
\begin{align*}
V_{00}^c & = V_{00} + 2v_{00}^l, \\
v_{01}^c & = v_{01}, \\
v_{10}^c & = v_{10}, \\
v_{11}^c & = v_{11} + 2v_{11}^l + \frac{4}{3}v_{11}^s, \\
v_{10}^q & = v_{10}, \\
v_{11}^q & = v_{11} - \frac{5}{12}v_{11}^s, \\
v_{10}^{ls} & = v_{10}^{ls} - 2v_{10}^l - 3v_{10}^s, \\
v_{11}^{ls} & = v_{11}^{ls} - \frac{1}{2}v_{11}^s,
\end{align*}
\tag{2.19}
\]

and does not take into account the small contribution of the remaining 4 operators. Hence the one-pion exchange and the phenomenological parts can be written as the sum of the eight terms in AV8', which consist of the central force, the tensor force, and the spin-orbit force

\[
V_{ij} = v_{ij}^c + v_{ij}^q + v_{ij}^{ls}, \tag{2.20}
\]

with the first eight operators in Eq. (2.16). Since AV8' does not contain the \( L^2 \) and \( (L \cdot S)^2 \) operators, it is relatively easy to use this potential for a variational calculation. However we should take of care the use of AV8' in the calculation beyond s-shell nuclei because the transformation of higher momentum \( L^2 \) and \( (L \cdot S)^2 \) terms into the central, tensor, and \( LS \) terms is performed in the lower partial waves of \( 2N \) system.

We do not include the Coulomb potential and three nucleon force in order to compare the results with the exact few-body calculations and the TOSM calculation for \(^4\text{He}\) nucleus. We will discuss three nucleon force in another chapter.

### 2.3.4 Parallelization calculation

We comment on the parallelization of the numerical calculation. To work out for larger mass nuclei beyond the s-shell in TOFM, we need a larger scale calculation using super computers. It is very important to reduce the computational time. Hence we require the method of parallelization of the calculation and apply it to the variational calculation of TOFM. Since in the parallel calculations we can calculate different things at a time with multiple computers, it is suitable for the situation where there are many long iterative calculations and the results do not change by the order of the calculation.

Thus, in TOFM, we apply the parallelization to the process of the selection of the optimal basis parameters like \( A_S \) and \( A_D \) in the variational calculation with SVM. In the SVM calculation, we have to search for the optimal base which gives the minimum energy in hundreds of trial and error calculations by preparing various base functions. We must calculate the matrix elements of all Hamiltonian components in hundreds times, which takes long computational time. Therefore, the present problem in SVM and TOFM is suitable to use the parallel calculation and share the time-consuming process with many parallel computers.
2.4 Numerical results with TOFM for s-shell nuclei

2.4.1 The total energy and various energy components for $^2$H and $^3$H, and $^4$He nucleus

We show the numerical results with TOFM and compare them with the rigorous few-body calculations for $A = 2$, 3- and 4-body systems. We also make a comparison with the TOSM calculation of Myo et al. for $^4$He [13], where they used UCOM in order to treat the strong short-range repulsion. We perform numerical calculations with TOFM using the AV8' potential for nuclei with $A = 2, 3$ and $4$. The results for the deuteron are shown in Table 1, where the importance of the tensor interaction with a large attraction of $-16.71$ MeV is clearly seen as we mentioned in the introduction. It is also important to point out that the kinetic energy of $19.97$ MeV is quite large, which is caused by the tensor interaction.

Figure 2.2: Energy convergence of triton in TOFM

![Figure 2.2: Energy convergence of triton in TOFM](image)

Fig. 2.2: Various energy components in the triton in TOFM with AV8' as functions of the number $N = N_S + N_D$ of the basis wave functions. Those energy components are the kinetic energy denoted by crosses, the spin-orbit energy denoted by pluses, the central interaction energy denoted by plus-crosses, the tensor interaction energy denoted by open squares and the total energy denoted by closed squares. The corresponding energy components obtained by rigorous calculations are shown by solid bars in the right-hand side.
As for \( A = 3 \) and 4, we include a \( Y_2 \) function for each Jacobi coordinate in the \( D \)-wave variational space, while the antisymmetrization \( \mathcal{A} \) takes care of all the necessary permutations. We obtain variationally suitable Gaussian ranges \( A_S \) and \( A_D \) and their amplitudes \( C_S \) in Eq. (2.5) and \( C_D \) in Eq. (2.10) by solving eigen value problem.

**Figure 2.3:** Energy convergence of \(^4\text{He}\) in TOFM

![Energy convergence of \(^4\text{He}\) in TOFM](image)

**Fig. 2.3:** Various energy components in \(^4\text{He}\) in TOFM with AV8' as functions of the number of basis wave functions. The notations are the same as those in Fig. 2.2.

In Fig. 2.2, we show the energy convergence for \(^3\text{H}\) as functions of the number \( N \) of the basis wave functions. The energy components and total energy are shown up to \( N = N_S + N_D = 150 \). We see that a good convergence is achieved at around \( N = 50 \). All the energies are compared with the SVM calculation shown in the right-hand side of the figure. We show in Fig. 2.3 the case of \(^4\text{He}\), which is obtained by using the \( l=2 \) component with \( \tilde{a}x = x_1 \) only. In this case, the convergence is slower and we ought to go up to about \( N = 300 \). The results of the rigorous few-body calculations for \(^4\text{He}\) are shown in the right-hand side of this figure. Although the convergence is achieved for the central and spin-orbit energies, the tensor interaction energy and also the kinetic energy are not yet converged.

We show in Table I the total energy and various energy components as the kinetic energy, the central interaction energy, the tensor interaction energy and the spin-orbit energy.
interaction energy for the deuteron, $^3$H and $^4$He. We do not include the Coulomb energy in this comparison. Although very small, we add other configurations of $D$-wave state including a $Y_2$ function for $\hat{x}_2 = x_2$ or $\hat{x}_3$ Jacobi coordinates in the calculation for this table. The present result is compared with the rigorous calculation. The results of various other few-body methods are essentially the same as the SVM result, and the Faddeev Yakubovsky (FY) results as shown explicitly in this table [10]. As for $^4$He, we also compare the results with TOFM.

We compare the TOFM result with the rigorous calculation for $^3$H and $^4$He. The TOFM results are compared almost perfectly with the SVM results for $^3$H. The good reproduction of the calculated result reminds us of the importance of the deuteron-like tensor correlation. The total energy is $-7.54$ MeV and compares almost perfectly with the rigorous result of $-7.76$ MeV. The tensor energy is $-30.47$ MeV as compared with the rigorous calculation of $-30.84$ MeV. The large kinetic energy of $46.67$ MeV is related with the strong tensor correlation. It is also interesting to point out that the central interaction provides a large fraction of attraction of $-21.98$ MeV and compares nicely again with the rigorous value of $-22.49$ MeV. This comparison indicates that the short-range correlation is also nicely described in TOFM. The TOFM is able to reproduce the rigorous results for the $A = 3$ system.

As for $^4$He, we find that the total energy is $-24.18$ MeV, which is close to the total energy of $-25.92$ MeV of SVM and $-25.94$ MeV of FY. The central interaction energy of $-54.67$ MeV is quite close to those values ($-55.23$ and $-55.26$) of SVM and FY. On the other hand, the tensor interaction energy of $-61.32$ MeV is to be compared with $-68.32$ MeV of SVM and $-68.35$ MeV of FY, which is about $10\%$ smaller than the rigorous results. Correspondingly, the kinetic energy of $95.53$ MeV is about $5\%$ smaller and the spin-orbit energy of $-4.09$ MeV is about $15\%$ smaller than the rigorous results. Although the tensor interaction energy and the kinetic energy are slightly underestimated, these two components largely cancel each other out to provide the small difference of $1.8$ MeV for the total energy.

From these results, we conclude that TOFM is good enough to use as an approximation in the variational calculation for the total energy. On the other hand, this model can reproduce the rigorous results with more than $90\%$ accuracy for the tensor correlation, which are indicated by the kinetic energy and spin-orbit energy. We expect that the
Table 2.3: Various energy components in unit of MeV with the AV8’ interaction. In the raw of double $Y_2$, we use the $Y_2$-functions into wave functions.

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Kinetic</th>
<th>Central</th>
<th>Tensor</th>
<th>LS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4\text{He}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOFM</td>
<td>-24.18</td>
<td>95.90</td>
<td>-54.67</td>
<td>-61.32</td>
<td>-4.09</td>
</tr>
<tr>
<td>double $Y_2$ [26]</td>
<td>-25.71</td>
<td>100.89</td>
<td>-54.73</td>
<td>-67.21</td>
<td>-4.66</td>
</tr>
<tr>
<td>SVM [10]</td>
<td>-25.92</td>
<td>102.35</td>
<td>-55.23</td>
<td>-68.32</td>
<td>-4.71</td>
</tr>
</tbody>
</table>

TOFM framework is able to provide good results for the total energy with a care that the tensor correlation is about 10% underestimated for a few-body analysis beyond s-shell nuclei.

We should now try to find the missing tensor strength of the TOFM approximation as compared with the rigorous result and understand the physical meaning of the residual 10% for the tensor energy component. We therefore searched for configurations where the missing strengths come from using the Gaussian expansion method of Hiyama et al. [26].

We include double $Y_2$ functions explicitly in all the Jacobi coordinates as $(l_1, l_2, l_3) = (2, 2, 0), (2, 0, 2), (0, 2, 2)$ of Fig. 1. Of course we also include the H type configurations with double $Y_2$ functions in the model space. We found that the inclusion of the double $Y_2$ functions in the coordinates brings most of the required energy. We show the results of the inclusion of double $Y_2$ functions in the variation calculation in Table 2.3. The total energy is now $-25.71$ MeV, the kinetic energy is $100.89$ MeV and the tensor interaction energy is $-67.21$ MeV. The spin-orbit energy is $-4.66$ MeV. These values are essentially the results of the rigorous calculations. We found that the H type coordinate contribute the most efficient configurations for the residual energy, which means two deuteron like states generate the large tensor interaction.

The $D$-state in TOFM is directly connected with the $S$-state by the tensor interaction, and the tensor energy comes from the transition between the $S$- and $D$-states. On the other hand, the double $Y_2$-state should be based on the $D$-state and the tensor energy comes from the transition between the $D$ state and the double $Y_2$-state. Since the $D$-state probability is about 10% in $^4\text{He}$, this additional contribution should be on the order of 10%. If we want to improve the TOFM description, we are able to obtain essentially the full tensor strength by having double $Y_2$ functions in the variational wave function. Of course, the full values are obtained by adding all the finite angular momentum components in the variational wave function [10,22,26].

As for the comparison with the TOSM calculation for $^4\text{He}$, the present calculation of the energy value is better than the TOSM result [13]. Large differences are found in the matrix elements of the kinetic energy and the tensor interaction. Since both the TOFM and TOSM wave functions take the state connected by one tensor operation, the difference in these results should come from the methods of treating the short-range correlation under the influence of the tensor correlation. This should mean that the TOSM calculation can be improved by taking a more general UCOM correlation function. The improvement of the short-range of $S$-wave in TOSM would refine the tensor interaction,
Table 2.4: Various energy components in unit of MeV with the G3RS interaction.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Energy</th>
<th>Kinetic</th>
<th>Central</th>
<th>Tensor</th>
<th>LS</th>
</tr>
</thead>
<tbody>
<tr>
<td>deuteron</td>
<td>-2.28</td>
<td>16.48</td>
<td>-7.29</td>
<td>-11.46</td>
<td>-0.00</td>
</tr>
<tr>
<td>$^3\text{H}$(TOFM)</td>
<td>-7.61</td>
<td>39.82</td>
<td>-26.70</td>
<td>-20.69</td>
<td>-0.04</td>
</tr>
<tr>
<td>SVM [22]</td>
<td>-7.73</td>
<td>40.24</td>
<td>-26.80</td>
<td>-21.13</td>
<td>-0.03</td>
</tr>
<tr>
<td>$^4\text{He}$(TOFM)</td>
<td>-25.22</td>
<td>84.83</td>
<td>-66.21</td>
<td>-43.66</td>
<td>-0.17</td>
</tr>
<tr>
<td>SVM [22]</td>
<td>-26.05</td>
<td>86.93</td>
<td>-66.24</td>
<td>-46.62</td>
<td>-0.13</td>
</tr>
</tbody>
</table>

Table 2.5: Various energy components for the second $0^+$ state of $^4\text{He}$ and $^3\text{H}$.

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Kinetic</th>
<th>Central</th>
<th>Tensor</th>
<th>LS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4\text{He} \ 0^+_2$</td>
<td>-6.27</td>
<td>45.09</td>
<td>-21.24</td>
<td>-28.32</td>
<td>-1.79</td>
</tr>
<tr>
<td>$^3\text{H}$ g.s.</td>
<td>-7.54</td>
<td>46.67</td>
<td>-21.98</td>
<td>-30.47</td>
<td>-1.95</td>
</tr>
</tbody>
</table>

because a large tensor interaction comes from the coupling $S$-wave and $D$-wave. Some defect in the short-range correlation may be seen also in the LS component, which is underestimated in TOSM. The tensor interaction induces finite angular momentum states and enhances the LS energy [18]. Hence, the improvement of the tensor component enhances not only the tensor interaction energy but also the LS energy.

To see the interaction dependence, we show the case of the Gaussian soft core potential with three-rages (G3RS) in Table 2.2 [27]. G3RS is the bare $NN$ interaction model. Here, we drop the contributions of small $L^2$ and $W_{12}$ terms to compare with the rigorous calculation. In G3RS potential, the tensor interaction is weaker than the case of the AV8' potential. Hence, the tensor energy of the TOFM calculation is smaller than that of the central energy in the same way of the rigorous calculation, and now the agreement of TOFM with the full calculation is impressive.

We have calculated the second $0^+$ state in $^4\text{He}$. The excitation energy comes out to be $E_x = 18.6$ MeV with AV8' potential, which is to be compared with the experimental data of 20.2 MeV. We show the energy and the various energy components of the second $0^+$ state and $^3\text{H}$ in Table 2.5. The calculated energy of the the second $0^+$ state $-6.27$ MeV and the triton ground state $-7.54$ MeV are very similar, and all matrix elements of the second $0^+$ state are very close to that of $^3\text{H}$. In addition the second $0^+$ state is located near the $^3\text{H} + n$ threshold ($E_x = 20.1\text{MeV}$) and the $^3\text{He} + p$ threshold ($E_x = 19.8\text{MeV}$). Hiyama et al suggests the $0^+$ state has the loose $3N+N$ structure [28] and our calculation of energy and matrix elements also indicate the $3N+N$ structure of the second $0^+$. In fact in the description of the $0^+$ state, we need a more spatial basis than that of the ground state, because of the inclusion of the $3N+N$ structure in the TOFM wave function. From these results our TOFM calculation is enough to describe not only the ground state but also the excited state for $^4\text{He}$ nucleus and confirm the loose $3N+N$ structure for the second $0^+$ state.
2.4.2 Correlation function

It is very important to study correlation functions of the $s$-shell nuclei in order to see the behavior of wave functions at a short distance. Moreover it is also important to compare with the correlation function obtained by UCOM, the function form of which is extracted from the short-range behavior of wave functions obtained with an effective interaction as the Malfliet-Tjon potential (MT-V) [20]. The MT-V interaction has a short-range repulsion without the tensor interaction [29]. We would like to calculate the correlation functions in order to see the behavior of the wave functions at short distances according to the description given in Ref. [39]. We define the correlation function as

$$C_{S,D}(r) = \frac{1}{4\pi} \int d\hat{r} \langle \Psi_{S,D} | \delta(r_1 - r_2 - r) | \Psi_{S,D} \rangle$$

(2.21)

In the first line, the matrix element is obtained first by integrating out over all the spatial and spin-isospin coordinates of all the particles. We then take the average over the angle $\hat{r}$ of the relative distance vector $r$.

This correlation function has the normalization of

$$4\pi \int_0^\infty drr^2 (C_S(r) + C_D(r)) = 1 .$$

(2.22)

Figure 2.4: Correlation function for $S$-wave and $D$-wave 1

![Correlation function graph](image)

Fig. 2.4 : Correlation function $C_S(r)$ of the $S$-wave component and $C_D(r)$ of the $D$-wave component for deuteron (dash-dotted curve), triton (dash curve) and $^4$He (solid curve) as functions of the relative distance $r$.  

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Fig. 2.5: Correlation functions $C_S$ and $C_D$ for deuteron and triton are normalized to those of $^4\text{He}$ as functions of the relative distance $r$ by fitting a peak of $^4\text{He}$. The normalization factors are 1.31 for deuteron and 1.21 for triton in the case of $S$-wave correlation function and 1.32 for deuteron and 1.57 for triton in the case of $D$-wave correlation function. The notations are the same as those in Fig. 2.4.

The total correlation functions for s-shell nuclei have already been shown in Ref. [10, 31]. We pay attention to the roles of the individual $S$- and $D$-wave components in the correlation function and show the correlation functions for the $S$-wave $C_S(r)$ and those for the $D$-wave $C_D(r)$. This separation of the correlation function was not performed in previous studies [31, 39].

We would like to show the $S$- and $D$-wave correlation functions $C_S(r)$ and $C_D(r)$ as functions of the relative distance $r$ of nucleons for $A = 2, 3$ and 4 nuclei in Fig. 2.4. As for the $S$-wave, we see a dip structure below 1 fm reflecting the presence of the strong short-range repulsion. The magnitude of the peak is the biggest for $^4\text{He}$ and smallest for the deuteron, which reflects the size of these nuclei. The $D$-wave components are found to be significant and similar among the three nuclei.

Since all the correlation functions among the three nuclei look similar, we normalize the correlation functions to those of $^4\text{He}$ and show them in Fig. 2.5. We see essentially the same short-range behaviors below 1 fm for these three nuclei for both the $S$-wave and $D$-wave components. Some difference among the three correlation functions appears in the behaviors at large distance ($r > 1$ fm), which reflects the size of these nuclei. The radii of the three nuclei are $\sqrt{\langle r^2 \rangle} = 1.95$ fm for the deuteron, 1.78 fm for the triton and 1.52 fm for $^4\text{He}$. This is an important finding for the study of heavier nuclei. We have to obtain correlation functions as those found here even for heavy nuclei to optimize the role of the tensor interaction and the short-range repulsion.

We comment on the difference in the correlation functions between the TOFM and
Figure 2.6: Correlation function for S-wave with AV8' and MT-V potential.

Fig. 2.6: Correlation functions $C_5(r)$ for the S-wave component calculated with the AV8' (solid curve) and MT-V potentials (dashed curve) for $^4$He as functions of the relative distance $r$. To compare the two correlation functions, we multiply 0.77 with $C_5(r)$ of MT-V so as to match its peak value with that of $C_5(r)$ of AV8'.

the exact calculations. There is the correlation function derived from the SVM wave functions shown in Fig. 2 of Ref. [22]. By using the figure directly, we can compare the result of TOFM with SVM. We see good agreement for the universality of the short-range behavior below 1 fm, which is caused by the strong short-range repulsion. We found that the tail part and amplitude of the peak of our results are changed from the SVM calculation, because our calculation of the root mean square radius is slightly larger than that of the rigorous SVM calculation. This is because the TOFM calculation give the smaller binding energy than the rigorous calculation.

We would like to discuss here the improvement of the TOSM results where the short-range correlation function is treated by UCOM. The function form of the central unitary correlation function was fixed by that obtained from the correlation function of the deuteron obtained with the MT-V interaction, and its validity in UCOM was checked with several nuclei [20].

Hence, we calculate $C_5(r)$ in the few-body framework with the MT-V interaction [29] for $^4$He and compare the result with $C_5(r)$ of the AV8' interaction in Fig. 2.6. We see quite a large difference between the two correlation functions particularly at a short distance ($r < 1$ fm). The short-range behaviors of the correlation function depend largely on the properties of interactions. We comment here that the correlation function calculated in TOSM using UCOM for $^4$He agrees almost perfectly with $C_5(r)$ of MT-V shown here [13]. It would be very interesting therefore to modify the correlation function in UCOM in the study of heavy nuclei to improve the TOSM results.
2.5 Summary of this chapter

We have formulated a tensor-optimized few-body model (TOFM) in the spirit of TOSM. TOSM can treat the tensor correlation efficiently and can apply to the study of the nuclear structure from the bare $NN$ interaction for medium and heavy nuclei. Both frameworks include the state connected by the one tensor operation. The TOFM wave function contains the $S$- and $D$-wave components. The $D$-wave component is connected directly by the tensor interaction with the $S$-wave component. Hence, the TOFM approximation makes not only the variational space much smaller than that of the rigorous calculation, but also the extension is less elaborate for a larger mass system.

We have calculated $A = 2$, 3- and 4-body systems in TOFM and compared with the rigorous calculations. As for $A = 3$, the inclusion of the $D$-wave component in the $x_1$ coordinate provides essentially the same results as the exact calculation. This result indicates that the $D$-wave component is enough to take care of the tensor interaction in the 3-body system. As for $A = 4$, we again obtain good reproduction of the rigorous results for various energy components and conclude that the TOFM wave function can treat the property of the $NN$ interaction, particularly the short-range correlation and the tensor interaction, and can be used for the discussion of the few-body analysis for beyond s-shell nuclei. On the other hand, TOFM slightly underestimates the tensor correlation, and we found that the difference between TOFM and the rigorous calculation comes essentially from additional $Y_2$ components in the other $p$-$n$ pairs. The inclusion of double $Y_2$ functions in the variational space using the Gaussian expansion method (GEM) brings the total energy very close to the rigorous results with AV8' for $^4$He [26].

We also compare the TOFM calculation with the TOSM one in $^4$He to investigate what should be done to improve the TOSM framework. We note the defect of TOSM is the treatment of the short-range correlation in UCOM, and the modification of the UCOM function form would improve the present TOSM result.

We have calculated correlation functions $C_S(r)$ and $C_D(r)$ for $A = 2$, 3- and 4-body systems. The short-range behaviors of the correlation functions for both the $S$- and $D$-wave components are very similar among the three nuclei. These similarities indicate that the short-range behaviors in the $S$- and $D$-wave components are essentially the same and should be reproduced even in calculations of heavy nuclei. We have compared the $S$-wave correlation function $C_S(r)$ of the MT-V interaction containing no tensor interaction and the corresponding one of the AV8' interaction containing the tensor interaction in $^4$He. We see quite a difference between these $S$-wave correlation functions. This suggests that the short-range correlation function used in UCOM should be studied further for the calculation in finite nuclei in the TOSM framework.

These results indicate that nuclei have large amounts of deuteron-like configurations and it is important to take the deuteron-like tensor correlations in the nuclear wave functions. The present study is very encouraging to extend our study to nuclei beyond s-shell in TOFM.
Chapter 3

Tensor optimized few-body model for p-shell nuclei

3.1 Tensor optimized few-body model for $^5$He nucleus

3.1.1 Unbound states of $^5$He

In the previous chapter we propose the tensor optimized few-body model and study the s-shell nuclei. The results are very encouraging to extend our study of TOFM to nuclei beyond s-shell. We would like to apply the TOFM framework to the p-shell nuclei. Hence we challenge to investigate the structure of $^5$He as the first p-shell nucleus.

It is well-known that there is no bound state with the mass number 5 and 8, which are also the most important nuclei for the understanding of nucleosynthesis processes to the construction of nuclei beyond the s-shell. Then we have to understand the structures of five-nucleon system as the first stage of the p-shell nuclei. It is also considered to be essential that we should understand the unbound system of $^5$He with the object of the relations between the $NN$ interaction and the development of nuclear structure from the s-shell closed nuclei $^4$He.

We show the energy diagram of mass number $A=5$ nuclei $^5$He and $^5$Li in Fig. 3.1. Since $^5$He and $^5$Li are the mirror nuclei, both structures are very similar and we focus on $^5$He in this study. $^5$He has two low-lying p-wave resonances with $J^P = 3/2^-$ and $J^P = 1/2^-$. The ground state of $3/2^-$ is a very narrow resonance with the energy peak at about $E_x = 0.8$ MeV and the width of about 0.6 MeV, while the excited state of $1/2^-$ is a broad resonance with energy peak at about $E_x = 2.1$ MeV and the width of about 5.5 MeV [33-35]. These differences of structures in two resonances between $3/2^-$ and $1/2^-$ are confirmed in the analysis of the phase shifts of the $n-^4$He scattering [34-36]. As shown in Fig 3.2, the $3/2^-$ state gets a larger attraction than the $1/2^-$ state. Hence it is interesting to shed light on the double splitting structure arisen from the $NN$ interaction in two states, and to find how the $^5$He structure changes from $^4$He by adding one outer nucleon.
Figure 3.1: Energy level diagrams of $^5\text{He}$ and $^5\text{Li}$.

Figure 3.2: $n^4\text{He}$ elastic scattering phase shift in Ref [36].
Figure 3.3: Calculation of the phase shifts for $n - ^4\text{He}$ scattering in GFMC [37].

3.1.2 Previous studies and purpose of this chapter

Firstly we would like to overview some previous studies on $^5\text{He}$. Only the Argonne group succeed the reproduction of the $n - ^4\text{He}$ phase shifts using the $NN$ interaction as the ab initio calculation, where they added a three-body interaction to the bare $NN$ interaction of AV18 [37,38]. Fig. 3.3 represents the theoretical calculation worked by the Argonne group. A light blue dotted line with diamonds shows the result of the phase-shift without a three-body force, while the red dotted line with points and blue dotted with squares are the results including a three-body force. The difference between red and blue curves is obtained by using the different three-body forces of Illinois or Urbana type. The black line is the R-matrix calculation corresponding the experimental data. Their study indicate necessity of three-body force to reproduce the phase shift for $3/2^-$ and $1/2^-$, and makes the double splitting well. Other ab initio calculation of $^5\text{He}$ without a three-body force also discusses the shortage of the attraction for $3/2^-$ [39]. However, it is not clear why the three-body interaction acts larger in the $3/2^-$ state than the $1/2^-$ state. Hence there are still problems remaining, and it is important to study the doublet splitting structure of $^5\text{He}$. In addition, it can be seen the doublet splitting using only the two-body interaction in Fig. 3.3. We should investigate how the differences between $3/2^-$ and $1/2^-$ are made by the coupling of $^4\text{He}$ with a last neutron. These points were not clarified in Ref. [37].

Therefore our purpose of this study is to perform the five-body calculation with the
NN interaction and to discuss the difference of structures between $3/2^-$ and $1/2^-$ states. Thus we have to develop our TOFM framework for p-shell nuclei. Since we also focus on the tensor interaction, we would like to discuss the role of the tensor interaction in $^5\text{He}$.

The role of tensor interaction in $^5\text{He}$ have been discussed at an early stage in the study of the nuclear physics [40–43]. Terasawa and Arima performed the pioneering shell model calculation with configuration mixing, where they treated the tensor force perturbatively. They suggested that the internal excitations of $^4\text{He}$ induce a Pauli blocking effect in $^5\text{He}$. Besides, the studies of p-shell nuclei with open TOSM have also shown that a pn pair correlated by the tensor interaction in $^4\text{He}$ are blocked selectively by a last neutron with the prp occupation, which causes a Pauli blocking effect [16,18]. Therefore it is important to treat the tensor interactions acting on a pn pair in $^4\text{He}$ and a pn pair between a last neutron and a proton in $^4\text{He}$. We investigate the behavior of coupling between $^4\text{He}$ and a last neutron, and also the difference of two resonances of $3/2^-$ and $1/2^-$ states quantitatively using a few-body method.

### 3.2 Formulation of TOFM for $^5\text{He}$

#### 3.2.1 TOFM for nuclei beyond s-shell nuclei

We would like to explain how to describe the p-shell nuclei schematically by taking $^5\text{He}$ as an example in the TOFM approximation. In order to describe the negative parity state, we add a p-wave function and take a $Y_1$ function in the last Jacobi coordinate in the TOFM basis state. Therefore, the TOFM wave function for $^5\text{He}$ is written as

$$|\Psi\rangle = |\Psi_{S+P}\rangle + |\Psi_{D+P}\rangle,$$

(3.1)

where the $|\Psi_{S+P}\rangle$ state consists of four s-wave functions and one p-wave function for $^5\text{He}$. For heavier p-shell nuclei, we add more p-states as $(A-4)$ P-wave functions. On the other hand, the $|\Psi_{D+P}\rangle$ state is defined as the 'D'-wave state, which is directly connected with the $S+P$- state by the tensor interaction. In the above wave function, we have to take the full antisymmetrization for all the 5 particles, which is a tedious work. In the same manner as the s-shell framework, the TOFM wave function of Eq. (3.1) corresponds to the open TOSM wave function,

$$|\Psi\rangle = \sum_{k_0} A_{00} |0p0h;k_0\rangle + \sum_{k_1} A_{11} |1p1h;k_1\rangle$$

$$+ \sum_{k_2} A_{22} |2p2h;k_2\rangle,$$

(3.2)

where the states $|0p0h;k_0\rangle$ are the $0p0h$ shell-model state. The configuration $|1p1h;k_1\rangle$ and $|2p2h;k_2\rangle$ are the $1p1h$ and $2p2h$ state with various radial components for particle state and involve high momentum components.

#### 3.2.2 Formulation of the TOFM wave function

In this section we would like to consider a formulation of the TOFM wave function for $^5\text{He}$ nucleus in more detail. As shown in Eq. (3.1), the TOFM wave function is
Figure 3.4: The $S + P$-wave and $D + P$-wave functions for the TOFM framework

\[ |\Psi_{S+P}\rangle \quad |\Psi_{D+P}\rangle \]

Fig. 3.2.2: The $S + P$-wave $|\Psi_{S+P}\rangle$ and $D$-wave $|\Psi_D\rangle$ functions for the case of $^5$He in TOFM is shown. In the $S + P$-wave the TOFM wave function takes a $Y_1$ function for the last coordinate, while the $D + P$-wave function has a $Y_2$ function for each Jacobi coordinate and a $Y_1$ function for the last coordinate.

constructed by the linear combination of $S + P$-state and $D + P$-state. Following the condition in Eq. (2.4), the components of the $S + P$-waves and the $D + P$-waves can be connected by the tensor operator of $S_{12}$,

\[ \langle \Psi_{S+P} | S_{12} | \Psi_{D+P} \rangle \neq 0. \] (3.3)

Since the $S + P$-wave function does not include $Y_2$ functions as shown in Fig. 3.2.2 (a), the tensor operator does not act in the matrix elements between the $S + P$-wave functions.

\[ \langle \Psi_{S+P} | S_{12} | \Psi_{S+P} \rangle = 0. \] (3.4)

Hence the $S + P$-wave function takes spin channels which does not have a finite matrix element for a tensor operator $S_{12}$. We would like to define the $S + P$-wave component in this manner.

Fig. 3.2.2 shows the TOFM model for $^5$He with the Jacobi coordinate of five particles system. A new additional Jacobi coordinate ($x_4$) corresponds to the coordinate between the center-of-mass for $^4$He and the fifth particle coordinate. We take a $p$-wave function ($Y_1$-function) in this coordinate. Regarding the $D + P$-wave state, we take $Y_1$-function in the same coordinate of $x_4$ and also add a $Y_2$ function in some Jacobi coordinate like $^4$He case. The rightest figure (e) means that the $Y_1$ function couples with $Y_2$ function in the $x_4$ coordinate to make the total angular momentum $L$. Furthermore we know from the results of $^4$He that the most efficient $Y_2$ configuration is operated on the $x_1$ coordinate for the $D + P$-wave to couple with the $S + P$-wave by the tensor operator. Hence these superpositions between the $S + P$-wave and the $D + P$-wave form the deuteron like state and generate large tensor correlations in nuclei.

Next we discuss the detail of the $S + P$-wave and $D + P$-wave components. It is necessary for both wave functions to take into account antisymmetrizations of the all particles including spatial and spin-isospin parts.
We denote the $S+P$-wave as,

$$|\Psi_{S+P}^{N_{S+P}}\rangle = \sum_{i=1}^{N_{S+P}} C_i^{S+P} \mathcal{A} [\psi_1(A_{S+P}, x) \chi_i(\sigma), \chi_i(\tau)].$$  \hspace{1cm} (3.5)$$

The isospin function is,

$$\chi_i(\tau) = [[[\chi_2^{P}(\tau_1) \times \chi_2^{P}(\tau_2)], n_{12} \times \chi_2^{P}(\tau_3)] n_{123} \chi_2^{P}(\tau_4)] n_{1234} \chi_2^{P}(\tau_5)].$$  \hspace{1cm} (3.6)$$

Here the total isospin is $t = \frac{1}{2}$. And also the spin function is,

$$\chi_s(\sigma) = [[[\chi_2^{S}(\sigma_1) \times \chi_2^{S}(\sigma_2)], s_{12} \times \chi_2^{S}(\sigma_3)] s_{123} \chi_2^{S}(\sigma_4)] s_{1234} \chi_2^{S}(\sigma_5)].$$  \hspace{1cm} (3.7)$$

We should take the total spin $s$ to make the total spin $J$ for $^5\text{He}$ by coupling with the angular momentum $L$. Since the $S+P$-wave state has the condition in Eq. (3.4), it qualify the the total spin $s$ as $\frac{1}{2}$.

Moreover the spatial part of the wave function is described by using the correlated Gaussian base,

$$\psi_1(A_{S+P}, x) = \exp \left( -\frac{1}{2} \tilde{X}_{S+P} x \right) |\tilde{u}x| Y_{1M}(\tilde{u}x).$$  \hspace{1cm} (3.8)$$

The definitions of the correlated Gaussian function $\tilde{X}_{S+P} x$ and the global vector $\tilde{u}x$ are same as we mentioned in the s-shell study. Since the $S+P$-wave state takes a $Y_1$ function for the last coordinate of $x_4$, we fix the global vector as $\tilde{u}x = x_4$.

On the other hand, the $D+P$-wave function is written as the same manner of the $S+P$-wave case,

$$|\Psi_{D+P}^{N_{D+P}}\rangle = \sum_{i=1}^{N_{D+P}} C_i^{D+P} \mathcal{A} [\psi_L(A_{D+P}, x) \chi_s(\sigma), \chi_i(\tau)].$$  \hspace{1cm} (3.9)$$

Here this wave function has total angular momentum $L$ and spin $s$ to make total spin of $^5\text{He}$ for $J = 3/2$ or $1/2$, and isospin $t$ is $1/2$. We require the enormous model space for the $D+P$-wave function. Hence it is important that we choose efficient configurations to obtain large tensor correlations involving high momentum components. Then we prepare two types of $D+P$-wave states. One type is constructed by choosing the configuration with the spin and angular momenta like the $D$-wave function of $^4\text{He},$

$$|\psi_L(A_{D+P}, x) \chi_s(\sigma)\rangle_J = \left[|\psi_2 \chi_{s4}(\sigma)\rangle_0 \times |\psi_1 \chi_2^{P}(\sigma_5)\rangle_J\right],$$  \hspace{1cm} (3.10)$$

In the first line, four-body system corresponding to $^4\text{He}$, takes the angular momentum of 2 and the spin $s_4$ for the same as a $^4\text{He}$ spin of 0, and the last nucleon has a $Y_1$ function and a spin to make the total spin of $J$. In the second line, we work out the recombination between angular momenta and spin functions. Here we take a spin wave function $\chi_{s4}(\sigma)$ as the spin function with $s_4 = 2$ in Eq. (2.6). Thus we determine the
intermediate configurations of the angular momentum and the spin like $^4$He. $(9j)_{L,S}$ is a nine-j coefficient and defined as

$$(9j)_{L,S} = \begin{pmatrix} 2 & s_4 & 0 \\ 1 & \frac{1}{2} & J \\ L & s & J \end{pmatrix} = \hat{\mathbf{J}} \hat{\mathbf{L}} \begin{pmatrix} 2 & s_4 & 0 \\ 1 & \frac{1}{2} & J \\ L & s & J \end{pmatrix}. \quad (3.11)$$

The other type allows us to take the remaining configurations, and the wave function is written as

$$[\psi_L(A_{D+P}, x)] \chi_s(\sigma) = [\psi_2 \psi_1]_L(A_{D+P}, x) \times \chi_s(\sigma) J. \quad (3.12)$$

Here the total angular momentum $L$ and the spin $s$ can be chosen to construct the total spin of $J$.

The spatial wave functions in Eq. (3.10) and Eq. (3.12) have the two angular components of $Y_2$- and $Y_1$-functions, and so we express those by using the double global vector description in Ref [22],

$$[\psi_2 \psi_1]_L(A_{D+P}, x) = \exp \left( -\frac{1}{2} \mathbf{A}_{D+P} \mathbf{x} \right) [\tilde{Y}_2(\tilde{u}_1 \mathbf{x}) \tilde{Y}_1(\tilde{u}_2 \mathbf{x})]_{LM}. \quad (3.13)$$

Here $\tilde{Y}_2(\tilde{u}_1 \mathbf{x})$ and $\tilde{Y}_1(\tilde{u}_2 \mathbf{x})$ are solid spherical harmonics

$$\tilde{Y}_{LM}(\tilde{u} \mathbf{x}) = |\tilde{u} \mathbf{x}|^L \tilde{Y}_{LM}(\tilde{u} \mathbf{x}). \quad (3.14)$$

Since we take the $Y_1$ function in the $x_4$ coordinate, the global vector $\tilde{u}_2 \mathbf{x}$ is fixed as $\tilde{u}_2 \mathbf{x} = x_4$. Another global vector $\tilde{u}_1 \mathbf{x}$ can be selected in the coordinate of $x_1$ or $x_2$ or $x_3$ or $x_4$, which we know the most efficient coordinate obtained by the tensor interaction is the $x_1$ coordinate.

### 3.2.3 Variational calculation with the radius constraint Hamiltonian

We perform the variational calculation for five-body system. The five-body Hamiltonian is given in the same form as Eq. (2.11), and the two-body $NN$ interaction is AV8'. In this calculation it is not enough to obtain a sufficient attraction for a construction of resonance states of $^5$He, because we do not introduce the effects of three-body interaction. However we would like to find the characteristics of $^5$He structure induced by the two-body interaction.

We employ the Stochastic Variational Method for the variational calculation in the same manner done for the s-shell study. The variational parameters of $A_{S+P}$ and $A_{D+P}$ are determined randomly, where the only guideline of determination for these parameters is to search the best parameters by trial and error in a direction toward the minimum of energy. In the case of unbound system such as $^5$He, this method continues choosing the variational parameters to reach the energy of $^4$He+$n$ threshold. Consequently, we have to prevent $^5$He from breaking into $^4$He+$n$ in the process of variational calculation.
In this study we would like to propose a method of taking a constraint on the radius in the total Hamiltonian. The radius constraint plays a role in restraining the large size of $^{5}\text{He}$. Then we have to solve the Hamiltonian in this way,

$$\hat{H} = H + \lambda \hat{r}^2,$$

(3.15)

where $\lambda$ is an appropriate value in unit of $\text{MeV}/\text{fm}^2$, and the $\hat{r}^2$ is a radius operator,

$$\hat{r}^2 = \frac{1}{5} \sum_{j>i=1}^{5} (\mathbf{r}_i - \mathbf{r}_j)^2.$$ 

(3.16)

We can obtain a true energy for the original Hamiltonian $H$ by taking away the matrix elements of the radius constraint,

$$E = \langle \hat{H} \rangle - \lambda \langle \hat{r}^2 \rangle.$$ 

(3.17)

If we take the large value of $\lambda$, the $^{5}\text{He}$ wave function is treated to be bounded at the small radius. If we take the small $\lambda$, $^{5}\text{He}$ has the large radius. Therefore we understand this constraint seems to correspond to the harmonic oscillator potential. We can provide with the energy of unbound state by giving a suitable value of $\lambda$, and discuss the structure of $^{5}\text{He}$ at a fixed radius.

### 3.3 Numerical results

In this section we present the numerical results for $^{5}\text{He}$ with TOFM framework. We obtained the energy at a certain radius by introducing the radius constraint in the total Hamiltonian. The calculated results of $^{5}\text{He}$ are shown in Fig. 3.3, where the energies are measured from $^{4}\text{He}$ ground state for two resonances $3/2^-$ and $1/2^-$ as the $\lambda$ value of the radius constraint. The red dot line denotes the results of the $3/2^-$ state and, the green one is the $1/2^-$ state. We reproduce that the $1/2^-$ state is located above the $3/2^-$ state for almost whole region of $\lambda$, which is consistently with the experimental situation. The strengthening of the constraint by increasing $\lambda$ makes a $^{5}\text{He}$ bound state with a compact radius.

The calculated results of $^{5}\text{He}$ with a small value of $\lambda$ have a large radius, for example, the radius is about 3.3 fm at $\lambda = 0.0 \text{ MeV}/\text{fm}^2$, where we cannot express the difference of two states in $3/2^-$ and $1/2^-$. These structures of $^{5}\text{He}$ do not have a correlation between $^{4}\text{He}$ and the last neutron, and seem to have the same structure of the isolated $^{4}\text{He}$ ground state due to the separation into the subsystem of $^{4}\text{He}+n$ in the unbound system. On the other hand we can see the energy difference in two resonances increases in the large value of $\lambda$, for example, the radius is about 1.8 fm at $\lambda = 0.10 \text{ MeV}/\text{fm}^2$. At this small radius region near $^{4}\text{He}$, the effects of the coupling between $^{4}\text{He}$ and the last neutron give rise to make the difference of the structures in two resonances of $^{5}\text{He}$.

Therefore we would like to analyze the matrix elements of $3/2^-$ and $1/2^-$ states at the region of small radius, and clarify the roles of tensor interaction in the first stage of the p-shell nuclei. Table 3.1 represents the calculated results of the radii of $^{5}\text{He}$.
Figure 3.5: $^5\text{He}$ energies measured from $^4\text{He}$ for two resonances of $3/2^-$ and $1/2^-$ as the values of the radius constraint.

and various energy components of $^5\text{He}$ with the difference from that of $^4\text{He}$ in TOFM shown in Table 2.2. We compare the structures of $3/2^-$ and $1/2^-$ states, by fixing the radius at $\lambda = 0.10$ MeV/fm$^2$. It is found that the $3/2^-$ state exhibits the larger tensor matrix element than the case of $1/2^-$ state, which provides the larger enhancement of kinetic energy in $3/2^-$. These tendencies seen in the tensor and kinetic matrix elements are also found in the study of TOSM [18]. In the TOSM calculation, the occupation of $p_{1/2}$ in the $1/2^-$ state disturbs the internal excitation of $^4\text{He}$, and induces the Pauli blocking effect. This blocking effect is, on the other hand, very weak in case of the $3/2^-$ state of $^5\text{He}$ due to the large occupation of the $p_{3/2}$ state of last neutron in $^5\text{He}$. From these results the tensor interaction plays important roles to create the different structures in two resonances. The central interaction of $3/2^-$ is also larger than the $1/2^-$ case because of the smaller radius of 1.8 fm in the $3/2^-$ state than the $1/2^-$ case. With respect to the LS interaction, the difference of the LS interaction acting on P-wave orbital makes an opposite sign of $3/2^-$ and $1/2^-$. Puting all these differences of components together, the energy of $3/2^-$ are lower than the $1/2^-$ case, which results in the doublet splitting energy of the p-wave resonances. It is also found that the total attraction between $^4\text{He}$ and the valence-nucleon are suppressed larger in the $1/2^-$ state than in $3/2^-$ state, which brings a more unbound nature as the resonance in $1/2^-$. The total energies and the splitting energy of two states are not enough in comparison with the experimental situation of $^5\text{He}$. One of the possible reasons is due to the absence of three-body interaction. Nevertheless we can understand how the two-body $NN$ interaction generates the p-shell structures from the double-closed shell of $^4\text{He}$ by adding one nucleon.
Table 3.1: Various matrix elements and constraint radius in the $3/2^-$ and $1/2^-$ states of $^5$He.

<table>
<thead>
<tr>
<th>$^5$He($J^p$)</th>
<th>Energy</th>
<th>Kinetic</th>
<th>Central</th>
<th>Tensor</th>
<th>LS</th>
<th>Radius[fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3/2^-$</td>
<td>5.12</td>
<td>18.00</td>
<td>-7.49</td>
<td>-4.30</td>
<td>-1.08</td>
<td>1.80</td>
</tr>
<tr>
<td>$1/2^-$</td>
<td>5.70</td>
<td>13.68</td>
<td>-5.47</td>
<td>-2.92</td>
<td>0.34</td>
<td>1.90</td>
</tr>
</tbody>
</table>

Table 3.1: Various matrix elements measured from those of the $^4$He ground state. Energy and other components are given in MeV.

3.4 Summary of this chapter

In this chapter we give the framework of the p-shell TOFM, and apply it to $^5$He as the first stage of p-shell nuclei. It is well-known that $^5$He has no bound state and has a p-wave doublet splitting of $3/2^-$ and $1/2^-$ states. There are characteristic difference between two states in a $n$-$^4$He scattering phase shifts. The previous studies succeeded in the reproduction of this phase shifts and have indicated the importance of the three-body interaction in $^5$He. However the relations between the bare $NN$ interaction and nuclear structure of p-shell has not been discussed well in previous studies so far.

In this study we investigate how the $NN$ interaction acts in the unbound states of $^5$He structure and determines the coupling behavior between the $^4$He nucleus and a last neutron. In particular we focus on the difference of two resonances from the point of view of the role of tensor force. We propose a method of the variational calculation with respect to the unbound states by adding a radius constraint in Hamiltonian for unbound system. This method enables us to discuss the $^5$He structure as the unbound state by fixing the radius. We can apply this approach to any unbound system. As for the framework of TOFM in the p-shell nuclei, we describe the negative parity states by adding a $Y_j$ function in a last Jacobi coordinate. We perform the five-body variational calculation using TOFM for $3/2^-$ and $1/2^-$ states of $^5$He.

We obtain the structure difference between $3/2^-$ and $1/2^-$ by constraining the system at a small radii, where the $3/2^-$ state is located lower than the $1/2^-$ state. Furthermore we find that the tensor force in $3/2^-$ are enhanced larger than in $1/2^-$ as well as the kinetic energy. This tendency is the same as obtained in the study of TOSM for the p-shell nuclei, and therefore, we understand that the Pauli blocking effect generates the difference of tensor force contributions in two states of $^5$He. Then the various matrix elements of $3/2^-$ state enhanced larger than $1/2^-$, which makes the energy of the doublet splitting.

From these results, it is important to investigate the roles of tensor force in the structures of p-shell nuclei beyond the double closed s-shell nucleus of $^4$He. As was discussed in the previous study, it is also important to take into account the three-body interaction. Hence we encourage not only to develop the TOFM framework for heavier p-shell nuclei, but also to deal with the effect of three-body force in our framework, which would be given in the next section.
Chapter 4

The explicit treatment of $\Delta$($1232$) excitation in nuclei and the role of tensor interaction

4.1 The importance of $\Delta$ degree of freedom and purpose of this chapter

The two-nucleon interaction models such as AV18 are constructed by fitting the two-nucleon scattering data and the deuteron properties. It is well known that the nuclear binding energies by using $NN$ interaction are less than the experimental values. The shortage of the energies has been handled by means of the effect of three-body interactions phenomenologically, and it has been possible to reproduce the various binding energies and energy levels of light nuclei by adding these interactions [3, 11, 12, 50]. The amount of the three-body force is typically about $10\% \sim 20\%$ of the total energy, and these attractions become enlarged as the mass of nuclei become large. It is therefore important to discuss the effect of three-body force on the nuclear structure and reaction.

A meson-exchange model for three-nucleon force was proposed by Fujita and Miyazawa [44]. It is a three-nucleon interaction due to the second order pion-exchange, where the intermediate nucleon is excited to the $P$-wave $\Delta$ resonance as shown in Fig. 4.1. The

![Figure 4.1: Fujita-Miyazawa type three-body force](image-url)
\(\Delta(1232)\) isobar has spin \(\frac{3}{2}\) and isospin \(\frac{3}{2}\). Hence the \(\Delta(1232)\) plays an important role to generates the intermediate-range attraction treated as the three-body force. On the basis of the Fujita-Miyazawa type, the series of the Urbana and Illinois models are constructed by including two-pion and three-pion exchange terms. These interactions have been developed as well as other three-body models [45–49]. However these models are phenomenologically constructed by fitting the observed binding energies and the nuclear densities of light nuclei, where the treatment of the \(\Delta\) excitation is effective for the description of the only nucleon space. It is not clear the direct relation between the three-body force and the \(\Delta\) resonance. Hence we would like to introduce explicitly \(\Delta(1232)\) degrees of freedom as well as the nucleon space to describe the nuclei. We study the origins of the three-body force from the two-body force including the \(\Delta\) resonance. Moreover it is important to develop the explicitly treatment of \(\Delta\) for various many-body framework.

To deal with the two-body interaction including explicitly \(\Delta\) degrees of freedom, we have to extend the model space of \(NN\) into \(NN\), \(N\Delta\) and \(\Delta\Delta\) channels due to the consideration of the transition between \(N\) and \(\Delta\) \((N \leftrightarrow \Delta)\). We show some examples of \(NN\), \(N\Delta\), and \(\Delta\Delta\) diagrams in Fig. 4.2. The model space extremely enlarges by including the \(\Delta\) degrees of freedom in comparison with the only \(NN\) space, and it is hard to extend the framework with the all configurations exactly in larger mass nuclei. Hence it is good idea to apply tensor optimized few-body model (TOFM) to this problem. In the TOFM framework, the most important configurations to obtain the strong tensor correlation treat in the model space, and it can makes the variational calculation efficiently. In particular it is important to pay attention to the roles of tensor force caused by the transitions between \(N\) and \(\Delta\). Because the tensor force induces high momentum components about 500 \(~\sim\) 600 MeV/c, it would be possible to provide a large tensor force through the excitation of the nucleon into the \(\Delta\) state, although the large mass difference of \(\Delta(1232)\) from the nucleon is about 300 MeV. Therefore we have to investigate the reasonable configurations to generates large tensor correlations so as to develop the treatment of \(\Delta\) in the framework of TOFM and TOSM. In the study of three- and four-body, it would be important to make clear the effects of phenomenological three-body force in many-body system by starting from the two-body interaction including the \(\Delta\) degrees of freedom.
4.2 The two-body interaction with $\Delta$ degrees of freedom

We need a two-body interaction with $\Delta$ degree of freedom. We start to use the Argonne v28 model which is developed by Wiringa et al [51]. In Ref. [51], Wiringa et al presented the $NN$ interaction with and without $\Delta(1232)$ degrees of freedom, Argonne v14 (AV14) and v28 (AV28) model respectively. Here it is important to introduce the constitutions of $NN$ interactions of AV28 together with AV14, because we should know the roles of $\Delta$ degrees of freedom and what new ingredients are necessary in TOFM to describe the $N \leftrightarrow \Delta$ transitions.

Firstly we note that the AV14 potential is constructed in the same manner as AV8' and AV18 potential. In fact the potential has the three radial components: the long-range OPE part $v_n$ and phenomenological intermediate-range and short-range part $v_I$ and $v_S$, and is written as a sum of 14 operator components,

$$v_{14,ij} = \sum_{p=1,14} [v_n^p(r_{ij}) + v_I^p(r_{ij}) + v_S^p(r_{ij})] \sigma_{ij}^p,$$

where the operators are

$$\sigma_{ij}^p = \begin{cases} 1, \tau \cdot \tau, \sigma_i \cdot \sigma_j, (\sigma_i \cdot \sigma_j)(\tau \cdot \tau), S_{ij}, S_{ij}(\tau \cdot \tau), (L \cdot S), (L \cdot S)(\tau \cdot \tau) \end{cases}$$

These operators have the same structure as Urbana model [52] and Paris potentials [53]. The $v_n^p$ contributes to the terms of $(\sigma_i \cdot \sigma_j)(\tau \cdot \tau)$ and $S_{ij}(\tau \cdot \tau)$ operators, and is formed by the usual Yukawa and tensor function,

$$v_n^\sigma(\tau) = \left[ \frac{f_{\pi NN}^2 m_\pi}{4\pi} \right] \frac{e^{-\mu r}}{\mu r} (1 - e^{-cr^2})$$

with coupling constant,

$$f_{\pi NN}^2/(4\pi) = 0.081.$$

Here the $\sigma \tau$ and $t \tau$ mean the short notation of a spin-isospin and a tensor-isospin operator, and the cutoff parameter $c$ is taken $c = 2 fm^{-2}$. The intermediate-range $v_I^p$ and short-range $v_S^p$ are given by the square of tensor function and the Wood-Saxon function phenomenologically, and the parameters in these functions are determined by fitting the nucleon-nucleon scattering data and the deuteron properties. The first 14 terms of the AV18 potential is the same construction of the AV14 potential. AV18 has also the additional 4 terms of the charge-dependent and charge-asymmetric, and the electromagnetic potential terms.

On the other hand, the AV28 potential allows the delta excitation. It contains the $v_{14}$ part for $NN$ channel and adds the operators of $N\Delta$ and $\Delta\Delta$ channels to represent all possible processes with $\pi N\Delta$ and $\pi \Delta\Delta$ vertices. All processes of these channels are
Fig. 4.3: All processes contained in AV28 potential.

Fig. 4.3: The diagrams represent the two-body potential of the AV28 model. Thin solid lines denote nucleons, thick lines denote Δ resonances, and dotted lines are the pion exchange. $p$ denotes the numbering of operator types.

expressed by 28 operators. Fig. 4.3 shows all processes including the delta excitation in the same order of the Ref. [51],

$$v_{28,ij} = \sum_{p=1,28} [v^3_N(r_{ij}) + v^p_f(r_{ij}) + v^p_\delta(r_{ij})] \phi^p_{ij}. \quad (4.5)$$

The $NN$ channels ($p = 1-14$) have 14 operators $\phi_{1-14}$ same as AV14 potential. The transition process of $N \leftrightarrow \Delta$ is written as the kind of operators $\phi_{15,16}$, $\phi_{17,18}$, $\phi_{19,20}$, and $\phi_{24,25}$, and shown as $p = 15-20, 24, 25$ in Fig. 4.3. Because the $N \leftrightarrow \Delta$ transition changes both spin and isospin from $1/2$ to $3/2$, those transition processes involve a $\pi N \Delta$ vertex with a spin-isospin or tensor-isospin operator and contribute only in $v^p_N$ part. The diagonal $\Delta \Delta$ ($p = 21-23$) and $\Delta \Delta$ ($p = 26-28$) channels correspond to the operator $\phi_{21-23}$ and $\phi_{26-28}$, and $p = 21-23, 26-28$ in Fig. 4.3. These channels contribute the $\pi$ exchange operator and the central operator just as in Eq. (4.5).

Here we show the operators $\phi_{15-28}$ except for $NN$ channel,
Here $S_i$ and $T_i$ are the transition operators of spin and isospin from $\frac{1}{2}$ to $\frac{3}{2}$ states, $\Sigma_j$ and $\theta_j$ are the spin and isospin operators for a $\frac{3}{2}$ state, and $1_{N_j}$ and $1_{\Delta}$ are the unit operators for a nucleon and delta particle. The reduced matrix elements of $S_i(T_i)$ and $\Sigma_j(\theta_j)$ are,

\[
\begin{align*}
\langle \frac{3}{2} | S_i | \frac{1}{2} \rangle &= -\langle \frac{1}{2} | S_i^\dagger | \frac{3}{2} \rangle = 2, \\
\langle \frac{3}{2} | \Sigma_j | \frac{3}{2} \rangle &= 2\sqrt{15}.
\end{align*}
\]
We show various tensor operators $S_{ij}^{(2)-(7)}$,

\begin{align}
S_{ij}^{(2)} &= 3(\sigma_i \cdot \hat{r}_i)(S_j \cdot r_j) - \sigma_i \cdot S_j,
\end{align}

\begin{align}
S_{ij}^{(3)} &= 3(S_i \cdot \hat{r}_i)(S_j \cdot \hat{r}_j) - S_i \cdot S_j,
\end{align}

\begin{align}
S_{ij}^{(4)} &= 3(S_i \cdot \hat{r}_i)(S_j \cdot \hat{r}_j) - S_i \cdot S_j,
\end{align}

\begin{align}
S_{ij}^{(5)} &= 3(\sigma_i \cdot \hat{r}_i)(\Sigma_j \cdot \hat{r}_j) - \sigma_i \cdot \Sigma_j,
\end{align}

\begin{align}
S_{ij}^{(6)} &= 3(S_i \cdot \hat{r}_i)(\Sigma_j \cdot \hat{r}_j) - S_i \cdot \Sigma_j,
\end{align}

\begin{align}
S_{ij}^{(7)} &= 3(\Sigma_i \cdot \hat{r}_i)(\Sigma_j \cdot \hat{r}_j) - \Sigma_i \cdot \Sigma_j.
\end{align}

Wiringa et al. assumed that the $N \leftrightarrow \Delta$ transition is caused only by the OPE $v_\pi$, while those in the short-range $v_\delta$ and intermediate-range $v_\rho$ part does not include this transition. For $N\Delta$ and $\Delta\Delta$ channels, the operators $p = 22, 23, 27, 28$ contribute in $v_\rho$ part, because the operators have a spin-isospin and tensor-isospin operators. The only unit operators $p = 21, 26$ of the diagonal $N\Delta$ and $\Delta\Delta$ channels contain the $v_\delta$ and $v_\rho$ part in the central force, these parameters are determined to be the same as those of the unit operator $p = 1$ for $NN$ channel.

The $v_\rho$ part forms the same functions in AV14 potential in Eq. (4.3) and the only difference is to change the coupling constants $f_{N\Delta}^2$ and $f_{\Delta\Delta}^2$. Wiringa et al. take the $f_{N\Delta}^2$ value by Chew-Low theory [54],

\begin{align}
(f_{N\Delta}^2/4\pi) &= 4(f_{NN}^2/4\pi) = 0.324,
\end{align}

and the $f_{\Delta\Delta}^2$ value by the quark and strong-coupling models,

\begin{align}
(f_{\Delta\Delta}^2/4\pi) &= \frac{1}{25}(f_{NN}^2/4\pi) = 0.00324.
\end{align}

Thus, the AV28 potential can express the $\Delta$ excitation process and enlarge the number of operators as compared to the AV14 potential. However the total number of parameters does not change in the AV28 potential. These short-range and intermediate-range parameters for $N\Delta$ and $\Delta\Delta$ channels are decided as the same value of the $NN$ channel. Then all phenomenological parameters should be determined to reproduce the experimental data as in the case of the AV14 potential. They can reproduced the $NN$ scattering data for all channels of $J \leq 5$ and the deuteron properties.

### 4.3 Deuteron including the $\Delta$ excitation explicitly

#### 4.3.1 Wave function of deuteron

First we would like to study the deuteron with AV28 potential. In Ref. [51], deuteron properties are presented such as the binding energy and probabilities of the wave function components. However, it is also important to discuss the tensor and other matrix elements, that are not shown in Ref. [51]. It is necessary to understand the roles of the tensor force derived from the $\Delta$ excitation.
Since the deuteron has a positive parity, total spin \( J = 1 \) and isospin \( T=0 \) (triplet-even channel), the \( N\Delta \) components are forbidden in wave function due to the isospin coupling. Hence the deuteron wave function is written as,

\[
|\Psi_{J=1}\rangle = |\Psi_{NN}\rangle + |\Psi_{\Delta}\rangle
\]

The \( NN \) wave function consists of the \( S \)-wave and \( D \)-wave component,

\[
|\Psi_{NN}\rangle = |^3S_1\rangle + |^3D_1\rangle.
\]

On the other hand the \( \Delta\Delta \) wave function, which we call double delta state, is

\[
|\Psi_{\Delta\Delta}\rangle = |^3S_1\rangle + |^3D_1\rangle + |^7D_1\rangle + |^7G_1\rangle.
\]

The state with spin \( S = 2 \) in the \( \Delta\Delta \) wave function is forbidden because of the antisymmetrization. Then the deuteron wave functions of \( NN \) channel and \( \Delta\Delta \) channel are obtained by the processes \( p = 17, 18, 26 - 28 \) in Fig. 4.3.

The two-body Hamiltonian is a sum of rest-mass and kinetic energy operators, and two-body interaction,

\[
H = (m_1 + m_2 - 2m_N) + \sum_{i=1,2} \frac{\mathbf{p_i}^2}{2m_i} - T_{cm} + V_{12}.
\]

We take the nucleon mass \( m_N = 938.9 \) MeV and the delta particle mass \( m_\Delta = 1232 \) MeV. Hence we solve this coupled channel equation using the variational method of SVM, where the variational parameters are given randomly. Here we explain which interactions act on the \( NN \) and \( \Delta\Delta \) channels in solving the coupled channel equation of deuteron. We show the operator number \( \theta^i \) interacting in two wave function components in the diagonal and off-diagonal matrix of Table 4.1, where \( \theta^i \) corresponds to the operator in Eq. (4.6) and (4.8). For example, in the calculation of the matrix elements between \( ^3S(NN) \) and \( ^3S(NN) \), we have to consider the diagonal matrix elements with the interaction operator from 1 to 4 (\( \theta^1 \)-4). Moreover in the calculation of the matrix elements between two \( \Delta\Delta \) states, the short- and intermediate-range and the one-pion exchange potentials with \( \theta^{26-28} \) act on these states. On the other hand, in the case of the transition process such as those between \( ^3S(NN) \) and \( ^3D(\Delta\Delta) \), the off-diagonal matrix element consists of only the pion exchange potentials which are presented as the operator number \( \theta^{17,18} \).

The matrix elements of the potential operators are calculated through the angular momentum algebra. The result for spin-spin (isospin-isospin) operators is,

\[
\langle s'_1 s'_2 | s_1 \cdot s_2 | s_1 s_2 S \rangle = (-1)^{s_1 + s'_1 + s S} \delta_{SS'} \left\{ \begin{array}{ccc} S & s'_2 & s'_1 \\ 1 & s_1 & s_2 \end{array} \right\} \langle s'_1 || s_1 \rangle \langle s'_2 || s_2 \rangle,
\]

where \( s_1 \) and \( s_2 \) correspond to a spin (isospin) of nucleon and delta, or a transition spin (isospin) operators.

We can also calculate the matrix elements for the tensor operator,

\[
\langle L'S'(s'_1 s'_2) | Y_2 | s_1 \times s_2 | LS(s_1 s_2) J \rangle = (-1)^{J + S' + L} \left\{ \begin{array}{ccc} L & S & J \\ S' & L' & 2 \end{array} \right\} \langle L'|| Y_2 || L \rangle \delta^{S' S} \delta \left\{ \begin{array}{ccc} s'_1 & s'_2 & S \\ 1 & 1 & 2 \end{array} \right\} \langle s'_1 || s_1 \rangle \langle s'_2 || s_2 \rangle \langle s_1 || s_2 \rangle.
Table 4.1: The operator number of AV28 interacting in the wave function components of deuteron.

<table>
<thead>
<tr>
<th>$J^\pi = 1^+$</th>
<th>$^3S(NN)$</th>
<th>$^3D(NN)$</th>
<th>$^3S(\Delta\Delta)$</th>
<th>$^3^7D(\Delta\Delta)$</th>
<th>$^7G(\Delta\Delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^3S(NN)$</td>
<td>$^6^1-4$</td>
<td>$^6^5-6$</td>
<td>$^6^0^7$</td>
<td>$^6^1^8$</td>
<td>$^6^0^0$</td>
</tr>
<tr>
<td>$^3D(NN)$</td>
<td>$^6^5-6$</td>
<td>$^6^1-14$</td>
<td>$^6^1^8$</td>
<td>$^6^1^7.18$</td>
<td>$^6^1^8$</td>
</tr>
<tr>
<td>$^3S(\Delta\Delta)$</td>
<td>$^6^0^1^7$</td>
<td>$^6^0^1^8$</td>
<td>$^6^0^2^6.27$</td>
<td>$^6^0^2^8$</td>
<td>$^6^0^2^8$</td>
</tr>
<tr>
<td>$^3^7D(\Delta\Delta)$</td>
<td>$^6^0^1^8$</td>
<td>$^6^0^1^7.18$</td>
<td>$^6^0^2^8$</td>
<td>$^6^0^2^6-28$</td>
<td>$^6^0^2^8$</td>
</tr>
<tr>
<td>$^7G(\Delta\Delta)$</td>
<td>0</td>
<td>$^6^0^1^8$</td>
<td>0</td>
<td>$^6^0^2^8$</td>
<td>$^6^0^2^6-28$</td>
</tr>
</tbody>
</table>

4.3.2 Numerical results for the deuteron

We show the numerical results of the deuteron using AV28 potential for comparison with AV14 potential without $\Delta$ excitation. Since the $\Delta\Delta$ channel has many components of the wave function, the model space and the basis function become much larger than those of the $NN$ channel. Table 4.2 represents the total energy and various matrix elements, and the probabilities of wave function components with AV14 and AV28. The third line means the matrix elements obtained from the $NN$ space with AV28.

The total energy almost reproduces the experimental deuteron binding energy for both the AV14 and AV28 potential. In Table 4.2, it is shown that the results of the kinetic energy, central force, and tensor force matrix elements with AV28 in comparison with those of AV14, where the parentheses in the results of AV28 show the occupancy of the $NN$ space of the matrix elements. These matrix elements of two potentials are significantly different owing to the inclusion of $\Delta$ excitation.

It should be noted that the tensor matrix elements give an extremely large attraction of -35.33 MeV in the AV28 potential with the $\Delta$ degrees of freedom, which is about twice that of -18.80 MeV in the AV14 potential without $\Delta$ excitation. Then it is found that the large contributions of the transition of nucleon to the $\Delta$ state amounts to about 30 % of the total tensor interaction. It is also important to see the large kinetic energy compensating the large tensor interactions. In case of AV14, the kinetic energy and the tensor interaction cancel out each other to provide the small binding energy in the same manner as AV8'. On the other hand in AV28, the kinetic energy cannot reach the large attraction caused by the tensor interaction, instead, the central interaction acts as a repulsion. As shown in AV28 of Table 4.2 and the Fig. 4.5, most of the kinetic energy is caused by the $NN$ space, and the contribution of the $\Delta$ space to the kinetic energy is very small. This is because the probabilities of $\Delta$ state is less than 0.5 % which less to the small matrix elements. These tendencies can also be seen in the matrix elements of the central force and the mass difference from two-nucleon. The value of the mass difference is presented as the label 'Mass' in Table 4.2. Although the $\Delta\Delta$ states have the large mass difference about 600 MeV from $NN$ states, these small probability and amplitude contribute to the matrix elements only 3.09 MeV. Hence the tensor forces derived from the $\Delta$ excitations generates high-momentum components without caring much about 600 MeV mass difference of $\Delta\Delta$ states in deuteron.
Table 4.2: Calculations of the deuteron with AV14 and AV28

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Kinetic+Mass</th>
<th>Central</th>
<th>Tensor</th>
<th>$LS$</th>
<th>$L^2$</th>
<th>$(LS)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AV14</td>
<td>-2.21</td>
<td>19.13</td>
<td>-1.94</td>
<td>-18.80</td>
<td>0.36</td>
<td>3.07</td>
<td>-4.03</td>
</tr>
<tr>
<td>AV28</td>
<td>-2.25</td>
<td>20.87+3.09</td>
<td>8.77</td>
<td>-35.33</td>
<td>0.84</td>
<td>3.55</td>
<td>-4.05</td>
</tr>
<tr>
<td>AV28(NN)</td>
<td>19.45</td>
<td>8.62</td>
<td>-25.42</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$P$ [%]</th>
<th>$^3S(\text{NN})$</th>
<th>$^3D(\text{NN})$</th>
<th>$^3S(\Delta\Delta)$</th>
<th>$^3S(\Delta\Delta)$</th>
<th>$^7D(\Delta\Delta)$</th>
<th>$^7G(\Delta\Delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AV14</td>
<td>93.93</td>
<td>6.07</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AV28</td>
<td>93.31</td>
<td>6.16</td>
<td>0.04</td>
<td>0.02</td>
<td>0.42</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 4.2: The upper table shows the various energy components in unit of MeV, and the lower one is the probabilities of the wave function for deuteron. We comparer these results between AV14 and AV28.

Figure 4.4: The correlation function for each $\Delta\Delta$ wave function components as the function of the relative distance $r$. 

\[
\text{Table 4.2: Calculations of the deuteron with AV14 and AV28}
\]

\[
\begin{array}{cccccccc}
\text{Energy} & \text{Kinetic+Mass} & \text{Central} & \text{Tensor} & \text{LS} & \text{L}^2 & (\text{LS})^2 \\
-2.21 & 19.13 & -1.94 & -18.80 & 0.36 & 3.07 & -4.03 \\
-2.25 & 20.87+3.09 & 8.77 & -35.33 & 0.84 & 3.55 & -4.05 \\
19.45 & 8.62 & -25.42 & & & & \\
\text{P [%]} & ^3S(\text{NN}) & ^3D(\text{NN}) & ^3S(\Delta\Delta) & ^3S(\Delta\Delta) & ^7D(\Delta\Delta) & ^7G(\Delta\Delta) \\
93.93 & 6.07 & & & & & \\
93.31 & 6.16 & 0.04 & 0.02 & 0.42 & 0.04 & \\
\end{array}
\]
Figure 4.5: Calculated results of the deuteron with AV14 and AV28.

The left hand side shows the results of AV14 and the right hand side shows those of AV28. Those energy components are the kinetic energy plus mass difference in AV28, while in AV14 kinetic energy, central force, and tensor force. The dotted lines in AV28 means the energy values given by the NN space.

We also study what kinds of the Δ states make significant contributions to the tensor matrix elements. As shown in Table 4.2 and the ΔΔ correlation function in Fig. 4.4, the probability of the $^7D$-state is 0.42% in particular larger than those of other states. Hence it is found the tensor matrix elements generated by the transition into $^7D(\Delta\Delta)$ from $^3S(\Delta\Delta)$ is the most important process. In comparison to the same D-state between $^7D$ with spin 3 and $^3D$ with spin 1, it is also found that the tensor interaction prefer the state which has the spin alignment.

Next, we discuss the differences of central forces between AV14 and AV28. As mentioned above, in case of AV28 the central force acts as a repulsion together with kinetic energy to counter balance the large attraction of the tensor force. Fig. 4.6 shows the central potential in the spin-isospin channel, where we compare the difference between AV14 and AV28 triplet even channel ($^3E$). In case of AV28, the attractive part is extremely shallow in comparison with that of AV14, and the central potential of AV28 is dominated the repulsive core. We can understand that these properties of the central
Figure 4.6: The central potential for each channel with AV14 and AV28.

![Figure 4.6: The central potential for each channel with AV14 and AV28.](image1)

force affect on the repulsive contribution in the deuteron. Also we show the distribution of the central matrix elements as function of the distance in Fig. 4.7, where we can see much larger repulsion comes from strong short-range repulsion in AV28 than in AV14 and AV8'. Thus, we find the additional of the Δ excitation in AV28 makes the central force contribution rather repulsive.

Figure 4.7: The distribution of the central matrix elements as functions of distance r.

![Figure 4.7: The distribution of the central matrix elements as functions of distance r.](image2)
Figure 4.8: The distribution of the \(LS\) matrix elements as functions of distance \(r\).

We would like to mention the \(LS\) force, which has an opposite sign to those results of \(AV8'\). Both of the \(AV14\) and \(AV28\) potentials provide the repulsive matrix elements as shown in Table 4.2. We find that the reason of this difference in the potentials is related to the radial form of the potential, that is, the \(LS\) potentials of \(AV28\) and \(AV14\) in the triplet even channel are opposite in sign to the \(AV8'\) \(LS\) potential. We can see these differences of the potential in the distribution of the \(LS\) matrix elements as the function of the distance in Fig. 4.8. We also have to consider the construction of the momentum term including \(L^2\) and \((LS)^2\) terms originated from the relativistic effects in the future.

4.4 Study of the \(\Delta\Delta\) and \(N\Delta\) channels in all two-body systems

4.4.1 Preparation for a three-body calculation

From the deuteron results of the previous section, we conclude that the transition processes of \(NN \leftrightarrow \Delta\Delta\) generate the large tensor contribution. It would be important that these deuteron like configurations including \(\Delta\) states play a significant role of the binding energy in three-body and larger mass systems. Hence we expect the \(\Delta\Delta\) channel, especially the \(^7D\) state, contributes to the tensor matrix elements.

For a three-body system such as triton, we should be take into account all \(\Delta\) channels, \(N\Delta\) and \(\Delta\Delta\) in the space of the wave function, because the \(T = 1\) isospin channel combined by two \(\frac{3}{2}\) isospin is permitted. It is necessary to perform the variational calculation for a three-body system with extremely large model space. In addition, we
have to investigate the role of the $N\Delta$ channels and the tensor force from the transition process $NN \leftrightarrow N\Delta$ in order to carry out the variational calculation efficiently. Hence as a preparation for a three-body calculation, we would like to study the properties of the effects of $N\Delta$ channel for all two-body systems, and bring out which states in $\Delta\Delta$ and $N\Delta$ channels are dominant to provide with the large tensor interaction through the transition from the $NN$ channel.

In a two-body system, there are four channels including the triplet even channel of the deuteron, which are,

- **Triplet-Even**: $L=$even, $S=$odd, $T=0$
- **Singlet-Even**: $L=$even, $S=$even, $T=1$
- **Singlet-Odd**: $L=$odd, $S=$odd, $T=0$
- **Triplet-Odd**: $L=$odd, $S=$even, $T=1$

where we have to take care of the $T=1$ channel for the $N\Delta$ state. Since these two-body channels except for the triplet even are unbound, it is difficult to see the effects of the $\Delta$ excitation in a phase-shift analysis.

Hence we would like to suggest a method to deal with these unbound states as the bound states by using a radius constraint Hamiltonian, which we have used in the study of $^5\text{He}$ nucleus. It enables us to find out the energies and the matrix elements for the unbound two-body systems. The Hamiltonian with a radius constraint given as,

$$\hat{H} = (m_1 + m_2 - 2m_N) + \sum_{i=1,2} \frac{p_i^2}{2m_i} - T_{cm} + V_{12} + \lambda \hat{r}^2,$$  \hspace{1cm} (4.15)

where $\hat{r}^2$ is the operator of the radius, and we can obtain the energy,

$$E = \langle \hat{H} \rangle - \lambda \langle \hat{r}^2 \rangle.$$  \hspace{1cm} (4.16)

We would like to estimate the various matrix elements for all-two-body channels with the same radius in order to know which $\Delta\Delta$ and $N\Delta$ channels bring the large tensor contributions.

#### 4.4.2 Analyses of the two-body $^1E$ channel

We study the single-even ($^1E$) channel for $J=0$ two-body system using the AV28 potential. In case of the singlet-even channel, the wave function consists of the $NN$, $N\Delta$, and $\Delta\Delta$ components,

$$|\Psi_{J=0}\rangle = |\Psi_{NN}\rangle + |\Psi_{N\Delta}\rangle + |\Psi_{\Delta\Delta}\rangle,$$  \hspace{1cm} (4.17)

and then,

$$|\Psi_{NN}\rangle = |^1S_0\rangle$$  \hspace{1cm} (4.18)

$$|\Psi_{N\Delta}\rangle = |^1S_0\rangle + |^5D_0\rangle$$

$$|\Psi_{\Delta\Delta}\rangle = |^5D_0\rangle.$$
Table 4.3: Calculated results of the $^1E$ channel with AV28

<table>
<thead>
<tr>
<th>Radius [fm]</th>
<th>Energy</th>
<th>Kinetic+Mass</th>
<th>Central</th>
<th>Tensor</th>
<th>$LS$</th>
<th>$L^2$</th>
<th>$(LS)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>4.09</td>
<td>7.55+1.58</td>
<td>0.49</td>
<td>-5.54</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(NN)</td>
<td>6.66</td>
<td>0.37</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.70</td>
<td>5.74</td>
<td>10.51+2.24</td>
<td>0.90</td>
<td>-7.92</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(NN)</td>
<td>9.23</td>
<td>0.70</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.40</td>
<td>8.5</td>
<td>15.87+3.49</td>
<td>1.86</td>
<td>-12.37</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(NN)</td>
<td>13.80</td>
<td>1.55</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: The upper table shows various energy components at the three different radii in unit of MeV, and the lower one is the probabilities of the wave function components.

We perform the variational calculation with the radius constraint and show the numerical result in Table 4.3. We fix the radius at 2.00 fm, 1.70 fm, and 1.40 fm, and present the total energy and various matrix elements, and probabilities of the wave function components.

For the singlet-even channel ($^1E$), since the $NN$ wave functions consists of only the $S$-wave component, there is no $LS$, $L^2$, and $(LS)^2$ matrix elements in this result. We would like to point out the tensor interaction, where all the matrix elements come from the $\Delta$ excitation. As the radius become compact, the tensor matrix elements are enhanced more together with the large kinetic energy. We investigated which state has the most significant role to bring the large tensor matrix elements. We find that the most of the tensor interaction are generated by the coupling of the $^1S(NN)$ state with the $^5D(N\Delta)$ state, and also the $^5D(N\Delta)$ state has the most dominant probability in the $N\Delta$ and $\Delta\Delta$ components in Table 4.3. In the same experience of the deuteron calculation, $^5D(N\Delta)$ with the spin alignment $S=2$ made by the two spin $\frac{3}{2}$ and $\frac{1}{2}$, prefer the more suitable state to $^5D(\Delta\Delta)$ with spin not-alignment by the two spin $\frac{3}{2}$. The kinetic energy and the small central matrix elements are obtained by $NN$ wave function component, which is reflected in the small probabilities of $N\Delta$ and $\Delta\Delta$ components. From these results, we conclude that the $^5D$ state of the $N\Delta$ channel is one of the most considerable state in the $^1E$ channel.

4.4.3 Analyses of the two-body $^1O$ channel

We deal with the singlet-odd channel ($^1O$) for $J=1$ two-body system. Since the singlet-odd channel has the isospin $T=0$, the $N\Delta$ states are forbidden and, the wave function is written by the $NN$ and $\Delta\Delta$ components,

$$ |\Psi_{J=1}\rangle = |\Psi_{NN}\rangle + |\Psi_{\Delta\Delta}\rangle, $$

(4.19)
Table 4.4: Calculated results of $^1O$ channel with AV28

<table>
<thead>
<tr>
<th>Radius [fm]</th>
<th>Energy</th>
<th>Kinetic+Mass</th>
<th>Central</th>
<th>Tensor</th>
<th>$LS$</th>
<th>$L^2$</th>
<th>($LS$)$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>18.01</td>
<td>16.57+0.19</td>
<td>1.64</td>
<td>-0.45</td>
<td>0.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.70</td>
<td>25.60</td>
<td>23.25+0.39</td>
<td>2.76</td>
<td>-0.91</td>
<td>0.11</td>
<td></td>
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<tr>
<td>(NN)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1.40</td>
<td>39.18</td>
<td>34.88+0.73</td>
<td>5.05</td>
<td>-1.69</td>
<td>0.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: The upper table shows various energy components at the three different radii in unit of MeV, and the lower one shows the probabilities of the wave function components.

where,

$$\Psi_{N\Delta} = |\Psi_{N\Delta}\rangle = |^1P\rangle + |^5P\rangle + |^5F\rangle.$$  \hspace{1cm} (4.20)

We present the result in Table 4.4. As the case of the singlet-even channel, the radius is fixed at 2.0, 1.7, and 1.4 fm, respectively. The singlet-odd channel becomes a quite unbound system in comparison with the triplet and singlet-even channels. This is because the kinetic energy are much enhanced by the odd-orbital of the $P$-wave component in the $NN$ space. Also this $P$-wave cannot have the matrix element by the tensor operator, then the tensor matrix elements are caused by the transition of $NN$ into $\Delta\Delta$ state. However this contribution is very small and we can learn the tensor force acts in the odd-orbital less than the even-orbital channel. Additionally we can understand this small tensor interaction from the spin configurations, that is, the $^5P$ and $^3F$ states in $\Delta\Delta$ wave function is not suitable configuration due to the not-alignment of two spin. These tendencies is seen in the small probabilities of the $\Delta\Delta$ state in Table 4.4.

In the analysis of the singlet-odd channel, we conclude that the states in this channel give the small contribution to gain the total energy and play a minor role in many-body system.

4.4.4 Analyses of the two-body $^3O$ channel

We study the triplet-odd ($^3O$) channel for $J = 0, 1, 2$. The wave function is written as,

$$\Psi_{J=0,1,2} = \Psi_{NN} + \Psi_{N\Delta} + \Psi_{\Delta\Delta},$$  \hspace{1cm} (4.21)
where for $J = 0$,

$$
|\Psi_{NN}\rangle = |^3P_0\rangle \quad (4.22)
$$

$$
|\Psi_{N\Delta}\rangle = |^3P_0\rangle \\
|\Psi_{\Delta\Delta}\rangle = |^3P_0\rangle + |^7F_0\rangle
$$

and for $J = 1$,

$$
|\Psi_{NN}\rangle = |^3P_1\rangle \quad (4.23)
$$

$$
|\Psi_{N\Delta}\rangle = |^3P_1\rangle + |^5P_1\rangle + |^5F_1\rangle \\
|\Psi_{\Delta\Delta}\rangle = |^3P_1\rangle + |^7F_1\rangle
$$

and for $J = 2$,

$$
|\Psi_{NN}\rangle = |^3P_2\rangle + |^3F_2\rangle \quad (4.24)
$$

$$
|\Psi_{N\Delta}\rangle = |^3P_2\rangle + |^3P_2\rangle + |^3F_2\rangle + |^5F_2\rangle \\
|\Psi_{\Delta\Delta}\rangle = |^3P_2\rangle + |^7P_2\rangle + |^3F_2\rangle + |^7F_2\rangle + |^7H_2\rangle.
$$

We show the numerical results for $J = 0$ in Table 4.5. Since the triplet-odd channel also has the odd orbital, the kinetic energies enhanced largely. We pay attention to the characteristics of the tensor energy in the results. We consider that the inclusion of the $^7F$-state in the $\Delta\Delta$ wave function enhances the large tensor interaction from the $^3P$-state in $NN$ function. Also it can be seen that the tensor interaction acting on the states between $^3P(N\Delta)$ and $^7F(\Delta\Delta)$ become large in the compact radius. The spin aligned configuration of $^7F$-state with spin $S = 3$ is favored in the tensor interaction.

We also show the calculated results for $J = 1$ and $J = 2$ in Table 4.6. As shown in both tables, there is no special state to contribute in the tensor matrix elements, however, the tensor matrix elements become large in the small radius size, which is caused by the increase of the coupling between the $N\Delta$ and $\Delta\Delta$ states. In $J = 2$ case the $NN$ wave function contains $P$ and $F$-states which can be connected by the tensor operator such as the $S - D$ coupling state in the deuteron, however, the additional $N\Delta$ and $\Delta\Delta$ space do not enhance the tensor matrix elements. It is considered that it is difficult for the odd-orbital configuration to obtain the tensor contributions form $\Delta$ states.

**4.4.5 Summary for the analysis of two-body system**

We have studied the roles of $N\Delta$ and $\Delta\Delta$ states in all the two-body systems for the preparation of three-body calculation. In the phase shift analysis it is difficult to discuss the role of $\Delta$ degrees of freedom in the NN scattering states. In this study we carry out it by constraining the size of the system at a fixed value. We would like to summarize the results of four channels including the triplet-even channel for the deuteron below.

- About $\Delta\Delta$ state
Table 4.5: Calculated results of $^3$O channel for $J = 0$ with AV28

<table>
<thead>
<tr>
<th>Radius [fm]</th>
<th>Energy</th>
<th>Kinetic+Mass</th>
<th>Central</th>
<th>Tensor</th>
<th>$LS$</th>
<th>$L^2$</th>
<th>$(LS)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>14.46</td>
<td>16.62+0.32</td>
<td>0.71</td>
<td>-4.15</td>
<td>0.57</td>
<td>-0.03</td>
<td>0.41</td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.70</td>
<td>20.07</td>
<td>23.18+0.60</td>
<td>1.27</td>
<td>-6.87</td>
<td>1.12</td>
<td>-0.07</td>
<td>0.82</td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.40</td>
<td>30.34</td>
<td>34.76+1.23</td>
<td>2.58</td>
<td>-12.44</td>
<td>2.49</td>
<td>-0.19</td>
<td>1.88</td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.5: The upper table shows the various energy components at the three different radii in units of MeV, and the lower one is the probabilities of the wave function components.

Table 4.6: Calculated results of $^3$O channel for $J = 1$ with AV28

<table>
<thead>
<tr>
<th>Radius [fm]</th>
<th>Energy</th>
<th>Kinetic+Mass</th>
<th>Central</th>
<th>Tensor</th>
<th>$LS$</th>
<th>$L^2$</th>
<th>$(LS)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>18.63</td>
<td>17.01+0.27</td>
<td>0.41</td>
<td>0.59</td>
<td>0.21</td>
<td>-0.009</td>
<td>0.13</td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.70</td>
<td>26.73</td>
<td>23.90+0.54</td>
<td>0.79 0.72</td>
<td>0.48</td>
<td>-0.02</td>
<td>0.31</td>
<td></td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.40</td>
<td>42.04</td>
<td>36.20+1.33</td>
<td>1.78</td>
<td>0.58</td>
<td>1.34</td>
<td>-0.08</td>
<td>0.88</td>
</tr>
<tr>
<td>(NN)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

51
Firstly the most important channel is the triplet-even channel for the deuteron. The large tensor interactions come from not only the \(NN\) space but also the coupling of the \(\Delta\Delta\) states. We find that the \(^7D_1\) state in the \(\Delta\Delta\) wave function play a significant role in the \(N \rightarrow \Delta\) transition. In other channels, we discuss the triplet-odd channel \((T=1)\). Because this channel has the odd-orbital, it is hard to take the tensor matrix elements in comparison with the even-orbital channels, however, we have to pay attention to the \(^7F_1\) state which is a preferable configuration for the tensor interaction.

- About \(N\Delta\) state
  We should take into account for the \(N\Delta\) state in only the \(T = 1\) isospin channel. In the calculation of the singlet channel, we show that the \(^5D_0\) state is important, where the coupling \(^5D_0(N\Delta)\) with \(^1S_0(NN)\) generates the large tensor matrix elements.

In the study of all the two-body channels, we specify the efficient channels of the \(\Delta\) and \(N\Delta\) states to obtain the large tensor matrix elements. We should take into account these configurations in the calculation of three-body system.

### 4.5 Triton including the \(\Delta\) excitation explicitly

#### 4.5.1 Previous study

In this section we investigate the triton including the \(\Delta\) degrees of freedom. First, we would like to overview two previous studies about the three-body calculation with the
Δ excitation.

The Hannover group has developed a framework to treat explicitly Δ isobar state in few-body nuclei, where they based on the CD Bonn potential [55–58], [5]. They expanded the model space up to the excitation of a single nucleon to a Δ so as to treat the additional Δ effect, and performed the coupled-channel calculation by using the Faddeev method. This means that their framework can take the NΔ interactions but not ΔΔ interactions. Then, they showed that the reproduction of the triton binding energy cannot reach the experimental data. Deltuva et al present the calculation results of the binding energy, where a half of the rest energies from two-body force is gained by the inclusion of the NΔ state. They mentioned the effects of double nucleon excitation to two Δ are small due to the large mass difference of ΔΔ states about 600 MeV, however, we consider the shortage of this energy would be recovered including the ΔΔ state. This is because in the deuteron study, we indicates the matrix element of the larger mass difference reduces to small about 3 MeV, and also the large tensor matrix elements come from the ΔΔ excitation. Hence it is important to take into account the model space by including the ΔΔ state.

The other one is Los Alamos group. Picklesimer et al studied the triton with Δ degrees of freedom using the AV28 potential [59–64]. They work out the Faddeev calculation and report that they cannot reproduce the triton binding energy in a series of their study. However we point out their truncation of the model space, in which they restrict angular momenta up to \( L(NΔ), L(ΔΔ) \leq 2 \) so as to reduce the size of the three-body calculations. This means that this model space does not take the G-wave component of the ΔΔ function whose probability in deuteron is 0.04%, very small. This contribution seems to be small, but we discuss how much their treatment is valid in the next section.

Argonne group proposing the AV28 potential has moved into treating the Δ effect as the three-body force in only the NN space without using AV28. We consider that this is because the negative conclusion by Los Alamos group and, the Δ channel requires quit a large configurations to calculate the wave functions of many-body systems. Therefore it is a present status that the reproduction of three-body system has not been successful so far by treating the explicitly Δ degrees of freedom. The relation between the phenomenological three-body force based on the Fujita-Miyazawa three-body force and the effects of Δ isobar has not been achieved from the microscopic view point. Hence we try to accomplish the triton calculation without the truncation of ΔΔ and NΔ space using AV28. This study is important to get the fundamental knowledge of the role of Δ in finite nuclei, also to get the foundation of the effective three-body force.

4.5.2 G-wave effects of the ΔΔ state in deuteron

As was mentioned in the last subsection, Picklesimer et al reduced the angular momentum up to \( L(NΔ), L(ΔΔ) \leq 2 \) in the calculation of triton. Therefore we would like to find out the validity of this assumption and the effect of G-wave in the deuteron case.

As shown in Eq. (4.13), there is a \( 7G_1 \) state in the ΔΔ wave function, which has the small probability, 0.04%. The G-wave state can be coupled with the D-wave state by a
Table 4.8: Calculated results of the deuteron with and without the $G$-wave component.

<table>
<thead>
<tr>
<th>$^2\text{H}$</th>
<th>Energy</th>
<th>Kinetic+Mass</th>
<th>Central</th>
<th>Tensor</th>
<th>$LS$</th>
<th>$L^2$</th>
<th>$(LS)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full</td>
<td>-2.25</td>
<td>20.87+3.09</td>
<td>8.77</td>
<td>-35.33</td>
<td>0.84</td>
<td>3.55</td>
<td>-4.05</td>
</tr>
<tr>
<td>Full(NN)</td>
<td></td>
<td>19.45</td>
<td>8.62</td>
<td>-25.42</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>w/o $G$</td>
<td>-1.66</td>
<td>17.03+2.39</td>
<td>6.94</td>
<td>-28.29</td>
<td>0.65</td>
<td>2.75</td>
<td>-3.14</td>
</tr>
<tr>
<td>w/oG(NN)</td>
<td></td>
<td>16.10</td>
<td>6.85</td>
<td>-20.95</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P\left[%\right]$</td>
<td>$^3\text{S}(NN)$</td>
<td>$^3\text{D}(NN)$</td>
<td>$^3\text{S}(\Delta\Delta)$</td>
<td>$^3\text{S}(\Delta\Delta)$</td>
<td>$^3\text{D}(\Delta\Delta)$</td>
<td>$^3\text{G}(\Delta\Delta)$</td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td>93.31</td>
<td>6.16</td>
<td>0.04</td>
<td>0.02</td>
<td>0.42</td>
<td>0.04</td>
<td></td>
</tr>
<tr>
<td>w/o $G$</td>
<td>94.38</td>
<td>5.21</td>
<td>0.03</td>
<td>0.01</td>
<td>0.36</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8 : The upper table shows various energy components in unit of MeV, and the lower one shows the probabilities of each wave function components for deuteron with AV28. The label " w/o $G$ " means the results of $G$-wave truncation for the deuteron wave function.

tensor operator, while it is not connected from the $S$-wave state by a tensor operator,

$$
\langle \Psi_S | S_{12} | \Psi_G \rangle = 0
$$

$$
\langle \Psi_D | S_{12} | \Psi_G \rangle \neq 0.
$$

Hence, we should see that the $G$-wave state gives the sizable effect on the deuteron binding energy and tensor matrix elements regardless of the small probability. We compare two results with and without $G$-wave component. We show the results of deuteron with the AV28 interaction in Table 4.8 and Fig. 4.9. The total energy without $G$-wave drops down to $-1.66$ MeV, which is not small. Furthermore, it can be seen that all the matrix elements vary dramatically as shown in Fig. 4.9. The tensor matrix elements is reduced to $-28.29$ MeV versus the full calculation of $-35.33$ MeV, where the part of the $NN$ space is also small value of $-20.95$ MeV. We note that the tensor matrix elements are not negligible, which generated from the coupling of the $D$-wave of $NN$ state with the $G$-wave of $\Delta\Delta$ state. The kinetic and central energies also change from $20.87$ MeV to $17.03$ MeV, and from $8.77$ MeV to $6.94$ MeV respectively. We found that the result without $G$-wave effects on the probability not only for the $D$-wave of $NN$ state but also for the $\Delta\Delta$ one. This means that the inclusion of $G$-wave improves the $D$-wave component of $NN$ state and the $\Delta\Delta$ wave functions to provide the exactly total binding energy. These roles of the $G$-wave enhance the tensor interaction as well as the kinetic energy and the other elements.

From these results, we understand the contributions of $G$-wave component are necessary to reproduce the binding energy. The role of $G$-wave should be important in the three-body system. Hence it is suspected that suspect the validity of $L \leq 2$ truncation is related to the $\Delta\Delta$ model space by Los Alamos group. We have to take care of all the partial wave components of $\Delta\Delta$ channel for the triton.
Figure 4.9: Calculated results of the deuteron with AV28.

Fig. 4.9: Various energy components for the deuteron with AV28. The left hand side shows the full space of deuteron wave function and the right hand side shows the results without the G-wave component. Those energy components are the kinetic energy plus mass difference, central force, and tensor force. The dotted lines means the energy values given by the NN space.

### 4.5.3 Framework of the triton calculation

We discuss the triton wave function including explicitly $\Delta$ excitation. The triton has spin $J = 1/2$ and isospin $T = 1/2$. The wave function consists of the N spaces, the single $\Delta$ excitation space, and the double $\Delta$ excitation space,

$$|\Psi_{J=1/2}\rangle = |\Psi_{NNN}\rangle + |\Psi_{N\Delta N}\rangle + |\Psi_{\Delta\Delta N}\rangle. \quad (4.26)$$

Although it is possible that the wave function has the triple $\Delta$ excitation ($|\Psi_{\Delta\Delta\Delta}\rangle$), we start with taking into account the $\Delta$ excitation up to double $\Delta$ states. Because it was reported that the contributions of the $\Delta\Delta\Delta$ component are negligibly small, which is about few keV in Ref. [62]. We may consider the $\Delta\Delta$ effects depending on the calculated results.
To solve the triton system, we perform the variational calculation, where the wave functions are described as,

\[
\Psi_{NNN} = \sum_i C_i^{NN} \psi_i(A_i^{NN}, x) \chi_i^{NN}(\sigma) \chi_i^{NN}(\tau) \tag{4.27}
\]

\[
\Psi_{N\Delta N} = \sum_i C_i^{N\Delta} \psi_i(A_i^{N\Delta}, x) \chi_i^{N\Delta}(\sigma, \Sigma) \chi_i^{N\Delta}(\tau, \theta) \tag{4.28}
\]

\[
\Psi_{\Delta\Delta N} = \sum_i C_i^{\Delta\Delta} \psi_i(A_i^{\Delta\Delta}, x) \chi_i^{\Delta\Delta}(\sigma, \Sigma) \chi_i^{\Delta\Delta}(\tau, \theta) .
\]

We should take care the antisymmetrized operator for all wave functions. Of course the NNN function should be taken into account the permutation of the three nucleons. On the other hand the antisymmetrized operator acts on the only two nucleons in the N\Delta N function, and the two deltas in the \Delta\Delta N function. The spin and isospin functions are expressed as,

\[
\chi_s^{NN} = \left[ \chi_i^1(\sigma_1) \times \chi_i^1(\sigma_2) \right]_{s_{12}} \times \chi_i^1(\sigma_3) \tag{4.28}
\]

\[
\chi_s^{N\Delta} = \left[ \chi_i^1(\sigma_1) \times \chi_i^1(\Sigma) \right]_{s_{12}} \times \chi_i^1(\sigma_3) ,
\]

\[
\chi_s^{\Delta\Delta} = \left[ \chi_i^1(\Sigma_1) \times \chi_i^1(\Sigma_2) \right]_{s_{12}} \times \chi_i^1(\sigma_3) ,
\]

and

\[
\chi_i^{NN} = \left[ \chi_i^1(\tau_1) \times \chi_i^1(\tau_2) \right]_{t_{12}} \times \chi_i^1(\tau_3) \tag{4.29}
\]

\[
\chi_i^{N\Delta} = \left[ \chi_i^1(\tau_1) \times \chi_i^1(\theta_2) \right]_{t_{12}} \times \chi_i^1(\tau_3) ,
\]

\[
\chi_i^{\Delta\Delta} = \left[ \chi_i^1(\theta_1) \times \chi_i^1(\theta_2) \right]_{t_{12}} \times \chi_i^1(\tau_3) .
\]

We show the Jacobi coordinates with vector \( x = (x_1, x_2) \) for each wave function in Fig. 4.10. In the study of G-wave effect, we mentioned the importance of the G-wave contribution to provide the exact deuteron binding energy. Hence it is important to include the angular momentum of \( \ell(N\Delta) \leq 4 \) and \( \ell(\Delta\Delta) \leq 4 \) in the vector \( x_1 \). For the spin and isospin wave function, there are lots of configuration in the N\Delta N and \Delta\Delta N states. However we have specified the efficient configurations in the N\Delta and \Delta\Delta two-body system to generate the large tensor interaction in the previous section. It should be a good idea that we make a priority to taking these configurations in three-body system.
so as to obtain the convergence of energy early. Then we would like to perform the variational calculation of triton with the AV28 potential, which includes all the processes of the explicit \( \Delta \) excitation with 28 operators as shown in Fig. 4.3. The total Hamiltonian is given as,

\[
H = (m_1 + m_2 + m_3 - 3m_N) + \sum_{i=1,2,3} \frac{P_i^3}{2m_i} - T_{cm} + V_{12} + V_{13} + V_{23}.
\]  

(4.30)

We have to determine the variational parameters \( (A_i^{NN}, A_i^{N\Delta}, A_i^{\Delta\Delta}) \) in Eq.( 4.27) as the same manner of Stochastic Variational Method. Now the calculation is in progress.

4.6 Summary of this chapter

We would like to treat the effect of the three-body force in our framework such as TOFM and TOSM to discuss nuclear structures in a more quantitative way. Since the pion exchange plays significant roles in the transition between nucleon and \( \Delta(1232) \) isobar, we develop the framework by treating the \( \Delta \) degrees of freedom explicitly in two-body correlation as the origin of three-body force. It is important to take into account the only reasonable configurations to obtain large tensor correlations for the difficulty of treating large model space of \( \Delta \) excitation, which is suitable for the TOFM framework. Hence we first examine the effect of the \( \Delta \) excitation in two-body systems, and study the role of \( \Delta \) and the relation with the tensor interaction in many-body system.

The two-body potential with the \( \Delta \) degrees of freedom has already proposed by Argonne group, which is named as AV28. We work out the role of \( \Delta \) by using AV28. First, we start from the deuteron. Since the deuteron has isospin \( T = 0 \), the \( \Delta\Delta \) wave function should be added to the \( NN \) space. We have shown the deuteron results not only for total energy and the probabilities of the wave function component, but also for the various energy components of Hamiltonian. We can reproduce the total energy and, obtain twice as large the tensor matrix elements in comparison with the AV14 results which does not include the \( \Delta \) excitation. This result shows that the \( N \leftrightarrow \Delta \) transitions play significant roles to enhance the tensor contribution. Hence we understand that the \( \Delta\Delta \) states have large effects on the nuclear structure by involving high-momentum components through the pion-exchange, even if the large mass difference of 600 MeV from two nucleon.

Second, we analyze all of the two-body channels with possible spin and isospin to find out the roles of the \( N\Delta \) and other \( \Delta\Delta \) states. Except for the triplet-even of the deuteron channel, these channels form only the unbound states. Therefore, we adopt the method of radius constraint for these systems to study various energy components at a fixed radius. As a result, we find that the \( N\Delta \) state in the singlet-even generates the large tensor interaction by the coupling with \( NN \) state, and the tensor interaction enhance larger in the even-orbital than in odd-orbital. Moreover we show that there is a favorable \( \Delta\Delta \) configuration for the tensor force in the triplet-odd channel. Taking these points carefully into account, we should give a priority to include these important channels for the calculation of three-body and many-body systems.

Next we start to deal with the triton calculation with the AV28 potential. The previous studies have not succeeded for the reproduction of the triton energy by including
the Δ isobar degrees of freedom. Then we note the importance of the ΔΔ state and the validation of the truncation of the model space related to Δ state. We discuss the $G$-wave truncation of the ΔΔ wave function in the results of deuteron, and conclude that the contribution of the $G$-wave is necessary to get the exact binding energy with large tensor interaction. From these results, we try to calculate the triton energy with the variational method, where the model space needs a lot of configurations by adding $N\Delta$ and ΔΔ states. We would like to develop the triton calculation in terms of the spirit of TOFM.
Chapter 5

Summary and Outlook

5.1 Summary

In this section we summarize this thesis. We study the nuclear structure with bare $NN$ interaction, where the difficulty of treating the nuclear interaction is the presence of strong short-range repulsion and strong tensor interaction. We state that the treatment of the tensor force is essential for the treatment of the one of the pion-exchange force, and review the effects of the bare $NN$ interaction. As discussed using the results of deuteron with the bare $NN$ interaction, the tensor force is the necessary elements to bind the nuclei, and also generates high-momentum components. In addition, it is necessary to include the effects of three-body force in many-body system. It is hard to describe these effects caused by the bare $NN$ interaction in a conventional nuclear model by using the effective interaction. In particular, we will make efforts to study high-momentum components in finite nuclei. Hence it is important to understand the nuclear structure with the bare $NN$ interaction, and to develop the theoretical framework which can treat the $NN$ interacting directly in medium and heavy mass nuclei. We propose tensor optimized few-body model (TOFM) to provide the foundation of the framework in terms of few-body system.

In this study, we apply the TOFM to the s-shell and the p-shell nuclei. The physical concept of TOFM comes from tensor optimized shell-model (TOSM) which is powerful method to study medium and heavy nuclei with the bare $NN$ interaction. The two methods of TOFM and TOSM can describe the deuteron-like tensor correlation bringing high momentum components in finite nuclei. TOFM is the efficient and simple model to obtain the tensor interaction using the few-body method.

It is shown that TOFM reproduces about 95% of the total binding energy and the tensor contribution of the few-body results for the s-shell nuclei. The universal properties of the spatial correlation functions of the S- and D-wave components, which are connected by the tensor interaction, are discussed among the s-shell nuclei. From these results we conclude that TOFM is good enough to use as an approximation in the variational calculation for the total binding energy and the TOFM wave function can treat the property of the $NN$ interaction, in particular the tensor interaction. This study is very encouraging to extend our TOFM framework to the nuclei beyond s-shell. We also compare the TOFM calculation with the TOSM one in $^4\text{He}$ to investigate what should
be done to improve the TOSM framework. We note the defect of TOSM is the treatment of the short-range correlation in UCOM, and the modification of the UCOM function form would improve the present TOSM result.

We further extend TOFM framework to the p-shell region, such as $^5$He, and show the TOFM results in case of $^5$He nucleus. We work out the variational calculation for $^5$He and discuss the structure difference between two resonance states $3/2^-$ and $1/2^-$. In particular, we shed light on the roles of the tensor interaction in two states, such as the $LS$ splitting energy. On the other hand we learned the importances of three-body force in the reproduction of the $^5$He structure. The effects of three-body force increase as the nuclear mass number becomes larger, and are not negligible to act on the nuclear structure.

We further study the effects of many-body forces to take into account these effects in our framework of TOFM and TOSM. We introduce the explicitly $\Delta(1232)$ isobar degrees of freedom in two-correlation levels, which can be the origin of the three-body forces via pion exchange. We find the additional $N\Delta$ and $\Delta\Delta$ states generate large tensor matrix elements, and change the various matrix elements from the results of only the $NN$ space. It turns out that $\Delta$ plays significant roles in nuclear structure through the pion exchange. TOFM and TOSM can also handle the tensor correlations generated from the $\Delta$ state as well as the two-correlation of nucleons. In the study of three-body system, we address $\Delta$ effects as the origin of three-body force from the explicit treatment of the $\Delta$ degrees of freedom.

5.2 Outlook

In this section we would like to discuss the extensions of this thesis. It is important to develop the TOFM framework and apply it to larger mass nuclei beyond the existing few-body calculation. In this study we studied up to $^5$He as the first stage of the p-shell nuclei. However the larger mass nuclei show various features of nuclear structures, such as shell structure and cluster structure. We would like to examine how these structures are constructed by the nuclear force.

For example in six-body system, it is well-known that $^6$Li makes a cluster structure of $^4$He and deuteron, while $^6$He has two-neutron halo structure. Those are the important nuclei as regarding the first appearance of the cluster and the halo structures. It is interesting to investigate the difference of these structures between $^6$Li and $^6$He in the few-body framework. In particular, the difference between the valence two-nucleon pair ($pn$ and $nn$), would play significant roles in these structures, where a $pn$ pair has tensor correlation, on the other hand a $nn$ pair does not have it. Hence it is important that we consider the six-body structure in terms of the relation between the internal structure of $^4$He and the valence two nucleons with the bare $NN$ interaction.

For the medium and heavy nuclei beyond the few-body framework, It is suitable for studying those nuclei by means of the TOSM framework. Due to the shell-model basis, TOSM is a powerful method to describe heavier nuclei. However TOSM requires the treatment of the short-range repulsion in some way, it is important to take the appropriate correlation function. We have to study further the short-range correlation function in UCOM to obtain the improvements of the TOSM calculation.
It should be noted the importance of the three-body force. In particular these effects would become large as the nuclear mass number increase. We would like to establish the new treatment of the many-body effects by including explicitly $\Delta$ excitations from more microscopic $NN$ interaction, and to develop the treatment of the explicit $\Delta$ degrees of freedom in the framework of TOFM, TOSM, and EBFH. In this study we used the AV28 potential model, however, it would be possible to consider whether this model is valid or not on the procedure of the $\Delta$ study. Furthermore it is interesting not only to discuss the roles of $\Delta$ in heavier nuclei but also to address the various problems of three-body force in terms of the $\Delta$ degrees of freedom [65–68].
Bibliography


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Appendix A

Various matrix elements for the base function

In this appendix we write the various matrix elements for the correlated Gaussian basis function and Ref [22, 25] discuss the method of calculation in more detail.

First we focus on the matrix elements of wavefunction for s-shell nuclei. For s-shell nuclei the TOFM wave function is written as a linear combination of S-wave and D-wave. Here we describe the spatial wave functions for S-wave and D-wave again. The S-wave function is written as

\[ f_0(x, \lambda) = \exp \left( -\frac{1}{2} \lambda x \right), \]

and the D-wave function has the global vector representation

\[ f_{2M}(x, A, u) = \exp \left( -\frac{1}{2} \lambda x \right) \tilde{Y}_{2M}(\tilde{u}x). \]

A.1 Overlap of the basis functions

First we show the overlap matrix elements. In the calculation of S-wave function, it become very simple

\[ \langle f_0(x, \lambda') | f_0(x, \lambda) \rangle = \left( \frac{2\pi}{\det B} \right)^{3/2} (2\pi)^{N-1} \]

with \( A + A' = B \) and \( N \) is the number of particles. Here the variational parameter \( \lambda \) includes the transformation of antisymmetrization.

In the case of D-wave function, the calculation of matrix elements is convenient by introducing the generating function

\[ g_{2M}(x, A, u) = \exp \left( -\frac{1}{2} \lambda x + \tilde{s} x \right), \]

which contains the column vector \( \tilde{s} = (s_1, s_2, ...) \) and \( s \) is defined by \( s = u\lambda e \) with unit vector \( e \). The relation between the correlated Gaussian basis function (A.2) and the
generating function (A.4) is provided as
\[ f_{2M}(x, A, u) = \frac{1}{B_2} \int de Y_{2M}(e) \left( \frac{\partial^L}{\partial \lambda L} g_{2M}(x, A, u) \right)_{\lambda=0, |e|=1} \] (A.5)

where
\[ B_2 = \frac{8\pi}{15} \] (A.6)

Using this relation the overlap matrix elements is given by
\[ \langle f_{2M}(x, A', u')|f_{2M}(x, A, u) \rangle = \left( \frac{1}{B_2} \right)^2 \int de d\lambda d\lambda' \left( \frac{\partial^{L+L'}}{\partial \lambda' \partial \lambda} \langle g_{2M}(x, A', u')|g_{2M}(x, A, u) \rangle \right)_{\lambda=\lambda'=0, |e|=|e'|=1} \] (A.7)

To calculate (A.7) we have to firstly calculate the matrix element of generating function \( g_{2M} \) and this calculation become simple due to the Gaussian form,
\[ \langle g_{2M}(x, A', u')|g_{2M}(x, A, u) \rangle = \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2} \exp \left( q\lambda^2 + q'\lambda'^2 + \rho \lambda \lambda' e \cdot e' \right) \] (A.8)

where
\[ q = \frac{1}{2} \bar{u} B^{-1} u, \quad q' = \frac{1}{2} \bar{u}' B^{-1} u', \quad \rho = \bar{u}' B^{-1} u. \] (A.9)

Then, we expand \( \exp \left( q\lambda^2 + q'\lambda'^2 + \rho \lambda \lambda' e \cdot e' \right) \) in power of \( \lambda \) and \( \lambda' \) terms in order to perform the differentiation with respect to \( \lambda \) and \( \lambda' \). In this expansion, the only \( \lambda^2 \lambda'^2 (e \cdot e')^2 \) term does not vanish and, we can integrate over the angle \( e \) and \( e' \) by following this relation,
\[ (e \cdot e')^2 = B_2 \sum_{m=-2}^2 Y_{2m}^*(e)Y_{2m}(e') + B_{10} Y_0^*(e)Y_0(e') \] (A.10)

where
\[ B_{10} = \frac{4\pi}{3} \] (A.11)

Finally the overlap matrix elements of D-wave function can be obtained as
\[ \langle f_{2M}(x, A', u')|f_{2M}(x, A, u) \rangle = \frac{1}{B_2} \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2} \rho^2 . \] (A.12)
A.2 Kinetic energy

We discuss the matrix element of the kinetic energy. The kinetic energy operator with the center-of-mass kinetic energy subtracted is expressed as

\[
\sum_{i=1}^{N} \frac{p_i^2}{2m_i} - T_{cm} = \sum_{i=1,j=1}^{N} \left( \frac{1}{2m_i} \delta_{ij} - \frac{1}{2m_1+2+\ldots+N} \right) p_i \cdot p_j
\]

\[
= \frac{1}{2} \sum_{i=1,j=1}^{N-1} \Lambda_{ij} \pi_i \cdot \pi_j = \frac{1}{2} \pi \Lambda \pi ,
\]

where,

\[
\pi_i = -i\hbar \frac{\partial}{\partial x_i} ,
\]

and the matrix \( \Lambda_{ij} \) is constructed by the matrix of the Jacobi coordinate set \( U_{ij} \)

\[
\Lambda_{ij} = \sum_{k=1}^{N} U_{ik} U_{jk} \frac{1}{m_k} ,
\]

where the matrix of Jacobi coordinate set of \( U \) is

\[
U = \begin{pmatrix}
\frac{1}{m_1} & -1 & 0 & \cdots & 0 \\
\frac{m_2}{m_1+m_2} & \frac{m_2}{m_1+m_2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\frac{m_N}{m_1+m_2+\ldots+m_N} & \frac{m_N}{m_1+m_2+\ldots+m_N} & \cdots & \frac{m_N}{m_1+m_2+\ldots+m_N} & \frac{m_N}{m_1+m_2+\ldots+m_N}
\end{pmatrix} .
\]

For \( S \)-wave function, the matrix element of the kinetic energy is

\[
\langle f_0(x,A') | \frac{1}{2} \pi \Lambda \pi | f_0(x,A) \rangle = \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2} \frac{3}{2} \hbar^2 Tr(AB^{-1}A') .
\]

For \( D \)-wave function, we can obtain the matrix element by following the similar way of overlap matrix elements. Firstly we calculate the matrix element of the generating function,

\[
\langle g_{2M}(x,A',u') | \frac{1}{2} \pi \Lambda \pi | g_{2M}(x,A,u) \rangle
\]

\[
= \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2} [R + P \lambda^2 + P' \lambda'^2 + R \lambda \lambda' \cdot e \cdot e']
\]

\[
\exp (q \lambda^2 + q' \lambda'^2 + \rho \lambda \lambda' \cdot e \cdot e') ,
\]

where

\[
R = 3Tr(AB^{-1}A' \Lambda) , \quad P = -\bar{u}B^{-1}A' \Lambda AB^{-1} u ,
\]

\[
P' = -\bar{u}B^{-1}A' \Lambda AB^{-1} u' , \quad Q = 2\bar{u}B^{-1}A' \Lambda AB^{-1} u .
\]
Then we can derive the matrix element of kinetic energy for $D$-wave function,

$$
\langle f_{2M}(x,A',u') | \frac{1}{2} \pi \Lambda \pi | f_{2M}(x,A,u) \rangle
$$

$$(A.19)
$$

$$
= \frac{1}{B^2} \frac{\bar{r}^2}{2} (R^2 + 2Qp) \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2}.
$$

A.3 Two-body interaction

Next we discuss the matrix element of two-body interaction, especially for the central interaction and the tensor interaction and the spin-orbit interaction, which are components of the AV8' potential.

Then we assume the interaction acting between particle $i$ and $j$ as $V(r_i - r_j)$. The relative distance vector $r_i - r_j$ is able to express as the linear combination of the relative coordinate vector $x$ with the column vector $w$,

$$
r_i - r_j = \sum_{k=1}^{N-1} w_k x_k = \tilde{w} x.
$$

(A.20)

Using this relation and $\delta$-function, the potential $V(r_i - r_j)$ can be described as,

$$
V(r_i - r_j) = \int d\bar{r} V(\bar{r}) \delta(\tilde{w} x - \bar{r}).
$$

(A.21)

Through the calculation of the matrix elements for the $\delta$-function, we can obtained the two-body matrix elements. Here we assume that the radial form of the potential $V(r)$ can be expressed as the sum of the Gaussian functions with the range of $c'$. In fact we expand the potential such as AV8' into the sum of the Gaussian functions $\exp(-c'r^2)$, or $r^2 \exp(-c'^2/2)$ for the numerical calculation of these matrix elements.

A.3.1 Central interaction

At first we discuss the matrix element of the central interaction for $S$-wave function. We assume that the central force is described as the Gaussian functions,

$$
\langle f_0(x,A') | V(r_i - r_j) | f_0(x,A) \rangle = \left( \frac{(2\pi)^{N-2}c}{\det B} \right)^{3/2} \int d\bar{r} e^{-\frac{1}{2}(c'+c)r^2}.
$$

(A.22)

$$
= \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2} \left( \frac{c}{c + c'} \right)^{3/2},
$$

where

$$
\frac{1}{c} = \tilde{w} B^{-1} w.
$$

(A.23)

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In the calculation of D-wave function, the matrix elements of the \( \delta \)-function is given as,

\[
\langle g_{2M}(x, A', u') | \delta(\tilde{w}x - r) | g_{2M}(x, A, u) \rangle = \left( \frac{(2\pi)^{N-2}c}{\det B} \right)^{3/2} \exp \left[ -\frac{1}{2} c (r - \tilde{w}B^{-1}(s + s'))^2 \right] \frac{(2\pi)^{N-2}c}{\det B} \exp \left( -\frac{1}{2} c r^2 + \tilde{q}\lambda^2 + q^2 + \tilde{\rho}\lambda\lambda' \cdot e' + \gamma\lambda \cdot e + \gamma'\lambda' \cdot e' \right),
\]

where

\[
\gamma = c\tilde{w}B - 1u, \quad \gamma' = c\tilde{w}B^{-1} u', \quad \tilde{q} = q - \frac{1}{2c}\gamma^2, \quad \tilde{q}' = q' - \frac{1}{2c}\gamma'^2, \quad \tilde{\rho} = \rho - \frac{1}{c}\gamma'\gamma.
\]

Hence we can get the matrix elements of the central interaction for D-wave,

\[
\langle f_{2M}(x, A', u') | V(\tilde{r}_i - r_j) | f_{2M}(x, A, u) \rangle = \frac{1}{B_2} \left( \frac{(2\pi)^{N-1}c}{\det B} \right)^{3/2} \left( \frac{c}{c + c'} \right)^{3/2} \left( \rho - \frac{c'}{c + c'} \gamma' \right)^2
\]

A.3.2 Tensor interaction

Next we discuss the matrix element of the tensor interaction, where the tensor operator is,

\[
S_{ij} = 3(\sigma_i \cdot \hat{r})(\sigma_j \cdot \hat{r}) - \sigma_i \cdot \sigma_j = \sqrt{\frac{24\pi}{5}} \langle Y_2(\hat{r})|\sigma_i \times \sigma_j|2 \rangle.
\]

As for the spatial part, the matrix elements of generating function is given as

\[
\langle g_{L'}(x, A', u') | \delta(\tilde{w}x - r)Y_{2u}(\tilde{w}x) | g_L(x, A, u) \rangle = \left( \frac{(2\pi)^{N-2}c}{\det B} \right)^{3/2} Y_{2u}(r) \exp \left[ -\frac{1}{2} c r^2 \right] \exp[q^2\lambda^2 + q'^2\lambda'^2 + \tilde{\rho}\lambda\lambda' \cdot e' + \gamma\lambda \cdot e + \gamma'\lambda' \cdot e']
\]

and then,

\[
\langle f_{LM}(x', A', u') | V(\tilde{w}x - r)Y_{2u}(\tilde{w}x) | f_{LM}(x, A, u) \rangle = \left( \frac{(2\pi)^{N-2}c}{\det B} \right)^{3/2} \frac{1}{B_L B_{L'}} \int dr \exp \left[ -\frac{1}{2} c r^2 \right] V(r) Y_{2u}(r) \int de d\epsilon Y_{L'M'}^*(e') Y_{LM}(\epsilon) \frac{\partial L \partial L'}{\partial \lambda \partial \lambda'} \exp[q^2\lambda^2 + q'^2\lambda'^2 + \tilde{\rho}\lambda\lambda' \cdot e' + \gamma\lambda \cdot e + \gamma'\lambda' \cdot e'] \bigg|_{\lambda = \lambda' = 0, |\epsilon| = |\epsilon'| = 1}.
\]
For example the matrix element of the tensor for between $D$-wave and $D$-wave can be obtained in this way,

$$
\langle f_{2M}(u', A', x) | V(\hat{\omega}x - r) Y_{2u}(\hat{\omega}x) | f_{2M}(u, A, x) \rangle
$$

$$
= \sum_{MM', u} \frac{1}{B_2} \left( \frac{2\pi}{\text{det} B} \right)^{3/2} [4\tilde{q} \gamma' B_1^3 (CG)_1 \left( \frac{3}{10\pi} \right)^{3/2} \int \text{d}rr^2 r^2 e^{-\frac{1}{2}cr^2} V(r)
\right.
$$

$$
\left. + \gamma^2 r^2 B_2^2 \sqrt{\frac{5}{14\pi}} (CG)_2 \int \text{d}rr^2 r^4 e^{-\frac{1}{2}cr^2} V(r) \right],
$$

here

$$(CG)_1 = \sum_{m_1} (-1)^{m_1 + M'} (1M - m_1 1 - M + m_1 | 2u)$$

(A.31)

$$(1M' - m_1 1m_1 | 2M')(1m_1 1M - m_1 | 2M)$$

$$(CG)_2 = (-1)^{M + 1} (2 - M' 2M | 2u),$$

and

$$B_1 = \frac{4\pi}{3}.$$  

(A.32)

### A.3.3 Spin-orbit interaction

Next we derive the matrix element of the spin-orbit interaction. The spin-orbit operator is

$$(L \cdot S)_{ij} = \left( \left( r_i - r_j \right) \times \frac{1}{2\hbar} (p_i - p_j) \right) \cdot \frac{1}{2} (\sigma_i + \sigma_j)$$

(A.33)

$$= \left( \frac{1}{\hbar} (r \times p) \cdot \frac{1}{2} (\sigma_i \cdot \sigma_j) \right),$$

and has the orbital angular momentum

$$\ell_u = \frac{1}{\hbar} (r \times p)_u$$

(A.34)

$$= \frac{1}{\hbar} (\hat{\omega}x \times \tilde{\zeta} \pi)_u,$$

where the column vector $\zeta$ is constructed by the matrix of the Jacobi coordinate set $U_{ij}$

$$\frac{1}{2} (p_i - p_j) = \hat{\omega} \pi = \frac{1}{2} \sum_{k=1}^{N-1} (U_{ki} - U_{kj}) \pi_k.$$  

(A.35)

Then the matrix elements of the generating function is

$$\langle g_{2M}(x, A', A') | \delta(\hat{\omega}x - r) (\hat{\omega}x \times \tilde{\zeta} \pi)_u | g_{2M}(x, A, u) \rangle$$

(A.36)

$$= \left( \frac{2\pi}{\text{det} B} \right)^{3/2} Y_{2u}(r) \exp \left[ -\frac{1}{2} cr^2 \right] (-i\hbar)
$$

$$\left( r \times \tilde{\zeta} (A'B^{-1} s - AB^{-1} s') + \tilde{\zeta} AB^{-1} wc (r \times \hat{\omega}B^{-1} (s + s')) \exp[\tilde{q}^2 \lambda^2 + \tilde{q}^2 \lambda'^2 + \tilde{p} \lambda' e' \cdot e' + \gamma \lambda e \cdot r + \gamma' \lambda' e' \cdot r] \right),$$

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and then we can calculate the matrix elements of the spin-orbit interaction

\[
\langle f_{2M}(x,A',u')|V(\hat{w}x - r)(\hat{w}x \times \hat{\gamma} \pi)|f_{2M}(x,A,u)\rangle = \left(\frac{(2\pi)^{N-2}}{\det B}\right)^{3/2} \frac{1}{B_2^2} \int dr V(r) \exp[-\frac{1}{2}cr^2] \int d\lambda e^{\gamma} Y_{2M}^\ast(\epsilon') Y_{2M}(\epsilon) \exp[g^2 \lambda^2 + \bar{g}^2 \lambda'^2 + \bar{p} \lambda \lambda' \epsilon \cdot \epsilon' + \gamma \lambda \epsilon \cdot r + \gamma' \lambda' \epsilon' \cdot r] \times \left[ \eta \lambda(\epsilon \times r)_u - \eta' \lambda'(\epsilon' \times r)_u \right]_{\lambda=\lambda'=0,|\epsilon|=|\epsilon'|=1}
\]

\[
\frac{\partial^2 L}{\partial \lambda^2 \partial \lambda'^2} \exp[g^2 \lambda^2 + \bar{g}^2 \lambda'^2 + \bar{p} \lambda \lambda' \epsilon \cdot \epsilon' + \gamma \lambda \epsilon \cdot r + \gamma' \lambda' \epsilon' \cdot r] \times i \left( \eta \lambda(\epsilon \times r)_u - \eta' \lambda'(\epsilon' \times r)_u \right)_{\lambda=\lambda'=0,|\epsilon|=|\epsilon'|=1}
\]

\[
= \sum_{M'} \frac{1}{B_2^{N-2}} \left(\frac{(2\pi)^{N-2}c}{\det B}\right)^{3/2} \frac{8\pi \sqrt{6}B_2(2M1u'|2M')}{2M1u|2M'}(\gamma\eta' + \gamma'\eta) \int dr^2 r V(r) e^{-\frac{1}{2}cr^2} \left(\frac{1}{3}q^2 + \frac{1}{15}r'^4\right),
\]

where

\[
\eta = \xi A'B^{-1}u, \quad \eta' = \xi AB^{-1}u'.
\]

We use this relation,

\[
i(\epsilon \times r)_u = \frac{4\sqrt{2}\pi}{3} r[Y_1(\epsilon) \times Y_1(r)]_u.
\]

### A.4 Root-mean-square radius

Using the wave function obtained by the variational calculation we can calculate a root-mean-square (r.m.s) radius. Therefore we discuss the matrix elements of the r.m.s radius operator when the masses of all the particles are the same. Here the r.m.s. radius operator is

\[
r^2 = \sum_{i=1}^N (r_i - x_N)^2 = \frac{1}{N} \sum_{j>i=1}^N (r_i - r_j)^2 = \frac{1}{N} \sum_{j>i=1}^N \hat{x}_{w\hat{w}} = \frac{1}{N} \sum_{j>i=1}^N \hat{x}_{w\hat{w}} x = \frac{1}{N} \hat{x}_{Qx},
\]

Then we have to know the matrix elements for the operator \(\hat{x}_{Qx}\) with the matrix \(Q = \hat{w}w\). For \(S\)-wave we can obtained:

\[
\langle f_0(x,A')|\hat{x}_{Qx}|f_0(x,A)\rangle = P \left(\frac{(2\pi)^{N-1}}{\det B}\right)^{3/2}.
\]

The matrix elements of the generating function for the \(D\)-wave is

\[
\langle g_{2M}(x,A',u')|\hat{x}_{Qx}|g_{2M}(x,A,u)\rangle = (P + \bar{v}B^{-1}Q^{-1}v) \left(\frac{(2\pi)^{N-1}}{\det B}\right)^{3/2} \exp(\frac{1}{2}\bar{v}B^{-1}v),
\]

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and then we can derive the matrix elements for the D-wave in this way,

\[
\langle f_{2M}(x,u',A')|\hat{Q}|f_{2M}(x,u,A)\rangle = \left( \frac{2}{B_2} \right) \left( \frac{(2\pi)^{N-1}}{\det B} \right)^{3/2} [Pq^2 + 4\hat{u}B^{-1}QB^{-1}u'q], \tag{A.43}
\]

where

\[
P = 3\text{Tr}(B^{-1}Q). \tag{A.44}
\]

### A.5 Correlation function

In this section we note the matrix elements of the correlation function. As we mentioned in the text, the correlation function is defined as,

\[
C(r) = \frac{1}{4\pi} \langle \Psi | \delta(|r_1 - r_2| - r) | \Psi \rangle. \tag{A.45}
\]

Since the TOFM wave function is constructed by the S-wave and D-wave, the total correlation function \( C(r) \) is able to be divided into two components for the S-wave and D-wave,

\[
C(r) = C_S(r) + C_D(r). \tag{A.46}
\]

In (A.45) we choose the distance \(|r_1 - r_2|\) as the certain distance of two particles. Here we can obtain the same result regardless of selecting two particles, due to the consideration of antisymmetrization for all particles in the wave function. As the definition of the correlation function, the matrix elements of correlation function needs the matrix elements of the \( \delta \)-function. We have already shown it in the previous section for two-body matrix element. Hence we note the correlation function for S-wave,

\[
C_S(r) = \frac{1}{4\pi} \langle \delta_0(x,A') | \delta(|r_1 - r_2| - r) | \delta_0(x,A) \rangle \tag{A.47}
\]

\[
= \left( \frac{(2\pi)^{N-1}c}{\det B} \right)^{3/2} e^{-\frac{1}{2}cr^2}.
\]

Since the D-wave function has the orientation for \( Y_2 \) function, we integrate the matrix element for all angle and average it divided by \( 4\pi \),

\[
C_D(r) = \frac{1}{4\pi} \int d\bar{r} \langle f_{2M}(x,A',u') | \delta(|r_1 - r_2| - r) | f_{2M}(x,A,x) \rangle \tag{A.48}
\]

\[
= \frac{1}{B_2} \left( \frac{(2\pi)^{N-1}c}{\det B} \right)^{3/2} e^{-\frac{1}{2}cr^2} \left[ 8\pi B_2 \rho^2 + 4B_1\gamma^2 \frac{3}{10\pi} \rho \gamma' + B_2^2 \gamma'^2 + \gamma^2 \gamma'' \right].
\]

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Appendix B

Figures of the bare $NN$ potential

We show figures of the bare $NN$ potential for AV8', AV14 and AV28 used in this study. These potentials are shown by the representations of spin-isospin channels.

Figure B.1: The central potentials of AV8' as functions of relative $NN$ distance $r$.

Figure B.2: The tensor (left figure) and $L\cdot S$ (right figure) potentials of AV8' as functions of relative $NN$ distance $r$. 
Figure B.3: The central potentials of AV14 (left figure) and AV28 (right figure) as functions of relative $NN$ distance $r$.

Figure B.4: The tensor potentials of AV14 (left figure) and AV28 (right figure) as functions of relative $NN$ distance $r$.

Figure B.5: The $L \cdot S$ potentials of AV14 (left figure) and AV28 (right figure) as functions of relative $NN$ distance $r$. 

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Figure B.6: The $L^2$ potentials of AV14 (left figure) and AV28 (right figure) as functions of relative $NN$ distance $r$.

Figure B.7: The $(L \cdot S)^2$ potentials of AV14 (left figure) and AV28 (right figure) as functions of relative $NN$ distance $r$. 