Title	Probabilistic Aspects of Besicovitch Almost Periodic Functions
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Citation	大阪大学, 2012, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/27473
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博士の専攻分野の名称	博　士（理学）
学　位　記　番　号	第　２５５７７　号
学位授与年月日	平成 24 年 6 月 20 日
学位授与の要件	学位規則第４条第１項該当 理学研究科数学専攻
学　位　論　文　名	Probabilistic Aspects of Besicovitch Almost Periodic Functions （ベシコビッチ概周期関数の確率的側面）
論　文　審　査　委　員	（主査） 教　授　杉田　　洋 （副査） 教　授　盛田　健彦　　准教授　森山　知則　　准教授　深澤　正彰

論　文　内　容　の　要　旨

The thesis is mainly taken from the author's five papers ([1, 2, 3, 4, 5]), and is divided into two parts corresponding to two classes of Besicovitch almost periodic functions.

Part I deals with Besicovitch almost periodic functions, that is, functions $f\colon \mathbb{R} \to \mathbb{C}$ which belong to the closure of trigonometric polynomials under some Besicovitch q-norm $(1 \leqslant q < \infty)$

$$\|f\|_q := \limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^q dt \right)^{1/q}.$$

It is well known that Besicovitch almost periodic functions possess mean values and limit distributions. Here the mean value of a function f is defined as $\mathbf{M}[f] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt$, provided that the limit exists, and the limit distribution is the probability distribution on $\mathbb{C}$ to which the sequence of probability measures $\nu_T\{\tau : f(\tau) \in A\}, A \in \mathcal{B}(\mathbb{C})$, converges as $T \to \infty$, ν_T being the uniform probability measure on $[-T, T]$.

For every Besicovitch almost periodic function f, the mean value $a(\lambda) = \mathbf{M}[f(t)e^{-i\lambda t}]$ exists for all $\lambda \in \mathbb{R}$, and those λ for which $a(\lambda) \neq 0$ are at most countable, called the Fourier exponents of f. The formal series

$$f(t) \sim \sum_{\lambda} a(\lambda) e^{i\lambda t}$$

is called the Fourier series of f.

In Chapter 1, we study Besicovitch functions with Fourier series of the forms

$$f(t) \sim \sum_{m=1}^{\infty} a_m e^{-i\lambda_m t},$$

where $\{\lambda_m\}$ is a strictly increasing sequence of non-negative numbers tending to infinity, called a Dirichlet sequence. We construct a suitable probability space where these functions f can be extended to random variables. For these functions, their Fourier series are shown to be convergent in norm with the usual order $(1 < q < \infty)$. This result is similar to the convergence in norm of classical Fourier series. Besides, a version of the Carleson-Hunt theorem is investigated.

Chapter 2 concerns with general Dirichlet series of the form

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad s = \sigma + it \in \mathbb{C},$$

where $a_m \in \mathbb{C}$, and $\{\lambda_m\}$ is a Dirichlet sequence. Suppose that the above series converges absolutely for $\sigma > \sigma_a$ and has the sum $f(s)$. Then $f(s)$ is an analytic function in the half-plane $D := \{s \in \mathbb{C} : \sigma > \sigma_a\}$.
Assume that the function $f(s)$ is meromorphically continuable to a wider half-plane $D_0 := \{s \in \mathbb{C} : \sigma > \sigma_0\}, \sigma_0 < \sigma_a$ and satisfies some mild conditions. Then for fixed $\sigma > \sigma_0$, $f(\sigma + it)$ is shown to be a Besicovitch almost periodic function with the Fourier series

$$f(\sigma + it) \sim \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} e^{-i\lambda_m t}.$$

Thus the limit distribution of $f(\sigma + it)$ is well identified by using the probability space developed in Chapter 1. Moreover, using this probability space, we can also identify limit distributions of general Dirichlet series in the space of analytic functions and in the space of meromorphic functions.

Part II deals with Besicovitch limit-periodic arithmetical functions, that is, functions $f\colon \mathbb{N} \to \mathbb{C}$ which belong to the closure of periodic functions under some Besicovitch q-norm $(1 \leqslant q < \infty)$

$$\|f\|_q := \limsup_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} |f(n)|^q \right)^{1/q}.$$

Besicovitch limit-periodic arithmetical functions also possess mean values and limit distributions. Here the mean value of a function f is defined as $\mathbf{M}[f] := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n)$, and the limit distribution is considered as follows; if the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp\left(i\, s \operatorname{Re} f(n) + i\, t \operatorname{Im} f(n)\right), \quad (s, t) \in \mathbb{R}^2,$$

exists and it coincides with the characteristic function of some probability distribution on $\mathbb{R}^2 \cong \mathbb{C}$, then we call it the limit distribution of f. Note that the space of periodic functions is spanned by $\{e_{a/r} : r = 1, 2, \ldots, 1 \leqslant a \leqslant r, \gcd(\mathrm{a}, \mathrm{r}) = 1\}$, where e_α stands for the function $e_\alpha \colon n \mapsto e^{2\pi i \alpha n}$. Thus, a limit-periodic function f has the following Fourier series

$$f(n) \sim \sum_{\substack{r \in \mathbb{N};\\ 1 \leqslant a \leqslant r; (a,r)=1}} c_{a/r} e_{a/r}(n),$$

where $c_{a/r} := \mathbf{M}[f(n)\overline{e_{a/r}(n)}]$ and (a, r) denotes the greatest common divisor $\gcd(a, r)$ of a and r.

Let $\widehat{\mathbb{Z}}$ be the ring of finite integral adeles with its normalized Haar measure λ. In Chapter 3, $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$ is shown to be a good probability space where limit-periodic functions can be considered as random variables. Dealing with the problem of convergence of Fourier series, for each $n \in \mathbb{N}$, we consider a finite Fourier expansion of a function $f \in D^q$ as

$$S_n(f) := \sum_{\substack{r | n;\\ 1 \leqslant a \leqslant r; (a,r)=1}} \langle f, e_{a/r} \rangle\, e_{a/r}.$$

Then our result is that, for $1 \leqslant q < \infty$, $\|S_n(f) - f\|_q \to 0$ as $n \to 0$ in $\widehat{\mathbb{Z}}$. This gives an approximation for limit-periodic functions by periodic functions. The natural extensions of additive and multiplicative functions are considered at the end of this chapter.

Chapter 4 deals with the distribution of k-th power free integers. Let $X^{(k)}(n)$ be the indicator function of the set of k-th power free integers. Then $X^{(k)}(n)$ is a multiplicative limit-periodic function. The mean value of $X^{(k)}(n)$ exists and it is equal to

$$\mathbf{M}[X^{(k)}] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X^{(k)}(n) = \frac{1}{\zeta(k)},$$

where ζ is the Riemann zeta function. Using a result in Chapter 3, we extend $X^{(k)}(n)$ to a random variable on the probability space $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$ in a natural way. Then investigating the rate of L^2-convergence of

$$S_N^{(k)}(x) := \frac{1}{N}\sum_{n=1}^{N} X^{(k)}(x+n), \quad x \in \widehat{\mathbb{Z}},$$

we obtain the estimate of the mean square convergence rate

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\left(N\left(S_N^{(k)}(m)-\frac{1}{\zeta(k)}\right)\right)^2 \sim const\cdot N^{1/k}.$$

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論 文 審 査 の 結 果 の 要 旨

本論文は，数論に現れる種々の重要な概周期数列および概周期関数について確率論的視点から研究したものである．個別の結果についてはすでに2編の論文が査読付き学術雑誌に出版済みであり，また3編が査読を経て出版が決定している．

数論に現れる数列の中には，加法的算術関数や乗法的算術関数のように，決定論的に定まる関数でありながら，平均値や極限分布と呼ばれるものが定まりあたかも確率変数と見なせるようなものがあることは古くから知らせていた．たとえば1930年代に Besicovitch が，今日，Besicovitch 概周期性と呼ばれる性質を定義し，平均値および極限分布について論じ，下って1960年代には Novoselov が整数環のコンパクト化である有限整アデール環とその上の加法に関する Haar 確率測度からなる確率空間を用いて同様の問題について新しいアプローチを示した．本論文では，この両者が本質的に同等であることを関数解析的手法を用いて示している．すなわち $1<q<\infty$ に対して q-Besicovitch 概周期数列全体の作る線形空間と有限整アデール環上の Haar 確率測度に基づく L^q-空間が等距離同型であることを証明して，Besicovitch 概周期数列を真正の確率変数として定式化することに成功した．興味深い応用として，k 乗の因子を含まない整数の分布の詳細について調べ，Riemann 予想を仮定した場合の結果が「平均的に」成り立つことを（Riemann 予想を用いずに）証明している．

Bohr の概周期関数理論は Riemann zeta 関数の虚軸に平行な直線上の振舞いを研究することを動機としている．これに関しても Besicovitch がある拡張された概周期性の定義（これもまた Besicovitch 概周期性と呼ばれる）を与え，zeta 関数が絶対収束しない領域での振舞いを研究するのに用いた．本論文では Besicovitch 概周期性を持つ複素関数を確率変数と見なすことが可能になるような確率空間の設定を与えることに成功した．それは同時に古典的 Fourier 級数理論を一般化することにもなっており，概周期関数に付随する Fourier 級数の L^q 収束や概収束について詳細に論じた．こうした理論の著しい応用として，広義の Dirichlet 級数の値分布論における非常に見通しの良い視点を与えたことがある．とくに有理型関数の空間における広義の Dirichlet 級数の値分布論は，この分野の大家である Laurincikas の証明にギャップが見つかり，ここ数年，暗礁に乗り上げていたが，本論文の手法によって救済されたことは特筆すべきである．

以上のように本論文は，古典を含む様々な数論の問題を確率論的に扱う枠組みを与え，その上で関数解析学の手法を駆使して幾多の新たな知見を得ている．よって，本論文は博士（理学）の学位論文として十分価値のあるものと認める．