Probabilistic Aspects of Besicovitch Almost Periodic Functions

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（ベシコビッチ概周期関数の確率的側面）

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Preface

The theory of almost periodic functions, established by H. Bohr, may be considered from two different points of view. On the one hand, an almost periodic function is a continuous function possessing a certain structural property which is a generalization of pure periodic functions, and on the other it is the limit of trigonometric polynomials under the uniform norm.

Corresponding to the two different points of view, further development of the theory of almost periodic functions to generalize this theory went in two different directions. The first direction was developed by W. Stepanov and H. Weyl, which led to two important classes of almost periodic functions: Stepanov almost periodic functions and Weyl almost periodic functions.

The second direction of generalizations was that followed by A.S. Besicovitch. Besicovitch enlarged the class of almost periodic functions by considering the convergence of sequences with respect to some Besicovitch q-(semi)norm ($1 \leqslant q < \infty$) rather than uniform convergence. Here the Besicovitch q-norm of a function $f\colon \mathbb{R} \to \mathbb{C}$ is defined as

$$\|f\|_q := \limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^q dt \right)^{1/q}.$$

Then a function is called Besicovitch almost periodic function if it is a limit of trigonometric polynomials under some Besicovitch q-norm. Besicovitch almost periodic functions look like random variables because they possess mean values and limit distributions. On the other hand, like periodic functions, they have "Fourier series" in the form of a general trigonometric series

$$f(t) \sim \sum a_m e^{i\lambda_m t}.$$

For arithmetical functions (functions defined on $\mathbb{N}$), a function $f\colon \mathbb{N} \to \mathbb{C}$ is called Besicovitch almost periodic arithmetical function if it belongs to the linear closure of $\{ e_\alpha : \alpha \in \mathbb{R}/\mathbb{Z} \}$ under some Besicovitch q-(semi)norm ($1 \leqslant q < \infty$)

$$\|f\|_q := \limsup_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} |f(n)|^q \right)^{1/q},$$

where e_α stands for the function $e_\alpha\colon n \mapsto e^{2\pi i\alpha n}$. Besicovitch almost periodic arithmetical functions also have mean values, limit distributions and "Fourier series".

This research is to study particular classes of Besicovitch functions: Besicovitch almost periodic functions with Fourier exponents belonging to a Dirichlet sequence and Besicovitch limit-periodic arithmetical functions. We will present recent results on the following problems.

(1) Constructing probability spaces where Besicovitch functions in these classes can be considered as random variables.

(2) Convergence of Fourier series.

(3) Some applications: value distributions of general Dirichlet series; natural extensions of additive/multiplicative arithmetical functions, and the distribution of k-th power free integers.

This thesis is mainly taken from the author's five papers ([10, 11, 12, 13, 14]), and is divided into two parts corresponding to two classes of Besicovitch functions.

Part I deals with Besicovitch almost periodic functions, that is, functions $f\colon \mathbb{R} \to \mathbb{C}$ which belong to the closure of trigonometric polynomials under some Besicovitch q-norm $(1 \leqslant q < \infty)$. In this case, the mean value of a function f is defined as

$$\mathbf{M}[f] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt,$$

provided that the limit exists, and the limit distribution is the probability distribution on $\mathbb{C}$ to which the sequence of probability measures

$$\nu_T\{\tau : f(\tau) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges as $T \to \infty$, where ν_T denotes the uniform probability measure on $[-T, T]$.

For every Besicovitch almost periodic function f, the mean value

$$a(\lambda) = \mathbf{M}[f(t)e^{-i\lambda t}] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t}dt$$

exists for all $\lambda \in \mathbb{R}$, and those λ for which $a(\lambda)$ is non-zero are at most countable, called the Fourier exponents of f. The formal series

$$f(t) \sim \sum_{\lambda} a(\lambda)e^{i\lambda t}$$

is called the Fourier series of f.

In Chapter 1, we study Besicovitch almost periodic functions with Fourier series of the forms

$$f(t) \sim \sum_{m=1}^{\infty} a_m e^{-i\lambda_m t},$$

where $\{\lambda_m\}$ is a strictly increasing sequence of non-negative numbers tending to infinity, called a Dirichlet sequence. We will construct a suitable probability space where these functions f can be extended to random variables. For these functions, their Fourier series are shown to be convergent in norm with the usual order $(1 < q < \infty)$. This result is similar to the convergence in norm of classical Fourier series. Besides, a version of the Carleson-Hunt theorem is investigated.

Chapter 2 concerns with general Dirichlet series of the form

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad s = \sigma + it \in \mathbb{C},$$

where $a_m \in \mathbb{C}$, and $\{\lambda_m\}$ is a Dirichlet sequence. Suppose that the above series converges absolutely for $\sigma > \sigma_a$ and has the sum $f(s)$. Then $f(s)$ is an analytic function in the half-plane $D := \{s \in \mathbb{C} : \sigma > \sigma_a\}$.

Assume that the function $f(s)$ is meromorphically continuable to a wider half-plane $D_0 := \{s \in \mathbb{C} : \sigma > \sigma_0\}, \sigma_0 < \sigma_a$ and satisfies some mild conditions. Then for fixed $\sigma > \sigma_0$, $f(\sigma + it)$ is a Besicovitch almost periodic function with the Fourier series

$$f(\sigma + it) \sim \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} e^{-i\lambda_m t}.$$

Thus the limit distribution of $f(\sigma + it)$ is well identified by using the probability space developed in Chapter 1. Moreover, using this probability space, we can also identify limit distributions of general Dirichlet series in the space of analytic functions and in the space of meromorphic functions.

Part II deals with Besicovitch limit-periodic arithmetical functions, that is, functions $f \colon \mathbb{N} \to \mathbb{C}$ which belong to the closure of periodic functions under some Besicovitch q-norm. The mean value of a function f is defined as

$$\mathbf{M}[f] := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n),$$

and the limit distribution is considered as follows; if the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp\big(i\, s \operatorname{Re} f(n) + i\, t \operatorname{Im} f(n)\big), \quad (s,t) \in \mathbb{R}^2,$$

exists and it coincides with the characteristic function of some probability distribution on $\mathbb{R}^2 \cong \mathbb{C}$, then we call it the limit distribution of f. Note that the space of periodic arithmetical functions is spanned by $\{e_{a/r} : r = 1, 2, \ldots, 1 \leqslant a \leqslant r, \gcd(\mathrm{a}, \mathrm{r}) = 1\}$. Thus, a limit-periodic arithmetical function f is a Besicovitch almost periodic arithmetical function with the following Fourier series

$$f(n) \sim \sum_{\substack{r \in \mathbb{N};\\ 1 \leqslant a \leqslant r; (a,r)=1}} c_{a/r} e_{a/r}(n),$$

where $c_{a/r} := \mathbf{M}[f(n)\overline{e_{a/r}(n)}]$ and (a, r) denotes the greatest common divisor $\gcd(a, r)$ of a and r.

Let $\mathfrak{D}^q$ denote the space of q-limit-periodic arithmetical functions and let D^q be the quotient space of $\mathfrak{D}^q$ with respect to the null-space $\mathcal{N}(\mathfrak{D}^q) := \{\, f \in \mathfrak{D}^q : \|f\|_q = 0 \,\}$. Let $\widehat{\mathbb{Z}}$ be the ring of finite integral adeles with its normalized Haar measure λ.

In Chapter 3, $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$ is shown to be a good probability space where limit-periodic arithmetical functions can be considered as random variables. In fact, every function in $\mathfrak{D}^q$ can be extended to a random variable in $L^q(\widehat{\mathbb{Z}}, \lambda)$. The limit distribution of the original function coincides with the distribution of the extended random variable. In addition, the space D^q is isometrically isomorphic to $L^q(\widehat{\mathbb{Z}}, \lambda)$.

Dealing with the problem of convergence of Fourier series, for each $n \in \mathbb{N}$, we define a finite Fourier expansion of a function $f \in D^q$ as

$$S_n(f) := \sum_{\substack{r \mid n;\\ 1 \leqslant a \leqslant r; (a,r)=1}} \langle f, e_{a/r} \rangle\, e_{a/r}.$$

Then our result is that, for $1 \leqslant q < \infty$,

$$\|S_n(f) - f\|_q \to 0 \quad \text{as} \quad n \to 0 \text{ in } \widehat{\mathbb{Z}}, \quad f \in D^q.$$

This gives an approximation for limit-periodic arithmetical functions by periodic arithmetical functions. The natural extensions of additive and multiplicative arithmetical functions will be considered at the end of this chapter.

Chapter 4 deals with the distribution of k-th power free integers. Let $X^{(k)}(n)$ be the indicator function of the set of k-th power free integers, that is,

$$X^{(k)}(n) := \begin{cases} 1, & (\forall p : \text{prime}, \ p^k \nmid n), \\ 0, & (\exists p : \text{prime}, \ p^k \mid n). \end{cases}$$

Then $X^{(k)}(n)$ is a multiplicative limit-periodic arithmetical function. The mean value of $X^{(k)}(n)$ exists and it is equal to

$$\mathbf{M}[X^{(k)}] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X^{(k)}(n) = \frac{1}{\zeta(k)},$$

where ζ is the Riemann zeta function. Using a result in Chapter 3, we extend $X^{(k)}(n)$ to a random variable on the probability space $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$ in a natural way. Then investigating the rate of L^2-convergence of

$$S_N^{(k)}(x) := \frac{1}{N} \sum_{n=1}^{N} X^{(k)}(x+n), \quad x \in \widehat{\mathbb{Z}},$$

we obtain the following result

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \left(N \left(S_N^{(k)}(m) - \frac{1}{\zeta(k)} \right) \right)^2 \sim \mathit{const} \cdot N^{1/k}.$$

Note that a conjecture that

$$\forall \varepsilon > 0, \quad N \left(S_N^{(k)}(m) - \frac{1}{\zeta(k)} \right) = O\left(N^{1/2k+\varepsilon} \right), \quad N \to \infty,$$

has not been proved yet. Our result may be called as a mean square version of this conjecture.

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Contents

Notations and symbols

$A := B$		A is defined by B ($B =: A$ as well).
$\Rightarrow$		weak convergence of probability measures.
$\xrightarrow{d}$		convergence in distribution of random elements.
$\xrightarrow{L^q}$		convergence in L^q.
$a\|b$		b is divisible by a.
L.H.S.		left-hand side.
R.H.S.		right-hand side.
i.i.d.		independent and identically distributed.
w.r.t.		with respect to.
a.e.		almost everywhere.
a.s.		almost surely.
$\mathbb{N}$	$:=$	$\{1, 2, \dots\}$, the set of all natural numbers.
$\mathbb{Z}$	$:=$	$\{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of all integers.
$\mathbb{Q}$	$:=$	the set of all rational numbers.
$\mathbb{R}$	$:=$	the set of all real numbers.
$\mathbb{C}$	$:=$	the set of all complex numbers.
$\gcd(\mathrm{a}, \mathrm{b})$	$:=$	the greatest common divisor of a and b.
$\mathrm{lcm}(\mathrm{a}, \mathrm{b})$	$:=$	the least common multiple of a and b.
$\sqrt{-1}$ or i	$:=$	the imaginary unit.
$\mathcal{B}(S)$	$:=$	the Borel (σ-)field of a topological space S.
$\mathbf{M}[f]$	$:=$	$\displaystyle\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt$, or $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} f(t)dt$, or $\displaystyle\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n)$, the mean value of function f.
$\mathbf{E}[X]$	$:=$	the mean (expectation) of random variable X, (the probability space is clear in the context).
$\mathbf{E}^{(\mathbf{P})}[X]$	$:=$	the mean (expectation) of random variable X with respect to the probability measure $\mathbf{P}$.
$\\|f\\|_{L^q(X,\mathcal{B},m)}$	$:=$	$\displaystyle\left(\int_X \|f(x)\|^q dm(x)\right)^{1/q}$ ($\\|f\\|_{L^q}$ or $\\|f\\|_{L^q(X,m)}$ as well), where $(X, \mathcal{B}, m)$ is a measure space.

Chapter 0

Preliminaries

0.1 Convergence of probability measures

0.1.1 Weak convergence in metric spaces

Let (S, ρ) be a metric space and let $\mathcal{B}(S)$ denote the *Borel σ-field* of S. Let $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and $\mathbf{P}$ be probability measures on $(S, \mathcal{B}(S))$.

Definition 0.1. We say that $\{\mathbf{P}_n\}$ converges weakly to $\mathbf{P}$ as $n \to \infty$, and write $\mathbf{P}_n \Rightarrow \mathbf{P}$, if for all bounded continuous functions $f\colon S \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_S f d\mathbf{P}_n = \int_S f d\mathbf{P}.$$

Since two probability measures $\mathbf{P}$ and $\mathbf{Q}$ coincide if $\int_S f d\mathbf{P} = \int_S f d\mathbf{Q}$ for all bounded, uniformly continuous real functions f, it follows that the sequence $\{\mathbf{P}_n\}$ cannot converge weakly to two different limits.

The Portmanteau theorem provides useful conditions equivalent to weak convergence.

Theorem 0.2. *The following five conditions are equivalent:*

(i) $\mathbf{P}_n \Rightarrow \mathbf{P}$;

(ii) $\int_S f d\mathbf{P}_n \to \int_S f d\mathbf{P}$ *for all bounded, uniformly continuous* f;

(iii) $\limsup_n \mathbf{P}_n(F) \leqslant \mathbf{P}(F)$ *for all closed* F;

(iv) $\liminf_n P_n(G) \geqslant \mathbf{P}(G)$ *for all open* G;

(v) $\mathbf{P}_n(A) \to \mathbf{P}(A)$ *for all* $\mathbf{P}$*-continuity sets* A. *Here a* $\mathbf{P}$*-continuity set is a set* A *whose boundary* ∂A *satisfies* $\mathbf{P}(\partial A) = 0$.

This is Theorem 2.1 from Billingsley [4].

Let (S', ρ') be another metric space and let $\mathcal{B}(S')$ be the Borel σ-field of S'. We consider a measurable (or $\mathcal{B}(S)/\mathcal{B}(S')$-measurable) mapping $h\colon S \to S'$, that is, a mapping h satisfies

$$h^{-1}(\mathcal{B}(S')) \subset \mathcal{B}(S).$$

Then each probability measure $\mathbf{P}$ on $(S, \mathcal{B}(S))$ induces on $(S', \mathcal{B}(S'))$ a probability measure $\mathbf{P}h^{-1}$ defined by $\mathbf{P}h^{-1}(A) = \mathbf{P}(h^{-1}A), A \in \mathcal{B}(S')$. Let D_h denote the set of discontinuities of h. Then $D_h \in \mathcal{B}(S)$ and we have the *mapping theorem.*

Theorem 0.3. *If* $\mathbf{P}_n \Rightarrow \mathbf{P}$ *and* $\mathbf{P}(D_h) = 0$, *then* $\mathbf{P}_n h^{-1} \Rightarrow \mathbf{P}h^{-1}$. *In particular, if* $h\colon S \to S'$ *is continuous and* $\mathbf{P}_n \Rightarrow \mathbf{P}$, *then* $\mathbf{P}_n h^{-1} \Rightarrow \mathbf{P}h^{-1}$.

This is Theorem 2.7 from Billingsley [4].

0.1.2 Convergence of random elements

A mapping X from a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ to a metric space (S, ρ) is said to be a random element if it is $\mathcal{F}/\mathcal{B}(S)$-measurable, that is,

$$X^{-1}(\mathcal{B}(S)) \subset \mathcal{F}.$$

An $\mathbb{R}$-valued or $\mathbb{C}$-valued random element is usually called a random variable. The *distribution* of X is the probability measure $\mathbf{P}_X = \mathbf{P}X^{-1}$ on $(S, \mathcal{B}(S))$ defined by

$$\mathbf{P}_X(A) = \mathbf{P}(X^{-1}A) = \mathbf{P}(\omega : X(\omega) \in A) = \mathbf{P}(X \in A).$$

Let $\{X_n\}_{n\in\mathbb{N}}$ and X be random elements.

Definition 0.4. We say that the sequence $\{X_n\}$ converges in distribution to X, and write $X_n \xrightarrow{d} X$, if the sequence of distributions $\{\mathbf{P}_{X_n}\}$ converges weakly to $\mathbf{P}_X$ as $n \to \infty$.

If X and Y are S-valued random elements defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then it makes sense to speak of the distance $\rho(X, Y)$. In the sequel, let (S, ρ) be a separable metric space. Then $\rho(X, Y)$ is a random variable.

Definition 0.5. We say that the sequence $\{X_n\}$ converges in probability to X if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}(\rho(X_n, X) \geqslant \varepsilon) = 0.$$

Theorem 0.6. *The convergence in probability implies the convergence in distribution.*

This is a consequence of Theorem 3.1 from Billingsley [4].

Theorem 0.7. *Let $\{Y_n\}_{n\in\mathbb{N}}$, $\{X_k\}_{k\in\mathbb{N}}$ and $\{X_{k,n}\}_{k,n\in\mathbb{N}}$ be S-valued random elements. Assume that*

(i)
$$X_{k,n} \xrightarrow{d} X_k \quad as \quad n \to \infty;$$

(ii)
$$X_k \xrightarrow{d} X \quad as \quad k \to \infty;$$

(iii) *for every $\varepsilon > 0$,*
$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbf{P}(\rho(X_{k,n}, Y_n) \geqslant \varepsilon) = 0.$$

Then $Y_n \xrightarrow{d} X$ as $n \to \infty$.

This is Theorem 3.2 from Billingsley [4].

0.1.3 Weak convergence in $\mathbb{R}^d$ and characteristic functions

The characteristic function $\varphi(\underline{\tau})$ of a probability measure $\mathbf{P}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is defined by

$$\varphi(\underline{\tau}) = \int_{\mathbb{R}^d} e^{i(\underline{\tau}, \underline{x})} d\mathbf{P}(\underline{x}),$$

where $(\underline{\tau}, \underline{x})$ denotes the inner product of $\underline{\tau}$ and $\underline{x}$ in $\mathbb{R}^d$. Note that the characteristic function φ uniquely determines the probability measure $\mathbf{P}$.

Theorem 0.8. *Let $\{\mathbf{P}_n\}$ and $\mathbf{P}$ be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let $\varphi_n(\underline{\tau})$ and $\varphi(\underline{\tau})$ be the corresponding characteristic functions. Then $\mathbf{P}_n \Rightarrow \mathbf{P}$, if and only if*

$$\varphi_n(\underline{\tau}) \to \varphi(\underline{\tau}) \text{ for all } \underline{\tau} \in \mathbb{R}^d.$$

The following theorem is Lévy's famous continuity theorem.

Theorem 0.9. *Let $\{\mathbf{P}_n\}$ be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let $\varphi_n(\underline{\tau})$ be the corresponding characteristic functions. Assume that*

$$\varphi_n(\underline{\tau}) \to \varphi(\underline{\tau}) \text{ for all } \underline{\tau} \in \mathbb{R}^d,$$

and that $\varphi(\underline{\tau})$ is continuous at the point $\underline{\tau} = (0, \dots, 0)$. Then there is a probability measure $\mathbf{P}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mathbf{P}_n \Rightarrow \mathbf{P}$, and $\varphi(\underline{\tau})$ is the characteristic function of $\mathbf{P}$.

Proofs of these theorems can be found in Section 29 of Billingsley [4].

0.1.4 Weak convergence in topological groups and Fourier transforms

A measure $\mathbf{P}$ on $(S, \mathcal{B}(S))$, S being a compact topological space, is said to be regular if for every $\varepsilon > 0$ and every $E \in \mathcal{B}(S)$, there is a compact set M and an open set U with $M \subset E \subset U$ and $\mathbf{P}(U \setminus M) < \varepsilon$. It is known that if S is metrizable, then any probability measure on $(S, \mathcal{B}(S))$ is regular. Let G be a compact topological group. A measure $\mathbf{P}$ on $(G, \mathcal{B}(G))$ is said to be invariant if

$$\mathbf{P}(A) = \mathbf{P}(xA) = \mathbf{P}(Ax)$$

for all $A \in \mathcal{B}(G)$ and all $x \in G$, where xA and Ax denote the sets $\{xy : y \in A\}$ and $\{yx : y \in A\}$, respectively.

Theorem 0.10. *Let G be a compact topological group. Then there is a unique invariant regular probability measure m_H on $(G, \mathcal{B}(G))$, called the normalized Haar measure.*

Now let G be a locally compact abelian group. Let $\widehat{G}$ denote the collection of all continuous homomorphisms of G into the unit circle $\gamma = \{z \in \mathbb{C} : |z| = 1\}$. The members of $\widehat{G}$ are called the characters of G. Under the operation of pointwise multiplication of functions, $\widehat{G}$ is an abelian group. With the compact open topology, $\widehat{G}$ becomes a locally compact abelian group.

We have the following results (see also Walters [46, Section 0.7] and its references).

(i) G is compact, if and only if $\widehat{G}$ is discrete.

(ii) (Duality theorem). $\widehat{(\widehat{G})}$ is naturally isomorphic (as a topological group) to G, the isomorphism being given by the mapping $G \ni g \mapsto X_g$, where $X_g(\chi) = \chi(g)$ for all $\chi \in \widehat{G}$.

(iii) If G_1, G_2 are locally compact abelian groups, then $\widehat{G_1 \times G_s} = \widehat{G}_1 \times \widehat{G}_2$. (Here "$\times$" denotes the direct product.) Hence all characters of $G_1 \times G_2$ are of the form $(g, h) \mapsto \chi(g)\delta(h)$, where $\chi \in \widehat{G}_1, \delta \in \widehat{G}_2$.

(iv) If Γ is a subgroup of $\widehat{G}$, then

$$H = \{g \in G : \chi(g) = 1, \forall \chi \in \Gamma\}$$

is a closed subgroup of G and $\widehat{(G/H)} = \Gamma$.

(v) If H is a closed subgroup of G and $H \neq G$, then there exists a character $\chi \in \widehat{G}, \chi \neq 1$ such that $\chi(h) = 1$ for all $h \in H$.

(vi) Let G be compact. Then finite linear combinations of characters are dense in $C(G)$, the space of complex-valued continuous functions on G. The members of $\widehat{G}$ form an orthonormal basis for $L^2(G, m_H)$.

We get back to the case G being compact abelian group. The Fourier transform of a probability measure $\mathbf{P}$ on $(G, \mathcal{B}(G))$ is a function defined on the characters group $\widehat{G}$,

$$g(\chi) = \int_G \chi(g) d\mathbf{P}(g), \quad \chi \in \widehat{G}.$$

By the property (vi) above, it is clear that $g(\chi)$ uniquely determines the probability measure $\mathbf{P}$. Let χ_0 denote the trivial character of G, the character that is identically equal to 1. Then the Fourier transform of the normalized Haar measure m_H is as follows

$$g(\chi) = \int_G \chi(g) dm_H(g) = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

We have the following continuity theorem.

Theorem 0.11. *Let G be a compact abelian group. Let $\{\mathbf{P}_n\}$ be probability measures on $(G, \mathcal{B}(G))$ and $g_n(\chi)$ be the corresponding Fourier transforms. Assume that*

$$g_n(\chi) \to g(\chi) \text{ for all } \chi \in \widehat{G}.$$

Then there is a probability measure $\mathbf{P}$ on $(G, \mathcal{B}(G))$ such that $\mathbf{P}_n \Rightarrow \mathbf{P}$, and $g(\chi)$ is the Fourier transform of $\mathbf{P}$.

This is a special case of Theorem 1.4.2 from Heyer [23].

0.2 Ergodic theory

0.2.1 Discrete time

This section is taken from Chapter 1 of Walters [46].

Let $(X, \mathcal{B}, m)$ be a probability space.

Definition 0.12. (i) A transformation $T \colon X \to X$ is said to be *measurable* if $T^{-1}(\mathcal{B}) \subset \mathcal{B}$.

(ii) A transformation $T \colon X \to X$ is said to be *measure-preserving* if T is measurable and $m(T^{-1}B) = m(B)$ for all $B \in \mathcal{B}$.

Let $T \colon X \to X$ be a measure-preserving transformation. If $T^{-1}B = B$ for $B \in \mathcal{B}$, then also $T^{-1}(X \backslash B) = X \backslash B$ and we could study T by studying the two simpler transformations $T|_B$ and $T|_{X\backslash B}$. If $0 < m(B) < 1$, this has simplified the study of T. If $m(B) = 0$ (or $m(B) = 1$), we can ignore B (or $X \setminus B$) and we have not significantly simplified T since neglecting a set of zero measure is allowed in measure theory. This raises the idea of studying those transformations that cannot be decomposed as above and of trying to express every measure-preserving transformation in terms of these indecomposable ones. The indecomposable transformations are called ergodic.

Definition 0.13. Let $(X, \mathcal{B}, m)$ be a probability space. A measure-preserving transformation T of $(X, \mathcal{B}, m)$ is called *ergodic* if the only members B of $\mathcal{B}$ with $T^{-1}B = B$ satisfy $m(B) = 0$ or $m(B) = 1$.

There are several other ways of stating the ergodicity condition and we present some of them in the next two theorems.

Theorem 0.14. *If $T\colon X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$, then the following statements are equivalent:*

(i) *T is ergodic;*

(ii) *the only members B of $\mathcal{B}$ with $m(T^{-1}B \triangle B) = 0$ are those with $m(B) = 0$ or $m(B) = 1$, where $T^{-1}B \triangle B := (T^{-1}B \setminus B) \cup (B \setminus T^{-1}B)$;*

(iii) *for every $A \in \mathcal{B}$ with $m(A) > 0$, we have $m(\cup_{m=1}^{\infty} T^{-n}A) = 1$;*

(iv) *for every $A, B \in \mathcal{B}$ with $m(A) > 0, m(B) > 0$, there exists an $n > 0$ with $m(T^{-n}A \cap B) > 0$.*

Theorem 0.15. *If $T\colon X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$, then the following statements are equivalent:*

(i) *T is ergodic;*

(ii) *whenever f is measurable and $(f \circ T)(x) = f(x), \forall x \in X$, then f is constant a.e.;*

(iii) *whenever f is measurable and $(f \circ T)(x) = f(x)$ a.e., then f is constant a.e.;*

(iv) *whenever $f \in L^2(X, m)$ and $(f \circ T)(x) = f(x), \forall x \in X$, then f is constant a.e.;*

(v) *whenever $f \in L^2(X, m)$ and $(f \circ T)(x) = f(x)$ a.e., then f is constant a.e.*

These are Theorem 1.5 and Theorem 1.6 from Walters [46].

Now we consider a rotation $T(x) = ax$ of a general compact group G. The measure involved is the normalized Haar measure m_H on $\mathcal{B}(G)$.

Theorem 0.16. *Let G be a compact group and let $T(x) = ax$ be a rotation of G. Then T is ergodic, if and only if $\{a^n\}_{n=-\infty}^{\infty}$ is dense in G. In particular, if T is ergodic, then G is abelian.*

This is Theorem 1.9 from Walters [46].

The following is the well-known Birkhoff ergodic theorem. The proof can be found in Walters [46, Theorem 1.14].

Theorem 0.17. *Suppose that $T\colon (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ is measure-preserving (where we allow $(X, \mathcal{B}, m)$ to be σ-finite) and $f \in L^1(X, m)$. Then*

$$\frac{1}{n} \sum_{k=1}^{n} f(T^k(x))$$

converges almost everywhere to a function $f^* \in L^1(X, m)$. Also $f^* \circ T = f$ a.e., and if $m(X) < \infty$, then $\int f^* dm = \int f dm$. In particular, if T is ergodic on a probability space $(X, \mathcal{B}, m)$, then for all $f \in L^1(X, m)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(x)) = \int f dm, \quad m\text{-a.e. } x \in X.$$

0.2.2 Continuous time

Let $(X, \mathcal{B}, m)$ be a measure space.

Definition 0.18. An *automorphism* of the measure space $(X, \mathcal{B}, m)$ is a one-to-one mapping T of the space M onto itself such that for all $A \in \mathcal{B}$ we have $TA, T^{-1}A \in \mathcal{B}$ and

$$m(A) = m(TA) = m(T^{-1}A).$$

The measure m is said to be an *invariant measure* for the automorphism T.

Definition 0.19. An *endomorphism* of the space X is a surjective (not necessarily one-to-one) mapping T of the space M onto itself such that for all $A \in \mathcal{B}$ we have $T^{-1}A \in \mathcal{B}$ and

$$m(A) = m(T^{-1}A).$$

Definition 0.20. Suppose $\{T^t\}$ is a one-parameter group of automorphisms of the measure space $(X, \mathcal{B}, m)$, $t \in \mathbb{R}$, that is, $T^{t+s}(x) = T^t(T^s(x))$ for all $t, s \in \mathbb{R}$ and $x \in X$. Then $\{T^t\}$ is said to be a *flow* if for any measurable function $f(x)$ on X, the function $f(T^t x)$ is measurable on the Cartesian product $X \times \mathbb{R}$.

The measurability condition appearing in this definition may also be stated in the following (equivalent) form: the mapping $\psi\colon X \times \mathbb{R} \to X$ given by the formula $\psi(x,t) = T^t x$ is measurable.

Definition 0.21. Suppose $\{T^t\}$ is a one-parameter semigroup of endomorphisms of the measure space $(X, \mathcal{B}, m)$, $t \in \mathbb{R}_+ := \{s : s \geqslant 0\}$, that is, $T^{t+s}(x) = T^t(T^s(x))$ for all $t, s \in \mathbb{R}_+$ and $x \in X$. Then $\{T^t\}$ is said to be a *semiflow* if for any measurable function $f(x)$ on X, the function $f(T^t x)$ is measurable on the Cartesian product $X \times \mathbb{R}_+$.

We have introduced four fundamental objects studied in ergodic theory: automorphisms, endomorphisms, flows and semiflows in measure spaces. Further the expression "dynamical system" stands for any of these objects. The measure space itself is said to be the *phase space* of the dynamical system.

Definition 0.22. The measurable function g is called *invariant* with respect to the automorphism T (endomorphism T, flow $\{T^t\}$, semiflow $\{T^t\}$) if for all $x \in X$, we have

$$\begin{aligned} &g(Tx) = g(x) = g(T^{-1}x) \\ &(g(Tx) = g(x), \\ &g(T^t x) = g(x) \text{ for all } t \in \mathbb{R}, \\ &g(T^t x) = g(x) \text{ for all } t \in \mathbb{R}_+). \end{aligned}$$

In other words, an invariant function assumes constant values on every trajectory of the dynamical system, that is, an invariant function is a function on trajectories.

Definition 0.23. The set $A \in \mathcal{B}$ is said to be *invariant* with respect to the automorphism T (endomorphism T, flow $\{T^t\}$, semiflow $\{T^t\}$) if its indicator $\mathbf{1}_A$ is an invariant function.

In the case of an automorphism T, the invariance of the set A means that $TA = A = T^{-1}A$; in the case of an endomorphism T or a flow (semiflow) $\{T^t\}$, the invariance means that $T^{-1}(A) = A$ or $T^{-t}A = A$ for all $t \in \mathbb{R}$ $(t \in \mathbb{R}_+)$.

Definition 0.24. The measurable function g is said to be *invariant* mod 0 with respect to the automorphism T (endomorphism T, flow or semiflow $\{T^t\}$), if

$$\begin{aligned}&g(Tx) = g(x) = g(T^{-1}x) \text{ almost everywhere}\\&(g(Tx) = g(x) \text{ almost everywhere},\\&g(T^t x) = g(x) \text{ for any } t \in \mathbb{R}(\mathbb{R}_+) \text{ for almost all } x).\end{aligned}$$

Lemma 0.25. *If g is an invariant* mod 0 *function, then there exists an invariant function g_1 such that $g = g_1$ almost everywhere.*

This is Lemma 1 (p. 13) from Cornfeld *et al.* [7].

Definition 0.26. The set $A \in \mathcal{B}$ is called *invariant* mod 0 with respect to the automorphism T (endomorphism T, flow $\{T^t\}$, semiflow $\{T^t\}$) if its indicator $\mathbf{1}_A$ is an invariant function mod 0 with respect to the automorphism T (endomorphism T, flow $\{T^t\}$, semiflow $\{T^t\}$).

Definition 0.27. A dynamical system on a probability space $(X, \mathcal{B}, m)$ is said to be *ergodic* if the measure $m(A)$ of any invariant set A equals 0 or 1.

Lemma 0.28. *If a dynamical system is ergodic, then any invariant function is constant on any set of full measure.*

This is Lemma 2 (p. 14) from Cornfeld *et al.* [7].

Now assume that $\{g_t : t \in \mathbb{R}\}$ is a continuous one-parameter subgroup of the abelian compact group G. Such a subgroup defines a flow $\{T^t\}$ on G by the formula

$$T^t x = g_t x, \quad x \in G.$$

It is obvious that this flow preserves the normalized Haar measure m_H.

Theorem 0.29. *The following conditions are equivalent:*

(i) *the flow $\{T^t\}$ is ergodic;*

(ii) *the one-parameter group of homeomorphisms $\{T^t\}$ is minimal, that is, for all $x \in G$, the trajectory $\{T^t x : t \in \mathbb{R}\}$ is dense in G.*

This is Theorem 1′ (p. 99) from Cornfeld *et al.* [7].

The following is the Birkhoff-Khinchin ergodic theorem.

Theorem 0.30. *Suppose that $(X, \mathcal{B}, m)$ is a probability space and $f \in L^1(X, m)$. Then for almost every (in the sense of the measure m) $x \in X$, the following limits exist and are equal to each other*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^{-k} x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} f(T^k x) =: f^*(x),$$

in the case of an automorphism T; for almost every $x \in X$, the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) =: f^*(x),$$

in the case of an endomorphism T; for almost every $x \in X$, the following limits exist and are equal to each other

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(T^\tau x)d\tau = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(T^{-\tau} x)d\tau = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^t f(T^\tau x)d\tau =: f^*(x),$$

in the case of a flow $\{T^t\}$; for almost every $x \in X$, the following limit exists

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(T^\tau x)d\tau =: f^*(x),$$

in the case of a semiflow $\{T^t\}$.

Further $f^*(Tx) = f^*(x)$ or $f^*(T^t x) = f^*(x)$ whenever the right-hand sides of these equations exist. Morever

$$f^* \in L^1(X, m) \text{ and } \int f^* dm = \int f dm.$$

In particular, if the dynamical system is ergodic, then $f^* = \int f dm$.

0.3 Martingales

This section is taken from Doob [8] and Durrett [9].

0.3.1 Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. In this section, we only consider $\mathbb{R}$-valued random variables. Let $Y\colon (\Omega, \mathcal{F}, \mathbf{P}) \to \mathbb{R}$ be an integrable random variable. Then the *conditional expectation of Y given $\mathcal{G}$*, denoted by $\mathbf{E}(Y|\mathcal{G})$, is defined to be any random variable Z that has

(i) $Z \in \mathcal{G}$, that is, Z is $\mathcal{G}$-measurable, and

(ii)
$$\int_A Z(\omega)d\mathbf{P}(\omega) = \int_A Y(\omega)d\mathbf{P}(\omega), \quad \text{for all } A \in \mathcal{G}.$$

It is clear that if the conditional expectation of Y given $\mathcal{G}$ exists, then it is unique $\mathbf{P}$-almost surely. Let us prove the existence of the conditional expectation. Since Y is integrable, the function φ defined by

$$\varphi(A) := \int_A Y(\omega)d\mathbf{P}(\omega), \quad A \in \mathcal{G},$$

is σ-additive on $\mathcal{G}$ and is absolutely continuous to the probability measure $\mathbf{P}$. Thus, according to the Radon-Nikodym theorem, there exists a $\mathcal{G}$-measurable function Z satisfying (ii). This proves the existence.

Now let $\{X_t\}_{t\in T}$ be any family of random variables. Let $\mathcal{G} = \sigma(X_t : t \in T)$ be the smallest σ-field of Ω such that X_t's are measurable. Then the *conditional expectation of Y given $\{X_t\}_{t\in T}$*, denoted by $\mathbf{E}(Y|X_t, t \in T)$, is defined to be

$$\mathbf{E}(Y|\mathcal{G}).$$

Example 0.31. (i) If $Y \in \mathcal{G}$, then $\mathbf{E}(Y|\mathcal{G}) = Y$.

(ii) Suppose that Y is independent of $\mathcal{G}$, that is, for all $B \in \mathcal{B}(\mathbb{R})$ and $A \in \mathcal{G}$,

$$\mathbf{P}(\{Y \in B\} \cap A) = \mathbf{P}(Y \in B)\mathbf{P}(A).$$

Then

$$\mathbf{E}(Y|\mathcal{G}) = \mathbf{E}[Y].$$

Here are some properties of the conditional expectation.

(i) Conditional expectation is linear:

$$\mathbf{E}(aY + Z|\mathcal{G}) = a\mathbf{E}(Y|\mathcal{G}) + \mathbf{E}(Z|\mathcal{G}).$$

(ii) If $Y \leqslant Z$, then

$$\mathbf{E}(Y|\mathcal{G}) \leqslant \mathbf{E}(Z|\mathcal{G}).$$

(iii) If $Y_n \geqslant 0$ and $Y_n \uparrow Y$ with $\mathbf{E}[Y] < \infty$, then

$$\mathbf{E}(Y_n|\mathcal{G}) \uparrow \mathbf{E}(Y|\mathcal{G}).$$

(iv) If φ is convex and $\mathbf{E}[|Y|], \mathbf{E}[|\varphi(Y)|] < \infty$, then

$$\varphi(\mathbf{E}(Y|\mathcal{G})) \leqslant \mathbf{E}(\varphi(Y)|\mathcal{G}).$$

(v) If $\mathcal{G}_1 \subset \mathcal{G}_2$, then

$$\mathbf{E}(\mathbf{E}(Y|\mathcal{G}_2)|\mathcal{G}_1) = \mathbf{E}(Y|\mathcal{G}_1).$$

(vi) If $Y \in \mathcal{G}$ and $\mathbf{E}[|Z|], \mathbf{E}[|YZ|] < \infty$, then

$$\mathbf{E}(YZ|\mathcal{G}) = Y\mathbf{E}(Z|\mathcal{G}).$$

(vii) Suppose $\mathbf{E}[|Y|^2] < \infty$. Then $\mathbf{E}(Y|\mathcal{G})$ is the random variable $Z \in \mathcal{G}$ that minimizes the "mean square error" $\mathbf{E}[|Y - Z|^2]$.

0.3.2 Martingales, definition and convergence theorems

Let $\{\mathcal{F}_n\}$ be a *filtration*, that is, an increasing sequence of sub-σ-fields of $\mathcal{F}$,

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}.$$

We first consider the case of real-valued martingale. Let $\{X_n\}_{n\in\mathbb{N}}$ be real-valued random variables. A sequence $\{X_n\}$ is said to be *adapted* to $\{\mathcal{F}_n\}$ if $X_n \in \mathcal{F}_n$ for all n.

Definition 0.32. A sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be a *martingale* (*suppermartingale* or *submartingale*) (with respect to $\mathcal{F}_n$) if

(i) $\mathbf{E}[|X_n|] < \infty$,

(ii) $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$,

(iii) $\mathbf{E}(X_{n+1}|\mathcal{F}_n) = X_n$ for all n ($\mathbf{E}(X_{n+1}|\mathcal{F}_n) \leqslant X_n$ or $\mathbf{E}(X_{n+1}|\mathcal{F}_n) \geqslant X_n$ for all n).

Example 0.33. (i) Consider a sequence $\{\xi_n\}$ of integrable independent random variables with $\mathbf{E}[\xi_n] = 0$. Let

$$X_n = \xi_1 + \cdots + \xi_n,$$

and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Then $\{X_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. Indeed, the fact that $\mathbf{E}[|X_n|] < \infty$ and $X_n \in \mathcal{F}_n$ are clear. Let us check the condition (iii). Since ξ_{n+1} is independent of $\mathcal{F}_n$, we have

$$\mathbf{E}(X_{n+1}|\mathcal{F}_n) = \mathbf{E}(X_n|\mathcal{F}_n) + \mathbf{E}(\xi_{n+1}|\mathcal{F}_n) = X_n + \mathbf{E}[\xi_{n+1}] = X_n.$$

Here we have just used the linearity of conditional expectation and Example 0.31(ii).

(ii) Consider a sequence $\{\xi_n\}$ of integrable independent random variables with $\mathbf{E}[\xi_n] = 1$. Let

$$X_n = \xi_1 \cdots \xi_n,$$

and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Then $\{X_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. As above, we need only to check the condition (iii). Using the property (vi) of conditional expectation and Example 0.31(ii), we have

$$\mathbf{E}(X_{n+1}|\mathcal{F}_n) = \mathbf{E}(X_n\xi_{n+1}|\mathcal{F}_n) = X_n\mathbf{E}(\xi_{n+1}|\mathcal{F}_n) = X_n\mathbf{E}[\xi_{n+1}] = X_n.$$

Theorem 0.34. *If $\{X_n\}$ is a martingale w.r.t. $\mathcal{F}_n$ and φ is a convex function with $\mathbf{E}[|\varphi(X_n)|] < \infty$ for all n, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently, if $p \geqslant 1$ and $\mathbf{E}[|X_n|^p] < \infty$ for all n, then $|X_n|^p$ is a submartingale w.r.t. $\mathcal{F}_n$.*

Theorem 0.35. *If $\{X_n\}$ is a submartingale w.r.t. $\mathcal{F}_n$ and φ is an increasing convex function with $\mathbf{E}[|\varphi(X_n)|] < \infty$ for all n, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently, (i) if $\{X_n\}$ is a submartingale, then $(X_n - a)^+$ is a submartingale; (ii) if $\{X_n\}$ is a suppermartingale, then $X_n \wedge a$ is a suppermartingale. Here a is a constant and $(X_n - a)^+ := \max\{X_n - a, 0\}, X_n \wedge a = \min\{X_n, a\}$.*

These are Theorem 5.2.3 and Theorem 5.2.4 from Durrett [9].

Theorem 0.36 (Martingale convergence theorem). *If X_n is a submartingale with*

$$\sup \mathbf{E}[X_n^+] < \infty,$$

then as $n \to \infty$, X_n converges a.s. to a limit X with $\mathbf{E}[|X|] < \infty$.

This is Theorem 5.2.8 from Durrett [9].

An important special case of this theorem is the following.

Theorem 0.37. *If $X_n \geqslant 0$ is a suppermartingale, then as $n \to \infty$, $X_n \to X$ a.s. and $\mathbf{E}[X] < \mathbf{E}[X_1]$.*

Theorem 0.38 (L^p maximum inequality). *Let $\{X_n\}$ be a submartingale and let*

$$\bar{X}_n := \max_{1 \leqslant m \leqslant n} X_m^+.$$

Then for $1 < p < \infty$,

$$\mathbf{E}[\bar{X}_n^p] \leqslant \left(\frac{p}{p-1}\right)^p \mathbf{E}[(X_n^+)^p].$$

Consequently, if $\{Y_n\}$ is a martingale and $Y_n^* := \max_{1 \leqslant m \leqslant n} |Y_m|$, then

$$\mathbf{E}[|Y_n^*|^p] \leqslant \left(\frac{p}{p-1}\right)^p \mathbf{E}[|Y_n|^p].$$

Theorem 0.39 (L^p convergence theorem). *If $\{X_n\}$ is a martingale with* $\sup \mathbf{E}[|X_n|^p] < \infty$, *where $p > 1$, then $X_n \to X$ a.s. and in L^p.*

These are Theorem 5.4.3 and Theorem 5.4.5 from Durrett [9].

We now turn to the case of complex-valued martingales. The definition of complex-valued martingale is the same as that of real-valued martingale. Note that $\{X_n = U_n + iV_n\}$, where U_n and V_n are real, is a martingale w.r.t. $\{\mathcal{F}_n\}$, if and only if $\{U_n\}$ and $\{V_n\}$ are martingales w.r.t. $\{\mathcal{F}_n\}$.

Theorem 0.40 (Doob's martingale convergence theorem). *Let $\{X_n\}$ be a (real or complex) martingale. Then $\{|X_n|\}$ is a submartingale and we have the following.*

(i) *If* $\lim_{n\to\infty} \mathbf{E}[|X_n|] = K < \infty$, *then* $\lim_{n\to\infty} X_n =: X_\infty$ *exists a.s. and* $\mathbf{E}[|X_\infty|] \leqslant K$. *In particular, $K < \infty$, if the X_n's are all real and $\geqslant 0$ or all real and $\leqslant 0$.*

(ii) *The following conditions are equivalent:*

  (a) *$K < \infty$, and the random variables $X_1, X_2, \ldots, X_\infty$ constitute a martingale;*

  (b) *the random variables $X_1, X_2, \ldots$ are uniformly integrable;*

  (c) *$K < \infty$ and $\mathbf{E}[|X_\infty|] = K$;*

  (d) *$K < \infty$ and $\lim_{n\to\infty} \mathbf{E}[|X_\infty - X_n|] = 0$.*

(iii) *If, for some $p > 1, \lim_{n\to\infty} \mathbf{E}[|X_n|^p] < \infty$, then the conditions of* (ii) *are satisfied, $\mathbf{E}[|X_\infty|^p] < \infty$, and*
$$\lim_{n\to\infty} \mathbf{E}[|X_\infty - X_n|^p] = 0.$$
Conversely, if the conditions of (ii) *are satisfied and if* $\mathbf{E}[|X_\infty|^q] < \infty$ *for some $q > 1$, then*
$$\mathbf{E}[|X_n|^q] \leqslant \mathbf{E}[|X_\infty|^q].$$

This is a part of Theorem 4.1 from Doob [8].

Part I

Besicovitch Almost Periodic Functions

Chapter 1

Besicovitch Almost Periodic Functions With Fourier Exponents Belonging to a Dirichlet Sequence

1.1 Introduction

A function $f\colon \mathbb{R} \to \mathbb{C}$ is called Besicovitch almost periodic if it is a limit of trigonometric polynomials under some Besicovitch q-(semi)norm $(1 \leqslant q < \infty)$,

$$\|f\|_q := \limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^q dt \right)^{1/q}.$$

Let B^q denote the quotient space of the q-Besicovitch almost periodic functions (B^q-a.p. for short) with respect to the null space $\mathcal{N}^q = \{f : \|f\|_q = 0\}$. Then B^q is a Banach space. It is well known that Besicovitch almost periodic functions possess mean values and limit distributions. Here the mean value of a function f is defined as

$$\mathbf{M}[f] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt,$$

provided that the limit exists, and the limit distribution is the probability distribution on $\mathbb{C}$ to which the sequence of probability measures

$$\nu_T\{\tau : f(\tau) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges as $T \to \infty$, where ν_T denotes the uniform probability measure on $[-T, T]$.

For every function $f \in B^q$, the mean value

$$a(\lambda) = \mathbf{M}[f(t)e^{-i\lambda t}] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t}dt$$

exists for all $\lambda \in \mathbb{R}$, and those λ for which $a(\lambda)$ is non-zero are at most countable, called the Fourier exponents of f. The formal series

$$f(t) \sim \sum_{\lambda} a(\lambda)e^{i\lambda t}$$

is called the Fourier series of f. Dealing with the problem of constructing trigonometric polynomials approximating Besicovitch almost periodic functions in norm, Besicovitch and Bohr [3] showed that a Bochner-Fejér sequence of a function f does converge to the function itself. Note that a Bochner-Fejér sequence contains trigonometric polynomials whose exponents are the Fourier exponents of f.

Let Λ be a subgroup of $\mathbb{R}_d$, the real line with the discrete topology. We consider the space $B^q(\Lambda)$ of Besicovitch almost periodic functions whose Fourier exponents belong to Λ,

$$B^q(\Lambda) := \{f \in B^q : a(\lambda) = 0 \text{ if } \lambda \notin \Lambda\}.$$

Let $\widehat{\Lambda}$ be the dual group of Λ and ν be the normalized Haar measure on $\widehat{\Lambda}$. Then the spaces $B^q(\Lambda)$ and $L^q(\widehat{\Lambda}, \nu)$ are isometrically isomorphic under the isomorphism T_q which maps a B^q-a.p. function

$$f(t) \sim \sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda t}$$

to an $L^q(\widehat{\Lambda}, \nu)$ function $T_q(f)$ with the usual Fourier series

$$T_q(f)(x) \sim \sum_{\lambda \in \Lambda} a(\lambda) \chi_\lambda(x),$$

where for $\lambda \in \Lambda$, $\chi_\lambda(x) = x(\lambda), (x \in \widehat{\Lambda})$ is a character of $\widehat{\Lambda}$. Moreover, the limit distribution of f coincides with the distribution of $T_q(f)$.

The main aim of this chapter is to study Besicovitch almost periodic functions whose Fourier series are of the forms

$$f(t) \sim \sum_{m=1}^{\infty} a_m e^{-i\lambda_m t}, \tag{1.1}$$

where $\{\lambda_m\}$ is a strictly increasing sequence of non-negative numbers tending to infinity, called a Dirichlet sequence. Let Λ be a subgroup of $\mathbb{R}_d$, generated by $\{\lambda_m\}$, and let $\widehat{\Lambda}$ be the dual group of Λ with the normalized Haar measure ν. Then the limit distribution of f coincides with the distribution of an $L^q(\widehat{\Lambda}, \nu)$ function

$$T_q(f)(x) \sim \sum_{m=1}^{\infty} a_m \chi_{-\lambda_m}(x). \tag{1.2}$$

For $q > 1$, we will prove in this chapter that the Fourier series (1.1) converges in norm with the usual order, which gives another way to approximate this kind of B^q-a.p. function by trigonometric polynomials. Equivalently, by the isometric property, the Fourier series (1.2) converges in $L^q(\widehat{\Lambda}, \nu)$ with the usual order. In addition, we will show that the Fourier series (1.2) converges almost everywhere (with respect to ν). This result is analogous to Carleson's theorem for classical Fourier series on $[0, 2\pi]$, and in fact is a consequence of Carleson's theorem in multi-dimensional case. However, to apply Carleson's theorem to our present case, we need to introduce another way to identify limit distributions of functions of the forms (1.1), as we will see later.

1.2 General theory of Besicovitch almost periodic functions

Recall that if a function f is B^q-a.p., then the mean value

$$a(\lambda) = \mathbf{M}[f(t)e^{-i\lambda t}] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t} dt$$

exists for all real values of λ, and is non-zero only for at most a countable set of values of λ, say,

$$\lambda_1, \lambda_2, \ldots.$$

Let $a_m = \mathbf{M}[f(t)e^{-i\lambda_m t}]$. Then

$$f(t) \sim \sum a_m e^{i\lambda_m t}$$

is the Fourier series of f.

Let us introduce an algorithm for the approximation of f by trigonometric polynomials [3]. If β's are $\mathbb{Q}$-linearly independent numbers, we consider Bochner-Fejér polynomials

$$\sigma_{\binom{n_1,n_2,\ldots,n_k}{\beta_1,\beta_2,\ldots,\beta_k}}(t) = \sum_{\substack{-n_1 \leqslant \nu_1 \leqslant n_1 \\ \cdots \\ -n_k \leqslant \nu_k \leqslant n_k}} \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_k|}{n_k}\right) a_m e^{i\lambda_m t},$$

where

$$\lambda_m = \nu_1\beta_1 + \cdots + \nu_p\beta_p,$$

and a_m is to be interpreted as zero when the above linear combination of β's does not belong to the Fourier exponents of f. We will use the notation $\sigma_B(t)$ instead of the detailed notation

$$\sigma_{\binom{n_1,n_2,\ldots,n_k}{\beta_1,\beta_2,\ldots,\beta_k}}(t).$$

Let $\alpha_1, \alpha_2, \ldots$ be a sequence of $\mathbb{Q}$-linearly independent positive numbers (which generally is infinite but in particular cases may be finite) such that every exponent λ_m may be expressed as a finite linear form in the α's with rational coefficients,

$$\lambda_m = r_{m,1}\alpha_1 + r_{m,2}\alpha_2 + \cdots + r_{m,q_m}\alpha_{q_m}.$$

We put

$$\beta_1 = \frac{\alpha_1}{N_1!}, \beta_2 = \frac{\alpha_2}{N_2!}, \ldots, \beta_k = \frac{\alpha_k}{N_k!},$$

where $N_1, N_2, \ldots, N_k$ are positive integers. The result on the approximation of B^q-a.p. function by Bochner-Fejér polynomials is as follows.

Theorem 1.1 ([3, Theorem II]). *The sum $\sigma_B(t)$ converges to $f(t)$ in the Besicovitch q-norm, as $k \to \infty$, $N_1 \to \infty$, $N_2 \to \infty, \ldots$, and $\frac{n_1}{N_1!} \to \infty$, $\frac{n_2}{N_2!} \to \infty, \ldots$.*

Remark 1.2. A sequence of Bochner-Fejér polynomials

$$\sigma_{B_1}(t), \sigma_{B_2}(t), \ldots$$

is called Bochner-Fejér sequence if the basic numbers $\beta_1, \ldots, \beta_k$ and the indices $n_1, \ldots, n_k$ satisfy the conditions of the above theorem.

1.3 Probability space associated with Besicovitch almost periodic functions

1.3.1 Besicovitch functions whose Fourier exponents belong to a subgroup

Let Λ be a subgroup of $\mathbb{R}_d$, the real line with the discrete topology. We consider the space $B^q(\Lambda)$ of Besicovitch almost periodic functions whose Fourier exponents belong to Λ,

$$\begin{aligned} B^q(\Lambda) &:= \{f \in B^q : a(\lambda) = 0 \text{ if } \lambda \notin \Lambda\} \\ &= \text{the linear closure of } \{e^{i\lambda t} : \lambda \in \Lambda\} \text{ with respect to the } \|\cdot\|_q \text{ norm.} \end{aligned}$$

The second identity follows from the convergence of a Bochner-Fejér sequence to the function itself (Theorem 1.1).

Let $\widehat{\Lambda}$ be the dual group of Λ. Since Λ is discrete, it follows that $\widehat{\Lambda}$ is a compact abelian group. Thus, there is a unique normalized Haar measure ν on $\widehat{\Lambda}$. For $\lambda \in \Lambda$, let χ_λ denote the character of $\widehat{\Lambda}$ which maps $x \in \widehat{\Lambda}$ to $x(\lambda)$,

$$\chi_\lambda(x) = x(\lambda), \quad x \in \widehat{\Lambda}.$$

Then $\{\chi_\lambda\}_{\lambda\in\Lambda}$ are the characters of $\widehat{\Lambda}$.

For $t \in \mathbb{R}$, let e_t be the element of $\widehat{\Lambda}$ defined by

$$e_t(\lambda) = e^{i\lambda t}, \quad \lambda \in \Lambda.$$

Then the mapping $\mathbb{R} \ni t \mapsto e_t \in \widehat{\Lambda}$ is continuous, and hence is Borel measurable. Consequently, probability measures ν_T on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ induce on $(\widehat{\Lambda}, \mathcal{B}(\widehat{\Lambda}))$ probability measures

$$Q_T(A) = \nu_T(\tau : e_\tau \in A), \quad A \in \mathcal{B}(\widehat{\Lambda}).$$

Theorem 1.3. *The sequence of probability measures $\{Q_T\}$ converges weakly to the normalized Haar measure ν as $T \to \infty$.*

Proof. For any character $\chi_\lambda, (\lambda \in \Lambda)$ of $\widehat{\Lambda}$, it is clear that

$$\begin{aligned}\int_{\widehat{\Lambda}} \chi_\lambda(x) dQ_T(x) &= \frac{1}{2T}\int_{-T}^{T} \chi_\lambda(e_t)dt = \frac{1}{2T}\int_{-T}^{T} e^{i\lambda t}dt \\ &= \begin{cases} 1, & \text{if } \lambda = 0, \\ \dfrac{e^{i\lambda T} - e^{-i\lambda T}}{2i\lambda T}, & \text{if } \lambda \neq 0, \end{cases} \\ &\longrightarrow \int_{\widehat{\Lambda}} \chi_\lambda(x) d\nu(x) \quad \text{as} \quad T \to \infty.\end{aligned}$$

Since the linear space spanned by the characters $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is dense in $C(\widehat{\Lambda})$, the space of continuous functions on $\widehat{\Lambda}$, the assertion of this theorem easily follows. (See also the continuity theorem for probability measures on compact abelian group (Theorem 0.11).) □

If Λ is dense in $\mathbb{R}$, then $\{e_t\}_{t\in\mathbb{R}}$ are distinct. The only subgroups of $\mathbb{R}$ that are not dense are isomorphic to the additive group of the integers. In this case, $\widehat{\Lambda}$ is the classical circle $\widehat{\Lambda} = \{z \in \mathbb{C} : |z| = 1\}$, $e_t = e_{t+T_0}$ for some $T_0 > 0$, and $\{e_t\}_{t\in\mathbb{R}} = \widehat{\Lambda}$.

Lemma 1.4. *Unless $\widehat{\Lambda}$ is a circle, the characters $\{e_t\}_{t\in\mathbb{R}}$ are distinct and form a dense one-parameter subgroup of $\widehat{\Lambda}$.*

Proof. Let S be the closure of $\{e_t\}_{t\in\mathbb{R}}$. Obviously, $Q_T(S) = 1$ for any $T > 0$. Since the set S is closed, it follows from the property of weak convergence that

$$\nu(S) \geqslant \limsup_{T\to\infty} Q_T(S) = 1.$$

Thus, the complement of S, $S^c = \widehat{\Lambda} \setminus S$ is open and has ν-measure zero, which implies that $S^c = \emptyset$ by the property of the Haar measure. The proof is complete. □

Theorem 1.5. *$T^t \colon \widehat{\Lambda} \to \widehat{\Lambda}, (t \in \mathbb{R})$, defined by $T^t(x) = xe_t$, is an ergodic flow (with respect to the Haar measure ν).*

Proof. We need to verify the measurability of the flow $\{T^t\}_{t\in\mathbb{R}}$. It is clear that the mapping

$$\begin{aligned}\psi\colon \mathbb{R}\times\widehat{\Lambda} &\to \widehat{\Lambda}\\ (t,x) &\mapsto e_t x\end{aligned}$$

is continuous (with respect to the product topology on $\mathbb{R}\times\widehat{\Lambda}$). Hence it is $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\widehat{\Lambda})/\mathcal{B}(\widehat{\Lambda})$-measurable. Consequently, for any measurable function $f(x)$ on $\widehat{\Lambda}$, the function $f(T^t x)$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\widehat{\Lambda})$-measurable.

The ergodic property of the flow $\{T^t\}_{t\in\mathbb{R}}$, which is equivalent to the denseness of $\{e_t\}_{t\in\mathbb{R}}$ in $\widehat{\Lambda}$, follows (see Theorem 0.29). □

We construct an isometric isomorphism $T_q\colon B^q(\Lambda)\to L^q(\widehat{\Lambda},\nu)$ as follows. For a trigonometric polynomial $p(t)$ whose exponents belong to Λ of the form

$$p(t)=\sum_{\lambda\in I} a(\lambda)e^{i\lambda t},\quad (I:\text{ finite subset of }\Lambda),$$

we define

$$T_q(p)(x)=\sum_{\lambda\in I} a(\lambda)\chi_\lambda(x)\in C(\widehat{\Lambda})\subset L^q(\widehat{\Lambda},\nu).$$

Then it follows from the property of weak convergence that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|p(t)|^q dt &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|T_q(p)(e_t)|^q dt\\ &= \lim_{T\to\infty}\int_{\widehat{\Lambda}}|T_q(p)(x)|^q dQ_T(x)=\int_{\widehat{\Lambda}}|T_q(p)(x)|^q d\nu(x),\end{aligned}$$

and hence,

$$\|p\|_q=\|T_q(p)\|_{L^q(\widehat{\Lambda},\nu)}. \tag{1.3}$$

Thus, T_q can be continuously extended to an isometric mapping from $B^q(\Lambda)$ to $L^q(\widehat{\Lambda},\nu)$. Indeed, let $f\in B^q(\Lambda)$. Then there exists a sequence of trigonometric polynomials $\{p_k\}_k$, whose exponents belong to Λ, which converges to f, that is, $\|p_k-f\|_q\to 0$ as $k\to\infty$. In particular, $\{p_k\}_k$ is a Cauchy sequence in $B^q(\Lambda)$. By (1.3), $\{T_q(p_k)\}_k$ is also a Cauchy sequence in $L^q(\widehat{\Lambda},\nu)$, and hence, the limit $\lim_{k\to\infty}T_q(p_k)$ exists in $L^q(\widehat{\Lambda},\nu)$. Let $T_q(f):=\lim_{k\to\infty}T_q(p_k)$. It is clear that T_q is well defined; moreover, it follows from (1.3) and from the continuity of norms that

$$\|f\|_q=\|T_q(f)\|_{L^q(\widehat{\Lambda},\nu)},\quad f\in B^q(\Lambda). \tag{1.4}$$

This implies that T_q is injective. On the other hand, T_q is surjective since the linear space spanned by the characters $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is dense in $L^q(K,\nu)$ and $B^q(\Lambda)$ is complete. Therefore T_q is an isometric isomorphism between $B^q(\Lambda)$ and $L^q(\widehat{\Lambda},\nu)$.

Theorem 1.6. *For* $1\leqslant q<\infty$,

$$B^q(\Lambda)\cong L^q(\widehat{\Lambda},\nu).$$

Theorem 1.7 ([17, Correspondence Theorem]). *The isomorphism T_q has the following properties.*

(i) *T_q is a linear isometric mapping from $B^q(\Lambda)$ onto $L^q(\widehat{\Lambda},\nu)$.*

(ii) $T_q(|f|)=|T_q(f)|$.

(iii) *If* $1/r = 1/p + 1/q$ $(r \geqslant 1)$, *and* $f \in B^p(\Lambda), g \in B^q(\Lambda)$, *then Hölder's inequality implies* $fg \in B^r(\Lambda)$ *and*

$$T_r(fg) = T_p(f)T_q(g).$$

(iv) *If* f *has the Fourier series* $\sum_{\lambda\in\Lambda} a(\lambda)e^{i\lambda t}$, *then* $T_q(f)$, *as a function in* $L^q(\widehat{\Lambda}, \nu)$, *has the Fourier series*

$$T_q(f)(x) \sim \sum_{\lambda\in\Lambda} a(\lambda)\chi_\lambda(x).$$

In particular, $\mathbf{M}[f] = \mathbf{E}[T_q(f)]$.

(v) *The limit distribution of* f *coincides with the distribution of* $T_q(f)$.

Proof. (i) and (iii) are clear. (ii) easily follows from the inequality $||a| - |b|| \leqslant |a - b|$ for any $a, b \in \mathbb{C}$. (iv) follows from the fact that the convergence in B^q or in $L^q(\widehat{\Lambda}, \nu)$ implies the convergence of each Fourier coefficient.

To prove (v), let $f \in B^q(\Lambda)$. Then there is a sequence of trigonometric polynomials $\{p_n\}$ whose exponents belong to Λ converging to f with respect to the $\|\cdot\|_q$ norm, say,

$$\lim_{n\to\infty} \|p_n - f\|_q = 0.$$

Let $\theta_T\colon (\widehat{\Omega}, \widehat{\mathbf{P}}) \to [-T, T]$ be a random variable uniformly distributed on $[-T, T]$. We put

$$X_{T,n} := p_n(\theta_T).$$

Note that the distribution of $X_{T,n}$ is the probability measure $\nu_T p_n^{-1}$. Thus, by the mapping theorem (Theorem 0.3),

$$X_{T,n} \xrightarrow{d} T_q(p_n) \quad \text{as} \quad T \to \infty, \tag{1.5}$$

because $T_q(p_n)$ is continuous. Moreover, by the isometric property of T_q, $T_q(p_n)$ converges to $T_q(f)$ in $L^q(\widehat{\Lambda}, \nu)$, which implies that

$$T_q(p_n) \xrightarrow{d} T_q(f) \quad \text{as} \quad n \to \infty. \tag{1.6}$$

In addition, for any $\varepsilon > 0$, using Chebyshev's inequality, we have

$$\widehat{\mathbf{P}}(|X_{T,n} - Y_T| \geqslant \varepsilon) \leqslant \frac{1}{\varepsilon}\frac{1}{2T}\int_{-T}^{T} |p_n(t) - f(t)|dt,$$

where $Y_T := f(\theta_T)$, and hence,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \widehat{\mathbf{P}}(|X_{T,n} - Y_T| \geqslant \varepsilon) \leqslant \frac{1}{\varepsilon} \lim_{n\to\infty} \|p_n - f\|_1 = 0. \tag{1.7}$$

Therefore, by using Theorem 0.7, we get the desired result

$$Y_T \xrightarrow{d} T_q(f) \quad \text{as} \quad T \to \infty. \qquad \square$$

1.3.2 Besicovitch functions whose Fourier exponents belong to a Dirichlet sequence

Consider a Dirichlet sequence $\{\lambda_m\}$,

$$0 \leqslant \lambda_1 < \lambda_2 < \cdots ; \quad \lambda_m \to \infty,$$

and let

$$B^q(\{\lambda_m\}) := \text{the linear closure of } \{e^{-i\lambda_m t}\} \text{ with respect to the } \|\cdot\|_q \text{ norm.}$$

Let Λ be a subgroup of $\mathbb{R}_d$, generated by $\{\lambda_m\}$, and let $\widehat{\Lambda}$ be the dual group of Λ with the normalized Haar measure ν as in the previous subsection. Then the limit distribution of a function $f \in B^q(\{\lambda_m\})$ with the Fourier series

$$f(t) \sim \sum_{m=1}^{\infty} a_m e^{-i\lambda_m t},$$

coincides with the distribution of an $L^q(\widehat{\Lambda}, \nu)$ function

$$T_q(f) \sim \sum_{m=1}^{\infty} a_m \chi_{-\lambda_m} \in L^q(\widehat{\Lambda}, \nu).$$

We now introduce another way to identify distributions of functions in $B^q(\{\lambda_m\})$. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological abelian group. For $T > 0$, we define a probability measure Q_T on $(\Omega, \mathcal{B}(\Omega))$ by

$$Q_T(A) := \nu_T\big(\tau : (e^{-i\tau\lambda_m})_{m\in\mathbb{N}} \in A\big), \quad A \in \mathcal{B}(\Omega).$$

Let

$$\mathcal{Z} = \bigoplus_{m\in\mathbb{N}} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}$. Then the dual group of Ω is isomorphic to $\mathcal{Z}$ and the dual group of $\mathcal{Z}$ is isomorphic to Ω. Each $\underline{k} = \{k_m : m \in \mathbb{N}\} \in \mathcal{Z}$, where only a finite number of k_m are non-zero, is identified with a character

$$\omega \mapsto \omega^{\underline{k}} = \prod_m \omega(m)^{k_m}$$

of Ω. Here $\omega(m)$ is the projection of $\omega \in \Omega$ onto γ_m. Conversely, each $\omega \in \Omega$ is identified with a character $(\underline{k} \mapsto \omega^{\underline{k}})$ of $\mathcal{Z}$.

Recall that for a probability measure Q on $(\Omega, \mathcal{B}(\Omega))$, the Fourier transform $g^{(Q)}(\underline{k})$ of Q is defined by

$$g^{(Q)}(\underline{k}) := \mathbf{E}^{(Q)}[\omega^{\underline{k}}] = \int_\Omega \left(\prod_m \omega(m)^{k_m}\right) dQ, \quad \underline{k} \in \mathcal{Z}. \tag{1.8}$$

Then we construct the probability measure $\mathbf{P}$ on $(\Omega, \mathcal{B}(\Omega))$ as follows.

Theorem 1.8. *There is a probability measure* $\mathbf{P}$ *on* $(\Omega, \mathcal{B}(\Omega))$ *such that the sequence of probability measures* $\{Q_T\}$ *converges weakly to* $\mathbf{P}$ *as* $T \to \infty$. *The Fourier transform of* $\mathbf{P}$ *is given by*

$$g(\underline{k}) = \begin{cases} 1, & \text{if } \sum_{m=1}^{\infty} \lambda_m k_m = 0, \\ 0, & \text{if } \sum_{m=1}^{\infty} \lambda_m k_m \neq 0. \end{cases}$$

Moreover if $\{\lambda_m\}$ *is* $\mathbb{Q}$*-linearly independent, then* $\mathbf{P}$ *coincides with the normalized Haar measure on* Ω.

Proof. The Fourier transform $g_T(\underline{k})$ of the measure Q_T is of the form

$$\begin{aligned} g_T(\underline{k}) &= \int_\Omega \left(\prod_{m=1}^{\infty} \omega^{k_m}(m) \right) dQ_T = \frac{1}{2T} \int_{-T}^{T} \left(\prod_{m=1}^{\infty} e^{-it\lambda_m k_m} \right) dt \\ &= \begin{cases} 1, & \text{if } \sum_{m=1}^{\infty} \lambda_m k_m = 0, \\ \dfrac{\exp\{-iT \sum_{m=1}^{\infty} \lambda_m k_m\} - \exp\{iT \sum_{m=1}^{\infty} \lambda_m k_m\}}{-2iT \sum_{m=1}^{\infty} \lambda_m k_m}, & \text{if } \sum_{m=1}^{\infty} \lambda_m k_m \neq 0. \end{cases} \end{aligned}$$

Hence, the limit

$$g(\underline{k}) := \lim_{T \to \infty} g_T(\underline{k}) = \begin{cases} 1, & \text{if } \sum_{m=1}^{\infty} \lambda_m k_m = 0, \\ 0, & \text{if } \sum_{m=1}^{\infty} \lambda_m k_m \neq 0, \end{cases}$$

exists for every $\underline{k}$. The continuity theorem (Theorem 0.11) implies that there exists a probability measure $\mathbf{P}$ on Ω such that $\{Q_T\}$ converges weakly to $\mathbf{P}$ as $T \to \infty$. Moreover, $g(\underline{k})$ is the Fourier transform of $\mathbf{P}$. Now, if $\{\lambda_m\}$ is $\mathbb{Q}$-linearly independent, then $\sum_{m=1}^{\infty} \lambda_m k_m = 0$ holds iff $k_m = 0$ for all m. It then follows that $\mathbf{P}$ coincides with the normalized Haar measure on Ω. □

Lemma 1.9. $\{\omega(m)\}_{m \in \mathbb{N}}$ *is an orthonormal system in* $L^2(\Omega, \mathbf{P})$, *that is,*

$$\mathbf{E}^{(\mathbf{P})} \left[\omega(m_1) \overline{\omega(m_2)} \right] = \begin{cases} 1, & \text{if } m_1 = m_2, \\ 0, & \text{if } m_1 \neq m_2. \end{cases}$$

Here $\mathbf{E}^{(\mathbf{P})}$ *denotes the expectation with respect to* $\mathbf{P}$.

Proof. Let $m_1 \neq m_2$. Take $\underline{k} = \{k_m : m \in \mathbb{N}\}$ such that $k_{m_1} = 1, k_{m_2} = -1$ and the others are zero. We have $\sum_{m=1}^{\infty} \lambda_m k_m = \lambda_{m_1} - \lambda_{m_2} \neq 0$. Therefore

$$\mathbf{E}^{(\mathbf{P})} [\omega(m_1) \overline{\omega(m_2)}] = g(\underline{k}) = 0,$$

which completes the proof. □

The following theorem is similar to Theorem 1.7(v).

Theorem 1.10. *Let* $f \in B^q(\{\lambda_m\})$ *with the Fourier series*

$$f \sim \sum_m a_m e^{-i\lambda_m t}.$$

Then the sequence of probability measures

$$\nu_T(\tau : f(\tau) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of $f(\omega) \sim \sum_m a_m \omega(m) \in L^q(\Omega, \mathbf{P})$. *Here* $f(\omega) \sim \sum_m a_m \omega(m)$ *means that* f *belongs to the linear closure of* $\{\omega(m)\}_{m \in \mathbb{N}}$ *with respect to the* $L^q(\Omega, \mathbf{P})$ *norm and*

$$a_m = \int_\Omega f(\omega) \overline{\omega(m)} d\mathbf{P}(\omega).$$

We are now in a position to characterize the dual group of Λ and the support of $\mathbf{P}$. Recall that Λ is a subgroup of $\mathbb{R}_d$ generated by $\{\lambda_m\}$. Hence, the mapping

$$\varphi\colon \mathcal{Z} \to \Lambda$$
$$\underline{k} = (k_m) \mapsto -\sum_m \lambda_m k_m$$

is an onto group homomorphism. Thus by the first isomorphism theorem in abstract algebra,

$$\Lambda \cong \mathcal{Z}/\ker\varphi.$$

Here $\ker\varphi = \{\underline{k} \in \mathcal{Z} : \sum_m \lambda_m k_m = 0\}$ is a (closed) subgroup of $\mathcal{Z}$. Let

$$K = \{\omega \in \Omega : \omega^{\underline{k}} = 1 \text{ for all } \underline{k} \in \ker\varphi\}.$$

Then K is a closed subgroup of Ω, called the *annihilator* of $\ker\varphi$. The result on the duality between subgroups and quotient groups ([42, Theorem 2.1.2]) gives the following.

Theorem 1.11. *The dual group of Λ is isomorphic to the subgroup K of Ω.*

Since K is a closed, and hence a measurable subset of Ω, the normalized Haar measure ν on K can be regarded as the probability measure $\mathbf{P}'$ on Ω defined by

$$\mathbf{P}'(A) := \nu(A \cap K), \quad A \in \mathcal{B}(\Omega).$$

Let us now calculate the Fourier transform of $\mathbf{P}'$. First, by the duality, $\ker\varphi$ is the annihilator of K because K is the annihilator of $\ker\varphi$ ([42, Lemma 2.1.3]), or $\ker\varphi$ can be rewritten as

$$\ker\varphi = \{\underline{k} : \omega^{\underline{k}} = 1 \text{ for all } \omega \in K\}.$$

This implies that a character $\underline{k}$ of Ω, restricted on K, is the trivial character of K if and only if $\underline{k} \in \ker\varphi$. Thus we obtain the Fourier transform of $\mathbf{P}'$,

$$g^{(\mathbf{P}')}(\underline{k}) = \int_\Omega \omega^{\underline{k}} d\mathbf{P}' = \int_K \omega^{\underline{k}} d\nu = \begin{cases} 1, & \text{if } \underline{k} \in \ker\varphi, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\mathbf{P} \equiv \mathbf{P}'$ because $\mathbf{P}$ and $\mathbf{P}'$ have the same Fourier transform. Moreover, by the property of Haar measure, $\operatorname{supp}(\nu) = K$. Hence $\operatorname{supp}(\mathbf{P}) = K$. Combining all the above, we get the following.

Theorem 1.12. (i) $\operatorname{supp}(\mathbf{P}) = K$.
(ii) $\mathbf{P}|_K \equiv \nu$, *the normalized Haar measure on K.*

1.4 Convergence results

1.4.1 Classical Fourier series

Let $L^q([0,2\pi])$ be the space of q-th power Lebesgue integrable functions $f\colon [0,2\pi] \to \mathbb{C}$ endowed with the norm

$$\|f\|_{L^q([0,2\pi])} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^q dx\right)^{1/q}.$$

For $f \in L^q([0,2\pi])$, the Fourier coefficients a_n are defined by the formula

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}.$$

Then the series

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

is called the Fourier series of f. For $N \in \mathbb{N}$, let

$$S_N(f)(x) = \sum_{|n| \leqslant N} a_n e^{inx}.$$

Theorem 1.13 ([26, Theorem II.1.5]). *For $1 < q < \infty$, the partial sums $S_N(f)$ of any $f \in L^q([0, 2\pi])$ converge to f in $L^q([0, 2\pi])$ as $N \to \infty$.*

This theorem is equivalent to the following.

Theorem 1.14 ([26, Theorem II.1.1]). *For $1 < q < \infty$, there is a constant K_q such that for any $f \in L^q([0, 2\pi])$*

$$\|S_N(f)\|_{L^q([0,2\pi])} \leqslant K_q \|f\|_{L^q([0,2\pi])}.$$

Indeed, let $\mathbf{S}_N$ be a continuous linear operator on $L^q([0, 2\pi])$ defined as $f \mapsto S_N(f)$, whose operator norm is denoted by $\|\mathbf{S}_N\|_q$. Now if the sequence $\{S_N(f)\}$ converges to f, then $\{S_N(f)\}$ is bounded for every $f \in L^q([0, 2\pi])$. Therefore, $\{\|\mathbf{S}_N\|_q\}$ is bounded by the uniform boundedness principle. Conversely, assume that there is a constant K_q such that for any $f \in L^q([0, 2\pi])$,

$$\|S_N(f)\|_{L^q([0,2\pi])} \leqslant K_q \|f\|_{L^q([0,2\pi])}.$$

Let $f \in L^q([0, 2\pi])$. Given any $\varepsilon > 0$, there is a trigonometric polynomial $P(x) = \sum_k b_k e^{ikx}$ satisfying $\|f - P\|_{L^q([0,2\pi])} \leqslant \varepsilon/(K_q + 1)$. For N greater than the degree of P, we have $S_N(P) = P$, and thus

$$\begin{aligned}
\|S_N(f) - f\|_{L^q([0,2\pi])} &= \|S_N(f) - S_N(P) + P - f\|_{L^q([0,2\pi])} \\
&\leqslant \|S_N(f) - S_N(P)\|_{L^q([0,2\pi])} + \|P - f\|_{L^q([0,2\pi])} \\
&\leqslant K_q \frac{\varepsilon}{K_q + 1} + \frac{\varepsilon}{K_q + 1} = \varepsilon,
\end{aligned}$$

which completes the proof of the equivalence between the above two theorems.

Carleson [6] showed that the Fourier series of an $L^2([0, 2\pi])$ function converges almost everywhere. Later on Hunt [24] generalized this to $L^p([0, 2\pi]), (1 < p < \infty)$. This result is now known as Carleson's theorem or the Carleson-Hunt theorem.

Theorem 1.15 (The Carleson-Hunt theorem). *For $1 < q < \infty$, the partial sums $S_N(f)$ of any $f \in L^q([0, 2\pi])$ converge almost everywhere to f as $N \to \infty$.*

To prove this, we consider the maximal function

$$\mathbb{M}f(x) := \sup_{N \geqslant 0} |S_N(f)(x)|.$$

The almost everywhere convergence is a consequence of the following maximal inequality.

Theorem 1.16 ([6, 24]). *For $1 < q < \infty$, there is a constant C_q such that for any $f \in L^q([0, 2\pi])$, we have*

$$\|\mathbb{M}f\|_{L^q([0,2\pi])} \leqslant C_q \|f\|_{L^q([0,2\pi])}.$$

The multi-dimensional version of this maximal inequality was investigated by Fefferman [16]. For our purpose, we only mention a special case of Fefferman's result. Let $f \in L^q([0,2\pi]^d)$ with Fourier coefficients $\{a_{\underline{k}}\}_{\underline{k}\in\mathbb{Z}^d}$, where $d \in \mathbb{N}, d \geqslant 2$ being fixed. For $\underline{v} = (v_1, \dots, v_d) \in (\mathbb{R}^+)^d$, we consider the maximal function

$$\mathbb{M}^{(\underline{v})} f(\underline{x}) := \sup_{b>0} \left| \sum_{\substack{\underline{k}\in\mathbb{N}^{*d} \\ (\underline{k},\underline{v})<b}} a_{\underline{k}} e^{i(\underline{k},\underline{x})} \right|.$$

Here $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $(\underline{x}, \underline{y})$ denotes the inner product of $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^d$,

$$(\underline{x}, \underline{y}) = \sum_{j=1}^{d} x_j y_j, \quad \underline{x} = (x_1, \dots, x_d), \underline{y} = (y_1, \dots, y_d) \in \mathbb{R}^d.$$

Lemma 1.17 ([16, 22]). *For $1 < q < \infty$, for any $\underline{v} = (v_1, \dots, v_d) \in (\mathbb{R}^+)^d$ and any $f \in L^q([0,2\pi]^d)$,*

$$\|\mathbb{M}^{(\underline{v})} f\|_{L^q([0,2\pi]^d)} \leqslant C_q \|f\|_{L^q([0,2\pi]^d)},$$

where C_q is the constant in Theorem 1.16.

1.4.2 Convergence of Fourier series of Besicovitch almost periodic functions

Recall that $\mathrm{supp}(\mathbf{P}) = K$ is isomorphic to the dual group of Λ. K itself is a compact abelian group and its normalized Haar measure ν coincides with $\mathbf{P}|_{\mathrm{supp}(\mathbf{P})}$. A summable function on K has Fourier series

$$f(\omega) \sim \sum_{\lambda\in\Lambda} a(\lambda)\chi_\lambda(\omega).$$

For $q \geqslant 1$, let $H^q(K, \{\lambda_m\})$ (or $H^q(\Omega, \{\lambda_m\})$) be the subspace of $L^q(K,\nu) \equiv L^q(\Omega, \mathbf{P})$ consisting of those functions f whose Fourier coefficients $a(\lambda)$ are zero except for $\lambda \in \{-\lambda_1, -\lambda_2, \dots\}$. A function $f \in H^q(K, \{\lambda_m\})$ has Fourier series of the form

$$f(\omega) \sim \sum_{m=1}^{\infty} a_m \chi_{-\lambda_m} = \sum_{m=1}^{\infty} a_m \omega(m).$$

Next, we will establish the maximal inequality for functions in $H^q(\Omega, \{\lambda_m\})$. Let

$$\mathbb{M} f(\omega) := \sup_{M>0} \left| \sum_{m=1}^{M} a_m \omega(m) \right|.$$

Theorem 1.18. *For $1 < q < \infty$, and for any $f \in H^q(\Omega, \{\lambda_m\})$,*

$$\|\mathbb{M} f\|_{L^q(\Omega,\mathbf{P})} \leqslant C_q \|f\|_{L^q(\Omega,\mathbf{P})},$$

where C_q is the constant in Theorem 1.16.

We need some preliminary results before proving Theorem 1.18.

Lemma 1.19. *Let $M \in \mathbb{N}$ and let $\{\lambda_m\}_{1\leqslant m\leqslant M}$ be positive numbers. Then there are $\mathbb{Q}$-linearly independent positive numbers $\{\mu_p\}_{1\leqslant p\leqslant P}$ such that*

$$\{\lambda_m\}_{1\leqslant m\leqslant M} \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{N}^* \mu_p.$$

Proof. We prove by induction on M. Of course, there is nothing to do if $M = 1$. Assume that this lemma holds for some $M \geqslant 1$. We now prove it for $M+1$. Let $\{\lambda_m\}_{1\leqslant m\leqslant M+1}$ be positive numbers. By the induction hypothesis, there are $\mathbb{Q}$-linearly independent positive numbers $\{\mu_p\}_{1\leqslant p\leqslant P}$ such that

$$\{\lambda_m\}_{1\leqslant m\leqslant M} \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{N}^*\mu_p.$$

Case 1: λ_{M+1} is $\mathbb{Q}$-linearly independent with $\{\mu_p\}_{1\leqslant p\leqslant P}$. Simply set $\mu_{P+1} = \lambda_{M+1}$, we obtain the desired result

$$\{\lambda_m\}_{1\leqslant m\leqslant M+1} \subset \bigoplus_{1\leqslant p\leqslant P+1} \mathbb{N}^*\mu_p.$$

Case 2: λ_{M+1} is $\mathbb{Q}$-linearly dependent with $\{\mu_p\}_{1\leqslant p\leqslant P}$; namely there are rational numbers $\{k_p\}_{1\leqslant p\leqslant P}$ such that

$$\lambda_{M+1} = \sum_{p=1}^{P} k_p\mu_p.$$

Let $\mathcal{P}_1 = \{p : k_p > 0\}$ and $\mathcal{P}_2 = \{p : k_p < 0\}$. If $\mathcal{P}_2 = \emptyset$, then

$$\lambda_{M+1} \in \bigoplus_{1\leqslant p\leqslant P} \mathbb{Q}^*\mu_p,$$

where $\mathbb{Q}^* = \mathbb{Q}^+ \cup \{0\}$. Thus with a suitable number $N \in \mathbb{N}$, by letting $\mu'_p = (1/N)\mu_p, p = 1, \ldots, P$, we get

$$\lambda_{M+1} \in \bigoplus_{1\leqslant p\leqslant P} \mathbb{N}^*\mu'_p.$$

Clearly,

$$\{\lambda_m\}_{1\leqslant m\leqslant M} \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{N}^*\mu'_p.$$

Therefore this lemma holds for $M+1$ if $\mathcal{P}_2 = \emptyset$. Next we consider the case $\mathcal{P}_2 \neq \emptyset$. Let $p_2 \in \mathcal{P}_2$. It is enough to construct positive numbers $\{\mu'_p\}_{1\leqslant p\leqslant P}$ that

$$\begin{cases} \{\mu'_p\}_{1\leqslant p\leqslant P} \text{ is } \mathbb{Q}\text{-linearly independent}, \\ \{\lambda_m\}_{1\leqslant m\leqslant M} \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{Q}^*\mu'_p, \\ \lambda_{M+1} = \sum_{p=1}^{P} k'_p\mu'_p, \quad k'_p \in \mathbb{Q}, (p = 1, \ldots, P); \mathcal{P}'_2 = \{p : k'_p < 0\} = \mathcal{P}_2 \setminus \{p_2\}. \end{cases} \tag{1.9}$$

To construct $\{\mu'_p\}_{1\leqslant p\leqslant P}$, let $\{r_p\}_{p\in\mathcal{P}_1}$ be positive rational numbers satisfying

$$\begin{cases} \sum_{p\in\mathcal{P}_1} r_p = -k_{p_2}, \\ 0 < r_p < k_p\mu_p/\mu_{p_2}, \quad p \in \mathcal{P}_1. \end{cases}$$

The existence of $\{r_p\}_{p\in\mathcal{P}_1}$ is ensured because

$$0 < \lambda_{M+1} = \sum_{p\in\mathcal{P}_1} k_p\mu_p + \sum_{p\in\mathcal{P}_2} k_p\mu_p \leqslant \sum_{p\in\mathcal{P}_1} k_p\mu_p + k_{p_2}\mu_{p_2}.$$

Let

$$\mu'_p = \begin{cases} \mu_p - (r_p/k_p)\mu_{p_2}, & p \in \mathcal{P}_1, \\ \mu_p, & p \notin \mathcal{P}_1. \end{cases}$$

Clearly, $\{\mu'_p\}_{1\leqslant p\leqslant P}$ is $\mathbb{Q}$-linearly independent and

$$\{\mu_p\}_{1\leqslant p\leqslant P} \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{Q}^* \mu'_p.$$

Thus

$$\{\lambda_m\}_{1\leqslant m\leqslant M} \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{N}^* \mu_p \subset \bigoplus_{1\leqslant p\leqslant P} \mathbb{Q}^* \mu'_p.$$

Let us consider the representation of λ_{M+1} with respect to $\{\mu'_p\}_{1\leqslant p\leqslant P}$,

$$\begin{aligned}\lambda_{M+1} &= \sum_{p\in\mathcal{P}_1} k_p\mu_p + \sum_{p\in\mathcal{P}_2} k_p\mu_p = \sum_{p\in\mathcal{P}_1} k_p(\mu_p - (r_p/k_p)\mu_{p_2}) + \sum_{p\in\mathcal{P}_2\setminus\{p_2\}} k_p\mu_p \\ &= \sum_{p\in\mathcal{P}_1} k_p\mu'_p + \sum_{p\in\mathcal{P}_2\setminus\{p_2\}} k_p\mu'_p.\end{aligned}$$

Hence the sequence $\{\mu'_p\}_{1\leqslant p\leqslant P}$ satisfies the condition (1.9). The proof is complete. □

For each $M \in \mathbb{N}$, let $\Omega_M = \prod_{m=1}^M \gamma_m$ (with the product topology and pointwise multiplication). Ω_M is also a compact topological abelian group. The projection pr_M from Ω onto Ω_M is continuous. Let $\mathbf{P}_M := \mathbf{P} \circ pr_M^{-1}$. We can check that

$$g_M(\underline{k}^{(M)}) = \begin{cases} 1, & \text{if } \sum_{m=1}^M \lambda_m k_m = 0, \\ 0, & \text{if } \sum_{m=1}^M \lambda_m k_m \neq 0, \end{cases} \quad \underline{k}^{(M)} = (k_1, \ldots, k_M) \in \bigoplus_{1\leqslant m\leqslant M} \mathbb{Z}_m,$$

is the Fourier transform of $\mathbf{P}_M$.

For the sequence $\{\lambda_m\}_{1\leqslant m\leqslant M}$, let $\{\mu_p\}_{1\leqslant p\leqslant P}$ be positive numbers as in Lemma 1.19 and let $\{k_{mp}\} \subset \mathbb{N}^*$ be the coordinates of $\{\lambda_m\}_{1\leqslant m\leqslant M}$ with respect to $\{\mu_p\}_{1\leqslant p\leqslant P}$, that is,

$$\lambda_m = \sum_{p=1}^P k_{mp}\mu_p, \quad (m = 1, \ldots, M).$$

Let $\Omega'_P = \prod_{p=1}^P \gamma_p$ (with the product topology, pointwise multiplication) and let m'_P be the normalized Haar measure on Ω'_P. Define $j_M : \Omega'_P \to \Omega_M$ as

$$(\omega'(p))_{1\leqslant p\leqslant P} \mapsto (\omega(m))_{1\leqslant m\leqslant M}, \quad \omega(m) = \prod_{p=1}^P \omega'(p)^{k_{mp}}.$$

Then the mapping j_M is also continuous. The following lemma easily follows from the calculation of the Fourier transform and from the continuity of j_M.

Lemma 1.20. (i) $\mathbf{P}_M = m'_P \circ j_M^{-1}$.
(ii) $\operatorname{supp}(\mathbf{P}_M) = j_M(\Omega'_P)$.

Proof of Theorem 1.18. *Step 1.* Assume that only a finite number of $\{a_m\}$ are non-zero, that is, an $M \in \mathbb{N}$ exists such that $a_m = 0$ if $m > M$. Since the maximal function in this case only depends on the first M coordinates, in what follows we identify Ω_M with Ω. For each $\underline{k} = (k_1, \ldots, k_P) \in \mathbb{N}^{*P}$, let

$$a_{\underline{k}} = \begin{cases} a_m, & \text{if } \sum_{p=1}^P k_p\mu_p = \lambda_m \text{ for some } m, (1 \leqslant m \leqslant M), \\ 0, & \text{otherwise.} \end{cases}$$

Then when $\omega = j_M(\omega')$, we have that for any $L \in \mathbb{N}$,

$$\sum_{m<L} a_m \omega(m) = \sum_{\substack{\underline{k}\in\mathbb{N}^{*P} \\ (\underline{k},\underline{\mu})<\lambda_L}} a_{\underline{k}} \omega'^{\underline{k}},$$

where $\underline{\mu} = (\mu_1, \ldots, \mu_P)$. Therefore

$$\mathbb{M}f(j_M(\omega')) = \sup_{b>0} \left| \sum_{\substack{\underline{k}\in\mathbb{N}^{*P} \\ (\underline{k},\underline{\mu})<b}} a_{\underline{k}} \omega'^{\underline{k}} \right|.$$

It follows from the multi-dimensional maximal inequality (Lemma 1.17) that

$$\left\|\mathbb{M}f(j_M(\omega'))\right\|_{L^q(\Omega'_P, m'_P)} \leqslant C_q \|f(j_M(\omega'))\|_{L^q(\Omega'_P, m'_P)}.$$

Moreover, by Lemma 1.20, we have

$$\|\mathbb{M}f(\omega)\|_{L^q(\Omega_M, \mathbf{P}_M)} = \left\|\mathbb{M}f(j_M(\omega'))\right\|_{L^q(\Omega'_P, m'_P)},$$
$$\|f(\omega)\|_{L^q(\Omega_M, \mathbf{P}_M)} = \|f(j_M(\omega'))\|_{L^q(\Omega'_P, m'_P)}.$$

Therefore

$$\|\mathbb{M}f(\omega)\|_{L^q(\Omega_M, \mathbf{P}_M)} \leqslant C_q \|f(\omega)\|_{L^q(\Omega_M, \mathbf{P}_M)}.$$

Step 2. We now prove the maximal inequality in general case. For each $M \in \mathbb{N}$, let

$$\mathbb{M}^{(M)} f(\omega) := \sup_{0<b\leqslant M} \left| \sum_{m<b} a_m \omega(m) \right|.$$

Then $\mathbb{M}^{(M)} f(\omega)$ increasingly converges to $\mathbb{M}f(\omega)$ as $M \to \infty$. Consequently,

$$\|\mathbb{M}f\|_{L^q(\Omega, \mathbf{P})} = \lim_{M\to\infty} \|\mathbb{M}^{(M)} f\|_{L^q(\Omega, \mathbf{P})}.$$

Applying Step 1 to a function $g(\omega)$ of the form

$$g(\omega) = \sum_{m=1}^{M} a_m \omega(m) + \sum_{m=M+1}^{M'} b_m \omega(m),$$

M' and $\{b_m\}$ being arbitrary, we obtain

$$\|\mathbb{M}g(\omega)\|_{L^q(\Omega, \mathbf{P})} \leqslant C_q \|g(\omega)\|_{L^q(\Omega, \mathbf{P})}.$$

Moreover, for $g(\omega)$ of that form, we have $\mathbb{M}^{(M)} f(\omega) \leqslant \mathbb{M}g(\omega)$, which implies that

$$\|\mathbb{M}^{(M)} f(\omega)\|_{L^q(\Omega, \mathbf{P})} \leqslant C_q \|g(\omega)\|_{L^q(\Omega, \mathbf{P})}.$$

By choosing a sequence of $g(\omega)$ converging to $f(\omega)$ in $L^q(\Omega, \mathbf{P})$, we arrive at

$$\|\mathbb{M}^{(M)} f(\omega)\|_{L^q(\Omega, \mathbf{P})} \leqslant C_q \|f(\omega)\|_{L^q(\Omega, \mathbf{P})}.$$

The proof is complete by letting $M \to \infty$. $\square$

Theorem 1.21. *For $1 < q < \infty$, we have the following.*
(i) *For any $f(t) \sim \sum a_m e^{-i\lambda_m t} \in B^q(\{\lambda_m\})$, the partial sums*

$$S_M(f)(t) = \sum_{m=1}^{M} a_m e^{-i\lambda_m t}$$

converge to f in the $\|\cdot\|_q$ norm as $M \to \infty$.
(ii) *Equivalently, for any $f(\omega) \sim \sum a_m \omega(m) \in H^q(\Omega, \{\lambda_m\})$, the partial sums*

$$S_M(f)(\omega) = \sum_{m=1}^{M} a_m \omega(m)$$

converge to f in $L^q(\Omega, \mathbf{P})$ as $M \to \infty$.

Theorem 1.21 is a consequence of the maximal inequality (Theorem 1.18) and the following result whose proof is similar to the proof of the equivalence between Theorem 1.13 and Theorem 1.14.

Lemma 1.22. *The following two conditions are equivalent:*
(i) *for any $f(\omega) \sim \sum a_m \omega(m) \in H^q(\Omega, \{\lambda_m\})$,*

$$S_M(f)(\omega) = \sum_{m=1}^{M} a_m \omega(m) \xrightarrow{L^q} f \quad \text{as} \quad M \to \infty;$$

(ii) *there is a constant $K_q > 0$ such that for any $f \in H^q(\Omega, \{\lambda_m\})$,*

$$\|S_M(f)\|_{L^q(\Omega,\mathbf{P})} \leqslant K_q \|f\|_{L^q(\Omega,\mathbf{P})}.$$

Theorem 1.23. *For $1 < q < \infty$, and for any $f(\omega) \sim \sum a_m \omega(m) \in H^q(\Omega, \{\lambda_m\})$, the partial sums $S_M(f)(\omega)$ converge* $\mathbf{P}$-a.e. to f as $M \to \infty$.

Proof. By applying Theorem 1.18 to the function $f - S_M(f)$, we obtain

$$\left\| \sup_{m \geqslant M} |S_m(f) - S_M(f)| \right\|_{L^q(\Omega,\mathbf{P})} \leqslant C_q \|f - S_M(f)\|_{L^q(\Omega,\mathbf{P})}.$$

Thus $\sup_{m \geqslant M} |S_m(f) - S_M(f)|$ converges to 0 in $L^q(\Omega, \mathbf{P})$ as $M \to \infty$. It follows that there is a subsequence $\{M_k\} \subset \mathbb{N}$ such that

$$\sup_{m \geqslant M_k} |S_m(f)(\omega) - S_{M_k}(f)(\omega)| \to 0 \quad \text{as} \quad k \to \infty \quad \text{for } \mathbf{P}\text{-a.e. } \omega \in \Omega.$$

Hence $\{S_M(f)(\omega)\}_M$ is a Cauchy sequence for $\mathbf{P}$-a.e. $\omega \in \Omega$. Consequently, the limit

$$f'(\omega) = \lim_{M \to \infty} S_M(f)(\omega)$$

exists for $\mathbf{P}$-a.e. $\omega \in \Omega$. On the other hand, the function f is the L^q-limit of $\{S_M(f)\}$ (Theorem 1.21). Therefore $f = f'$ for $\mathbf{P}$-a.e. $\omega \in \Omega$, which completes the proof of Theorem 1.23. $\square$

1.4.3 The linearly independent case

In this section, we consider the case when $\{\lambda_m\}$ is $\mathbb{Q}$-linearly independent and give another proofs of Theorem 1.18, Theorem 1.21 and Theorem 1.23. Assume that $\{\lambda_m\}$ is $\mathbb{Q}$-linearly independent. Then the probability measure $\mathbf{P}$ coincides with the normalized Haar measure on Ω. Under $\mathbf{P}$, the sequence $\{\omega(m)\}$ becomes independent.

Proof of Theorem 1.21 *and Theorem* 1.23. Let

$$f(\omega) \sim \sum_{m=1}^{\infty} a_m\omega(m) \in H^q(\Omega, \{\lambda_m\}).$$

Then $\{S_M\}_{M\in\mathbb{N}}$ is a martingale with respect to the filtration

$$\mathcal{F}_M = \sigma(\omega(1), \ldots, \omega(M)),$$

because $\{\omega(m)\}$ is a sequence of independent random variables with means zero, where $S_M := S_M(f)$. On the other hand, fixing M, we consider $\{Y_L\}_{L\geqslant M}$ of the form

$$Y_L = \sum_{m=1}^{M} a_m\omega(m) + \sum_{M<m\leqslant L} b_m\omega(m), \tag{1.10}$$

where $\{b_m\}$ is an arbitrary sequence of complex numbers. Then $\{Y_L\}_{L\geqslant M}$ is also a martingale, and hence, $\{|Y_L|^q\}_{L\geqslant M}$ is a submartingale. Consequently,

$$\mathbf{E}^{(\mathbf{P})}[|S_M|^q] = \mathbf{E}^{(\mathbf{P})}[|Y_M|^q] \leqslant \mathbf{E}^{(\mathbf{P})}[|Y_L|^q].$$

Since there is a sequence of $\{Y_L\}$ of the above form which converges to f in $L^q(\Omega, \mathbf{P})$, it follows that

$$\mathbf{E}^{(\mathbf{P})}[|S_M|^q] \leqslant \mathbf{E}^{(\mathbf{P})}[|f|^q] < \infty. \tag{1.11}$$

Thus, if $q > 1$, then by Doob's martingale convergence theorem (see Theorem 0.40(iii)), the sequence $\{S_M\}$ converges $\mathbf{P}$-a.e. and converges in $L^q(\Omega, \mathbf{P})$ to a random variable S_∞. It is clear that

$$S_\infty(\omega) \sim \sum_{m=1}^{\infty} a_m\omega(m)$$

is the Fourier series of S_∞. This implies that $S_\infty = f, \mathbf{P}$-a.e., which complete the proofs of Theorem 1.21 and Theorem 1.23. □

Proof of Theorem 1.18. Let

$$S_M^* := \max_{1\leqslant m\leqslant M} |S_m|.$$

Then by L^q maximum inequality (see Theorem 0.38), we obtain

$$\mathbf{E}^{(\mathbf{P})}[|S_M^*|^q] \leqslant \left(\frac{q}{q-1}\right)^q \mathbf{E}^{(\mathbf{P})}[|S_M|^q] \leqslant \left(\frac{q}{q-1}\right)^q \mathbf{E}^{(\mathbf{P})}[|f|^q] < \infty,$$

where the second inequality follows from (1.11). Note that $S_M^*(\omega)$ increasingly converges to $\mathbb{M}f(\omega)$. Consequently

$$\mathbf{E}^{(\mathbf{P})}[|\mathbb{M}f|^q] = \lim_{M\to\infty} \mathbf{E}^{(\mathbf{P})}[|X_M^*|^q] \leqslant \left(\frac{q}{q-1}\right)^q \mathbf{E}^{(\mathbf{P})}[|f|^q]. \qquad \square$$

Theorem 1.24. *Let $f \in H^1(\Omega, \{\lambda_m\})$ with the Fourier series $f(\omega) \sim \sum_{m=1}^{\infty} a_m \omega(m)$. Then the partial sums $\{S_M(f)(\omega)\}$ converge $\mathbf{P}$-a.e. and converge in $L^1(\Omega, \mathbf{P})$ to f as $M \to \infty$.*

Proof. Similarly as in the proof of Theorem 1.21 and Theorem 1.23, we have

$$\mathbf{E}^{(\mathbf{P})}[|S_M|] \leqslant \mathbf{E}^{(\mathbf{P})}[|f|] < \infty.$$

Moreover, taking the sequence $\{Y_L\}$ of the form (1.10) converging to f in $L^1(\Omega, \mathbf{P})$, then the sequence $\{\mathbf{E}^{(\mathbf{P})}(Y_L|\mathcal{F}_M)\}$ also converges in $L^1(\Omega, \mathbf{P})$ to $\mathbf{E}^{(\mathbf{P})}(f|\mathcal{F}_M)$. Consequently,

$$\mathbf{E}^{(\mathbf{P})}(f|\mathcal{F}_M) = S_M.$$

This implies that the sequence $\{S_M\}$ is uniformly integrable. Therefore, by Doob's martingale convergence theorem (see Theorem 0.40(ii)), $\{S_M\}$ converges to a limit S_∞ almost everywhere and converges in L^1. It then follows that the Fourier series of S_∞ coincides with that of f, and hence $S_\infty = f, \mathbf{P}$-a.e. $\square$

Chapter 2

Value Distributions of General Dirichlet Series

2.1 Introduction

A general Dirichlet series is a series of the form

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad s = \sigma + it \in \mathbb{C}, \tag{2.1}$$

where $a_m \in \mathbb{C}$, and $\{\lambda_m\}$ is a Dirichlet sequence,

$$0 \leqslant \lambda_1 < \lambda_2 < \cdots; \quad \lambda_m \to \infty.$$

Suppose that the series (2.1) converges absolutely for $\sigma > \sigma_a$ and has the sum $f(s)$. Then $f(s)$ is an analytic function in the half-plane $D := \{s \in \mathbb{C} : \sigma > \sigma_a\}$.

Limit theorems for general Dirichlet series on the complex plane, in the space of analytic functions as well as in the space of meromorphic functions have been studied relatively completely through papers [18, 19, 31, 32, 33, 34, 35]. Let us mention here the most recent results. For $T > 0$, denote by ν_T the uniform probability measure on $[0, T]$. Let $H(D)$ be the space of analytic functions on D equipped with the topology of uniform convergence on compacta. Then the limit theorem for the absolutely convergent general Dirichlet series in the space of analytic functions was proved in [31].

Theorem 2.1. *There exists a probability measure P on $(H(D), \mathcal{B}(H(D)))$ such that the sequence of probability measures*

$$\nu_T(\tau : f(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P as $T \to \infty$.

Suppose that $f(s)$ is meromorphically continuable to a wider half-plane $D_0 := \{s \in \mathbb{C} : \sigma > \sigma_0\}, \sigma_0 < \sigma_a$. Moreover, we require that all poles of $f(s)$ in D_0 are included in a compact set and that the following two conditions are satisfied.

(i) $f(s)$ is of finite order in any half-plane $\sigma \geqslant \sigma_1 (\sigma_1 > \sigma_0)$, that is, there exist constants $a > 0$ and $t_0 \geqslant 0$ such that the estimate

$$f(\sigma + it) = O(|t|^a), \quad |t| \geqslant t_0, \tag{2.2}$$

holds uniformly for $\sigma \geqslant \sigma_1$.

(ii) For $\sigma > \sigma_0$ such that $\{\sigma + it : t \in \mathbb{R}\}$ does not contain any pole of $f(s)$,

$$\int_{-T}^{T} |f(\sigma + it)|^2 dt = O(T), \quad T \to \infty. \tag{2.3}$$

Let $\mathbb{C}_\infty$ be the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and d be the sphere metric on $\mathbb{C}_\infty$ defined by

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \ d(s, \infty) = \frac{2}{\sqrt{1+|s|^2}}, \ d(\infty, \infty) = 0, (s, s_1, s_2 \in \mathbb{C}).$$

This metric is compatible with the topology of $\mathbb{C}_\infty$. Let $M(D_0)$ denote the space of meromorphic functions $g\colon D_0 \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. Then the limit theorem in the space of meromorphic functions was obtained in [32].

Theorem 2.2. *Suppose that conditions* (2.2) *and* (2.3) *are satisfied. Then there exists a probability measure P on $(M(D_0), \mathcal{B}(M(D_0)))$ such that the sequence of probability measures*

$$\nu_T(\tau : f(s + i\tau) \in A), \quad A \in \mathcal{B}(M(D_0)),$$

converges weakly to P as $T \to \infty$.

The limit theorem on the complex plane was obtained in [33].

Theorem 2.3. *Suppose that conditions* (2.2) *and* (2.3) *are satisfied. Then for each $\sigma > \sigma_0$, there exists a probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the sequence of probability measures*

$$\nu_T(\tau : f(\sigma + i\tau) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$. In other words, for each $\sigma > \sigma_0$, the limit distribution of $f(\sigma + it)$ exists.

To identify the limit probability measures in the above three theorems, some additional conditions are necessary. Suppose that the sequence of exponents $\{\lambda_m\}$ is $\mathbb{Q}$-linearly independent. Let $\Omega = \prod_{m=1}^{\infty} \gamma_m$ be the infinite-dimensional torus as defined in Section 1.3 and m_H be the normalized Haar measure on Ω. Assume further that, for $\sigma > \sigma_0$,

$$\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} (\log m)^2 < \infty. \tag{2.4}$$

Then it was proved in [34] that for $\sigma > \sigma_0$, the series

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m \sigma}$$

converges almost everywhere, and hence is a complex-valued random variable on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Moreover, the limit probability measure P_σ in Theorem 2.3 coincides with the distribution of the random variable $f(\sigma, \omega)$. In addition, under conditions (2.2)–(2.4), it was proved in [19] that $f(s, \omega)$ defined by

$$f(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m s}$$

is an $H(D_0)$-valued random element and the limit probability measure in Theorem 2.2 coincides with the distribution of $f(s, \omega)$.

This chapter is devoted to identify the limit probability measures without assumption of linear independence of $\{\lambda_m\}$. Under conditions (2.2) and (2.3), we will show that for fixed $\sigma > \sigma_0$, $f(\sigma + it)$ is a B^2-almost periodic function with the Fourier series

$$f(\sigma + it) \sim \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} e^{-i\lambda_m t}.$$

Therefore, the limit distribution of $f(\sigma + it)$ exists and coincides with the distribution of an $L^2(\Omega, \mathbf{P})$ function

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} \omega(m).$$

Here $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$ is the probability space developed in Section 1.3. Moreover, as we proved in Chapter 1, the series $f(\sigma, \omega)$ converges $\mathbf{P}$-almost everywhere without any further assumption. Consequently,

$$f(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m s}$$

is a well-defined $H(D_0)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$, and its distribution coincides with the limit probability measure in Theorem 2.2.

2.2 General theory

The main aim of this section is to approximate the function $f(s)$ by a sequence of absolutely convergent Dirichlet series. If the function $f(s)$ is analytic in D_0, we can find this kind of result in [35, 18]. We begin with a result on the mean value of absolutely convergent Dirichlet series.

Theorem 2.4 (cf. [45, §9.5]). *For any $\sigma_1 > \sigma_a$, uniformly in $\sigma \geqslant \sigma_1$, we have*

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt = \sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma}.$$

The following formula is known as Perron's formula . We will use the Dirichlet series defined in that formula to approximate the function $f(s)$.

Lemma 2.5 (cf. [45, §9.43]). *For $\delta > 0, \lambda > 0$, and $c > 0, c > \sigma_a - \sigma$, we have*

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s} e^{-(e^{\lambda_m} \delta)^{\lambda}} = \frac{1}{2\pi i \lambda} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{w}{\lambda}\right) f(s+w) \delta^{-w} dw, \tag{2.5}$$

where Γ denotes the Gamma function.

Let

$$g_{\lambda,\delta}(s) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} e^{-(e^{\lambda_m} \delta)^{\lambda}}, \quad (\lambda > 0, \delta > 0).$$

It is clear that the Dirichlet series $g_{\lambda,\delta}(s)$ is absolutely convergent for any $s \in \mathbb{C}$. The sequence $\{g_{\lambda,\delta}(s)\}_\delta$ approximates the function $f(s)$ in the following sense.

Corollary 2.6. *Let K be a compact subset in D. Then for fixed $\lambda > 0$,*

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt = 0.$$

Proof. Let L be a simple closed contour lying in D and enclosing the set K and let δ_K denote the distance of L from the set K. It follows from Cauchy's integral formula that

$$\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)| \leqslant \frac{1}{2\pi\delta_K}\int_L |f(z+it)-g_{\lambda,\delta}(z+it)||dz|,$$

then by the Cauchy-Schwarz inequality,

$$\left(\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|\right)^2 \leqslant \frac{|L|}{(2\pi\delta_K)^2}\int_L |f(z+it)-g_{\lambda,\delta}(z+it)|^2|dz|.$$

Here $|L|$ denotes the length of the contour L. Thus when $T > \max_{z\in L}|\operatorname{Im} z|$,

$$\begin{aligned}
&\left(\frac{1}{T}\int_0^T \sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt\right)^2\\
&\leqslant \frac{1}{T}\int_0^T\left(\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|\right)^2 dt\\
&\leqslant \frac{1}{T}\int_0^T\left(\frac{|L|}{(2\pi\delta_K)^2}\int_L |f(z+it)-g_{\lambda,\delta}(z+it)|^2|dz|\right)dt\\
&= \frac{|L|}{(2\pi\delta_K)^2}\int_L\left(\frac{1}{T}\int_0^T |f(z+it)-g_{\lambda,\delta}(z+it)|^2dt\right)|dz|\\
&\leqslant \frac{|L|}{(2\pi\delta_K)^2}\int_L\left(\frac{1}{T}\int_{-2T}^{2T} |f(\operatorname{Re} z+it)-g_{\lambda,\delta}(\operatorname{Re} z+it)|^2dt\right)|dz|\\
&\leqslant \frac{4|L|^2}{(2\pi\delta_K)^2}\sup_{\sigma\geqslant\sigma_1}\frac{1}{4T}\int_{-2T}^{2T}|f(\sigma+it)-g_{\lambda,\delta}(\sigma+it)|^2dt,
\end{aligned}$$

where $\sigma_1 = \min_{z\in L}\operatorname{Re} z > \sigma_a$. Now, uniformly in $\sigma \geqslant \sigma_1$,

$$\lim_{T\to\infty}\frac{1}{4T}\int_{-2T}^{2T}|f(\sigma+it)-g_{\lambda,\delta}(\sigma+it)|^2 = \sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma}(1-e^{-(e^{\lambda_m}\delta)^{\lambda}})^2,$$

by applying Theorem 2.4 to the function $f(s)-g_{\lambda,\delta}(s)$. Therefore,

$$\begin{aligned}
&\limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T \sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt\right)^2\\
&\leqslant \frac{4|L|^2}{(2\pi\delta_K)^2}\sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma_1}(1-e^{-(e^{\lambda_m}\delta)^{\lambda}})^2.
\end{aligned}$$

The above series is dominated by $\sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma_1} < \infty$ for any $\delta > 0$, and each term converges to 0 as $\delta \to 0$. Thus by the dominated convergence theorem, we arrive at

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt = 0.$$

The proof is complete. $\square$

If there is no pole in D_0 or $f(s)$ is analytic in D_0, we have the following version of Theorem 2.4.

Theorem 2.7 (cf. [45, §9.51]). *Let $f(s)$ denote the analytic continuation of the function $f(s), \sigma > \sigma_a$ to the half-plane $\sigma \geqslant \alpha$. Assume that $f(s)$ is regular and of finite order for $\sigma \geqslant \alpha$, and that*

$$\int_{-T}^{T} |f(\alpha+it)|^2 dt = O(T), \quad T \to \infty. \tag{2.6}$$

Then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 dt = \sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma}, \tag{2.7}$$

for $\sigma > \alpha$, and uniformly in any strip $\alpha < \sigma_1 \leqslant \sigma \leqslant \sigma_2$.

Consequently, if $f(s)$ is analytic in D_0, the statement of Corollary 2.6 is still true for any compact subset K of D_0. We are now in a position to extend Corollary 2.6 to our considering case in which all poles of $f(s)$ in D_0 are included in a compact set. It then follows that the number of poles are finite. The poles and their orders are denoted by $s_1, \dots, s_r$ and $n_1, \dots, n_r$, respectively.

Proposition 2.8. *Let K be a compact subset in D_0. Then for fixed $\lambda > \sigma_a - \sigma_0 + 1$,*

$$\lim_{\delta\to 0} \limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \sup_{s\in K} |f(s+it) - g_{\lambda,\delta}(s+it)| dt = 0,$$

where t_0 is a positive real number satisfying

$$\min_{s\in K}\{\operatorname{Im} s\} + t_0 > \max\{\operatorname{Im} s_1, \dots, \operatorname{Im} s_r\}.$$

Proof. From Corollary 2.6, we can assume without loss of generality that the compact subset K is included in the strip $\sigma_0 < \sigma < \sigma_a + 1$. Let L be a simple closed contour lying in the strip $\sigma_0 < \sigma < \sigma_a + 1$, enclosing the set K and

$$\min_{s\in L}\{\operatorname{Im} s\} + t_0 > \max\{\operatorname{Im} s_1, \dots, \operatorname{Im} s_r\}.$$

Then L lies in the strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, where

$$\sigma_1 = \min_{s\in L} \operatorname{Re} s > \sigma_0, \quad \sigma_2 = \max_{s\in L} \operatorname{Re} s < \sigma_a + 1.$$

Choose $\alpha \in (\sigma_0, \sigma_1)$ such that all poles $s_1, \dots, s_r$ lie in the half-plane $\sigma > \alpha$.

For $s = \sigma + it$ with $\sigma \in [\sigma_1, \sigma_2]$ and $s \notin \{s_1, \dots, s_r\}$, by moving c in the formula (2.5) to $c = \alpha - \sigma$, we pass a pole at $w = 0$, with residue $\lambda f(s)$, poles at $w = s_1 - s, \dots, w = s_r - s$. Since $\lambda > \sigma - \alpha$, no other pole is passed. Therefore, by the residue theorem, we obtain

$$\begin{aligned} g_{\lambda,\delta}(s) - f(s) &= \frac{1}{2\pi i\lambda} \int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty} \Gamma(\frac{w}{\lambda}) f(s+w) \delta^{-w} dw \\ &\quad + \frac{1}{\lambda} \sum_{j=1}^{r} \operatorname{Res}\left(\Gamma(\frac{w}{\lambda}) f(s+w)\delta^{-w}, s_j - s\right) \\ &=: I(s) + J(s). \end{aligned} \tag{2.8}$$

Observe that

$$\operatorname{Res}\left(\Gamma(\frac{w}{\lambda}) f(s+w)\delta^{-w}, s_j - s\right) = \sum_{k=0}^{n_j-1} \frac{a(k,s_j)}{k!} \left(\Gamma(\frac{w}{\lambda})\delta^{-w}\right)^{(k)} \Big|_{s_j-s},$$

where $^{(k)}$ denotes the k-th derivative with respect to w and

$$a(k, s_j) = \frac{1}{(n_j - k - 1)!} \lim_{w \to s_j - s} \left[\left(f(s+w)(w - (s_j - s))^{n_j} \right)^{(n_j - k - 1)} (w) \right].$$

Thus, for fixed $\delta > 0$,

$$J(s) = O\left(\sum_{k=0}^{n} \left| \Gamma^{(k)}\left(\frac{s_j - s}{\lambda}\right) \right| \right), \quad n = \max\{n_1, \dots, n_r\} - 1. \tag{2.9}$$

Now, an argument similar to the one used in the proof of Corollary 2.6 shows that

$$\begin{aligned} &\left(\frac{1}{T} \int_{t_0}^{T} \sup_{s \in K} |f(s+it) - g_{\lambda,\delta}(s+it)| dt \right)^2 \\ &= O\left(\sup_{\sigma \in [\sigma_1, \sigma_2]} \frac{1}{4T} \int_{t_1}^{2T} |f(\sigma+it) - g_{\lambda,\delta}(\sigma+it)|^2 dt \right), \end{aligned}$$

where $t_1 = \min_{s \in L}\{\operatorname{Im} s\} + t_0 > \max\{\operatorname{Im} s_1, \dots, \operatorname{Im} s_r\}$. For $\sigma \in [\sigma_1, \sigma_2]$ and $t \geqslant t_1$, the point $s = \sigma + it$ does not belong to the set $\{s_1, \dots, s_r\}$, thus the relation (2.8) implies

$$|g_{\lambda,\delta}(\sigma+it) - f(\sigma+it)|^2 \leqslant 2\Big(|I(\sigma+it)|^2 + |J(\sigma+it)|^2 \Big).$$

Note that in the proof of Theorem 2.7 (see [45, §9.51]), we have the following

$$\frac{1}{2T} \int_{-T}^{T} |I(\sigma+it)|^2 dt = O(\delta^{2\sigma - 2\alpha})$$

uniformly with respect to T and $\sigma \in [\sigma_1, \sigma_2]$. It follows that

$$\begin{aligned} &\lim_{\delta \to 0} \limsup_{T \to \infty} \sup_{\sigma \in [\sigma_1, \sigma_2]} \frac{1}{4T} \int_{t_1}^{2T} |I(\sigma+it)|^2 dt \\ &\leqslant \lim_{\delta \to 0} \limsup_{T \to \infty} \sup_{\sigma \in [\sigma_1, \sigma_2]} \frac{1}{4T} \int_{-2T}^{2T} |I(\sigma+it)|^2 dt = \lim_{\delta \to 0} O(\delta^{2\sigma_1 - 2\alpha}) = 0. \end{aligned}$$

On the other hand, by Stirling's formula, there is a constant $A > 0$ such that uniformly in the strip $\sigma' \leqslant \sigma \leqslant \sigma''$, we have

$$|\Gamma(\sigma+it)| = O(e^{-A|t|}), \quad t \to \infty,$$

where $\sigma' < \min_j\{\operatorname{Re}((s_j - \sigma_2)/\lambda)\}$ and $\sigma'' > \max_j\{\operatorname{Re}((s_j - \sigma_1)/\lambda)\}$ being chosen beforehand. This, together with (2.9), implies that

$$\limsup_{T \to \infty} \sup_{\sigma \in [\sigma_1, \sigma_2]} \frac{1}{4T} \int_{t_1}^{2T} |J(\sigma+it)|^2 dt = O\left(\limsup_{T \to \infty} \frac{1}{4T} \int_{t_1}^{2T} e^{-At} dt \right) = 0.$$

The proof is complete by combining the above two estimates. □

As a consequence of Proposition 2.8, we have the following.

Proposition 2.9. *Let K be a compact subset in D_0. Then for fixed $\lambda > \sigma_a - \sigma_0 + 1$*

$$\lim_{\delta \to 0} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K} d(f(s+it), g_{\lambda,\delta}(s+it)) dt = 0.$$

2.3 Limit theorems for general Dirichlet series on the complex plane

Let $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$ be the probability space defined in Section 1.3. We consider $B^q_{t_0}$, $t_0 \geqslant 0$ being fixed, the space of one-sided Besicovitch almost periodic functions, that is, functions $f\colon [0,\infty) \to \mathbb{C} \cup \{\infty\}$ which belong to the linear closure of $\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$ with respect to the one-sided Besicovitch q-norm $(1 \leqslant q < \infty)$,

$$\|f\|_{q,t_0} := \limsup_{T\to\infty} \left(\frac{1}{T} \int_{t_0}^{T} |f(t)|^q dt \right)^{1/q}.$$

Then the following results for one-side Besicovitch almost periodic functions are similar to those of Besicovitch almost periodic functions. They are taken from Theorem 1.8, Lemma 1.9, Theorem 2.10 and Theorem 1.23. Note that in this chapter ν_T denotes the uniform probability measure on $[0,T]$ while in the previous chapter, ν_T denotes the uniform probability measure on $[-T,T]$.

Theorem 2.10. (i) *The sequence of probability measures $\{Q_T\}$ converges weakly to $\mathbf{P}$ as $T \to \infty$, where*

$$Q_T(A) := \nu_T\big(\tau : (e^{-i\tau\lambda_m})_{m\in\mathbb{N}} \in A\big), \quad A \in \mathcal{B}(\Omega).$$

(ii) *$\{\omega(m)\}_{m\in\mathbb{N}}$ is an orthonormal system in $L^2(\Omega, \mathbf{P})$.*

(iii) *Let $f\colon [0,\infty) \to \mathbb{C}$ be a one-sided $B^2_{t_0}$-a.p. function with Fouries series of the form*

$$f(t) \sim \sum_{m=1}^{\infty} a_m e^{-i\lambda_m t}.$$

Then

$$f(\omega) = \sum_{m=1}^{\infty} a_m \omega(m)$$

converges for $\mathbf{P}$-a.e. $\omega \in \Omega$ and converges in $L^2(\Omega, \mathbf{P})$. Moreover, the sequence of probability measures

$$\nu_T(\tau : f(\tau) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of $f(\omega)$.

Lemma 2.11. *For $\sigma > \sigma_0$, we have $f(\sigma + it) \in B^2_{t_0}$ with the Fourier series*

$$f(\sigma + it) \sim \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} e^{-i\lambda_m t},$$

where $t_0 \geqslant 0$ is a number such that $\{\sigma + it : t \geqslant t_0\}$ does not contain any pole of $f(s)$. In particular,

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} |f(\sigma + it)|^2 dt = \sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} < \infty. \tag{2.10}$$

Proof. Fix $\lambda > \sigma_a - \sigma_0 + 1$. For each $n \in \mathbb{N}$, we define

$$g_n(s) := g_{\lambda, e^{-\lambda_n}}(s) = \sum_{m=1}^{\infty} a_m \exp\{-e^{(\lambda_m - \lambda_n)\lambda}\} e^{-\lambda_m s} = \sum_{m=1}^{\infty} a_m v(m,n) e^{-\lambda_m s}.$$

The Dirichlet series g_n is absolutely convergent for $s \in \mathbb{C}$. Thus, it is clear that $g_n(\sigma+it) \in B^2_{t_0}$ for any $t_0 \geqslant 0$, which has the following Fourier series

$$g_n(\sigma+it) \sim \sum_{m=1}^{\infty} a_m v(m,n) e^{-\lambda_m \sigma} e^{-i\lambda_m t}.$$

For $\sigma > \sigma_0$, let $t_0 \geqslant 0$ be a number such that $\{\sigma + it : t \geqslant t_0\} \cap \{s_1, \dots, s_r\} = \emptyset$. Then in view of the proof of Proposition 2.8 with $K = \{\sigma\}$, we have

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} |f(\sigma+it) - g_n(\sigma+it)|^2 dt = 0.$$

Note that for each $m \in \mathbb{N}$, $v(m,n) \to 0$ as $n \to \infty$. Thus $f(\sigma+it) \in B^2_{t_0}$ and

$$f(\sigma+it) \sim \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} e^{-i\lambda_m t}$$

is the Fourier series of $f(\sigma+it)$. □

As a consequence of Theorem 2.10 and Lemma 2.11, we have the following.

Theorem 2.12. *For $\sigma > \sigma_0$,*

$$f(\sigma,\omega) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} \omega(m)$$

converges for $\mathbf{P}$-a.e. $\omega \in \Omega$ and converges in $L^2(\Omega, \mathbf{P})$. Moreover, the sequence of probability measures

$$\nu_T\big(\tau : f(\sigma+i\tau) \in A\big), \quad A \in \mathcal{B}(\mathbb{C})$$

converges weakly to the distribution of $f(\sigma,\omega)$ as $T \to \infty$.

2.4 Limit theorems for general Dirichlet series in functional spaces

2.4.1 Absolutely convergent case

Recall that $D = \{s \in \mathbb{C} : \sigma > \sigma_a\}$ and $H(D)$ denotes the space of analytic functions on D equipped with the topology of uniform convergence on compacta. For $s \in D, \omega \in \Omega$, let

$$f(s,\omega) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \omega(m).$$

Then $f(s,\omega)$ is an $H(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$. In addition, we will prove that $f(s,\omega)$ is continuous as a mapping from Ω to $H(D)$. Indeed, let $\{\omega^{(n)}\}$ be a sequence converging to ω in Ω. We need to prove that $\{f(s,\omega^{(n)})\}$ converges to $f(s,\omega)$ in $H(D)$. Given a compact subset $K \subset D$, let $\sigma_1 = \min_{s\in K} \operatorname{Re} s > \sigma_a$. We have

$$\sup_{s\in K} |f(s,\omega^{(n)}) - f(s,\omega)| \leqslant \sum_{m=1}^{\infty} |a_m| e^{-\lambda_m \sigma_1} |\omega^{(n)}(m) - \omega(m)|.$$

Since $\sum_{m=1}^{\infty} |a_m| e^{-\lambda_m \sigma_1} < \infty$, it follows from the dominated convergence theorem that

$$\sup_{s\in K} |f(s,\omega^{(n)}) - f(s,\omega)| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, the mapping $f(s,\omega)$ is continuous. Consequently, the sequence of probability measures $\{Q_T f(s,\omega)^{-1}\}_T$ on $(H(D), \mathcal{B}(H(D)))$ converges weakly to $\mathbf{P}f(s,\omega)^{-1}$ as $T \to \infty$. Obviously, we have

$$\nu_T(\tau : f(s+i\tau) \in A) = Q_T(\omega : f(s,\omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Therefore, we have just proved the following theorem.

Theorem 2.13. *The sequence of probability measures*

$$\nu_T(\tau : f(s+i\tau) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $\mathbf{P}_f$ *as* $T \to \infty$, *where* $\mathbf{P}_f$ *denotes the distribution of the* $H(D)$*-valued random element* $f(s,\omega)$.

2.4.2 General case

Recall that $D_0 = \{s \in \mathbb{C} : \sigma > \sigma_0\}$. There is a sequence $\{K_n\}$ of compact subsets of D_0 such that (i) $D_0 = \bigcup_{n=1}^{\infty} K_n$; (ii) $K_n \subset K_{n+1}$; (iii) if K is a compact set and $K \subset D_0$, then $K \subset K_n$ for some n. Then for $f, g \in H(D_0)$, let

$$\rho(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{s\in K_n} |f(s) - g(s)|}{1 + \sup_{s\in K_n} |f(s) - g(s)|}.$$

The topological space $H(D_0)$ becomes a complete separable metric space. Similarly, for $f, g \in M(D_0)$, let

$$\bar{\rho}(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{s\in K_n} d(f(s), g(s))}{1 + \sup_{s\in K_n} d(f(s), g(s))}.$$

Then $M(D_0)$ also becomes a separable metric space.

Lemma 2.14.

$$f(s,\omega) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \omega(m), \quad s \in D_0$$

is a well-defined $H(D_0)$*-valued random element. Besides, for any fixed* $s \in D_0$,

$$f(s,\omega) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \omega(m)$$

converges for $\mathbf{P}$*-a.e.* $\omega \in \Omega$ *and converges in* $L^2(\Omega, \mathbf{P})$.

Proof. By Theorem 2.12, for any $\sigma_1 > \sigma_0$, the series

$$f(\sigma_1,\omega) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma_1} \omega(m)$$

converges for $\mathbf{P}$-a.e. $\omega \in \Omega$. Therefore by a fundamental property of general Dirichlet series, for $\mathbf{P}$-a.e. $\omega \in \Omega$, the series

$$f(s,\omega) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \omega(m)$$

converges uniformly on each compact subset of the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_1\}$. Let A_n denote the set of $\omega \in \Omega$ for which the series $f(s,\omega)$ converges uniformly on compact subsets of the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_0 + 1/n\}$. Obviously, $\mathbf{P}(A_n) = 1$ for all $n \in \mathbb{N}$. Now if we take

$$A = \bigcap_{n=1}^{\infty} A_n,$$

then $\mathbf{P}(A) = 1$, and, for $\omega \in A$, the series $f(s,\omega)$ converges uniformly on compact subsets of D_0. It follows that $f(s,\omega)$ is an $H(D_0)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$. □

For each $n \in \mathbb{N}$, we define a random element $g_n(s,\omega)\colon \Omega \to H(D_0)$ as

$$g_n(s,\omega) := \sum_{m=1}^{\infty} a_m v(m,n) e^{-\lambda_m s} \omega(m).$$

Lemma 2.15.

$$\lim_{n\to\infty} \mathbf{E}^{(\mathbf{P})}\left[\rho(g_n(\cdot,\omega), f(\cdot,\omega))^2\right] = 0.$$

Proof. Let K be a compact subset in D_0. We will show that

$$\lim_{n\to\infty} \mathbf{E}^{(\mathbf{P})}[|h_n(\omega)|^2] = 0, \tag{2.11}$$

where

$$h_n(\omega) := \sup_{s\in K} |g_n(s,\omega) - f(s,\omega)|.$$

To prove (2.11), let L be a simple closed contour lying in D_0 and enclosing the set K and let δ denote the distance of L from the set K. For $\omega \in \Omega$ for which $f(s,\omega) \in H(D_0)$, Cauchy's integral formula implies that

$$h_n(\omega) = \sup_{s\in K} |g_n(s,\omega) - f(s,\omega)| \leqslant \frac{1}{2\pi\delta} \int_L |g_n(z,\omega) - f(z,\omega)||dz|,$$

then by the Cauchy-Schwarz inequality, we obtain

$$|h_n(\omega)|^2 \leqslant \frac{|L|}{(2\pi\delta)^2} \int_L |g_n(z,\omega) - f(z,\omega)|^2 |dz|.$$

Let $\sigma_1 = \min_{z\in L} \operatorname{Re} z > \sigma_0$. For $z = \sigma + it (\sigma \geqslant \sigma_1)$, from Lemma 2.14, we have

$$f(z,\omega) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m z} \omega(m), \quad (\text{in } L^2(\Omega, \mathbf{P})),$$

which implies

$$\begin{aligned} \mathbf{E}^{(\mathbf{P})}[|g_n(z,\omega) - f(z,\omega)|^2] &= \sum_{m=1}^{\infty} |a_m|^2 |1 - v(m,n)|^2 e^{-2\lambda_m \sigma} \\ &\leqslant \sum_{m=1}^{\infty} |a_m|^2 |1 - v(m,n)|^2 e^{-2\lambda_m \sigma_1} < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}^{(\mathbf{P})}[|h_n(\omega)|^2] &\leqslant \mathbf{E}^{(\mathbf{P})}\left[\frac{|L|}{(2\pi\delta)^2}\int_L |g_n(z,\omega)-f(z,\omega)|^2|dz|\right] \\ &= \frac{|L|}{(2\pi\delta)^2}\int_L \mathbf{E}^{(\mathbf{P})}[|g_n(z,\omega)-f(z,\omega)|^2]|dz| \\ &\leqslant \frac{|L|^2}{(2\pi\delta)^2}\sum_{m=1}^{\infty}|a_m|^2|1-v(m,n)|^2 e^{-2\lambda_m\sigma_1}.\end{aligned}$$

Our desired result (2.11) follows from the dominated convergence theorem. The above result holds for any compact subset K in D_0. Therefore, taking the definition of the metric ρ into account, we obtain

$$\lim_{n\to\infty}\mathbf{E}^{(\mathbf{P})}\left[\rho(g_n(\cdot,\omega),f(\cdot,\omega))^2\right]=0.$$

The proof is complete. □

Corollary 2.16. (i) *For any* $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbf{P}\Big(\rho(g_n(\cdot,\omega),f(\cdot,\omega))\geqslant\varepsilon\Big)=0.$$

(ii)

$$\mathbf{P}_{g_n}\xrightarrow{d}\mathbf{P}_f \quad \text{as} \quad n\to\infty, \tag{2.12}$$

where $\mathbf{P}_{g_n}$ *and* $\mathbf{P}_f$ *denote the distributions of the* $H(D_0)$*-valued or* $M(D_0)$*-valued random elements* g_n *and* f*, respectively.*

Proof. (i) follows from Lemma 2.15 by Chebyshev's inequality. (ii) follows from (i) by Theorem 0.6. □

For every compact subset K of D_0, Proposition 2.9 claims that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}d(g_n(s+it),f(s+it))dt=0.$$

Thus, by Chebyshev's inequality, for any $\varepsilon>0$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\nu_T\Big(\tau:\bar{\rho}(g_n(\cdot+i\tau),f(\cdot+i\tau))\geqslant\varepsilon\Big)=0. \tag{2.13}$$

Since the Dirichlet series $g_n(s)$ is absolutely convergent in D_0, it follows from Theorem 2.13 that

$$\nu_T(\tau:g_n(s+i\tau)\in\cdot)\xrightarrow{d}\mathbf{P}_{g_n} \quad \text{as} \quad T\to\infty, \tag{2.14}$$

where the weak convergence still holds in the space of meromorphic functions $M(D_0)$. Therefore, (2.12)–(2.14) imply the limit theorem for $f(s)$ in the space of meromorphic functions $M(D_0)$.

Theorem 2.17. *Suppose that conditions* (2.2) *and* (2.3) *are satisfied. Then*

$$f(s,\omega)=\sum_{m=1}^{\infty}a_m e^{-\lambda_m s}\omega(m), \quad s\in D_0$$

is a well-defined $H(D_0)$*-valued random element. Moreover, the sequence of probability measures*

$$\nu_T(\tau:f(s+i\tau)\in A), \quad A\in\mathcal{B}(M(D_0)),$$

converges weakly to $\mathbf{P}_f$ *as* $T\to\infty$*, where* $\mathbf{P}_f$ *denotes the distribution of* $f(s,\omega)$.

Remark 2.18. If the strip $D_1 = \{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2\}(\sigma_0 < \sigma_1 < \sigma_2 \leqslant \infty)$ contains no pole of $f(s)$, then in view of the proof of Theorem 2.17 we can assert the following.

"The sequence of probability measures

$$\nu_T(\tau : f(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_1)),$$

converges weakly to $\mathbf{P}_f$ *as* $T \to \infty$*, where* $\mathbf{P}_f$ *denotes the distribution of the* $H(D_1)$*-valued random element* $f(s, \omega)$*."*

Remark 2.19. Theorem 2.12 and Theorem 2.17 are extensions of the main results in [34] and [19], respectively. Comparing with proofs in [34, 19], the basic idea does not change but a number of arguments are reduced. For instance, we use L^2-convergence instead of using the tightness of measures and ergodic theory.

Part II

Besicovitch Limit-periodic Arithmetical Funtions

Chapter 3

Limit-periodic Arithmetical Functions and The Ring of Finite Integral Adeles

3.1 Introduction

Chapter 3 and Chapter 4 mainly concern with arithmetical functions, real or complex valued functions defined on the set of all natural numbers $\mathbb{N}$. For simplicity, we sometimes omit the term "arithmetical".

An arithmetical function $f\colon \mathbb{N} \to \mathbb{C}$ is called limit-periodic if it is a limit of periodic arithmetical functions under some Besicovitch q-(semi)norm $(1 \leqslant q < \infty)$,

$$\|f\|_q := \limsup_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} |f(n)|^q \right)^{1/q} .$$

Limit-periodic arithmetical functions, in some sense, look like random variables because they possess mean values and limit distributions. Here the limit distribution of an arithmetical function f is considered as follows; if the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp\left(\sqrt{-1}\, s \operatorname{Re} f(n) + \sqrt{-1}\, t \operatorname{Im} f(n)\right), \quad (s,t) \in \mathbb{R}^2,$$

exists and it coincides with the characteristic function of some probability distribution on $\mathbb{R}^2 \cong \mathbb{C}$, then we call it the limit distribution of f.

This chapter deals with the problem of finding an appropriate probability space where limit-periodic functions can be considered as random variables. The ring of finite integral adeles $\widehat{\mathbb{Z}}$, together with the Borel field $\mathcal{B}(\widehat{\mathbb{Z}})$ and the normalized Haar measure λ, is shown to be a good candidate. Indeed, let $\mathfrak{D}^q$ denote the space of q-limit-periodic functions. Then every function in $\mathfrak{D}^q$ can be extended to a random variable in $L^q(\widehat{\mathbb{Z}}, \lambda)$. The limit distribution of the original function coincides with the distribution of the extended random variable. In addition, the quotient space D^q of $\mathfrak{D}^q$ with respect to the null-space $\mathcal{N}(\mathfrak{D}^q) := \{\, f \in \mathfrak{D}^q : \|f\|_q = 0 \,\}$ is isomorphic to $L^q(\widehat{\mathbb{Z}}, \lambda)$, which means $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$ is a good candidate. In fact, the ring of finite integral adeles $\widehat{\mathbb{Z}}$, which was initiated by [39] and has been studied by several papers and books [10, 28, 36, 37, 44], has become an useful tool for studying probabilistic properties of limit-periodic functions. It is called the limit-periodic compactification of $\mathbb{Z}$. Besides, to investigate the probabilistic properties of arithmetical functions, there are other compactifications, for example, the almost even compactification used to investigate almost even functions [27]; the Bohr-compactification used to investigate almost periodic functions [37] and the Stone-Čech compactification used to investigate a wilder class of arithmetical functions which contains the Möbius function [25].

This chapter also concerns with the convergence of Fourier expansions of limit-periodic functions. First of all, we define the scalar product $\langle f, g\rangle$ of functions $f, g\colon \mathbb{N} \to \mathbb{C}$ to be the limit

$$\langle f, g\rangle := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n)\overline{g(n)},$$

provided that it exists. Then $\{e_\alpha\}_{\alpha\in\mathbb{Q}/\mathbb{Z}}$ becomes an orthonormal system, where e_α stands for the function $e_\alpha\colon n \mapsto \exp(2\pi\sqrt{-1}\,\alpha n)$. Note that the space of limit periodic functions is spanned by $\{e_\alpha\}_{\alpha\in\mathbb{Q}/\mathbb{Z}}$. Moreover, for each $f \in D^q$, the Fourier coefficients $\{\langle f, e_\alpha\rangle\}_{\alpha\in\mathbb{Q}/\mathbb{Z}}$ exist and the formal series

$$f \sim \sum_{\alpha\in\mathbb{Q}/\mathbb{Z}} \langle f, e_\alpha\rangle\, e_\alpha$$

is called the Fourier series of f. Next, for each $n \in \mathbb{N}$, we consider a finite Fourier expansion of a function $f \in D^q$ as

$$S_n(f) := \sum_{\substack{r\mid n;\\ 1\leqslant a\leqslant r;(a,r)=1}} \langle f, e_{a/r}\rangle\, e_{a/r},$$

where (a, r) denotes the greatest common divisor $\gcd(a, r)$ of a and r. Then, we have the following result about the convergence of Fourier expansions of limit-periodic functions

$$\|S_n(f) - f\|_q \to 0 \quad \text{as} \quad n \to 0 \text{ in } \widehat{\mathbb{Z}}, \quad f \in D^q.$$

From this a similar result for almost-even arithmetical functions easily follows. These results give an approximation for limit-periodic functions (resp. almost-even functions) by periodic functions (resp. even functions), and they are the generalizations of [43, Theorem VI.5.1].

The convergence of Fourier expansions in the special case $q = 2$ is easily seen since $\{e_\alpha\}_{\alpha\in\mathbb{Q}/\mathbb{Z}}$ is an orthonormal basis of the Hilbert space D^2. To extend this result to general q, the idea here is to apply the interpolation of norms and of linear operators tool, a major tool in harmonic analysis, which was successfully used by Bochner to prove the convergence of Fourier series on $L^q([0, 2\pi]), 1 < q < \infty$ (see [26, Chapter IV]). The isomorphism between D^q and $L^q(\widehat{\mathbb{Z}}, \lambda)$ makes it easy to apply this tool as we will see in Section 3.3.

This chapter is organized as follows. In Section 3.2, the probability space $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$ is introduced. Then the natural isometric isomorphism T_q between D^q and $L^q(\widehat{\mathbb{Z}}, \lambda)$ is defined. Moreover, the inverse of T_q can be obtained by Lebesgue's density theorem (Theorem 3.21). The beginning of Section 3.3 deals with the convergence of Fourier expansions in $L^1(\widehat{\mathbb{Z}}, \lambda)$. By using the interpolation of norms and of linear operators tool and the dual property, we obtain the convergence of Fourier expansions in $L^q(\widehat{\mathbb{Z}}, \lambda)$ for all $1 \leqslant q < \infty$. At the end of this section, we deduce the above convergence in D^q (Theorem 3.27) by the isometric property of T_q and its consequence on the convergence of Ramanujan expansions of almost-even functions. The natural extensions of additive and multiplicative functions will be considered in Section 3.4.

3.2 The ring of finite integral adeles: basic properties and connection to limit-periodic arithmetical functions

3.2.1 The ring of finite integral adeles and some basic properties

This section deals with the construction and properties of the ring of finite integral adeles. Results are taken from [29, 44]. For a prime p, the p-adic metric d_p is defined by

$$d_p(x,y) := \inf\{p^{-l} : p^l|(x-y)\}, \quad x,y \in \mathbb{Z},$$

where $p^l|(x-y)$ means that $(x-y)$ is divisible by p^l. The completion of $\mathbb{Z}$ by d_p is denoted by $\mathbb{Z}_p$. By extending the algebraic operations '+' and '×' in $\mathbb{Z}$ continuously to those in $\mathbb{Z}_p$, the compact metric space $(\mathbb{Z}_p, d_p)$ becomes a ring, called the ring of p-adic integers. In particular, $(\mathbb{Z}_p, d_p)$ is a compact abelian group with respect to '+'. According to the general theory of compact groups, there is a unique normalized Haar measure λ_p with respect to '+' on the measurable space $(\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p))$.

Definition 3.1. (i) Let $\{p_i\}_{i=1}^{\infty}, 2 = p_1 < p_2 < \cdots$, be the sequence of all primes.

(ii) Put

$$\widehat{\mathbb{Z}} := \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}, \quad \lambda := \prod_{i=1}^{\infty} \lambda_{p_i}.$$

For $x = (x_i), y = (y_i) \in \widehat{\mathbb{Z}}$, we define

$$\rho(x,y) := \sum_{i=1}^{\infty} \frac{1}{2^i} d_{p_i}(x_i, y_i), \quad x + y := (x_i + y_i), \quad xy := (x_i y_i).$$

By these definitions, $\widehat{\mathbb{Z}}$ becomes a ring, called *the ring of finite integral adeles.* $(\widehat{\mathbb{Z}}, \rho)$ is again a compact metric space, and both '+' and '×' are continuous. In particular, $(\widehat{\mathbb{Z}}, \rho)$ is a compact abelian group with respect to '+' and its normalized Haar measure on the Borel field $\mathcal{B}(\widehat{\mathbb{Z}})$ is nothing but λ.

Definition 3.2. (i) We identify $\mathbb{Z}$ with the diagonal set $\{(n,n,\ldots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots\} \subset \widehat{\mathbb{Z}}$.

(ii) For $\mathbb{N} \ni m \geqslant 2$ and $l \in \{0,1,\ldots,m-1\}$, we define $m\widehat{\mathbb{Z}} + l := \{mx + l : x \in \widehat{\mathbb{Z}}\}$. Then we have $\widehat{\mathbb{Z}} = \bigcup_{l=0}^{m-1}(m\widehat{\mathbb{Z}} + l)$, which is a disjoint union (Lemma 3.6 (iii)). So, for $x \in \widehat{\mathbb{Z}}$ and $\mathbb{N} \ni m \geqslant 2$, there exists a unique $l \in \{0,1,\ldots,m-1\}$ such that $x - l \in m\widehat{\mathbb{Z}}$. This l is denoted by $x \bmod m$. For $m = 1$, we always set $x \bmod m := 0$. Obviously, if $x \in \mathbb{Z}$, this definition coincides with the usual modulo operation.

(iii) For $x, y \in \widehat{\mathbb{Z}}$, we define the greatest common divisor of x and y by

$$\gcd(x,y) := \sup\{m \in \mathbb{N} : (x \bmod m) = (y \bmod m) = 0\}.$$

Obviously, for $x, y \in \mathbb{Z}$, this definition coincides with the usual gcd.

Let us give some fundamental properties of $\widehat{\mathbb{Z}}$ and λ.

Lemma 3.3 (Chinese remainder theorem, cf. [21, Theorem 121]). *Assume $m_1, \ldots, m_l \in \mathbb{N}$ to be co-prime. Then, for any $a_1, \ldots, a_l \in \mathbb{Z}$, there exists an $n \in \mathbb{Z}$ such that $n = a_i \pmod{m_i}$, $i = 1, \ldots, l$. This n is unique up to* mod $\prod_{i=1}^{l} m_i$.

Lemma 3.4. $\mathbb{N}' := \{(n,n,\ldots) \in \widehat{\mathbb{Z}} : n \in \mathbb{N}\}$ *is dense in* $\widehat{\mathbb{Z}}$.

Proof. The Chinese remainder theorem implies that for any $l, m \in \mathbb{N}$ and any $n_1, \ldots, n_l \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $n = n_i \bmod p_i^m, i = 1, \ldots, l$. This means that $\mathbb{N}'$ is dense in $\mathbb{Z} \times \mathbb{Z} \times \cdots$ with respect to the metric ρ. □

As we identify $\mathbb{Z}$ with $\mathbb{Z}' = \{(n, n, \ldots) \in \widehat{\mathbb{Z}} : n \in \mathbb{Z}\}$ (Definition 3.2 (i)) by a bijection $\mathbb{Z} \ni n \mapsto (n, n, \ldots) \in \mathbb{Z}'$, Lemma 3.4 implies that $\mathbb{Z}$ is a dense subring of $\widehat{\mathbb{Z}}$. Thus $\widehat{\mathbb{Z}}$ is a compactification of $\mathbb{Z}$. It is called the limit-periodic compactification of $\mathbb{Z}$ or $\mathbb{N}$.

Lemma 3.5. (i) *Let p be a prime and $j \in \mathbb{N}$. Then $p^j\mathbb{Z}_p$ is closed and open.*

(ii) *Let p, q be distinct primes and $j \in \mathbb{N}$. Then we have $p^j\mathbb{Z}_q = \mathbb{Z}_q$.*

Proof. (i) It is easy to see that $p^j\mathbb{Z}_p = \{x \in \mathbb{Z}_p : d_p(x, 0) \leqslant p^{-j}\}$, and hence it is closed. Since $d_p(x, 0) \in \{p^{-a} : a = 0, 1, \ldots, \infty\}$ for any $x \in \mathbb{Z}_p$, we may write $p^j\mathbb{Z}_p = \{x \in \mathbb{Z}_p : d_p(x, 0) < p^{-j+1}\}$, which implies it is open.
(ii) $p^j\mathbb{Z}_q \subset \mathbb{Z}_q$ is obvious, so let us prove $p^j\mathbb{Z}_q \supset \mathbb{Z}_q$. To this end, it is enough to show that there is an $x \in \mathbb{Z}_q$ such that $p^j x = 1$. For any $m \in \mathbb{N}$, there exists an $x_m \in \mathbb{N}$ such that $x_m p^j = 1 \bmod q^m$. Then for any $n > m$, we have $(x_n - x_m)p^j = 0 \bmod q^m$. Since $\gcd(p^j, q^m) = 1$, we see $x_n - x_m = 0 \bmod q^m$, which means that $\{x_m\}_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{Z}_q$. Then putting $x := \lim_{m\to\infty} x_m$, we have $p^j x = 1$ in $\mathbb{Z}_q$. □

Lemma 3.6. *Let $m \in \mathbb{N}$ and $l \in \{0, 1, \ldots, m-1\}$.*

(i) *The set $(m\widehat{\mathbb{Z}} + l)$ is closed and open.*

(ii) *$\rho_m \colon \widehat{\mathbb{Z}} \to \{0, 1\}$ is continuous, where* $\rho_m(x) = \begin{cases} 1, & \textit{if } x \bmod m = 0, \\ 0, & \textit{otherwise.} \end{cases}$

(iii) *$\widehat{\mathbb{Z}} = \bigcup_{l=0}^{m-1}(m\widehat{\mathbb{Z}} + l)$, which is a disjoint union.*

Proof. (i) Let $m = \prod_p p^{\alpha_p(m)}$ be the factorization of m into primes, where $\alpha_p(m) = 0$ except for finitely many primes p. Then, Lemma 3.5 implies that

$$m\widehat{\mathbb{Z}} = \prod_p m\mathbb{Z}_p = \prod_p p^{\alpha_p(m)}\mathbb{Z}_p,$$

and that each $p^{\alpha_p(m)}\mathbb{Z}_p$ is closed and open. Therefore, $m\widehat{\mathbb{Z}}$ is also closed and open in $\widehat{\mathbb{Z}}$. Finally, since the shift $\widehat{\mathbb{Z}} \ni x \mapsto (x + l) \in \widehat{\mathbb{Z}}$ is a homeomorphism, $m\widehat{\mathbb{Z}} + l$ is also closed and open.
(ii) Since (i) implies that $\rho_m^{-1}(\{1\}) = m\widehat{\mathbb{Z}}$ is closed and open, the statement is obvious.
(iii) From the denseness of $\mathbb{Z}$ in $\widehat{\mathbb{Z}}$, and from the continuity and closedness of the mapping $x \mapsto mx + l$, it follows that $\overline{m\mathbb{Z} + l} = m\widehat{\mathbb{Z}} + l$. Since $\mathbb{Z} = \bigcup_{l=0}^{m-1}(m\mathbb{Z} + l)$, this implies

$$\widehat{\mathbb{Z}} = \bigcup_{l=0}^{m-1}(m\widehat{\mathbb{Z}} + l).$$

Next we check the disjointness of this union. Let $m \geqslant 2$ and $l, l' \in \{0, \ldots, m-1\}$ be distinct integers. By (i), $A := (m\widehat{\mathbb{Z}} + l) \cap (m\widehat{\mathbb{Z}} + l')$ is open. If $A \neq \emptyset$, then $\mathbb{Z} \cap A \neq \emptyset$, because $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$. But then, taking an $n \in \mathbb{Z} \cap A$, we see from the observation of (i) that

$$d_p(n - l, 0) \leqslant p^{-\alpha_p(m)}, \quad d_p(n - l', 0) \leqslant p^{-\alpha_p(m)}, \forall p : \text{prime}.$$

This implies that $p^{\alpha_p(m)} \mid (l - l')$ for each prime p, that is, $m \mid (l - l')$, which is impossible. Thus A should be empty. □

Corollary 3.7. *For any $l \in \mathbb{Z}$, the mapping*

$$\widehat{\mathbb{Z}} \ni x \mapsto \frac{(l+x) \bmod m}{m} \in [0,1)$$

is continuous.

Lemma 3.8. *For any $l \in \mathbb{Z} \setminus \{0\}$ and any $A \in \mathcal{B}(\widehat{\mathbb{Z}})$, we have $lA \in \mathcal{B}(\widehat{\mathbb{Z}})$ and*

$$\lambda(lA) = \frac{1}{|l|}\lambda(A). \tag{3.1}$$

Proof. Since $\widehat{\mathbb{Z}}$ is a complete separable metric space and the map $\widehat{\mathbb{Z}} \ni x \mapsto lx \in \widehat{\mathbb{Z}}$ is one-to-one and measurable, we have $lA \in \mathcal{B}(\widehat{\mathbb{Z}})$ (cf. [41, Theorem I.3.9]). Let ν be a Borel probability measure on $\widehat{\mathbb{Z}}$ defined by

$$\nu(A) = \frac{\lambda(lA)}{\lambda(|l|\widehat{\mathbb{Z}})}, \quad A \in \mathcal{B}(\widehat{\mathbb{Z}}).$$

Then ν is clearly shift invariant, and hence $\nu = \lambda$, so that $\lambda(lA) = \lambda(|l|\widehat{\mathbb{Z}})\lambda(A)$. By Lemma 3.6 and the shift invariance of λ, we see

$$1 = \lambda(\widehat{\mathbb{Z}}) = \sum_{i=0}^{|l|-1} \lambda(|l|\widehat{\mathbb{Z}} + i) = |l|\lambda(|l|\widehat{\mathbb{Z}}),$$

from which, (3.1) immediately follows. □

Lemma 3.9. (i) *Let $f\colon \widehat{\mathbb{Z}} \to \mathbb{C}$ be a continuous function. Then $\{f(n)\}_{n\in\mathbb{Z}}$ is a uniformly limit-periodic sequence, that is,*

$$\forall\varepsilon > 0, \exists l_0,\ m_0 \in \mathbb{N} \text{ such that} \left| f(n) - f\left(n \bmod \prod_{i=1}^{l_0} p_i^{m_0}\right)\right| < \varepsilon, \quad \forall n \in \mathbb{Z}. \tag{3.2}$$

(ii) *Conversely, if $\{f(n)\}_{n\in\mathbb{Z}}$ is a uniformly limit-periodic sequence, then there is a unique continuous function $\widehat{f}\colon \widehat{\mathbb{Z}} \to \mathbb{C}$ such that $\widehat{f}(n) = f(n)$ for each $n \in \mathbb{Z}$.*

Proof. (i) Obvious by the definition of the metric of $\widehat{\mathbb{Z}}$.
(ii) If f is a periodic sequence with period $m \in \mathbb{N}$, it is of the form $f(n) = \sum_{i=1}^m f(i)\rho_m(n-i), n \in \mathbb{Z}$. Then $\widehat{f}(x) := \sum_{i=1}^m f(i)\rho_m(x-i), x \in \widehat{\mathbb{Z}}$, is the continuous function with the property $\widehat{f}|_{\mathbb{Z}} = f$. Note that a general f satisfying (3.2) is a uniformly convergent limit of a sequence of periodic sequences, and hence it has again a continuous extension $\widehat{f}$. Since $\mathbb{Z}$ is densely embedded in $\widehat{\mathbb{Z}}$, the uniqueness of $\widehat{f}$ is obvious. □

For a periodic sequence $\{g(n)\}_{n\in\mathbb{Z}}$ with period m and its unique continuous extension $\widehat{g}(x)$, it is easy to see that

$$\begin{aligned}\int_{\widehat{\mathbb{Z}}} \widehat{g}(x)\lambda(dx) &= \sum_{n=1}^m g(n) \int_{\widehat{\mathbb{Z}}} \rho_m(x-m)\lambda(dx) \\ &= \frac{1}{m}\sum_{n=1}^m g(n) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N g(n).\end{aligned} \tag{3.3}$$

In general, we have the following lemma.

Lemma 3.10. *If $f\colon \widehat{\mathbb{Z}} \to \mathbb{C}$ is continuous, then*

$$\int_{\widehat{\mathbb{Z}}} f(x)\lambda(dx) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} f(n), \quad \forall n_0 \in \mathbb{Z}. \tag{3.4}$$

The convergence is uniform in $n_0 \in \mathbb{Z}$.

Proof. Let $f\colon \widehat{\mathbb{Z}} \to \mathbb{C}$ be continuous and set $f_{n_0}(x) = f(n_0 + x), n_0 \in \widehat{\mathbb{Z}}$. By the uniform continuity of f, a family $\{f_{n_0}; n_0 \in \mathbb{Z}\}$ satisfies (3.2). For simplicity, set $m := \prod_{i=1}^{l_0} p_i^{m_0}$. Then we see for any $n_0 \in \mathbb{Z}$,

$$\left| \int_{\widehat{\mathbb{Z}}} f_{n_0}(x)\lambda(dx) - \int_{\widehat{\mathbb{Z}}} f_{n_0}(x \bmod m)\lambda(dx) \right| \leqslant \varepsilon,$$
$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n) - \frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n \bmod m) \right| < \varepsilon, \quad \forall N \in \mathbb{N}.$$

By (3.3),

$$\int_{\widehat{\mathbb{Z}}} f_{n_0}(x \bmod m)\lambda(dx) = \frac{1}{m} \sum_{r=0}^{m-1} f_{n_0}(r).$$

Also, by a simple calculation

$$\begin{aligned}
\frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n \bmod m) &= \frac{1}{N} \left(\left\lfloor \frac{N-1}{m} \right\rfloor \sum_{r=0}^{m-1} f_{n_0}(r) + \sum_{r=0}^{(N-1) \bmod m} f_{n_0}(r) \right) \\
&= \frac{1}{m} \sum_{r=0}^{m-1} f_{n_0}(r) - \frac{1}{N} \left(\frac{1}{m} + \frac{(N-1) \bmod m}{m} \right) \sum_{r=0}^{m-1} f_{n_0}(r) \\
&\quad + \frac{1}{N} \sum_{r=0}^{(N-1) \bmod m} f_{n_0}(r).
\end{aligned}$$

In the above and in what follows, the symbol $\lfloor t \rfloor$ stands for the largest integer not exceeding $t \in \mathbb{R}$.

From these, it follows that

$$\left| \int_{\widehat{\mathbb{Z}}} f_{n_0}(x \bmod m)\lambda(dx) - \frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n \bmod m) \right| \leqslant \frac{2}{N} m \|f\|_\infty.$$

Therefore, choosing an $N_0 \in \mathbb{N}$ so large that $(2/N_0)m\|f\|_\infty \leqslant \varepsilon$, we have that for any $N \geqslant N_0$ and any $n_0 \in \mathbb{N}$,

$$\left| \int_{\widehat{\mathbb{Z}}} f(x)\lambda(dx) - \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} f(n) \right| < 3\varepsilon. \qquad \square$$

Remark 3.11. As a matter of fact, (3.4) is a consequence of the following general theorem.

Theorem 3.12. *Let G be a compact group, and let $x \in G$. Then, if the sequence $\{x^n\}_{n=1}^\infty$ is dense in G, it is uniformly distributed; that is, $N^{-1} \sum_{n=1}^N \delta_{x^n}$ converges weakly to the normalized Haar measure of G as $N \to \infty$, where δ_{x^n} denotes the Dirac measure at x^n.*

For details, see [30, Theorem IV.4.2]. In the present case, setting $G := \widehat{\mathbb{Z}}$ and $x := 1$, we see the desired weak convergence

$$\frac{1}{N}\sum_{n=1}^{N}\delta_n \Rightarrow \lambda \quad \text{as} \quad N \to \infty, \tag{3.5}$$

because $\mathbb{N}'$, which is generated by $x = 1$, is dense in $\widehat{\mathbb{Z}}$.

Theorem 3.13. *The shifts $x \mapsto x+1$ and $x \mapsto x-1$ on the compact metric group $\widehat{\mathbb{Z}}$ are ergodic.*

Proof. The denseness of $\mathbb{Z}$ in $\widehat{\mathbb{Z}}$ (Lemma 3.4) implies the ergodic properties of these shifts (see Theorem 0.16). □

3.2.2 Connection to limit-periodic arithmetical functions

Let $\mathfrak{D}$ be the space of periodic arithmetical functions, that is,

$$\mathfrak{D} := \mathrm{Lin}_{\mathbb{C}}\left[e_{a/r} : r = 1, 2, \dots, 1 \leqslant a \leqslant r, \gcd(a, r) = 1\right],$$

and let $\mathfrak{D}^u$ be the linear closure of $\mathfrak{D}$ with respect to the uniform norm

$$\|f\|_u := \sup_{n\in\mathbb{N}} |f(n)|, \quad f\colon \mathbb{N} \to \mathbb{C}.$$

Functions in $\mathfrak{D}^u$ are called uniformly limit-periodic functions or uniformly limit-periodic sequences. Recall that the space of q-limit-periodic functions $\mathfrak{D}^q$ is just the linear closure of $\mathfrak{D}$ with respect to the Besicovitch q-norm.

Denote by $C(\widehat{\mathbb{Z}})$ the space of continuous functions on $\widehat{\mathbb{Z}}$ endowed with the supremum norm

$$\|\widehat{f}\| := \sup_{x\in\widehat{\mathbb{Z}}} |\widehat{f}(x)|, \quad \widehat{f} \in C(\widehat{\mathbb{Z}}).$$

Then the spaces $\mathfrak{D}^u$ and $C(\widehat{\mathbb{Z}})$ are isomorphic under the isomorphism $T_u\colon \mathfrak{D}^u \to C(\widehat{\mathbb{Z}})$, which maps $f \in \mathfrak{D}^u$ to the unique continuous extension of f from $\mathbb{N}$ to $\widehat{\mathbb{Z}}$. The preimage of the function $\widehat{f} \in C(\widehat{\mathbb{Z}})$ under T_u is just the restriction of $\widehat{f}$ to $\mathbb{N}$ (Lemma 3.9). Let $f \in \mathfrak{D}^u$ with $\widehat{f} := T_u(f) \in C(\widehat{\mathbb{Z}})$. Then it follows from Lemma 3.10 that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|f(n)|^q = \int_{\widehat{\mathbb{Z}}}|\widehat{f}(x)|^q\lambda(dx),$$

and hence for any $f \in \mathfrak{D}^u$,

$$\|f\|_q = \|T_u(f)\|_{L^q}, \tag{3.6}$$

where

$$\|\widehat{f}\|_{L^q} := \left(\int_{\widehat{\mathbb{Z}}}|\widehat{f}(x)|^q\lambda(dx)\right)^{1/q}, \quad \widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda).$$

The continuous linear operator T_u on $\mathfrak{D}^u$ can now be continuously extended to a continuous linear operator $T_q\colon D^q \to L^q(\widehat{\mathbb{Z}}, \lambda)$ since $\mathfrak{D}^u$ is dense in D^q. For instance, let $f \in D^q$. Then there exists a sequence of periodic functions $\{f_k\}_k$ converging to f, that is, $\|f_k - f\|_q \to 0$ as $k \to \infty$. In particular, $\{f_k\}_k$ is a Cauchy sequence in D^q. By (3.6), $\{T_u(f_k)\}_k$ is also a Cauchy sequence in $L^q(\widehat{\mathbb{Z}}, \lambda)$, and hence the limit $\lim_{k\to\infty} T_u(f_k)$ exists in $L^q(\widehat{\mathbb{Z}}, \lambda)$. Let $T_q(f) := \lim_{k\to\infty} T_u(f_k)$. It is clear that T_q is well defined; moreover it follows from (3.6) and from the continuity of norms that

$$\|f\|_q = \|T_q(f)\|_{L^q}, \quad f \in D^q. \tag{3.7}$$

This implies that T_q is injective. On the other hand, T_q is surjective since $C(\widehat{\mathbb{Z}})$ is dense in $L^q(\widehat{\mathbb{Z}}, \lambda)$ and D^q is complete. Thus we have proved the following theorem.

Theorem 3.14 ([36, Theorem 2.1]). *For* $1 \leqslant q < \infty$,

$$D^q \cong L^q(\widehat{\mathbb{Z}}, \lambda).$$

Remark 3.15. (i) Let $\chi\colon \widehat{\mathbb{Z}} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be a character of the compact group $\widehat{\mathbb{Z}}$, that is, χ is a continuous function satisfying

$$\chi(x+y) = \chi(x)\chi(y), \quad x, y \in \widehat{\mathbb{Z}}.$$

Then $\chi = T_u(e_\alpha)$ for some $\alpha \in \mathbb{Q}/\mathbb{Z}$. Conversely, for each $\alpha \in \mathbb{Q}/\mathbb{Z}$, the image $\widehat{e}_\alpha := T_u(e_\alpha)$ is a character of the compact group $\widehat{\mathbb{Z}}$. Consequently, Theorem 3.14 implies that $T_u(\mathfrak{D}) = \mathrm{Lin}_{\mathbb{C}}[\,\widehat{e}_\alpha : \alpha \in \mathbb{Q}/\mathbb{Z}\,]$ is dense in $L^q(\widehat{\mathbb{Z}}, \lambda)$. Functions in $\mathrm{Lin}_{\mathbb{C}}[\,\widehat{e}_\alpha : \alpha \in \mathbb{Q}/\mathbb{Z}\,]$ are periodic functions. In addition, $\{\widehat{e}_\alpha\}_{\alpha\in\mathbb{Q}/\mathbb{Z}}$ is an orthonormal basis of the Hilbert space $L^2(\widehat{\mathbb{Z}}, \lambda)$ with respect to the scalar product $\langle \widehat{f}, \widehat{g}\rangle := \int_{\widehat{\mathbb{Z}}} \widehat{f}(x)\overline{\widehat{g}(x)}\lambda(dx)$.

(ii) For a periodic function f, we always think of $T_q(f) = T_u(f)$ although $T_q(f)$ is unique only almost surely.

The following are some properties of T_q.

(i) T_q is shift invariant and multiplicatively shift invariant

$$T_q(f(n+\cdot)) = T_q(f)(n+\cdot), \quad (n \in \mathbb{N}),$$
$$T_q(f_{(b\cdot)}) = T_q(f)_{(b\cdot)}, \quad (b \in \mathbb{N}),$$

where $f_{(b\cdot)}(n) = f(bn), n \in \mathbb{N}$ and $T_q(f)_{(b\cdot)}(x) = T_q(f)(bx), x \in \widehat{\mathbb{Z}}$.

(ii)
$$T_q(|f|) = |T_q(f)|.$$
This follows from the inequality $||a| - |b|| \leqslant |a-b|, a, b \in \mathbb{C}$.

(iii) If $1/r = 1/p + 1/q$ $(r \geqslant 1)$ and $f \in D^p, g \in D^q$, then Hölder's inequality implies $fg \in D^r$ and
$$T_r(fg) = T_p(f)T_q(g).$$

(iv) If $f \in D^q$, then $|f|^q \in D^1$ ([43, Theorem VI.2.9]) and
$$T_1(|f|^q) = |T_q(f)|^q.$$

Proof of (iv). Without loss of generality, assume that $f \geqslant 0$. This proof is somewhat similar to the proof of [43, Theorem VI.2.9]. Let $\widehat{g} := T_1(f^q)^{1/q}$. We will show that

$$\widehat{g} = T_q(f).$$

Given an $\varepsilon > 0$, first choose a real-valued periodic function h such that $\|f^q - h\|_1 \leqslant (\varepsilon/2)^q$ and then choose a polynomial Q with the property

$$\left|Q(t) - \{\max(0,t)\}^{1/q}\right| \leqslant \frac{\varepsilon}{2} \quad \text{in } |t| \leqslant \|h\|_u.$$

The composition $Q \circ h$ is also a periodic function, and moreover,

$$T_q(Q \circ h) = Q \circ T_q(h) =: Q \circ \widehat{h}.$$

Next we use inequalities

$$\begin{array}{ll} \text{(i)} & |a-b|^q \leqslant |a^q - b^q| \quad \text{in } a, b \geqslant 0, \\ \text{(ii)} & (a+b)^q \leqslant 2^{q-1}(a^q + b^q) \quad \text{in } a, b \geqslant 0, \\ \text{(iii)} & |\max(0,a) - \max(0,b)| \leqslant |a-b|, \quad \text{if } a, b \in \mathbb{R}, \end{array}$$

to estimate the following

$$\begin{aligned} |f(n) - Q(h(n))|^q &\leqslant \left\{ \left| f(n) - \max(0, h(n))^{1/q} \right| + \left| \max(0, h(n))^{1/q} - Q(h(n)) \right| \right\}^q \\ &\leqslant 2^{q-1} \left\{ |f(n)^q - h(n)| + \left| \max(0, h(n))^{1/q} - Q(h(n)) \right|^q \right\} \\ &\leqslant 2^{q-1} \left\{ |f(n)^q - h(n)| + \left(\frac{\varepsilon}{2}\right)^q \right\}. \end{aligned}$$

Now taking the Besicovitch q-norm, we have

$$\|f - Q \circ h\|_q \leqslant \varepsilon,$$

which implies

$$\|T_q(f) - T_q(Q \circ h)\|_{L^q} \leqslant \varepsilon.$$

The same argument as above, with noting that $|\widehat{h}(x)| \leqslant \|h\|_u, x \in \widehat{\mathbb{Z}}$, yields

$$|\widehat{g}(x) - Q(\widehat{h}(x))|^q \leqslant 2^{q-1} \left\{ \left| \widehat{g}(x)^q - \widehat{h}(x) \right| + \left(\frac{\varepsilon}{2}\right)^q \right\}.$$

Then, integrating the above with respect to λ, we see

$$\|\widehat{g} - Q \circ \widehat{h}\|_{L^q}^q \leqslant 2^{q-1} \left\{ \|\widehat{g}^q - \widehat{h}\|_{L^1} + \left(\frac{\varepsilon}{2}\right)^q \right\} = 2^{q-1} \left\{ \|f^q - h\|_1 + \left(\frac{\varepsilon}{2}\right)^q \right\} \leqslant \varepsilon^q.$$

Hence,

$$\|T_q(f) - \widehat{g}\|_{L^q} \leqslant \|T_q(f) - T_q(Q \circ h)\|_{L^q} + \|Q \circ \widehat{h} - \widehat{g}\|_{L^q} \leqslant 2\varepsilon.$$

The proof is complete by letting $\varepsilon \to 0$. □

For each $n \in \mathbb{N}$, let $r_n(x), x \in \widehat{\mathbb{Z}}$, denote the smallest positive residue of x modulo n, that is, $r_n(x) = x \bmod n$, if $x \bmod n > 0$, and $r_n(x) = n$, if $x \bmod n = 0$.

Proposition 3.16. *The function $f\colon \mathbb{N} \to \mathbb{C}$ is q-limit-periodic, if and only if for every sequence $\{n_k\}_k \subset \mathbb{N}$ converging to 0 in $\widehat{\mathbb{Z}}$, $\|f - f(r_{n_k}(\cdot))\|_q \to 0$ as $k \to \infty$.*

Proof. The sufficient condition is obvious since the functions $f(r_{n_k}(\cdot))$ are periodic. For the necessary condition, assume that f is q-limit-periodic. Let $\{n_k\}_k \subset \mathbb{N}$ be a sequence converging to 0 in $\widehat{\mathbb{Z}}$. This means that for every fixed $m \in \mathbb{N}$, all except finitely many n_k are multiples of m. Hence it is enough to show that for any given $\varepsilon > 0$, there is a positive integer m satisfying

$$\|f - f(r_{lm}(\cdot))\|_q < \varepsilon \quad \text{for all } l \geqslant 1.$$

To see this, first choose a periodic function g near f, say, $\|f - g\|_q \leqslant \varepsilon/3$. Let a be a period of g. Then by the definition of the Besicovitch q-norm, we have

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |f(n) - g(n)|^q = \|f - g\|_q^q \leqslant \left(\frac{\varepsilon}{3}\right)^q.$$

Therefore, there exists an integer $m \in \mathbb{N}$, which can be chosen as a multiple of a, such that

$$\frac{1}{lm}\sum_{n=1}^{lm}|f(n)-g(n)|^q \leqslant 2^q\left(\frac{\varepsilon}{3}\right)^q \quad \text{for all } l \geqslant 1.$$

Since both $f(r_{lm}(\cdot))$ and g are periodic with period lm, it follows that

$$\|f(r_{lm}(\cdot))-g\|_q = \left(\frac{1}{lm}\sum_{n=1}^{lm}|f(n)-g(n)|^q\right)^{1/q} \leqslant \frac{2\varepsilon}{3} \quad \text{for all } l \geqslant 1.$$

Consequently,

$$\|f(r_{lm}(\cdot))-f\|_q \leqslant \|f(r_{lm}(\cdot))-g\|_q + \|g-f\|_q \leqslant \varepsilon \quad \text{for all } l \geqslant 1,$$

which completes the proof. □

Let $\mathfrak{H}^r$ be the space of functions $f\colon \widehat{\mathbb{Z}} \to \mathbb{C}$ for which $f(r_{n_k}(x)) \xrightarrow{L^r} f(x)$ for some sequence $\{n_k\}_k \subset \mathbb{N}$ converging to 0 in $\widehat{\mathbb{Z}}$ [39, Proposition 10]. From our viewpoint, for $r \geqslant 1$, the space $\mathfrak{H}^r$ is just the space $L^r(\widehat{\mathbb{Z}},\lambda)$ with restricted condition that $\widehat{f}|_{\mathbb{N}} = T_r^{-1}(\widehat{f})$. More exactly, let $\widehat{f} \in L^r(\widehat{\mathbb{Z}},\lambda)$ and $f := T_r^{-1}(\widehat{f}) \in \mathfrak{D}^r$. Since $\lambda(\mathbb{N}) = 0$, there is a function $\widehat{f}'$ in the class of equivalent functions of $\widehat{f}$ whose values on $\mathbb{N}$ are assumed to coincide with f. Then, by Proposition 3.16, $\widehat{f}' \in \mathfrak{H}^r$. Therefore, Proposition 20 and Proposition 26 in [39] can now be rewritten as follows (see also [28, 36]).

Proposition 3.17. *Let $f \in D^q$ and $\widehat{f} = T_q(f)$. Then the following three statements hold.*

(i)

$$\mathbf{M}[f] := \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n) = \mathbf{E}[\widehat{f}].$$

(ii) *For $(s,t) \in \mathbb{R}^2$,*

$$\mathbf{M}\big[\exp\big(\sqrt{-1}\,s\operatorname{Re} f + \sqrt{-1}\,t\operatorname{Im} f\big)\big] = \mathbf{E}\big[\exp\big(\sqrt{-1}\,s\operatorname{Re}\widehat{f} + \sqrt{-1}\,t\operatorname{Im}\widehat{f}\big)\big].$$

(iii)

$$\langle f, e_\alpha\rangle = \langle \widehat{f}, \widehat{e}_\alpha\rangle, \quad \alpha \in \mathbb{Q}/\mathbb{Z}. \tag{3.8}$$

Remark 3.18. Proposition 3.17(ii) claims that every limit-periodic arithmetical function $f \in D^q$ has limit distribution and its limit distribution coincides with the distribution of $\widehat{f} = T_q(f) \in L^q(\widehat{\mathbb{Z}},\lambda)$.

Theorem 3.19. *Let $\widehat{f} \in L^q(\widehat{\mathbb{Z}},\lambda)$. Then there exists a full measure shift invariant set $\Omega \subset \widehat{\mathbb{Z}}$ with the following properties.*

(i) *$f_x \in D^q$ for all $x \in \Omega$, where $f_x(n) := \widehat{f}(x+n), n \in \mathbb{N}$.*

(ii) *$T_q(f_x) = \widehat{f}(x+\cdot)$ for all $x \in \Omega$.*

(iii)

$$\lim_{\rho(x,y)\to 0;\, x,y\in\Omega}\|f_x - f_y\|_q = 0 \textit{ uniformly in } \Omega.$$

Proof. Let $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$ and $\{\widehat{f}^{(m)}\}_m$ be a sequence of periodic functions converging to $\widehat{f}$. Recall that the shift $\widehat{\mathbb{Z}} \ni x \mapsto x+1$ is ergodic (Theorem 3.13). Then for each $m \in \mathbb{N}$, by the ergodic theorem, there exists a full measure shift invariant set Ω_m such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\widehat{f}^{(m)}(x+n) - \widehat{f}(x+n)|^q = \mathbf{E}[|\widehat{f}^{(m)} - \widehat{f}|^q], \quad x \in \Omega_m. \tag{3.9}$$

Let $\Omega = \bigcap_m \Omega_m$. It is clear that Ω is shift invariant and $\lambda(\Omega) = 1$. It now follows from (3.9) that

$$\|f_x^{(m)} - f_x\|_q = \|\widehat{f}^{(m)} - \widehat{f}\|_{L^q}, \quad x \in \Omega, \tag{3.10}$$

where $f_x^{(m)}(\cdot) := \widehat{f}^{(m)}(x+\cdot)$. Moreover, $f_x^{(m)}$ is periodic and $T_u(f_x^{(m)}) = \widehat{f}^{(m)}(x+\cdot)$ because $\widehat{f}^{(m)}$ is periodic. Therefore, letting $m \to \infty$ in (3.10), we see $f_x \in D^q$ and $T_q(f_x) = \widehat{f}(x+\cdot)$ for all $x \in \Omega$.

Next we prove (iii). Given an $\varepsilon > 0$, there exists an m such that

$$\|\widehat{f}^{(m)} - \widehat{f}\|_{L^q} < \frac{\varepsilon}{3}.$$

Now, due to the shift invariant property of the Haar measure λ,

$$\|\widehat{f}^{(m)}(x+\cdot) - \widehat{f}(x+\cdot)\|_{L^q} = \|\widehat{f}^{(m)} - \widehat{f}\|_{L^q} < \frac{\varepsilon}{3}, \quad x \in \widehat{\mathbb{Z}}.$$

On the other hand, the function $\widehat{f}^{(m)}$ is uniformly continuous since $\widehat{\mathbb{Z}}$ is compact. Thus, there is a $\delta > 0$ such that $|\widehat{f}^{(m)}(x) - \widehat{f}^{(m)}(y)| < \varepsilon/3$ when $\rho(x,y) < \delta$. It follows that $|f_x^{(m)}(n) - f_y^{(m)}(n)| < \varepsilon/3$ for all $n \in \mathbb{N}$, and hence $\|f_x^{(m)} - f_y^{(m)}\|_u \leqslant \varepsilon/3$, if $\rho(x,y) < \delta$. Finally, for $x, y \in \Omega$ with $\rho(x,y) < \delta$, we have

$$\begin{aligned}\|f_x - f_y\|_q &\leqslant \|f_x - f_x^{(m)}\|_q + \|f_x^{(m)} - f_y^{(m)}\|_q + \|f_y^{(m)} - f_y\|_q \\ &\leqslant \|\widehat{f} - \widehat{f}^{(m)}\|_{L^q} + \|f_x^{(m)} - f_y^{(m)}\|_u + \|\widehat{f}^{(m)} - \widehat{f}\|_{L^q} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Theorem 3.19 is thus proved. □

Remark 3.20. Let $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$. Then by Theorem 3.19(iii), the limit

$$\lim_{\Omega \ni x \to 0} f_x =: f \in D^q$$

exists. Moreover, we have $T_q(f) = \widehat{f}$.

Theorem 3.21. *Let $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$ and $f = T_q^{-1}(\widehat{f})$. For each $\delta > 0$, put $B(\delta) := \{\, x \in \widehat{\mathbb{Z}} : \rho(0,x) < \delta \,\}$. Let*

$$f_\delta(n) := \frac{1}{\lambda(B(\delta))} \int_{B(\delta)} \widehat{f}(n+x)\lambda(dx), \quad n \in \mathbb{N}.$$

Then, $f_\delta \in D^q$ and f_δ converges to f in D^q as $\delta \to 0$.

Proof. For each $\delta > 0$, using Hölder's inequality, we have

$$\begin{aligned}|f_\delta(n)| &= \frac{1}{\lambda(B(\delta))} \left| \int_{B(\delta)} \widehat{f}(n+x)\lambda(dx) \right| \\ &\leqslant \frac{1}{\lambda(B(\delta))} \left(\int_{B(\delta)} |\widehat{f}(n+x)|^q \lambda(dx) \right)^{1/q} \lambda(B(\delta))^{(q-1)/q}.\end{aligned}$$

It follows that

$$\begin{aligned}|f_\delta(n)|^q &\leqslant \frac{1}{\lambda(B(\delta))}\int_{B(\delta)} |\widehat{f}(n+x)|^q \lambda(dx) = \frac{1}{\lambda(B(\delta))}\int_{\widehat{\mathbb{Z}}} |\widehat{f}(n+x)|^q \mathbf{1}_{B(\delta)}(x)\lambda(dx)\\ &= \frac{1}{\lambda(B(\delta))}\int_{\widehat{\mathbb{Z}}} |\widehat{f}(x)|^q \mathbf{1}_{B(\delta)}(x-n)\lambda(dx),\end{aligned}$$

and hence,

$$\frac{1}{N}\sum_{n=1}^{N} |f_\delta(n)|^q \leqslant \frac{1}{\lambda(B(\delta))}\int_{\widehat{\mathbb{Z}}} |\widehat{f}(x)|^q \left(\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{B(\delta)}(x-n)\right)\lambda(dx). \tag{3.11}$$

Note that the integrand of the above integral is bounded by $|\widehat{f}(x)|^q$. Moreover,

$$\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{B(\delta)}(x-n) \to \lambda(B(\delta)) \quad \text{as} \quad N\to\infty,$$

by the ergodic theorem ($x \mapsto x-1$ is ergodic, Theorem 3.13). Thus, letting $N\to\infty$ in (3.11) and using Lebesgue's dominated convergence theorem, we arrive at

$$\|f_\delta\|_q \leqslant \|\widehat{f}\|_{L^q}.$$

The above inequality guarantees that f_δ is a q-limit-periodic function. Indeed, let $\{\widehat{f}^{(m)}\}_m$ be a sequence of periodic functions converging to $\widehat{f}$ in $L^q(\widehat{\mathbb{Z}},\lambda)$. Then $f_\delta^{(m)}$ is also periodic and $\|f_\delta^{(m)} - f_\delta\|_q \leqslant \|\widehat{f}^{(m)} - \widehat{f}\|_{L^q} \to 0$ as $m\to\infty$.

For each m, from the uniform continuity of $\widehat{f}^{(m)}$ it follows that

$$\begin{aligned}|f_\delta^{(m)}(n) - \widehat{f}^{(m)}(n)| &\leqslant \frac{1}{\lambda(B(\delta))}\int_{B(\delta)} |\widehat{f}^{(m)}(n+x) - \widehat{f}^{(m)}(n)|\lambda(dx)\\ &\to 0 \quad \text{as} \quad \delta\to 0 \text{ uniformly in } n.\end{aligned}$$

Consequently, $\|f_\delta^{(m)} - f^{(m)}\|_q \leqslant \|f_\delta^{(m)} - f^{(m)}\|_u \to 0$ as $\delta\to 0$. In addition, we have the following estimate

$$\begin{aligned}\|f_\delta - f\|_q &\leqslant \|f_\delta - f_\delta^{(m)}\|_q + \|f_\delta^{(m)} - f^{(m)}\|_q + \|f^{(m)} - f\|_q\\ &\leqslant 2\|\widehat{f} - \widehat{f}^{(m)}\|_{L^q} + \|f_\delta^{(m)} - f^{(m)}\|_q.\end{aligned}$$

Therefore,

$$\limsup_{\delta\to 0} \|f_\delta - f\|_q \leqslant 2\|\widehat{f} - \widehat{f}^{(m)}\|_{L^q}.$$

The proof is complete by letting $m\to\infty$. □

Example 3.22. Let $k \in \{2,3,\ldots\}$. Let $f\colon \mathbb{N} \to \{0,1\}$ be the indicator function of the set of k-th power free integers, that is,

$$f(n) := \begin{cases} 1, & (\forall p : \text{prime},\ n \bmod p^k \neq 0),\\ 0, & (\exists p : \text{prime},\ n \bmod p^k = 0),\end{cases}$$

and let $\widehat{f}\colon \widehat{\mathbb{Z}} \to \{0,1\}$ be a natural extension of f defined by

$$\widehat{f}(x) := \begin{cases} 1, & (\forall p : \text{prime},\ x \bmod p^k \neq 0),\\ 0, & (\exists p : \text{prime},\ x \bmod p^k = 0).\end{cases}$$

Then $\widehat{f} = \mathbf{1}_B$, where $B = \bigcap_p(\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \in \mathcal{B}(\widehat{\mathbb{Z}})$. It is well known that $f \in D^q$ for all $1 \leqslant q < \infty$. Moreover, $T_q(f) = \widehat{f}$, as we will see in Section 3.4. For $m \in \mathbb{N}$, put $A_m := m\widehat{\mathbb{Z}} = \{ mx : x \in \widehat{\mathbb{Z}} \}$. For any sequence $\{n_l\}_l$ converging to 0 in $\widehat{\mathbb{Z}}$, by changing $B(\delta)$ to A_{n_l} in the proof of Theorem 3.21, we also have

$$f_{n_l}(n) := \frac{1}{\lambda(A_{n_l})} \int_{A_{n_l}} \widehat{f}(n+x)\lambda(dx) \xrightarrow{\|\cdot\|_q} f \quad \text{as} \quad l \to \infty.$$

Next, we calculate $\{f_{n_l}\}_l$ for a special sequence $n_l := \prod_{p \leqslant p_l} p^l := \prod_{i=1}^l p_i^l$, where p_l denotes the l-th prime. We have

$$f_{n_l}(n) = \frac{\lambda((B-n) \cap A_{n_l})}{\lambda(A_{n_l})} = \frac{\lambda(B \cap (A_{n_l}+n))}{\lambda(A_{n_l})}.$$

The numerator can now be expressed as

$$\begin{aligned}
\lambda(B \cap (A_{n_l}+n)) &= \lambda\left(\bigcap_p (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \cap (A_{n_l}+n)\right) \\
&= \lambda\left(\Big(\bigcap_{p \leqslant p_l} (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \cap (A_{n_l}+n)\Big) \cap \bigcap_{p > p_l} (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}})\right) \\
&= \lambda\left(\bigcap_{p \leqslant p_l} (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \cap (A_{n_l}+n)\right) \prod_{p > p_l} \lambda\left(\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}\right) \\
&= \lambda\left(\bigcap_{p \leqslant p_l} (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \cap (A_{n_l}+n)\right) \prod_{p > p_l} \left(1 - \frac{1}{p^k}\right).
\end{aligned}$$

The second last equality holds since the sets $\left(\bigcap_{p \leqslant p_l}(\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \cap (A_{n_l}+n)\right)$ and $\{(\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}})\}_{p > p_l}$ are independent. Fix $l \geqslant k$. For each $p \leqslant p_l$, we see

$$A_{n_l} + n = n_l\widehat{\mathbb{Z}} + n \subset p^k\widehat{\mathbb{Z}} + n.$$

It follows that $A_{n_l} + n \subset (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}})$, if $n \bmod p^k \neq 0$, while $(A_{n_l} + n) \cap (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) = \emptyset$, otherwise. Therefore,

$$\bigcap_{p \leqslant p_l} (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \cap (A_{n_l}+n) = \begin{cases} A_{n_l} + n, & (\forall p \leqslant p_l, p^k \nmid n), \\ \emptyset, & (\exists p \leqslant p_l, p^k \mid n). \end{cases}$$

Consequently,

$$f_{n_l}(n) = \begin{cases} \prod_{p > p_l} \left(1 - \frac{1}{p^k}\right), & (\forall p \leqslant p_l, p^k \nmid n), \\ 0, & (\exists p \leqslant p_l, p^k \mid n). \end{cases}$$

In particular, $f_{n_l}(n) = \prod_{p > p_l}\left(1 - \frac{1}{p^k}\right)$, if n is k-th power free. Thus, $\{f_{n_l}(n)\}_l$ converges to $f(n)$ uniformly on the set of k-th power free integers. However, the convergence is not uniform on $\mathbb{N}$ at all. Indeed, given any n_l, take $n = p_{l+1}^k$, for example. Then $f_{n_l}(n) = \prod_{p > p_l}\left(1 - \frac{1}{p^k}\right)$, which is not near $f(n) = 0$. However, $\{f_{n_l}\}_l$ converges to f in D^q for all $1 \leqslant q < \infty$ as stated above.

3.3 Convergence of Fourier expansions

3.3.1 Convergence of Fourier expansions in $L^q(\widehat{\mathbb{Z}}, \lambda)$

For each $n \in \mathbb{N}$, define an operator $\widehat{S}_n \colon L^q(\widehat{\mathbb{Z}}, \lambda) \to L^q(\widehat{\mathbb{Z}}, \lambda)$ as

$$\widehat{S}_n(\widehat{f}) := \sum_{\substack{r|n; \\ 1\leqslant a\leqslant r; (a,r)=1}} \langle \widehat{f}, \widehat{e}_{a/r} \rangle \, \widehat{e}_{a/r}, \quad \widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda).$$

It is easy to see that $\widehat{S}_n$ is a continuous linear operator and has the finite operator norm $\|\widehat{S}_n\|_q$. The following lemma is similar to [26, Theorem II.1.1].

Lemma 3.23. *Let $\{n_k\}_k \subset \mathbb{N}$ be a sequence converging to 0 in $\widehat{\mathbb{Z}}$. Then the following two conditions are equivalent:*

(i) *for all $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$,*

$$\|\widehat{S}_{n_k}(\widehat{f}) - \widehat{f}\|_{L^q} \to 0 \quad \textit{as} \quad k \to \infty; \tag{3.12}$$

(ii) *there exists a constant K_q such that $\|\widehat{S}_{n_k}\|_q \leqslant K_q$ for all k.*

Proof. (i) $\Rightarrow$ (ii). If the sequence $\{\widehat{S}_{n_k}(\widehat{f})\}_k$ converges to $\widehat{f}$, then $\{\widehat{S}_{n_k}(\widehat{f})\}_k$ is bounded for every $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$. Therefore, $\|\widehat{S}_{n_k}\|_q$ is uniformly bounded by the uniform boundedness principle.
(ii) $\Rightarrow$ (i). Let $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$. Given an $\varepsilon > 0$, there is a periodic function $\widehat{g}$ such that $\|\widehat{f} - \widehat{g}\|_{L^q} \leqslant \varepsilon/(K_q + 1)$. Let m be a period of $\widehat{g}$. Recall that the convergence of $\{n_k\}_k$ implies that all except finitely many n_k are multiples of m. Now, if n_k is a multiple of m, then $\widehat{S}_{n_k}(\widehat{g}) = \widehat{g}$, and hence

$$\begin{aligned}
\|\widehat{S}_{n_k}(\widehat{f}) - \widehat{f}\|_{L^q} &= \|\widehat{S}_{n_k}(\widehat{f}) - \widehat{S}_{n_k}(\widehat{g}) + \widehat{g} - \widehat{f}\|_{L^q} \\
&\leqslant \|\widehat{S}_{n_k}(\widehat{f}) - \widehat{S}_{n_k}(\widehat{g})\|_{L^q} + \|\widehat{g} - \widehat{f}\|_{L^q} \\
&\leqslant K_q \frac{\varepsilon}{K_q + 1} + \frac{\varepsilon}{K_q + 1} = \varepsilon,
\end{aligned}$$

which completes the proof. □

Lemma 3.24. *For all $n \in \mathbb{N}$,*

$$\|\widehat{S}_n\|_1 \leqslant 6,$$
$$\|\widehat{S}_n\|_2 \leqslant 1.$$

In other words, $K_1 = 6$ and $K_2 = 1$ satisfy the condition of Lemma 3.23(ii) *for every sequence $\{n_k\}_k$ converging to 0 in $\widehat{\mathbb{Z}}$.*

Proof. Since $\{\widehat{e}_\alpha\}_{\alpha \in \mathbb{Q}/\mathbb{Z}}$ is an orthonormal basis of $L^2(\widehat{\mathbb{Z}}, \lambda)$, it holds that

$$\|\widehat{S}_n(\widehat{f})\|_{L^2} \leqslant \|\widehat{f}\|_{L^2}, \quad \widehat{f} \in L^2(\widehat{\mathbb{Z}}, \lambda), \quad n \in \mathbb{N}.$$

Therefore, $\|\widehat{S}_n\|_2 \leqslant 1$ for all $n \in \mathbb{N}$.

Let us now show that $\|\widehat{S}_n\|_1 \leqslant 6$ for all $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$ and let $\widehat{f} \in L^1(\widehat{\mathbb{Z}}, \lambda)$. It follows from the definition of $\widehat{S}_n$ that

$$\langle \widehat{f}, \widehat{e}_{a/n} \rangle = \langle \widehat{S}_n(\widehat{f}), \widehat{e}_{a/n} \rangle, \quad 1 \leqslant a \leqslant n,$$

and hence, for any periodic function g with period n, we have

$$\langle \widehat{f}, \widehat{g} \rangle = \langle \widehat{S}_n(\widehat{f}), \widehat{g} \rangle.$$

Assume first that the function $\widehat{f}$ is real. Then $\widehat{S}_n(\widehat{f})$ is real, too. Let $A = \{x \in \widehat{\mathbb{Z}} : \widehat{S}_n(\widehat{f})(x) \geqslant 0\}$. It is clear that $\mathbf{1}_A(x)$ is a periodic function with period n. Hence, we have the following estimate

$$\begin{aligned}
&\int_A \left|\widehat{S}_n(\widehat{f})(x) - \widehat{f}(x)\right| \lambda(dx) \\
&= \int_{A\bigcap\{\widehat{S}_n(\widehat{f})-\widehat{f}\geqslant 0\}} \left(\widehat{S}_n(\widehat{f})(x) - \widehat{f}(x)\right) \lambda(dx) \\
&\quad + \int_{A\bigcap\{\widehat{S}_n(\widehat{f})-\widehat{f}<0\}} \left(\widehat{f}(x) - \widehat{S}_n(\widehat{f})(x)\right) \lambda(dx) \\
&= \int_A \left(\widehat{S}_n(\widehat{f})(x) - \widehat{f}(x)\right) \lambda(dx) + 2\int_{A\bigcap\{\widehat{S}_n(\widehat{f})-\widehat{f}<0\}} \left(\widehat{f}(x) - \widehat{S}_n(\widehat{f})(x)\right) \lambda(dx) \\
&\leqslant \langle \widehat{S}_n(\widehat{f}) - \widehat{f}, \mathbf{1}_A \rangle + 2\int_{A\bigcap\{\widehat{S}_n(\widehat{f})-\widehat{f}<0\}} \widehat{f}(x)\lambda(dx) \\
&\leqslant 0 + 2\int_A \left|\widehat{f}(x)\right| \lambda(dx) = 2\int_A \left|\widehat{f}(x)\right| \lambda(dx).
\end{aligned}$$

By the same argument, we also have

$$\int_{A^c} \left|\widehat{S}_n(\widehat{f})(x) - \widehat{f}(x)\right| \lambda(dx) \leqslant 2\int_{A^c} \left|\widehat{f}(x)\right| \lambda(dx).$$

Combining the above inequalities, we see

$$\begin{aligned}
\|\widehat{S}_n(\widehat{f}) - \widehat{f}\|_{L^1} &= \int_A \left|\widehat{S}_n(\widehat{f})(x) - \widehat{f}(x)\right| \lambda(dx) + \int_{A^c} \left|\widehat{S}_n(\widehat{f})(x) - \widehat{f}(x)\right| \lambda(dx) \\
&\leqslant 2\left(\int_A \left|\widehat{f}(x)\right| \lambda(dx) + \int_{A^c} \left|\widehat{f}(x)\right| \lambda(dx)\right) = 2\|\widehat{f}\|_{L^1}.
\end{aligned}$$

Consequently,

$$\|\widehat{S}_n(\widehat{f})\|_{L^1} \leqslant \|\widehat{S}_n(\widehat{f}) - \widehat{f}\|_{L^1} + \|\widehat{f}\|_{L^1} \leqslant 3\|\widehat{f}\|_{L^1}.$$

In the general case, write the complex-valued function $\widehat{f}$ as $\widehat{f} = \widehat{g} + \sqrt{-1}\,\widehat{h}$, where $\widehat{g}$ and $\widehat{h}$ are real-valued functions. Then $\widehat{S}_n(\widehat{f}) = \widehat{S}_n(\widehat{g}) + \sqrt{-1}\,\widehat{S}_n(\widehat{h})$, and hence the above estimate implies

$$\|\widehat{S}_n(\widehat{f})\|_{L^1} \leqslant \|\widehat{S}_n(\widehat{g})\|_{L^1} + \|\widehat{S}_n(\widehat{h})\|_{L^1} \leqslant 3(\|\widehat{g}\|_{L^1} + \|\widehat{h}\|_{L^1}) \leqslant 6\|\widehat{f}\|_{L^1}.$$

This means that $\|\widehat{S}_n\|_1 \leqslant 6$. The proof is complete. □

Next, we will use the theory of interpolation of norms and of linear operators to show the existence of a constant K_q satisfying the condition of Lemma 3.23(ii) for any $1 < q < 2$. We will apply the following two results.

Lemma 3.25 ([26, Theorem IV.1.2]). *Let B (resp. B') be a normed linear space with two consistent norms $\|\cdot\|_0$ and $\|\cdot\|_1$ (resp. $\|\cdot\|_0'$ and $\|\cdot\|_1'$). Let $\|\cdot\|_\alpha$ (resp. $\|\cdot\|_\alpha'$) denote the interpolating norm, $0 < \alpha < 1$. Let $\mathbf{S}$ be a linear transformation from B to B' which is bounded as*

$$(B, \|\cdot\|_j) \xrightarrow{\mathbf{S}} (B', \|\cdot\|_j'), \quad j = 0, 1.$$

Then $\mathbf{S}$ *is bounded as*

$$(B, \|\cdot\|_\alpha) \xrightarrow{\mathbf{S}} (B', \|\cdot\|'_\alpha),$$

and its norm $\|\mathbf{S}\|_\alpha$ *satisfies*

$$\|\mathbf{S}\|_\alpha \leqslant \|\mathbf{S}\|_0^{1-\alpha}\|\mathbf{S}\|_1^{\alpha}.$$

Lemma 3.26 ([26, Theorem IV.1.3]). *Let* $(\mathfrak{X}, \nu)$ *be a measure space,* $B = L^1 \bigcap L^\infty(\nu)$, *and let* $1 \leqslant p_0 < p_1 \leqslant \infty$. *Let* $\|\cdot\|_j$ *denote the norms induced on* B *by* $L^{p_j}(\nu)$, *and let* $\|\cdot\|_\alpha$ *denote the interpolating norms. Then* $\|\cdot\|_\alpha$ *coincides with the norm induced on* B *by* $L^{p_\alpha}(\nu)$ *where*

$$p_\alpha = \frac{p_0 p_1}{p_0\alpha + p_1(1-\alpha)} \qquad \left(= \frac{p_0}{1-\alpha}, \ \textit{if } p_1 = \infty\right).$$

Let $(\mathfrak{X}, \nu) = (\widehat{\mathbb{Z}}, \lambda)$ and $B = L^\infty(\widehat{\mathbb{Z}}, \lambda)$ in Lemma 3.26, moreover let $p_0 = 1$ and $p_1 = 2$. Then for $1 < q < 2$, the interpolating norm $\|\cdot\|_\alpha$ coincides with the L^q norm $\|\cdot\|_{L^q}$, where $\alpha = 2(1 - 1/q)$.

Let $B = B' = L^\infty(\widehat{\mathbb{Z}}, \lambda)$ in Lemma 3.25 and $\|\cdot\|_0 = \|\cdot\|'_0 = \|\cdot\|_{L^1}; \|\cdot\|_1 = \|\cdot\|'_1 = \|\cdot\|_{L^2}$, respectively. Then the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are consistent [26, IV.3.1]. Fix an $n \in \mathbb{N}$, and let $\mathbf{S} = \widehat{S}_n|_B$. Lemma 3.24 claims that $\|\mathbf{S}\|_0 \leqslant 6$ and $\|\mathbf{S}\|_1 \leqslant 1$. Now, applying Lemma 3.25 to $\alpha = 2(1 - 1/q)$, we see

$$\|\mathbf{S}\|_\alpha \leqslant 6^{1-\alpha} \cdot 1^\alpha = 6^{2/q-1}.$$

Since the interpolating norms $\|\cdot\|_\alpha = \|\cdot\|'_\alpha$ coincide with the L^q norm $\|\cdot\|_{L^q}$, it follows that

$$\|\widehat{S}_n(\widehat{f})\|_{L^q} = \|\mathbf{S}(\widehat{f})\|_\alpha \leqslant \|\mathbf{S}\|_\alpha \|\widehat{f}\|_{L^q} \leqslant 6^{2/q-1}\|\widehat{f}\|_{L^q}, \quad \widehat{f} \in L^\infty(\widehat{\mathbb{Z}}, \lambda).$$

This implies $\|\widehat{S}_n\|_q \leqslant 6^{2/q-1}$ because $L^\infty(\widehat{\mathbb{Z}}, \lambda)$ is dense in $L^q(\widehat{\mathbb{Z}}, \lambda)$. Therefore, for $1 < q < 2$, constants $K_q = 6^{2/q-1}$ satisfy the condition of Lemma 3.23(ii).

Next, consider $p, q > 1$ such that $1/p + 1/q = 1$. It is well known that the dual space of $L^p(\widehat{\mathbb{Z}}, \lambda)$ coincides with $L^q(\widehat{\mathbb{Z}}, \lambda)$. Moreover, it is easy to verify that the dual map of $\widehat{S}_n^{(p)}$ coincides with $\widehat{S}_n^{(q)}$, where $\widehat{S}_n^{(p)}$ and $\widehat{S}_n^{(q)}$ denote the same operator $\widehat{S}_n$ but on different spaces $L^p(\widehat{\mathbb{Z}}, \lambda)$ and $L^q(\widehat{\mathbb{Z}}, \lambda)$. It now follows from the duality that

$$\|\widehat{S}_n\|_p = \|\widehat{S}_n\|_q.$$

Therefore, constants K_q in Lemma 3.23(ii) exist for all $1 \leqslant q < \infty$. Consequently, for any sequence $\{n_k\}_k \subset \mathbb{N}$ converging to 0 in $\widehat{\mathbb{Z}}$ and for any $\widehat{f} \in L^q(\widehat{\mathbb{Z}}, \lambda)$ $(1 \leqslant q < \infty)$, we have

$$\|\widehat{S}_{n_k}(\widehat{f}) - \widehat{f}\|_{L^q} \to 0 \quad \textit{as} \quad k \to \infty. \tag{3.13}$$

3.3.2 Convergence of Fourier expansions of limit-periodic functions

Now we turn to the study of Fourier expansions of limit-periodic functions. For each $n \in \mathbb{N}$, define $S_n \colon D^q \to D^q$ as

$$S_n(f) := \sum_{\substack{r|n;\\ 1\leqslant a\leqslant r;(a,r)=1}} \langle f, e_{a/r}\rangle \, e_{a/r}, \quad f \in D^q.$$

Let $f \in D^q$ and $\widehat{f} := T_q(f) \in L^q(\widehat{\mathbb{Z}}, \lambda)$. Recall that by Proposition 3.17, we have

$$\langle f, e_{a/r}\rangle = \langle \widehat{f}, \widehat{e}_{a/r}\rangle, \quad r, a \in \mathbb{N}.$$

Consequently, $T_q(S_n(f)) = \widehat{S}_n(\widehat{f})$. Therefore, for any $n \in \mathbb{N}$,

$$\|S_n(f) - f\|_q = \|\widehat{S}_n(\widehat{f}) - \widehat{f}\|_{L^q}.$$

The above equality, together with (3.13), gives us the convergence of Fourier expansions of limit-periodic functions, namely;

Theorem 3.27. *Let $\{n_k\}_k \subset \mathbb{N}$ be a sequence converging to 0 in $\widehat{\mathbb{Z}}$. Then for every function f in D^q $(1 \leqslant q < \infty)$, we have*

$$\lim_{k\to\infty} \|S_{n_k}(f) - f\|_q = 0. \tag{3.14}$$

3.3.3 Convergence of Fourier expansions of almost-even functions

Let $c_r, r = 1, 2, \ldots$, be the Ramanujan sums,

$$c_r(n) := \sum_{1\leqslant a\leqslant r,(a,r)=1} e_{a/r}(n).$$

Let $\mathfrak{B}$ denote the space of even functions. Then, by [43, Theorem IV.1.1],

$$\mathfrak{B} = \mathrm{Lin}_{\mathbb{C}}\,[\, c_r : r = 1, 2, \ldots]\,.$$

Denote by $\mathfrak{B}^q$ the linear closure of $\mathfrak{B}$ with respect to $\|\cdot\|_q$. Functions in $\mathfrak{B}^q$ are called q-almost-even arithmetical functions. Moreover, $\{\varphi(r)^{-1/2}c_r\}_{r=1,2,\ldots}$ is an orthonormal basis of the "Hilbert space" $\mathfrak{B}^2$, where $\varphi(r)$ is the Euler function, $\varphi(r) = \#\{1 \leqslant a \leqslant r : (a, r) = 1\}$. For almost-even functions, the following result is a version of the convergence of Fourier expansions.

Corollary 3.28. *Let $\{n_k\}_k \subset \mathbb{N}$ be a sequence converging to 0 in $\widehat{\mathbb{Z}}$. Then for every function f in $\mathfrak{B}^q$ $(1 \leqslant q < \infty)$, we have*

$$\lim_{k\to\infty} \Big\| \sum_{r|n_k} a_r(f)\, c_r - f \Big\|_q = 0, \tag{3.15}$$

where $a_r(f) = \{\varphi(r)\}^{-1}\langle f, c_r\rangle, r = 1, 2, \ldots$, denote the Ramanujan coefficients of the function f.

Proof. Since $\mathfrak{B}^q \subset \mathfrak{D}^q$, the proof is complete, if we can show that for any $f \in \mathfrak{B}^q$,

$$\sum_{r|n_k} a_r(f)\, c_r = S_{n_k}(f). \tag{3.16}$$

Let us first show that (3.16) holds for $f \in \mathfrak{B}$. Take $f \in \mathfrak{B}$ of the form

$$f = \sum_{r\in I} b_r\, c_r,$$

where $I \subset \mathbb{N}$ is a finite set and $\{b_r\}_{r\in I} \subset \mathbb{C}$. Then $a_r(f) = b_r$, if $r \in I$, and $a_r(f) = 0$, otherwise. Thus,

$$\text{L.H.S. of (3.16)} = \sum_{I\ni r|n_k} b_r\, c_r.$$

For the R.H.S. of (3.16), observe that for $1 \leqslant a \leqslant r, (a, r) = 1$, we have

$$\langle f, e_{a/r}\rangle = \begin{cases} b_r, & \text{if } r \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}\text{R.H.S. of (3.16)} &= \sum_{I\ni r|n_k} \sum_{1\leqslant a\leqslant r,(a,r)=1} b_r\, e_{a/r} = \sum_{I\ni r|n_k} b_r \sum_{1\leqslant a\leqslant r,(a,r)=1} e_{a/r} \\ &= \sum_{I\ni r|n_k} b_r\, c_r = \text{L.H.S. of (3.16)}.\end{aligned}$$

This means that (3.16) holds for any $f \in \mathfrak{B}$. Since both sides of (3.16) are continuous operators with respect to f, it follows that (3.16) holds for any $f \in \mathfrak{B}^q$. □

Example 3.29. Consider again the indicator function f of the set of k-th power free integers. Define $f'\colon \mathbb{N} \to \mathbb{R}$ as

$$f'(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right),$$

where μ denotes the Möbius function. Recall that f is multiplicative; $f(p^l) = 1$, if $l < k$, and $f(p^l) = 0$, otherwise. Then f' is also multiplicative. Moreover, for each prime p and $l \geqslant 1$,

$$f'(p^l) = \mu(1)f(p^l) + \mu(p)f(p^{l-1}) = f(p^l) - f(p^{l-1}) = \begin{cases} -1, & \text{if } l = k, \\ 0, & \text{otherwise.}\end{cases}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{|f'(n)|}{n} = \prod_p \left(1 + \frac{|f'(p)|}{p} + \frac{|f'(p^2)|}{p^2} + \cdots\right) = \prod_p \left(1 + \frac{1}{p^k}\right) < \infty. \tag{3.17}$$

Therefore, by [43, Theorem VIII.2.1], the Ramanujan coefficients $a_r(f)$ are equal to

$$a_r(f) = \sum_{\substack{1\leqslant d<\infty;\\ d\equiv 0 \bmod r}} \frac{f'(d)}{d}.$$

Let $\{n_l\} \subset \mathbb{N}$ be a sequence converging to 0 in $\widehat{\mathbb{Z}}$. Since $f \in \mathfrak{B}^q(1 \leqslant q \leqslant \infty)$ (see Lemma 3.33), for each n_l, the Fourier expansion coincides with the Ramanujan expansion as we have shown in the proof of Corollary 3.28,

$$S_{n_l}(f)(n) = \sum_{r|n_l} a_r(f)\, c_r(n) = \sum_{r|n_l} c_r(n) \sum_{\substack{1\leqslant d<\infty;\\ d\equiv 0 \bmod r}} \frac{f'(d)}{d} = \sum_{1\leqslant d<\infty} \frac{f'(d)}{d} \sum_{r|n_l;r|d} c_r(n).$$

We have

$$\sum_{r|d} c_r(n) = \begin{cases} d, & \text{if } d \mid n, \\ 0, & \text{otherwise.}\end{cases} \tag{3.18}$$

Now, the Möbius inversion formula implies that

$$f(n) = \sum_{d|n} f'(d) = \sum_{1\leqslant d<\infty} \frac{f'(d)}{d} \sum_{r|d} c_r(n).$$

Therefore

$$f(n) - S_{n_l}(f)(n) = \sum_{1\leqslant d<\infty} \frac{f'(d)}{d} \left(\sum_{r|d} c_r(n) - \sum_{r|n_l;r|d} c_r(n)\right). \tag{3.19}$$

Recall that we have the convergence of Ramanujan expansions as follows

$$\|S_{n_l}(f) - f\|_q \to 0 \quad \text{as} \quad l \to \infty.$$

On the other hand, we can show that $\{S_{n_l}(f)(n)\}_l$ converges pointwise to $f(n)$ as $l \to \infty$. Indeed, fix $n \in \mathbb{N}$. It follows easily from (3.18) that

$$\left| \sum_{r|d} c_r(n) \right| \leqslant n \quad \text{for all } d.$$

Let $D(n_l) = \min_d \{d \neq \gcd(n_l, d)\}$. Then $D(n_l) \to \infty$ as $l \to \infty$. Moreover, we have

$$\begin{aligned} |f(n) - S_{n_l}(f)(n)| &= \left| \sum_{D(n_l) \leqslant d < \infty} \frac{f'(d)}{d} \Big(\sum_{r|d} c_r(n) - \sum_{r|n_l; r|d} c_r(n) \Big) \right| \\ &\leqslant \sum_{D(n_l) \leqslant d < \infty} \frac{|f'(d)|}{d} 2n \longrightarrow 0 \quad \text{as} \quad l \to \infty. \end{aligned}$$

Note that the condition (3.17) is enough to ensure the pointwise convergence of $S_{n_l}(f)(n)$. Comparing with [43, Theorem VIII.2.1(iv)], we need an additional condition to ensure

$$\sum_{r=1}^{\infty} a_r(f)\, c_r(n) = f(n), \qquad n \in \mathbb{N}.$$

3.4 Limit-periodic additive and multiplicative arithmetical functions

A function $f\colon \mathbb{N} \to \mathbb{C}$ is called additive (resp. multiplicative) if

$$f(mn) = f(m) + f(n) \quad (\text{resp. } f(mn) = f(m)f(n)) \quad \text{whenever } (m, n) = 1.$$

Let f be an additive (resp. a multiplicative) function. For each prime p, define a function f_p as

$$f_p(n) = f(p^k), \quad \text{if} \quad n \in p^k\mathbb{N} \setminus p^{k+1}\mathbb{N}, \quad k = 0, 1, \ldots.$$

Then the function f can be expressed by

$$f(n) = \sum_p f_p(n) \quad (\text{resp. } f(n) = \prod_p f_p(n)). \tag{3.20}$$

Note that for each $n \in \mathbb{N}$, the infinite sum (resp. product) is actually a finite one. Now, extend f_p to a random variable $\widehat{f}_p\colon \widehat{\mathbb{Z}} \to \mathbb{C}$ in a natural way as

$$\widehat{f}_p(x) = f(p^k), \quad \text{if} \quad x \in p^k\widehat{\mathbb{Z}} \setminus p^{k+1}\widehat{\mathbb{Z}}, \quad k = 0, 1, \ldots.$$

Then $\{\widehat{f}_p\}_p$ is a sequence of independent random variables. It is obvious that if we replace $\{f_p(n)\}_p$ in the expression (3.20) by $\{\widehat{f}_p(x)\}_p$, then the sum (resp. product), in general, is not finite at all. Does this sum (resp. product) converge? For additive function, Novoselov [39] proved the following results.

Theorem 3.30 (cf. [39, Proposition 46, 47]). *Let $f \in \mathfrak{D}^q$ be an additive function. Then the sum*

$$\widehat{f}(x) := \sum_p \widehat{f}_p(x)$$

converges for λ-a.e. $x \in \widehat{\mathbb{Z}}$. Moreover, $T_q(f) = \widehat{f}$.

The tool used to deal with additive functions is Kolmogov's three series theorem. For multiplicative functions, Novoselov [39] only gave some sufficient conditions for which $\prod_p \widehat{f}_p$ converges. This section deals with multiplicative case and our result is the following.

Theorem 3.31. *Let $f \in \mathfrak{D}^q$ be a multiplicative function with $\mathbf{M}[f] \neq 0$. Then the product*

$$\widehat{f}(x) := \prod_p \widehat{f}_p(x)$$

converges for λ-a.e. $x \in \widehat{\mathbb{Z}}$. Moreover, $T_q(f) = \widehat{f}$.

The following is a key lemma to show Theorem 3.31.

Lemma 3.32 (cf. [5, Theorem 1]). (i) *Let $\{X_n\}_n$ be a sequence of independent complex random variables with $\mathbf{E}[X_n] \neq 0$. Assume that $\prod_n \mathbf{E}[|X_n|]$ converges and $\prod_n \mathbf{E}[X_n]$ converges to a non-zero limit. Then $X := \prod_n X_n$ converges almost surely. Moreover, X is integrable and*

$$\mathbf{E}[|X|] \leqslant \prod_n \mathbf{E}[|X_n|].$$

(ii) *If, in addition, $\prod_n \mathbf{E}[|X_n|^q]$ converges for $q > 1$, then*

$$\mathbf{E}[X] = \prod_n \mathbf{E}[X_n], \quad \mathbf{E}[|X|^q] = \prod_n \mathbf{E}[|X_n|^q].$$

Proof. (i) Let

$$M_n := \frac{\prod_{k=1}^n X_k}{\prod_{k=1}^n \mathbf{E}[X_k]}.$$

Then $\{M_n\}_n$ is a martingale. It follows from our assumptions that

$$\mathbf{E}[|M_n|] = \frac{\prod_{k=1}^n \mathbf{E}[|X_k|]}{|\prod_{k=1}^n \mathbf{E}[X_k]|}, \quad n \in \mathbb{N},$$

is bounded. Therefore, by Doob's martingale convergence theorem (Theorem 0.40), the limit

$$M := \lim_{n\to\infty} M_n$$

exists almost surely. Consequently, the limit $X := \prod_n X_n$ also exists almost surely. Next, applying Fatou's lemma, we see

$$\mathbf{E}[|X|] \leqslant \liminf_{n\to\infty} \mathbf{E}[|\prod_{k=1}^n X_k|] = \lim_{n\to\infty} \prod_{k=1}^n \mathbf{E}[|X_k|] = \prod_n \mathbf{E}[|X_n|].$$

(ii) If, in addition, $\prod_n \mathbf{E}[|X_n|^q]$ converges for $q > 1$, then $\{\mathbf{E}[|M_n|^q]\}_n$ is bounded. It follows from Theorem 0.40(iii) that $\{M_n\}_n$ converges to M in L^q. Consequently,

$$\begin{aligned} \mathbf{E}[M] &= \lim_{n\to\infty} \mathbf{E}[M_n] = 1, \\ \mathbf{E}[|M|^q] &= \lim_{n\to\infty} \mathbf{E}[|M_n|^q] = \frac{\prod_n \mathbf{E}[|X_n|^q]}{|\prod_n \mathbf{E}[X_n]|^q}. \end{aligned}$$

The proof is complete by noting that $X = M \prod_n \mathbf{E}[X_n]$. □

For each prime p, it is easy to see that the random variable $\widehat{f}_p$ is integrable, if and only if

$$\sum_{k=0}^{\infty} |f(p^k)| \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) = \left(1 - \frac{1}{p}\right) \sum_{k=0}^{\infty} \frac{|f(p^k)|}{p^k} < \infty.$$

The expectation can be expressed by

$$\mathbf{E}[\widehat{f}_p] = \sum_{k=0}^{\infty} f(p^k) \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) = \left(1 - \frac{1}{p}\right) \sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}.$$

The following properties of multiplicative functions with non-zero mean value can be found in [43, Chapter VII].

Lemma 3.33. *Let $f \in \mathfrak{D}^q$ be a multiplicative function with $\mathbf{M}[f] \neq 0$. Then the following holds.*

(i) $f \in \mathfrak{B}^q$.

(ii)

$$\mathbf{M}[f] = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right) = \prod_p \mathbf{E}[\widehat{f}_p].$$

In particular, for every prime, $\mathbf{E}[\widehat{f}_p] \neq 0$.

Proof of Theorem 3.31. Since $\mathfrak{D}^q \subset \mathfrak{D}^1$, it is enough to prove Theorem 3.31 for the case $q = 1$. Let $f \in \mathfrak{D}^1$ be a multiplicative function with $\mathbf{M}[f] \neq 0$. Then $|f| \in \mathfrak{D}^1$ with $\mathbf{M}[|f|] \geqslant |\mathbf{M}[f]| > 0$. Moreover, the function $|f|$ is also multiplicative and $\widehat{|f|}_p = |\widehat{f}_p|$. Now, applying Lemma 3.33(ii) to functions f and $|f|$, we have the following.

(i) The product $\prod_p \mathbf{E}[|\widehat{f}_p|]$ converges.

(ii) The product $\prod_p \mathbf{E}[\widehat{f}_p]$ converges to the non-zero value $\mathbf{M}[f]$.

These mean that the sequence $\{\widehat{f}_p\}_p$ satisfies the condition of Lemma 3.32(i). Therefore,

$$\widehat{f}(x) := \prod_p \widehat{f}_p(x), \quad \lambda\text{-a.e. } x \in \widehat{\mathbb{Z}}.$$

Next, we will prove that $T_1(f) = \widehat{f}$. This proof is divided into three steps.
Step 1. When $q > 1$, $\mathbf{M}[|f|^q] = \mathbf{E}[|\widehat{f}|^q]$ and the Ramanujan coefficients of f coincide with those of $\widehat{f}$, that is,

$$\langle f, c_r \rangle = \langle \widehat{f}, \widehat{c}_r \rangle, \quad r \in \mathbb{N}, \tag{3.21}$$

where $\widehat{c}_r := T_u(c_r)$. Indeed, $f \in \mathfrak{D}^q$ implies $|f|^q \in \mathfrak{D}^1$. Then Lemma 3.33(ii) applying for $|f|^q$ implies the convergence of the product $\prod_p \mathbf{E}[|\widehat{f}_p|^q]$. Therefore, by Lemma 3.32(ii),

$$\mathbf{E}[|\widehat{f}|^q] = \prod_p \mathbf{E}[|\widehat{f}_p|^q] = \mathbf{M}[|f|^q]. \tag{3.22}$$

Next, we will prove (3.21). It follows from the following expression of the Ramanujan sum $c_r(n) = \sum_{d|n;d|r} \mu(r/d)\, d = \sum_{d|r} \mu(r/d)\, d\, \mathbf{1}_{d\mathbb{N}}(n)$, that

$$\widehat{c}_r(x) = \sum_{d|r} \mu(r/d)\, d\, \mathbf{1}_{d\widehat{\mathbb{Z}}}(x).$$

Therefore, (3.21) will hold, if

$$\langle f, \mathbf{1}_{d\mathbb{N}}\rangle = \langle \widehat{f}, \mathbf{1}_{d\widehat{\mathbb{Z}}}\rangle, \quad d \in \mathbb{N}. \tag{3.23}$$

Let us prove (3.23). Fix $d \in \mathbb{N}$ and write $d = \prod_{p\in I} p^{\alpha(p)}$, where I is the set of primes p such that $\alpha(p) > 0$. Then $\langle f, \mathbf{1}_{d\mathbb{N}}\rangle$ is represented as ([43, Theorem VIII.4.4])

$$\langle f, \mathbf{1}_{d\mathbb{N}}\rangle = \mathbf{M}[f] \prod_{p\in I} \left(\frac{f(p^{\alpha(p)})}{p^{\alpha(p)}} + \frac{f(p^{\alpha(p)+1})}{p^{\alpha(p)+1}} + \cdots \right) \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots \right)^{-1},$$

or it can be rewritten as

$$\langle f, \mathbf{1}_{d\mathbb{N}}\rangle = \prod_{p\in I} \mathbf{E}[\widehat{f}_p \mathbf{1}_{p^{\alpha(p)}\widehat{\mathbb{Z}}}] \prod_{p\notin I} \mathbf{E}[\widehat{f}_p].$$

The sequence $\{\widehat{f}_p\}_{p\notin I}$ also satisfies the condition of Lemma 3.32(ii) because I is a finite set. Therefore,

$$\mathbf{E}[\prod_{p\notin I} \widehat{f}_p] = \prod_{p\notin I} \mathbf{E}[\widehat{f}_p].$$

This implies that

$$\begin{aligned}
\langle \widehat{f}, \mathbf{1}_{d\widehat{\mathbb{Z}}}\rangle &= \mathbf{E}\Big[\prod_{p} \widehat{f}_p \prod_{p\in I} \mathbf{1}_{p^{\alpha(p)}\widehat{\mathbb{Z}}}\Big] = \mathbf{E}\Big[\prod_{p\in I} \widehat{f}_p \mathbf{1}_{p^{\alpha(p)}\widehat{\mathbb{Z}}} \prod_{p\notin I} \widehat{f}_p\Big] \\
&= \mathbf{E}\Big[\prod_{p\in I} \widehat{f}_p \mathbf{1}_{p^{\alpha(p)}\widehat{\mathbb{Z}}}\Big]\mathbf{E}\Big[\prod_{p\notin I} \widehat{f}_p\Big] = \prod_{p\in I} \mathbf{E}[\widehat{f}_p \mathbf{1}_{p^{\alpha(p)}\widehat{\mathbb{Z}}}] \prod_{p\notin I} \mathbf{E}[\widehat{f}_p] \quad = \langle f, \mathbf{1}_{d\mathbb{N}}\rangle.
\end{aligned}$$

Step 2. We will prove $T_q(f) = \widehat{f}$ in the special case $q = 2$. Since the function $f \in \mathfrak{D}^2$ is a multiplicative function with non-zero mean value, Lemma 3.33(i) implies that $f \in \mathfrak{B}^2$. Moreover, $\{\varphi(r)^{-1/2} c_r\}_{r\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $\mathfrak{B}^2$, thus we have Parseval's identity as follows

$$\sum_{r=1}^{\infty} \frac{|\langle f, c_r\rangle|^2}{\varphi(r)} = \|f\|_2^2 = \mathbf{M}[|f|^2].$$

It then follows that

$$\sum_{r=1}^{\infty} \frac{|\langle \widehat{f}, \widehat{c}_r\rangle|^2}{\varphi(r)} = \|\widehat{f}\|_{L^2}^2 = \mathbf{E}[|\widehat{f}|^2].$$

Note that $\{\varphi(r)^{-1/2} \widehat{c}_r\}_{r\in\mathbb{N}}$ is an orthonormal system in $L^2(\widehat{\mathbb{Z}}, \lambda)$. Now Parseval's identities for f and $\widehat{f}$ imply

$$\sum_{r=1}^{R} \frac{\langle f, c_r\rangle}{\varphi(r)} c_r \overset{\|\cdot\|_2}{\longrightarrow} f \quad \text{as} \quad R \to \infty,$$

$$\sum_{r=1}^{R} \frac{\langle \widehat{f}, \widehat{c}_r\rangle}{\varphi(r)} \widehat{c}_r \overset{\|\cdot\|_{L^2}}{\longrightarrow} \widehat{f} \quad \text{as} \quad R \to \infty.$$

In addition, it is clear that

$$T_u\left(\sum_{r=1}^{R} \frac{\langle f, c_r\rangle}{\varphi(r)} c_r \right) = \sum_{r=1}^{R} \frac{\langle \widehat{f}, \widehat{c}_r\rangle}{\varphi(r)} \widehat{c}_r,$$

and hence, letting $R \to \infty$, we obtain $T_2(f) = \widehat{f}$.
Step 3. Now we turn back to the case $q = 1$. Following the first proof of [43, Theorem VII.4.1], we define the multiplicative function $f^{(K)}$ by truncation of f for $K \geqslant 2$

$$f^{(K)}(p^k) = \begin{cases} f(p^k), & \text{if } |f(p^k)| \leqslant K, \\ 1, & \text{if } |f(p^k)| > K. \end{cases}$$

Then the following two statements hold.

(i) $f^K \in \mathfrak{D}^2$.

(ii) $\|f^{(K)} - f\|_1 \to 0$ as $K \to \infty$.

The statement(ii) implies that $\mathbf{M}[f^{(K)}] \neq 0$ when K is large enough because $\mathbf{M}[f] \neq 0$. Therefore, when K is large enough, the natural extension $\widehat{f^{(K)}} = \prod_p \widehat{f_p^{(K)}}$ is well defined and moreover by Step 2, $T_2(f^{(K)}) = \widehat{f^{(K)}}$. We also have

(ii)′ $\|\widehat{f^{(K)}} - \widehat{f}\|_{L^1} \to 0$ as $K \to \infty$.

Let us prove statement(ii)′. It follows from the definition of $\widehat{f^{(K)}}$ that $|\widehat{f^{(K)}}(x)| \leqslant |\widehat{f}(x)|$, and hence $|\widehat{f^{(K)}}(x) - \widehat{f}(x)| \leqslant 2|\widehat{f}(x)|$ for λ-a.e. $x \in \widehat{\mathbb{Z}}$. Put

$$A^{(K)} := \{\, x \in \widehat{\mathbb{Z}} : \widehat{f^{(K)}}(x) \neq \widehat{f}(x) \,\}.$$

It is clear that

$$A^{(K)} \subset \bigcup_{\substack{p;\ k \geqslant 1; \\ |f(p^k)| > K}} \left(p^k \widehat{\mathbb{Z}} \setminus p^{k+1} \widehat{\mathbb{Z}} \right) \quad (\lambda\text{-a.e.}).$$

Consequently,

$$\begin{aligned} \lambda(A^{(K)}) &\leqslant \sum_{\substack{p;\ k \geqslant 1; \\ |f(p^k)| > K}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \leqslant \frac{1}{K} \left(1 - \frac{1}{p} \right) \sum_{\substack{p;\ k \geqslant 1; \\ |f(p^k)| > K}} \frac{|f(p^k)|}{p^k} \\ &\leqslant \frac{1}{K} \left(1 - \frac{1}{p} \right) \sum_{\substack{p;\ k \geqslant 1; \\ |f(p^k)| > 2}} \frac{|f(p^k)|}{p^k} \longrightarrow 0 \quad \text{as} \quad K \to \infty, \end{aligned}$$

where, we have used the fact that

$$\sum_{\substack{p;\ k \geqslant 1; \\ |f(p^k)| > 2}} \frac{|f(p^k)|}{p^k} < \infty, \quad \text{([43, Theorem VII.5.1, Definition VII.1.2])}.$$

Therefore,

$$\begin{aligned} \|\widehat{f^{(K)}} - \widehat{f}\|_{L^1} &= \int_{A^{(K)}} |\widehat{f^{(K)}}(x) - \widehat{f}(x)| \lambda(dx) \\ &\leqslant \int_{A^{(K)}} 2|\widehat{f}(x)| \lambda(dx) \to 0 \quad \text{as} \quad K \to \infty. \end{aligned}$$

Thus, the statement(ii)′ has been proved. Finally, the statement(ii) and the statement(ii)′, together with the continuity of T_1, imply that $T_1(f) = \widehat{f}$. The proof of Theorem 3.31 is complete. □

Chapter 4

The Distribution of k-th Power Free Integers

4.1 Introduction

For $k \in \{2, 3, \ldots\}$, let $X^{(k)}(n), n \in \mathbb{Z}$, be the indicator function of the set of k-th power free integers, that is,

$$X^{(k)}(n) := \begin{cases} 1, & (\forall p : \text{prime}, \ p^k \nmid n\,), \\ 0, & (\,\exists p : \text{prime}, \ p^k \mid n\,), \end{cases}$$

and let $S_N^{(k)}(m), m \in \mathbb{Z}$, denote the frequency of k-th power free integers between $m+1$ and $m+N$, that is,

$$S_N^{(k)}(m) := \frac{1}{N} \sum_{n=1}^{N} X^{(k)}(m+n).$$

Then it is well known that for each $m \in \mathbb{Z}$,

$$\lim_{N \to \infty} S_N^{(k)}(m) = \frac{1}{\zeta(k)}, \tag{4.1}$$

where ζ is the Riemann zeta function.

Many researchers have been interested in estimating the error $S_N^{(k)}(m) - 1/\zeta(k)$. Under the Riemann hypothesis, there is a conjecture about this;

$$\forall \varepsilon > 0, \quad N\left(S_N^{(k)}(m) - \frac{1}{\zeta(k)}\right) = O\left(N^{1/2k+\varepsilon}\right), \quad N \to \infty. \tag{4.2}$$

As is mentioned in [40], this conjecture should hold, but it is quite unlikely that it will be proved in near future, because it is related to the Riemann hypothesis so closely. In particular, in the case of $k = 2$, there have been many challenges to this conjecture, assuming the Riemann hypothesis, such as [1, 2, 20, 38]. Refer to [40] for an overview of this topic.

In this chapter, we study the probabilistic aspects of this problem. We take here a compactification method which has been developed in Chapter 3. Let us give an overview of this chapter.

Recall that $\widehat{\mathbb{Z}}$ denotes the ring of finite integral adeles and we consider the probability space $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$. Since $X^{(k)}(n)$ is a multiplicative limit-periodic function, it has the natural extension $X^{(k)}(x)$ as a random variable on $(\widehat{\mathbb{Z}}, \mathcal{B}(\widehat{\mathbb{Z}}), \lambda)$. As a consequence of ergodic property of the shift $x \mapsto x+1$ on $\widehat{\mathbb{Z}}$, we get the following law of large numbers

$$\lim_{N\to\infty} S_N^{(k)}(x) := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} X^{(k)}(x+n) = \mathbf{E}[X^{(k)}] = \frac{1}{\zeta(k)}, \qquad \lambda\text{-a.e. } x \in \widehat{\mathbb{Z}}, \tag{4.3}$$

which is the adelic version of (4.1).

The main aim of this chapter is to study the convergence rate of the law of large numbers (4.3). With the help of the explicit formula for the random variable $S_N^{(k)}$ given in Section 4.2, we can estimate the rate of convergence in Section 4.3 as follows;

$$\mathbf{E}\left[\left(N\left(S_N^{(k)} - \frac{1}{\zeta(k)}\right)\right)^2\right] \sim const \cdot N^{1/k}.$$

Finally, in Section 4.4, the last estimate is translated into the language of $\mathbb{Z}$ as

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\left(N\left(S_N^{(k)}(m) - \frac{1}{\zeta(k)}\right)\right)^2 \sim const \cdot N^{1/k} \quad \text{(Corollary 4.17).}$$

This may be called as a mean square version of the conjecture (4.2). It should be noted that we do not need the Riemann hypothesis to prove this and nevertheless get the same exponent as in the conjecture.

4.2 Explicit formula for S_N

In what follows, we fix an integer $k \geqslant 2$. It is known that $X^{(k)}(n)$ is multiplicative and $X^{(k)}(n) \in D^q$ for all $1 \leqslant q < \infty$. However, for the sake of completeness, let us give the proof here. It follows from the definition of $X^{(k)}(n)$ that $X^{(k)}(n)$ is multiplicative and

$$X^{(k)}(n) = \prod_p (1 - \rho_{p^k}(n)).$$

For each $L \in \mathbb{N}$, let $X_L^{(k)} := \prod_{p \leqslant p_L}(1 - \rho_{p^k}(n))$. Then $X_L^{(k)}$ converges to $X^{(k)}$ in D^q. Indeed, observe that

$$|X^{(k)}(n) - X_L^{(k)}(n)| \leqslant \sum_{p>p_L} \rho_{p^k}(n),$$

which implies,

$$\frac{1}{N}\sum_{n=1}^{N}|X^{(k)}(n) - X_L^{(k)}(n)| \leqslant \frac{1}{N}\sum_{n=1}^{N}\sum_{p>p_L}\rho_{p^k}(n) = \sum_{p>p_L}\frac{1}{N}\sum_{n=1}^{N}\rho_{p^k}(n) \leqslant \sum_{p>p_L}\frac{1}{p^k}.$$

Thus,

$$\begin{aligned} \|X^{(k)} - X_L^{(k)}\|_q &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|X^{(k)}(n) - X_L^{(k)}(n)|^q = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|X^{(k)}(n) - X_L^{(k)}(n)| \\ &\leqslant \sum_{p>p_L}\frac{1}{p^k} \to 0 \quad \text{as} \quad L \to \infty. \end{aligned}$$

Since the functions $X_L^{(k)}$ are periodic and converge to $X^{(k)}$ in the $\|\cdot\|_q$ norm, it follows that $X^{(k)} \in D^q$ for all $1 \leqslant q < \infty$. Hence as we proved in Section 3.4,

$$X^{(k)}(x) = \prod_p (1 - \rho_{p^k}(x)), \quad x \in \widehat{\mathbb{Z}},$$

is the natural extension of $X^{(k)}(n)$.

If we put

$$B^{(k)} := \bigcap_p (\widehat{\mathbb{Z}} \setminus p^k\widehat{\mathbb{Z}}) \in \mathcal{B}(\widehat{\mathbb{Z}}),$$

then it is clear that $X^{(k)} = \mathbf{1}_{B^{(k)}}$, and thus,

$$\mathbf{E}[X^{(k)}] = \lambda(B^{(k)}) = \prod_p \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k)}.$$

Recall that the shift $\widehat{\mathbb{Z}} \ni x \mapsto x+1$ is ergodic (Theorem 3.13). Then applying the ergodic theorem to $X^{(k)}$, we obtain

$$S_N^{(k)}(x) := \frac{1}{N}\sum_{n=1}^{N} X^{(k)}(x+n) \overset{N\to\infty}{\longrightarrow} \mathbf{E}[X^{(k)}] = \frac{1}{\zeta(k)} \quad (\lambda\text{-a.e. } x \in \widehat{\mathbb{Z}}),$$

which is the adelic version of (4.1).

We are now in a position to give the explicit formula for $S_N^{(k)}(x)$. For each $L \in \mathbb{N}$, let

$$X_L^{(k)}(x) := \prod_{p \leqslant p_L} (1 - \rho_{p^k}(x)),$$

$$S_{N,L}^{(k)}(x) := \frac{1}{N}\sum_{n=1}^{N} X_L^{(k)}(x+n),$$

$$\mathbb{M}_L := \{\, u = p_1^{\alpha_1} \cdots p_L^{\alpha_L} \in \mathbb{N} : 0 \leqslant \alpha_1, \ldots, \alpha_L \leqslant L \,\}.$$

Remark 4.1. From now on, if there is no confusion, we will omit ${}^{(k)}$ in formulae. For example, X will be considered as $X^{(k)}$ and so on.

Lemma 4.2. *For each* $N \in \mathbb{N}$,

$$S_{N,L}(x) \overset{L\to\infty}{\longrightarrow} S_N(x) \ \textit{(pointwise convergence)}, \tag{4.4}$$

$$S_{N,L}(x) = \sum_{u \in \mathbb{M}_L} \mu(u) \left(\frac{1}{u^k} - \frac{1}{N}\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right)\right), \tag{4.5}$$

where μ denotes the Möbius function.

Proof. The convergence (4.4) is obvious. We now prove (4.5). The definition of $S_{N,L}(x)$ gives

$$\begin{aligned}
S_{N,L}(x) &= \frac{1}{N}\sum_{n=1}^{N}\prod_{p \leqslant p_L}(1 - \rho_{p^k}(x+n)) \\
&= \frac{1}{N}\sum_{n=1}^{N}\left(1 + \sum_{r=1}^{L}\sum_{1 \leqslant i_1 < \cdots < i_r \leqslant L} (-1)^r \rho_{p_{i_1}^k}(x+n) \cdots \rho_{p_{i_r}^k}(x+n)\right) \\
&= \frac{1}{N}\sum_{n=1}^{N}\left(1 + \sum_{r=1}^{L}\sum_{1 \leqslant i_1 < \cdots < i_r \leqslant L} (-1)^r \rho_{p_{i_1}^k \cdots p_{i_r}^k}(x+n)\right) \\
&= \frac{1}{N}\sum_{n=1}^{N}\sum_{u \mid p_1 \cdots p_L} \mu(u)\rho_{u^k}(x+n)
\end{aligned}$$

$$= \sum_{u \in \mathbb{M}_L} \mu(u) \left(\frac{1}{N} \sum_{n=1}^{N} \rho_{u^k}(x+n) \right). \tag{4.6}$$

Here we have

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^{N} \rho_{u^k}(x+n) &= \frac{1}{N} \left\lfloor \frac{N + x \bmod u^k}{u^k} \right\rfloor \\ &= \frac{1}{N} \left(\frac{N + x \bmod u^k}{u^k} - \frac{(N+x) \bmod u^k}{u^k} \right) \\ &= \frac{1}{u^k} - \frac{1}{N} \left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k} \right). \end{aligned} \tag{4.7}$$

Therefore, substituting (4.7) into (4.6), we obtain (4.5). The lemma is proved. □

The following is a key lemma in this chapter.

Lemma 4.3 (cf. [44, Lemma 8]). *For $u, v \in \mathbb{N}$ and $y, z \in \widehat{\mathbb{Z}}$, we have*

$$\begin{aligned} &\mathbf{E}\left[\left(\frac{(y+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \left(\frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v} \right) \right] \\ &= \frac{(y \bmod (u,v)) \wedge (z \bmod (u,v))}{\{u,v\}} \left(1 - \frac{(y \bmod (u,v)) \vee (z \bmod (u,v))}{(u,v)} \right), \end{aligned}$$

where the expectation $\mathbf{E}$ works on x, and

$$\begin{aligned} (u,v) &= \gcd(u,v), \\ \{u,v\} &= \operatorname{lcm}(u,v) = \textit{the least common multiple of } u \textit{ and } v. \end{aligned}$$

Proof. We divide the proof into four steps.
Step 1. For $a, b, c \in \mathbb{N}$ with $(b,c) = 1$ and for $x \in \mathbb{Z}$, it holds that

$$\frac{1}{b} \sum_{s=0}^{b-1} \frac{(x + sac) \bmod ab}{ab} = \frac{x \bmod a}{ab} + \frac{b-1}{2b}. \tag{4.8}$$

This is shown in the following way. Since $(b,c) = 1$, by a similar argument of [21, Theorem 56], we have

$$\begin{aligned} &\{ (x + sac) \bmod ab : s = 0, 1, \ldots, b-1 \} \\ &= \{ (x + sa) \bmod ab : s = 0, 1, \ldots, b-1 \}. \end{aligned}$$

Thus, it is enough to prove (4.8) only for $c = 1$. Moreover, we have

$$\begin{aligned} &\{ (x + sa) \bmod ab : s = 0, 1, \ldots, b-1 \} \\ &= \{ (x + a + sa) \bmod ab : s = 0, 1, \ldots, b-1 \}, \end{aligned}$$

so that we have only to prove (4.8) for $x = 0, 1, \ldots, a-1$. But then, for $s = 0, 1, \ldots, b-1$, we have $(x + sa) \bmod ab = x + sa$, consequently,

$$\frac{1}{b} \sum_{s=0}^{b-1} \frac{(x+sa) \bmod ab}{ab} = \frac{1}{b} \sum_{s=0}^{b-1} \frac{x + sa}{ab} = \frac{x}{ab} + \frac{b-1}{2b}.$$

Thus (4.8) is valid.

Step 2. By the fact that for $z \in \widehat{\mathbb{Z}}$,

$$(z + sac) \bmod ab = (z \bmod ab + sac) \bmod ab,$$
$$(z \bmod ab) \bmod a = z \bmod a,$$

and by Step 1, it is easy to see that for $a, b, c \in \mathbb{N}$ with $(b, c) = 1$ and $x, y \in \widehat{\mathbb{Z}}$,

$$\begin{aligned}
&\frac{1}{b}\sum_{s=0}^{b-1}\left(\frac{(y+x+sac) \bmod ab}{ab} - \frac{(x+sac) \bmod ab}{ab}\right) \\
&= \frac{1}{b}\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right).
\end{aligned}$$

Therefore, for any periodic function $f\colon \widehat{\mathbb{Z}} \to \mathbb{R}$ with period ac, we have

$$\begin{aligned}
&\mathbf{E}\left[\left(\frac{(y+x) \bmod ab}{ab} - \frac{x \bmod ab}{ab}\right) f(x)\right] \\
&\quad [\text{by the shift invariance of } \lambda] \\
&= \frac{1}{b}\sum_{s=0}^{b-1}\mathbf{E}\left[\left(\frac{(y+x+sac) \bmod ab}{ab} - \frac{(x+sac) \bmod ab}{ab}\right) f(x+sac)\right] \\
&= \mathbf{E}\left[\frac{1}{b}\sum_{s=0}^{b-1}\left(\frac{(y+x+sac) \bmod ab}{ab} - \frac{(x+sac) \bmod ab}{ab}\right) f(x)\right] \\
&= \frac{1}{b}\mathbf{E}\left[\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right) f(x)\right].
\end{aligned}$$

Step 3. Set $a := (u, v), b := u/a, c := v/a$ and f to be

$$f(x) := \frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v}.$$

Then Step 2 implies that

$$\begin{aligned}
&\mathbf{E}\left[\left(\frac{(y+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\left(\frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v}\right)\right] \\
&= \mathbf{E}\left[\left(\frac{(y+x) \bmod ab}{ab} - \frac{x \bmod ab}{ab}\right)\left(\frac{(z+x) \bmod ac}{ac} - \frac{x \bmod ac}{ac}\right)\right] \\
&= \frac{1}{b}\mathbf{E}\left[\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)\left(\frac{(z+x) \bmod ac}{ac} - \frac{x \bmod ac}{ac}\right)\right].
\end{aligned}$$

By letting y, b, c and $f(x)$ in Step 2 be $z, c, 1$ and

$$\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a},$$

respectively, we see that the last line above is equal to

$$\frac{1}{bc}\mathbf{E}\left[\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)\left(\frac{(z+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)\right]. \tag{4.9}$$

Step 4. Without loss of generality, we assume that $y \bmod a \leqslant z \bmod a$. By Corollary 3.7, the integrand of (4.9) is continuous, and it is periodic with period a. Therefore, Lemma 3.10 implies that

$$(4.9) = \frac{1}{bc}\frac{1}{a}\sum_{s=0}^{a-1}\left(\frac{(y+s) \bmod a}{a} - \frac{s \bmod a}{a}\right)\left(\frac{(z+s) \bmod a}{a} - \frac{s \bmod a}{a}\right). \tag{4.10}$$

Moreover, it is clear that

$$\frac{(y+s) \bmod a}{a} - \frac{s \bmod a}{a} = \begin{cases} \dfrac{y \bmod a}{a}, & \text{if } 0 \leqslant s < a - y \bmod a, \\ \dfrac{y \bmod a}{a} - 1, & \text{if } a - y \bmod a \leqslant s < a, \end{cases}$$

and that

$$\frac{(z+s) \bmod a}{a} - \frac{s \bmod a}{a} = \begin{cases} \dfrac{z \bmod a}{a}, & \text{if } 0 \leqslant s < a - z \bmod a, \\ \dfrac{z \bmod a}{a} - 1, & \text{if } a - z \bmod a \leqslant s < a. \end{cases}$$

Finally, dividing the sum (4.10) into three parts and using the above expressions, we arrive at

$$\begin{aligned}
(4.10) =& \frac{1}{bc}\frac{1}{a}\Bigg(\sum_{0 \leqslant s < a - z \bmod a} \frac{y \bmod a}{a}\frac{z \bmod a}{a} \\
&+ \sum_{a - z \bmod a \leqslant s < a - y \bmod a} \frac{y \bmod a}{a}\left(\frac{z \bmod a}{a} - 1\right) \\
&+ \sum_{a - y \bmod a \leqslant s < a} \left(\frac{y \bmod a}{a} - 1\right)\left(\frac{z \bmod a}{a} - 1\right)\Bigg) \\
=& \frac{1}{bc}\frac{1}{a}\Bigg(\frac{y \bmod a}{a}\frac{z \bmod a}{a}(a - z \bmod a) \\
&+ \frac{y \bmod a}{a}\left(\frac{z \bmod a}{a} - 1\right)(z \bmod a - y \bmod a) \\
&+ \left(\frac{y \bmod a}{a} - 1\right)\left(\frac{z \bmod a}{a} - 1\right)(y \bmod a)\Bigg) \\
=& \frac{1}{bc}\frac{1}{a}(y \bmod a)\left(1 - \frac{z \bmod a}{a}\right) \\
=& \frac{y \bmod (u,v)}{\{u,v\}}\left(1 - \frac{z \bmod (u,v)}{(u,v)}\right) \\
=& \frac{(y \bmod (u,v)) \wedge (z \bmod (u,v))}{\{u,v\}}\left(1 - \frac{(y \bmod (u,v)) \vee (z \bmod (u,v))}{(u,v)}\right).
\end{aligned}$$

The lemma is proved. □

A small modification of [44, Lemma 9] gives the following.

Lemma 4.4. *For any bounded function $H\colon \mathbb{N} \to \mathbb{R}$, it holds that*

$$\sum_{u,v=1}^{\infty} \frac{|\mu(u)\mu(v)|}{\{u,v\}^k}|H((u,v))| = \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^k}\prod_{p \nmid n}\left(1 + \frac{2}{p^k}\right) < \infty,$$

$$\sum_{u,v=1}^{\infty} \frac{\mu(u)\mu(v)}{\{u,v\}^k}H((u,v)) = \sum_{n=1}^{\infty} \frac{|\mu(n)|H(n)}{n^k}\prod_{p \nmid n}\left(1 - \frac{2}{p^k}\right).$$

Lemma 4.5. *For each $N \in \mathbb{N}$,*

$$\sum_{u=1}^{\infty} \mu(u)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right) =: T(x,N) \tag{4.11}$$

is convergent in $L^2(\widehat{\mathbb{Z}}, \lambda)$.

Proof. Fix an $N \in \mathbb{N}$. For finite sets $\mathbb{L}$ and $\mathbb{M}$ such that $\mathbb{L} \subset \mathbb{M} \subset \mathbb{N}$, Lemma 4.3 and Lemma 4.4 imply that

$$\begin{aligned}
&\mathbf{E}\Bigg[\Bigg(\sum_{u\in\mathbb{M}}\mu(u)\bigg(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\bigg)\\
&\quad -\sum_{u\in\mathbb{L}}\mu(u)\bigg(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\bigg)\Bigg)^2\Bigg]\\
&=\sum_{u,v\in\mathbb{M}\setminus\mathbb{L}}\mu(u)\mu(v)\mathbf{E}\Bigg[\bigg(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\bigg)\\
&\qquad\qquad \times\bigg(\frac{(N+x) \bmod v^k}{v^k}-\frac{x \bmod v^k}{v^k}\bigg)\Bigg]\\
&=\sum_{u,v\in\mathbb{M}\setminus\mathbb{L}}\mu(u)\mu(v)\frac{N \bmod (u,v)^k}{\{u,v\}^k}\bigg(1-\frac{N \bmod (u,v)^k}{(u,v)^k}\bigg)\\
&\leqslant N\sum_{u,v\in\mathbb{M}\setminus\mathbb{L}}\frac{|\mu(u)\mu(v)|}{\{u,v\}^k}\\
&\leqslant N\sum_{u,v\in\mathbb{N}\setminus\mathbb{L}}\frac{|\mu(u)\mu(v)|}{\{u,v\}^k}\to 0 \quad\text{as}\quad \mathbb{L}\nearrow\mathbb{N}.
\end{aligned}$$

The lemma is proved. □

By letting $\mathbb{M} \nearrow \mathbb{N}$ in the proof of Lemma 4.5, and then $\mathbb{L} \nearrow \mathbb{N}$, it follows that

$$\sum_{u\in\mathbb{L}}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\right)\xrightarrow{L^2} T(x,N) \quad\text{as}\quad \mathbb{L}\nearrow\mathbb{N}. \tag{4.12}$$

On the other hand,

$$\sum_{u=1}^{\infty}\frac{\mu(u)}{u^k}=\frac{1}{\zeta(k)} \quad \text{(absolute convergence)}, \tag{4.13}$$

and $S_{N,L} \xrightarrow{L^2} S_N$ by the bounded convergence theorem. Therefore, using these convergences in the formula (4.5), we have an explicit formula for $S_N^{(k)}$ as in the following theorem.

Theorem 4.6. *For each $N \in \mathbb{N}$, as an equality in $L^2(\widehat{\mathbb{Z}},\lambda)$, the following holds;*

$$S_N^{(k)}(x)=\frac{1}{\zeta(k)}-\frac{1}{N}\sum_{u=1}^{\infty}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\right). \tag{4.14}$$

4.3 Estimate of the L^2-norm and limit points in L^2

In Section 4.2, we proved that $N\left(S_N(x)-\dfrac{1}{\zeta(k)}\right)=-T(x,N)$ and

$$T(x,N)=\sum_{u=1}^{\infty}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\right) \quad (\text{in } L^2(\widehat{\mathbb{Z}},\lambda)).$$

Let us calculate the explicit formula for the L^2-norm of $T(x,N)$. By using Lemma 4.3 and Lemma 4.4, we have

$$\begin{aligned}
\mathbf{E}[|T(x,N)|^2] &= \lim_{U\to\infty} \mathbf{E}\left[\left(\sum_{u\leqslant U}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right)\right)^2\right] \\
&= \lim_{U\to\infty} \mathbf{E}\left[\sum_{u,v\leqslant U}\mu(u)\mu(v)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right)\right. \\
&\qquad\qquad \left.\times\left(\frac{(N+x) \bmod v^k}{v^k} - \frac{x \bmod v^k}{v^k}\right)\right] \\
&= \lim_{U\to\infty}\sum_{u,v\leqslant U}\frac{\mu(u)\mu(v)}{\{u,v\}^k}\left(N \bmod (u,v)^k\right)\left(1-\frac{N \bmod (u,v)^k}{(u,v)^k}\right) \\
&= \sum_{u,v\in\mathbb{N}}\frac{\mu(u)\mu(v)}{\{u,v\}^k}\left(N \bmod (u,v)^k\right)\left(1-\frac{N \bmod (u,v)^k}{(u,v)^k}\right) \\
&= \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^k}\left(N \bmod n^k\right)\left(1-\frac{N \bmod n^k}{n^k}\right)\prod_{p\nmid n}\left(1-\frac{2}{p^k}\right), \qquad (4.15)
\end{aligned}$$

where in the last line, we have applied Lemma 4.4 to

$$H(n) = H_N(n) := \left(N \bmod n^k\right)\left(1-\frac{N \bmod n^k}{n^k}\right).$$

Let

$$f(n) := \frac{|\mu(n)|}{\prod_{p\mid n}\left(1-\frac{2}{p^k}\right)}.$$

Then f is multiplicative, and moreover the following conditions hold:

(i) $\displaystyle\sum_p \frac{f(p)-1}{p} < \infty$; (ii) $\displaystyle\sum_p \frac{|f(p)-1|^2}{p} < \infty$;

(iii) $\displaystyle\sum_p\sum_{l\geqslant 2}\frac{|f(p^l)|^2}{p^l} < \infty$; (iv) $\displaystyle\sum_{l=1}^{\infty} f(p^l)p^{-l} \neq -1$.

Indeed, by definition,

$$f(p) = \frac{1}{1-\frac{2}{p^k}} = \frac{p^k}{p^k-2}; \quad f(p^l) = 0, (l = 2,3,\ldots).$$

Therefore,

$$\begin{aligned}
\sum_p \frac{f(p)-1}{p} &= \sum \frac{2}{p(p^k-2)} < \infty, \\
\sum_p \frac{|f(p)-1|^2}{p} &= \sum \frac{4}{p(p^k-2)^2} < \infty,
\end{aligned}$$

which proves (i) and (ii). The series in (iii) is 0 since $f(p^l) = 0, (l = 2,3,\ldots)$. (iv) is obvious since f is positive. Since f satisfies (i)–(iv), the mean value $\mathbf{M}[f]$ exists and is non-zero (see [15]). Moreover, by [43, Lemma VII.1.6],

$$\mathbf{M}[f] = \prod_p\left(1+\frac{f(p)-1}{p}-\frac{f(p)}{p^2}\right) = \prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{p^{k-1}}{p^k-2}\right).$$

Proposition 4.7. *As $N \to \infty$,*

$$\sum_{n=1}^{\infty} f(n)\left\{\frac{N}{n^k}\right\}\left(1-\left\{\frac{N}{n^k}\right\}\right) = N^{1/k}\left(\mathbf{M}[f]\frac{1}{k}\int_0^{\infty}\frac{\{x\}(1-\{x\})}{x^{1/k+1}}dx + o(1)\right),$$

where $\{x\}$ denotes the fractional part of x.

Note that

$$\mathbf{E}[|T(x,N)|^2] = \prod_p\left(1-\frac{2}{p^k}\right)\sum_{n=1}^{\infty} f(n)\left\{\frac{N}{n^k}\right\}\left(1-\left\{\frac{N}{n^k}\right\}\right).$$

Consequently, Proposition 4.7 gives us the main result of this section.

Theorem 4.8.

$$\lim_{N\to\infty} N^{-1/k}\mathbf{E}[|T(x,N)|^2] = \left(\prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{2}{p^k}\right)\right)\frac{\zeta(2-\frac{1}{k})}{(2\pi)^{1-\frac{1}{k}}\Gamma(\frac{1}{k})\sin\frac{\pi}{2k}} =: C_k.$$

It remains to calculate the value of the integral in Proposition 4.7.

Lemma 4.9.

$$\frac{1}{k}\int_0^{\infty}\frac{\{x\}(1-\{x\})}{x^{1/k+1}}dx = \frac{\zeta(2-\frac{1}{k})}{(2\pi)^{1-\frac{1}{k}}\Gamma(\frac{1}{k})\sin\frac{\pi}{2k}}.$$

Proof. The Fourier series of the function $\{x\}(1-\{x\})$ is as follows

$$\begin{aligned}
\{x\}(1-\{x\}) &= \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}\\
&= \frac{1}{\pi^2}\left(\frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}\right)\\
&= \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos 2n\pi x}{n^2} \qquad \left[\text{because } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\right]\\
&= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi x}{n^2}.
\end{aligned}$$

Termwise integration yields that

$$\begin{aligned}
\frac{1}{k}\int_0^{\infty}\frac{\{x\}(1-\{x\})}{x^{1/k+1}}dx &= \frac{1}{k}\int_0^{\infty}\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi x}{n^2}\frac{1}{x^{\frac{1}{k}+1}}dx\\
&= \frac{1}{k}\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}\frac{\sin^2 n\pi x}{x^{\frac{1}{k}+1}}dx\\
&= \frac{1}{k}\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}\frac{\sin^2 y}{(\frac{y}{n\pi})^{\frac{1}{k}+1}}\frac{dy}{n\pi}\\
&= \frac{1}{k}\frac{2}{\pi^{2-\frac{1}{k}}}\sum_{n=1}^{\infty}\frac{1}{n^{2-\frac{1}{k}}}\int_0^{\infty}\frac{\sin^2 y}{y^{\frac{1}{k}+1}}dy\\
&= \frac{2}{\pi^{2-\frac{1}{k}}}\left(\frac{1}{k}\int_0^{\infty}\frac{\sin^2 y}{y^{\frac{1}{k}+1}}dy\right)\zeta\left(2-\frac{1}{k}\right).
\end{aligned}$$

We here note that from a formula: $\int_0^\infty \frac{\sin vx}{x^u}dx = \frac{\pi v^{u-1}}{2\Gamma(u)\sin\frac{u\pi}{2}}(0 < u < 2, v > 0)$

$$\begin{aligned}\frac{1}{k}\int_0^\infty \frac{\sin^2 y}{y^{\frac{1}{k}+1}}dy &= \int_0^\infty \frac{1}{k}y^{-\frac{1}{k}-1}\sin^2 ydy \\ &= \int_0^\infty (-y^{-\frac{1}{k}})'\sin^2 ydy \\ &= -y^{-\frac{1}{k}}\sin^2 y\Big|_0^\infty - \int_0^\infty (-y^{-\frac{1}{k}})2\sin y\cos ydy \\ &= \int_0^\infty \frac{\sin 2y}{y^{\frac{1}{k}}}dy = \frac{\pi 2^{\frac{1}{k}-1}}{2\Gamma(\frac{1}{k})\sin\frac{\pi}{2k}}.\end{aligned}$$

Substituting this into the above, we have

$$\begin{aligned}\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}dx &= \frac{2}{\pi^{2-\frac{1}{k}}}\frac{\pi 2^{\frac{1}{k}-1}}{2\Gamma(\frac{1}{k})\sin\frac{\pi}{2k}}\zeta\Big(2-\frac{1}{k}\Big) \\ &= \frac{\zeta(2-\frac{1}{k})}{(2\pi)^{1-\frac{1}{k}}\Gamma(\frac{1}{k})\sin\frac{\pi}{2k}}.\end{aligned}$$

The proof is complete. □

In order to prove Proposition 4.7, we need the following lemma.

Lemma 4.10. *Let $\{a_n\}_n$ be a complex sequence. Put $s_n := a_1 + \cdots + a_n$. Assume that there exists a constant $c \in \mathbb{C}$ such that*

$$\frac{s_N}{N} \to c \quad as \quad N \to \infty. \tag{4.16}$$

Then, for any $s \in (0, \infty)$,

$$N^s \sum_{n=N}^{\infty} \frac{a_n}{n^{s+1}} \to \frac{c}{s} \quad as \quad N \to \infty. \tag{4.17}$$

Proof. Let $s_x = \sum_{l\leqslant x} a_l, (x \in \mathbb{R}^+)$, be an extension of s_n as a function on $\mathbb{R}^+$. Clearly $\lim_{x\to\infty} s_x/x = c$. First, we check the convergence of $\sum_n \frac{a_n}{n^{s+1}}$. For $N, M \in \mathbb{N}, N < M$,

$$\begin{aligned}\sum_{N\leqslant n\leqslant M} \frac{a_n}{n^{s+1}} &= \sum_{N\leqslant n\leqslant M} \frac{s_n - s_{n-1}}{n^{s+1}} \\ &= \sum_{N\leqslant n\leqslant M-1} s_n\left(\frac{1}{n^{s+1}} - \frac{1}{(n+1)^{s+1}}\right) - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}} \\ &= \sum_{N\leqslant n\leqslant M-1} s_n\int_n^{n+1}\left(-\frac{1}{x^{s+1}}\right)'dx - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}} \\ &= \sum_{N\leqslant n\leqslant M-1} s_n\int_n^{n+1}\frac{s+1}{x^{s+2}}dx - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}} \\ &= (s+1)\int_N^M \frac{s_x}{x^{s+2}}dx - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}} \\ &= (s+1)\int_N^M \frac{s_x}{x}\frac{dx}{x^{s+1}} - \frac{s_{N-1}}{N-1}\frac{N-1}{N}\frac{1}{N^s} + \frac{s_M}{M}\frac{1}{M^s}.\end{aligned}$$

This tells us that $\sum_n \frac{a_n}{n^{s+1}}$ is convergent. Next, letting $M \to \infty$ in the above, and then multiplying this by N^s yield that

$$\begin{aligned}
N^s \sum_{n=N}^{\infty} \frac{a_n}{n^{s+1}} &= (s+1)N^s \int_N^{\infty} \frac{s_x}{x} \frac{dx}{x^{s+1}} - \frac{s_{N-1}}{N-1} \frac{N-1}{N} \\
&= (s+1) \int_N^{\infty} \frac{s_x}{x} \frac{1}{(x/N)^{s+1}} \frac{dx}{N} - \frac{s_{N-1}}{N-1} \frac{N-1}{N} \\
&= (s+1) \int_1^{\infty} \frac{s_{Ny}}{Ny} \frac{dy}{y^{s+1}} - \frac{s_{N-1}}{N-1} \frac{N-1}{N}.
\end{aligned}$$

From Lebesgue's dominated convergence theorem, the assertion follows immediately. □

Proof of Proposition 4.7. Let us consider

$$\begin{aligned}
&\sum_{n=1}^{\infty} f(n) \left\{\frac{N}{n^k}\right\} \left(1 - \left\{\frac{N}{n^k}\right\}\right) \\
&= \sum_{n \leqslant N^{1/k}} f(n) \left\{\frac{N}{n^k}\right\} \left(1 - \left\{\frac{N}{n^k}\right\}\right) + \sum_{n > N^{1/k}} f(n) \left\{\frac{N}{n^k}\right\} \left(1 - \left\{\frac{N}{n^k}\right\}\right) \\
&= \sum_{n \leqslant N^{1/k}} f(n) \left\{\frac{N}{n^k}\right\} \left(1 - \left\{\frac{N}{n^k}\right\}\right) + N \sum_{n > N^{1/k}} \frac{f(n)}{n^k} - N^2 \sum_{n > N^{1/k}} \frac{f(n)}{n^{2k}}.
\end{aligned}$$

By Lemma 4.10, as $N \to \infty$,

$$N \sum_{n > N^{1/k}} \frac{f(n)}{n^k} = N^{1/k} \left((N^{1/k})^{k-1} \sum_{n > N^{1/k}} \frac{f(n)}{n^k} \right) = N^{1/k} \left(\frac{\mathbf{M}[f]}{k-1} + o(1) \right), \tag{4.18}$$

and

$$N^2 \sum_{n > N^{1/k}} \frac{f(n)}{n^{2k}} = N^{1/k} \left((N^{1/k})^{2k-1} \sum_{n > N^{1/k}} \frac{f(n)}{n^{2k}} \right) = N^{1/k} \left(\frac{\mathbf{M}[f]}{2k-1} + o(1) \right). \tag{4.19}$$

It will follow from Lemma 4.11 and Lemma 4.12 below that

$$\sum_{n \leqslant N^{1/k}} f(n) \left\{\frac{N}{n^k}\right\} \left(1 - \left\{\frac{N}{n^k}\right\}\right) = N^{1/k} \left(\mathbf{M}[f] \frac{1}{k} \int_1^{\infty} \frac{\{x\}(1-\{x\})}{x^{1/k+1}} dx + o(1) \right). \tag{4.20}$$

The proof is complete by combining the estimates (4.18)–(4.20). □

Lemma 4.11. *As* $N \to \infty$,

$$\sum_{n \leqslant N^{1/k}} \left\{\frac{N}{n^k}\right\} \left(1 - \left\{\frac{N}{n^k}\right\}\right) = N^{1/k} \left(\frac{1}{k} \int_1^{\infty} \frac{\{x\}(1-\{x\})}{x^{1/k+1}} dx + o(1) \right).$$

Proof. In the proof, we will us Euler-Maclaurin's formula: for any $\varphi \in C^1([a,b])$, we have

$$\sum_{a < n \leqslant b} \varphi(n) = \int_a^b \varphi(x)dx - \left(\{x\} - \frac{1}{2}\right) \varphi(x) \Big|_a^b + \int_a^b \left(\{x\} - \frac{1}{2}\right) \varphi'(x)dx.$$

Note that Euler-Maclaurin's formula still holds for functions which are continuous on $[a,b]$ and have continuous derivative except at finitely many points. Let

$$\varphi(x) := \left\{\frac{N}{x^k}\right\} \left(1 - \left\{\frac{N}{x^k}\right\}\right), \quad x \geqslant 1. \tag{4.21}$$

Then φ is continuous. Now, if $\left(\frac{N}{l+1}\right)^{1/k} < x \leqslant \left(\frac{N}{l}\right)^{1/k}$ for some positive integer l, then

$$\varphi(x) = \left(\frac{N}{x^k} - l\right)\left(1 - \left(\frac{N}{x^k} - l\right)\right) = -\left(\frac{N}{x^k} - l\right)^2 + \left(\frac{N}{x^k} - l\right),$$

and hence

$$\varphi'(x) = 2\left(\frac{N}{x^k} - l\right)\frac{Nk}{x^{k+1}} - \frac{Nk}{x^{k+1}} = \left(2\left\{\frac{N}{x^k}\right\} - 1\right)\frac{Nk}{x^{k+1}}.$$

It follows that φ has continuous derivative except at finitely many points in any bounded interval. For $L \in \mathbb{N}, L \leqslant N$, by applying Euler-Maclaurin's formula to the function φ with $a = \left(\frac{N}{L}\right)^{1/k}$ and $b = N^{1/k}$, and noting that $\varphi(a) = \varphi(b) = 0$, we obtain

$$\begin{aligned}
&\sum_{(\frac{N}{L})^{1/k} < n \leqslant N^{1/k}} \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right) \\
&= \int_a^b \left\{\frac{N}{x^k}\right\}\left(1 - \left\{\frac{N}{x^k}\right\}\right) dx + \int_a^b \left(\{x\} - \frac{1}{2}\right)\left(2\left\{\frac{N}{x^k}\right\} - 1\right)\frac{Nk}{x^{k+1}} dx \\
&\text{[change variable } y = N/x^k; x = a \to y = L; x = b \to y = 1; dx = -\frac{1}{k}\frac{N^{1/k}}{y^{1/k+1}} dy \text{]} \\
&= \frac{N^{1/k}}{k}\int_1^L \frac{\{y\}(1 - \{y\})}{y^{1/k+1}} dy + \int_1^L \left(\left\{\left(\frac{N}{y}\right)^{1/k}\right\} - \frac{1}{2}\right)(2\{y\} - 1) dy \\
&= \frac{N^{1/k}}{k}\int_1^L \frac{\{y\}(1 - \{y\})}{y^{1/k+1}} dy + O(L).
\end{aligned}$$

Note that the above estimate $O(L)$ is uniform for L, N. Choose $L = L(N)$ satisfying $L(N) \leqslant N; L(N) \to \infty; L(N) = o(N^{1/k})$, we see that

$$\begin{aligned}
&\sum_{n \leqslant N^{1/k}} \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right) \\
&= \sum_{n \leqslant (\frac{N}{L(N)})^{1/k}} \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right) + \sum_{(\frac{N}{L(N)})^{1/k} < n \leqslant N^{1/k}} \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right) \\
&= O\left(\frac{N^{1/k}}{L(N)^{1/k}}\right) + \frac{N^{1/k}}{k}\int_1^{L(N)} \frac{\{y\}(1 - \{y\})}{y^{1/k+1}} dy + O(L(N)) \\
&= N^{1/k}\left(\frac{1}{k}\int_1^{L(N)} \frac{\{y\}(1 - \{y\})}{y^{1/k+1}} dy + O\left(\frac{1}{L(N)^{1/k}}\right) + O\left(\frac{L(N)}{N^{1/k}}\right)\right).
\end{aligned}$$

The proof is complete by letting $N \to \infty$. □

Lemma 4.12. *As $N \to \infty$,*

$$\sum_{n \leqslant N^{1/k}} (f(n) - \mathbf{M}[f])\left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right) = o(N^{1/k}).$$

Proof. Put

$$\begin{cases} a_n = f(n) - \mathbf{M}[f], \\ S(x) = \sum_{n \leqslant x} a_n, \quad (x \in \mathbb{R}^+). \end{cases}$$

Then it is not so difficult to check this formula: for $\varphi \in C^1([a,b]), 1 \leqslant a < b < \infty$, we have

$$\sum_{a<n\leqslant b} a_n\varphi(n) = -\int_a^b S(x)\varphi'(x)dx + S(a)\varphi(a) - S(b)\varphi(b).$$

Of course, the above formula still holds if φ is continuous and has continuous derivative except at finitely many points in $[a,b]$. Now, for $L \in \mathbb{N}, L \leqslant N$, let $a = \left(\frac{N}{L}\right)^{1/k}$ and $b = N^{1/k}$. Then by applying the above formula to the function φ defined in (4.21) and noting that $f(a) = f(b) = 0$, we get

$$\sum_{a<n\leqslant b} a_n \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right) = -\int_a^b S(x)\left(2\left\{\frac{N}{x^k}\right\} - 1\right)\frac{Nk}{x^{k+1}}dx.$$

Given an $\varepsilon' > 0$, there exists an N_0 such that $|S_x| \leqslant \varepsilon' x$ for all $x \geqslant N_0$. If $a > N_0$, we have

$$\begin{aligned}
\left|\sum_{a<n\leqslant b} a_n \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right)\right| &= \left|\int_a^b S(x)\left(2\left\{\frac{N}{x^k}\right\} - 1\right)\frac{Nk}{x^{k+1}}dx\right| \\
&\leqslant \varepsilon' Nk\int_a^b \frac{1}{x^k}dx \qquad \text{(because } |S_x| \leqslant \varepsilon' x\text{)} \\
&= \frac{\varepsilon' Nk}{k-1}(a^{-k+1} - b^{-k+1}) \\
&\leqslant \frac{\varepsilon' Nk}{k-1}a^{-k+1} \\
&= \frac{\varepsilon' Nk}{k-1}\left(\frac{L}{N}\right)^{(k-1)/k} \\
&= \varepsilon' N^{1/k}\frac{k}{k-1}L^{(k-1)/k}.
\end{aligned}$$

In addition,

$$\left|\sum_{n\leqslant a} a_n \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right)\right| \leqslant cy = c\left(\frac{N}{L}\right)^{1/k},$$

where $c = \sup|a_n| < \infty$. Now, given $\varepsilon > 0$, choose an L such that $c/L^{1/k} < \varepsilon/2$. Next, choose ε' satisfying $\varepsilon'\frac{k}{k-1}L^{(k-1)/k} < \varepsilon/2$. Then there exists an N_0 as above. For N being large enough such that $a = \left(\frac{N}{L}\right)^{1/k} > N_0$, we have

$$\begin{aligned}
&\left|\sum_{n\leqslant N^{1/k}} a_n \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right)\right| \\
&\leqslant \left|\sum_{n\leqslant a} a_n \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right)\right| + \left|\sum_{a<n\leqslant b} a_n \left\{\frac{N}{n^k}\right\}\left(1 - \left\{\frac{N}{n^k}\right\}\right)\right| \\
&\leqslant c\left(\frac{N}{L}\right)^{1/k} + \varepsilon' N^{1/k}\frac{k}{k-1}L^{(k-1)/k} \\
&\leqslant \frac{\varepsilon}{2}N^{1/k} + \frac{\varepsilon}{2}N^{1/k} = \varepsilon N^{1/k}.
\end{aligned}$$

The lemma is proved. □

Let

$$Y_N(x) := -\frac{1}{N^{1/2k}}T(x,N) = \frac{1}{N^{1/2k}}\sum_{n=1}^{N}\left(X(x+n) - \frac{1}{\zeta(k)}\right).$$

By Theorem 4.8, $\lim_{N\to\infty}\mathbf{E}[|Y_N|^2] = C_k > 0$. From this, we may expect that the sequence $\{Y_N\}_N$ converges in distribution. We could not prove this yet. However, considering the L^2-limit, we get the following result.

Theorem 4.13. *$\{Y_N\}_{N=1,2,\ldots}$ has no limit point in $L^2(\widehat{\mathbb{Z}},\lambda)$.*

Lemma 4.14. *For fixed $N \in \mathbb{N}$,*

$$\lim_{M\to\infty}\mathbf{E}[Y_M Y_N] = 0.$$

Proof. Similarly as in showing the equality (4.15), by using Lemma 4.3 and Lemma 4.4 again, we have

$$\begin{aligned}
\mathbf{E}[T(x,M)T(x,N)] &= \lim_{U\to\infty}\mathbf{E}\Bigg[\left(\sum_{u\leqslant U}\mu(u)\left(\frac{(M+x)\bmod u^k}{u^k} - \frac{x\bmod u^k}{u^k}\right)\right) \\
&\qquad\times\left(\sum_{v\leqslant U}\mu(v)\left(\frac{(N+x)\bmod v^k}{v^k} - \frac{x\bmod v^k}{v^k}\right)\right)\Bigg] \\
&= \lim_{U\to\infty}\sum_{u,v\leqslant U}\mu(u)\mu(v)\mathbf{E}\Bigg[\left(\frac{(M+x)\bmod u^k}{u^k} - \frac{x\bmod u^k}{u^k}\right) \\
&\qquad\times\left(\frac{(N+x)\bmod v^k}{v^k} - \frac{x\bmod v^k}{v^k}\right)\Bigg] \\
&= \lim_{U\to\infty}\sum_{u,v\leqslant U}\frac{\mu(u)\mu(v)}{\{u,v\}^k}H_{M,N}((u,v)) \\
&= \sum_{u,v\in\mathbb{N}}\frac{\mu(u)\mu(v)}{\{u,v\}^k}H_{M,N}((u,v)) \\
&= \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^k}H_{M,N}(n)\prod_{p\nmid n}\left(1-\frac{2}{p^k}\right),
\end{aligned}$$

where

$$H_{M,N}(n) := \left((M\bmod n^k)\wedge(N\bmod n^k)\right)\left(1-\frac{(M\bmod n^k)\vee(N\bmod n^k)}{n^k}\right)$$

is a bounded function. It is easy to see that

$$0 \leqslant H_{M,N}(n) \leqslant H_N(n), \quad \forall n\in\mathbb{N}.$$

Thus,

$$\begin{aligned}
0 \leqslant \mathbf{E}[Y_M Y_N] &= \frac{1}{M^{1/2k}N^{1/2k}}\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^k}H_{M,N}(n)\prod_{p\nmid n}\left(1-\frac{2}{p^k}\right) \\
&\leqslant \frac{1}{M^{1/2k}N^{1/2k}}\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^k}H_N(n)\prod_{p\nmid n}\left(1-\frac{2}{p^k}\right)
\end{aligned}$$

$$= \frac{N^{1/2k}}{M^{1/2k}} \frac{1}{N^{1/k}} \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^k} H_N(n) \prod_{p \nmid n} \left(1 - \frac{2}{p^k}\right)$$
$$= \frac{N^{1/2k}}{M^{1/2k}} \mathbf{E}[|Y_N|^2] \longrightarrow 0 \quad \text{as} \quad M \to \infty.$$

The lemma is proved. □

Proof of Theorem 4.13. For $0 < N < M$, we consider

$$\mathbf{E}[|Y_M - Y_N|^2] = \mathbf{E}[|Y_M|^2] + \mathbf{E}[|Y_N|^2] - 2\mathbf{E}[Y_M Y_N].$$

From Lemma 4.14 and Theorem 4.8, it follows that

$$\lim_{M\to\infty} \mathbf{E}[|Y_M - Y_N|^2] = \mathbf{E}[|Y_M|^2] + C_k \geqslant C_k > 0.$$

This implies that $\{Y_N\}_N$ has no limit point in $L^2(\widehat{\mathbb{Z}}, \lambda)$. The theorem is proved. □

Remark 4.15. Since $\{\|Y_N\|_2\}_N$ is bounded, the sequence of probability measures $\{\lambda \circ Y_N^{-1}\}_N$ on $\mathbb{R}$ is tight. Therefore, for any subsequence $\{N_j\}_j$ there exists a subsubsequence $\{N_j'\}$ such that $\{\lambda \circ Y_{N_j'}^{-1}\}_j$ converges weakly, or $\{Y_{N_j'}\}_j$ converges in distribution.

4.4 Mean square convergence rate

Recall that since $X(n)$ is multiplicative and $X(n) \in D^q$ for all $1 \leqslant q < \infty$, $T_q(X(n)) = X(x)$ for all $1 \leqslant q < \infty$. In particular, $T_2(X(n)) = X(x)$. Property(i) of T_q (Section 3.2) implies that for each $N \in \mathbb{N}$,

$$T_2\left(S_N(n) - \frac{1}{\zeta(k)}\right) = \left(S_N(x) - \frac{1}{\zeta(k)}\right).$$

Thus, by the isometric of T_2, we get the following result.

Lemma 4.16. *For each $N \in \mathbb{N}$,*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \left(S_N(m) - \frac{1}{\zeta(k)}\right)^2 = \mathbf{E}\left[\left(S_N - \frac{1}{\zeta(k)}\right)^2\right]. \tag{4.22}$$

The convergence (4.22) together with the estimate in Theorem 4.8 gives us the estimate of the mean square convergence rate, namely;

Corollary 4.17. *As $N \to \infty$,*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \left(N\left(S_N^{(k)}(m) - \frac{1}{\zeta(k)}\right)\right)^2 \sim C_k N^{1/k},$$

where C_k is the constant in Theorem 4.8.

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