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非線形分散型波動方程式の解の漸近解析

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CONTENTS

PROLOGUE

This thesis is concerned about the author's study about nonlinear dispersive wave equations during the doctor course of Department of Mathematics of Science in Osaka University. He has been studying well-posedness for the equations and an asymptotic behavior of solutions for them. More specifically, he has been interested in the nonlinear Schrödinger equation, Klein-Gordon equation, Dirac equation and their system. He has obtained some results about Scattering or Blow-up phenomena for these equations so far. Some of them are organized in this thesis.

In Chapter 1, the author studies a scattering problem for the time-dependent Hartree-Fock equation (HF). This system appears in the quantum mechanics as an approximation to a Fermionic multi-body system. His aim in that chapter is to show existence of the modified scattering operator for HF (for the definition of the operator see Chapter 1). To obtain this operator, one has to improve the domain and range of the modified wave operator (for the definition of the operator see also Chapter 1) obtained in [70] by Takeshi Wada. The author uses a different approximate solution to HF, which differs from that in [70], and succeeds in improving the domain and range of the modified wave operator. By combining this improvement and existence of the inverse wave operator, which was already obtained in [70], the author proves existence of the modified scattering operator to HF. The author notes that how to construc the approximate solution was based on paper [21] by Nakao Hayashi and Pavel I. Naumkin.

In Chapter 2 and Chapter 3, the author studies a scattering problem for the Dirac-Klein-Gordon system (DKG), which is the couple of the Dirac equation and the Klein-Gordon equation with the Yukawa type interaction and plays an important role in quantum mechanics. It is well know that solutions for Dirac equation satisfy a reduced Klein-Gordon equation. From this fact, solutions for DKG also satisfy a Klein-Gordon system (KG), to which many mathematicians has studied existence of global solution. Among them, in [71], existence of the scattering operator (for the difinition of this operator see Chapter 2) for the reduced KG system was proved in three space dimensions in lower order Sobolev spaces. However, existence of the scattering operator for DKG itself is not so clear even from the previous result [71].

In Chapter 2, which is based on a joint work [20] with Nakao Hayashi and Pavel. I. Naumkin, the author proves existence of the scattering operator for DKG itself in lower order Sobolev space. In this chapter, DKG itself is treated without reducing it into the KG system. Moreover, by using their estimates, one can improve the domain and range of the scattering operator for the KG system obtained in [71].

In Chapter 3, the author considers existence of the wave operator for DKG in two space dimensions. The author notes that 2d case is more difficult than the 3d one, since as the dimension is lower, an expected time decay property of solutions is slower. In fact, 2d case is delicate one and the borderline between the short range scattering and the long range one. To overcome the insufficient time decay property, the author uses an algebraic normal form transformation, which one is permitted to use under the non-resonance mass condition, developed by Hideaki Sunagawa in [59] and the decomposition of the Klein-Gordon operator into a product of the Dirac operators (which is essential). Moreover, as one see in Chapter 2, one meets derivative loss difficulty for the Dirac part. To defeat the derivative loss, the author uses a special structure of the nonlinear term to the Dirac part (for more detail, see Chapter 3). By combining these two facts, existence of the wave operator for DKG is obtained in two space dimensions in lower order Sobolev space under the non-resonance mass condition. The author notes that by using the first method, existence of the inverse wave operator was also proved in [33].

In Chapter 4 and Chapter 5, the author discusses Blow-up phenomena of solutions for the nonlinear Schrödinger equation with a non-gauge invariant power nonlinearity:

$$i\partial_t u + \Delta u = \lambda |u|^p, \quad (t,x) \in [0,T) \times \mathbb{R}^n. \tag{0.1}$$

Blow-up phenomena to the corresponding heat equation and the wave equation has been studied extensively. However, there are few results about (0.1). When it comes to NLS, the one with a gauge invariant power nonlinearity

$$i\partial_t\varphi + \Delta\varphi = \mu\,|\varphi|^{p-1}\varphi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \tag{0.2}$$

where $\mu \in \mathbb{R}$, has been studying extensively. It is well known that large data local well-posedness holds for (0.2) in L^2-sense under $1 < p < p_0$, where $p_0 = 1 + 4/n$ is called L^2-critical exponent (see e.g. [65]). Moreover, L^2-norm of those solutions for (0.2) conserves

$$\|\varphi(t)\|_{L^2} = \|\varphi(0)\|_{L^2}, \text{ for any } t \in \mathbb{R}. \tag{0.3}$$

Thus L^2-conservation law and the local well-posedness imply large data global well-posedness in L^2-sense of (0.2) in L^2-subcritical, i.e. $1 < p < p_0$.

In the present (0.1) case, large data local well-posedness also holds in L^2-sense under $1 < p < p_0$. However, L^2-conservation law (0.3) for (0.1) can not be expected. Thus global well-posedness results for (0.1) are not trivial in L^2-subcritical case. The author notes that in [58], when $(n,p) = (2,2)$, non-existence of the usual wave operator was shown and some mathematicians had expected to get a small data global existence result for (0.1) in the case $(n,p) = (2,2)$.

On the contrary, in Chapter 4, a small data blow-up result will be shown in the case $1 < p \leq 1 + 2/n$, which includes $(n,p) = (2,2)$. This is a joint work with Mr. Yuta Wakasugi. The method in this Chapter is based on a test-function method [72, 73] used by Qis. Zhang, who proved the same result for some parabolic equations and the damped wave equation respectively. This test-function method was extensively used to obtain small data blow-up result for the various damped wave equations.

In Chapter 5, proceeding Chapter 4, the author considers (0.1) and discusses estimates of the "lifespan" for L^2-solution. The method in Chapter 4 is based on a contradiction argument to construct the blow-up solution. Therefore the mechanism of the blow-up solution, such as estimates of the lifespan and the blow-up rate, can not be understood. To avoid contradiction argument, the author uses the idea of paper [46] (for the detail, see Chapter 4). By combining this and the test-function method, he succeeds in proving an upper bound of the lifespan for (0.1) in the case $1 < p < 1 + 2/n$.

Finally, the author notes that both results in Chapter 4 and Chapter 5 were extended to the wider case $1 < p < p_0$ in the recent author's paper [32] by the same method as in Chapter 5, after submitting this doctoral thesis. This is a joint work with Mr. Takahisa Inui. The author also notes that the method in Chapter 5 can be applicable to the damped wave equation (see [35]), to which the lifespan of solutions has not been well studied.

1. Existence of the modified scattering operator for the Hartree-Fock system

1.1. Introduction. In this chapter, we study scattering problem for the nonlinear Schrödinger equation with nonlocal interaction:

$$\text{(HF)} \qquad i\partial_t u + (1/2)\,\Delta u = f(u), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,$$

where space dimension is $n \geq 2$, Δ denotes the Laplace operator in x, $u = {}^t(u_1, \cdots, u_N)$ is a $\mathbb{C}^N$ $(N \geq 2)$-valued unknown function of (t,x) and $f(u)$ denotes a nonlinear term. The j-th element of $f(u) = {}^t(f_1(u), \cdots, f_N(u))$ is defined by

$$f_j(u) = \int_{\mathbb{R}^n} V(x-y) \sum_{k=1}^{N} \left\{ |u_k(y)|^2 u_j(x) - u_j(y)\,\bar{u}_k(y)\,u_k(x) \right\} dy, \tag{1.1}$$

where $V(x)$ is called a Coulomb potential given by

$$V(x) = \lambda\,|x|^{-1}, \quad (x \in \mathbb{R}^n \setminus \{0\}) \tag{1.2}$$

and λ is a non-zero real constant. The system (HF) is called a time-dependent Hartree-Fock equation and appears in the quantum mechanics as an approximation to a Fermionic N-body system. Our aim is to show existence of the modified scattering operator for the system (HF). To do so, we will improve domain and range of a modified wave operator obtained in T. Wada [70]. As for a modified inverse wave operator, we will use results obtained by T. Wada [70].

We introduce an $N\times N$ matrix $F(u,v) = \{F_{ij}(u,v)\}_{1\leq i,j\leq N}$ whose (i,j)-element is defined by

$$F_{ij}(u,v) = V * \left\{ \left(\sum_{k=1}^{N} u_k \bar{v}_k \right) \delta_{ij} - u_i \bar{v}_j \right\}, \tag{1.3}$$

where "$*$" denotes the convolution for space variables, δ_{ij} is Kronecker's delta i.e. $\delta_{ii} = 1$, $\delta_{ij} = 0$ $(i \neq j)$. Furthermore we define an $N \times N$ matrix $F(u) = F(u,u)$ and then we can express nonlinear term $f(u)$ as

$$f(u) = F(u)\,u.$$

We note that $F(u)$ is an N-dimensional Hermitian matrix.

Let u_+ be a given final state. $A = A(t,\xi)$ is an $N \times N$ matrix-valued function and the solution of the Cauchy problem

$$i\partial_t A = t^{-1} F(A\hat{u}_+)\,A, \quad t \geq 1, \quad \xi \in \mathbb{R}^n \tag{1.4}$$

$$A(1,\xi) = I_N, \quad \xi \in \mathbb{R}^n, \tag{1.5}$$

where I_N is the $N \times N$ unit matrix.

Our purpose can be formulated as follows. We assume that the final data

$$u_+ \in H^{0,\alpha} \text{ with } 1/2 < \beta < \alpha < 1$$

and the norm $\|u_+\|_{H^{0,\alpha}}$ is sufficiently small. Then we will find a unique global solution $u \in C\left([0,\infty); H^{0,\beta}\right)$ of (HF) satisfying

$$\lim_{t\to+\infty} \left(u(t) - (it)^{-\frac{n}{2}}\, e^{\frac{i|x|^2}{2t}} A(t,x/t)\,\hat{u}_+(x/t) \right) = 0, \quad \text{in } H^{0,\delta} \tag{1.6}$$

with $1/2 < \delta < \beta$. This means that the modified wave operator for the system (HF) is well-defined from a neighborhood at the origin in the space $H^{0,\alpha}$ to a neighborhood at the origin in the space $H^{0,\beta}$.

Finally, we introduce several notations used in this chapter. $\mathcal{U}(t)$ denotes the free Schrödinger evolution group defined by

$$\begin{aligned} \mathcal{U}(t)\phi &\equiv \mathcal{F}^{-1}e^{-\frac{it}{2}|\xi|^2}\mathcal{F} = (2\pi i t)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{\frac{i}{2t}|x-y|^2}\phi(y)\,dy \\ &= \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t), \end{aligned}$$

where $\mathcal{M}=\mathcal{M}(t)=\exp\left(\frac{i|x|^2}{2t}\right)$ is a multiplication operator and $\mathcal{D}(t)$ is a dilation operator defined by

$$(\mathcal{D}(t)\phi)(x) = (it)^{-\frac{n}{2}}\phi(x/t).$$

We note that

$$\mathcal{U}(-t) = \mathcal{M}(-t)\,i^n\mathcal{F}^{-1}\mathcal{D}(1/t)\,\mathcal{M}(-t),$$

since $(\mathcal{D}(t))^{-1} = i^n\mathcal{D}(1/t)$. By using the above identities, we easily see that

$$\mathcal{J}(t) \equiv \mathcal{U}(t)\,x\,\mathcal{U}(-t) = \mathcal{M}(t)\,it\nabla\mathcal{M}(-t) = x + it\nabla \quad (t\in\mathbb{R}). \tag{1.7}$$

For $\beta \geq 0$, we define

$$|\mathcal{J}|^\beta = |\mathcal{J}(t)|^\beta \equiv \mathcal{U}(t)\,|x|^\beta\,\mathcal{U}(-t) = t^\beta\mathcal{M}(t)(-\Delta)^{\beta/2}\mathcal{M}(-t), \quad (t\in\mathbb{R}). \tag{1.8}$$

Then the commutation relation

$$\left[i\partial_t + (1/2)\Delta, |\mathcal{J}|^\beta\right] = 0$$

holds, where $[A,B]=AB-BA$.

1.2. **Existence of the modified scattering operator.**

We now state our results in this chapter.

Theorem 1.1. *Let $1/2<\beta<\alpha<1$. We assume that $u_+\in H^{0,\alpha}$ and $\|u_+\|_{H^{0,\alpha}}=\varepsilon$, where ε is sufficiently small. Then there exists a unique global solution u of the system (HF) satisfying*

$$u\in C\left([0,\infty);L^2\right), \quad |\mathcal{J}|^\beta u \in C\left([0,\infty);L^2\right).$$

Moreover the following estimate

$$\left\|\mathcal{U}(-t)\left(u(t) - (it)^{-n/2}e^{\frac{i|x|^2}{2t}}A(t,\cdot/t)\,\hat{u}_+(\cdot/t)\right)\right\|_{H^{0,\delta}} \lesssim t^{-\frac{\beta-\delta}{2}-\mu} \tag{1.9}$$

is true for all $t\geq 1$, where $0\leq\delta\leq\beta$ and $0<\mu<(\alpha-\beta)/2$.

By the above Theorem, we get existence of the modified wave operator

$$\mathcal{W}^+ : u_+ \to u(0)$$

for the system (HF) as follows.

Corollary 1.2. *The modified wave operator $\mathcal{W}^+$ for the system (HF) is well-defined from a neighborhood at the origin in the space $H^{0,\alpha}$ to a neighborhood at the origin in the space $H^{0,\beta}$.*

We state a result of existence of modified inverse wave operator $(\mathcal{W}^-)^{-1}$ obtained in T. Wada [70]. He studied the initial value problem

$$\begin{cases} i\partial_t u + (1/2)\Delta u = f(u), \\ u(0)=u_0, \end{cases} \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n,\ n\geq 2 \tag{1.10}$$

where u_0 be a given initial data and then he got the following results:

Theorem 1.3. *(see [70]) Let $1/2 < \delta < \beta$, $\eta = \min(1, (\beta - \delta)/2)$ and $0 < \nu < \eta/3$. We assume that $u_0 \in H^{0,\beta}$ and $\|u_0\|_{H^{0,\beta}} = \varepsilon$, where ε is sufficiently small. Then there exists a unique global solution u of the system (HF) satisfying*

$$u \in C\left((-\infty, 0]; L^2\right), \quad |\mathcal{J}|^\beta u \in C\left((-\infty, 0]; L^2\right).$$

Moreover there exists a scattering state $u_- \in H^{0,\delta}$ such that the estimate

$$\left\| \mathcal{U}(-t)\left(u(t) - (it)^{-n/2} e^{\frac{i|x|^2}{2t}} A(t, \cdot/t)\, \hat{u}_+(\cdot/t)\right)\right\|_{H^{0,\delta}} \lesssim |t|^{-\eta+2\nu},$$

is true for all $t \le -1$, where $A(t,\xi)$ is the solution of the following Cauchy problem

$$i\partial_t A = t^{-1} F(A\hat{u}_+)\, A, \quad t \le -1, \quad \xi \in \mathbb{R}^n$$
$$A(-1, \xi) = I_N, \quad \xi \in \mathbb{R}^n.$$

By the above Theorem, we obtain existence of the modified inverse wave operator

$$\left(\mathcal{W}^-\right)^{-1} : u_0 \to u_-$$

for the system (HF) as follows.

Corollary 1.4. *The modified inverse wave operator $\left(\mathcal{W}^-\right)^{-1}$ for the system (HF) is well-defined from a neighborhood at the origin in the space $H^{0,\beta}$ to a neighborhood at the origin in the space $H^{0,\delta}$.*

As a consequence of Corollaries 1.2 and 1.4, we can define the modified scattering operator

$$\mathcal{S}^+ = \left(\mathcal{W}^-\right)^{-1} \mathcal{W}^+ : u_+ \to u_-.$$

Theorem 1.5. *The modified scattering operator $\mathcal{S}^+ = \left(\mathcal{W}^-\right)^{-1} \mathcal{W}^+$ for the system (HF) is well-defined from a neighborhood at the origin in the space $H^{0,\alpha}$ to a neighborhood at the origin in the space $H^{0,\delta}$.*

Theorem 1.1 is improvement of Theorem 1.1 obtained in [70]. In Theorem 1 of paper [70], it was shown that for any $u_+ \in H^{0,2}$ with smallness condition on $\|\hat{u}_+\|_{L^{p_i}}$ $(i = 1, 2)$ where p_1, p_2 be the numbers such that $2 < p_1 < \frac{2n}{n-1} < p_2 < \frac{2n}{n-2}$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1 - \frac{1}{n}$ (see Lemma 1.9), the system (HF) has a unique global solution $u \in C\left([0,\infty); L^2\right) \cap L^q_t\left([0,\infty); L^p_x\right)$ such that the estimate

$$\left\| u(t) - (it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{2t}} A(t, \cdot/t)\, \hat{u}_+ \right\|_{L^q_t\left([t,\infty); L^p_x\right)} \lesssim t^{-b}$$

is true for any $t > 0$, where $1/4 < b < 1$, $n \ge 2$ and $0 \le 2/q = n/2 - n/p < 1$. This means that the modified wave operator $\mathcal{W}^+$ for the system (HF) is well-defined from a neighborhood at the origin in the space $H^{0,2}$ to a neighborhood at the origin in the space L^2. His result requires more smoothness for the final data u_+ $\left(H^{0,2}\right)$ than ours $\left(H^{0,\alpha}\right.$ with $\left.\alpha > 1/2\right)$ and the value $u(0)$ of solution obtained in [70] belongs to wider class $\left(L^2\right)$ than ours $\left(H^{0,\beta}\right.$ with $\left.1/2 < \beta < \alpha\right)$. It is not clear whether modified scattering operator for the system (HF) can be obtained or not. His method is based on that of J. Ginibre and T. Ozawa [11] to study the Hartree equation

$$i\partial_t u + (1/2)\,\Delta u = \left(V * |u|^2\right) u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \ n \ge 2,$$

where $u = u(t,x)$ is a $\mathbb{C}$-valued unknown function and $V(x) = \lambda |x|^{-1}$ $(\lambda \in \mathbb{R})$ is a Coulomb potential. Their method is based on Strichartz estimate (see [?]) and the use of an approximate solution $\tilde{u}_1$

$$\tilde{u}_1 = (it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{2t}} \hat{u}_+(x/t) \exp\left(-i\left\{\left(V * |\hat{u}_+|^2\right)(x/t)\right\} \log t\right)$$

to the free Hartree equation. In [70], T. Wada put

$$u_1 \equiv (it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{2t}} A(t, x/t)\, \hat{u}_+(x/t)$$

as an approximate solution to the free Hartree-Fock equation following from the paper [11] and showed that

$$\begin{aligned}\tilde{R} &\equiv (i\partial_t + (1/2)\,\Delta)\,u_1 - f(u_1) \\ &= (1/2)\,t^{-2}\mathcal{M}(t)\,\mathcal{D}(t)\,\Delta A(t)\,\hat{u}_+\end{aligned}$$

is the remainder term which implies the second differentiability of $\hat{u}_+$. On the other hand, in order to get our Theorem 1.1, we use the factorization of $\mathcal{U}(t) = \mathcal{M}(t)\,\mathcal{D}(t)\,\mathcal{F}\mathcal{M}(t)$, take $\mathcal{U}(t)\,\mathcal{F}^{-1}A(t)\,\hat{u}_+$ as an approximate solution of u and utilize the operator $\mathcal{J}$ given by (1.7). This method was used by N. Hayashi and P. I. Naumkin [21] to study nonlinear Schrödinger equations with a critical power nonlinearity

$$i\partial_t u + (1/2)\,\Delta u = \lambda\,|u|^{2/n}\,u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

where $\lambda \in \mathbb{R}$ and $n = 1, 2$ or 3. By the identity

$$\mathcal{U}(t)\,\mathcal{F}^{-1}A(t)\,\hat{u}_+ = u_1 + \mathcal{M}(t)\,\mathcal{D}(t)\,\mathcal{F}(\mathcal{M}(t) - 1)\,\mathcal{F}^{-1}A(t)\,\hat{u}_+,$$

we can see that the difference between the two approximate solutions is

$$\mathcal{M}(t)\,\mathcal{D}(t)\,\mathcal{F}(\mathcal{M}(t) - 1)\,\mathcal{F}^{-1}A(t)\,\hat{u}_+.$$

We show that this term is a remainder term in L^2 (see (1.36)).

The rest of this chapter is organized as follows. In Section 1.3, we state several Sobolev type inequalities and unitarity of $A(t,\xi)$. In Section 1.4, we lead integral equation corresponding to the system (HF) and the final data condition (1.6). In subection 1.5, we introduce several propositions used in the proof of Theorem 1.1. In subsection 1.6, we prove Theorem 1.1.

1.3. **Sobolev type inequalities.** First we state the Gagliardo-Nirenberg-Sobolev inequality.

Lemma 1.6. *Let q, r be any numbers satisfying $1 \le q, r \le \infty$, and let j, m be any real numbers satisfying $0 \le j < m$. If $u \in \dot{H}^{m,0}_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, then the following inequality is valid:*

$$\left\|(-\Delta)^{j/2}\,u\right\|_{L^p} \le C\left\|(-\Delta)^{m/2}\,u\right\|_{L^r}^{a}\,\|u\|_{L^q}^{1-a}, \tag{1.11}$$

where C is a constant depending only on n, m, j, q, r and a. Here $p \ge 1$ is such that $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-a}{q}$ and the parameter a is any from the interval $\frac{j}{m} \le a \le 1$, with the following exception: if the value $m - j - \frac{n}{r}$ is a nonnegative integer, then the parameter a is any from the interval $\frac{j}{m} \le a < 1$.

For the proof of Lemma 1.6, see, e.g [16]. This lemma is used to obtain the estimates (1.10) and (1.24) in this paper. Next we state the Sobolev inequality which immediately follows from Lemma 1.6 with $j = 0$ and $a = 1$.

Corollary 1.7. *Let $1 < r < \infty$, $0 < m < n/r$ and $1/p = 1/r - m/n$. Then there exists a positive constant $C > 0$ such that for any $u \in \dot{H}^m_r$*

$$\|u\|_{L^p} \le C\left\|(-\Delta)^{m/2}\,u\right\|_{L^r}.$$

Next we state the Hardy-Littlewood-Sobolev inequality. This one also follows from Lemma 1.6 with $j = 0$, $m = n - \gamma$, $a = 1$ and $u = (-\Delta)^{-(n-\gamma)/2}\,\phi$.

Lemma 1.8. *Let $0 < \gamma < n, 1 < p, q < \infty$ and $1 + 1/p = \gamma/n + 1/q$. Then there exists a constant $C > 0$ such that*

$$\left\||\cdot|^{-\gamma} * \phi\right\|_{L^p} = C\left\|(-\Delta)^{-(n-\gamma)/2}\,\phi\right\|_{L^p} \le C\,\|\phi\|_{L^q}. \tag{1.12}$$

Here C is independent of ϕ.

Lemma 1.8 means the embedding $\dot{H}_p^{n-\gamma}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ holds. For the proof of the equality in (1.12), see, e.g [62].

When we estimate L^∞-norm of the term such as $|\cdot|^{-\gamma} * \phi$, we use the following Lemma (see Remark 1.1).

Lemma 1.9. *Let $n \geq 2$, $0 < \gamma < n$, $2 < p < 2n/(n-\gamma) < q < 2n/(n-2)$ (∞ if $n = 2$) and $1/p + 1/q = 1 - \gamma/n$. Then there exists a constant $C > 0$ such that*

$$\left\| |\cdot|^{-\gamma} * (\phi\bar{\psi}) \right\|_{L^\infty} \leq C \left(\|\phi\|_{L^p} \|\psi\|_{L^p} \|\phi\|_{L^q} \|\psi\|_{L^q} \right)^{1/2}, \tag{1.13}$$

provided that the right hand side is finite.

We can prove Lemma 1.9 exactly in the same way as in the proof of Lemma 2.4 of paper T. Wada [69] and see also A. Simomura [56].

Remark 1.1. *We explain the reason why we need Lemma 1.9. Let $p = \infty$, $j = 0$, $m = n-1$, $r = \frac{n}{n-1}$ and $a = 1$. Then $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-a}{q}$ is satisfied, but unfortunately, the equality*

$$m - j - n/r = 0$$

also holds. Therefore we can not use Lemma 1.6 in this case.

Next we introduce a commutator estimate for fractional derivatives.

Lemma 1.10. *Let $s \in (0,1)$, $s_1, s_2 \in [0,s]$ with $s = s_1 + s_2$, $1 < p, q_1, q_2 < \infty$ and $1 < r_1, r_2 \leq \infty$ with $1/p = 1/q + 1/r$. Then the following estimate holds:*

$$\left\| (-\Delta)^{s/2}(\phi\psi) - \phi(-\Delta)^{s/2}\psi - \psi(-\Delta)^{s/2}\phi \right\|_{L^p} \leq C \|\phi\|_{\dot{H}_q^{s_1}} \|\psi\|_{\dot{H}_r^{s_2}}. \tag{1.14}$$

For the proof of (1.14) see, Kenig-Ponce-Vega [42].

1.4. **Application of the Dollard decomposition.** We write the integral equation associated with the system (HF) and the final state condition (1.6). Define a first approximation for the solution of (HF) by

$$u_1(t) = \mathcal{M}(t)\mathcal{D}(t)\hat{w}(t), \; \hat{w}(t) = A(t)\hat{u}_+. \tag{1.15}$$

We note that the free Schrödinger evolution group $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it|\xi|^2/2}\mathcal{F}$ can be decomposed as

$$\begin{aligned} \mathcal{U}(t)\phi &= \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t)\phi \\ &= \mathcal{M}(t)\mathcal{D}(t)\hat{\phi} + \mathcal{R}(t)\hat{\phi}, \end{aligned} \tag{1.16}$$

where

$$\mathcal{R}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}(\mathcal{M}(t) - 1)\mathcal{F}^{-1}. \tag{1.17}$$

Multiplying both sides of (HF) by $\mathcal{F}\mathcal{U}(-t)$, we obtain

$$i\partial_t(\mathcal{F}\mathcal{U}(-t)u) = \mathcal{F}\mathcal{U}(-t)f(u). \tag{1.18}$$

Note that $\hat{w}(t,\xi) = A(t,\xi)\hat{u}_+(\xi)$ satisfies the equation

$$i\partial_t\hat{w}(t) = t^{-1}f(\hat{w}(t)), \quad t \geq 1, \; \xi \in \mathbb{R}^n, \tag{1.19}$$

since $A(t,\xi)$ is the solution of (1.4). Due to the difference of (1.18) and (1.19), we get

$$\begin{aligned} & i\partial_t(\mathcal{F}\mathcal{U}(-t)u - \hat{w}) \\ =\; & \mathcal{F}\mathcal{U}(-t)\left(f(u) - t^{-1}\mathcal{U}(t)\mathcal{F}^{-1}f(\hat{w})\right) \\ =\; & \mathcal{F}\mathcal{U}(-t)\left(f(u) - t^{-1}\mathcal{M}(t)\mathcal{D}(t)f(\hat{w}) - t^{-1}\mathcal{R}f(\hat{w})\right) \\ =\; & \mathcal{F}\mathcal{U}(-t)(f(u) - f(u_1)) - t^{-1}\mathcal{F}\mathcal{U}(-t)\mathcal{R}f(\hat{w}), \end{aligned} \tag{1.20}$$

here we have used the decomposition (1.16) and the identity

$$t^{-1}\mathcal{M}(t)\mathcal{D}(t)f(\hat{w}) = f(u_1).$$

We note that by using (1.16) again, the equalities

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)u-\hat{w} &= \mathcal{F}\mathcal{U}(-t)\left(u-\mathcal{U}(t)\mathcal{F}^{-1}\hat{w}\right)\\ &= \mathcal{F}\mathcal{U}(-t)\left(u-u_1-\mathcal{R}\hat{w}\right)\end{aligned}$$

hold. Thus integrating (1.20) in time over $[t,\infty)$ and using the final condition (1.6), we obtain

$$\begin{aligned}u(t)-u_1(t) &= i\int_t^\infty \mathcal{U}(t-\tau)\left(f(u)-f(u_1)\right)d\tau\\ &\quad+\mathcal{R}\hat{w}-i\int_t^\infty \mathcal{U}(t-\tau)\mathcal{R}(\tau)f(\hat{w})\frac{d\tau}{\tau}.\end{aligned} \tag{1.21}$$

The system (1.21) is the integral equation for (HF) with the final condition (1.6). Let us consider the linearized version of (1.21)

$$\begin{aligned}u(t)-u_1(t) &= i\int_t^\infty \mathcal{U}(t-\tau)\left(f(v)-f(u_1)\right)d\tau\\ &\quad+\mathcal{R}\hat{w}-i\int_t^\infty \mathcal{U}(t-\tau)\mathcal{R}(\tau)f(\hat{w})\frac{d\tau}{\tau}.\end{aligned} \tag{1.22}$$

For $T\geq 1$, where T is sufficiently large, we define the following function space:

$$X=\left\{v\in C\left([T,\infty);L^2\right);\|v-u_1\|_X<\infty\right\}$$

with the norm

$$\|v\|_X=\sup_{t\in[T,\infty)}\left(t^{\beta/2+\mu}\|v(t)\|_{L^2}+t^{\mu}\left\||\mathcal{J}|^{\beta}v(t)\right\|_{L^2}\right),$$

where $1/2<\beta<\alpha<1$ and $(\alpha-\beta)/2>\mu>0$ is sufficiently small. We will show the map $v\in X_\rho\longmapsto u$ defined by (1.22) is a contraction mapping in subsection 1.6, where $X_\rho\equiv\{v\in X;\|v-u_1\|_X\leq\rho\}$ $(\rho=C\|u_+\|_{H^{0,\alpha}})$ be a closed ball in X.

1.5. **Properties of solutions.** We state several Propositions used in the proof of Theorem 1.1.

First we remember results to the Cauchy problem (1.10) of the Hartree-Fock equation (HF).

Proposition 1.11. *Let $n\geq 2$ and $u_0\in H^{0,\beta}$ with $\beta\geq 0$. Then the following statements hold.*

(i) There exists a unique global solution u of (1.10) which belongs to

$$C\left(\mathbb{R};L^2\right)\cap L_t^q\left(\mathbb{R};L_x^p\right)$$

where (p,q) is any pair of numbers such that $0<2/q=n/2-n/p<1$. Furthermore if $\beta>0$, then

$$|\mathcal{J}|^{\beta}u\in C\left(\mathbb{R};L^2\right).$$

(ii) If $\beta>0$, then the solution $u\in C\left(\mathbb{R};L^2\right)$ with $|\mathcal{J}|^{\beta}u\in C\left(\mathbb{R};L^2\right)$ is unique.

(iii) If the norm $\|u_0\|_{\mathbb{H}^{0,\beta}}$ is sufficiently small, then the inequality

$$\sup_{|t|\leq 1}\left\||\mathcal{J}|^{\beta}u(t)\right\|_{L^2}\leq 2\|u_0\|_{H^{0,\beta}}$$

holds.

We can prove Proposition 1.11 by applying the method in [4], [5], [14], [26] and [38] and see also [69].

Second we state unitarity of the matrix-valued function $A(t,\xi)$ defined by (1.4) and (1.5).

Lemma 1.12. *When $t\geq 1$ and $\xi\in\mathbb{R}^n$, the solution $A(t,\xi)$ to the Cauchy problem (1.4)-(1.5) is an $N\times N$ unitary matrix. Therefore for $u\in\mathbb{C}^N$, the equality*

$$|A(t,\xi)u|_{\mathbb{C}^N}=|u|_{\mathbb{C}^N}$$

is valid.

We can find the proof of Lemma 1.12 in that of Lemma 3.1 of paper T. Wada [70] and see also A. Simomura [56].

Next we show time decay estimates for the first approximation u_1, the difference $u_1 - v$ and $v \in X_\rho$.

Proposition 1.13. *Let $n \geq 2$ and $u_+ \in H^{0,\alpha}$ with $0 < \beta < \alpha < n/2$.*

(i) For $2 \leq p \leq 2n/(n-2\beta)$, the inequality

$$\|u_1\|_{L^p} \lesssim t^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \|u_+\|_{H^{0,\alpha}} \tag{1.23}$$

is valid for all $t \geq 1$, where u_1 is defined by (1.15).

(ii) Let $1/2 < \beta < n/2$ and $v \in X_\rho$. For $2 \leq p \leq 2n/(n-2\beta)$, the inequalities

$$\|u_1 - v\|_{L^p} \lesssim t^{-(\beta/2+\mu)-n/2(1/2-1/p)} \|u_+\|_{H^{0,\alpha}}, \tag{1.24}$$

$$\|v\|_{L^p} \lesssim t^{-n/2(1-2/p)} \|u_+\|_{H^{0,\alpha}} \tag{1.25}$$

are valid for all $t \geq T$, where T is sufficiently large.

Proof. First we prove the estimate (1.23). Let $2 \leq p \leq 2n/(n-2\beta)$. By the definition of $u_1, |\mathcal{M}(t)| = 1, |\hat{w}(t)|_{\mathbb{C}^N} = |A(t)\hat{u}_+|_{\mathbb{C}^N} = |\hat{u}_+|_{\mathbb{C}^N}$ (see Lemma 1.12) and Corollary 1.7 with $r = 2$ and $m = \sigma = n(1/2 - 1/p)$, we have

$$\begin{aligned} \|u_1\|_{L^p} &= t^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \|\hat{w}(t)\|_{L^p} = t^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \|\hat{u}_+\|_{L^p} \\ &\lesssim t^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \|\hat{u}_+\|_{H^\sigma} \lesssim t^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \|\hat{u}_+\|_{H^\alpha}, \end{aligned}$$

since $0 \leq \sigma \leq \beta < \alpha$. This completes the proof of (1.23).

Next we show the estimate (1.24). Let $2 \leq p \leq 2n/(n-2\beta)$ and $t \geq T$ where T is sufficiently large. By Lemma 1.6 with $j = 0, m = \beta, q = r = 2$ and $a = n/\beta(1/2 - 1/p)$, $|\mathcal{M}(-t)| = 1$ and the identity (1.8), we have

$$\begin{aligned} \|u_1 - v\|_{L^p} &= \|\mathcal{M}(-t)(u_1 - v)\|_{L^p}, \\ &\lesssim \left\|(-\Delta)^{\frac{\beta}{2}} \mathcal{M}(-t)(u_1 - v)\right\|_{L^2}^{\frac{n}{\beta}\left(\frac{1}{2}-\frac{1}{p}\right)} \|\mathcal{M}(-t)(u_1 - v)\|_{L^2}^{1-\frac{n}{\beta}\left(\frac{1}{2}-\frac{1}{p}\right)} \\ &= t^{-n\left(\frac{1}{2}-\frac{1}{p}\right)} \left\||\mathcal{J}|^\beta (u_1 - v)\right\|_{L^2}^{\frac{n}{\beta}\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_1 - v\|_{L^2}^{1-\frac{n}{\beta}\left(\frac{1}{2}-\frac{1}{p}\right)}, \\ &\lesssim t^{-(\beta/2+\mu)-n/2(1/2-1/p)} \rho, \end{aligned}$$

since $m - j - n/r = \beta - n/2 < 0$ and $v \in X_\rho$.

Finally we prove the estimate (1.25). Let $2 \leq p \leq 2n/(n-2\beta)$ and $t \geq T$ where T is sufficiently large. By (1.23) and (1.24), we have

$$\begin{aligned} \|v\|_{L^p} &\leq \|v - u_1\|_{L^p} + \|u_1\|_{L^p} \\ &\lesssim t^{-(\beta/2+\mu)-n/2(1/2-1/p)} \rho + t^{-n/2(1-2/p)} \|u_+\|_{H^{0,\alpha}} \\ &\lesssim t^{-n/2(1-2/p)} \|u_+\|_{H^{0,\alpha}}, \end{aligned}$$

since

$$(\beta/2 + \mu) + n/2(1/2 - 1/p) > n/2(1 - 2/p).$$

This completes the proof of (1.25). □

Next we show the estimate of $\hat{w}(t,\xi) = A(t,\xi)\hat{u}_+(\xi)$ in $\dot{H}^\alpha$, where $A(t,\xi)$ is the solution of

$$i\partial_t A = t^{-1} F(A\hat{u}_+) A, \quad t \geq 1, \quad \xi \in \mathbb{R}^n, \tag{1.26}$$

$$A(1,\xi) = I_N, \quad \xi \in \mathbb{R}^n. \tag{1.27}$$

Unfortunately, we don't have explicit representation of $A(t,\xi)$.

Proposition 1.14. *Let $1/2 < \alpha < 1$ and $u_+ \in H^{0,\alpha}$. Then the inequality*

$$\|\hat{w}(t)\|_{\dot{H}^\alpha} \lesssim \|u_+\|_{H^{0,\alpha}} \left(1 + \|u_+\|_{H^{0,\alpha}}^2 \log t\right).$$

is true for all $t \geq 1$.

Proof. Multiplying both sides of (1.26) by $(-\Delta)^{\alpha/2}$, we have

$$i\partial_t \left((-\Delta)^{\alpha/2} \hat{w}(t)\right) = t^{-1} (-\Delta)^{\alpha/2} f(\hat{w}(t)) \tag{1.28}$$

Taking a scalar product $(-\Delta)^{\alpha/2} \hat{w}$ in $\mathbb{C}^N$ to both sides of (1.28) and imaginary part, we obtain

$$\partial_t \left|(-\Delta)^{\alpha/2} \hat{w}\right|_{\mathbb{C}^N}^2 = 2t^{-1} \operatorname{Im} \left((-\Delta)^{\alpha/2} F(\hat{w}) \hat{w}, (-\Delta)^{\alpha/2} \hat{w}\right)_{\mathbb{C}^N}, \tag{1.29}$$

where $(\cdot,\cdot)_{\mathbb{C}^N}$ denotes the scalar product in $\mathbb{C}^N$ and we have used $f(\hat{w}) = F(\hat{w})\hat{w}$. By integrating (1.30) over $\mathbb{R}^n$, we get

$$\partial_t \left\|(-\Delta)^{\alpha/2} \hat{w}\right\|_{L^2}^2 = 2t^{-1} \operatorname{Im} \left((-\Delta)^{\alpha/2} (F(\hat{w}) \hat{w}), (-\Delta)^{\alpha/2} \hat{w}\right)_{L^2}. \tag{1.30}$$

Since $F(\hat{w})$ is an N-dimensional Hermitian matrix (see (1.3)),

$$\operatorname{Im} \left(F(\hat{w}) (-\Delta)^{\alpha/2} \hat{w}, (-\Delta)^{\alpha/2} \hat{w}\right)_{L^2} = 0. \tag{1.31}$$

By (1.30) and (1.31), the equality

$$\partial_t \left\|(-\Delta)^{\alpha/2} \hat{w}\right\|_{L^2}^2 = 2t^{-1} \operatorname{Im} \left((-\Delta)^{\frac{\alpha}{2}} (F(\hat{w}) \hat{w}) - F(\hat{w}) (-\Delta)^{\frac{\alpha}{2}} \hat{w}, (-\Delta)^{\frac{\alpha}{2}} \hat{w}\right)_{L^2}$$

holds. By Schwarz's inequality and the commutator estimate (1.14) with $s = s_1 = \alpha$, $s_2 = 0$, $p = 2$, $q = n/\alpha\frac{n}{\alpha}$ and $r = 2n/(n - 2\alpha)$, we have

$$\begin{aligned} \partial_t \|\hat{w}\|_{\dot{H}^\alpha}^2 &\leq 2t^{-1} \left\|(-\Delta)^{\alpha/2} (F(\hat{w}) \hat{w}) - F(\hat{w}) (-\Delta)^{\alpha/2} \hat{w}\right\|_{L^2} \|\hat{w}\|_{\dot{H}^\alpha} \\ &\lesssim t^{-1} \|\hat{w}\|_{L^{2n/(n-2\alpha)}} \left\|(-\Delta)^{\alpha/2} (F(\hat{w}))\right\|_{L^{\frac{n}{\alpha}}} \|\hat{w}\|_{\dot{H}^\alpha} \end{aligned} \tag{1.32}$$

(see Lemma 2.3 of paper [25]). By the equalities $|\hat{w}(t)|_{\mathbb{C}^N} = |A(t)\hat{u}_+|_{\mathbb{C}^N} = |\hat{u}_+|_{\mathbb{C}^N}$ (see Lemma 1.12) and Corollary 1.7 with $p = 2n/(n - 2\alpha)$, $r = 2$ and $m = \alpha$, we have

$$\|\hat{w}\|_{L^{2n/(n-2\alpha)}} = \|A(t)\hat{u}_+\|_{L^{2n/(n-2\alpha)}} = \|\hat{u}_+\|_{L^{2n/(n-2\alpha)}} \lesssim \|u_+\|_{H^{0,\alpha}}. \tag{1.33}$$

Let $\tilde{q} = n/(n-1)$. We note that $|x|^{-1} * f = C(-\Delta)^{-(n-1)/2} f$ (see Lemma 1.8). By Lemma 1.8 with $\gamma = 1 + \alpha$, $p = n/\alpha$ and $q = \tilde{q}$, we get

$$\begin{aligned} \left\|(-\Delta)^{\alpha/2} (F(\hat{w}))\right\|_{L^{n/\alpha}} &\lesssim \left\|(-\Delta)^{-(n-1-\alpha)/2} (\hat{w}\overline{\hat{w}})\right\|_{L^{n/\alpha}} \lesssim \|\hat{w}\|_{L^{2\tilde{q}}}^2 \\ &= \|A(t)\hat{u}_+\|_{L^{2\tilde{q}}}^2 = \|\hat{u}_+\|_{L^{2\tilde{q}}}^2 \\ &\lesssim \|\hat{u}_+\|_{H^{1/2}}^2 \leq \|u_+\|_{H^{0,\alpha}}^2, \end{aligned} \tag{1.34}$$

here we have also used the equalities $|\hat{w}(t)|_{\mathbb{C}^N} = |A(t)\hat{u}_+|_{\mathbb{C}^N} = |\hat{u}_+|_{\mathbb{C}^N}$ and Corollary 1.7 with $p = 2\tilde{q}$, $r = 2$ and $m = 1/2$. By combining (1.32)-(1.34), we have

$$\partial_t \|\hat{w}\|_{\dot{H}^\alpha} \lesssim t^{-1} \|u_+\|_{H^{0,\alpha}}^3. \tag{1.35}$$

Integrating (1.35) over $[1, t]$ and using the initial condition $A(1,\xi) = I_N$ (see (1.27)), we have desired estimate. This completes the proof of the proposition. □

Next we state two estimates of remainder term involving operator $\mathcal{R}(t)$ given by (1.17).

Proposition 1.15. *Let $1/2 < \beta < \alpha < 1$ and $0 \le \delta \le \beta$. The following two estimates*

$$\left\|\mathcal{R}(t)(-\Delta)^{\delta/2}\hat{w}\right\|_{L^2} \lesssim t^{-(\alpha-\delta)/2}\|u_+\|_{H^{0,\alpha}}\left(1+\|u_+\|_{H^{0,\alpha}}^2\log t\right) \tag{1.36}$$

and

$$\left\|\mathcal{R}(\tau)(-\Delta)^{\delta/2}f(\hat{w})\right\|_{L^2} \lesssim t^{-(\alpha-\delta)/2}\|u_+\|_{H^{0,\alpha}}^3\left(1+\|u_+\|_{H^{0,\alpha}}^2\log t\right) \tag{1.37}$$

hold for any $t \ge 1$, where $\hat{w}(t,\xi) = A(t,\xi)\hat{u}_+$.

Proof. First we show the estimate (1.36). By the definition of $\mathcal{R}(t)$, $|\mathcal{M}(t)| = 1$ and $\|\mathcal{D}(t)\phi\|_{L^2} = \|\phi\|_{L^2}$, we have

$$\left\|\mathcal{R}(t)(-\Delta)^{\delta/2}\hat{w}\right\|_{L^2} = \left\|(\mathcal{M}(t)-1)\mathcal{F}^{-1}(-\Delta)^{\delta/2}\hat{w}\right\|_{L^2}. \tag{1.38}$$

We note that for any $\mu \in [0,1]$, we have

$$|\mathcal{M}(t)-1| \lesssim t^{-\mu}|x|^{2\mu}, \tag{1.39}$$

for any $t \ge 1$ and $x \in \mathbb{R}^n$. By (1.38)-(1.39) with $\mu = (\alpha-\delta)/2$, we have

$$\left\|\mathcal{R}(t)(-\Delta)^{\delta/2}\hat{w}\right\|_{L^2} \lesssim t^{-(\alpha-\delta)/2}\left\||x|^{\alpha-\delta}\mathcal{F}^{-1}(-\Delta)^{\alpha/2}\hat{w}\right\|_{L^2} = t^{-(\alpha-\delta)/2}\|\hat{w}\|_{\dot{H}^\alpha}. \tag{1.40}$$

Applying Proposition 1.14, the desired estimate (1.36) is obtained.

Next we prove the estimate (1.37). In the same proof as (1.40), we get

$$\left\|\mathcal{R}(\tau)(-\Delta)^{\frac{\delta}{2}}f(\hat{w})\right\|_{L^2} \lesssim t^{-(\alpha-\delta)/2}\|f(\hat{w})\|_{\dot{H}^\alpha}. \tag{1.41}$$

By the identity

$$(-\Delta)^{\alpha/2}f(\hat{w}) = (-\Delta)^{\alpha/2}(F(\hat{w})\hat{w}) - F(\hat{w})(-\Delta)^{\alpha/2}\hat{w} + F(\hat{w})(-\Delta)^{\alpha/2}\hat{w},$$

we have

$$\begin{aligned}\|f(\hat{w})\|_{\dot{H}^\alpha} \le{}& \left\|(-\Delta)^{\alpha/2}(F(\hat{w})\hat{w}) - F(\hat{w})(-\Delta)^{\alpha/2}\hat{w}\right\|_{L^2}\\ &+\left\|F(\hat{w})(-\Delta)^{\alpha/2}\hat{w}\right\|_{L^2}.\end{aligned} \tag{1.42}$$

We estimate the first term of the right hand side of (1.42). By the commutator estimate (1.14) with $s = s_1 = \alpha$, $s_2 = 0$, $q = n/\alpha$ and $r = 2n/(n-2\alpha)$, we get

$$\begin{aligned}&\left\|(-\Delta)^{\alpha/2}(F(\hat{w})\hat{w}) - F(\hat{w})(-\Delta)^{\alpha/2}\hat{w}\right\|_{L^2}\\ \lesssim{}& \|\hat{w}\|_{L^{2n/(n-2\alpha)}}\left\|(-\Delta)^{\alpha/2}(F(\hat{w}))\right\|_{L^{n/\alpha}}.\end{aligned}$$

By the estimates (1.33) and (1.34), we have

$$\|\hat{w}\|_{L^{2n/(n-2\alpha)}}\left\|(-\Delta)^{\alpha/2}(F(\hat{w}))\right\|_{L^{n/\alpha}} \lesssim \|u_+\|_{H^{0,\alpha}}^3.$$

Therefore we obtain

$$\left\|(-\Delta)^{\alpha/2}(F(\hat{w})\hat{w}) - F(\hat{w})(-\Delta)^{\alpha/2}\hat{w}\right\|_{L^2} \lesssim \|u_+\|_{H^{0,\alpha}}^3. \tag{1.43}$$

Next we consider the second term of the right hand side of (1.41). By Hölder's inequality, we have

$$\left\|F(\hat{w})(-\Delta)^{\alpha/2}\hat{w}\right\|_{L^2} \le \|F(\hat{w})\|_{L^\infty}\|\hat{w}\|_{\dot{H}^\alpha}.$$

We put $q^* = 2n/(n-2\beta)$ and $\tilde{q}^* = 2n/\{n-2(1-\beta)\}$. Then the following relations

$$2 < \tilde{q}^* < 2n/(n-1) < q^* < 2n/(n-2)$$

and $1/\tilde{q}^* = 1 - 1/n - 1/q^*$ are valid. Thus by Lemma 1.9 with $\gamma = 1$, $p = \tilde{q}^*$ and $q = q^*$, we get

$$\begin{aligned} \|F(\hat{w})\|_{L^\infty} &\lesssim \left\| |\cdot|^{-1} * (\hat{w}\overline{\hat{w}}) \right\|_{L^\infty} \lesssim \|\hat{w}\|_{L^{\tilde{q}^*}} \|\hat{w}\|_{L^{q^*}} \\ &= \|A(t)\hat{u}_+\|_{L^{\tilde{q}^*}} \|A(t)\hat{u}_+\|_{L^{q^*}} = \|\hat{u}_+\|_{L^{\tilde{q}^*}} \|\hat{u}_+\|_{L^{q^*}} \\ &\lesssim \|\hat{u}_+\|_{H^{1-\beta}} \|\hat{u}_+\|_{H^\beta} \le \|u_+\|^2_{H^{0,\alpha}}, \end{aligned}$$

here we have also used the equality $|\hat{w}(t)|_{\mathbb{C}^N} = |A(t)\hat{u}_+|_{\mathbb{C}^N} = |\hat{u}_+|_{\mathbb{C}^N}$ and the Sobolev inequality with $1/\tilde{q}^* = 1/2 - (1-\beta)/n$ and $1/q^* = 1/2 - \beta/n$ (see Corollary 1.7). Therefore we have

$$\left\| F(\hat{w})(-\Delta)^{\frac{\alpha}{2}} \hat{w} \right\|_{L^2} \lesssim \|u_+\|^2_{H^{0,\alpha}} \|\hat{w}\|_{\dot{H}^\alpha}. \tag{1.44}$$

By combining Proposition 1.14, (1.41)-(1.44), we get the desired estimate (1.37). This completes the proof of the proposition. □

1.6. **Proof of existence of the modified wave operator.** In this subsection, we give a proof of Theorem 1.1.

Proof. Let $t \ge T$ where T is sufficiently large. We take L^2-norm of (1.22) to get

$$\begin{aligned} \|u(t) - u_1\|_{L^2} \le& \int_t^\infty \|f(v) - f(u_1)\|_{L^2} d\tau \\ &+ \|\mathcal{R}\hat{w}\|_{L^2} + \int_t^\infty \|\mathcal{R}(\tau) f(\hat{w})\|_{L^2} \frac{d\tau}{\tau}. \end{aligned} \tag{1.45}$$

We consider the first term of the right hand side of (1.45). Let $p^* = n/\beta$ and $q^* = 2n/(n-2\beta)$. By the decomposition

$$\begin{aligned} & f(u_1) - f(v) \\ =\ & F(u_1)(u_1 - v) + F(u_1, u_1 - v) v + F(u_1 - v, v) v \end{aligned}$$

and Hölder's inequality with $1/2 = 1/p^* + 1/q^*$, we have

$$\begin{aligned} & \|f(u_1) - f(v)\|_{L^2} \\ \le\ & \|F(u_1)(u_1 - v)\|_{L^2} + \|F(u_1, u_1 - v) v\|_{L^2} + \|F(u_1 - v, v) v\|_{L^2} \\ \le\ & \|F(u_1)\|_{\mathbf{L}^\infty} \|u_1 - v\|_{L^2} + \|F(u_1, u_1 - v)\|_{L^{p^*}} \|v\|_{L^{q^*}} \\ & + \|F(u_1 - v, v)\|_{L^{p^*}} \|v\|_{L^{q^*}}. \end{aligned} \tag{1.46}$$

We estimate the first term of the right hand side of (1.46). Let $\tilde{q}^* = 2n/\{n - 2(1-\beta)\}$. Then the following relations

$$2 < \tilde{q}^* < 2n/(n-1) < q^* < 2n/(n-2) \text{ and } 1/\tilde{q}^* + 1/q^* = 1 - 1/n$$

hold, since $1/2 < \beta < 1$. Applying Lemma 1.9 with $\gamma = 1$, $p = \tilde{q}^*$ and $q = q^*$, we have

$$\|F(u_1)\|_{L^\infty} \lesssim \left\| |\cdot|^{-1} * (u_1 \overline{u_1}) \right\|_{L^\infty} \lesssim \|u_1\|_{L^{\tilde{q}^*}} \|u_1\|_{L^{q^*}}.$$

By (1.23) with $p = q^*$ and $\tilde{q}^*$, we obtain

$$\|F(u_1)\|_{L^\infty} \lesssim \tau^{-1} \|u_+\|^2_{H^{0,\alpha}}, \tag{1.47}$$

since $2 \le \tilde{q}^* < q^* \le 2n/(n-2\beta)$. Next we consider the second term of the right hand side of (1.46). Let $1/p_3 = 1 + 1/p^* - 1/n = 1 + (\beta - 1)/n$. By Lemma 1.8 with $\gamma = 1$, $p = p^*$ and $q = p_3$, we obtaind

$$\begin{aligned} \|F(u_1, u_1 - v)\|_{L^{p^*}} &\lesssim \left\| |\cdot|^{-1} * \left\{ u_1 \overline{(u_1 - v)} \right\} \right\|_{L^{p^*}} \\ &\lesssim \|u_1 (u_1 - v)\|_{L^{p_3}}. \end{aligned} \tag{1.48}$$

We choose p_4 such that $1/p_4 = 1/p_3 - 1/2 = 1/2 + (\beta - 1)/n$. By Hölder's inequality with $1/p_3 = 1/p_4 + 1/2$, we get

$$\|u_1 (u_1 - v)\|_{L^{p_3}} \leq \|u_1\|_{L^{p_4}} \|u_1 - v\|_{L^2}. \tag{1.49}$$

By (1.48)-(1.49), we have

$$\|F(u_1, u_1 - v)\|_{L^{p^*}} \|v\|_{L^{q^*}} \lesssim \|u_1\|_{L^{p_4}} \|u_1 - v\|_{L^2} \|v\|_{L^{q^*}}.$$

By (1.23) with $p = p_4$ and (1.25) with $p = q^*$, we obtain

$$\|F(u_1, u_1 - v)\|_{L^{p^*}} \|v\|_{L^{q^*}} \lesssim \rho\tau^{-(\beta/2+\mu)-1} \|u_+\|^2_{H^{0,\alpha}}, \tag{1.50}$$

since $2 \leq p_4, q^* \leq 2n/(n - 2\beta)$ and $v \in X_\rho$. In the same manner as in the proof of (1.50), we can estimate the third term of the right hand side of (1.46) as follows:

$$\|F(u_1 - v, v)\|_{L^{p^*}} \|v\|_{L^{q^*}} \lesssim \rho\tau^{-(\beta/2+\mu)-1} \|u_+\|^2_{H^{0,\alpha}}. \tag{1.51}$$

By combining (1.46), (1.47), (1.50) and (1.51), we get

$$\|f(v) - f(u_1)\|_{L^2} \lesssim \rho\tau^{-(\beta/2+\mu)-1} \|u_+\|^2_{H^{0,\alpha}}, \tag{1.52}$$

since $v \in X_\rho$. By Proposition 1.15 with $\delta = 0$ and (1.52), we obtain

$$\begin{aligned} \|u(t) - u_1\|_{L^2} &\lesssim \rho \|u_+\|^2_{H^{0,\alpha}} \int_t^\infty \tau^{-(\beta/2+\mu)-1} d\tau \\ &\quad + t^{-\alpha/2} \|u_+\|_{H^{0,\alpha}} \left(1 + \|u_+\|^2_{H^{0,\alpha}} \log t\right) \\ &\quad + \int_t^\infty \tau^{-1-\alpha/2} \|u_+\|^3_{H^{0,\alpha}} \left(1 + \|u_+\|^2_{H^{0,\alpha}} \log \tau\right) d\tau \\ &\lesssim t^{-(\beta/2+\mu)} \|u_+\|_{H^{0,\alpha}}, \end{aligned} \tag{1.53}$$

for any $t \geq T$ if T is sufficiently large and $\|u_+\|_{H^{0,\alpha}} \leq 1$. Precisely we take $T > 0$ such that the estimates

$$1 \leq \|u_+\|^2_{H^{0,\alpha}} \log t$$

and

$$t^{-\alpha/2} \log t \lesssim t^{-(\beta/2+\mu)}$$

are satisfied for any $\tau \geq t \geq T$, since $\alpha/2 > \beta/2 + \mu$.

Note that $|\mathcal{J}|^\beta \mathcal{R}(\tau) = \mathcal{R}(\tau)(-\Delta)^{\beta/2}$. Multiplying both sides of (1.22) by $|\mathcal{J}|^\beta$, we obtain

$$\begin{aligned} &|\mathcal{J}|^\beta (u(t) - u_1(t)) \\ &= i \int_t^\infty \mathcal{U}(t-\tau) |\mathcal{J}|^\beta (f(v) - f(u_1)) d\tau \\ &\quad + \mathcal{R}(t)(-\Delta)^{\frac{\beta}{2}} \hat{w} - i \int_t^\infty \mathcal{U}(t-\tau) \mathcal{R}(\tau)(-\Delta)^{\frac{\beta}{2}} f(\hat{w}) \frac{d\tau}{\tau}. \end{aligned} \tag{1.54}$$

We take L^2-norm of (1.54) to get

$$\begin{aligned} &\left\| |\mathcal{J}|^\beta (u(t) - u_1(t)) \right\|_{L^2} \\ &\leq \int_t^\infty \left\| |\mathcal{J}|^\beta (f(v) - f(u_1)) \right\|_{L^2} d\tau \\ &\quad + \left\| \mathcal{R}(t)(-\Delta)^{\frac{\beta}{2}} \hat{w} \right\|_{L^2} + \int_t^\infty \left\| \mathcal{R}(\tau)(-\Delta)^{\frac{\beta}{2}} f(\hat{w}) \right\|_{L^2} \frac{d\tau}{\tau}. \end{aligned}$$

By Proposition 1.15 with $\delta = \beta$, we have

$$\begin{aligned} & \left\| |\mathcal{J}|^{\beta} (u(t) - u_1(t)) \right\|_{L^2} \\ \le & \int_t^{\infty} \left\| |\mathcal{J}|^{\beta} (f(v) - f(u_1)) \right\|_{L^2} d\tau \\ & + t^{-(\alpha-\beta)/2} \|u_+\|_{H^{0,\alpha}} \left(1 + \|u_+\|_{H^{0,\alpha}}^2 \log t \right) \\ & + \|u_+\|_{H^{0,\alpha}}^3 \int_t^{\infty} \tau^{-1-(\alpha-\beta)/2} \left(1 + \|u_+\|_{H^{0,\alpha}}^2 \log \tau \right) d\tau. \end{aligned} \tag{1.55}$$

We consider the first term of the right hand side of (1.55). By the factorization of $|\mathcal{J}|^{\beta}(\tau) = \tau^{\beta} \mathcal{M}(\tau) (-\Delta)^{\beta/2} \mathcal{M}(-\tau)$, we can estimate

$$\begin{aligned} & \left\| |\mathcal{J}|^{\beta} (f(v) - f(u_1)) \right\|_{L^2} \\ \le & \tau^{\beta} \|F(u_1) \mathcal{M}(-t)(u_1 - v)\|_{\dot{H}^{\beta}} \\ & + \tau^{\beta} \|F(u_1, u_1 - v) \mathcal{M}(-t) v\|_{\dot{H}^{\beta}} \\ & + \tau^{\beta} \|F(u_1 - v, v) \mathcal{M}(-t) v\|_{\dot{H}^{\beta}}. \end{aligned} \tag{1.56}$$

We consider the first term of the right hand side of (1.56). We remember the definition of $p^* = n/\beta$ and $q^* = 2n/(n - 2\beta)$. By the fractional Leibniz rule (Lemma 3.5) with $\kappa = \beta$, $p = 2$, $q_1 = 2$, $q_2 = p^*$, $r_1 = \infty$ and $r_2 = q^*$, we have

$$\begin{aligned} & \|F(u_1) \mathcal{M}(-\tau)(u_1 - v)\|_{\dot{H}^{\beta}} \\ \lesssim & \|F(u_1)\|_{L^{\infty}} \|\mathcal{M}(-\tau)(u_1 - v)\|_{\dot{H}^{\beta}} \\ & + \|F(u_1)\|_{\dot{H}^{\beta}_{p^*}} \|u_1 - v\|_{L^{q^*}}. \end{aligned} \tag{1.57}$$

By combining (1.57), (1.8), (1.47) and (1.24), we obtain

$$\begin{aligned} & \|F(u_1) \mathcal{M}(-\tau)(u_1 - v)\|_{\dot{H}^{\beta}} \\ \lesssim & \tau^{-\beta} \|F(u_1)\|_{L^{\infty}} \left\| |\mathcal{J}|^{\beta} (u_1 - v) \right\|_{L^2} \\ & + \|F(u_1)\|_{\dot{H}^{\beta}_{p^*}} \|u_1 - v\|_{L^{q^*}} \\ \lesssim & \rho \tau^{-\beta-\mu} \|F(u_1)\|_{L^{\infty}} + \rho \tau^{-\beta-\mu} \|F(u_1)\|_{\dot{H}^{\beta}_{p^*}} \\ \lesssim & \rho \tau^{-1-\beta-\mu} \|u_+\|_{H^{0,\alpha}}^2 + \rho \tau^{-\beta-\mu} \|F(u_1)\|_{\dot{H}^{\beta}_{p^*}}, \end{aligned} \tag{1.58}$$

since $2 \le q^* \le 2n/(n - 2\beta)$ and $v \in X_{\rho}$. We need to estimate $\|F(u_1)\|_{\dot{H}^{\beta}_{p^*}}$. We put $\tilde{q} = n/(n-1)$ for simplicity. Then the equality

$$1 + 1/p^* = (\beta + 1)/n + 1/\tilde{q}$$

holds. By Lemma 1.8 with $\gamma = \beta + 1$, $p = p^*$ and $q = \tilde{q}$, we have

$$\begin{aligned} \|F(u_1)\|_{\dot{H}^{\beta}_{p^*}} & \lesssim \left\| |\cdot|^{-1} * (u_1 \overline{u_1}) \right\|_{\dot{H}^{\beta}_{p^*}} \\ & = \left\| (-\Delta)^{-(n-\beta-1)/2} (u_1 \overline{u_1}) \right\|_{L^{p^*}} \lesssim \|u_1\|_{L^{2\tilde{q}}}^2. \end{aligned} \tag{1.59}$$

Applying (1.23) with $p = 2\tilde{q}$, we get

$$\|u_1\|_{L^{2\tilde{q}}}^2 \lesssim \tau^{-1} \|u_+\|_{H^{0,\alpha}}^2, \tag{1.60}$$

since $2 \le 2\tilde{q} \le 2n/(n - 2\beta)$. By combining (1.58)-(1.60), we have

$$\|F(u_1) \mathcal{M}(-\tau)(u_1 - v)\|_{\dot{H}^{\beta}} \lesssim \rho \tau^{-1-\beta-\mu} \|u_+\|_{H^{0,\alpha}}^2. \tag{1.61}$$

Next we consider the second term of the right hand side of (1.56). We note that $p^* = n/\beta$ and $q^* = 2n/(n-2\beta)$. By the fractional Leibniz rule (Lemma 3.5) with $\kappa = \beta$, $p = 2$, $q_1 = p^*$, $q_2 = 2$, $r_1 = q^*$ and $r_2 = \infty$, we have

$$\begin{aligned} & \|F(u_1, u_1 - v)\mathcal{M}(-\tau)v\|_{\dot{H}^\beta} \\ \lesssim\ & \|F(u_1, u_1 - v)\|_{\dot{H}^\beta_{p^*}} \|v\|_{L^{q^*}} \\ & + \|F(u_1, u_1 - v)\|_{L^\infty} \|\mathcal{M}(-\tau)v\|_{\dot{H}^\beta}. \end{aligned} \tag{1.62}$$

We estimate the first term of the right hand side of (1.62). By Lemma 1.8 with $\gamma = \beta + 1$, $p = p^*$ and $q = \tilde{q}$ and Hölder's inequality with $1/\tilde{q} = 1/2\tilde{q} + 1/2\tilde{q}$, we have

$$\begin{aligned} \|F(u_1, u_1 - v)\|_{\dot{H}^\beta_{p^*}} &\lesssim \left\| |\cdot|^{-1} * \left\{ u_1 \overline{(u_1 - v)} \right\} \right\|_{\dot{H}^\beta_{p^*}} \\ &= \left\| (-\Delta)^{-(n-1-\beta)/2} \left\{ u_1 \overline{(u_1 - v)} \right\} \right\|_{L^{p^*}} \\ &\lesssim \|u_1(u_1 - v)\|_{L^{\tilde{q}}} \lesssim \|u_1\|_{L^{2\tilde{q}}} \|u_1 - v\|_{L^{2\tilde{q}}}. \end{aligned}$$

We note that $2 \le 2\tilde{q} < q^* \le 2n/(n-2\beta)$. By using (1.23), (1.24) with $p = 2\tilde{q}$ and (1.25) with $p = q^*$, we obtain

$$\begin{aligned} & \|F(u_1, u_1 - v)\|_{\dot{H}^\beta_{p^*}} \|v\|_{L^{q^*}} \\ \lesssim\ & \|u_1\|_{L^{2\tilde{q}}} \|u_1 - v\|_{L^{2\tilde{q}}} \|v\|_{L^{q^*}} \lesssim \rho\tau^{-3/4-3\beta/2-\mu} \|u_+\|^2_{H^{0,\alpha}}. \end{aligned} \tag{1.63}$$

Next we estimate the second term of the right hand side of (1.62). By (1.8), we get

$$\|\mathcal{M}(-\tau)v\|_{\dot{H}^\beta} = \tau^{-\beta} \left\| |\mathcal{J}|^\beta v \right\|_{L^2}. \tag{1.64}$$

By (1.15), $(-\Delta)^{\frac{\beta}{2}} \mathcal{D}(\tau) = \tau^{-\beta} \mathcal{D}(\tau)(-\Delta)^{\frac{\beta}{2}}$, $\|\mathcal{D}(\tau)\phi\|_{L^2} = \|\phi\|_{L^2}$ and (1.8), we have

$$\begin{aligned} \left\| |\mathcal{J}|^\beta u_1 \right\|_{L^2} &= \tau^\beta \left\| (-\Delta)^{\beta/2} \mathcal{D}(\tau) \hat{w}(\tau) \right\|_{L^2} \\ &= \left\| \mathcal{D}(\tau)(-\Delta)^{\beta/2} \hat{w}(\tau) \right\|_{L^2} = \|\hat{w}\|_{\dot{H}^\beta}. \end{aligned} \tag{1.65}$$

By applying Proposition 1.14 with $\alpha = \beta$ to (1.65), we obtain

$$\begin{aligned} \left\| |\mathcal{J}|^\beta v \right\|_{L^2} &\le \left\| |\mathcal{J}|^\beta (u_1 - v) \right\|_{L^2} + \left\| |\mathcal{J}|^\beta u_1 \right\|_{L^2} \\ &\le \rho\tau^{-\mu} + \|\hat{w}\|_{\dot{H}^\beta} \\ &\lesssim \rho\tau^{-\mu} + \|u_+\|_{H^{0,\alpha}} \left(1 + \|u_+\|^2_{H^{0,\alpha}} \log\tau \right) \\ &\lesssim \|u_+\|_{H^{0,\alpha}} \log\tau, \end{aligned} \tag{1.66}$$

for all $\tau \ge t \ge T$, if T is sufficiently large and $\|u_+\|_{H^{0,\alpha}}$ is sufficiently small, since $v \in X_\rho$. Precisely we take $T > 0$ and $\|u_+\|_{H^{0,\alpha}}$ such that the inequalities $\tau^{-\mu} \le \log\tau$, $1 \le \|u_+\|^2_{H^{0,\alpha}} \log\tau$ and $\|u_+\|_{H^{0,\alpha}} \le 1$ since $\mu > 0$. We remember the definition of $q^* = 2n/(n-2\beta)$ and $\tilde{q}^* = 2n/\{n - 2(1-\beta)\}$. We note that the relations

$$2 < \tilde{q}^* < 2n/(n-1) < q^* < 2n/(n-2)$$

and $1/\tilde{q}^* = 1 - 1/n - 1/q^*$ hold. Thus by Lemma 1.9 with $\gamma = 1$, $p = \tilde{q}^*$ and $q = q^*$, we have

$$\begin{aligned} \|F(u_1, u_1 - v)\|_{L^\infty} &\lesssim \left\| |\cdot|^{-1} * \left\{ u_1 \overline{(u_1 - v)} \right\} \right\|_{L^\infty} \\ &\lesssim \left(\|u_1 - v\|_{L^{q^*}} \|u_1\|_{L^{q^*}} \|u_1 - v\|_{L^{\tilde{q}^*}} \|u_1\|_{L^{\tilde{q}^*}} \right)^{1/2}. \end{aligned} \tag{1.67}$$

Applying (1.23) and (1.24) to (1.67), we obtain

$$\|F(u_1, u_1 - v)\|_{L^\infty} \lesssim \rho\tau^{-3/4-\beta/2-\mu} \|u_+\|^2_{H^{0,\alpha}}, \tag{1.68}$$

since $2 \le \tilde{q}, \tilde{q}^* \le 2n/(n-2\beta)$. By combining (1.64), (1.66) and (1.68), we have

$$\|F(u_1, u_1 - v)\|_{L^\infty} \|\mathcal{M}(-\tau)v\|_{\dot{H}^\beta} \lesssim \rho\tau^{-3/4-3\beta/2-\mu} (\log\tau) \|u_+\|^2_{H^{0,\alpha}} \tag{1.69}$$

if $\|u_+\|_{H^{0,\alpha}} \leq 1$. Therefore by (1.62), (1.63) and (1.69), we obtain

$$\|F(u_1, u_1 - v)\mathcal{M}(-\tau)v\|_{\dot{H}^\beta} \lesssim \rho\tau^{-3/4-3\beta/2-\mu}(\log\tau)\|u_+\|^2_{H^{0,\alpha}}, \tag{1.70}$$

for all $\tau \geq t \geq T$, if T is sufficiently large.

In the same way as in the proof of (1.70), we can estimate the third term of the right hand side of (1.56) as follows:

$$\|F(u_1 - v, v)\mathcal{M}(-\tau)v\|_{\dot{H}^\beta} \lesssim \rho\tau^{-3/4-3\beta/2-\mu}(\log\tau)\|u_+\|^2_{H^{0,\alpha}}. \tag{1.71}$$

By combining (1.56), (1.61), (1.70) and (1.71), we have

$$\left\||\mathcal{J}|^\beta(f(v) - f(u_1))\right\|_{L^2} \lesssim \rho\tau^{-1-\mu}\|u_+\|^2_{H^{0,\alpha}}, \tag{1.72}$$

for $\tau \geq t \geq T$, if T is so large that

$$\tau^{-3/4-3\beta/2-\mu}(\log\tau) \lesssim \tau^{-1-\mu}$$

for all $\tau \geq t \geq T$, since $\beta > 1/2$.

Then by virtue of (1.55) and (1.72), we have

$$\begin{aligned}\left\||\mathcal{J}|^\beta(u(t) - u_1(t))\right\|_{L^2} &\lesssim \rho\|u_+\|^2_{H^{0,\alpha}}\int_t^\infty \tau^{-1-\mu}d\tau + t^{-(\alpha-\beta)/2}(\log t)\|u_+\|_{H^{0,\alpha}} \\ &\lesssim t^{-\mu}\|u_+\|_{H^{0,\alpha}},\end{aligned} \tag{1.73}$$

for $t \geq T$, if $\|u_+\|_{H^{0,\alpha}} \leq 1$ and T is so large that the estimate

$$t^{-(\alpha-\beta)/2}(\log t) \lesssim t^{-\mu},$$

for any $t \geq T$, since $(\alpha - \beta)/2 > \mu$.

By (1.53) and (1.73), there exists a large time $T > 0$ such that

$$\|u - u_1\|_X \leq \|u_+\|_{H^{0,\alpha}}.$$

Furthermore if $\|u_+\|_{H^{0,\alpha}} \leq \rho$, then $u \in X_\rho$. In the same manner, we can prove the estimate

$$\|u - \tilde{u}\|_X \lesssim \|u_+\|_{H^{0,\alpha}}\|v - \tilde{v}\|_X,$$

for large $T > 0$, where $\tilde{u}$ is defined by (1.22) with (u, v) replaced by $(\tilde{u}, \tilde{v})$. From this inequality, we can obtain

$$\|u - \tilde{u}\|_X \leq \frac{1}{2}\|v - \tilde{v}\|_X,$$

if $\|u_+\|_{H^{0,\alpha}} \leq 1/2$ is satisfied. Therefore (1.22) defines a contraction mapping. Hence there exists a unique solution $u \in X_\rho$ of the integral equation (1.21). Therefore we see $u \in C([T,\infty); L^2)$, $|\mathcal{J}|^\beta u \in C([T,\infty); L^2)$ and the following inequalities

$$\|u(t) - u_1(t)\|_{L^2} \leq \rho t^{-\beta/4-\mu} \tag{1.74}$$

$$\left\||\mathcal{J}|^\beta(u(t) - u_1(t))\right\|_{L^2} \leq \rho t^{-\mu} \tag{1.75}$$

hold for any $t \geq T$. The estimate (1.9) for $t \geq T$ follows from (1.74), (1.75) and Lemma 1.6 with $p = q = r = 2$, $j = \delta$, $m = \beta$ and $a = \delta/\beta$.

Let $t > t_0 \geq T$. Using the integral equation (1.21), we can see that the equation

$$u(t) = \mathcal{U}(t - t_0)u(t_0) - i\int_{t_0}^t \mathcal{U}(t-\tau)f(u)d\tau \tag{1.76}$$

holds. By Proposition 1.11 with $u_0 = u(t_0) \in H^{0,\beta}$, we can extend the existence time to zero. Theorem 1.1 is proved □

2. Scattering problem for the Dirac-Klein-Gordon system (DKG)

2.1. **Introduction.** In this chapter, we study the Dirac-Klein-Gordon system (DKG) in three space dimensions.

$$\text{(DKG)} \qquad \begin{cases} (\partial_t + \alpha \cdot \nabla + iM\beta)\,\psi = \lambda\phi\beta\psi, \\ (\partial_t^2 - \Delta + m^2)\,\phi = \mu\psi^*\beta\psi, \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n$$

where M, m are positive constant and denote the masses of the spinor field and the scalar field respectively, ψ is a $\mathbb{C}^{2\left[\frac{n+1}{2}\right]}$-valued unknown function of (t,x) and ϕ is a $\mathbb{R}$-valued unknown function and stands a scalar field, $\lambda \in \mathbb{C}, \mu \in \mathbb{R}, \psi^* = {}^t\bar{\psi}$ denotes a transposed conjugate to the vector ψ. Here $\alpha_j, \beta \in M_{\left[\frac{n+1}{2}\right]}(\mathbb{C})$ are called Dirac matrices and defined by

$$\text{(DM)} \qquad \begin{gathered} \beta^2 = \alpha_j^2 = I,\ \alpha_j\beta + \beta\alpha_j = O, \\ \alpha_j\alpha_k + \alpha_k\alpha_j = O, \end{gathered}$$

for $j,k = 1,\dots n,\ j \neq k,\ I = [\delta_{jk}]_{1\le j,k\le 4}$. with the Kronecker symbols $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$.

Our purpose in this chapter is to prove existence of the scattering operator for (DKG) in theree space dimensions. The DKG system (DKG) was studied by reducing it to a nonlinear Klein-Gordon (KG) system (see [1, 71]). Denote $\mathcal{D}_\pm = \mathcal{D}_{\pm,M} = \partial_t + \pm(\alpha\cdot\nabla + iM\beta)$. We note that in view of the properties of the matrices α_j, β we have

$$\mathcal{D}_-\mathcal{D}_+\psi = \left\{\partial_t^2 - (\alpha\cdot\nabla + iM\beta)(\alpha\cdot\nabla + iM\beta)\right\}\psi = \left(\partial_t^2 + \langle\nabla\rangle_M^2\right)\psi,$$

where $\langle\nabla\rangle_M^2 = M^2 - \Delta$. Hence multiplying both sides of the Dirac equation $\mathcal{D}_+\psi = \lambda\phi\beta\psi$ by $\mathcal{D}_-$, we obtain

$$\left(\partial_t^2 + \langle\nabla\rangle_M^2\right)\psi = \lambda\mathcal{D}_-(\phi\beta\psi). \tag{2.1}$$

Solutions for (2.1) also satisfies (DKG). In fact since $\mathcal{D}_-(\mathcal{D}_+\psi - \lambda(\phi\beta\psi)) = 0$, by L^2-conservation law

$$\|\mathcal{D}_+\psi(t) - \lambda(\phi\beta\psi)(t)\|_{L^2} = \|(\mathcal{D}_+\psi)(0) - \lambda(\phi\beta\psi)(0)\|_{L^2} = 0,$$

if $\partial_t\psi(0) = -(\alpha\cdot\nabla + iM\beta)\psi(0) + \lambda(\phi\beta\psi)(0)$. Thus (DKG) is equivalent to (2.1). Moreover, solutions for (2.1) also satisfies

$$\begin{aligned} \left(\partial_t^2 + \langle\nabla\rangle_M^2\right)\psi &= \lambda(\mathcal{D}_-\phi)\beta\psi - iM\phi I\psi + \phi(\mathcal{D}_+\psi) \\ &= \lambda\left((\mathcal{D}_-\phi)\beta - iM\phi I + \lambda\phi^2 I\right)\psi, \end{aligned} \tag{2.2}$$

where we have used the fact that ψ is the solution of Dirac equation. Here we can see that equation (2.2) does not necesarily involve the derivative of ψ. However, solutions (2.2) does not satisfy (2.1) unfortunately. Thus especially, whether existence of the wave operator (local Cauchy problem at $t = +\infty$) can be constructed or not is not clear because of this fact and the derivative loss difficulty.

We recall the problem of existence of the wave operator for DKG system. Define the free Dirac evolution group by

$$\mathcal{V}_D(t) = \mathcal{V}_{D,M}(t) = I\cos(t\langle\nabla\rangle_M) - (\alpha\cdot\nabla + iM\beta)\langle\nabla\rangle_M^{-1}\sin(t\langle\nabla\rangle_M),$$

and the free KG evolution group by

$$\mathcal{V}_{KG}(t) = \mathcal{V}_{KG,m}(t) = \begin{pmatrix} \cos(\langle\nabla\rangle_m t) & \sin(\langle\nabla\rangle_m t) \\ -\sin(\langle\nabla\rangle_m t) & \cos(\langle\nabla\rangle_m t) \end{pmatrix}.$$

Then we look for the solutions of DKG system, which obey the following final state conditions

$$\lim_{t\to\infty}\left\|\mathcal{V}_D(-t)\,\psi(t)-\psi^+\right\|_{X_1}=0 \tag{2.3}$$

$$\lim_{t\to\infty}\left\|\mathcal{V}_{KG}(-t)\begin{pmatrix}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix}-\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{X_2}=0 \tag{2.4}$$

for the given final data $\psi^+\in X_1$, $(\phi_1^+,\phi_2^+)\in X_2$ with some Hilbert spaces X_1 and X_2 which are defined explicitly later.

The problem of the existence of the wave operator can be formulated in the form of the integral equations

$$\mathcal{V}_D(-t)\,\psi(t)=\psi^+-\int_t^\infty \mathcal{V}_D(-s)\,\phi\beta\psi(s)\,ds, \tag{2.5}$$

$$\mathcal{V}_{KG}(-t)\begin{pmatrix}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} \tag{2.6}$$

$$=\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}-\int_t^\infty \mathcal{V}_{KG}(-s)\begin{pmatrix}0\\ \langle\nabla\rangle_m^{-1}\psi^*\beta\psi(s)\end{pmatrix}ds \tag{2.7}$$

for the given final data $\psi^+\in\mathbf{X}_1$, $(\phi_1^+,\phi_2^+)\in\mathbf{X}_2$. If there exists a unique solution

$$\left(\psi(t),\phi(t),\langle\nabla\rangle_m^{-1}\partial_t\phi(t)\right)$$

of system (DKG) for the given final data $(\psi^+,\phi_1^+,\phi_2^+)$, then the wave operator $\mathcal{W}^+$ for the system (DKG) denotes the mapping

$$\left(\psi(t),\phi(t),\langle\nabla\rangle_m^{-1}\partial_t\phi(t)\right)=\mathcal{W}^+\left(\psi^+,\phi_1^+,\langle\nabla\rangle_m^{-1}\phi_2^+\right).$$

We introduce some notations.

If we can show that $D(\mathcal{W}^+)=R(\mathcal{W}^+)$, where $D(\mathcal{W}^+)$ is the domain and $R(\mathcal{W}^+)$ is the range of the wave operator $\mathcal{W}^+$, then we can easily construct the scattering operator. The existence of solutions for the cubic nonlinear Klein-Gordon equations in the low energy space along with the property $D(\mathcal{W}^+)=R(\mathcal{W}^+)$ were obtained in [60] by using the L^p-L^q method and the Strichartz type estimates. The cubic nonlinear Dirac equation $(\partial_t+\alpha\cdot\nabla+iM\beta)\,\psi=\lambda\,(\psi^*\beta\psi)\,\psi$ was studied in paper [49], where the scattering operator was obtained in H^s with $s>1$.

We now survey some works concerning KG system. The existence of the global small solutions to the Cauchy problem for the quadratic nonlinear KG including (DKG)-(2.2) was shown in [43] by applying the time decay estimates through the operators $(\partial_j,\partial_t,x_j\partial_t+t\partial_j)_{1\le j\le 3}$ and using the hyperbolic coordinates. The use of the hyperbolic coordinates implies the consideration of the problem inside of the light cone and so yields the compactness condition on the initial data. In papers [1], [8], [10], [28] the method of [43] was improved and the compactness condition on the data was removed however the higher order Sobolev spaces for the initial data were implemented. The global in time existence of small solutions to the Cauchy problem for quadratic nonlinear KG system including (DKG)-(2.2) was shown in [27] for the case of small initial data $(\psi(0),\partial_t\psi(0),\phi(0),\partial_t\phi(0))$ in the space $\left(H^{4,3}\times H^{3,3}\right)^5$, moreover the inverse wave operator was constructed from the neighborhood at the origin in the space $\left(H^{4,3}\times H^{3,3}\right)^5$ to the neighborhood at the origin in the space $\left(H^{4,1}\times H^{3,1}\right)^5$. In paper [71], it was shown the existence of the scattering operators for (DKG)-(2.2) from the neighborhood at the origin in the space $\left(H^{5/2,1}\times H^{3/2,1}\right)^4\times\left(H^{3,1}\times H^{2,1}\right)$ to the neighborhood at the origin in the same space.

Our main result is the following. Denote $90/37<q<6$ and $\mu=5/4-5/2q$. Note that we can choose $\mu=1/4$ when we take $q=5/2$.

Theorem 2.1. *Let $n = 3$ and the final data $\psi^+ \in \left(H^{3/2+\mu,1}\right)^4$, $\left(\phi_1^+, \phi_2^+\right) \in H^{2+\mu,1} \times H^{1+\mu,1}$. Then there exists $\varepsilon > 0$ such that for any final data $\left(\psi^+, \phi_1^+, \phi_2^+\right)$ satisfying estimate*

$$\left\|\psi^+\right\|_{H^{\frac{3}{2}+\mu,1}} + \left\|\phi_1^+\right\|_{H^{2+\mu,1}} + \left\|\phi_2^+\right\|_{H^{1+\mu,1}} \le \varepsilon,$$

there exists a unique global solution

$$\begin{aligned}&\left(\mathcal{V}_D(-t)\psi(t), \mathcal{V}_{KG}(-t)\begin{pmatrix}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix}\right)\\ \in\ &\left(C\left([0,\infty); H^{3/2+\mu,1}\right)\right)^4 \times \left(C\left([0,\infty); H^{2+\mu,1}\right)\right)^2\end{aligned}$$

for (DKG) with the final state condition

$$\begin{aligned}&\left\|\mathcal{V}_D(-t)\psi(t) - \psi^+\right\|_{H^{3/2+\mu,1}}\\ &+\left\|\mathcal{V}_{KG}(-t)\begin{pmatrix}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} - \begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H^{2+\mu,1}} \to 0\end{aligned} \tag{2.8}$$

as $t \to \infty$. Moreover the estimate is true

$$\left\|\mathcal{V}_D(-t)\psi(t)\right\|_{H^{\frac{3}{2}+\mu,1}} + \left\|\mathcal{V}_{KG}(-t)\begin{pmatrix}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix}\right\|_{H^{2+\mu,1}} \le C\varepsilon$$

for all $t \ge 0$.

Corollary 2.2. *The wave operator $\mathcal{W}^+$ for (DKG) is defined from the neighborhood at the origin in the space $\left(H^{3/2+\mu,1}\right)^4 \times \left(H^{2+\mu,1} \times H^{1+\mu,1}\right)$ to the neighborhood at the origin in the same space.*

In the same way as in the proof of Theorem 2.1 below, we can solve the initial value problem in the same function space, therefore we have the following result.

Corollary 2.3. *The inverse wave operator $\left(\mathcal{W}^-\right)^{-1}$ is defined from the neighborhood at the origin in the space $\left(H^{3/2+\mu,1}\right)^4 \times \left(H^{2+\mu,1} \times H^{1+\mu,1}\right)$ to the neighborhood of the origin in the same space.*

As a consequence of Corollaries 2.2 and 2.3 we get.

Corollary 2.4. *The scattering operator $\mathcal{S} = \left(\mathcal{W}^-\right)^{-1}\mathcal{W}^+$ is defined from the neighborhood at the origin in the space $\left(H^{3/2+\mu,1}\right)^4 \times \left(H^{2+\mu,1} \times H^{1+\mu,1}\right)$ to the neighborhood at the origin in the same space.*

We now explain our strategy of the proof of Theorem 2.1. Denote the free evolution group

$$\mathcal{U}_\pm(t) = \mathcal{U}_{\pm,M}(t) = e^{\pm it\langle\nabla\rangle_M} = \mathcal{F}^{-1}e^{\pm it\langle\xi\rangle_M}\mathcal{F}.$$

We can decompose $\mathcal{V}_D(t)$ as

$$\mathcal{V}_D(t) = \sum_{\pm}\mathcal{U}_\pm(t)\mathcal{A}_{\pm,M}, \tag{2.9}$$

where the operators

$$\mathcal{A}_\pm = \mathcal{A}_{\pm,M} = \frac{1}{2}\left(I \pm i\langle\nabla\rangle_M^{-1}(\alpha\cdot\nabla + iM\beta)\right).$$

Note that $(\alpha\cdot\nabla + iM\beta)^2 = -\langle\nabla\rangle_M^2 I$, due to the properties (DM). Hence by a direct calculation we get

$$\mathcal{A}_\pm\mathcal{A}_\mp = 0,\ \sum_{\pm}\mathcal{A}_\pm = I \text{ and } \mathcal{A}_\pm^2 = \mathcal{A}_\pm, \tag{2.10}$$

which show that the operators $\mathcal{A}_\pm$ are the projectors. A simple calculation shows that

$$\mathcal{A}_\pm(\alpha\cdot\nabla + iM\beta) = \frac{1}{2}\left(\alpha\cdot\nabla + iM\beta \mp i\langle\nabla\rangle_M I\right) = \mp i\langle\nabla\rangle_M\mathcal{A}_\pm. \tag{2.11}$$

Note that $\mathcal{A}_{\pm}$ commutes with operators ∂_t, $\langle i\nabla\rangle_M$ and $(\alpha \cdot \nabla + iM\beta)$, therefore we obtain

$$\mathcal{A}_{\pm}\mathcal{D}_{+}\psi = \mathcal{L}_{\pm}\mathcal{A}_{\pm}\psi,$$

where the Klein-Gordon operators $\mathcal{L}_{\pm} = \mathcal{L}_{\pm,M} = \partial_t \mp i\langle\nabla\rangle_M$. We multiply both sides of the Dirac equation (DKG) by the operator $\mathcal{A}_{\pm,M}$ to get

$$\left\{\begin{array}{l} \mathcal{L}_{\pm}\mathcal{A}_{\pm}\psi = \lambda\mathcal{A}_{\pm}(\phi\beta\psi), \\ \left(\partial_t^2 + \langle\nabla\rangle_m^2\right)\phi = \mu\psi^*\beta\psi. \end{array}\right. \tag{2.12}$$

Then we can reconstruct the solution of the Dirac equation (DKG) by the formula $\psi = \sum_{\pm}\mathcal{A}_{\pm}\psi$. Therefore to prove the existence of the wave operator we look for the solutions of system (2.12) which obey the following final state conditions

$$\left\|\mathcal{U}_{\pm}(-t)\mathcal{A}_{\pm}\psi(t) - \mathcal{A}_{\pm}\psi^{+}\right\|_{H^{\frac{3}{2}+\mu,1}} \to 0 \tag{2.13}$$

and

$$\left\|\mathcal{V}_{KG}(-t)\left(\begin{array}{c}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{array}\right) - \left(\begin{array}{c}\phi_1^{+}\\ \langle\nabla\rangle_m^{-1}\phi_2^{+}\end{array}\right)\right\|_{H^{2+\mu,1}} \to 0 \tag{2.14}$$

as $t \to \infty$ for given final data $\psi^{+} \in H^{\frac{3}{2}+\mu,1}$, $(\phi_1^{+}, \phi_2^{+}) \in H^{2+\mu,1}$. In order to show that problem (2.12)-(2.14) is equivalent to the original one, we must prove that the final state conditions $\left\|\mathcal{U}_{\pm}(-t)\mathcal{A}_{\pm}\psi(t) - \mathcal{A}_{\pm}\psi^{+}\right\|_{H^{3/2+\mu,1}} \to 0$ are equivalent to $\left\|\mathcal{V}_D(-t)\psi(t) - \psi^{+}\right\|_{H^{3/2+\mu,1}} \to 0$.

Through the paper, we write $A \simeq B$ if there exist some positive constants $C_1, C_2 > 0$ such that $C_1 B \le A \le C_2 B$ and we also write $A \lesssim B$ if there exists a positive constant $C > 0$ such that $A \le CB$.

Lemma 2.5. *The final state condition*

$$\lim_{t\to\infty}\sum_{\pm}\left\|\mathcal{U}_{\pm}(-t)\mathcal{A}_{\pm}f - \mathcal{A}_{\pm}f^{+}\right\|_{H^{3/2+\mu,1}} = 0$$

holds if and only if

$$\lim_{t\to\infty}\left\|\mathcal{V}_D(-t)f - f^{+}\right\|_{H^{3/2+\mu,1}} = 0.$$

Proof. We have by decomposition (2.9)

$$\left\|\mathcal{V}_D(-t)f - f^{+}\right\|_{H^{m,s}} = \left\|\sum_{\pm}\left(\mathcal{U}_{\pm}(-t)\mathcal{A}_{\pm}f - \mathcal{A}_{\pm}f^{+}\right)\right\|_{H^{m,s}}.$$

Hence by the triangle inequality

$$\left\|\mathcal{V}_D(-t)f - f^{+}\right\|_{H^{m,s}} \le \sum_{\pm}\left\|\mathcal{U}_{\pm}(-t)\mathcal{A}_{\pm}f - \mathcal{A}_{\pm}f^{+}\right\|_{H^{m,s}}.$$

On the other hand, since the operators $\mathcal{A}_{\pm}$ are the projectors by (2.10) we have

$$(\mathcal{A}_{\pm}g, \mathcal{A}_{\mp}h)_{L^2} = (\mathcal{A}_{\mp}\mathcal{A}_{\pm}g, h)_{L^2} = 0 \tag{2.15}$$

which implies

$$\left\|\mathcal{V}_{D,M}(-t)f - f^{+}\right\|_{H^{m,0}}^2 = \sum_{\pm}\left\|\mathcal{U}_{\pm,M}(-t)\mathcal{A}_{\pm,M}f - \mathcal{A}_{\pm,M}f^{+}\right\|_{H^{m,0}}^2.$$

This yields the result for $H^{m,0}$ spaces. We next consider the case of the weighted space. By (2.9), (2.10), and the Cauchy-Schwarz inequality we find

$$\begin{aligned} & \left\| x \langle \nabla \rangle^m \left(\mathcal{V}_D(-t) f - f^+ \right) \right\|_{L^2}^2 \\ = & \sum_{\pm} \left\| \mathcal{A}_\pm x \langle \nabla \rangle^m \left(\sum_{\pm} \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right) \right) \right\|_{L^2}^2 \\ \geq & \frac{1}{2} \sum_{\pm} \left\| x \mathcal{A}_\pm \langle \nabla \rangle^m \left(\sum_{\pm} \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right) \right) \right\|_{L^2}^2 \\ & - \sum_{\pm} \left\| [\mathcal{A}_\pm, x] \langle \nabla \rangle^m \left(\sum_{\pm} \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right) \right) \right\|_{L^2}^2 . \end{aligned}$$

By (2.10), we find

$$\begin{aligned} & x \mathcal{A}_\pm \langle \nabla \rangle^m \left(\sum_{\pm} \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right) \right) \\ = & \ x \langle \nabla \rangle^m \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right). \end{aligned}$$

Then using the estimate

$$\begin{aligned} & \left\| [\mathcal{A}_\pm, x] \langle \nabla \rangle^m \sum_{\pm} \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right) \right\|_{L^2}^2 \\ \lesssim & \left\| \sum_{\pm} \left(\mathcal{U}_\pm(-t) \mathcal{A}_\pm f - \mathcal{A}_\pm f^+ \right) \right\|_{H^{m-1}}^2 \lesssim \left\| \mathcal{V}_D(-t) f - f^+ \right\|_{H^{m-1}}^2 \end{aligned}$$

we get

$$\begin{aligned} & \left\| x \langle \nabla \rangle^m \left(\mathcal{U}_{\pm,M}(-t) \mathcal{A}_{\pm,M} f - \mathcal{A}_{\pm,M} f^+ \right) \right\|_{L^2}^2 \\ \lesssim & \left\| x \langle \nabla \rangle^m \left(\mathcal{V}_{D,M}(-t) f - f^+ \right) \right\|_{L^2}^2 + \left\| \mathcal{V}_{D,M}(-t) f - f^+ \right\|_{H^{m-1}}^2 . \end{aligned}$$

Therefore

$$\lim_{t \to \infty} \sum_{\pm} \left\| \mathcal{A}_\pm \left(\mathcal{U}_\pm(-t) f - f^+ \right) \right\|_{H^{m,1}} = 0$$

holds if

$$\lim_{t \to \infty} \left\| \mathcal{V}_D(-t) f - f^+ \right\|_{H^{m,1}} = 0.$$

Lemma 2.5 is proved. □

We use the Strichartz type estimates to treat the problem in the lower order Sobolev spaces, however it seems difficult to apply the Strichartz estimates to (2.10) due to the derivative loss of the nonlinear term. To obtain better differentiability properties of the nonlinear term we apply the operators

$$\mathcal{B}_\pm = \mathcal{B}_{\pm,M} = \frac{1}{2} \left(1 \mp i \langle \nabla \rangle_M^{-1} \partial_t \right) = \mp \frac{i}{2} \langle \nabla \rangle_M^{-1} \mathcal{L}_\mp$$

instead of $\mathcal{A}_\pm$, (which were used previously in [22], [23] to make a factorization of the Klein-Gordon operator $\partial_t^2 + \langle \nabla \rangle_M^2 = \mathcal{L}_+ \mathcal{L}_-$). Note that in view of the properties of the matrices α_j, β the identities are true

$$\mathcal{A}_\pm - \mathcal{B}_\pm = \pm \frac{i}{2} \langle \nabla \rangle_M^{-1} \mathcal{D}_+ \tag{2.16}$$

and

$$\mathcal{L}_\pm \mathcal{B}_\pm = \mp \frac{i}{2} \langle \nabla \rangle_M^{-1} \left(\partial_t^2 + \langle \nabla \rangle_M^2 \right) = \mp \frac{i}{2} \langle \nabla \rangle_M^{-1} \mathcal{D}_- \mathcal{D}_+ .$$

We will construct the desired solutions by the iterative procedure such that

$$\left\{ \begin{array}{l} \mathcal{D}_+\psi^{(k+1)} = \lambda\phi^{(k)}\beta\psi^{(k)}, \\ \left(\partial_t^2 + \langle\nabla\rangle_m^2\right)\phi^{(k+1)} = \mu\left(\psi^{(k)}\right)^*\beta\psi^{(k)}, \end{array} \right. \quad k \geq 1, \tag{2.17}$$

and

$$\left\{ \begin{array}{l} \mathcal{D}_+\psi^{(0)} = 0, \\ \left(\partial_t^2 + \langle\nabla\rangle_m^2\right)\phi^{(0)} = 0. \end{array} \right.$$

Also by virtue of the Dirac equation $\mathcal{D}_+\psi^{(k+1)} = \lambda\phi^{(k)}\beta\psi^{(k)}$ we obtain

$$\begin{aligned} & \mathcal{D}_-\left(\phi^{(k)}\beta\psi^{(k)}\right) \\ = & \left(\mathcal{D}_-\phi^{(k)}\right)\beta\psi^{(k)} - iM\phi^{(k)}I\psi^{(k)} + \phi^{(k)}\beta\mathcal{D}_+\psi^{(k)} \\ = & \left(\mathcal{D}_-\phi^{(k)}\right)\beta\psi^{(k)} - iM\phi^{(k)}I\psi^{(k)} + \lambda\phi^{(k)}\phi^{(k-1)}I\psi^{(k-1)}. \end{aligned}$$

Thus in view of the Dirac equation (DKG) it follows that

$$\begin{aligned} & \mathcal{L}_\pm\mathcal{B}_\pm\psi^{(k+1)} = \mp\frac{i}{2}\langle\nabla\rangle_M^{-1}\mathcal{D}_-\mathcal{D}_+\psi^{(k+1)} \\ = & \mp\frac{i}{2}\lambda\langle\nabla\rangle_M^{-1}\mathcal{D}_-\left(\phi^{(k)}\beta\psi^{(k)}\right) \\ = & \mp\frac{i}{2}\lambda\langle\nabla\rangle_M^{-1}\left(\left(\mathcal{D}_-\phi^{(k)}\right)\beta\psi^{(k)} - iM\phi^{(k)}I\psi^{(k)}\right. \\ & \left. +\lambda\phi^{(k)}\phi^{(k-1)}I\psi^{(k-1)}\right) \end{aligned} \tag{2.18}$$

for $k \geq 1$. In the case of $k = 1$, we have

$$\begin{aligned} & \mathcal{L}_\pm\mathcal{B}_\pm\psi^{(1)} = \mp\frac{i}{2}\lambda\langle\nabla\rangle_M^{-1}\mathcal{D}_\mp\left(\phi^{(0)}\beta\psi^{(0)}\right) \\ = & \mp\frac{i}{2}\lambda\langle\nabla\rangle_M^{-1}\left(\left(\mathcal{D}_-\phi^{(0)}\right)\beta\psi^{(0)} - iM\phi^{(0)}I\psi^{(0)}\right). \end{aligned}$$

Thus instead of (2.12), we can study the following system

$$\left\{ \begin{array}{l} \mathcal{L}_\pm\mathcal{B}_\pm\psi^{(k+1)} = \langle\nabla\rangle_M^{-1}F_{\pm,k}, \\ \left(\partial_t^2 + \langle\nabla\rangle_m^2\right)\phi^{(k+1)} = \pm 2iG_{\pm,k}, \end{array} \right. \tag{2.19}$$

where

$$\begin{aligned} F_{\pm,k} &= \mp\frac{i}{2}\lambda\left(\left(\mathcal{D}_{-,M}\phi^{(k)}\right)\beta\psi^{(k)} - iM\phi^{(k)}I\psi^{(k)}\right. \\ & \qquad \left. +\lambda\phi^{(k)}\phi^{(k-1)}I\psi^{(k-1)}\right), \\ G_{\pm,k} &= \mp\frac{i}{2}\mu\left(\psi^{(k)}\right)^*\beta\psi^{(k)}, \end{aligned}$$

for $k \geq 0$, here and below we use definitions $\phi^{(-l)} \equiv 0$, $\psi^{(-l)} \equiv 0$, so that, in particular, we define $F_{\pm,0} = 0$ and $G_{\pm,0} = 0$.

Note that the nonlinear terms of system (2.19) do not involve the derivatives of the unknown function ψ. This fact enables us to use the Strichartz type estimates. To treat the second equation of system (2.19) as in [22], [23] we introduce the new dependent variables $\mathcal{B}_\pm\phi^{(k)}$, then we have

$$\mathcal{L}_\pm\mathcal{B}_\pm\phi^{(k+1)} = \mp\frac{i}{2}\langle\nabla\rangle_m^{-1}\left(\partial_t^2 + \langle\nabla\rangle_m^2\right)\phi^{(k+1)} = \langle\nabla\rangle_m^{-1}G_{\pm,k},$$

where $\mathcal{L}_\pm = \mathcal{L}_{\pm,m} = \partial_t \mp i\langle\nabla\rangle_m$.

Thus we have a system

$$\begin{cases} \mathcal{L}_\pm \mathcal{B}_\pm \psi^{(k+1)} = \langle \nabla \rangle_M^{-1} F_{\pm,k}, \\ \mathcal{L}_\pm \mathcal{B}_\pm \phi^{(k+1)} = \langle \nabla \rangle_m^{-1} G_{\pm,k}, \end{cases} \tag{2.20}$$

for $k \geq 0$. Our purpose is to prove that the sequence

$$\left(\mathcal{V}_D(-t)\, \psi^{(k)}, \mathcal{V}_{KG}(-t) \begin{pmatrix} \phi^{(k)} \\ \langle i\nabla \rangle_m^{-1} \partial_t \phi^{(k)} \end{pmatrix} \right)$$

is the Cauchy sequence in the space

$$\left(C\left([0,\infty) ; H^{3/2+\mu,1} \right) \right)^4 \times \begin{pmatrix} C\left([0,\infty) ; H^{2+\mu,1}\right) \\ C\left([0,\infty) ; H^{2+\mu,1}\right) \end{pmatrix}$$

under the final state condition

$$\begin{aligned} &\lim_{t\to\infty} \left\| \mathcal{V}_D(-t)\, \psi^{(k)}(t) - \psi^+ \right\|_{H^{\frac{3}{2}+\mu,1}} \\ &+ \lim_{t\to\infty} \left\| \mathcal{V}_{KG}(-t) \begin{pmatrix} \phi^{(k)}(t) \\ \langle \nabla \rangle_m^{-1} \partial_t \phi^{(k)}(t) \end{pmatrix} - \begin{pmatrix} \phi_1^+ \\ \langle \nabla \rangle_m^{-1} \phi_2^+ \end{pmatrix} \right\|_{H^{2+\mu,1}} = 0. \end{aligned} \tag{2.21}$$

In order to write the integral equation associated with (2.20), (2.21), we study the asymptotic behavior of $\mathcal{B}_\pm \psi^{(k)}$. By Lemma 2.5 we find that

$$\sum_\pm \left\| \mathcal{U}_\pm(-t)\, \mathcal{A}_\pm \psi^{(k)}(t) - \mathcal{A}_\pm \psi^+ \right\|_{H^{3/2+\mu,1}} \lesssim \left\| \mathcal{V}_D(-t)\, \psi^{(k)}(t) - \psi^+ \right\|_{H^{3/2+\mu,1}}.$$

By the identity $\mathcal{B}_\pm \psi^{(k)} = \mathcal{A}_\pm \psi^{(k)} \mp \frac{i}{2}\lambda \langle \nabla \rangle_M^{-1} \phi^{(k-1)} \beta \psi^{(k-1)}$, we obtain

$$\begin{aligned} &\left\| \mathcal{U}_\pm(-t)\, \mathcal{A}_\pm \psi^{(k)}(t) - \mathcal{A}_\pm \psi^+ \right\|_{H^{3/2+\mu,1}} \\ = &\left\| \mathcal{U}_\pm(-t)\, \mathcal{B}_\pm \psi^{(k)}(t) - \mathcal{A}_\pm \psi^+ \pm \frac{i}{2}\lambda \langle \nabla \rangle_M^{-1} \mathcal{U}_\div(-t)\, \phi^{(k-1)} \beta \psi^{(k-1)} \right\|_{H^{3/2+\mu,1}} \end{aligned}$$

from which it follows that

$$\begin{aligned} &\left\| \mathcal{U}_\pm(-t)\, \mathcal{B}_\pm \psi^{(k)}(t) - \mathcal{A}_\pm \psi^+ \pm \frac{i}{2}\lambda \langle \nabla \rangle_M^{-1} \mathcal{U}_\pm(-t)\, \phi^{(k-1)} \beta \psi^{(k-1)} \right\|_{H^{3/2+\mu,1}} \\ = &\left\| \mathcal{U}_\pm(-t)\, \mathcal{A}_\pm \psi^{(k)}(t) - \mathcal{A}_\pm \psi^+ \right\|_{H^{3/2+\mu,1}} \lesssim \left\| \mathcal{V}_D(-t)\, \psi^{(k)}(t) - \psi^+ \right\|_{H^{3/2+\mu,1}}. \end{aligned}$$

Therefore from the first equation of (2.20) by virtue of the final state condition we have the integral equation

$$\begin{aligned} &\mathcal{U}_\pm(-t)\, \mathcal{B}_\pm \psi^{(k+1)}(t) - \mathcal{A}_\pm \psi^+ \\ = &-\int_t^\infty \langle \nabla \rangle_M^{-1} \mathcal{U}_\pm(-s)\, F_{\pm,k} ds \mp \frac{i}{2}\lambda \langle \nabla \rangle_M^{-1} \lim_{t\to\infty} \mathcal{U}_\pm(-t)\, \phi^{(k)} \beta \psi^{(k)} \end{aligned} \tag{2.22}$$

with given final data $\psi^+ \in H^{3/2+\mu,1}$. Denote $\Phi_\pm^+ = \frac{1}{2}\left(\phi_1^+ \mp i \langle\nabla\rangle_m^{-1} \phi_2^+\right)$. By the identity

$$\begin{aligned}
&\left\| \mathcal{V}_{KG}(-t) \begin{pmatrix} \phi^{(k)}(t) \\ \langle\nabla\rangle_m^{-1} \partial_t \phi^{(k)}(t) \end{pmatrix} - \begin{pmatrix} \phi_1^+ \\ \langle\nabla\rangle_m^{-1} \phi_2^+ \end{pmatrix} \right\|_{H^{2+\mu,1}}^2 \\
= & \left\| \sum_\pm \mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k)}(t) - \phi_1^+ \right\|_{H^{2+\mu,1}}^2 \\
& + \left\| \sum_\pm \pm \mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k)}(t) + i \langle\nabla\rangle_m^{-1} \phi_2^+ \right\|_{H^{2+\mu,1}}^2 \\
= & \left\| \sum_\pm \left(\mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k)}(t) - \Phi_\pm^+ \right) \right\|_{H^{2+\mu,1}}^2 \\
& + \left\| \sum_\pm \pm \left(\mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k)}(t) - \Phi_\pm^+ \right) \right\|_{H^{2+\mu,1}}^2 \\
= & \; 2 \sum_\pm \left\| \mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k)}(t) - \Phi_\pm^+ \right\|_{H^{2+\mu,1}}^2
\end{aligned}$$

via the parallelogram property. Thus we see that the latter of (2.21) is equivalent to

$$\lim_{t\to\infty} \sum_\pm \left\| \mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k)}(t) - \Phi_\pm^+ \right\|_{H^{2+\mu,1}} = 0.$$

Thus the second equation of system (2.20) can be written in the form of the integral equation

$$\mathcal{U}_\pm(-t) \mathcal{B}_\pm \phi^{(k+1)}(t) - \Phi_\pm^+ = -\int_t^\infty \langle\nabla\rangle_m^{-1} \mathcal{U}_\pm(-s) G_{\pm,k} ds. \tag{2.23}$$

In what follows we study the integral equations (2.22) and (2.23).

We introduce the operators $\mathcal{Z} = (\mathcal{Z}_k)_{1\le k\le n}, \mathcal{J}_\pm = \mathcal{J}_{\pm,M} = (\mathcal{J}_{k,\pm,M})_{1\le k\le n}$, where $\mathcal{Z}_k = x_k\partial_t + t\partial_k$, $\mathcal{J}_{k,\pm} = \mathcal{J}_{k,\pm,M} = t\partial_k \pm i x_k \langle\nabla\rangle_M$, which were used previously in [43] and in [23], respectively. We easily see that

$$x_k \mathcal{U}_\pm(-t) = \mathcal{F}^{-1} i\partial_k e^{\mp it\langle\xi\rangle_M} \mathcal{F} = \mathcal{U}_\pm(-t)\left(x_k \mp it \langle\nabla\rangle_M^{-1} \partial_k\right)$$

from which by the commutation relation $[x_k, \langle\nabla\rangle_M^a] = a\langle\nabla\rangle_M^{a-2}\partial_k$ it follows that

$$\langle\nabla\rangle_M \mathcal{U}_\pm(t) x_k \mathcal{U}_\pm(-t) = \langle\nabla\rangle_M x_k \mp it\partial_k = \mp i\mathcal{J}_{k,\pm} - \langle\nabla\rangle_M^{-1}\partial_k. \tag{2.24}$$

Let us also compute the commutation relations for the operators $\mathcal{L}_\pm$, $\mathcal{Z}_k$ and $\mathcal{J}_{k,\pm}$

$$\begin{aligned}
\mathcal{L}_\pm \mathcal{Z}_k = \mathcal{Z}_k \mathcal{L}_\pm + \partial_k \pm i [x_k, \langle\nabla\rangle_M] \partial_t = \left(\mathcal{Z}_k \pm i\partial_k \langle\nabla\rangle_M^{-1}\right) \mathcal{L}_\pm, \\
[\mathcal{J}_{k,\pm}, \mathcal{L}_\pm] = [x_k, \langle\nabla\rangle_M] \langle\nabla\rangle_M - \partial_k = 0.
\end{aligned} \tag{2.25}$$

Also we have the relations $\left[\mathcal{Z}_k, \langle\nabla\rangle_M^{-1}\right] = -\langle\nabla\rangle_M^{-3}\partial_k\partial_t$ and

$$\mathcal{Z}_k - \mathcal{J}_{k,\pm} = x_k \mathcal{L}_\pm. \tag{2.26}$$

2.2. Time decay estimate and Strichartz estimate.

We first state the time decay estimates through the operators $\mathcal{J}_{\pm,m}$ for any smooth and decaying function (see [27] for the proof).

Lemma 2.6. *Let $m > 0$ and the space dimension $n \ge 2$. Then the estimate*

$$\|\phi\|_{L^p} \le C \langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \left(\|\phi\|_{H^\nu}^{1-\frac{n}{2}+\frac{n}{p}} \|\mathcal{J}_{\pm,m}\phi\|_{H^{\nu-1}}^{\frac{n}{2}-\frac{n}{p}} + \|\phi\|_{H^\nu} \right)$$

is valid for all $t \geq 0$, *where* $\nu = \frac{n+2}{2}\left(1-\frac{2}{p}\right)$, $2 \leq p < 2n/(n-2)$, *provided that the right-hand side is finite.*

Denote the space-time norm

$$\|\phi\|_{L_t^r(I;L_x^q)} = \left\|\|\phi(t)\|_{L_x^q}\right\|_{L_t^r(I)},$$

where I is a bounded or unbounded time interval. Define

$$\mathcal{G}_{\pm,m}[g](t) = \int_T^t \mathcal{U}_{\pm,m}(t-\tau)\langle\nabla\rangle^{-1} g(\tau)\,d\tau$$

for any $T \in \overline{I}$, where $m > 0$. By the duality argument of [68] along with the $L^p - L^q$ time decay estimates of [48] we have the Strichartz type estimates.

Lemma 2.7. *Let* $2 \leq q < 2n/(n-2)$ *and* $\frac{2}{r} = \frac{n}{2}\left(1-\frac{2}{q}\right)$. *Then for any time interval* I *the following estimates are true*

$$\|\mathcal{G}_{\pm,m}[g]\|_{L_t^r(I;L_x^q)} \lesssim \|g\|_{L_t^{r'}\left(I;H_{q'}^{2\mu-1}\right)},$$

$$\|\mathcal{G}_{\pm,m}[g]\|_{L_t^\infty(I;L_x^2)} \lesssim \|g\|_{L_t^{r'}\left(I;H_{q'}^{\mu-1}\right)}$$

and

$$\|\mathcal{U}_{\pm,m}(t)\phi\|_{L_t^r(I;L_x^q)} \lesssim \|\phi\|_{H^\mu},$$

where $r' = r/(r-1)$, $q' = q/(q-1)$ *and* $\mu = \frac{n+2}{4}\left(1-\frac{2}{q}\right)$.

2.3. **Proof of existence of the scattering operator for DKG in 3d.** We define the vectors

$$v^{(k)} = \left(\mathcal{B}_+\psi^{(k)}, \mathcal{B}_-\psi^{(k)}, \langle\nabla\rangle_m^{\frac{1}{2}}\mathcal{B}_+\phi^{(k)}, \langle\nabla\rangle_m^{\frac{1}{2}}\mathcal{B}_-\phi^{(k)}\right),$$

$$F^{(k)} = \left(\langle\nabla\rangle_M^{-1}F_{+,k}, \langle\nabla\rangle_M^{-1}F_{-,k}, \langle\nabla\rangle_m^{-\frac{1}{2}}G_{+,k}, \langle\nabla\rangle_m^{-\frac{1}{2}}G_{-,k}\right)$$

and the matrix-operator

$$\mathcal{L} = \mathrm{diag}\left(\mathcal{L}_{+,M}, \mathcal{L}_{-,M}, \mathcal{L}_{+,m}, \mathcal{L}_{-,m}\right).$$

Then system (2.20) can be rewritten as

$$\mathcal{L}v^{(k+1)} = F^{(k)}. \tag{2.27}$$

Also we denote the matrices

$$\mathcal{U}(t) = \mathrm{diag}\left(\mathcal{U}_{+,M}(t), \mathcal{U}_{-1,M}(t), \mathcal{U}_{+1,m}(t), \mathcal{U}_{-1,m}(t)\right),$$
$$\mathbf{b} = \mathrm{diag}(1,1,0,0)$$

and the vector of the final data

$$\mathbf{v}^+ = \mathrm{diag}\left(\mathcal{A}_+\psi^+, \mathcal{A}_-\psi^+, \langle i\nabla\rangle_m^{\frac{1}{2}}\Phi_+^+, \langle i\nabla\rangle_m^{\frac{1}{2}}\Phi_-^+\right),$$

where $\Phi_\pm^+ = \frac{1}{2}\left(\phi_1^+ \mp i\langle\nabla\rangle_m^{-1}\phi_2^+\right)$, then the integral equations (2.22) and (2.23) can be written as

$$\mathcal{U}(-t)v^{(k+1)}(t) - \mathbf{v}^+ = -\int_t^\infty \mathcal{U}(-s)F^{(k)}ds \mp \frac{i}{2}\lambda\langle\nabla\rangle_M^{-1}\lim_{t\to\infty}\mathcal{U}(-t)\mathbf{b}\phi^{(k)}\beta\psi^{(k)} \tag{2.28}$$

for $k \geq 0$. We also have $\mathcal{U}(-t)v^{(0)}(t) - \mathbf{v}^+ = 0$, i.e. $v^{(0)}(t) = \mathcal{U}(t)\mathbf{v}^+$.

We introduce the function space for the final data $v^+ \in \left(H^{3/2+\mu,1}\right)^{10}$ and for the successive approximations $v^{(k)}$

$$X_I = \left\{v \in C\left(I;\left(H^{\frac{3}{2}+\mu,1}\right)^{10}\right);\ \|v\|_{X_I} < \infty\right\}$$

with the norm

$$\|v\|_{X_I} = \sum_{|\alpha|\le 1}\left(\|\mathcal{P}^\alpha v\|_{L_t^r\left(I;H_q^{3/2-|\alpha|}\right)} + \|\mathcal{P}^\alpha v\|_{L_t^\infty\left(I;H^{3/2+\mu-|\alpha|}\right)}\right),$$

where $I=[t,\infty)$, $\frac{2}{r} = \frac{3}{2}\left(1-\frac{2}{q}\right)$, and $\mu = \frac{5}{4} - \frac{5}{2q}$, $\frac{90}{37} < q < 6$, and $\mathcal{P} = (\mathcal{Z}, \mathcal{J})$,

$$\mathcal{J} = \operatorname{diag}\left(\mathcal{J}_{+1,M}, \mathcal{J}_{-1,M}, \mathcal{J}_{+1,m}, \mathcal{J}_{-1,m}\right).$$

For example, we can choose $q = 5/2$, then $\mu = 1/4$.

Then the result of Theorem 2.1 follows from Lemma 2.5 and the following theorem.

Theorem 2.8. *Assume that* $\mathbf{v}^+ \in \left(H^{3/2+\mu,1}\right)^{10}$, *and* $\|\mathbf{v}^+\|_{H^{3/2+\mu,1}} \le \varepsilon$ *for some* $\varepsilon > 0$. *Then there exists a unique global solution* $v^{(\infty)} \in X_{[0,\infty)}$ *to the integral equation (2.28). Moreover, the estimate* $\|v^{(\infty)}\|_{X_{[0,\infty)}} \le \varepsilon^{3/4}$ *is true.*

Proof. It is easy to see that $v^{(0)} = \mathcal{U}(t)\,\mathbf{v}^+ \in X_{[0,\infty)}$ and $\|v^{(0)}\|_{X_{[0,\infty)}} \le \varepsilon^{3/4}$. By induction we assume that for some $k \ge 1$

$$v^{(l)} \in X_{[0,\infty)} \text{ and } \left\|v^{(l)}\right\|_{X_{[0,\infty)}} \le \varepsilon^{3/4}$$

for all $0 \le l \le k$. Let us prove that $v^{(k+1)} \in X_{[0,\infty)}$ and $\|v^{(k+1)}\|_{X_{[0,\infty)}} \le \varepsilon^{3/4}$.

By Lemma 2.6 we have the estimates

$$\begin{aligned}\left\|v^{(l)}\right\|_{H_p^a} &\lesssim \langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}\left(\left\|\mathcal{J}v^{(l)}\right\|_{H^{a+\nu-1}} + \left\|v^{(l)}\right\|_{H^{a+\nu}}\right)\\ &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}\left\|v^{(l)}\right\|_{X_{[0,\infty)}} \le C\varepsilon^{3/4}\langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}, \end{aligned}\tag{2.29}$$

where $2 \le p < 6$, $a \le 3/2 + \mu - \nu = 5/p + \mu - 1$, $1 \le l \le k$. In particular, since

$$\psi^{(l)} = v_1^{(l)} + v_2^{(l)},\ \partial_t\psi^{(l)} = i\langle\nabla\rangle_M\left(v_1^{(l)} - v_2^{(l)}\right),$$

and

$$\phi^{(l)} = \langle\nabla\rangle_m^{-\frac{1}{2}}\left(v_3^{(l)} + v_4^{(l)}\right),\ \partial_t\phi^{(l)} = i\langle\nabla\rangle_m^{\frac{1}{2}}\left(v_3^{(l)} - v_4^{(l)}\right),$$

we can write

$$\left\|\psi^{(l)}\right\|_{H_p^a} + \left\|\partial_t\psi^{(l)}\right\|_{H_p^{a-1}} + \left\|\phi^{(l)}\right\|_{H_p^{a+1/2}} + \left\|\partial_t\phi^{(l)}\right\|_{H_p^{a-1/2}} \lesssim \varepsilon^{3/4}\langle t\rangle^{\gamma-1}$$

with $\gamma = 3/p - 1/2 > 0$, $2 \le p < 6$, $a \le 5/p + \mu - 1$, $1 \le l \le k$. So using the identity $x = \mp i\mathcal{J}_\pm\langle i\nabla\rangle_M^{-1} \pm it\nabla\langle\nabla\rangle_M^{-1}$, we find the estimate

$$\begin{aligned}&\left\|x\psi^{(l)}\right\|_{H_p^a} + \left\|x\langle\nabla\rangle_m^{\frac{1}{2}}\phi^{(k)}\right\|_{H_p^a}\\ &\lesssim \sum_\pm\left(\left\|\mathcal{J}_\pm\langle\nabla\rangle_M^{-1}\mathcal{B}_\pm\psi^{(k)}\right\|_{H_p^a} + t\left\|\mathcal{B}_\pm\psi^{(k)}\right\|_{H_p^a}\right)\\ &\quad+\sum_\pm\left(\left\|\mathcal{J}_\pm\langle\nabla\rangle_m^{-\frac{1}{2}}\mathcal{B}_\pm\phi^{(k)}\right\|_{H_p^a} + t\left\|\langle\nabla\rangle_m^{\frac{1}{2}}\mathcal{B}_\pm\phi^{(k)}\right\|_{H_p^a}\right)\\ &\lesssim \langle t\rangle^{3/p-1/2}\left\|v^{(k)}\right\|_{X_{[0,\infty)}}\end{aligned}\tag{2.30}$$

for $2 \le p < 6$, $a \le 5/p + \mu - 1$, $1 \le l \le k$. Then by (2.29), (2.30), the commutation relation $[x_l, \langle\nabla\rangle_M^a] = a\langle\nabla\rangle_M^{a-2}\partial_l$, we obtain by the Sobolev and Hölder inequalities with

$1/q = 1/p_1 + 1/p_2$, $\mu = 5/4 - 5/(2q)$. (When we choose $q = 5/2$, $\mu = 1/4$, then we can take $p_1 = 4$ and $p_2 = 20/3$)

$$\begin{aligned} & \left\| x_l \mathcal{L}_\pm \mathcal{B}_\pm \psi^{(k+1)} \right\|_{H_q^{1/2}} = \left\| x_l \langle \nabla \rangle_M^{-1} F_{\pm,k} \right\|_{H_q^{1/2}} \\ \lesssim & \left(\left\| \mathcal{D}_- \phi^{(k)} \right\|_{L^{p_1}} + \left\| \phi^{(k)} \right\|_{L^{p_1}} \right) \left\| \langle x \rangle \psi^{(k)} \right\|_{L^{p_2}} \\ & + \left\| \phi^{(k)} \right\|_{L^\infty} \left\| \phi^{(k)} \right\|_{L^{p_1}} \left\| \langle x \rangle \psi^{(k-1)} \right\|_{L^{p_2}} \\ \lesssim & \langle t \rangle^{-\gamma} \left(\left\| v^{(k)} \right\|_{X_{[0,\infty)}}^2 + \left\| v^{(k-1)} \right\|_{X_{[0,\infty)}}^3 \right) \end{aligned}$$

with $\gamma = 2 - 3\left(\frac{1}{p_1} + \frac{1}{p_2}\right) > 0$, and similarly by the Hölder inequality with $1/q = 1/p_1 + 1/p_2$ (When we choose $q = 5/2$, $\mu = 1/4$, then we can take $p_1 = 4$ and $p_2 = 20/3$)

$$\begin{aligned} & \left\| x_l \mathcal{L}_\pm \langle \nabla \rangle_m^{\frac{1}{2}} \mathcal{B}_\pm \phi^{(k+1)} \right\|_{H_q^{1/2}} \lesssim \left\| x_l \langle \nabla \rangle_m^{-\frac{1}{2}} G_{\pm,k} \right\|_{H_q^{1/2}} \lesssim \left\| \langle x \rangle G_{\pm,k} \right\|_{L^q} \\ \lesssim & \left\| \psi^{(k)} \right\|_{L^{p_2}} \left\| \langle x \rangle \psi^{(k)} \right\|_{L^{p_1}} \lesssim \langle t \rangle^{-\gamma} \left\| v^{(k)} \right\|_{X_{[0,\infty)}}^2, \end{aligned}$$

with $\gamma = 2 - 3(1/p_1 + 1/p_2) > 0$. In the same manner by the Sobolev and Hölder inequalities with $1/2 = 1/p_1 + 1/p_2$, $\mu = 5/4 - 5/2q$, we obtain (When we choose $q = 5/2$, $\mu = 1/4$, then we can choose $p_1 = 3$ and $p_2 = 6$)

$$\begin{aligned} & \left\| x_l \mathcal{L}_\pm \mathcal{B}_\pm \psi^{(k+1)} \right\|_{H^{1/2+\mu}} \\ \lesssim & \left\| \langle x \rangle F_{\pm,k} \right\|_{H^{\mu-1/2}} \\ \lesssim & \left(\left\| \mathcal{D}_- \phi^{(k)} \right\|_{L^{p_1}} + \left\| \phi^{(k)} \right\|_{L^{p_1}} \right) \left\| \langle x \rangle \psi^{(k)} \right\|_{L^{p_2}} \\ & + \left\| \phi^{(k)} \right\|_{L^\infty} \left\| \phi^{(k-1)} \right\|_{L^{p_1}} \left\| \langle x \rangle \psi^{(k-1)} \right\|_{L^{p_2}} \\ \lesssim & \langle t \rangle^{-\gamma} \left(\left\| v^{(k)} \right\|_{X_{[0,\infty)}}^2 + \left\| v^{(k-1)} \right\|_{X_{[0,\infty)}}^3 \right) \end{aligned}$$

with $\gamma = 2 - 3(1/p_1 + 1/p_2) > 0$, and by the Hölder inequality with $1/2 = 1/p_1 + 1/p_2$ (When we choose $q = 5/2$, $\mu = 1/4$, then we can choose $p_1 = 3$ and $p_2 = 6$)

$$\begin{aligned} & \left\| x_l \mathcal{L}_\pm \langle \nabla \rangle_m^{\frac{1}{2}} \mathcal{B}_\pm \phi^{(k+1)} \right\|_{H^{1/2+\mu}} \\ \lesssim & \left\| \langle x \rangle G_{\pm,k} \right\|_{H^\mu} \lesssim \left\| \psi^{(k)} \right\|_{H_{p_2}^\mu} \left\| \langle x \rangle \psi^{(k)} \right\|_{H_{p_1}^\mu} \lesssim \langle t \rangle^{-\gamma} \left\| v^{(k)} \right\|_{X_{[0,\infty)}}^2 \end{aligned}$$

with $\gamma = 2 - 3\left(\frac{1}{p_1} + \frac{1}{p_2}\right) > 0$. Therefore

$$\begin{aligned} & \left\| x \mathcal{L} v^{(k+1)} \right\|_{L_t^r\left(I; H_q^{1/2}\right)} + \left\| x \mathcal{L} v^{(k+1)} \right\|_{L_t^\infty\left(I; H^{1/2+\mu}\right)} \\ \lesssim & \left(\left\| v^{(k)} \right\|_{X_{[0,\infty)}}^2 + \left\| v^{(k+1)} \right\|_{X_{[0,\infty)}}^3 \right) \end{aligned}$$

with $\gamma > 0$. By (2.26), we have

$$\begin{aligned} &\left\|\mathcal{J}v^{(k+1)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)} + \left\|\mathcal{J}v^{(k+1)}\right\|_{L_t^\infty\left(I;H^{1/2+\mu}\right)} \\ \leq\ &\left\|\mathcal{Z}v^{(k+1)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)} + \left\|\mathcal{Z}v^{(k+1)}\right\|_{L_t^\infty\left(I;H^{1/2+\mu}\right)} \\ &+ \left\|x\mathcal{L}v^{(k+1)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)} + \left\|x\mathcal{L}v^{(k+1)}\right\|_{L_t^\infty\left(I;H^{1/2+\mu}\right)} \\ \lesssim\ &\left\|\mathcal{Z}v^{(k+1)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)} + \left\|\mathcal{Z}v^{(k+1)}\right\|_{L_t^\infty\left(I;H^{1/2+\mu}\right)} \\ &+ \left(\left\|v^{(k)}\right\|^2_{X_{[0,\infty)}} + \left\|v^{(k-1)}\right\|^3_{X_{[0,\infty)}}\right). \end{aligned}$$

So we need to estimate only the operators $\mathcal{Z}$. Multiplying both sides of (2.27) by $\mathcal{Z}^\alpha$, and using (2.25) we obtain

$$\mathcal{L}\mathcal{Z}^\alpha v^{(k+1)} = \left(\mathcal{Z}^\alpha + \mathcal{Q}^\alpha\right) F^{(k)} \tag{2.31}$$

where $\mathcal{Q}^\alpha = i\nabla^\alpha\left(+\langle\nabla\rangle_M^{-1}, -\langle\nabla\rangle_M^{-1}, +\langle\nabla\rangle_m^{-1}, -\langle\nabla\rangle_m^{-1}\right)^t$. Note that the operator $\mathcal{Q}^\alpha$ acts as a zeroth order operator. From (2.31) we find

$$\frac{d}{dt}\mathcal{U}(-t)\,\mathcal{Z}^\alpha v^{(k+1)} = \mathcal{U}(-t)\left(\mathcal{Z}^\alpha + \mathcal{Q}^\alpha\right) F^{(k)} \tag{2.32}$$

for $|\alpha| \leq 1$. By our final conditions, we have

$$\lim_{t\to\infty}\mathcal{U}(-t)\,v^{(k+1)}(t) = \mathbf{v}^+ \mp \frac{i}{2}\lambda\langle\nabla\rangle_M^{-1}\lim_{t\to\infty}\mathcal{U}(-t)\,\mathbf{b}\phi^{(k)}\beta\psi^{(k)}.$$

By (2.26) and (2.27), we obtain

$$\mathcal{Z}_l\mathcal{B}_\pm\langle\nabla\rangle_m^{\frac{1}{2}}\phi^{(k+1)} = \mathcal{J}_{l,\pm}\mathcal{B}_\pm\langle\nabla\rangle_m^{\frac{1}{2}}\phi^{(k+1)} + x_l\langle\nabla\rangle_m^{-\frac{1}{2}}G_{\pm,k}$$

from which in view of (2.25) it follows

$$\mathcal{U}_\pm(-t)\,\mathcal{Z}_l\mathcal{B}_\pm\langle\nabla\rangle_m^{\frac{1}{2}}\phi^{(k+1)} = \mathcal{K}_{l,\pm}\mathcal{U}_\pm(-t)\,\mathcal{B}_\pm\langle\nabla\rangle_m^{\frac{1}{2}}\phi^{(k+1)} + \mathcal{U}_\pm(-t)\,x_l\langle\nabla\rangle_m^{-\frac{1}{2}}G_{\pm,k}$$

where $\mathcal{K}_{l,\pm} = \mathcal{K}_{l,\pm,m} = \pm i\left(\langle\nabla\rangle_m x_l + \langle\nabla\rangle_m^{-1}\partial_l\right)$. Note that

$$\begin{aligned} &\left\|x_l\langle\nabla\rangle_m^{-\frac{1}{2}}G_{\pm,k}\right\|_{H^{\frac{1}{2}+\mu}} \lesssim \left\|G_{\pm,k}\right\|_{H^{\mu,1}} \\ \lesssim\ &\left\|x\psi^{(k)}\right\|_{H_6^\mu}\left\|\psi^{(k)}\right\|_{H_3^\mu} \lesssim \langle t\rangle^{-\gamma}\left\|v^{(k)}\right\|^2_{X_{[0,\infty)}}. \end{aligned}$$

Therefore

$$\lim_{t\to\infty}\mathcal{U}_\pm(-t)\,\mathcal{Z}^\alpha\mathcal{B}_\pm\langle\nabla\rangle_m^{\frac{1}{2}}\phi^{(k+1)} = \mathcal{K}_\pm^\alpha\langle\nabla\rangle_m^{\frac{1}{2}}\Phi_\pm$$

for $|\alpha| \leq 1$, where $\mathcal{K}_\pm^\alpha = \mathcal{K}_{\pm,m}^\alpha = \pm i\left(\langle\nabla\rangle_m x^\alpha + \langle\nabla\rangle_m^{-1}\nabla^\alpha\right)$. In the same way as above by (2.26) and (2.27), we get

$$\mathcal{Z}_l\mathcal{B}_\pm\psi^{(k+1)} = \mathcal{J}_{l,\pm}\mathcal{B}_{\pm,M}\psi^{(k+1)} + x_l\langle\nabla\rangle_M^{-1}F_{\pm,k}.$$

Therefore by (2.25), it follows

$$\begin{aligned} &\mathcal{U}_\pm(-t)\,\mathcal{Z}_l\mathcal{B}_\pm\psi^{(k+1)} \\ =\ &\mathcal{K}_{l,\pm}\mathcal{U}_\pm(-t)\,\mathcal{B}_\pm\psi^{(k+1)} + \mathcal{U}_\pm(-t)\,x_l\langle\nabla\rangle_M^{-1}F_{\pm,k}. \end{aligned}$$

Note that

$$\begin{aligned}&\left\|x_l \langle\nabla\rangle_M^{-1} F_{\pm,k}\right\|_{H^{1/2+\mu}} \leq \|F_{\pm,k}\|_{H^{-1/2+\mu,1}}\\ &\lesssim \left(\left\|\mathcal{D}_-\phi^{(k)}\right\|_{L^6}+\left\|\phi^{(k)}\right\|_{L^6}\right)\left\|x\psi^{(k)}\right\|_{L^3}\\ &\quad+\left\|\phi^{(k)}\right\|_{L^\infty}\left\|\phi^{(k-1)}\right\|_{L^6}\left\|x\psi^{(k-1)}\right\|_{L^3}\\ &\lesssim \langle t\rangle^{-\gamma}\left(\left\|v^{(k)}\right\|^2_{X_{[0,\infty)}}+\left\|v^{(k-1)}\right\|^3_{X_{[0,\infty)}}\right).\end{aligned}$$

Then

$$\lim_{t\to\infty}\mathcal{U}_\pm(-t)\,\mathcal{Z}^\alpha\mathcal{B}_\pm\psi^{(k)}=\mathcal{K}^\alpha_\pm\mathcal{A}_\pm\psi^+\mp\frac{i}{2}\lambda\mathcal{K}^\alpha_\pm\langle\nabla\rangle_M^{-1}\lim_{t\to\infty}\mathcal{U}(-t)\,\mathbf{b}\phi^{(k)}\beta\psi^{(k)}$$

for $|\alpha|\leq 1$. Thus we have

$$\lim_{t\to\infty}\mathcal{U}(-t)\,\mathcal{Z}^\alpha v^{(k)}=\mathcal{K}^\alpha\mathbf{v}^+\mp\frac{i}{2}\lambda\mathcal{K}^\alpha_{\pm,M}\langle\nabla\rangle_M^{-1}\lim_{t\to\infty}\mathcal{U}(-t)\,\mathbf{b}\phi^{(k)}\beta\psi^{(k)}. \tag{2.33}$$

Then integrating (2.32) in time over (t,∞) in view of (2.33), we find

$$\begin{aligned}\mathcal{U}(-t)\,\mathcal{Z}^\alpha v^{(k+1)}&=\mathcal{K}^\alpha\mathbf{v}^++\int_\infty^t\mathcal{U}(-s)\left(\mathcal{Z}^\alpha+\mathcal{Q}^\alpha\right)F^{(k)}ds\\ &\mp\frac{i}{2}\lambda\mathcal{K}^\alpha_\pm\langle\nabla\rangle_M^{-1}\lim_{t\to\infty}\mathcal{U}(-t)\,\mathbf{b}\phi^{(k)}\beta\psi^{(k)},\end{aligned} \tag{2.34}$$

for $|\alpha|\leq 1$.

We take the $L_t^r\left(I;H_q^{\frac{1}{2}}\right)$ and $L_t^\infty\left(I;H^{1/2+\mu}\right)$ norm of (2.34) , where $I=[t,\infty)$, and use the estimates of Lemma 2.7 to obtain with $2\leq q<6$, $2/r=3/2\,(1-2/q)$, and $\mu=5/4-5/(2q)$

$$\begin{aligned}&\left\|\mathcal{Z}^\alpha v^{(k+1)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)}+\left\|\mathcal{Z}^\alpha v^{(k+1)}\right\|_{L_t^\infty\left(I;H^{1/2+\mu}\right)}\\ &\lesssim \left\|\mathbf{v}^+\right\|_{H^{3/2+\mu,1}}+\begin{pmatrix}\left\|\left(\mathcal{Z}^\alpha+\mathcal{Q}^\alpha\right)F_{\pm,k}\right\|_{L_t^{\frac{r}{r-1}}\left(I;H^{2\mu-1/2}_{q/(q-1)}\right)}\\ \left\|\left(\mathcal{Z}^\alpha+\mathcal{Q}^\alpha\right)G_{\pm,k}\right\|_{L_t^{\frac{r}{r-1}}\left(I;H^{2\mu}_{q/(q-1)}\right)}\end{pmatrix}.\end{aligned}$$

Since $1/2-2\mu\geq 0$ by the Sobolev inequality with $\frac{1}{p_1}=1-\frac{1}{q}+\frac{1/2-2\mu}{3}$, we find

$$\begin{aligned}&\left\|\left(\mathcal{Z}^\alpha+\mathcal{Q}^\alpha\right)F_{\pm,k}\right\|_{L_t^{\frac{r}{r-1}}\left(I;H^{2\mu-1/2}_{q/(q-1)}\right)}\lesssim\left\|\left(\mathcal{Z}^\alpha+\mathcal{Q}^\alpha\right)F_{\pm,k}\right\|_{L_t^{\frac{r}{r-1}}(I;L^{p_1})}\\ &\lesssim\left(\left\|\mathcal{P}\phi^{(k)}\right\|_{L_t^r\left(I;H_q^1\right)}+\left\|\partial_t\phi^{(k)}\right\|_{L_t^r(I;L^q)}\right)\left\|\psi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_2})}\\ &\quad+\left\|\mathcal{P}\psi^{(k)}\right\|_{L_t^r\left(I;L^{l_3}\right)}\left(\left\|\phi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}\left(I;H^1_{p_3}\right)}+\left\|\partial_t\phi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_3})}\right)\\ &\quad+\left\|\mathcal{P}\phi^{(k)}\right\|_{L_t^r(I;L^q)}\left\|\phi^{(k-1)}\right\|_{L_t^\infty(I;L^\infty)}\left\|\psi^{(k-1)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_2})}\\ &\quad+\left\|\mathcal{P}\psi^{(k-1)}\right\|_{L_t^r\left(I;L^{l_3}\right)}\left\|\phi^{(k-1)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_3})}\left\|\phi^{(k-1)}\right\|_{L_t^\infty(I;L^\infty)}\\ &\quad+\left\|\mathcal{P}\phi^{(k-1)}\right\|_{L_t^r(I;L^q)}\left\|\phi^{(k-1)}\right\|_{L_t^\infty(I;L^\infty)}\left\|\psi^{(k-1)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_2})},\end{aligned}$$

where by the Hölder inequality $\frac{1}{q}+\frac{1}{p_2}=\frac{1}{l_3}+\frac{1}{p_3}=\frac{1}{p_1}=\frac{7}{6}-\frac{1}{q}-\frac{2\mu}{3}$.

We have $\frac{1}{p_2}=\frac{7}{6}-\frac{2}{q}-\frac{2}{3}\mu$ (when we choose $q=5/2$, $\mu=1/4$, then we can take $p_1=5/3$ and $p_2=5$). We only show the following inequality . $q>90/37$ comes from this.

$$\left\|(\mathcal{D}_{-,M}\phi^{(k)})(\mathcal{Z}\psi^{(k)})\right\|_{L_t^{\frac{r}{r-1}}(I;L^{p_1})}\lesssim\left\|v^{(k)}\right\|^2_{X_{[0,\infty)}}$$

We apply the Hölder inequality $1/p_1 = 1/p_2 + 1/p_3$

$$\left\|(\mathcal{D}_-\phi^{(k)})(\mathcal{Z}\psi^{(k)})\right\|_{L^{p_1}} \leq \left\|\mathcal{D}_-\phi^{(k)}\right\|_{L^{p_2}} \left\|\mathcal{Z}\psi^{(k)}\right\|_{L^{p_3}}$$

We apply the Sobolev inequality with $1/p_3 = 1/q - a_3/3$, $a_3 \geq 0$

$$\left\|\mathcal{Z}\psi^{(k)}\right\|_{L^{p_3}} \lesssim \left\|\mathcal{Z}\psi^{(k)}\right\|_{H_q^{a_3}}$$

We also have

$$\left\|\mathcal{Z}\psi^{(k)}\right\|_{H_q^{a_3}} \lesssim \left\|\mathcal{Z}\psi^{(k)}\right\|_{H_q^{\frac{1}{2}}}$$

if $a_3 \leq 1/2$ and

$$\left\|\mathcal{Z}\psi^{(k)}\right\|_{H_q^{a_3}} \lesssim \left\|\mathcal{Z}\mathcal{B}_\pm\psi^{(k)}\right\|_{H_q^{\frac{1}{2}}}$$

By Sobolev's inequality with $1/p_2 = 1/l_2 - a_2/3$, $a_2 \geq 0$

$$\left\|\mathcal{D}_-\phi^{(k)}\right\|_{L^{p_2}} \lesssim \left\|\mathcal{D}_-\phi^{(k)}\right\|_{H_{l_2}^{a_2}}$$

By lemma 2.6, the following estimate holds

$$\left\|\mathcal{D}_-\phi^{(k)}\right\|_{H_{l_2}^{a_2}} \lesssim \varepsilon^{3/4}\langle t\rangle^{-3/2(1-2/l_2)} = \varepsilon^{3/4}\langle t\rangle^{3/l_2-3/2}$$

if $2 \leq l_2 < 6$, $a_2 + 1/2 \leq 5/l_2 + \mu - 1$. Thus the estimate holds

$$\left\|\mathcal{D}_-\phi^{(k)}\right\|_{L^{p_2}} \lesssim \varepsilon^{3/4}\langle t\rangle^{3/l_2-3/2}$$

By Hölder's inequality $(r-1)/r = 1/r + (r-2)/r$, we get the above estimate, if $(3/l_2 - 3/2)\frac{r}{r-2} < -1$. These inequalities is valid if $97/30 < q < 6$.(When we choose $q = 5/2$, $\mu = 1/4$, then we can take $p_1 = 5/3$, $p_2 = l_2 = 20/3$, $a_2 = 0$)

We now estimate the next term

$$\begin{aligned} &\left\|(\mathcal{Z}^\alpha + \mathcal{Q}^\alpha) G_{\pm,k}\right\|_{L_t^{\frac{r}{r-1}}\left(I;H^{2\mu}_{\frac{q}{q-1}}\right)} \\ \lesssim\ &\left\|\mathcal{P}\psi^{(k)}\right\|_{L_t^r\left(I;H_{l_4}^{2\mu}\right)}\left\|\psi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_4})} \\ &+\left\|\mathcal{P}\psi^{(k)}\right\|_{L_t^r(I;L^{l_5})}\left\|\psi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}\left(I;H_{p_5}^{2\mu}\right)} \end{aligned}$$

where $1/l_4 + 1/p_4 = 1/l_5 + 1/p_5 = 1 - 1/q$. By the Sobolev inequality with $1/l_4 = 1/q - 1/6 + 2\mu/3 = 2/3(1 - 1/q)$ we obtain

$$\left\|\mathcal{P}\psi^{(k)}\right\|_{L_t^r\left(I;H_{l_4}^{2\mu}\right)} \lesssim \sum_\pm \left\|\mathcal{Z}\mathcal{B}_\pm\psi^{(k)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)}.$$

As above we apply the Sobolev inequality with $b = 3/p - 3/p_4 \geq 0$, where $1/p_4 = 7/6 - 2/q - 2\mu/3 = 1/p_2 = 1/3(1 - 1/q)$ and after that Lemma 2.6

$$\left\|\mathcal{B}_\pm\psi^{(k)}\right\|_{L^{p_2}} \lesssim \langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}\left(\left\|\mathcal{J}_\pm\mathcal{B}_\pm\psi^{(k)}\right\|_{H^{b+\nu-1}} + \left\|\mathcal{B}_\pm\psi^{(k)}\right\|_{H^{b+\nu}}\right),$$

where $\nu = 5/2 - 5/p$. Hence as above

$$\left\|\psi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}(I;L^{p_4})} \lesssim \left\|v^{(k)}\right\|_{X_{[0,\infty)}}.$$

And finally the Sobolev inequality with $1/l_5 = 1/q - 1/6$ gives

$$\left\|\mathcal{P}\psi^{(k)}\right\|_{L_t^r(I;L^{l_5})} \lesssim \sum_\pm \left\|\mathcal{Z}\mathcal{B}_\pm\psi^{(k)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)}.$$

We show the following estimate.

$$\left\|\mathcal{P}\psi^{(k)}\right\|_{L_t^r(I;L^{l_5})}\left\|\psi^{(k)}\right\|_{L_t^{\frac{r}{r-2}}(I;H_{p_5}^{2\mu})}\lesssim\left\|v^{(k)}\right\|_{X_{[0,\infty)}}^2.$$

We apply the Sobolev inequality with $1/l_5=1/q-a_5/3$, $a_5\geq 0$

$$\left\|\mathcal{P}\psi^{(k)}\right\|_{L^{l_5}}\lesssim\left\|\mathcal{P}\psi^{(k)}\right\|_{H_q^{a_5}}$$

We also have

$$\left\|\mathcal{P}\psi^{(k)}\right\|_{H_q^{a_5}}\lesssim\left\|\mathcal{P}\psi^{(k)}\right\|_{H_q^{1/2}}$$

if $a_5\leq 1/2$ and

$$\left\|\mathcal{P}\psi^{(k)}\right\|_{L^{l_5}}\lesssim\sum_{\pm}\left\|\mathcal{P}\mathcal{B}_{\pm}\psi^{(k)}\right\|_{H_q^{1/2}}$$

By Sobolev's inequality with $1/p_5=1/p_6-a_6/3$, $a_6\geq 0$

$$\left\|\psi^{(k)}\right\|_{H_{p_5}^{2\mu}}\lesssim\left\|v^{(k)}\right\|_{H_{p_6}^{2\mu+a_6}}$$

By lemma 2.6, the following estimate holds

$$\left\|v^{(k)}\right\|_{H_{p_6}^{2\mu+a_6}}\lesssim\varepsilon^{\frac{3}{4}}\langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{p_6}\right)}=\varepsilon^{\frac{3}{4}}\langle t\rangle^{\frac{3}{p_6}-\frac{3}{2}}$$

if $2\leq p_6<6$, $2\mu+a_6\leq 5/p_6+\mu-1$. By Hölder's inequality $(r-1)/r=1/r+(r-2)/r$, we get the above estimate, if $\left(\frac{3}{p_6}-\frac{3}{2}\right)\frac{r}{r-2}<-1$. These inequalities is valid if $97/30<q<6$.(When we choose $q=5/2$, $\mu=1/4$, then we can take $l_5=20/7$, $p_5=p_6=4$, $a_5=3/20$, $a_6=0$)

Therefore

$$\begin{aligned}&\left\|\mathcal{Z}^{\alpha}v^{(k+1)}\right\|_{L_t^r\left(I;H_q^{1/2}\right)}+\left\|\mathcal{Z}^{\alpha}v^{(k+1)}\right\|_{L_t^\infty(I;H^{1/2+\mu})}\\ \lesssim\ &\left\|\mathbf{v}^{+}\right\|_{H^{3/2+\mu,1}}+\left\|v^{(k)}\right\|_{X_{[0,\infty)}}^2+\left\|v^{(k-1)}\right\|_{X_{[0,\infty)}}^3\lesssim\varepsilon+\varepsilon^{\frac{3}{2}}.\end{aligned}$$

Thus we have

$$\left\|v^{(k+1)}\right\|_{X_{[0,\infty)}}\leq C\varepsilon.$$

In the same manner we consider the difference $v^{(k)}-v^{(k-1)}$ and prove the estimate

$$\left\|v^{(k)}-v^{(k-1)}\right\|_{X_{[0,\infty)}}\leq\frac{1}{2}\left\|v^{(k-1)}-v^{(k-2)}\right\|_{X_{[0,\infty)}}$$

which shows that the sequence $\left\{v^{(k)}\right\}$ defined in (2.28) is a Cauchy sequence in the space $X_{[0,\infty)}$. Thus the result of the theorem follows. Theorem 2.8 is proved. □

3. Wave operator for DKG in 2D

3.1. **Difficulty in 2d case.** In this chapter, we consider the final state problem for DKG in two space dimentions. As we mentioned previous chapter, we meet with two problems in considering the final value problem. First, we are not allowed to start with the Klein-Gordon system (2.2), which does not involve the derivative of ψ. Second, though (DKG) is equivalent to (2.1) not (2.2), equation (2.1) includes the derivative of ψ.

Moreover, two dimensional case is known as critical, i.e. borderline case between the long range scattering and the short range one. More precisely, time decay property for solutions of the DKG system is slower than three dimensional case. Therefore, previous section's argument does not work to two dimensional case directly.

To overcome the lack of time decay property, we will use the *algebraic normal form transformation* developed in paper [59] and the *decomposition of the Klein-Gordon operator into the product of Dirac operators:*

$$\partial_t^2 + \langle\nabla\rangle_M^2 = \mathcal{D}_+\mathcal{D}_-. \tag{3.1}$$

This combination allows us to find a suitable second approximate solution to ψ.

3.2. **Several Notations and Main Results.** We state our main results in this chapter. We introduce the function space

$$D_q \equiv H^{4-\frac{4}{q}}_{\frac{q}{q-1}} \cap H^{\frac{5}{2},1} \cap H^2_1. \tag{3.2}$$

Theorem 3.1. *Let $m, M > 0$, $m \neq 2M$, $4 < q \leq \infty$ and $\left(\psi^+, \left(\langle\nabla\rangle \phi_1^+, \phi_2^+\right)\right) \in (D_q)^4$. If the norm $\rho \equiv \left\|\left(\psi^+, \left(\langle\nabla\rangle \phi_1^+, \phi_2^+\right)\right)\right\|_{H_1^2}$ is sufficiently small, then there exist a positive constant $T > 0$ and a unique solution*

$$\left(\psi(t), \begin{pmatrix} \langle\nabla\rangle_m^{\frac{1}{2}} \phi(t) \\ \langle\nabla\rangle_m^{-\frac{1}{2}} \partial_t\phi(t) \end{pmatrix}\right) \in \left(C\left([T,\infty); H^{\frac{1}{2}}\right)\right)^4$$

for the system (DKG). Moreover, there exists a positive constant $C > 0$ such that the following estimate

$$\|\psi(t) - \psi_0(t)\|_{H^{1/2}} + \left\| \begin{pmatrix} \phi(t) \\ \langle\nabla\rangle_m^{-1} \partial_t\phi(t) \end{pmatrix} - \begin{pmatrix} \phi_0(t) \\ \langle\nabla\rangle_m^{-1} \partial_t\phi_0(t) \end{pmatrix} \right\|_{H^1} \leq Ct^{-\mu}$$

is true for all $t \geq T$, where $\frac{1}{2} < \mu < 1 - \frac{2}{q}$, where

$$\left(\psi_0, \phi_0, \langle\nabla\rangle_m^{-1} \partial_t\phi_0\right)(t) = \left(\mathcal{V}_D(t)\psi^+, \mathcal{V}_{KG}(t)\left(\phi_1^+, \langle\nabla\rangle_m^{-1}\phi_2^+\right)\right) \tag{3.3}$$

.

By Theorem 3.1, we can get existence of the wave operator for (DKG) as follows:

Corollary 3.2. *Let $m, M > 0$, $m \neq 2M$ and $4 \leq q < \infty$. Then the wave operator $\mathcal{W}^+$ for (DKG) is well-defined from a neighborhood at the origin in the space $(D_q)^2 \times \left(\langle\nabla\rangle^{-1} D_q \times D_q\right)$ to the space $\left(H^{1/2}\right)^2 \times \left(H^1 \times L^2\right)$.*

The rest of this chapter is organized as follows. In subsection 3.3, we state some basic estimates for free solutions of the DKG system and we introduce "null forms" and state their properties. In Section 3.4, we decompose two harmful terms by the algebraic normal form transformation and we find a second approximation for ψ through the decomposition of the Klein-Gordon operator by the Dirac one. In Section 3.5, following paper [20], we will also change the transformed DKG system into another form in order to apply the Strichartz type estimates to the Dirac part. In Section 3.6, we will prove Theorem 3.1 by a iteration scheme based on paper [24].

3.3. **Elementary Estimates and Null forms.** We state $L^p - L^q$ time decay estimates through the free evolution groups $\mathcal{U}_{\pm,m}(t)$ obtained in paper [48].

Lemma 3.3. *Let $m \neq 0$ and $2 \leq p \leq \infty$. Then the estimate*

$$\|\mathcal{U}_{\pm,m}(t)\phi\|_{L^p} \lesssim t^{2/p-1}\|\phi\|_{H_q^{2(1-2/p)}}$$

is true for any $t>0$, where q is a conjugate exponent of p: $1/p+1/q=1$.

By the lemma, we can easily get $L^p - L^q$ time decay estimates to free solutions for the DKG system.

Corollary 3.4. *Under the same assumption of Lemma 3.3 and $M>0$, the following estimates*

$$\begin{aligned}\left\|\mathcal{V}_D(t)\psi^+\right\|_{L^p} &\lesssim t^{2/p-1}\left\|\psi^+\right\|_{H_q^{2(1-2/p)}},\\ \left\|\mathcal{V}_K(t)\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{L^p} &\lesssim t^{2/p-1}\left\|\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H_q^{2(1-2/p)}},\end{aligned}$$

are valid for any $t>0$, where q is a conjugate exponent of p: $1/p+1/q=1$.

Remark 3.1. *Let $\kappa \in \mathbb{R}$, $M, m \neq 0$ and $2 \leq p < \infty$. Then the following estimates*

$$\begin{aligned}\left\|\mathcal{V}_D(t)\psi^+\right\|_{H_p^\kappa} &\lesssim t^{2/p-1}\left\|\psi^+\right\|_{H^{\kappa+2-4/p,1}}\\ \left\|\mathcal{V}_K(t)\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H_p^\kappa} &\lesssim t^{2/p-1}\left\|\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H^{\kappa+2-4/p,1}},\end{aligned}$$

hold for any $t>0$.

Next, we introduce the Leibniz rule for fractional derivatives.

Lemma 3.5. *Let $\kappa>0$, $1<p,q_1,q_2<\infty$, $1<r_1,r_2\leq\infty$ and $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{r_1}=\frac{1}{q_2}+\frac{1}{r_2}$. Then the following estimate holds:*

$$\|uv\|_{H_p^\kappa} \lesssim \|u\|_{H_{q_1}^\kappa}\|v\|_{L^{r_1}} + \|v\|_{H_{q_2}^\kappa}\|u\|_{L^{r_2}}. \tag{3.4}$$

For the proof of (3.4) see, e.g. [40].

Let $\mathcal{Z}^\gamma = \mathcal{Z}_1^{\gamma_1}\cdots\mathcal{Z}_n^{\gamma_n}$ for a multi-index $\gamma=(\gamma_1,\cdots,\gamma_n)\in(\mathbb{N}\cup\{0\})^n$. We can see the commutation relations (see [1] and [64]):

$$[\mathcal{D}_+, \mathcal{Z}_k - (1/2)\alpha_k] = \alpha_k\mathcal{D}_+, \tag{3.5}$$

$$[\partial_t^2 - \Delta + m^2, \mathcal{Z}_k] = 0, \tag{3.6}$$

for $k=1,2$, where $[A,B]\equiv AB-BA$.

We introduce the quadratic (null) forms:

$$\mathcal{Q}_0(f,g) \equiv (\partial_t f)(\partial_t g) - (\nabla f)\cdot(\nabla g), \tag{3.7}$$

$$\mathcal{Q}_{j,k}(f,g) \equiv (\partial_j f)(\partial_k g) - (\partial_k f)(\partial_j g), \tag{3.8}$$

for $0\leq j<k\leq 2$, where $\partial \equiv (\partial_0,\nabla)\equiv(i\partial_t,\partial_1,\partial_2)$. Especially $\mathcal{Q}_{j,k}$ is called strong null form and has an additional time decay property through the operator $\mathcal{Z}_k$, obtained in [44] (see also [24], [36] and [59] etc).

Lemma 3.6. *Let $j,k=1,2$. Then for any smooth function f,g, the identities*

$$\mathcal{Q}_{0,j}(f,g) = t^{-1}(\partial_0 f)(\mathcal{Z}_j g) - t^{-1}(\mathcal{Z}_j f)(\partial_0 g), \tag{3.9}$$

$$\begin{aligned}\mathcal{Q}_{j,k}(f,g) = {} & t^{-2}(\mathcal{Z}_j g)(\mathcal{Z}_k f) - t^{-2}(\mathcal{Z}_j f)(\mathcal{Z}_k g) + t^{-1}(\partial_j f)(\mathcal{Z}_k g)\\ & - t^{-1}(\partial_j g)(\mathcal{Z}_k f) + t^{-1}(\mathcal{Z}_j f)(\partial_k g) - t^{-1}(\mathcal{Z}_j g)(\partial_k f),\end{aligned} \tag{3.10}$$

are valid for any $t\in\mathbb{R}\setminus\{0\}$.

3.4. **Decomposition of critical terms.** We study a structure of some harmful terms of (DKG). By the difference of (DKG) and the free DKG system, it follows that

$$(3.11)\qquad \begin{cases} \mathcal{D}_+(\psi-\psi_0)=(\phi-\phi_0)\beta\psi+\phi_0\beta(\psi-\psi_0)+\phi_0\beta\psi_0, \\ (\square+m^2)(\phi-\phi_0)=(\psi-\psi_0)^*\beta\psi+\psi_0^*\beta(\psi-\psi_0)+\psi_0^*\beta\psi_0, \end{cases}$$

where $\square=\partial_t^2-\Delta$. The last two terms $\phi_0\beta\psi_0$, $\psi_0^*\beta\psi_0$ are critical, both of which have the worst time decay property. Especially, since

$$\phi_0\beta\psi_0,\ \psi_0^*\beta\psi_0=O\left(t^{-1}\right)\quad \text{in } L^2 \text{ as } t\to+\infty$$

(see Corollary 3.4), the L^2-norm of these terms are not integrable with respect to time t over $[1,\infty)$. Therefore, it can not be expected that usual pertubation technique is applicable to (3.11). To overcome this lack of time decay property, we will decompose them into an image of a Klein-Gordon operator and a remainder term following paper [59], based on papers [36], [45] and [66].

Let (v_1,v_2) be a solution for the following homogeneous KG system with masses M_1, $M_2>0$,

$$(3.12)\qquad \left(\square+M_j^2\right)v_j=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}^2,\ \text{for } j=1,2.$$

By the masses M_1, M_2, we introduce the symmetric matrix

$$\mathcal{M}=\mathcal{M}(M_1,M_2)=\begin{pmatrix} M_1^2+M_2^2 & 2M_1M_2 \\ 2M_1M_2 & M_1^2+M_2^2\end{pmatrix}.$$

We have the following:

Lemma 3.7. *(see [59]) Let $\tilde{m}>0$ with $\det\left(\tilde{m}^2I-\mathcal{M}\right)\neq0$. Then the quadratic term v_1v_2 can be decomposed as*

$$(3.13)\qquad v_1v_2=\frac{1}{\det\left(\tilde{m}^2I-\mathcal{M}\right)}\left\{\left(\square+\tilde{m}^2\right)f-4\mathcal{R}\right\},$$

where

$$(3.14)\qquad f\ =\ f(v_1,v_2)\equiv\left(-M_1^2-M_2^2+\tilde{m}^2\right)v_1v_2-2\mathcal{Q}_0(v_1,v_2),$$

$$(3.15)\qquad \begin{aligned}\mathcal{R}\ =\ \mathcal{R}(v_1,v_2)\equiv&\sum_{m=1}^{2}\mathcal{Q}_{0,m}(\partial_tv_1,\partial_mv_2)+\sum_{m=1}^{2}\mathcal{Q}_{0,m}(\partial_tv_2,\partial_mv_1)\\ &-\mathcal{Q}_{1,2}(\partial_1v_1,\partial_2v_2)-\mathcal{Q}_{2,1}(\partial_2v_1,\partial_1v_2).\end{aligned}$$

Under the nonresonance mass condition $m,M>0$ and $m\neq2M$, we can apply Lemma 3.7 to the critical terms $\phi_0\beta\psi_0$, $\psi_0^*\beta\psi_0$. Before doing so, we prepare for several notations. We put

$$(3.16)\qquad \tilde{\mathcal{M}}\equiv\frac{1}{m^2(2M+m)(m-2M)}$$

which is well defined if $m,M>0$ and $m\neq2M$. For a real-valued function ϕ and a $\mathbf{C}^2$-valued function $\psi=(\psi_1,\psi_2)^t$, we define $\mathbf{C}^2$-valued functions of bilinear form:

$$(3.17)\qquad \begin{cases} f_D=f_D(\phi,\psi)\equiv\left(f(\phi,\psi_1),f(\phi,\psi_2)\right)^t, \\ \mathcal{R}_D=\mathcal{R}_D(\phi,\psi)\equiv\left(\mathcal{R}(\phi,\psi_1),\mathcal{R}(\phi,\psi_2)\right)^t, \\ \mathcal{Q}_0^D=\mathcal{Q}_0^D(\phi,\psi)\equiv\left(\mathcal{Q}_0(\phi,\psi_1),\mathcal{Q}_0(\phi,\psi_2)\right)^t, \end{cases}$$

Moreover, for $\mathbf{C}^2$-valued functions $\varphi=(\varphi_1,\varphi_2)^t$, $\psi=(\psi_1,\psi_2)^t$, we put the bilinear forms:

$$(3.18)\qquad \begin{cases} f_K=f_K\left(\varphi^t,\psi\right)\equiv\sum_{j=1}^2f\left(\varphi_j,\psi_j\right), \\ \mathcal{R}_K=\mathcal{R}_K\left(\varphi^t,\psi\right)\equiv\sum_{j=1}^2\mathcal{R}\left(\varphi_j,\psi_j\right), \\ \mathcal{Q}_0^K=\mathcal{Q}_0^K\left(\varphi^t,\psi\right)\equiv\sum_{j=1}^2\mathcal{Q}_0\left(\varphi_j,\psi_j\right). \end{cases}$$

We have the following:

Corollary 3.8. *Let $m, M > 0$, $m \neq 2M$ and (ψ_0, ϕ_0) be a free solution for the Dirac-Klein-Gordon equations. Then the quadratic terms $\phi_0\beta\psi_0$, $\psi_0^*\beta\psi_0$ can be expressed as*

$$(3.19)\qquad \begin{cases} \phi_0\beta\psi_0 = \tilde{\mathcal{M}}\left\{\left(\Box + M^2\right) f_D\left(\phi_0, \beta\psi_0\right) - 4\mathcal{R}_D\left(\phi_0, \beta\psi_0\right)\right\}, \\ \psi_0^*\beta\psi_0 = \tilde{\mathcal{M}}\left\{\left(\Box + m^2\right) f_K\left(\psi_0^*, \beta\psi_0\right) - 4\mathcal{R}_K\left(\psi_0^*, \beta\psi_0\right)\right\}. \end{cases}$$

Proof. We consider the Dirac part of (3.11). Multiplying by $\mathcal{D}_-$ both hand sides of $\mathcal{D}_+\psi_0 = 0$, we get

$$(3.20)\qquad \mathcal{D}_-\mathcal{D}_+\psi_0 = \left(\Box + M^2\right)\psi_0 = 0,$$

which implies $\psi_0 = \left(\psi_{0,1}, \psi_{0,2}\right)^t$ is also solution of the free KG equation. Note that by the condition $m, M > 0$ and $m \neq 2M$, we can apply Lemma 3.7 with $\tilde{m} = M$, $v_1 = \phi_0$ and $v_2 = \psi_{0,k}$ to get for $k = 1, 2$,

$$\phi_0\psi_{0,k} = \tilde{\mathcal{M}}\left\{\left(\Box + M^2\right) f\left(\phi_0, \psi_{0,k}\right) - 4\mathcal{R}\left(\phi_0, \psi_{0,k}\right)\right\}.$$

Thus by a simple calculation, we obtain (3.19). Next, note that from the equality (3.20), we see that $\overline{\psi_0}$ satisfies the free KG equation. Thus in the same manner as the proof of the Dirac part, we can prove the KG part, which completes the proof of the corollary. □

Next, we will change the DKG equations into another form without critical nonlinearities. We introduce a new unknown function (Ψ, Φ) as follows:

$$(3.21)\qquad \Psi \equiv \psi - \psi_0 - \tilde{f}_D \equiv \tilde{\psi} - \tilde{f}_D, \quad \Phi \equiv \phi - \phi_0 - \tilde{f}_K \equiv \tilde{\phi} - \tilde{f}_K,$$

where (ψ_0, ϕ_0) is defined by (3.3) and

$$(3.22)\qquad \begin{aligned} \tilde{f}_D &= \tilde{f}_D\left(\phi_0, \psi_0\right) \equiv \tilde{\mathcal{M}}\mathcal{D}_- f_D\left(\phi_0, \beta\psi_0\right) \\ &= \tilde{\mathcal{M}}\left(f_D\left(\mathcal{D}_-\phi_0, \beta\psi_0\right) - iMf_D\left(\phi_0, \psi_0\right)\right), \end{aligned}$$

$$(3.23)\qquad \tilde{f}_K = \tilde{f}_K\left(\psi_0\right) \equiv \tilde{\mathcal{M}} f_K\left(\psi_0^*, \beta\psi_0\right),$$

are the second approximate solution to (ψ, ϕ), where we have used the properties (DM) and $\mathcal{D}_+\psi_0 = 0$ to obtain the third equality in (3.22).

Here we remember that by the anti-commutation relations (DM) of the Dirac matrices, we can decompose the KG operator as follows:

$$(3.24)\qquad \Box + M^2 = \mathcal{D}_+\mathcal{D}_-.$$

By combining Corollary 3.8 and this decomposition, we can rewrite (DKG) as follows:

Lemma 3.9. *Let $m, M > 0$ and $m \neq 2M$. Then (ψ, ϕ) satisfies (DKG) if and only if the new variable (Ψ, Φ) defined by (3.21) is a solution of*

$$(3.25)\qquad \begin{cases} \mathcal{D}_+\Psi = F, \\ \left(\Box + m^2\right)\Phi = G, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2,$$

where

$$(3.26)\qquad \begin{cases} F = F\left(\tilde{\phi}, \tilde{\psi}\right) \equiv \tilde{\phi}\beta\tilde{\psi} + \tilde{\phi}\beta\psi_0 + \phi_0\beta\tilde{\psi} - 4\tilde{\mathcal{M}}\mathcal{R}_D\left(\phi_0, \beta\psi_0\right), \\ G = G\left(\tilde{\psi}\right) \equiv \tilde{\psi}^*\beta\tilde{\psi} + \tilde{\psi}^*\beta\psi_0 + \psi_0^*\beta\tilde{\psi} - 4\tilde{\mathcal{M}}\mathcal{R}_K\left(\psi_0^*, \beta\psi_0\right), \end{cases}$$

and $\tilde{\mathcal{M}}$, $\mathcal{R}_D$ and $\mathcal{R}_K$ are defined by (3.16), (3.17) and (3.18), respectively.

The first identity of (3.25) is new and enables us to treat the Dirac equation in two space dimensions.

Proof. From (3.11), we see that (ψ, ϕ) is a solution of (DKG) if and only if the new variable $\left(\tilde{\psi}, \tilde{\phi}\right)$ satisfies the following DKG equations:

$$(3.27)\qquad \begin{cases} \mathcal{D}_+\tilde{\psi} = \tilde{\phi}\beta\tilde{\psi} + \tilde{\phi}\beta\psi_0 + \phi_0\beta\tilde{\psi} + \phi_0\beta\psi_0, \\ \left(\Box + m^2\right)\tilde{\phi} = \tilde{\psi}^*\beta\tilde{\psi} + \tilde{\psi}^*\beta\psi_0 + \psi_0^*\beta\tilde{\psi} + \psi_0^*\beta\psi_0, \end{cases}$$

We consider the Dirac part of (3.27) only, since it is easier to handle the KG part. Note that by the assumption $m, M > 0$ and $m \neq 2M$, we can apply Corollary 3.8 to $\phi_0\beta\psi_0$. Thus we have

$$\phi_0\beta\psi_0 = \tilde{\mathcal{M}} \left\{ \left(\Box + M^2 \right) f_D \left(\phi_0, \beta\psi_0 \right) - 4\mathcal{R}_D \left(\phi_0, \beta\psi_0 \right) \right\}. \tag{3.28}$$

Moreover, by the decomposition (3.24), we can transform the first term of the right hand side of (3.25) as follows:

$$\lambda\tilde{\mathcal{M}} \left(\Box + M^2 \right) f_D \left(\phi_0, \beta\psi_0 \right) = \tilde{\mathcal{M}}\mathcal{D}_+\mathcal{D}_- f_D \left(\phi_0, \beta\psi_0 \right) = \mathcal{D}_+\tilde{f}_D, \tag{3.29}$$

where we have used the definition of $\tilde{f}_D$ given by (3.22). Inserting (3.27)-(3.29) into the Dirac part of (3.27), we obtain the Dirac part of (3.25), which completes the proof of the lemma. □

3.5. **Reduction DKG to a first order system.** To construct a solution for the final value problem of the DKG system, we will use the Strichartz type estimates (Lemma 2.7). However, it seems difficult to apply these estimates to the Dirac part for (3.25) due to a derivative loss difficulty. To gain first order differentiability properties of nonlinear term, we use the matrix operators $\mathcal{B}_\pm, \mathcal{L}_\pm$ as in the previous chapter 2, though we do not necessarily need the operator $\mathcal{B}$ in dealing with the initial value problem for the DKG system (see [33]). We will construct the desired solution (ψ, ϕ) for the DKG system by the iteration scheme. Let $\left\{ \left(\psi^{(k)}, \phi^{(k)} \right) \right\}_{l \geq 0}$ be a sequence such that (2.17) and

$$\left(\psi^0, \phi^0 \right) = \left(\psi_0, \phi_0 \right), \tag{3.30}$$

under the final conditions

$$\lim_{t\to\infty} \left\| \psi^{(k)}(t) - \psi_0(t) \right\|_{H^{1/2}} = 0, \tag{3.31}$$

$$\lim_{t\to\infty} \left\| \begin{pmatrix} \phi^{(k)}(t) \\ \langle\nabla\rangle_m^{-1} \partial_t\phi^{(k)}(t) \end{pmatrix} - \begin{pmatrix} \phi_0(t) \\ \langle\nabla\rangle_m^{-1} \partial_t\phi_0(t) \end{pmatrix} \right\|_{H^1} = 0, \tag{3.32}$$

for $k \geq 0$, where (ψ_0, ϕ_0) is given by (3.3). It suffices to prove that the sequence

$$\left\{ \psi^{(k)}, \left(\langle\nabla\rangle_m^{\frac{1}{2}} \phi^{(k)}, \langle\nabla\rangle_m^{-\frac{1}{2}} \partial_t\phi^{(k)} \right) \right\}_{k\geq 0}$$

is a Cauchy one in the Banach space $\left(C\left([T, \infty) ; H^{1/2} \right) \right)^4$ for some $T > 0$.

As the previous section, we introduce the new sequence $\left\{ \left(\Psi^{(k)}, \Phi^{(k)} \right) \right\}$ as follows:

$$\Psi^{(k)} \equiv \psi^{(k)} - \psi_0 - \tilde{f}_D \equiv \tilde{\psi}^{(k)} - \tilde{f}_D, \quad \Phi^{(k)} \equiv \phi^{(k)} - \phi_0 - \tilde{f}_K \equiv \tilde{\phi}^{(k)} - \tilde{f}_K. \tag{3.33}$$

By Lemma 3.9, the sequence $\left\{ \left(\psi^{(k)}, \phi^{(k)} \right) \right\}$ is a solution of (2.17), (3.30)-(3.32) if and only if the new one $\left\{ \left(\Psi^{(k)}, \Phi^{(k)} \right) \right\}$ satisfies the transformed DKG equations

$$\begin{cases} \mathcal{D}_+\Psi^{(k+1)} = F^{(k)}, \\ \left(\Box + m^2 \right) \Phi^{(l+1)} = G^{(k)}, \end{cases} \quad k \geq 1, \tag{3.34}$$

$$\left(\Psi^0, \Phi^0 \right) = -\left(\tilde{f}_D, \tilde{f}_K \right),$$

where

$$F^{(k)} \equiv F\left(\tilde{\phi}^{(k)}, \tilde{\psi}^{(k)} \right), \quad G^{(k)} \equiv G\left(\tilde{\psi}^{(k)} \right),$$

for $k \geq 0$ ($\tilde{f}_D$, $\tilde{f}_K$, F and G are defined by (3.22), (3.23) and (3.26), respectively).

By the decomposition of the Klein-Gordon operator by the Dirac operator, we have

$$\mathcal{L}_\pm\mathcal{B}_\pm = \mp\frac{i}{2} \langle\nabla\rangle_M^{-1} I \left(\partial_t^2 + \langle\nabla\rangle_M^2 \right) = \mp\frac{i}{2} \langle\nabla\rangle_M^{-1} \mathcal{D}_-\mathcal{D}_+.$$

Thus from the Dirac part for (3.34), we can deduce the following:

$$\mathcal{L}_\pm \mathcal{B}_\pm \Psi^{(k+1)} = \mp \frac{i}{2} \langle \nabla \rangle_M^{-1} \mathcal{D}_- \mathcal{D}_+ \Psi^{(k+1)} = \langle \nabla \rangle_M^{-1} F_\pm^{(k)}, \tag{3.35}$$

for $l \geq 0$, where $F_\pm^{(k)} \equiv \mp \frac{i}{2} \mathcal{D}_- F^{(k)}$. Therefore from Dirac part of (3.34), we have:

$$\begin{cases} \mathcal{L}_\pm \mathcal{B}_\pm \Psi^{(k+1)} = \langle \nabla \rangle_M^{-1} F_\pm^k, \ k \geq 0, \\ \mathcal{B}_\pm \Psi^0 = -\mathcal{B}_\pm \tilde{f}_D. \end{cases} \tag{3.36}$$

Remark 3.2. *By the properties (DM) of the Dirac matrices, we can transform $F_\pm^l$ into another form without any derivatives of $\tilde{\psi}$ or the free solution ψ_0 (see (3.44)-(3.45) precisely). This fact enables us to use the Strichartz estimates to (3.36).*

Next we will also transform the KG part of (3.34) as in [20], [24]. We can see that the sequence $\{\Phi^{(k)}\}$ is a solution of the KG part for (3.34) if and only if the sequence $\{\mathcal{B}_\pm \Phi^{(k)}\}$ satisfies

$$\begin{cases} \mathcal{L}_\pm \mathcal{B}_\pm \Phi^{(k+1)} = \langle \nabla \rangle_m^{-1} G_\pm^k, \text{ for } k \geq 0, \\ \mathcal{B}_\pm \Phi^{(0)} = -\mathcal{B}_\pm \tilde{f}_K, \end{cases} \tag{3.37}$$

where $G_\pm^k \equiv G_\pm^k \left(\tilde{\psi}^{(k)}\right) \equiv \mp \frac{i}{2} G^k$.

Therefore by (3.36) and (3.37), we get

$$\begin{cases} \mathcal{L}_\pm \mathcal{B}_\pm \Psi^{(k+1)} = \langle \nabla \rangle_M^{-1} F_\pm^k, \\ \mathcal{L}_\pm \mathcal{B}_\pm \Phi^{(k+1)} = \langle \nabla \rangle_m^{-1} G_\pm^k, \end{cases} \text{for } k \geq 0, \tag{3.38}$$
$$\left(\mathcal{B}_\pm \Psi^{(0)}, \mathcal{B}_\pm \Phi^{(0)}\right) = -\left(\mathcal{B}_\pm \tilde{f}_D, \mathcal{B}_\pm \tilde{f}_K\right).$$

Remark 3.3. *The identity $\sum_\pm \mathcal{B}_\pm^* = I$ holds, which enables us to reconstruct a solution (Ψ, Φ) for (3.34) from $(\mathcal{B}_\pm \Psi, \mathcal{B}_\pm \Phi)$.*

Inserting the identities

$$\tilde{\psi}^{(k)} = \sum_\pm \mathcal{B}_\pm \Psi^{(k)} + \tilde{f}_D, \quad \tilde{\phi}^{(k)} = \sum_\pm \mathcal{B}_\pm \Phi^{(k)} + \tilde{f}_K, \tag{3.39}$$

into the nonlinearities $F_\pm^k$, $G_\pm^k$, we can express (3.38) by the new variable $(\mathcal{B}_\pm \Psi^l, \mathcal{B}_\pm \Phi^l)$ only without $\left(\tilde{\phi}^l, \tilde{\psi}^l\right)$.

At the end of this section, we will lead the integral equations associated with (3.38). We introduce a new unknown function sequence $\{v^{(k)}\}$ whose components are defined by

$$v^{(k)} \equiv \left(\mathcal{B}_+ \Psi^{(k)}, \mathcal{B}_- \Psi^{(k)}, \langle \nabla \rangle_m^{\frac{1}{2}} \mathcal{B}_+ \Phi^{(k)}, \langle \nabla \rangle_m^{\frac{1}{2}} \mathcal{B}_- \Phi^{(k)}\right)^t,$$

a nonlinear term

$$\mathcal{N} = \mathcal{N}\left(v^{(k)}\right) \equiv \left(\langle \nabla \rangle_M^{-1} F_+^k, \langle \nabla \rangle_M^{-1} F_-^k, \langle \nabla \rangle_m^{-\frac{1}{2}} G_+^k, \langle \nabla \rangle_m^{-\frac{1}{2}} G_-^k\right)^t$$

for $k \geq 0$. Then by using these notations, (3.38) can be simplified as

$$\mathcal{L} v^{(k+1)} = \mathcal{N}\left(v^{(k)}\right) \quad \text{for } k \geq 0. \tag{3.40}$$

To lead the integral equations for (3.40), we need to study the asymptotic behavior of the new variable $v^{(k)}$. We can obtain the following:

Lemma 3.10. *Let $(\psi^+, (\langle \nabla \rangle \phi_1^+, \phi_2^+)) \in \left(H^{5/2,1}\right)^4$. The function $\left(\psi^{(k)}, \phi^{(k)}\right)$ satisfies (2.17), (3.30)-(3.32) for any $k \geq 0$ if and only if the new function $v^{(k)}$ satisfies (3.40) and*

$$\lim_{t \to \infty} \left\| v^{(k)} \right\|_{H^{1/2}} = 0, \ \textit{for } k \geq 0. \tag{3.41}$$

The proof of the lemma will be given in Appendix. From Lemma 3.10, we can lead the integral equations associated with (3.40) as follows:

$$v^{(k+1)}(t) = -\int_t^\infty \mathcal{U}(t-s)\mathcal{N}\left(v^{(k)}\right) ds. \tag{3.42}$$

3.6. **Proof of existence of the wave operator for DKG in 2d.** In this subsection, we give a proof of Theorem 3.1. Note that the identities

$$\partial_t \Psi^{(k)} = i\langle\nabla\rangle_M\left(v_1^{(k)} - v_2^{(k)}\right),\ \partial_t \Phi^{(k)} = i\langle\nabla\rangle_m^{1/2}\left(v_3^{(k)} - v_4^{(k)}\right). \tag{3.43}$$

hold, the nonlinearity $\mathcal{N}\left(v^{(k)}\right)$ can be expressed in terms of the space derivatives of $v^{(k)}$ (so excluding the time derivatives).

For $T > 1$, where T is sufficiently large, we introduce the following function space:

$$X_T = \left\{v \in \left(C\left([T,\infty); H^{1/2}\right)\right)^6;\ \|v\|_{X_T} < \infty\right\},$$

with the norm

$$\|v\|_{X_T} \equiv \sup_{t\in[T,\infty)} t^\mu\left(\|v\|_{L_t^4(I;L^4)} + \|v\|_{L_t^\infty(I;H^{1/2})}\right)$$

where $1/2 < \mu < 1 - 2/q$, $4 < q \le \infty$ and $I = [t,\infty)$. We define

$$A \equiv C\left\|\left(\psi^+, \left(\langle\nabla\rangle\phi_1^+, \phi_2^+\right)\right)\right\|_{H^{4-4/q}_{q/(q-1)}\cap H^{5/2,1}}.$$

In order to obtain the theorem, we will show that the sequence $\left\{v^{(k)}\right\}$ is a Cauchy one in a closed ball $X_{T,A}$ for appropriate T and ρ, where $X_{T,A} \equiv \left\{v \in X_T; \|v\|_{X_T} \le A\right\}$.

Hereafter, we will use the notation $L_t^r X = L_t^r(I;X)$, $\mathcal{D} = \mathcal{D}_-$ and

$$\mathcal{B}\Psi = \mathcal{B}_\pm\Psi,\quad \mathcal{B}\Phi = \mathcal{B}_\pm\Phi,$$

for simplicity if it does not cause a confusion.

Proof. We will prove $v^{(k)} \in X_{T,A}$ for any $k \ge 0$ by induction. In the case of $k = 0$, it is easy to see that $v^{(0)} \in X_{T,A}$ for some T and ρ. We omit the detail. For $k \ge 1$, we assume that $v^{(k)} \in X_{T,A}$ for $0 \le l \le k$. We will show $v^{(k+1)} \in X_{T,A}$ for some T and ρ.

First, by the identities $\mathcal{D}_+\psi_0 = 0$ and $\mathcal{D}_+\psi^{(k)} = \lambda\phi^{(k-1)}\beta\psi^{(k-1)}$ for $k \ge 1$, we get for $k \ge 1$,

$$\begin{aligned}\mathcal{D}_-\left(\tilde\phi^{(k)}\beta\tilde\psi^{(k)}\right) &= \left(\mathcal{D}_-\tilde\phi^{(k)}\right)\beta\tilde\psi^{(k)} - iM\tilde\phi^{(k)}I\tilde\psi^{(k)} + \lambda\tilde\phi^{(k)}\tilde\phi^{(k-1)}I\tilde\psi^{(k-1)}\\ &\quad+\lambda\tilde\phi^{(k)}\tilde\phi^{(k-1)}I\psi_0 + \lambda\tilde\phi^{(k)}\phi_0 I\tilde\psi^{(k-1)} + \lambda\tilde\phi^{(k)}\phi_0 I\psi_0\end{aligned}$$

and

$$\mathcal{D}_-\mathcal{R}_D(\phi_0,\beta\psi_0) = \mathcal{R}_D(\mathcal{D}_-\phi_0,\beta\psi_0) - iM\mathcal{R}_D(\phi_0,\beta\psi_0).$$

From these identities, we can express $F_\pm^k$ as follows

$$F_\pm^k = \mp\frac{i}{2}\sum_{j=1}^3 F_j^k + \text{"remainder" for } k \ge 1, \tag{3.44}$$

where

$$\begin{aligned}F_1^k &\equiv \left(\mathcal{D}_-\tilde\phi^{(k)}\right)\beta\tilde\psi^{(k)},\ F_2^k \equiv (\mathcal{D}_-\phi_0)\beta\tilde\psi^{(k)} + \left(\mathcal{D}_-\tilde\phi^{(k)}\right)\beta\psi_0,\\ F_3^k &\equiv 4i\tilde{\mathcal{M}}\mathcal{R}_D(\mathcal{D}_-\phi_0,\beta\psi_0).\end{aligned} \tag{3.45}$$

Here we note that "remainder" (given by (3.44)) can be handled in the same manner as either F_j^k $(j=1,2$ or $3)$. Thus we will omit the estimate of them. We also decompose $G_\pm^k$ as $G_\pm^k = \mp\frac{i}{2}\sum_{j=1}^3 G_j^k$, where

$$G_1^k = \left(\tilde{\psi}^{(k)}\right)^* \beta\tilde{\psi}^{(k)}, \quad G_2^k = \left(\tilde{\psi}^{(k)}\right)^* \beta\psi_0 + \psi_0^*\beta\tilde{\psi}^{(k)}, \quad G_3^k = 4i\tilde{\mathcal{M}}\mathcal{R}_K(\psi_0^*, \beta\psi_0). \tag{3.46}$$

Taking $L_t^4L_x^4$-norm and $L_t^\infty H^{\frac{1}{2}}$-norm of (3.42) and applying Lemma 2.7 with $(q,r,\gamma) = (4,4,1/2)$ and $(2,\infty,0)$, we have

$$\left\|v^{(k+1)}\right\|_{L_t^4L_x^4} + \left\|v^{(k+1)}\right\|_{L_t^\infty H^{\frac{1}{2}}} \lesssim \left\|F_1^k\right\|_{L_t^{4/3}L_x^{4/3}} + \left\|G_1^k\right\|_{L_t^{4/3}H_{4/3}^{1/2}} + \sum_{j=2,3}\left(\left\|F_j^k\right\|_{L_t^1H^{-1/2}} + \left\|G_j^k\right\|_{L_t^1L_x^2}\right). \tag{3.47}$$

Moreover, we remember that $\left(\tilde{\phi}^{(k)}, \tilde{\psi}^{(k)}\right)$ is expressed as (3.39).

Now we will estimate F_1^k. By the Hölder inequality, we have

$$\begin{aligned}\left\|\left(\mathcal{D}\mathcal{B}\Phi^{(k)}\right)\mathcal{B}\Psi^{(k)}\right\|_{L_t^{4/3}L_x^{4/3}} &\lesssim \left\|\left\|\mathcal{B}\Phi^{(k)}(s)\right\|_{H^1}\left\|\mathcal{B}\Psi^{(k)}(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|s^{-\mu}\left\|\mathcal{B}\Psi^{(k)}(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\le A\left\|\mathcal{B}\Psi^{(k)}\right\|_{L_t^4L_x^4}\left\|s^{-\mu}\right\|_{L_t^2(I)} \lesssim A^2t^{1/2-2\mu},\end{aligned} \tag{3.48}$$

for any $t \ge T$ since $v^{(l)} \in X_{T,A}$ for $0 \le l \le k$. By the Hölder inequality and Remark 3.1 with $p=8$, we obtain

$$\left\|\tilde{f}_D(s)\right\|_{L^4} \lesssim \|\phi_0(s)\|_{H_8^2}\|\psi_0(s)\|_{H_8^1} \lesssim A^2s^{-\frac{3}{2}}. \tag{3.49}$$

for any $s \ge t$. In the same manner as the proof of the estimate (3.48), we also obtain

$$\left\|\left(\mathcal{D}\mathcal{B}\Phi^{(k)}\right)\tilde{f}_D\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim A^3t^{-3/4-\mu}, \tag{3.50}$$

for all $t \ge T$, due to $v^{(l)} \in X_{T,A}$ for $0 \le l \le k$ and (3.49). By the Hölder inequality and Remark 3.1 with $p = 8/3, 8$, we obtain

$$\left\|\mathcal{D}\tilde{f}_K(s)\right\|_{L^2} \lesssim \|\psi_0(s)\|_{H_{8/3}^2}\|\psi_0(s)\|_{H_8^1} \lesssim s^{-1}\left\|\psi^+\right\|_{H_{8/3}^{5/2}}\left\|\psi^+\right\|_{\mathbf{H}_{8/7}^{5/2}} \lesssim A^2s^{-1} \tag{3.51}$$

for any $s \ge t$, where we have used the properties (DM) and $\mathcal{D}_+\psi_0 = 0$. Thus in the same manner as the proof of the estimate (3.50), we obtain

$$\left\|\left(\mathcal{D}\tilde{f}_K\right)\mathcal{B}\Psi^{(k)}\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim A^3t^{-1/2-\mu}, \tag{3.52}$$

for all $t \ge T$ due to $v^{(k)} \in \mathbf{X}_{T,A}$ and (3.51). By the Hölder inequality and the estimates (3.49) and (3.52), we get

$$\left\|\left(\mathcal{D}\tilde{f}_K\right)\tilde{f}_D\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim \left\|\left\|\mathcal{D}\tilde{f}_K(s)\right\|_{L_x^2}\left\|\tilde{f}_D(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}(I)} \lesssim A^4t^{-7/4}, \tag{3.53}$$

for all $t \ge T$. Thus by combining (3.48), (3.50) and (3.52)-(3.53), we obtain

$$\left\|F_1^k\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim A^2t^{1/2-2\mu}, \tag{3.54}$$

for $t \geq T \geq 1$ since $\mu < 1$. Next we consider F_2^k. We have

$$\left\| F_2^k \right\|_{L_t^1 H^{-1/2}} \leq \left\| \left(\mathcal{D}\tilde{\phi}^{(k)}\right) \psi_0 \right\|_{L_t^1 L_x^2} + \left\| (\mathcal{D}\phi_0) \tilde{\psi}^{(k)} \right\|_{L_t^1 L_x^2}. \tag{3.55}$$

By Corollary 3.4 with $p = \infty$, we have

$$\left\| \left(\mathcal{D}\mathcal{B}\Phi^{(k)}\right) \psi_0 \right\|_{L_t^1 L_x^2} \lesssim \left\| \left\| \mathcal{B}\Phi^{(k)}(s) \right\|_{H^1} \left\| \psi_0(s) \right\|_{L_x^\infty} \right\|_{L_t^1} \lesssim \rho A t^{-\mu}, \tag{3.56}$$

for all $t \geq T$ since $v^{(l)} \in X_{T,A}$ for $0 \leq l \leq k$. In the same manner as the estimate (3.57), we get

$$\left\| \left(\mathcal{D}\tilde{f}_K\right) \psi_0 \right\|_{L_t^1 L_x^2} \lesssim \left\| \left\| \mathcal{D}\tilde{f}_K(s) \right\|_{L_x^2} \left\| \psi_0(s) \right\|_{L_x^\infty} \right\|_{L_t^1} \leq \rho A^2 t^{-1}, \tag{3.57}$$

for any $t \geq T$, where we have used the estimate (3.51). Moreover, we also have

$$\left\| (\mathcal{D}\phi_0) \mathcal{B}\Psi^{(k)} \right\|_{L_t^1 L_x^2} \lesssim \left\| \left\| \phi_0(s) \right\|_{H_\infty^1} \left\| \mathcal{B}\Psi^{(k)}(s) \right\|_{L_x^2} \right\|_{L_t^1} \leq A\rho t^{-\mu}, \tag{3.58}$$

for all $t \geq T$ since $v^{(k)} \in X_{T,A}$. In the same proof as the estimate (3.49), by the Hölder inequality and Remark 3.1 with $p = 4$, we get

$$\left\| \tilde{f}_D(s) \right\|_{L^2} \lesssim \left\| \phi_0(s) \right\|_{H_4^2} \left\| \psi_0(s) \right\|_{H_4^1} \lesssim A^2 s^{-1}, \tag{3.59}$$

for any $s \geq t$. By the estimate (3.59) and Corollary 3.4 with $p = \infty$, we have

$$\left\| (\mathcal{D}\phi_0) \tilde{f}_D \right\|_{L_t^1 L_x^2} \lesssim \left\| \left\| \phi_0(s) \right\|_{H_\infty^1} \left\| \tilde{f}_D(s) \right\|_{L_x^2} \right\|_{L_t^1} \lesssim \rho A^2 t^{-1}, \tag{3.60}$$

for all $t \geq T$. Therefore by combining the estimates (3.55)-(3.58) and (3.60), we obtain

$$\left\| F_2^k \right\|_{L_t^1 H^{-1/2}} \lesssim \rho A t^{-\mu}, \tag{3.61}$$

for any $t \geq T \geq 1$ since $\mu < 1$. Next we consider F_3^k. By the definition of $\mathcal{R}_D$, we have

$$\left\| \mathcal{R}_D(\mathcal{D}\phi_0, \psi_0) \right\|_{L_t^1 H^{-1/2}} \lesssim \sum_{j=1,2} \left\| \mathcal{R}(\mathcal{D}\phi_0, \psi_{0,j}) \right\|_{L_t^1 L_x^2}, \tag{3.62}$$

where we put $\psi_0 = (\psi_{0,1}, \psi_{0,2})^t$. By Lemma 3.6, we can express $\mathcal{R}$ as

$$\mathcal{R}(\mathcal{D}\phi_0, \psi_{0,j}) \equiv s^{-1} Z_1 + s^{-2} Z_2, \tag{3.63}$$

for $s \in \mathbb{R} \setminus \{0\}$, where

$$Z_1 \equiv (\partial_0 \partial_t \mathcal{D}\phi_0)(\mathcal{Z}_1 \partial_1 \psi_{0,j}) - (\mathcal{Z}_1 \partial_t \mathcal{D}\phi_0)(\partial_0 \partial_1 \psi_{0,j}) + \text{similar},$$

$$Z_2 \equiv -(\mathcal{Z}_1 \partial_2 \psi_{0,j})(\mathcal{Z}_2 \partial_1 \mathcal{D}\phi_0) + (\mathcal{Z}_1 \partial_1 \mathcal{D}\phi_0)(\mathcal{Z}_2 \partial_2 \psi_{0,j}) + \text{similar}.$$

By applying the Hölder inequality, we have

$$\left\| s^{-1} Z_1 \right\|_{L_t^1 L_x^2} \lesssim \int_t^\infty s^{-1} \left(\left\| \phi_0 \right\|_{H_q^3} \left\| \mathcal{Z}\psi_0 \right\|_{H^1_{\frac{2q}{q-2}}} + \left\| \psi_0 \right\|_{H_q^2} \left\| \mathcal{Z}\phi_0 \right\|_{H^2_{\frac{2q}{q-2}}} \right) ds. \tag{3.64}$$

By Corollary 3.4 with $p = q$, we get

$$\left\| \phi_0(s) \right\|_{H_q^3} \lesssim s^{-1+2/q} \left\| \left(\langle \nabla \rangle \phi_1^+, \phi_2^+ \right) \right\|_{H^{4-4/q}_{q/(q-1)}} \lesssim A s^{-1+2/q}, \tag{3.65}$$

$$\left\| \psi_0(s) \right\|_{H_q^2} \lesssim s^{-1+2/q} \left\| \psi^+ \right\|_{H^{4-4/q}_{q/(q-1)}} \lesssim A s^{-1+2/q}, \tag{3.66}$$

for any $s \geq t$. On the other hand, note that the commutation relations (3.5)-(3.6) hold, by applying the Sobolev inequality and the charge and energy conservation laws, we obtain

$$\left\| \mathcal{Z}\psi_0 \right\|_{H^1_{2q/(q-2)}} \lesssim \left\| \mathcal{Z}\psi_0 \right\|_{H^{1+2/q}} \lesssim \left\| \mathcal{Z}\psi_0 \right\|_{H^{3/2}} \lesssim \left\| (\mathcal{Z}\psi_0)(0) \right\|_{H^{3/2}} \lesssim A, \tag{3.67}$$

$$\left\| \mathcal{Z}\phi_0 \right\|_{H^2_{2q/(q-2)}} \lesssim \left\| \mathcal{Z}\phi_0 \right\|_{H^{2+2/q}} \lesssim \left\| \mathcal{Z}\phi_0 \right\|_{H^{5/2}} \lesssim \left\| (\mathcal{Z}\phi_0)(0) \right\|_{H^{5/2}} \lesssim A, \tag{3.68}$$

since $q > 4$. Thus by combining (3.64)-(3.68), we get

$$\left\|s^{-1}Z_1\right\|_{L_t^1L_x^2} \lesssim A^2t^{-1+2/q}, \tag{3.69}$$

for any $t \geq T$. By the Hölder inequality, we have

$$\left\|s^{-2}Z_2\right\|_{L_t^1L_x^2} \lesssim \int_t^\infty s^{-2}\left\|\mathcal{Z}\psi_0(s)\right\|_{H_4^1}\left\|\mathcal{Z}\phi_0(s)\right\|_{H_4^2}ds \lesssim A^2t^{-1}, \tag{3.70}$$

since in the same manner as the proof of the estimates (3.67)-(3.68), we obtain

$$\left\|\mathcal{Z}\psi_0(s)\right\|_{H_4^1} + \left\|\mathcal{Z}\phi_0(s)\right\|_{H_4^2} \lesssim A,$$

for any $s \geq t$. Therefore combining (3.61)-(3.63), (3.69) and (3.70), we have

$$\left\|F_3^k\right\|_{L_t^1H^{-1/2}} \lesssim A^2t^{-1+2/q}, \tag{3.71}$$

for all $t \geq T \geq 1$ since $q > 4$.

Next, we will estimate G_1^k. By the Leibniz formula (3.4) with $\kappa = 1/2$, $p = 4/3$, $q_1 = q_2 = 2$ and $r_1 = r_2 = 4$ and the Hölder inequality, we obtain

$$\begin{aligned}\left\|\left(\mathcal{B}\Psi^{(k)}\right)^*\mathcal{B}\Psi^{(k)}\right\|_{L_t^{4/3}H_{4/3}^{1/2}} &\lesssim \left\|\left\|\mathcal{B}\Psi^{(k)}(s)\right\|_{H^{1/2}}\left\|\mathcal{B}\Psi^{(k)}(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|s^{-\mu}\left\|\mathcal{B}\Psi^{(k)}(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|s^{-\mu}\right\|_{L_t^2(I)}\left\|\mathcal{B}\Psi^{(k)}\right\|_{L_t^4L_x^4} \lesssim A^2t^{1/2-2\mu},\end{aligned} \tag{3.72}$$

for any $t \geq T$ since $v^{(k)} \in X_{T,A}$. By the fractional Leibniz rule (3.4) again and Remark 3.1 with $p = 4$, we have

$$\left\|\tilde{f}_D(s)\right\|_{H^{1/2}} \lesssim \left\|\phi_0(s)\right\|_{H_4^{5/2}}\left\|\psi_0(s)\right\|_{H_4^{3/2}} \lesssim A^2s^{-3/2}, \tag{3.73}$$

for any $s \geq t$. In the same manner as the proof of the estimate (3.72), we obtain

$$\left\|\left(\mathcal{B}\Psi^{(k)}\right)^*\tilde{f}_D\right\|_{L_t^{4/3}H_{4/3}^{1/2}} \lesssim \left\|\left\|\mathcal{B}\Psi^{(k)}(s)\right\|_{H^{1/2}}\left\|\tilde{f}_D(s)\right\|_{H^{1/2}}\right\|_{L_t^{4/3}} \lesssim A^3t^{-3/4-\mu}, \tag{3.74}$$

for any $t \geq T$ due to $v^{(k)} \in X_{T,A}$ and (3.73). In the same manner as the proof of the estimate (3.75), we get

$$\left\|\left(\tilde{f}_D\right)^*\tilde{f}_D\right\|_{L_t^{4/3}H_{4/3}^{1/2}} \lesssim A^4t^{-7/4}, \tag{3.75}$$

for all $t \geq T$. Thus by combining the estimates (3.72) and (3.74)-(3.75), we obtain

$$\left\|G_1^k\right\|_{L_t^{4/3}H_{4/3}^{1/2}} \lesssim A^2t^{1/2-2\mu}, \tag{3.76}$$

for $t \geq T \geq 1$ since $\mu < 1$. In the same manner as the proof of the estimates (3.61) and (3.71), we obtain

$$\left\|G_2^k\right\|_{L_t^1L_x^2} \lesssim \rho At^{-\mu}, \quad \left\|G_3^k\right\|_{L_t^1L_x^2} \lesssim A^2t^{-1+2/q}, \tag{3.77}$$

for any $t \geq T$. Finally, by combining (3.54), (3.61), (3.71), (3.76) and (3.76)-(3.77), we obtain

$$\left\|v^{(k+1)}\right\|_{X_T} \lesssim A\left(AT^{1/2-\mu} + \rho + AT^{-1+\mu+2/q}\right), \tag{3.78}$$

for $T \geq 1$. By the estimate (3.78) and $1/2 < \mu < 1 - 2/q$, there exist a large $T > 0$ and a small $\rho > 0$ such that $v^{(k+1)} \in X_{T,A}$. In the same manner as the proof of (3.78), we can prove the estimate

$$\left\|v^{(k+1)} - v^{(k)}\right\|_{X_T} \leq \frac{1}{2}\left\|v^{(k)} - v^{(k-1)}\right\|_{X_T}, \tag{3.79}$$

for $l \geq 1$ if $T > 1$ is sufficiently large and $\rho > 0$ is suffuciently small, which implies that $\{v^{(k)}\}_{k\geq 0}$ is a Cauchy sequence in $X_{T,A}$. Theorem 3.1 is proved. □

3.7. **Appendix.** In this subsection, we give a proof of Lemma 3.10. By Lemma DKG and a decay property of $\tilde{f}_D$ given by (3.22), we also have the following:

Corollary 3.11. *Let* $(\psi^+, (\langle\nabla\rangle \phi_1^+, \phi_2^+)) \in (H^{5/2,1})^4$. *The final state condition (2.3)-(2.4) with* $\mathbf{X}_1 = H^{1/2}$, $\mathbf{X}_2 = H^1$*holds if and only if the identity*

$$\lim_{t\to\infty} \sum_{\pm} \|\mathcal{A}_\pm \Psi(t)\|_{H^{1/2}} = 0 \tag{3.80}$$

is valid, where Ψ *is defined by (3.33).*

We put $B = \|(\psi^+, (\langle\nabla\rangle \phi_1^+, \phi_2^+))\|_{H^{5/2,1}}$.

Proof. By Lemma 2.5, we see that (2.3) with $\mathbf{X}_1 = H^{1/2}$ is equivalent to

$$\lim_{t\to\infty} \sum_{\pm} \|\mathcal{A}_\pm \psi(t) - \mathcal{U}_\pm(t) \mathcal{A}_\pm \psi^+\|_{H^{1/2}} = 0. \tag{3.81}$$

By the decomposition (2.9) and the identities (2.10), we have

$$\|\mathcal{A}_\pm \Psi(t)\|_{H^{1/2}} = \left\|\mathcal{A}_\pm \psi(t) - \mathcal{U}_\pm(t) \mathcal{A}_\pm \psi^+ - \mathcal{A}_\pm \tilde{f}_D\right\|_{H^{1/2}}.$$

By the property of $\mathcal{A}_\pm$, the fractional Leibniz rule (3.4) with $p = 2$ and $q_i = r_i = 4$ $(i = 1, 2)$ and Remark 3.1 with $p = 4$, we get

$$\left\|\mathcal{A}_\pm^D \tilde{f}_D\right\|_{H^{1/2}} \lesssim \|\phi_0\|_{H_4^{5/2}} \|\psi_0\|_{H_4^{3/2}} \lesssim t^{-1}B^2,$$

for all $t > 0$, which completes the proof of the corollary. □

Next we will prove Lemma 3.10.

Proof of Lemma 3.10. First we prove the Dirac part. By Corollary 3.11, we see that (3.31) is equivalent to

$$\lim_{t\to\infty} \sum_{\pm} \left\|\mathcal{A}_\pm \Psi^{(k)}(t)\right\|_{H^{1/2}} = 0 \quad \text{for } k \geq 0. \tag{3.82}$$

By (2.16) and the Dirac part of (3.34), we have

$$\mathcal{B}_\pm \Psi^{(k+1)} = \mathcal{A}_\pm \Psi^{(k+1)} - \langle\nabla\rangle_M^{-1} F^k \quad \text{for } k \geq 0. \tag{3.83}$$

Thus it is sufficient to show that

$$\lim_{t\to\infty} \left\|F^k\right\|_{H^{-1/2}} = 0 \quad \text{for } k \geq 0. \tag{3.84}$$

By the Sobolev inequality and the Hölder inequality, we have for $k \geq 1$,

$$\begin{aligned} \left\|F^k\right\|_{H^{-1/2}} &\lesssim \left\|\tilde{\phi}^k\right\|_{H^{1/2}} \left\|\tilde{\psi}^k\right\|_{H^{1/2}} + \left\|\tilde{\phi}^k\right\|_{H^{1/2}} \|\psi^+\|_{H^{1/2}} \\ &\quad + \left(\|\phi_1^+\|_{H^{1/2}} + \|\phi_2^+\|_{H^{-1/2}}\right) \left\|\tilde{\psi}^l\right\|_{H^{1/2}} + \|\phi_0\|_{H_8^2} \|\psi_0\|_{H_{8/3}^2}. \end{aligned} \tag{3.85}$$

By Remark 3.1 with $p = 8, 8/3$, we get

$$\|\phi_0\|_{H_8^2} \lesssim t^{-3/4}B, \quad \|\psi_0\|_{H_{8/3}^2} \lesssim t^{-1/2}B. \tag{3.86}$$

Thus by the assumptions and the estimates (3.85)-(3.86), we obtain (3.84) for $k \geq 1$. In the case of $k = 0$, it is easy to see (3.84). We omit the detail. Conversely, we assume (3.41) and we will prove (3.31). By the decomposition $I = \sum_\pm \mathcal{B}_\pm$, we have only to show

$$\lim_{t\to\infty} \sum_{\pm} \left\|\mathcal{B}_\pm^D \tilde{f}_D\right\|_{H^{1/2}} = 0. \tag{3.87}$$

We have

$$\left\|\mathcal{B}\tilde{f}_D\right\|_{H^{1/2}} \lesssim \left\|\mathcal{B}\mathcal{Q}_0^D\left(\mathcal{D}\phi_0,\psi_0\right)\right\|_{H^{1/2}} + \text{remainder}, \tag{3.88}$$

$$\left\|\mathcal{B}\mathcal{Q}_0^D\left(\mathcal{D}\phi_0,\psi_0\right)\right\|_{H^{1/2}} \lesssim \left\|\mathcal{Q}_0^D\right\|_{H^{1/2}} + \left\|\partial_t\mathcal{Q}_0^D\right\|_{H^{1/2}}. \tag{3.89}$$

By the Hölder inequality and Remark 3.1 with $p = 8, 8/3$, we obtain

$$\left\|\partial_t\mathcal{Q}_0\left(\mathcal{D}\phi_0,\psi_{0,j}\right)\right\|_{H^{-1/2}} \lesssim \|\phi_0\|_{H^3_{8/3}}\|\psi_0\|_{H^1_8} + \|\phi_0\|_{H^2_8}\|\psi_0\|_{H^2_{8/3}} \lesssim t^{-1}B^2. \tag{3.90}$$

Since the remainder terms in (3.88) can be estimated in the same manner as the proof of (3.90), we obtain

$$\left\|\mathcal{B}\tilde{f}_D\right\|_{H^{1/2}} \lesssim t^{-1}B^2, \tag{3.91}$$

from which (3.87) follows.

Next we consider the KG part. By the identity

$$\|f+g\|_{H^\kappa}^2 + \|f-g\|_{H^\kappa}^2 = 2\left(\|f\|_{H^\kappa}^2 + \|g\|_{H^\kappa}^2\right),$$

we can see that (2.4) with $\mathbf{X}_2 = H^1$ is equivalent to

$$\sum_{\pm}\left\|\mathcal{B}_\pm\left(\phi^{(k)}(t) - \phi_0(t)\right)\right\|_{H^1}.$$

In the same manner as the proof of the estimate (3.91), we can obtain

$$\left\|\mathcal{B}\tilde{f}_K\right\|_{H^1} \lesssim t^{-1}B^2,$$

which completes the proof of the lemma. □

4. Small data blow-up of L^2-solution for the nonlinear Schrödinger equation (NLS) without gauge invariance

4.1. **Introduction.** We study existence of a blow-up solution for the nonlinear Schrödinger equation (NLS) with non-gauge invariant power nonlinearity

$$i\partial_t u + \Delta u = \lambda |u|^p, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \tag{4.1}$$

with the initial condition

$$u(0,x) = f(x), \quad x \in \mathbb{R}^n, \tag{4.2}$$

where $T > 0$, $1 < p \le 1 + 2/n$, $u = u(t,x)$ is a complex-valued unknown function, $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} \setminus \{0\}$, $\lambda_j \in \mathbb{R}$ $(j = 1,2)$, $f = f(x) = f_1(x) + if_2(x)$ and $f_j = f_j(x) \in L^1_{loc}(\mathbb{R}^n)$ $(j = 1,2)$ are real-valued functions. This chapter is based on a joint work with Yuta Wakasugi.

It is well known that local well-posedness holds for (4.1) in several Sobolev spaces H^s $(s \ge 0)$ (see e.g. [3, 65] and the references therein). However, there had been no results of global existence of the solution of (4.1)-(4.2). In this paper, we will prove a small data blow up result for (4.1). More precisely, we will show that if the initial data f in L^2 satisfies a certain condition related to its sign, then the L^2-norm of the solution u of (4.1)-(4.2) blows up in finite time, even if the data is sufficiently small (see subsection 4.2). We note that when $p \ge p_s$, where p_s is the well-known Strauss exponent, which is greater than $1 + 2/n$, global existence results are known (see [3]). Thus the following natural questions arises: What happens in the case of $1+2/n < p \le p_s$?. This questions was addressed in recent paper [32].

Our result implies that the nonlinear effect of $\lambda |u|^p$ is quite different from that of $\lambda_0 |u|^{p-1} u$ $(\lambda_0 \in \mathbb{R})$, since the L^2-norm of solutions for

$$i\partial_t u + \Delta u = \lambda_0 |u|^{p-1} u \tag{4.3}$$

conserves for any $t \in \mathbb{R}$. Tsutsumi [65] proved global existence of L^2-solution of (4.3) when $1 < p < 1+4/n$. It is also well known that for (4.3), the exponent $p = 1+2/n$ is the threshold between the short range scattering and the long range one (see [2, 67, 52, 11, 18, 17]). We also mention that when $p \ge 1 + 4/n$, blow-up of H^1-solution of (4.3) is proved by Glassey [15] (see also [51]). However, their results require that the data are large as contrast with our result.

Back to our problem (4.1), in the short range critical case $(n,p) = (2,2)$, Shimomura [57] and Shimomura-Tsutsumi [58] studied the asymptotic behavior of solutions of (4.1). Especially, Shimomura-Tsutsumi [58] proved nonexistence of the wave operator for (4.1). On the other hand, Hayashi-Naumkin [21] considered the final state problem for NLS with the quadratic nonlinearity $\mu u^2 + \nu \bar{u}^2 + \lambda |u|^2$, which includes the term $\lambda |u|^2$, in two space dimension. They proved existence of the global solution which behaves unlike the free one in L^2. We note that their result requires that $\mu, \nu \ne 0$ and is not applicable to (4.1).

From these results, some people might think that the non-gauge invariant nonlinearity $\lambda |u|^p$ with $1 < p \le 1 + 2/n$ may act as a long range effect such as $\lambda_0 |u|^{p-1} u$. However, our result gives a negative conclusion to such an expectation.

4.2. **Main Result.** We first recall the well-known fact about local existence of the solution in L^2 for the integral equation

$$u(t) = U(t) f - i\lambda \int_0^t U(t-s) |u|^p \, ds \tag{4.4}$$

associated with (4.1)-(4.2), where $U(t) = \exp(it\Delta)$ is the evolution group of the free Schrödinger operator.

Proposition 4.1 (Tsutsumi [65]). *Let $1 < p < 1 + 4/n, \lambda \in \mathbb{C} \setminus \{0\}$ and $f \in L^2$. Then there exist a positive time $T = T(\|f\|_{L^2}) > 0$ and a unique solution $u \in C([0,T); L^2) \cap L^r_t(0,T; L^\rho_x)$ of the integral equation (4.4), where r, ρ are defined by $\rho = p+1$ and $2/r = n/2 - n/\rho$.*

We call the solution u in the above proposition "L^2-solution". Let T_m be the maximal existence time of local L^2-solution, that is,

$$T_m \equiv \sup\{T \in (0,\infty]\,;\ \text{there exists the unique solution } u \text{ to (4.4)} \\ \text{such that } u \in C\left([0,T)\,; L^2\right) \cap L_t^r\left(0,T; L_x^\rho\right)\},$$

where r, ρ are as in the above proposition. To state our result, we put the following assumption on the data:

$$\text{"}f_1 \in L^1(\mathbb{R}^n),\ \lambda_2 \int_{\mathbb{R}^n} f_1(x)dx > 0\text{" or "}f_2 \in L^1(\mathbb{R}^n),\ \lambda_1 \int_{\mathbb{R}^n} f_2(x)dx < 0\text{".} \tag{4.5}$$

Our main result is the following:

Theorem 4.2. *Let $1 < p \leq 1 + 2/n$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in L^2$. If the initial data f satisfies (4.5), then T_m must be finite. Moreover, we have*

$$\lim_{t \to T_m - 0} \|u(t)\|_{L^2} = +\infty. \tag{4.6}$$

We note that we put no restriction on the size of the data. In order to prove Theorem 4.2, in the next section, we introduce a weak solution of (4.1)-(4.2) and the result of nonexistence of a global weak solution.

4.3. **Reduction of the integral equation to a weak form.** To prove Theorem 4.2, we define a weak solution of (4.1)-(4.2).

Definition 4.1. *Let $T > 0$. We mean u is a weak solution of NLS (4.1)-(4.2) on $[0,T)$ if u belongs to $L^p_{loc}([0,T) \times \mathbb{R}^n)$ and satisfies*

$$\begin{aligned} &\int_{[0,T)\times\mathbb{R}^n} u\left(-i\partial_t\psi + \Delta\psi\right) dxdt \\ &= i\int_{\mathbb{R}^n} f(x)\,\psi(0,x)\,dx + \lambda \int_{[0,T)\times\mathbb{R}^n} |u|^p\,\psi dxdt \end{aligned} \tag{4.7}$$

for any $\psi \in C_0^2([0,T) \times \mathbb{R}^n)$. Moreover, if $T > 0$ can be chosen as any positive number, u is called a global weak solution for (4.1)-(4.2).

We note that an L^2-solution as in Proposition 4.1 is always a weak solution in the sense of Definition 4.1:

Proposition 4.3. *Let $T > 0$. If u is an L^2-solution for the equation (4.4) on $[0,T)$, then u is also a weak solution on $[0,T)$ in the sense of Definition 4.1.*

We will give a proof of this proposition in Appendix.

Next, we mention nonexistence of a nontrivial global weak solution for (4.1)-(4.2) with the condition (4.5).

Proposition 4.4. *Let $1 < p \leq 1 + 2/n$, $\lambda \in \mathbb{C}\setminus\{0\}$ and let f satisfy (4.5). If there exists a global weak solution u of (4.1)-(4.2), then $u = 0$.*

Combining Proposition 4.3 and 4.4, we obtain Theorem 4.2. Indeed, let $f \in L^2$ satisfy (4.5) and u be the L^2-solution of (4.4). Suppose that $T_m = \infty$. By Proposition 4.3, u is also a global weak solution of (4.1)-(4.2) in the sense of Definition 4.1. Thus, we can apply Proposition 4.4 and have $u = 0$. However, by noting $u \in C([0,\infty); L^2(\mathbb{R}^n))$, it contradicts $f \neq 0$. Therefore, we have $T_m < \infty$.

Next, we prove (4.6). First we suppose

$$\liminf_{t \to T_m - 0} \|u(t)\|_{L^2} < \infty.$$

Then there exist a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, T_m)$ and a positive constant $M > 0$ such that

$$\lim_{k\to\infty} t_k = T_m \tag{4.8}$$

$$\sup_{k\in\mathbb{N}} \|u(t_k)\|_{L^2} \le M. \tag{4.9}$$

By (4.9) and Proposition 4.1, there exists a positive constant $T(M)$ such that we can construct a solution

$$u \in C([t_k, t_k + T(M)); L^2) \cap L_t^r([t_k, t_k + T(M)); L_x^\rho)$$

of (4.4) for all $k \in \mathbb{N}$. However, by (4.8), when k is sufficiently large, the inequality $t_k + T(M) > T_m$ holds and it contradicts the definition of T_m. Therefore, we obtain

$$\liminf_{t\to T_m-0} \|u(t)\|_{L^2} = \infty,$$

which completes the proof of Theorem 4.2.

At the end of this subsection, we mention the strategy of the proof of Proposition 4.4. We apply a test-function method used by Zhang [72, 73] to NLS (4.1). By using some test-functions and space-time sets cleverly, he obtained some blow-up results for nonlinear parabolic equations (see [72]). By the same method, he also proved a blow-up result for the nonlinear damped wave equation:

$$\begin{cases} v_{tt} - \Delta v + v_t = |v|^p, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $1 < p \le 1 + 2/n$, $v = v(t,x)$ is a real-valued unknown function, $v_0(x)$ and $v_1(x)$ are compactly supported given functions (see [73]). However, since this method needs a positivity of the nonlinear term $|v|^p$, it can not be applicable to NLS (4.1) directly, because solutions for NLS are generally complex-valued and the constant λ in front of the nonlinearity is a complex number. To overcome these difficulties, we make a little modification to this method by introducing an appropriate positive function (see (4.12)) related to $\lambda |u|^p$.

For the nonlinear heat equation and the damped wave equation with the same type nonlinearity as $|u|^p$, it is well known that the exponent $p = 1 + 2/n$, which is often referred to as the "Fujita exponent", is the threshold between the small data global existence and blow-up of solutions (see [47, 7, 63] and the references therein).

4.4. **Proof of non-existence for non-trivial global weak solution.**

In this subsection, we give a proof of Proposition 4.4.

Proof. First we introduce two cut-off functions $\eta = \eta(t) \in C_0^\infty([0,\infty))$ and $\phi = \phi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \eta, \phi \le 1$,

$$\eta(t) \equiv \begin{cases} 1 & \text{if } t \le 1/2 \\ 0 & \text{if } t \ge 1 \end{cases}, \quad \phi(x) \equiv \begin{cases} 1 & \text{if } |x| \le 1/2 \\ 0 & \text{if } |x| \ge 1 \end{cases}.$$

Furthermore, it is possible to take ϕ satisfying the inequality

$$\frac{|(\nabla\phi)(x)|^2}{\phi(x)} \le C \quad \text{for } |x| \le 1, \tag{4.10}$$

with some constant C independent of x. Let $R > 0$ be large parameter. Using the above cut-off functions, we also put three cut-off functions dependent on R:

$$\eta_R(t) \equiv \eta\left(\frac{t}{R^2}\right) \quad \text{for } t \in \mathbb{R}, \ \phi_R(x) \equiv \phi\left(\frac{x}{R}\right) \quad \text{for } x \in \mathbb{R}^n,$$
$$\psi_R(t,x) \equiv \eta_R(t)\,\phi_R(x) \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^n. \tag{4.11}$$

Let $B_R \equiv \{x \in \mathbb{R}^n;\ |x| \le R\}$ be a ball at the origin. We also define the time-space set $Q_R \equiv [0, R^2] \times B_R$. We note that Q_R includes the support of ψ_R. Denote $q \equiv p/(p-1) \in [1 + n/2, \infty)$. We consider the case $\lambda_1 > 0$ and $\lambda_1 \int f_2 dx < 0$ only, since the other cases can be treated almost in the same way (see Remark 4.1). In this case, we may assume $f_2 \in L^1$ and $\int_{\mathbb{R}^n} f_2(x)\,dx < 0$ by the assumption (4.5). We define a positive function of R by

$$I_R \equiv \operatorname{Re} \int_{Q_R} \lambda |u|^p \psi_R^q dxdt. \tag{4.12}$$

We note that $\psi_R^q \in C_0^2([0, R^2+1) \times \mathbb{R}^n)$. Since u is a global weak solution of (4.1) (see Definition 4.1), we can use the identity (4.7) with $T = R^2 + 1$ and have

$$\begin{aligned} I_R &= \int_{B_R} f_2(x)\,\phi_R^q(x)\,dx + q\int_{Q_R} (\operatorname{Im} u)\,\psi_R^{q-1}\partial_t(\psi_R)\,dxdt \\ &\quad + \int_{Q_R} (\operatorname{Re} u)\,\Delta\left(\psi_R^q\right) dxdt. \end{aligned} \tag{4.13}$$

By the assumption on f_2, the first term of the right hand side of (4.13) is negative for sufficiently large $R > 0$. In fact, by $f_2 \in L^1$ and Lebesgue's convergence theorem, there exists $R_1 > 0$ such that for any $R > R_1$,

$$\int_{B_R} f_2(x)\,\phi_R^q(x)\,dx < 0.$$

Thus, we have for $R > R_1$,

$$\begin{aligned} I_R &< q\int_{Q_R} (\operatorname{Im} u)\,\psi_R^{q-1}(\partial_t\psi_R)\,dxdt + \int_{Q_R} (\operatorname{Re} u)\,\Delta\left(\psi_R^q\right) dxdt \\ &\lesssim \int_{Q_R} |u|\,\psi_R^{q-1}\,|\partial_t(\psi_R)|\,dxdt + \int_{Q_R} |u|\,\left|\Delta\left(\psi_R^q\right)\right| dxdt \\ &\equiv J_{1,R} + J_{2,R}. \end{aligned} \tag{4.14}$$

First we will estimate $J_{1,R}$. By a simple calculation, we get

$$\partial_t\psi_R(t,x) = \frac{1}{R^2}\phi_R(x)\,(\partial_t\eta)\left(\frac{t}{R^2}\right).$$

By noting $\partial_t\eta(t) = 0$ if $t \in [0, 1/2]$ and the Hölder inequality, we obtain

$$\begin{aligned} J_{1,R} &\lesssim \frac{1}{R^2}\int_{R^2/2}^{R^2}\int_{B_R} |u|\psi_R^{q-1} dxdt \\ &\lesssim \frac{1}{R^2}\left(\int_{R^2/2}^{R^2}\int_{\mathbf{B}_R} |u|^p\psi_R^q dxdt\right)^{1/p}\left(\int_{R^2/2}^{R^2}\int_{B_R} dxdt\right)^{1/q} \\ &\simeq I_{1,R}^{1/p} R^{(n+2-2q)/q}, \end{aligned} \tag{4.15}$$

where

$$I_{1,R} \equiv \operatorname{Re} \int_{R^2/2}^{R^2}\int_{B_R} \lambda|u|^p\psi_R^q dxdt.$$

We note that $n + 2 - 2q \leq 0$, since $1 < p \leq 1 + 2/n$. Next we consider $J_{2,R}$. By a direct computation, we have

$$\begin{aligned} \Delta\left(\psi_R^q\right) &= \frac{1}{R^2}q(q-1)\,\eta_R^q(t)\,\phi_R^{q-2}(x)\,|\nabla\phi|^2\left(\frac{x}{R}\right) \\ &\quad + \frac{1}{R^2}q\eta_R^q(t)\,\phi_R^{q-1}(x)\,(\Delta\phi)\left(\frac{x}{R}\right). \end{aligned}$$

Using this and (4.10), in the same manner as above, we obtain

$$\begin{aligned} J_{2,R} &\lesssim \frac{1}{R^2}\left(\int_0^{R^2}\int_{B_R\setminus B_{R/2}} |u|^p\psi_R^q dxdt\right)^{1/p}\left(\int_0^{R^2}\int_{B_R\setminus B_{R/2}} dxdt\right)^{1/q} \\ &\simeq I_{2,R}^{1/p} R^{(n+2-2q)/q}, \end{aligned} \tag{4.16}$$

where we put

$$I_{2,R} \equiv \operatorname{Re} \int_0^{R^2} \int_{B_R \setminus B_{R/2}} \lambda |u|^p \psi_R^q dx dt.$$

By combining (4.14)-(4.16), we have

$$I_R \lesssim \left(I_{1,R}^{1/p} + I_{2,R}^{1/p} \right) R^{(n+2-2q)/q}, \tag{4.17}$$

for $R > R_1$. Since it is clear that $I_{j,R} \le I_R \, (j = 1, 2)$, we obtain

$$I_R \lesssim R^{n+2-2q} \le C, \tag{4.18}$$

with some constant C independent of R, since $n + 2 - 2q \le 0$. Here we note that only in the critical case $p = 1 + 2/n$, the identity $n + 2 - 2q = 0$ holds. By (4.18) and letting $R \to +\infty$, we have

$$\operatorname{Re} \int_{[0,\infty) \times \mathbb{R}^n} \lambda |u|^p \, dt dx < \infty,$$

that is, $u \in L^p([0, \infty) \times \mathbb{R}^n)$. Noting this and the integral region of $I_{1,R}$ and $I_{2,R}$, we have

$$\lim_{R \to +\infty} I_{j,R} = 0, \quad \text{for } j = 1, 2. \tag{4.19}$$

Therefore by the inequality (4.17) and (4.19), we get

$$\lim_{R \to +\infty} I_R = 0,$$

which implies $u = 0$. This completes the proof. □

Remark 4.1. *In the different cases from $\lambda_1 > 0$, putting*

$$I_R \equiv \begin{cases} -\operatorname{Re} \displaystyle\int_{Q_R} \lambda |u|^p \psi_R^q dx dt & \text{if } \lambda_1 < 0, \ \lambda_1 \int f_2 dx < 0, \\ \operatorname{Im} \displaystyle\int_{Q_R} \lambda |u|^p \psi_R^q dx dt & \text{if } \lambda_2 > 0, \ \lambda_2 \int f_1 dx > 0, \\ -\operatorname{Im} \displaystyle\int_{Q_R} \lambda |u|^p \psi_R^q dx dt & \text{if } \lambda_2 < 0, \ \lambda_2 \int f_1 dx > 0, \end{cases}$$

we can prove the same conclusion in the same manner as above.

4.5. **Appendix.** In this subsection, we give a proof of Proposition 4.3. The main difficulty of the proof lies in the fact that if p is close to 1, then the nonlinear term $|u|^p$ does not have twice differentiability with respect to space variables. To avoid differentiating twice, we use appropriate changing variables and differentiate with regard to time variable (see (4.27)). As the result, we can derive an H^2-estimate (see also [3]).

We first recall the well-known Strichartz estimates for the Schrödinger equation (see [68]). Let

$$\begin{cases} 2 \le \rho_j < 2n/(n-2) & \text{if } n \ge 3 \\ 2 \le \rho_j < \infty & \text{if } n = 2 \\ 2 \le \rho_j \le \infty & \text{if } n = 1 \end{cases} \quad \text{and} \quad \frac{2}{r_j} = \frac{n}{2} - \frac{n}{\rho_j} \ (j = 1, 2).$$

Then the following estimates hold:

Lemma 4.5. *For any time interval I, the estimates*

$$\|U(t)f\|_{L_t^{r_1}(I; L_x^{\rho_1})} \lesssim \|f\|_{L^2},$$

$$\left\| \int_0^t U(t-s) F ds \right\|_{L_t^{r_1}(I; L_x^{\rho_1})} \lesssim \|F\|_{L_t^{r_2'}(I; L_x^{\rho_2'})} \tag{4.20}$$

are true, where $r_2' = r_2/(r_2 - 1)$ and $\rho_2' = \rho_2/(\rho_2 - 1)$.

Now we give a proof of Proposition 4.3. Denote the nonlinear term by $F(u) = \lambda |u|^p$ and the time interval by $I = [0, T)$ for simplicity.

Proof. Let $T > 0, \rho = p+1, 2/r = n/2 - n/\rho$, $\psi \in C_0^2([0,T) \times \mathbb{R}^n)$ and let u be an L^2-solution of (4.4) on $[0,T)$. It is easy to see that $u \in L^p_{loc}([0,T) \times \mathbb{R}^n)$. We decompose u into $u = u_1 + u_2$, where $u_1 \equiv U(t) f$ is the homogeneous part and

$$u_2 \equiv -i \int_0^t U(t-s) F(u)\, ds$$

is the inhomogeneous one. The homogeneous part u_1 can be treated easily. In fact, by a standard density argument, we can obtain the identity

$$\int_{I\times\mathbb{R}^n} u_1(-i\partial_t\psi + \Delta\psi)dxdt = i\int_{\mathbb{R}^n} f(x)\psi(0,x)dx.$$

Thus, it suffices to prove

$$\int_{I\times\mathbb{R}^n} u_2(-i\partial_t\psi + \Delta\psi)dxdt = \int_{I\times\mathbb{R}^n} F(u)\psi dxdt, \tag{4.21}$$

which must be dealt with somewhat carefully because of involving the non-smooth nonlinearity $|u|^p$. we split the left-hand-side of (4.21) as

$$\begin{aligned} & -i\int_{I\times\mathbb{R}^n} u_2(\partial_t\psi)dxdt + \int_{I\times\mathbb{R}^n} u_2\Delta\psi dxdt \\ \equiv\ & K_1 + K_2. \end{aligned} \tag{4.22}$$

Hereafter we use the notation $L_t^r L_x^\rho \equiv L_t^r(I; L_x^\rho)$ for simplicity. Since $u \in L_t^r L_x^\rho$ and $C_0^\infty(I\times\mathbb{R}^n)$ is dense in $L_t^r L_x^\rho$, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_0^\infty(I\times\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{L_t^r L_x^\rho} = 0. \tag{4.23}$$

We also introduce an approximate function sequence $\{u_{2,k}\}_{k\in\mathbb{N}}$ to the inhomogeneous part u_2, whose component is given by

$$u_{2,k} \equiv -i\int_0^t U(t-s)F(u_k)ds.$$

Let $\alpha \equiv \frac{n}{4}\left(1 + \frac{4}{n} - p\right) > 0$. By the Strichartz estimate (4.20) and the Hölder inequality with $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{p-1}{p+1}$ and $\frac{1}{r'} = \frac{1}{r} + \frac{p-1}{r} + \alpha$, we can estimate

$$\begin{aligned} \|u_2 - u_{2,k}\|_{L_t^\infty L_x^2} &\lesssim \left\| |u|^p - |u_k|^p \right\|_{L_t^{r'} L_x^{\rho'}} \\ &\lesssim \left\| \left(\|u\|_{L_x^\rho}^{p-1} + \|u_k\|_{L_x^\rho}^{p-1} \right) \|u - u_k\|_{L_x^\rho} \right\|_{L_t^{r'}} \\ &\lesssim T^\alpha \left(\|u\|_{L_t^r L_x^\rho}^{p-1} + \|u_k\|_{L_t^r L_x^\rho}^{p-1} \right) \|u - u_k\|_{L_t^r L_x^\rho}. \end{aligned} \tag{4.24}$$

By (4.23), (4.24) , noting $u_{2,k}(0,x) = 0$ and integration by parts, we have

$$\begin{aligned} K_1 &= -i\lim_{n\to\infty}\int_{I\times\mathbb{R}^n} u_{2,k}(\partial_t\psi)dxdt \\ &= \lim_{n\to\infty} i\int_{I\times\mathbb{R}^n} (\partial_t u_{2,k})\psi dxdt. \end{aligned} \tag{4.25}$$

By the almost same argument as in (4.24), we find that $u_{2,k} \in C\left(I; H^1\right)$ and there exists a time derivative $\partial_t u_{2,k} \in C\left(I; H^{-1}\right)$ such that the identity

$$\partial_t u_{2,k} = i\Delta u_{2,k} - iF(u_k) \tag{4.26}$$

is valid. From this identity, we can show that $\Delta u_{2,k} \in C\left(I; L^2\right)$. In fact, changing variables with $t - s = s'$, we have

$$\partial_t u_{2,n}(t) = -i\partial_t \int_0^t U(s')F(u_k)(t-s')ds'$$

$$= -iU(t)F(u_k)(0) - i\int_0^t U(s)\partial_t(F(u_k))(t-s)d\dot{s}. \tag{4.27}$$

Applying the Strichartz estimate (4.20) to (4.27), we have

$$\|\partial_t u_{2,k}(t)\|_{L^2} \lesssim \|u_k(0)\|_{L^{2p}}^p + \|\partial_t(F(u_k))\|_{L_t^{r'}L_x^{\rho'}}. \tag{4.28}$$

By the same way as in (4.24), we also have

$$\|\partial_t(F(u_k))\|_{L_t^{r'}L_x^{\rho'}} \lesssim T^\alpha \|u_k\|_{L_t^rL_x^\rho}^{p-1} \|\partial_t u_k\|_{L_t^rL_x^\rho} \tag{4.29}$$

and the right-hand-side is finite due to $u_k \in C_0^\infty(I \times \mathbb{R}^n)$. Therefore by combining (4.28)-(4.29), we obtain

$$\|\partial_t u_{2,k}(t)\|_{L^2} \lesssim \|u_k\|_{L_t^\infty L_x^{2p}}^p + T^\alpha \|u_k\|_{L_t^rL_x^\rho}^{p-1} \|\partial_t u_k\|_{L_t^rL_x^\rho} < \infty$$

for any $k \in \mathbb{N}$, from which we can see $\partial_t u_{2,k} \in C\left(I; L^2\right)$. Thus by the equation (4.26) again, we also find $u_{2,k} \in C\left(I; H^2\right)$ for any $k \in \mathbb{N}$. Therefore we have the identity

$$(\Delta u_{2,k}, \psi)_{L_x^2} = (u_{2,k}, \Delta\psi)_{L_x^2}. \tag{4.30}$$

Thus by combining the identities (4.25), (4.26) and (4.30), we obtain

$$K_1 = \lim_{k\to\infty}\left(\int_{I\times\mathbb{R}^n} F(u_k)\psi dxdt - \int_{I\times\mathbb{R}^n} u_{2,k}\Delta\psi dxdt\right)$$

$$= \int_{I\times\mathbb{R}^n} F(u)\psi dxdt - K_2. \tag{4.31}$$

In fact, by the same way as in (4.24), we obtain

$$\left|\int_{I\times\mathbb{R}^n}(F(u_k) - F(u))\psi dxdt\right| \lesssim T^\alpha\left(\|u_k\|_{L_t^rL_x^\rho}^{p-1} + \|u\|_{L_t^rL_x^\rho}^{p-1}\right)\|u_k - u\|_{L_t^rL_x^\rho}\|\psi\|_{L_t^rL_x^\rho}$$

and

$$\left|\int_{I\times\mathbb{R}^n}(u_{2,k} - u_2)\Delta\psi dxdt\right| \lesssim T\|u_{2,k} - u_2\|_{L_t^\infty L_x^2}\|\Delta\psi\|_{L_t^\infty L_x^2}.$$

Therefore, combining (4.22) and (4.31), we obtain (4.7). This completes the proof. □

5. Lifespan of solutions for NLS without gauge invariance

5.1. **Introduction.** In this chapter, we continue to study the initial value problem for NLS:

$$i\partial_t u + \Delta u = \lambda |u|^p, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \tag{5.1}$$

with the initial condition

$$u(0,x) = \varepsilon f(x), \quad x \in \mathbb{R}^n, \tag{5.2}$$

where $T > 0$, $1 < p < 1 + 2/n$, u is a complex-valued unknown function of (t,x), $\lambda \in \mathbb{C}$, f is a given complex-valued function, $\varepsilon > 0$ is a small parameter.

In the previous chapter, a blow-up solution for (5.1)-(5.2) was constructed in the case of $1 < p \leq 1+2/n$ under a suitable initial data. But since a contradiction argument to construct a blow-up solution was used, the mechanism of the blow-up solution (e.g.estimate of the lifespan, blow-up speed etc.) was not understandable. In this chapter, by the modification of the method, we will prove an upper bound of the lifespan in the case $1 < p < 1 + 2/n$. This result was extended to the wider case $1 < p < 1 + 4/n$ in the recent paper [32]. (For more information of blow-up results of NLS, see e.g. [15], [51], [50], [54] and the references therein.)

5.2. **Known Results and Main Result.** Our concern in this chapter is the estimate of the lifespan. The lower bound of the lifespan follows from Proposition 4.1 immediately.

Corollary 5.1. *Under the same assumptions as in Proposition 4.1 and $\varepsilon > 0$. the estimate is valid*

$$T_\varepsilon \geq C\varepsilon^{1/\omega},$$

where $\omega = n/4 - 1/(p-1)$ and $C = C(n,p,\|f\|_{L^2})$ is a positive constant.

The next interest is an upper bound of the lifespan.

Remark 5.1. *In* [34], *in order to prove $T_\varepsilon < \infty$, a contradiction argument based on papers* [72], [73] *was used. Therefore, an upper bound of the lifespan was not obtained.*

Next, we state our main result in this chapter, which gives an upper bound of the lifespan. Then the following is valid;

Theorem 5.2. *Let $1 < p < 1 + 2/n$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in L^2$. If f satisfies (4.5), then there exist $\varepsilon_0 > 0$ and positive constant $C = C(p,\lambda)$ such that*

$$T_\varepsilon \leq C\varepsilon^{1/\kappa}$$

for any $\varepsilon \in (0,\varepsilon_0)$ where $\kappa \equiv n/2 - 1/(p-1)$.

Remark 5.2. *There is a gap between the lower bound (see Corollary 5.1) and the upper bound in L^2-framework, that is $\kappa > \omega$. Recently, this result was extended in* [32].

Finally, we mention the strategy of the proof of Theorem 5.2. We will use a test-function method based on papers [46], [61]. In [46], [61], upper bounds of lifespan for some parabolic equations were obtained. However, their arguement does not be applicable to the present NLS directly. Since solutions for NLS are complex-valued, the constant λ in front of the nonlinearity is a complex number and especially, the appropriate function spaces for NLS differs from that of those parabolic equations. To overcome these difficulties, we will consider the real part or imaginary part for the equation and reconsider the problem under the suitable function spaces L^2 to NLS, so that we can use the local existence theorem.

5.3. **Integral inequalities by appropriate test-functions.** In this subsection, we prepare some integral inequalities. Before doing so, we introduce the non-negative smooth function ϕ as follows, which was constructed in the papers [6], [9]:

$$\phi(x)=\phi(|x|),\ \phi(0)=1,\ 0<\phi(x)\le 1 \text{ for } |x|>0,$$

where $\phi(|x|)$ is decreasing of $|x|$ and $\phi(|x|)\to 0$ as $|x|\to\infty$ sufficiently fast. Moreover, there exists $\mu>0$ such that

$$|\Delta\phi|\le\mu\phi,\quad x\in\mathbb{R}^n, \tag{5.3}$$

and $\|\phi\|_{L^1}=1$. This can be done by letting $\phi(r)=e^{-r^\nu}$ for $r\gg 1$ with $\nu\in(0,1]$ and extending ϕ to $[0,\infty)$ by a smooth approximation. Let θ be suffuciently large and

$$\eta(t)=\eta_T(t)=\begin{cases}0, & \text{if } t>T,\\ (1-t/T)^\theta, & \text{if } 0\le t\le T,\end{cases}$$

where $0\le S<T$. Furthermore, set $\eta_R(t)=\eta(t/R^2)$, $\phi_R(x)=\phi(x/R)$ and $\psi_R(t,x)=\eta_R(t)\phi_R(x)$ for $R>0$.

First, we reduce the integral equation (4.4) into the weak form.

Lemma 5.3. *Let u be an L^2-solution of (5.1)-(5.2) on $[0,T_\varepsilon)$. Then u satisfies*

$$\begin{aligned}&\int_{[0,TR^2)\times\mathbb{R}^n}u(-i\partial_t(\psi_R)+\Delta(\psi_R))dxdt\\ &\quad= i\varepsilon\int_{\mathbb{R}^n}f(x)\psi_R(0,x)\,dx+\lambda\int_{[0,TR^2)\times\mathbb{R}^n}|u|^p\psi_R dxdt,\end{aligned} \tag{5.4}$$

for any $T,R>0$ with $TR^2<T_\varepsilon$.

This lemma can be proved in the same manner as the proof of Proposition 3.1 in [34] and Proposition 4.3.

Next, we will lead a integral inequality. Hereafter we only consider the case of $\lambda_1>0$ for simplicity. The other cases can be treated in the almost same way (see Remark 5.3).

We introduce some functions:

$$\begin{aligned}I_R(T)&=\int_{[0,TR^2)\times\mathbb{R}^n}|u|^p\psi_R dxdt,\\ J_R&=\varepsilon\int_{\mathbb{R}^n}-f_2(x)\phi(x/R)\,dx\end{aligned}$$

and

$$\begin{aligned}A(T)&=\left(\int_{[0,T)\times\mathbb{R}^n}|\partial_t\eta(t)|^q\eta(t)^{-1/(p-1)}\phi(x)\,dxdt\right)^{1/q},\\ B(T)&=\left(\int_{[0,T)\times\mathbb{R}^n}\eta_T(t)\phi(x)\,dxdt\right)^{1/q},\end{aligned}$$

where $q=p/(p-1)$. By the direct computation, we have

$$A(T)=\theta\{\theta-1/(p-1)\}^{-1/q}T^{-1/p},\ B(T)=\left(\frac{T}{\theta+1}\right)^{1/q}. \tag{5.5}$$

We have the following:

Lemma 5.4. *Let u be an L^2-solution of (5.1)-(5.2) on $[0,T_\varepsilon)$. Then the inequality holds*

$$\lambda_1 I_R(T)+J_R\le R^s\left\{I_R(T)^{1/p}A(T)+\mu I_R(T)^{1/p}B(T)\right\} \tag{5.6}$$

for any $0<T$ and $R>0$ with $TR^2<T_\varepsilon$, where $s=-2+(2+n)/q$.

Proof. Since u is an L^2-solution on $[0, T_\varepsilon)$ and $TR^2 < T_\varepsilon$, by Lemma 5.3, we have

$$\lambda \int_{[0,TR^2)\times\mathbb{R}^n} |u|^p \psi_R dxdt + i\varepsilon \int_{\mathbb{R}^n} f(x)\psi_R(0,x)\,dx$$

$$= \int_{[0,TR^2)\times\mathbb{R}^n} u(-i\partial_t(\psi_R) + \Delta(\psi_R))dxdt. \tag{5.7}$$

Note that $\lambda_1 > 0$, by taking real part as the above identity, we obtain

$$\begin{aligned} \lambda_1 I_R(T) + J_R &= \int_{[0,TR^2)\times\mathbb{R}^n} \mathrm{Re}\, u\left(-i\partial_t(\psi_R) + \Delta(\psi_R)\right)dxdt \\ &\leq \int_{[0,TR^2)\times\mathbb{R}^n} |u|\left\{|\partial_t(\psi_R)| + |\Delta(\psi_R)|\right\}dxdt \\ &\equiv K_R^1 + K_R^2 \end{aligned} \tag{5.8}$$

We note that $(\partial_t\eta)(t) = 0$ except on $(0, T)$. By using the identity

$$\partial_t\psi_R(t,x) = R^{-2}\phi_R(x)(\partial_t\eta)(t/R^2)$$

and the Hölder inequality, we can get

$$\begin{aligned} K_R^1 &= R^{-2}\int_{[0,TR^2)\times\mathbb{R}^n} |u|\,\eta_R^{1/p}\left|(\partial_t\eta)(t/R^2)\right|\eta_R^{-1/p}\phi_R dxdt \\ &\leq R^{-2} I_R(T)^{1/p}\left(\int_{[0,TR^2)\times\mathbb{R}^n} \left|(\partial_t\eta)(t/R^2)\right|^q \eta_R^{-1/(p-1)}\phi_R dxdt\right)^{1/q} \\ &= I_R(T)^{1/p} A(T) R^s, \end{aligned} \tag{5.9}$$

where we have used the changing variables with $t/R^2 = t'$ and $x/R = x'$ to obtain the last identity. Next, by the identity $\Delta(\phi(x/R)) = R^{-2}(\Delta\phi)(x/R)$, the Hölder inequality and the estimate (5.3), we have

$$\begin{aligned} K_R^2 &= R^{-2}\int_{[0,TR^2)\times\mathbb{R}^n} |u|\,\eta(t/R^2)\left|(\Delta\phi)(x/R)\right|dxdt \\ &\leq \mu R^{-2}\int_{[0,TR^2)\times\mathbb{R}^n} |u|\,\psi_R dxdt \\ &\leq \mu R^{-2} I_R(T)^{1/p}\left(\int_{[0,TR^2)\times\mathbb{R}^n} \psi_R dxdt\right)^{1/q} \\ &= \mu I_R(T)^{1/p} B(T) R^s, \end{aligned} \tag{5.10}$$

where we have used the changing variables again. By combining the estimates (5.8)-(5.10), we have the conclusion. □

Remark 5.3. *We remark the other cases different from $\lambda_1 > 0$. For example, when $\lambda_2 > 0$, by taking the imaginary part as (5.7), an estimate similar to (5.6) can be obtained.*

Next, we give the upper bound of J_R. Let $\sigma > 0$ and $0 < \omega < 1$. We introduce the function

$$\Psi(\sigma,\omega) \equiv \max_{x\geq 0}(\sigma x^\omega - x) = (1-\omega)\,\omega^{\frac{\omega}{1-\omega}}\sigma^{\frac{1}{1-\omega}}. \tag{5.11}$$

We denote

$$D(T) = A(T) + \mu B(T),$$

for simplicity. The following estimates are valid:

Lemma 5.5. *Let u be an L^2-solution of (5.1)-(5.2) on $[0, T_\varepsilon)$. Then the estimate*

$$J_R \leq \lambda_1 \Psi(D(T) R^s/\lambda_1, 1/p) \tag{5.12}$$

holds for any $T > 0, R > 0$ *with* $TR^2 < T_\varepsilon$, *where* $s = -2 + (2+n)/q$. *Moreover, if* $T_\varepsilon = \infty$, *that is* u *is a global solution, then the inequality is valid:*

$$\limsup_{R\to\infty} R^{-sq} J_R \le (\mu/\lambda_1)^{1/(p-1)}. \tag{5.13}$$

The proof of this lemma was based on that of Theorem 3.3 in [46] and Theorem 2.2 in [61].

Proof. Since u is an L^2-solution on $[0, T_\varepsilon)$, by using (5.6) with $S = 0$, we obtain

$$J_R \le R^s D(T) I_R(T)^{1/p} - \lambda_1 I_R(T) \le \lambda_1 \Psi\left(D(T) R^s/\lambda_1, 1/p\right),$$

which is exactly (5.12).

Next, we will prove (5.13) under the assumption $T_\varepsilon = \infty$. By (5.11) and (5.12), we have

$$\begin{aligned} J_R &\le \lambda_1 \Psi\left(D(T) R^s/\lambda_1, 1/p\right) \\ &= \lambda_1 (1-1/p)(1/p)^{\frac{1/p}{1-1/p}} \left\{D(T) R^s/\lambda_1\right\}^{\frac{1}{1-1/p}} \\ &= C_1 R^{sq} D(T)^q, \end{aligned} \tag{5.14}$$

for any $T > 0, R > 0$, where $C_1 = \lambda_1^{-1/(p-1)} (p-1)(1/p)^q$. This inequality implies

$$\limsup_{R\to\infty} R^{-sq} J_R \le C_1 \left\{\inf_{T>0} D(T)\right\}^q. \tag{5.15}$$

Next, we will estimate $D(T)$. Set

$$a_p = \frac{\theta}{\{\theta - 1/(p-1)\}^{1/q}}, \quad b_p = \frac{\mu}{(\theta+1)^{1/q}}.$$

Remembering the identities (5.5), we can rewrite $D(T)$ as

$$D(T) = a_p T^{-1/p} + b_p T^{1/q}. \tag{5.16}$$

Since

$$\begin{aligned} \min_{T>0} D(T) &= p(p-1)^{-1/q} a_p^{1/q} b_p^{1/p} \\ &= \frac{\mu^{1/p} p (p-1)^{-1/q} \theta^{1/q}}{\{\theta - 1/(p-1)\}^{1/q^2} (1+\theta)^{1/(pq)}}, \end{aligned} \tag{5.17}$$

we have

$$\lim_{\theta\to\infty} \min_{T>0} D(T) = \mu^{1/p} p (p-1)^{-1/q}. \tag{5.18}$$

Finally, by combining (5.15)-(5.18), we obtain (5.13), which completes the proof of the lemma. □

5.4. **Upper bound of lifespan.** In this subsection, we give a proof of Theorem 5.2, which implies an upper bound of the lifespan for the local L^2-solution. We also consider the case of $\lambda_1 > 0$ only. The other cases can be treated in the almost same manner. When $\lambda_1 > 0$, we may assume that f_2 satisfies

$$f_2 \in L^1, \int_{\mathbb{R}^n} f_2(x)\, dx < 0.$$

Proof. First, we note that by Corollary 5.1, there exists $\varepsilon_0 > 0$ such that $T_\varepsilon > T_0$ for any $\varepsilon \in (0, \varepsilon_0)$, where T_0 is defined later. Moreover, since $1 < p < 1 + 2/n$ and f satisfies (4.5), by Theorem 4.2, we also find $T_\varepsilon < \infty$.

Next, we consider the lower bound of J_R. By $f_2 \in L^1$ and Lebesgue's convergence theorem, there exists $R_0 > 0$ such that for any $R > R_0$, $J_R \ge C_0$. Set $T_0 = a_p b_p^{-1} R_0^2$. On the other hand, let $\tau \in (T_0, T_\varepsilon)$ and $R > R_0$. By using (5.12) with $T = \tau R^{-2}$, we have

$$\varepsilon \le C_0^{-1} C_1 \left\{R^s D\left(\tau R^{-2}\right)\right\}^q \equiv C_2 H(\tau, R), \tag{5.19}$$

where $C_2 = C_0^{-1}C_1$. By (5.16), we can rewrite H as

(5.20) $$H(\tau, R) = \left\{D\left(\tau R^{-2}\right) R^s\right\}^q = \left\{a_p\tau^{-1/p}R^{\alpha_1} + b_p\tau^{1/q}R^{-\alpha_2}\right\}^q,$$

where $\alpha_1 = n/q$, $\alpha_2 = 2 - n/q$. We solve the equation $a_p\tau^{-1/p}R^{\alpha_1} = b_p\tau^{1/q}R^{-\alpha_2}$ and we put

$$R_\tau = \left\{a_p^{-1}b_p\tau\right\}^{1/2}.$$

Note that $R_\tau > R_0$, by substituting R_τ into R of the inequality (5.20), we obtain

(5.21) $$\varepsilon \le C_2 H(\tau, R_\tau) = C_3\tau^\kappa$$

where $\kappa = n/2 - 1/(p-1)$ and $C_3 = C_3(\theta, p) > 0$ is constant dependent only on θ, p. From the assumption $n < 2/(p-1)$, we obtain $\kappa < 0$. Therefore, by (5.21), we can get

$$\tau \le C\varepsilon^{1/\kappa}$$

for any $\tau \in (T_0, T_\varepsilon)$, with some $C > 0$. Finally, we can get $T_\varepsilon \le C\varepsilon^{1/\kappa}$, which completes the proof of the theorem. □

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ACKNOWLEGEMENTS

My deepest appreciation goes to my supervisor, Professor Nakao Hayashi for his insightful advice, constant encouragement and enormous support. He has always taught me a lot of analysis methods and interesting problems. Especially, he taught me how to analyse the Dirac equation as well as how to use the vector field method though the joint work of Chapter 2. Not only that, without his offerings, I would not be able to complete any chapters of this thesis.

I am deeply grateful to Professor Hideaki Sunagawa for his constructive advice and a lot of comments about my papers. Without a joint work [33] with him, I would not be able to find the structure of the DKG system.

I also would like to thank Mr.Yuta Wakasugi for joint works [34, 35]. Without discussion with him, I would not be able to complete Chapter 4 and Chapter 5. I am also grateful to him for reading the first draft of paper [30] and teaching me some improvements.

My appreciation also goes to Mr.Takahisa Inui for reading my papers carefully. Without his pointing out, I would not be able to realize the improvement of Chapter 5. I also would like to thank him for a joint work [32].

I am grateful to Professor Pavel. I. Naumkin and Professor Akihiro Shimomura for a joint work of Chapter 2 and paper [33] respectively.

I am also thankful to Professor Tatsuo Nishitani for his fruitful advice, comments and enormous support. And I also would like to express deep gratitude to Professor Soichiro Katayama for his fruitful comments, advice and discussion. My deep appriciation goes to Professor Yoshio Tsutsumi for his fruitful comments and encouragement. Without his encouragement, I would not be able to complete Chapter 5. I am also deep gratitude to Professor Kenji Nakanishi for his constructive comments, encouragement and enormous support. Without his offering, I would not be able to realize the extension of Chapter 5.

I am also grateful to Professors Akitaka Matsumura, Shinichi Doi, Naohito Tomita and Hideyuki Miura for their many constructive comments, advice and enormous support.

I would like to express deep gratitude to Mr.Kiyotaka Suzaki for his enormous constructive advice and comments. Without his suggestion, I would not be able to complete Chapter 5.

My appreciation also goes to Professors Ratno Bagus Edy Wibowo, Chunhua Li and Mr.Seishiro Kobayashi and Mr.Tomoyuki Niizato for their useful comments, advice, suggestions and enormous support.

My thanks would go to other professors, secrateries and my all friends at Mathematical Department of Science in Osaka University and other supportes for my study.

Finally, my deep appriciation goes to my family for their enormous support and constant encouragement.

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