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<th>Probabilistic Aspects of Besicovitch Almost Periodic Functions</th>
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<td><strong>Author(s)</strong></td>
<td>Trinh, Khanh Duy</td>
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The thesis is mainly taken from the author's five papers (1, 2, 3, 4, 5), and is divided into two parts corresponding to two classes of Besicovitch almost periodic functions.

Part I deals with Besicovitch almost periodic functions, that is, functions \( f: \mathbb{R} \to \mathbb{C} \) which belong to the closure of trigonometric polynomials under some Besicovitch \( q \)-norm \((1 < q < \infty)\)

\[ \|f\|_q := \limsup_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |f(t)|^q dt \right)^{1/q}. \]

It is well known that Besicovitch almost periodic functions posses mean values and limit distributions. Here the mean value of a function \( f \) is defined as \( M[f] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) \), provided that the limit exists, and the limit distribution is the probability distribution on \( \mathbb{C} \) to which the sequence of probability measures \( \nu_T : \{ f : f(t) \in A \}, A \subset \mathbb{C} \), converges as \( T \to \infty, \nu_T \) being the uniform probability measure on \([-T,T] \).

For every Besicovitch almost periodic function \( f \), the mean value \( \bar{a}(\lambda) = M[f(e^{-\lambda x})] \) exists for all \( \lambda \in \mathbb{R} \), and those \( \lambda \) for which \( \bar{a}(\lambda) \neq 0 \) are at most countable, called the Fourier exponents of \( f \). The formal series

\[ f(t) \sim \sum_{\lambda} \bar{a}(\lambda)e^{\lambda t} \]

is called the Fourier series of \( f \).

In Chapter 1, we study Besicovitch functions with Fourier series of the form

\[ f(t) \sim \sum_{n=1}^{\infty} a_n e^{\alpha n t} \]

where \( \{a_n \} \) is a strictly increasing sequence of non-negative numbers tending to infinity, called a Dirichlet sequence. We construct a suitable probability space where these functions can extend to random variables. For these functions, their Fourier series are shown to be convergent in norm with the usual order \((1 < q < \infty)\). This result is similar to the convergence in norm of classical Fourier series. Besides, a version of the Carleson-Hunt theorem is investigated.

Chapter 2 concerns with general Dirichlet series of the form

\[ \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}, \quad s = \sigma + it \in \mathbb{C}, \]

where \( a_n \in \mathbb{C} \), and \( \{\lambda_n\} \) is a Dirichlet sequence. Suppose that the above series converges absolutely for \( \sigma > \sigma_0 \) and has the sum \( f(t) \). Then \( f(t) \) is an analytic function in the half-plane \( D := \{ s \in \mathbb{C} : \sigma > \sigma_0 \} \).

Assume that the function \( f(t) \) is meromorphically extendable to a wider half-plane \( D_0 := \{ s \in \mathbb{C} : \sigma > \sigma_0 \}, \sigma_0 < \sigma_0 \) and satisfies some mild conditions. Then for \( \sigma > \sigma_0 \), \( f(t) \) is shown to be a Besicovitch almost periodic function with the Fourier series

\[ f(t) \sim \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \]

Thus the limit distribution of \( f(t) \) is well identified by using the probability space developed in Chapter 1. Moreover, using this probability space, we can also identify limit distributions of general Dirichlet series in the space of analytic functions and in the space of meromorphic functions.

Part II deals with Besicovitch limit-periodic arithmetical functions, that is, functions \( f: \mathbb{N} \to \mathbb{C} \) which belong to the closure of periodic functions under some Besicovitch \( q \)-norm \((1 < q < \infty)\)

\[ \|f\|_q := \limsup_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} |f(n)|^q \right)^{1/q}. \]

Besicovitch limit-periodic arithmetical functions also possess mean values and limit distributions. Here the mean value of a function \( f \) is defined as \( M[f] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) \), and the limit distribution is considered as follows; if the limit

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp \{ i s \Re f(n) + it \Im f(n) \}, \quad (s, t) \in \mathbb{R}^2, \]

exists and it coincides with the characteristic function of some probability distribution on \( \mathbb{R}^2 \supset \mathbb{C} \), then we call it the limit distribution of \( f \). Note that the space of periodic functions is spanned by \( (e_{r/q} : r = 1, 2, \ldots, 1 \leq a \leq r, \gcd(a, r) = 1) \), where \( e_{r/q} \) stands for the function \( e_{r/q} : n \mapsto e^{r \pi i q n} \). Then, a limit-periodic function \( f \) has the following Fourier series

\[ f(t) \sim \sum_{1 < r \mid \lambda} c_{r/q} e^{r/t_0}, \]

where \( c_{r/q} := M[f(n) e_{r/q}(n)] \) and \( (a, r) \) denotes the greatest common divisor \( \gcd(a, r) \) of \( a \) and \( r \).

Let \( \mathbb{Z} \) be the ring of finite integral ideals with its normalized Haar measure \( \lambda \). In Chapter 3, \( \mathbb{Z}, \mathbb{R}(\mathbb{Z}), \lambda \) is shown to be a good probability space where limit-periodic functions can be considered as random variables. Dealing with the problem of convergence of Fourier series, for each \( n \in \mathbb{N} \), we consider a finite Fourier expansion of a function \( f \in D^2 \) as

\[ S_n(f) := \sum_{1 < r \mid \lambda \cap C(a, r)} (f, e_{r/q}) e_{r/q}. \]

Then our result is that, for \( 1 < q < \infty \), \( \|S_n(f) - f\|_q \to 0 \) as \( n \to \infty \) in \( \mathbb{Z} \). This gives an approximation for limit-periodic functions by periodic functions. The natural extensions of additive and multiplicative functions are considered at the end of this chapter.

Chapter 4 deals with the distribution of \( k \)-th power free integers. Let \( X^{(k)}(n) \) be the indicator function of the set of \( k \)-th power free integers. Then \( X^{(k)}(n) \) is a multiplicative limit-periodic function. The mean value of \( X^{(k)}(n) \) exists and it is equal to

\[ M[X^{(k)}] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X^{(k)}(n) = \frac{1}{(k)!}, \]

where \( \zeta \) is the Riemann zeta function. Using a result in Chapter 3, we extend \( X^{(k)}(n) \) to a random variable on the probability space \( (\mathbb{Z}, \mathbb{R}(\mathbb{Z}), \lambda) \) in a natural way. Then investigating the rate of \( L^2 \)-convergence of
we obtain the estimate of the mean square convergence rate

\[
\lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} \left( N \left( S_N^{(b)}(m) - \frac{1}{\zeta(b)} \right) \right)^2 \sim \text{const} \cdot N^{1/b}.
\]

References