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Applications of Order Statistics to Estimation Problems

Masao Futatsuya

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Abstract

This thesis deals with applications of order statistics to ranked set sampling and interval estimation. The ranked set sampling for estimating a finite population mean is studied when sampling is carried out without replacement. Some confidence intervals for the mean of a normal distribution are studied when samples are censored.

In Chapter 2, we first prove the positive likelihood ratio dependence between order statistics of one sample and the negative regression dependence between order statistics of two samples without replacement drawn from a finite population. By using these results, we show that, when samples are drawn without replacement from a finite population, the relative precision of the ranked set sampling estimator of the population mean relative to a simple random sample estimator with the same number of units quantified is never smaller than 1, and is greater than 1 unless $N-1$ elements of the population of size N have the same value.

In Chapter 3, we consider the finite population of maximum relative precision under given population size and sample size. Unlike the case of sampling from an infinite population, here the discrete uniform distribution does not lead to the maximum relative precision.

In Chapter 4, we discuss the problem of interval estimation for the mean of a normal distribution based on censored samples. Five kinds of confidence intervals of the mean from Type I and Type II censored samples are compared by simulation. Two of the confidence intervals are based on the maximum likelihood estimators of parameters, two others are based on the best linear unbaised estimators of parameters, and the last one is the confidence interval proposed by Halperin (1961). The main conclusion is that the ML-T intervals seem to be practicable in the case of the Type I and the LU-T intervals in the case of Type II.

Contents

Chapter 1

Introduction

Order statistics and functions of these statistics play an important role in numerous practical applications. The smallest and the largest order statitistics (extremes) arise in floods and droughts, as well as in breaking strength and fatigue failure studies. The sample range is widely employed in the field of quality control as a quick esitimator of standard deviation. The studentized range is useful in the detection of outliers. Order statistics appear in a natural way in inference procedures when the sample is censored. We take up in Chapter 4 the problem of censored sample. Further, an ingenious application of order statistics can be found in the ranked set sampling procedure. Ranked set sampling will be studied in Chapter 2 and 3. Other applications of order statistics arise in the study of system reliability and in the area of data compression, furthermore, order statistics play an important supporting role in multiple comparisons and multiple decision procedures such as the ranking of treatment means.

In this thesis, ranked set sampling (RSS) for estimating a population mean is studied when sampling is without replacement from a finite population. In sampling field work, research workers encounter a situation where the exact measurement (or quantification) of a selected unit is either difficult or expensive in terms of time, money or labor, but where the ranking of selected units can be done with reasonable success on the basis of visual inspection or other rough methods not requiring actual measurement. For example, suppose that it is required to estimate the mean volume of trees in an area. Then, the measurement of a selected tree will be difficult, but the ranking of a small set of selected trees will be relatively easy. In such a situation, RSS is used to obtain an improved estimate of the mean volume. The concept of RSS was first introduced by McIntyre (1952) for estimating mean pasture yields. Takahasi and Wakimoto (1968) considered the much needed mathematical foundation for RSS and proposed independently the same estimator as McIntyre. The RSS procedure consists of drawing n (called the *set-size*) random samples from the population, each of size n, and ranking each of them. Then the smallest unit from the first sample is chosen for measurement, as is the second smallest unit from the second sample. The process continues in this way until the largest unit from the nth sample is measured, for a total of n measured units, one from each order class. The entire cycle is repeated r times (called the *number of cycles*) until a total of rn^2 units have been drawn from the population but only rn have been measured. The arithmetical mean of these rn mesurements is called the RSS estimator. The RSS estimator is an unbiased estimator of the population mean. McIntyre (1952) examined the precision of the RSS estimator relative to the estimator from a simple random sample (SRS) of the same size rn, and defined the relative precision as

$$RP = \frac{\text{variance of SRS estimator}}{\text{variance of RSS estimator}}.$$

Takahasi and Wakimoto (1968) have shown that, for any continuous distribution with finite variance, RP is bounded below by 1 and above by $(n+1)/2$, and that the upper bound is achieved only by the uniform distribution. Because of this potential for observational economy, the RSS method has received growing attention both from statisticians and research workers. Patil *et al.* (1994) have reviewed the theory, methods, and applications of RSS. However, these researches have been mostly concerned with sampling from infinite (continuous) populations. In this thesis, we consider RSS estimators of population means when sampling is without replacement from a finite population. Here, the corresponding SRS estimator in the denominator of the equation RP consists of drawing a simple random sample of size nr without replacement from the finite population.

In Chapter 2, we show that RP is never smaller than 1, and is greater than 1 unless $N-1$ units of a population of size N have the same value [see Takahasi and Futatsuya (1997)]. We begin by deciding some mathematical interests between order statistics from a finite population. In Section 2.2, we prove the positive likelihood ratio dependence between order statistics of a sample. The corresponding result in the conventional case of an infinite population was proved by Lehmann (1966). As a first step of the proof, we assume that the population values are labeled 1 through N; that is, when the population values are all distinct. Using a multivariate hypergeometric type argument, we prove this dependence. For the case of a general finite population whose values are not neccessarily distinct, the positive likelihood ratio dependence can be obtained by considering the track of the number of ties among the population values and by the preceding result. In Section 2.3, it follows that, in contrast to the continuous case, joint distributions between order statistics of two samples are negatively regression dependent. As in Section 2.2, we first assume that the population values are labeled 1 through N. Then the negative regression dependence be given by delicate combinatrial calculation and by applying an inequality related to hypergeometric cumulative distribution functions. For the case of a general finite population, the negative regression dependence can be obtain by considering the more complicated track of the number of ties among the population values. In Section 2.4, we show that, from Section 2.2 and Lehmann (1966), covariances of two order statistics from one sample are nonnegative and that, from Section 2.3 and Lehmann (1966), covariances of two order statistics from two samples are nonpositive. Further, we give the conditions under which covariances are zero. In Section 2.5, we show from Section 2.4 that the relative precision RP is bounded below by 1 and that the lower bound 1 is achieved by some finite populations.

One of important problems in RSS from a finite population is this: Given the population size N, the set-size n and the number of cycles r, what is the maximum RP, and what is the population that has the maximum RP? In Chapter 3, we describe how to calculate the possible extremal finite populations maximizing RP for any N, n and r [see Futatsuya and Takahasi (1990)]. Section 3.2 shows that the difference between the variance of the RSS estimator and the variance of the SRS estimator is given by a quadratic form $-\theta'\Gamma\theta/n^2r$ in the variables of population values $\theta = (x_1, x_2, \cdots, x_N)'$, where $x_1 \leq x_2 \leq \cdots, x_N$. Here, Γ is an $N \times N$ matrix whose elements are functions of the population size N and the set-size n, but Γ does not depend on the number of cycles r. If the eigenvector $(v_1, v_2, \cdots, v_N)'$ corresponding to the maximum eigenvalue of Γ satisfy the order condition $v_1 \leq v_2 \leq \cdots \leq v_N$, then we can say that the elements of the eigenvector construct the extremal population maximizing RP and that the largest eigenvalue becomes the maximum RP. We computed the eigenvectors corresponding to the largest eigenvalues of Γ for $4 \leq N \leq 100$ and $2 \leq n \leq 5$. Consequently, everything we computed satisfied the order condition. In Figure 3.3.1 of Section 3.3, we show numerically some extremal finite populations. Table 3.3.2 gives the maximum RP and, for comparison, the values of RP for the discrete uniform distributions for some N and n. As the result, we see that, unlike the case of sampling from an infinite population, the extremal populations do not have the discrete uniform distributions.

Let us leave the subject of ranked set sampling and turn to that of censored samples. Censored samples occur quite frequently in many practical problems in engineering and in biological and behavioral sciences. In engineering, a typical experiment in life testing of equipment consists of installing a sample of n similar units on appropriate devices and subjecting the units to operation under specified conditions until failure of the equipment is observed. Suppose that the life lengths of these n units are independent and identically distributed random variables with a normal distribution (possibly after transformation of the data). Note, however, that these values are recorded in increasing order of magnitude; that is, the data appear as vector of order statistics in a natural way. For some reasons or other, suppose that we have to terminate the experiment before all units have failed. We would then have a censored sample in which order statistics play an important role. Let us now look at two prominent types of censored samples discussed in the literature. A *Type I censored sample* is one in which the terminus or point of censoring is fixed, while a *Type II censored sample* is one in which the number of censored observations is fixed. Type I censoring is more common in practice; Type II censoring is more common in the literature, as it is mathematically more tractable. Previous work on censored samples has dealt mainly with point estimation. We study some kinds of confidence intervals for the mean of a normal distribution based on censored samples.

In Chapter 4, five kinds of confidence intervals for the mean of a normal distribution from Type I and Type II censored samples are compared through a simulation study [see Futatsuya and Takahasi (1978)]. Section 4.2 presents five kinds of confidence intervals for the mean. We call the usual large-sample confidence interval based on the maximum likelihood estimates *ML interval* and, by replacing the quantile of the standard normal distribution which determines the confidence coefficient of the ML interval by that of Student's t distribution, we obtain *ML-T interval.* Furthermore, by replacing the maximum likelihood estimates in the ML interval by the best linear unbiased estimates, we obtain *LU interval.* We can also get *LU-T interval* in the same manner as the ML-T interval. The last one is *H interval* proposed by Halperin (1961). This H interval is based on a joint confidence region for the mean and standard deviation. In Section 4.3, five kinds of these approximate confidence intervals for the mean are compared by simulation. The main conclusion is that the ML-T intervals seem to be practicable in the case of Type I, and the LU-T intervals in the case of Type II.

Chapter 2

Ranked Set Sampling from Finite Population

2.1 Introduction

Let $X_1, X_2, \ldots, X_m$ be independently distributed according to a univariate distribution and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ be their order statistics. In this case, Lehmann (1966) has shown that the joint distribution of two order statistics $X_{(i)}$ and $X_{(j)}$ is positively likelihood ratio dependent. We consider the case where $X_1, X_2, \ldots, X_m$ is a simple random sample without replacement from a finite population and, therefore, $X_1, X_2, \ldots, X_m$ are not independent. It is shown that the joint distribution of $X_{(i)}$ and $X_{(j)}$ is positively likelihood ratio dependent also in our case. We prove this result in Section 2.2.

Next, let $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$ be a simple random sample of size $m+n$ without replacement from a finite population and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ be two sets of the order statistics of $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$, respectively. We consider possible dependence between $X_{(i)}$ and $Y_{(j)}$. It is shown that $Y_{(j)}$ is negatively regression dependent on $X_{(i)}$. We prove this result in Section 2.3. In Section 2.4, we show that $\mathrm{Cov}(X_{(i)}, X_{(j)}) \geq 0$ and $\mathrm{Cov}(X_{(i)}, Y_{(j)}) \leq 0$ and we give the conditions for the equality to hold.

Finally, using these results, we shall prove a theorem on ranked set sampling (RSS) in finite populations. As it was pointed out by Patil *et al.* (1995), most of researches in RSS have been concerned with sampling from infinite (continuous) populations. Takahasi and Futatsuya (1988), Futatsuya and Takahasi (1990) and Patil *et al.* (1995) studied RSS for estimating a population mean when sampling is without replacement from a finite population. Takahasi and Futatsuya (1988) gave an expression for the variance of the RSS estimator ($\hat{\mu}_{RSS}$) and Patil *et al.* (1995) obtained explicit expressions for the variance of $\hat{\mu}_{RSS}$ and the corresponding relative savings. Performance of the RSS estimator is generally benchmarked against that of the simple random sampling (SRS) estimator ($\hat{\mu}_{SRS}$) with the same number of quantifications. For this purpose, we use the relative precision (RP), $RP = \mathrm{Var}(\hat{\mu}_{SRS})/\mathrm{Var}(\hat{\mu}_{RSS})$. Futatsuya and Takahasi (1990) considered the extremal finite populations maximizing relative precision. In Section 2.5, we show that RP is never smaller than 1, and is greater than 1 unless $N-1$ elements of the population of size N have the same value.

2.2 Positive likelihood ratio dependence between order statistics of a sample

Let $\mathbf{\Omega}$ be the finite population $\{x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_l, \cdots, x_l\}$ $(x_1 < x_2 < \cdots < x_l)$ of size N. Let ν_a be the number of x_a in $\mathbf{\Omega}$ $(a = 1, 2, \ldots, l)$, $f_a = \nu_1 + \nu_2 + \cdots + \nu_a$ $(a = 1, 2, \ldots, l)$, $f_0 = 0$ and $\tilde{f}_a = f_{a-1} + 1$ $(a = 1, 2, \cdots, l)$. We assume that $\nu_a > 0$ $(a = 1, 2, \ldots, l)$. Let $X_1, X_2, \ldots, X_n$ be a simple random sample of size n without replacement from $\mathbf{\Omega}$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics of this sample. In this section, we prove that the joint distribution $X_{(i)}$ and $X_{(j)}$ $(i \neq j)$ is positively likelihood ratio dependent.

Let us first consider the case $\mathbf{\Omega} = \mathbf{\Omega}_N = \{1, 2, \cdots, N\}$. Let $Z_1, Z_2, \cdots, Z_n$ be a simple random sample of size n without replacement from $\mathbf{\Omega}_N$. Let $Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ be the order statistics of this sample. The set of all points (s, t) satisfying

$\Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\} > 0$ is denoted by $\mathbf{S}$. For $1 \le i < j \le n$, we have

$$\mathbf{S} = \{(s,t) \mid i \le s \le N - n + i, j \le t \le N - n + j, j - i \le t - s\}. \tag{2.2.1}$$

We prove the following.

LEMMA 2.2.1 *Let $1 \le i < j \le n$. Then the joint distribution of $Z_{(i)}$ and $Z_{(j)}$ is positively likelihood ratio dependent; that is, if $s < s'$ and $t < t'$, then*

$$\begin{aligned}\Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\} &\Pr\left\{Z_{(i)} = s', Z_{(j)} = t'\right\} \\ &\ge \Pr\left\{Z_{(i)} = s, Z_{(j)} = t'\right\} \Pr\left\{Z_{(i)} = s', Z_{(j)} = t\right\},\end{aligned} \tag{2.2.2}$$

with equality holding if and only if $(s,t) \notin \mathbf{S}$ or $(s',t') \notin \mathbf{S}$.

PROOF. From (2.2.1), if $(s,t) \notin \mathbf{S}$ or $(s',t') \notin \mathbf{S}$, then $(s,t') \notin \mathbf{S}$ or $(s',t) \notin \mathbf{S}$ and, therefore, equality holds in (2.2.2). If $(s,t) \in \mathbf{S}$ and $(s',t') \in \mathbf{S}$, and $(s,t') \notin \mathbf{S}$ or $(s',t) \notin \mathbf{S}$, then

$$\begin{aligned}\Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\} &\Pr\left\{Z_{(i)} = s', Z_{(j)} = t'\right\} \\ &> 0 = \Pr\left\{Z_{(i)} = s, Z_{(j)} = t'\right\} \Pr\left\{Z_{(i)} = s', Z_{(j)} = t\right\}.\end{aligned}$$

Finally, if $(s,t), (s',t'), (s,t'), (s',t) \in \mathbf{S}$, then, since

$$\Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\} = \frac{1}{\binom{N}{n}} \binom{s-1}{i-1} \binom{t-1-s}{j-1-i} \binom{N-t}{n-j},$$

we have

$$\begin{aligned}&\frac{\Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\} \Pr\left\{Z_{(i)} = s', Z_{(j)} = t'\right\}}{\Pr\left\{Z_{(i)} = s, Z_{(j)} = t'\right\} \Pr\left\{Z_{(i)} = s', Z_{(j)} = t\right\}} \\ &\quad = \frac{(t-s-1)(t-s-2)\cdots(t-s-j+i+1)}{(t'-s-1)(t'-s-2)\cdots(t'-s-j+i+1)} \\ &\qquad \times \frac{(t'-s'-1)(t'-s'-2)\cdots(t'-s'-j+i+1)}{(t-s'-1)(t-s'-2)\cdots(t-s'-j+i+1)}.\end{aligned} \tag{2.2.3}$$

Because $(t-s-k)(t'-s'-k) - (t'-s-k)(t-s'-k) = (t'-t)(s'-s) > 0$, we get (2.2.3)$> 1$. This completes the proof of the lemma.

Now we consider the general case $\mathbf{\Omega} = \{x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_l, \cdots, x_l\}$. Let

$$\mathbf{G}_{ab} = \left\{(s,t) \mid \tilde{f}_a \le s \le f_a, \tilde{f}_b \le t \le f_b, s, t \in \mathbf{\Omega}_N\right\}.$$

We have the following lemma.

LEMMA 2.2.2 *For* $1 \le i, j \le n$, $i \ne j$ *and* $1 \le a, b \le l$,

$$\Pr\left\{X_{(i)} \le x_a, X_{(j)} \le x_b\right\} = \Pr\left\{Z_{(i)} \le f_a, Z_{(j)} \le f_b\right\}$$

and

$$\Pr\left\{X_{(i)} = x_a, X_{(j)} = x_b\right\} = \sum_{(s,t)\in \mathbf{G}_{ab}} \Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\}.$$

PROOF. Define $t(k)$ $(k = 1, 2, \cdots, N)$ by

$$t(k) = x_a \qquad \text{if } \tilde{f}_a \le k \le f_a.$$

Since the joint distribution of $X_1, X_2, \ldots, X_n$ is the same as that of $t(Z_1), t(Z_2), \cdots, t(Z_n)$, we can assume that $X_i = t(Z_i)$, $i = 1, 2, \cdots, n$. Then, we have

$$\begin{aligned} X_{(i)} \le x_a &\Leftrightarrow \sharp\{j \mid X_j \le x_a\} \ge i \\ &\Leftrightarrow \sharp\{j \mid t(Z_j) \le x_a\} \ge i \\ &\Leftrightarrow \sharp\{j \mid Z_j \le f_a\} \ge i \\ &\Leftrightarrow Z_{(i)} \le f_a \end{aligned}$$

and

$$\begin{aligned} X_{(i)} = x_a &\Leftrightarrow \sharp\{j \mid X_j \le x_a\} \ge i \text{ and } \sharp\{j \mid X_j \ge x_a\} \ge n - i + 1 \\ &\Leftrightarrow \sharp\{j \mid t(Z_j) \le x_a\} \ge i \text{ and } \sharp\{j \mid t(Z_j) \ge x_a\} \ge n - i + 1 \\ &\Leftrightarrow \sharp\{j \mid Z_j \le f_a\} \ge i \text{ and } \sharp\{j \mid Z_j \ge \tilde{f}_a\} \ge n - i + 1 \\ &\Leftrightarrow \tilde{f}_a \le Z_{(i)} \le f_a, \end{aligned}$$

where $\sharp\mathcal{X}$ denotes the number of elements in the set $\mathcal{X}$. The proof of this lemma is immediate from these relations.

The following theorem shows positively likelihood ratio dependence between two order statistics in general finite population.

THEOREM 2.2.1 *Let $1 \le i < j \le n$. The joint distribution of $X_{(i)}$ and $X_{(j)}$ is positively likelihood ratio dependent; that is, if $1 \le a < a' \le l$ and $1 \le b < b' \le l$, then*

$$\Pr\left\{X_{(i)} = x_a, X_{(j)} = x_b\right\} \Pr\left\{X_{(i)} = x_{a'}, X_{(j)} = x_{b'}\right\} \\ \ge \Pr\left\{X_{(i)} = x_a, X_{(j)} = x_{b'}\right\} \Pr\left\{X_{(i)} = x_{a'}, X_{(j)} = x_b\right\} \tag{2.2.4}$$

with equality holding if and only if $\mathbf{G}_{ab} \cap \mathbf{S} = \emptyset$ or $\mathbf{G}_{a'b'} \cap \mathbf{S} = \emptyset$.

PROOF. Let $(s,t) \in \mathbf{G}_{ab}$ and $(s',t') \in \mathbf{G}_{a'b'}$. Then, $s < s'$ and $t < t'$. Therefore, from Lemma 2.2.1, we have

$$\Pr\left\{Z_{(i)} = s, Z_{(j)} = t\right\} \Pr\left\{Z_{(i)} = s', Z_{(j)} = t'\right\} \\ \ge \Pr\left\{Z_{(i)} = s, Z_{(j)} = t'\right\} \Pr\left\{Z_{(i)} = s', Z_{(j)} = t\right\}.$$

Summing both sides over $(s,t) \in \mathbf{G}_{ab}$ and $(s',t') \in \mathbf{G}_{a'b'}$, and, applying Lemma 2.2.2, we get (2.2.4). The condition for equality in (2.2.4) also comes from Lemma 2.2.1. This completes the proof of the theorem.

2.3 Negative regression dependence between order statistics of two samples

Let $\mathcal{M} = \{X_1, X_2, \ldots, X_m\}$ be a simple random sample of size m without replacement from $\mathbf{\Omega} = \{x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_l, \cdots, x_l\}$ given in Section 2.2. Put $\mathbf{\Phi} = \mathbf{\Omega} - \mathcal{M}$. Let $\mathcal{N} = \{Y_1, Y_2, \ldots, Y_n\}$ be a simple random sample of size n without replacement from $\mathbf{\Phi}$ and let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$ and $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$ be the order statistics of these samples, respectively. In this section, we prove that $Y_{(j)}$ is negatively regression dependent on $X_{(i)}$.

Let us first consider the case $\mathbf{\Omega} = \mathbf{\Omega}_N = \{1, 2, \cdots, N\}$. In this case, let us denote $\mathcal{M} = \{X_1, X_2, \ldots, X_m\}$ and $\mathcal{N} = \{Y_1, Y_2, \ldots, Y_n\}$ by $\mathcal{M} = \{U_1, U_2, \cdots, U_m\}$

and $\mathcal{N} = \{V_1, V_2, \cdots, V_n\}$, respectively. Let $U_{(1)} < U_{(2)} < \cdots < U_{(m)}$ and $V_{(1)} < V_{(2)} < \cdots < V_{(n)}$ be the order statistics of $\{U_1, U_2, \cdots, U_m\}$ and $\{V_1, V_2, \cdots, V_n\}$, respectively. Let us assume that $1 \le i \le m$, $1 \le j \le n$ and, of course, $N \ge m + n$. Put $\tilde{N} = N - m - n + i + j - 1$. We define several subsets of $\mathbf{\Omega}_N \times \mathbf{\Omega}_N$ as follows:

$$\mathbf{O} = \left\{(u,v) \mid \Pr\left\{U_{(i)} = u\right\} > 0 \quad \text{and} \quad \Pr\left\{V_{(j)} = v\right\} > 0\right\},$$

$$\mathbf{A} = \left\{(u,v) \mid \Pr\left\{U_{(i)} = u, V_{(j)} = v\right\} > 0 \quad \text{and} \quad u > v\right\},$$

$$\mathbf{C} = \left\{(u,v) \mid \Pr\left\{U_{(i)} = u, V_{(j)} = v\right\} > 0 \quad \text{and} \quad u < v\right\},$$

$$\mathbf{B} = \left\{(u,v) \mid i + j \le u \le \tilde{N}, u = v\right\},$$

$$\mathbf{D} = \{(u,v) \mid i \le u \le i + j - 1, j \le v \le i + j - 1\}$$

and

$$\mathbf{E} = \left\{(u,v) \mid \tilde{N} + 1 \le u \le N - m + i, \tilde{N} + 1 \le v \le N - n + j\right\}.$$

Then we have the following lemma.

Lemma 2.3.1 *For the sets defined above, we have*

$$\mathbf{O} = \{(u,v) \mid i \le u \le N - m + i, j \le v \le N - n + j\},$$

$$\mathbf{A} = \left\{(u,v) \mid i + j \le u \le N - m + i, j \le v \le \tilde{N}, u > v\right\}, \tag{2.3.1}$$

$$\mathbf{C} = \left\{(u,v) \mid i \le u \le \tilde{N}, i + j \le v \le N - n + j, u < v\right\}, \tag{2.3.2}$$

$$\left\{(u,v) \mid \Pr\left\{U_{(i)} = u, V_{(j)} = v\right\} > 0\right\} = \mathbf{A} \cup \mathbf{C} \tag{2.3.3}$$

and

$$\mathbf{O} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D} \cup \mathbf{E},$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ *are mutually disjoint.*

Proof. It is sufficient to prove (2.3.2), because the proof of (2.3.1) is similar to the proof of (2.3.2) and others are obvious. Let us consider Table 2.3.1. Under the assumption $u < v$, $\Pr\left\{U_{(i)} = u, V_{(j)} = v\right\} > 0$ if and only if there are non-negative integers x and y such that all the entries of the table are non-negative. Using this fact, it is easy to obtain (2.3.2). Fig. 2.3.1 gives an example of the partition of $\mathbf{O}$ into $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$, where the points of $\mathbf{A}$ are the lattice points in area $\mathbf{A}$ in Fig. 2.3.1 and so on.

The following lemma is used in proving Theorem 2.3.2.

Table 2.3.1: Conditions for $\mathbf{C}$

	$1, \cdots, u-1$	u	$u+1, \cdots, v-1$	v	$v+1, \cdots, N$	Total
$\mathcal{M}$	$i-1$	1	y	0	$m-i-y$	m
$\mathcal{N}$	x	0	$j-1-x$	1	$n-j$	n
$\Phi - \mathcal{N}$	$u-i-x$	0	$v-u-j+x-y$	0	$N-m-n+i+j-v+y$	$N-m-n$
Total	$u-1$	1	$v-1-u$	1	$N-v$	N

LEMMA 2.3.2 *Let* $(u, v) \in \mathbf{O}$. *Then,*

$$\Pr\left\{U_{(i)} = u, V_{(j)} = v\right\} = \Pr\left\{U_{(m-i+1)} = N-u+1, V_{(n-j+1)} = N-v+1\right\},$$

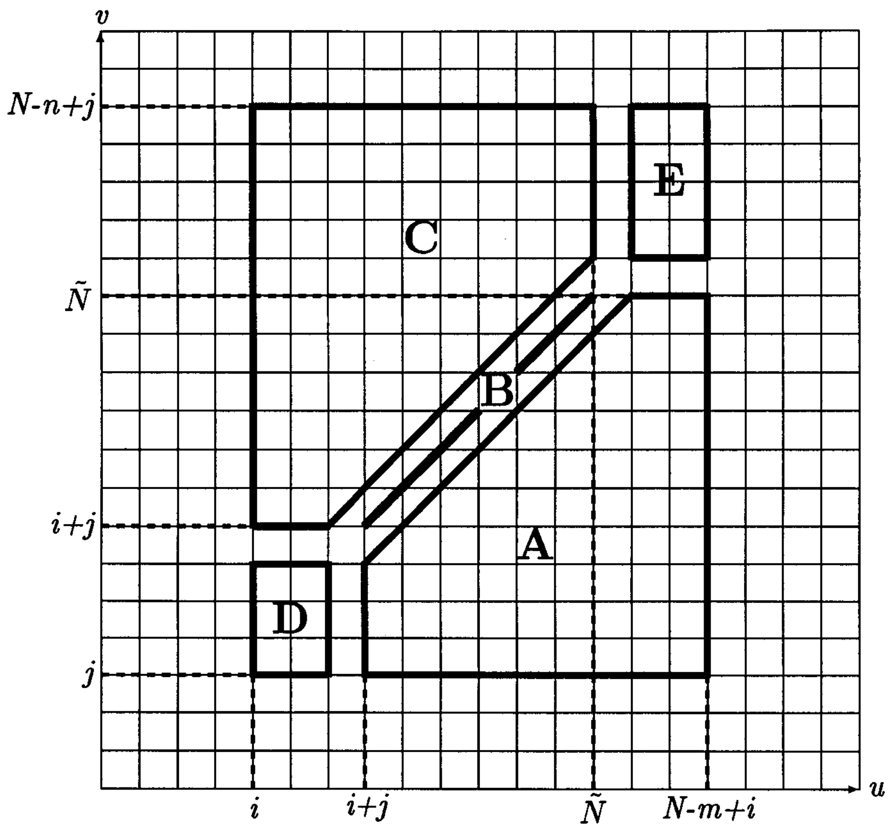
$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} = \Pr\left\{U_{(m-i+1)} = N-u+1\right\}$$
$$-\Pr\left\{U_{(m-i+1)} = N-u+1, V_{(n-j+1)} \le N-v\right\}$$

and

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} = 1 - \Pr\left\{V_{(n-j+1)} \le N-v \mid U_{(m-i+1)} = N-u+1\right\}.$$

PROOF. Put $U'_i = N+1-U_i$ $(i = 1, 2, \cdots, m)$ and $V'_j = N+1-V_j$ $(j = 1, 2, \cdots, n)$. Let $U'_{(1)} < U'_{(2)} < \cdots < U'_{(m)}$ and $V'_{(1)} < V'_{(2)} < \cdots < V'_{(n)}$ be the order statistics of $U'_1, U'_2, \cdots, U'_m$ and $V'_1, V'_2, \cdots, V'_n$, respectively. It is obvious that the joint distribution of $U'_1, U'_2, \cdots, U'_m$ is the same as that of $U_1, U_2, \cdots, U_m$, the joint distribution of $V'_1, V'_2, \cdots, V'_n$ is the same as that of $V_1, V_2, \cdots, V_n$ and $U'_{(i)} = N+1-U_{(m+1-i)}$, $V'_{(j)} = N+1-V_{(n+1-j)}$. Thus, we have

Figure 2.3.1: Example of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ ($N = 20, m = 8, n = 3, i = 4, j = 3$)



$$\begin{aligned}
\Pr\left\{U_{(i)} = u, V_{(j)} = v\right\} &= \Pr\left\{U'_{(i)} = u, V'_{(j)} = v\right\} \\
&= \Pr\left\{N+1-U_{(m-i+1)} = u, N+1-V_{(n-j+1)} = v\right\} \\
&= \Pr\left\{U_{(m-i+1)} = N+1-u, V_{(n-j+1)} = N+1-v\right\},
\end{aligned}$$

$$\begin{aligned}
\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} &= \Pr\left\{U'_{(i)} = u, V'_{(j)} \le v\right\} \\
&= \Pr\left\{U_{(m-i+1)} = N+1-u, V_{(n-j+1)} \ge N+1-v\right\} \\
&= \Pr\left\{U_{(m-i+1)} = N+1-u\right\} - \Pr\left\{U_{(m-i+1)} = N-u+1, V_{(n-j+1)} \le N-v\right\}
\end{aligned}$$

and

$$\begin{aligned}
\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} &= \frac{\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\}}{\Pr\left\{U_{(i)} = u\right\}} \\
&= 1 - \frac{\Pr\left\{U_{(m-i+1)} = N+1-u, V_{(n-j+1)} \le N-v\right\}}{\Pr\left\{U_{(m-i+1)} = N+1-u\right\}} \\
&= 1 - \Pr\left\{V_{(n-j+1)} \le N-v \mid U_{(m-i+1)} = N+1-u\right\}.
\end{aligned}$$

Let $T_v = \sharp\{a \in \mathbf{\Phi} \mid a \le v\}$ and $R_{(j)} = \sharp\{h \in \mathbf{\Phi} \mid h \le V_{(j)}\}$. The following lemma is important for deriving the joint distribution of $U_{(i)}$ and $V_{(j)}$.

LEMMA 2.3.3 *Let* $(u, v) \in \mathbf{O}$. *Then,*

$$\begin{aligned}
\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} &= \sum_{a=j}^{v} \Pr\left\{U_{(i)} = u, T_v = a\right\} \Pr\left\{R_{(j)} \le a\right\} \\
&= \sum_{a=j}^{v} \left[\Pr\left\{U_{(i)} = u, T_v = a\right\} \sum_{h=j}^{\min\{a, N-m-n+j\}} \Pr\left\{R_{(j)} = h\right\}\right]
\end{aligned}$$

PROOF. It is easily checked that

$$\begin{aligned}
V_{(j)} \le v &\iff T_v \ge j \text{ and } R_{(j)} \le T_v \\
&\iff \bigvee_{a=j}^{v} \left[T_v = a, R_{(j)} \le a\right],
\end{aligned}$$

where $\mathcal{X}_1 \vee \mathcal{X}_2$ means $\mathcal{X}_1$ or $\mathcal{X}_2$. Because $(U_{(i)}, T_v)$ and $R_{(j)}$ are independent, we have

$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} = \sum_{a=j}^{v} \Pr\left\{U_{(i)} = u, T_v = a, R_{(j)} \le a\right\}$$

$$= \sum_{a=j}^{v} \Pr\left\{U_{(i)} = u, T_v = a\right\} \Pr\left\{R_{(j)} \leq a\right\}$$

$$= \sum_{a=j}^{v} \left[\Pr\left\{U_{(i)} = u, T_v = a\right\} \sum_{h=j}^{\min\{a, N-m-n+j\}} \Pr\left\{R_{(j)} = h\right\} \right].$$

Now we can give an expression of the joint distribution of $U_{(i)}$ and $V_{(j)}$. Let

$$\mathbf{A_1} = \{(u,v) \mid (u,v) \in \mathbf{A}, v = \tilde{N}\},$$

$$\mathbf{A_2} = \mathbf{A} - \mathbf{A_1} = \{(u,v) \mid i+j \leq u \leq N-m+i, j \leq v < \tilde{N}, u > v\}, \tag{2.3.4}$$

$$\mathbf{C_1} = \{(u,v) \mid (u,v) \in \mathbf{C}, v = N-n+j\}$$

and

$$\mathbf{C_2} = \mathbf{C} - \mathbf{C_1}.$$

THEOREM 2.3.1 *Let* $1 \leq i \leq m$, $1 \leq j \leq n$ *and* $(u,v) \in \mathbf{O}$.
Then, $\Pr\left\{U_{(i)} = u, V_{(j)} \leq v\right\}$ *is given as follows:*

$$\Pr\left\{U_{(i)} = u, V_{(j)} \leq v\right\} = \frac{\binom{u-1}{i-1}\binom{N-u}{m-i}}{\binom{N}{m}}, \qquad (u,v) \in \mathbf{A_1} \cup \mathbf{C_1} \cup \mathbf{E}$$

$$\Pr\left\{U_{(i)} = u, V_{(j)} \leq v\right\} = \frac{\binom{N-u}{m-i}}{\binom{N}{m}\binom{N-m}{n}} \sum_{h=j}^{\min\{u-i,v,N-m-n+j\}} \binom{h-1}{j-1}\binom{N-m-h}{n-j}$$
$$\times \sum_{a=\max\{h,v+1-i\}}^{\min\{u-i,v\}} \binom{v}{a}\binom{u-1-v}{u-i-a}, \qquad (u,v) \in \mathbf{A_2} \tag{2.3.5}$$

$$\Pr\left\{U_{(i)} = u, V_{(j)} \leq v\right\} = \frac{\binom{N-u}{m-i}\binom{u-1}{u-i}}{\binom{N}{m}\binom{N-m}{n}} \sum_{h=j}^{u-i} \binom{h-1}{j-1}\binom{N-m-h}{n-j}, \qquad (u,v) \in \mathbf{B}$$

$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} = \frac{\binom{N-u}{m-i}\binom{u-1}{i-1}}{\binom{N}{m}}$$

$$- \frac{\binom{u-1}{i-1}}{\binom{N}{m}\binom{N-m}{n}} \sum_{h=n-j+1}^{\min\{N-v, N-u-m+i, N-m-j+1\}} \binom{h-1}{n-j}\binom{N-m-h}{j-1}$$

$$\times \sum_{a=\max\{h, N-v-m+i\}}^{\min\{N-v, N-u-m+i\}} \binom{N-v}{a}\binom{v-u}{N-u-m+i-a}, \qquad (u,v) \in \mathbf{C_2} \tag{2.3.6}$$

$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} = 0, \qquad (u,v) \in \mathbf{D}.$$

PROOF. By elementary combinatorial calculations, we have the following (i),(ii) and (iii).

(i) $$\Pr\left\{R_{(j)} = h\right\} = \begin{cases} \dfrac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}}, & j \le h \le N-m-n+j \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $$\Pr\left\{U_{(i)} = u\right\} = \begin{cases} \dfrac{\binom{u-1}{i-1}\binom{N-u}{m-i}}{\binom{N}{m}}, & i \le u \le N-m+i \\ 0, & \text{otherwise.} \end{cases}$$

(iii) If $v = u$ or $v = u-1$, then

$$\Pr\left\{U_{(i)} = u, T_v = a\right\} = \begin{cases} 0, & a \ne u-i \\ \Pr\left\{U_{(i)} = u\right\}, & a = u-i. \end{cases}$$

If $1 \le v \le u-1$, then

$$\Pr\left\{U_{(i)} = u, T_v = a\right\} = \begin{cases} \dfrac{1}{\binom{N}{m}}\binom{v}{a}\binom{u-1-v}{u-i-a}\binom{N-u}{m-i}, & \max\{0, v+1-i\} \le a \le \min\{u-i, v\} \\ 0, & \text{otherwise.} \end{cases}$$

Case 1; $(u,v) \in \mathbf{A_1} \cup \mathbf{C_1} \cup \mathbf{E}$. In this case, we have $\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} = \Pr\left\{U_{(i)} = u\right\}$. The desired result comes from (ii).

Case 2; $(u,v) \in \mathbf{D}$. From Fig. 2.3.1, this is obvious.

Case 3; $(u,v) \in \mathbf{B}$. In this case, we have

$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\}$$

$$\begin{aligned}
&= \Pr\left\{U_{(i)} = u, V_{(j)} \le u-1\right\} \\
&= \sum_{a=j}^{u-1} \Pr\left\{U_{(i)} = u, T_v = a\right\} \Pr\left\{R_{(j)} \le a\right\} && \text{(by Lemma 2.3.3)} \\
&= \Pr\left\{U_{(i)} = u\right\} \Pr\left\{R_{(j)} \le u-i\right\} && (U_{(i)} = u \text{ implies } T_v = u-i\ ) \\
&= \frac{\binom{u-1}{i-1}\binom{N-u}{m-i}}{\binom{N}{m}} \sum_{h=j}^{\min\{u-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} && \text{(by (ii) and (i))} \\
&= \frac{\binom{N-u}{m-i}\binom{u-1}{i-1}}{\binom{N}{m}\binom{N-m}{n}} \sum_{h=j}^{\min\{u-i,N-m-n+j\}} \binom{h-1}{j-1}\binom{N-m-h}{n-j}.
\end{aligned}$$

Case 4; $(u,v) \in \mathbf{A_2}$. From Lemma 2.3.3, (ii) and (i), we have

$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\}$$

$$\begin{aligned}
&= \sum_{a=j}^{v} \Pr\left\{U_{(i)} = u, T_v = a\right\} \sum_{h=j}^{\min\{a,N-m-n+j\}} \Pr\left\{R_{(j)} = h\right\} \\
&= \sum_{a=\max\{j,v+1-i\}}^{\min\{u-i,v\}} \frac{1}{\binom{N}{m}} \binom{v}{a}\binom{u-1-v}{u-i-a}\binom{N-u}{m-i} \sum_{h=j}^{\min\{a,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \\
&= \frac{\binom{N-u}{m-i}}{\binom{N}{m}\binom{N-m}{n}} \sum_{a=\max\{j,v+1-i\}}^{\min\{u-i,v\}} \binom{v}{a}\binom{u-1-v}{u-i-a} \sum_{h=j}^{\min\{a,N-m-n+j\}} \binom{h-1}{j-1}\binom{N-m-h}{n-j} \\
&= \frac{\binom{N-u}{m-i}}{\binom{N}{m}\binom{N-m}{n}} \sum_{h=j}^{\min\{u-i,v,N-m-n+j\}} \binom{h-1}{j-1}\binom{N-m-h}{n-j} \sum_{a=\max\{h,v+1-i\}}^{\min\{u-i,v\}} \binom{v}{a}\binom{u-1-v}{u-i-a}.
\end{aligned}$$

Case 5; $(u,v) \in \mathbf{C_2}$. This case can be obtained by using Case 4. By Lemma 2.3.2, we have

$$\Pr\left\{U_{(i)} = u, V_{(j)} \le v\right\} = \Pr\left\{U_{(m-i+1)} = N-u+1\right\}$$

$$-\Pr\left\{U_{(m-i+1)} = N-u+1, V_{(n-j+1)} \le N-v\right\}. \qquad (2.3.7)$$

Put $i' = m - i + 1$, $j' = n - j + 1$, $u' = N - u + 1$, $v' = N - v$ and $\tilde{N}' = N - m - n + i' + j' - 1$. Then,(2.3.7) $= \dfrac{\binom{N-u}{m-i}\binom{u-1}{i-1}}{\binom{N}{m}} - \Pr\left\{U_{(i')} = u', V_{(j')} \leq v'\right\}$. Since $(u,v) \in \mathbf{C_2}$, we have $i' + j' \leq u' \leq N - m + i'$, $j' \leq v' < \tilde{N}'$ and $u' > v'$. From these inequalities, it can be said that (u', v') belongs to $\mathbf{A_2}$ in the case of $(i, j, \tilde{N}) = (i', j', \tilde{N}')$ in (2.3.4). Therefore, we can apply Case 4 to $\Pr\left\{U_{(i')} = u', V_{(j')} \leq v'\right\}$ and obtain an expression corresponding to (2.3.5). Substituting $i' = m - i + 1$, $j' = n - j + 1$, $u' = N - u + 1$ and $v' = N - v$ for this expression, we obtain (2.3.6).

This completes the proof.

From this theorem, we obtain the following corollary.

COROLLARY 2.3.1 *Suppose* $(u, v) \in \mathbf{O}$. *Then,*

$$\Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u\right\}$$

$$
\begin{aligned}
&= 1, && (u,v) \in \mathbf{A_1} \cup \mathbf{C_1} \cup \mathbf{E} \\
&= \sum_{h=j}^{\min\{u-i,v,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \sum_{a=\max\{h,v+1-i\}}^{\min\{u-i,v\}} \frac{\binom{v}{a}\binom{u-1-v}{u-i-a}}{\binom{u-1}{u-i}}, && (u,v) \in \mathbf{A_2} \\
&= \sum_{h=j}^{u-i} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}}, && (u,v) \in \mathbf{B} \\
&= 1 - \sum_{h=n-j+1}^{\min\{N-u-m+i,N-v,N-m-j+1\}} \frac{\binom{h-1}{n-j}\binom{N-m-h}{j-1}}{\binom{N-m}{n}} \\
&\qquad \times \sum_{a=\max\{h,N-v-m+i\}}^{\min\{N-u-m+i,N-v\}} \frac{\binom{N-v}{a}\binom{v-u}{N-u-m+i-a}}{\binom{N-u}{N-u-m+i}}, && (u,v) \in \mathbf{C_2} \\
&= 0, && (u,v) \in \mathbf{D}.
\end{aligned}
$$

The following lemma shows a stochastic ordering between hyper geometric distributions and is essential for the proof of Theorem 2.3.2.

Lemma 2.3.4 *Suppose that r, M and L are positive integers and $r < M + L$. For any k satisfying* $\max\{0, r-L\} < k \le \min\{r, M\}$,

$$\sum_{a=k}^{\min\{r,M\}} \frac{\binom{M}{a}\binom{L}{r-a}}{\binom{M+L}{r}} < \sum_{a=k}^{\min\{r+1,M\}} \frac{\binom{M}{a}\binom{L+1}{r+1-a}}{\binom{M+L+1}{r+1}}.$$

Proof. Let $c = \max\{0, n-L\}, d_1 = \min\{r, M\}$ and $d_2 = \min\{r+1, M\}$. Define

$$p_1(a) = \begin{cases} \frac{\binom{M}{a}\binom{L}{r-a}}{\binom{M+L}{r}}, & c \le a \le d_1 \\ 0, & \text{otherwise,} \end{cases}$$

$$p_2(a) = \begin{cases} \frac{\binom{M}{a}\binom{L+1}{r+1-a}}{\binom{M+L+1}{r+1}}, & c \le a \le d_2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p(a) = p_2(a) - p_1(a).$$

For $c \le a \le d_1$,

$$p(a) = \frac{\binom{M}{a}\binom{L}{r-a}}{\binom{M+L}{r}} \frac{\{(M+L+1)a - M(r+1)\}}{(M+L+1)(r+1-a)}.$$

Put $g = \dfrac{M(r+1)}{M+L+1}$. It is easily checked that $c < g < d_2$. Therefore, we have that

$$\begin{aligned} p(a) &< 0, \quad \text{if } c \le a < g, \\ p(a) &= 0, \quad \text{if } a = g, \\ p(a) &> 0, \quad \text{if } g < a \le d_1, \end{aligned} \tag{2.3.8}$$

and particularly,

$$p(c) < 0,$$

$$p(d_2) > 0 \quad \text{if } d_2 = d_1.$$

If $d_2 > d_1$, then $p_2(d_2) > 0$ and $p_1(d_2) = 0$ and, therefore, $p(d_2) > 0$. We can, now, replace d_1 in (2.3.8) by d_2. Then, it is obvious that

$$\sum_{a=k}^{d_2} p(a) > 0, \quad \text{if } k \ge g$$

and, noting that $\sum_{a=c}^{d_2} p(a) = 0$,

$$\sum_{a=k}^{d_2} p(a) = -\sum_{a=c}^{k-1} p(a) > 0, \quad \text{if } c < k < g.$$

This completes the proof.

Before proceeding further, we define several subsets of $\mathbf{O}$ as follows:

$$\begin{aligned}
\widetilde{\mathbf{O}} &= \{(u,v) \in \mathbf{O} \mid u \neq N - m + i\}, \\
\widetilde{\mathbf{A}} &= \mathbf{A} \cap \widetilde{\mathbf{O}}, \\
\tilde{\mathbf{E}} &= \mathbf{E} \cap \widetilde{\mathbf{O}}, \\
\widetilde{\mathbf{A}_1} &= \{(u,v) \in \widetilde{\mathbf{A}} \mid v = \tilde{N}\}, \\
\widetilde{\mathbf{A}_2} &= \widetilde{\mathbf{A}} - \widetilde{\mathbf{A}_1}, \\
\widetilde{\mathbf{C}_2} &= \{(u,v) \in \mathbf{C} - \mathbf{C}_1 \mid u \neq \tilde{N}\}, \\
\mathbf{C}_3 &= \mathbf{C} - \mathbf{C}_1 - \widetilde{\mathbf{C}_2}, \\
\mathbf{D}_1 &= \{(u,v) \in \mathbf{D} \mid u = i + j - 1\}, \\
\mathbf{D}_2 &= \mathbf{D} - \mathbf{D}_1
\end{aligned}$$

and

$$\mathbf{F} = \tilde{\mathbf{E}} \cup \mathbf{C}_1 \cup \widetilde{\mathbf{A}_1}.$$

Then,

$$\widetilde{\mathbf{O}} = \widetilde{\mathbf{A}_2} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D} \cup \tilde{\mathbf{E}},$$

where $\widetilde{\mathbf{A}_2}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and $\tilde{\mathbf{E}}$ are mutually disjoint.

THEOREM 2.3.2 *$V_{(j)}$ is negatively regression dependent on $U_{(i)}$; that is, for $(u,v) \in \widetilde{\mathbf{O}}$,*

$$\Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u\right\} \leq \Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u + 1\right\}.$$

Furthermore, we obtain the following results on when equality will hold.

(i) *For* $i = 1$, *equality holds if and only if* $(u, v) \in \mathbf{D_2} \cup \mathbf{F} \cup \widetilde{\mathbf{A_2}}$.

(ii) *For* $i = m$, *equality holds if and only if* $(u, v) \in \mathbf{D_2} \cup \mathbf{F} \cup \widetilde{\mathbf{C_2}}$.

(iii) *For* $1 < i < m$, *equality holds if and only if* $(u, v) \in \mathbf{D_2} \cup \mathbf{F}$.

PROOF. From Lemma 2.3.1, it follows that

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} = \Pr\left\{V_{(j)} \le v \mid U_{(i)} = u+1\right\} = 0 \qquad \text{for } (u,v) \in \mathbf{D_2},$$

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} = \Pr\left\{V_{(j)} \le v \mid U_{(i)} = u+1\right\} = 1 \qquad \text{for } (u,v) \in \mathbf{F},$$

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} = 0 < \Pr\left\{V_{(j)} \le v \mid U_{(i)} = u+1\right\} \qquad \text{for } (u,v) \in \mathbf{D_1}$$

and

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} < 1 = \Pr\left\{V_{(j)} \le v \mid U_{(i)} = u+1\right\} \qquad \text{for } (u,v) \in \mathbf{C_3}.$$

(a) The case of $(u, v) \in \mathbf{B}$. Note that $u = v$ in this case. If $u = \tilde{N}$, then, by Lemma 2.3.1, we have

$$\Pr\left\{V_{(j)} \le \tilde{N} \mid U_{(i)} = \tilde{N}\right\} < 1 = \Pr\left\{V_{(j)} \le \tilde{N}+1 \mid U_{(i)} = \tilde{N}+1\right\}.$$

Suppose that $u < \tilde{N}$. From Lemma 2.3.1, we have

$$\Pr\left\{V_{(j)} \le u \mid U_{(i)} = u+1\right\} = \Pr\left\{V_{(j)} \le u+1 \mid U_{(i)} = u+1\right\}.$$

From Corollary 2.3.1,

$$\Pr\left\{V_{(j)} \le u \mid U_{(i)} = u\right\} = \sum_{h=j}^{u-i} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-n}{n}}$$

$$< \Pr\left\{V_{(j)} \le v \mid U_{(i)} = u+1\right\} = \sum_{h=j}^{u+1-i} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-n}{n}}.$$

(b) The case of $(u, v) \in \widetilde{\mathbf{A_2}}$. First, note that $(u+1, v) \in \mathbf{A_2}$. By Corollary 2.3.1, we have

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u\right\} = \sum_{h=j}^{\min\{v,u-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \times \sum_{a=\max\{h,v+1-i\}}^{\min\{v,u-i\}} \frac{\binom{v}{a}\binom{u-1-v}{u-i-a}}{\binom{u-1}{u-i}} \qquad (2.3.9)$$

and

$$\Pr\left\{V_{(j)} \le v \mid U_{(i)} = u+1\right\} = \sum_{h=j}^{\min\{v,u+1-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}}$$

$$\times \sum_{a=\max\{h,v+1-i\}}^{\min\{v,u+1-i\}} \frac{\binom{v}{a}\binom{u+1-1-v}{u-i-a}}{\binom{u+1-1}{u+1-i}}. \tag{2.3.10}$$

If $i = 1$, then it is easily seen that

$$(2.3.9) = (2.3.10) = \sum_{h=j}^{\min\{v,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}}.$$

Now, suppose that $i \ge 2$. For $v = j$, we have

$$\frac{(2.3.10)}{(2.3.9)} = \frac{(u-j)(u+1-i)}{u(u+1-i-j)} > 1.$$

Let $v \ge j+1$. First, we consider the case of $v \le i+j-1$. In this case, we have

$$(2.3.9) = \sum_{h=j}^{\min\{v,u-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \left\{ \sum_{a=h}^{\min\{v,u-i\}} \frac{\binom{v}{a}\binom{u-1-v}{u-i-a}}{\binom{u-1}{u-i}} \right\} \tag{2.3.11}$$

and

$$(2.3.10) = \sum_{h=j}^{\min\{v,u+1-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}}$$

$$\times \left\{ \sum_{a=h}^{\min\{v,u+1-i\}} \frac{\binom{v}{a}\binom{u+1-1-v}{u+1-i-a}}{\binom{u+1-1}{u+1-i}} \right\}. \tag{2.3.12}$$

If $u = v+1$, then we have $u = i+j$ and $v = i+j-1$, and, therefore, we can show (2.3.10) < (2.3.11) by direct calculations. If $u > v+1$, then, putting $M = v$, $L = u-1-v$ and $r = u-i$ in Lemma 2.3.4, we have

$$\sum_{a=h}^{\min\{v,u-i\}} \frac{\binom{v}{a}\binom{u-1-v}{u-i-a}}{\binom{u-1}{u-i}} < \sum_{a=h}^{\min\{v,u+1-i\}} \frac{\binom{v}{a}\binom{u-1-v+1}{u-i+1-a}}{\binom{u-1+1}{u-i+1}}, \quad \text{for } j \le h \le \min\{v, u-i\}.$$

And, therefore, (2.3.12) > (2.3.11). Now, we consider the case of $v \ge i+j$. Then, (2.3.9) can be decomposed into two terms;

$$(2.3.9) = \sum_{h=j}^{v+1-i} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \left\{ \sum_{a=v+1-i}^{\min\{v,u-i\}} \frac{\binom{v}{a}\binom{u-1-v}{u-i-a}}{\binom{u-1}{u-i}} \right\}$$

$$+\sum_{h=v+1-i+1}^{\min\{v,u-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \left[\sum_{a=h}^{\min\{v,u-i\}} \frac{\binom{v}{a}\binom{u-1-v}{u-i-a}}{\binom{u-1}{u-i}}\right], \qquad (2.3.13)$$

where the second term of the right hand side of (2.3.13) disappears for $v = u+1$. Similarly, (2.3.10) can be written as

$$(2.3.10) = \sum_{h=j}^{v+1-i} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \left\{\sum_{a=v+1-i}^{\min\{v,u+1-i\}} \frac{\binom{v}{a}\binom{u+1-1-v}{u+1-i-a}}{\binom{u+1-1}{u+1-i}}\right\}$$

$$+\sum_{h=v+1-i+1}^{\min\{v,u+1-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \left[\sum_{a=h}^{\min\{v,u+1-i\}} \frac{\binom{v}{a}\binom{u+1-1-v}{u+1-i-a}}{\binom{u+1-1}{u+1-i}}\right]. \qquad (2.3.14)$$

Because the quantities in { } of (2.3.13) and (2.3.14) are 1, the first terms of (2.3.13) and (2.3.14) are equal. If $v = u - 1$, then the second term of (2.3.13) $= 0$ and the second term of (2.3.14) > 0. Therefore, we get (2.3.14) $>$ (2.3.13) for $v = u - 1$. Now, assume that $v \le u - 2$. Put $M = v$, $L = u - 1 - v$, $r = u - i$ and $k = h$ for the expression in [ ] of (2.3.13). By Lemma 2.3.4, we have

the second term of (2.3.13)

$$< \sum_{h=v+1-i+1}^{\min\{v,u+1-i,N-m-n+j\}} \frac{\binom{h-1}{j-1}\binom{N-m-h}{n-j}}{\binom{N-m}{n}} \sum_{a=h}^{\min\{u+1-i,v\}} \frac{\binom{v}{a}\binom{u+1-1-v}{u+1-i-a}}{\binom{u+1-1}{u+1-i}}$$

$\le$ the second term of (2.3.14).

Thus,we have (2.3.13) $<$ (2.3.14).

(c) The case of $(u, v) \in \widetilde{\mathbf{C_2}}$. Let $i' = m-i+1$, $j' = n-j+1$, $u' = N-u$, $v' = N-v$ and $\tilde{N}' = N - m - n + i' + j' - 1$. Since $i \le u < \tilde{N}$, $i+j \le v \le N-n+j-1$ and $u < v$, we have $i' + j' \le u' \le N - m + i' - 1$,$j' \le v' \le \tilde{N}' - 1$ and $v' < u'$. Therefore, we can use the result of the case (b) for (i', j', u', v'). Hence, we have

$$\Pr\left\{V_{(j')} \le v' \mid U_{(i')} = u'\right\} \le \Pr\left\{V_{(j')} \le v' \mid U_{(i')} = u' + 1\right\}$$

with equality holding if and only if $i' = 1$; that is,

$$\Pr\left\{V_{(n-j+1)} \le N - v \mid U_{(m-i+1)} = N - u\right\}$$

$$\leq \Pr\left\{V_{(n-j+1)} \leq N - v \mid U_{(m-i+1)} = N - u + 1\right\}$$

with equality holding if and only if $m - i + 1 = 1$. By Lemma 2.3.2 , we have

$$\Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u\right\} = \Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u + 1\right\} \quad \text{for } i = m$$

and

$$\Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u\right\} < \Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u + 1\right\} \quad \text{for } i \neq m.$$

This completes the proof of the theorem.

REMARK 2.3.1 If $(u, v) \in \mathbf{D_1} \cup \mathbf{B} \cup \mathbf{C_3}$, then

$$\Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u\right\} < \Pr\left\{V_{(j)} \leq v \mid U_{(i)} = u + 1\right\} \quad \text{for any } 1 \leq i \leq m.$$

Now let us consider the general case $\mathbf{\Omega} = \{x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_l, \cdots, x_l\}$. We easily obtain the following lemma.

LEMMA 2.3.5 *For* $1 \leq a, b \leq l$,

$$\Pr\left\{X_{(i)} \leq x_a, Y_{(j)} \leq x_b\right\} = \Pr\left\{U_{(i)} \leq f_a, V_{(j)} \leq f_b\right\}$$

and

$$\Pr\left\{X_{(i)} = x_a, Y_{(j)} \leq x_b\right\} = \sum_{s=\tilde{f}_a}^{f_a} \Pr\left\{U_{(i)} = s, V_{(j)} \leq f_b\right\}.$$

Let $\mathbf{I}_{ab} = \{(u, f_b) \mid \tilde{f}_a \leq u < f_{a+1}\}$.

THEOREM 2.3.3 *Let* $1 \leq i \leq m$ *and* $1 \leq j \leq n$. *Then,* $Y_{(j)}$ *is negatively regression dependent on* $X_{(i)}$*; that is,*

$$\Pr\left\{Y_{(j)} \leq x_b \mid X_{(i)} = x_a\right\} \leq \Pr\left\{Y_{(j)} \leq x_b \mid X_{(i)} = x_{a+1}\right\}, \tag{2.3.15}$$

where $i \leq f_a < f_{a+1} \leq N - m + i$ *and* $j \leq f_b \leq N - j + 1$. *Furthermore, the following results with equality hold.*

(i) *For* $i = 1$, *equality holds if and only if* $\mathbf{I}_{ab} \subseteq \mathbf{D_2} \cup \mathbf{F} \cup \widetilde{\mathbf{A_2}}$.

(ii) *For* $i = m$, *equality holds if and only if* $\mathbf{I}_{ab} \subseteq \mathbf{D_2} \cup \mathbf{F} \cup \mathbf{C_2}$.

(iii) *For* $1 < i < m$, *equality holds if and only if* $\mathbf{I}_{ab} \subseteq \mathbf{D_2} \cup \mathbf{F}$.

PROOF. First, note that $i \leq f_a < f_{a+1} \leq N - m + i \Leftrightarrow \Pr\left\{X_{(i)} = x_a\right\} > 0$ and $\Pr\left\{X_{(i)} = x_{a+1}\right\} > 0$, and $j \leq f_b \leq N - j + 1 \Leftrightarrow \Pr\left\{Y_{(j)} = x_b\right\} > 0$. Using Lemma 2.3.5 , it is seen that (2.3.15) is equivalent to

$$\sum_{s=\tilde{f}_a}^{f_a} \sum_{t=\tilde{f}_{a+1}}^{f_{a+1}} \Pr\left\{U_{(i)} = s, V_{(j)} \leq f_b\right\} \Pr\left\{U_{(i)} = t\right\}$$

$$\leq \sum_{s=\tilde{f}_a}^{f_a} \sum_{t=\tilde{f}_{a+1}}^{f_{a+1}} \Pr\left\{U_{(i)} = t, V_{(j)} \leq f_b\right\} \Pr\left\{U_{(i)} = s\right\}. \tag{2.3.16}$$

From Theorem 2.3.2 , we have

$$\Pr\left\{U_{(i)} = u_1, V_{(j)} \leq f_b\right\} \Pr\left\{U_{(i)} = u_2\right\}$$
$$\leq \Pr\left\{U_{(i)} = u_2, V_{(j)} \leq f_b\right\} \Pr\left\{U_{(i)} = u_1\right\}, \tag{2.3.17}$$

for $i \leq u_1 < u_2 \leq N - m + i$. For $1 \leq u_1 \leq i - 1$ or $u_2 \geq N - m + 1$, the both sides of (2.3.17) are 0. Summing the inequalities (2.3.17) over $\tilde{f}_a \leq u_1 \leq f_a$, $\tilde{f}_{a+1} \leq u_2 \leq f_{a+1}$, we obtain (2.3.16). It remains only to check the conditions for equality. From (2.3.16) and Theorem 2.3.2 , it follows that

the equality holds in (2.3.15)

$\Leftrightarrow$ the equality holds in (2.3.16) for all (s,t) such that $\tilde{f}_a \leq s \leq f_a$ and $\tilde{f}_{a+1} \leq t \leq f_{a+1}$

$\Leftrightarrow$ $(s, f_b), (s+1, f_b), \cdots, (t, f_b) \in \chi$ for all (s,t) such that $\tilde{f}_a \leq s \leq f_a$ and $\tilde{f}_{a+1} \leq t \leq f_{a+1}$, where χ denotes $\mathbf{D_2} \cup \mathbf{F} \cup \widetilde{\mathbf{A_2}}$ for $i = 1$, $\mathbf{D_2} \cup \mathbf{F} \cup \mathbf{C_2}$ for $i = m$ and $\mathbf{D_2} \cup \mathbf{F}$ for $1 < i < m$

$\Leftrightarrow$ $(s, f_b) \in \chi$ for $\tilde{f}_a \leq s \leq f_{a+1} - 1$

$\Leftrightarrow$ $\mathbf{I}_{ab} \subseteq \chi$.

This completes the proof of the theorem.

2.4 Covariances of two order statistics from one sample and from two samples

Let $\{Y_1, Y_2, \cdots, Y_n\}$ be a simple random sample of size n without replacement from $\mathbf{\Omega} = \{x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_l, \cdots, x_l\}$ of size N. Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ be the order statistics of $\{Y_1, Y_2, \cdots, Y_n\}$. We define $\mathcal{A}_{n:i}$ and $\mathcal{B}$ by $\{s \mid i \leq s < N - n + i\}$ and $\{f_k \mid 1 \leq k \leq l\}$, respectively.

THEOREM 2.4.1 *Let $1 \leq i < j \leq n$. Then we have*

$$\mathrm{Cov}(Y_{(i)}, Y_{(j)}) \geq 0$$

with equality holding if and only if $\mathcal{A}_{n:i} \cap \mathcal{B} = \emptyset$ or $\mathcal{A}_{n:j} \cap \mathcal{B} = \emptyset$.

PROOF. If $\mathcal{A}_{n:i} \cap \mathcal{B} = \emptyset$, then $Y_{(i)}$ is constant and, therefore, $\mathrm{Cov}(Y_{(i)}, Y_{(j)}) = 0$. Similarly, if $\mathcal{A}_{n:j} \cap \mathcal{B} = \emptyset$, then $\mathrm{Cov}(Y_{(i)}, Y_{(j)}) = 0$. Suppose that $\mathcal{A}_{n:i} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{A}_{n:j} \cap \mathcal{B} \neq \emptyset$. Let $f_a = \min\{\mathcal{A}_{n:i} \cap \mathcal{B}\}$ and let $f_b = \max\{\mathcal{A}_{n:j} \cap \mathcal{B}\}$. Then, $(i, f_b) \in \mathbf{G}_{ab} \cap \mathbf{S}$ and $(f_a + 1, N - n + j) \in \mathbf{G}_{(a+1)(b+1)} \cap \mathbf{S}$. By Theorem 2.2.1, we get

$$\begin{aligned} &\Pr\left\{Y_{(i)} = x_a, Y_{(j)} = x_b\right\} \Pr\left\{Y_{(i)} = x_{a+1}, Y_{(j)} = x_{b+1}\right\} \\ &\quad > \Pr\left\{Y_{(i)} = x_a, Y_{(j)} = x_{b+1}\right\} \Pr\left\{Y_{(i)} = x_{a+1}, Y_{(j)} = x_b\right\}. \end{aligned}$$

Hence, $Y_{(i)}$ and $Y_{(j)}$ are not independent. Since, by Theorem 2.2.1, $(Y_{(i)}, Y_{(j)})$ is positive likelihood ratio dependent, $(Y_{(i)}, Y_{(j)})$ is positive quadrant dependent (See Lehmann (1959), p.74 and Lehmann (1966), p.1144). By Lemma 3 of Lehmann (1966), we have

$$\mathrm{Cov}(Y_{(i)}, Y_{(j)}) > 0.$$

This completes the proof of the theorem.

Let $\{X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_n\}$ be a simple random sample of size $m + n$ without replacement from $\mathbf{\Omega}$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ be the order statistics of $X_1, X_2, \cdots, X_m$ and $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$ the order statistics of $Y_1, Y_2, \cdots, Y_n$.

THEOREM 2.4.2 *Let $1 \le i \le m$ and $1 \le j \le n$. Then we have*

$$\mathrm{Cov}(X_{(i)}, Y_{(j)}) \le 0,$$

with equality holding if and only if $\mathcal{A}_{m:i} \cap \mathcal{B} = \emptyset$ or $\mathcal{A}_{n:j} \cap \mathcal{B} = \emptyset$.

PROOF. If $\mathcal{A}_{m:i} \cap \mathcal{B} = \emptyset$ or $\mathcal{A}_{n:j} \cap \mathcal{B} = \emptyset$, then $X_{(i)}$ or $Y_{(j)}$ is constant and, therefore, $\mathrm{Cov}(X_{(i)}, Y_{(j)}) = 0$. Assume that $\mathcal{A}_{m:i} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{A}_{n:j} \cap \mathcal{B} \neq \emptyset$. Since, by Theorem 2.3.3, $Y_{(j)}$ is negatively regression dependent on $X_{(i)}$, $(X_{(i)}, Y_{(j)})$ is negatively quadrant dependent (See Lehmann (1966), p.1144). By Lemma 3 of Lehmann (1966), we have

$$\mathrm{Cov}(X_{(i)}, Y_{(j)}) \le 0$$

and if $X_{(i)}$ and $Y_{(j)}$ are not independent, then we have

$$\mathrm{Cov}(X_{(i)}, Y_{(j)}) < 0.$$

Now,we shall prove that $X_{(i)}$ and $Y_{(j)}$ are not independent.Suppose that $\mathbf{I}_{ab} \cap (\mathbf{B} \cup \mathbf{C_3} \cup \mathbf{D_1}) \neq \emptyset$. Since $(\mathbf{B} \cup \mathbf{C_3} \cup \mathbf{D_1}) \cap (\mathbf{A_2} \cup \mathbf{C_2} \cup \mathbf{D_2} \cup \mathbf{F}) = \emptyset$, we have $\mathbf{I}_{ab} \nsubseteq \mathbf{A_2} \cup \mathbf{C_2} \cup \mathbf{D_2} \cup \mathbf{F}$. By Theorem 2.3.3, we have

$$\Pr\left\{Y_{(j)} \le x_b \mid X_{(i)} = x_a\right\} < \Pr\left\{Y_{(j)} \le x_b \mid X_{(i)} = x_{a+1}\right\}.$$

This implies that $X_{(i)}$ and $Y_{(j)}$ are not independent. Therefore, it suffices to show the existence of (a, b) such that $\mathbf{I}_{ab} \cap (\mathbf{B} \cup \mathbf{C_3} \cup \mathbf{D_1}) \neq \emptyset$. Let $\mathcal{J} = \mathcal{A}_{m:i} \cap \mathcal{A}_{n:j} \cap \mathcal{B}$, $\mathcal{C}_1 = \{k \mid \max\{i, j\} \le k < i + j\}$, $\mathcal{C}_2 = \{k \mid i + j \le k \le \tilde{N}\}$ and $\mathcal{C}_3 = \{k \mid \tilde{N} < k < \min\{N - m + i, N - n + j\}\}$.

Case(i): $\mathcal{J} \neq \emptyset$.

(a): $\mathcal{C}_2 \cap \mathcal{B} \neq \emptyset$. Let $f_a \in \mathcal{C}_2 \cap \mathcal{B}$. Then, $i + j \le f_a \le \tilde{N}$. It follows that $(f_a, f_a) \in \mathbf{B}$. Thus, $\mathbf{I}_{aa} \cap \mathbf{B} \neq \emptyset$.

(b): $\mathcal{C}_2 \cap \mathcal{B} = \emptyset$ and $\mathcal{C}_1 \cap \mathcal{B} \neq \emptyset$. Let $f_a = \max\{\mathcal{C}_1 \cap \mathcal{B}\}$. Then, $\max\{i, j\} \leq f_a < i+j$ and $\tilde{f}_a \leq i+j-1 < f_{a+1}$. It follows that $(i+j-1, f_a) \in \mathbf{I}_{aa} \cap \mathbf{D_1}$. Thus, $\mathbf{I}_{aa} \cap \mathbf{D_1} \neq \emptyset$.

(c): $\mathcal{C}_1 \cap \mathcal{B} = \emptyset$, $\mathcal{C}_2 \cap \mathcal{B} = \emptyset$ and $\mathcal{C}_3 \neq \emptyset$. Let $f_a = \min\{\mathcal{C}_3 \cap \mathcal{B}\}$. We have $\tilde{N} + 1 \leq f_a \leq \min\{N - m + i, N - n + j\} - 1$ and $\tilde{f}_a \leq \tilde{N} < f_{a+1}$. It follows that $(\tilde{N}, f_a) \in \mathbf{I}_{aa} \cap \mathbf{C_3}$. Thus, $\mathbf{I}_{aa} \cap \mathbf{C_3} \neq \emptyset$.

Case(ii): $\mathcal{J} = \emptyset$. Note that, under the assumption $\mathcal{A}_{m:i} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{A}_{n:j} \cap \mathcal{B} \neq \emptyset$, $i = j$ implies $\mathcal{J} \neq \emptyset$.

(a): $i < j$. There exists $f_a \in \mathcal{B}$ such that $i \leq f_a < j$ and $N - m + i \leq f_{a+1} < N - n + j$. It is seen that $(\tilde{N}, f_a) \in \mathbf{I}_{a(a+1)} \cap \mathbf{C_3}$. Thus, $\mathbf{I}_{a(a+1)} \cap \mathbf{C_3} \neq \emptyset$.

(b): $i > j$. There exists $f_a \in \mathcal{B}$ such that $j \leq f_a < i$ and $N - n + j \leq f_{a+1} < N - m + i$. It is seen that $(i + j - 1, f_a) \in \mathbf{I}_{aa} \cap \mathbf{D_1}$. Thus, $\mathbf{I}_{aa} \cap \mathbf{D_1} \neq \emptyset$.

This completes the proof of the theorem.

2.5 Lower bound of relative precision of RSS from a finite population

Let n and r have the same meaning as the set-size and the number of cycles in Chapter 1. We draw a simple random sample $\{X_{ijk} \mid i, j = 1, 2, \ldots, n; k = 1, 2, \ldots, r\}$ of size $n^2 r$ without replacement from a finite population $\mathbf{\Omega}$ of size N with mean μ and variance σ^2. For each i and k, the n units $\{X_{i1k}, X_{i2k}, \ldots, X_{ink}\}$ are ranked by a visual inspection and the unit with the i-th smallest rank is quantified. This procedure involves the quantification of nr units out of the $n^2 r$ units originally drawn.

Let $X_{[i]k}$ be the i-th smallest order statistics of $\{X_{i1k}, X_{i2k}, \ldots, X_{ink}\}$. Then, the ranked set sample obtained by the above procedure can be written as

$\left\{X_{[i]k} \mid i = 1, 2, \ldots, n; k = 1, 2, \ldots, r\right\}$ and the ranked set estimator $\hat{\mu}_{RSS}$ of μ is the average of $X_{[i]k}$ $(i = 1, 2, \ldots, n; k = 1, 2, \ldots, r)$:

$$\hat{\mu}_{RSS} = \overline{X}_{[n]r} = \frac{1}{nr} \sum_{k=1}^{r} \sum_{i=1}^{n} X_{[i]k}.$$

This is an unbiased estimator of μ (Takahasi and Futatsuya (1988), Patil *et al.* (1995)). Let $\overline{X}_{nr}$ be the sample mean of a simple random sample of size nr drawn without replacement from $\mathbf{\Omega}$. The definition of relative precision (RP) of $\overline{X}_{[n]r}$ (Patil *et al.* (1994)) is

$$RP = \frac{\mathrm{Var}(\overline{X}_{nr})}{\mathrm{Var}(\overline{X}_{[n]r})}.$$

In this section, we show that $RP > 1$ for almost all populations and $RP = 1$ for very exceptional populations. The corresponding result in infinite population can be found in Takahasi and Wakimoto (1968).

Let $\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$ be a simple random sample of size $2n$ without replacement from $\mathbf{\Omega}$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics of $X_1, X_2, \ldots, X_n$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ be the order statistics of $Y_1, Y_2, \ldots, Y_n$. We shall use the following notations:

$$\mu_{n:i} = \mathrm{E}(X_{(i)}) = \mathrm{E}(Y_{(i)}),$$

$$\tilde{\alpha}_{n:ij} = \mathrm{Cov}(X_{(i)}, X_{(j)}) = \mathrm{Cov}(Y_{(i)}, Y_{(j)}),$$

and

$$\tilde{\gamma}_{n:ij} = \mathrm{Cov}(X_{(i)}, Y_{(j)}).$$

LEMMA 2.5.1 *Let $n^2 \leq N$. Then,*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{\gamma}_{n:ij} = -\frac{n^2 \sigma^2}{N-1} \tag{2.5.1}$$

and

$$\sum_{i=1}^{n} \tilde{\alpha}_{n:ii} = \frac{N-n}{N-1} n\sigma^2 - \sum_{i \neq j}\sum \tilde{\alpha}_{n:ij}. \tag{2.5.2}$$

PROOF. It is well known that

$$\mathrm{Cov}(\overline{X},\overline{Y}) = -\frac{\sigma^2}{N-1} \tag{2.5.3}$$

and

$$\mathrm{Var}\overline{X} = \frac{N-n}{N-1}\frac{\sigma^2}{n}, \tag{2.5.4}$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. On the other hand, we have

$$\begin{aligned}
\mathrm{Cov}(\overline{X},\overline{Y}) &= \mathrm{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{j=1}^{n} Y_j\right) \\
&= \mathrm{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} X_{(i)}, \frac{1}{n}\sum_{j=1}^{n} Y_{(j)}\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}\left(X_{(i)}, Y_{(j)}\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{\gamma}_{n:ij}
\end{aligned} \tag{2.5.5}$$

and

$$\begin{aligned}
\mathrm{Var}\overline{X} &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_{(i)}\right) \\
&= \frac{1}{n^2}\mathrm{E}\left[\left(\sum_{i=1}^{n}(X_{(i)} - \mu_{n:i})\right)^2\right] \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n}\mathrm{E}\left((X_{(i)} - \mu_{n:i})^2\right) + \sum\sum_{i\neq j}\mathrm{E}\left((X_{(i)} - \mu_{n:i})(X_{(j)} - \mu_{n:j})\right)\right] \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n}\tilde{\alpha}_{n:ii} + \sum\sum_{i\neq j}\tilde{\alpha}_{n:ij}\right].
\end{aligned} \tag{2.5.6}$$

From (2.5.3) and (2.5.5), we have (2.5.1) and, from (2.5.4) and (2.5.6), we have (2.5.2).

THEOREM 2.5.1 *Let $n^2 r \leq N$. Then,*

$$\mathrm{Var}(\overline{X}_{[n]r}) = \mathrm{Var}(\overline{X}_{nr}) - \frac{1}{n^2 r}\left(\sum\sum_{i\neq j}\tilde{\alpha}_{n:ij} - \sum\sum_{i\neq j}\tilde{\gamma}_{n:ij}\right). \tag{2.5.7}$$

Proof. We see that

$$\begin{aligned}
\mathrm{Var}(\overline{X}_{[n]r}) &= \mathrm{E}\left(\left(\frac{1}{nr}\sum_{k=1}^{r}\sum_{i=1}^{n}X_{[i]k}-\mu\right)^2\right)\\
&= \mathrm{E}\left(\left(\frac{1}{nr}\sum_{k=1}^{r}\sum_{i=1}^{n}(X_{[i]k}-\mu_{n:i})\right)^2\right)\\
&= \frac{1}{n^2r^2}\sum_{k=1}^{r}\sum_{h=1}^{r}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_{[i]k},X_{[j]h})\\
&= \frac{1}{n^2r^2}\left(\sum_{k=1}^{r}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_{[i]k},X_{[j]k})+\sum\sum_{k\neq h}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_{[i]k},X_{[j]h})\right)\\
&= \frac{1}{n^2r^2}\left(\sum_{k=1}^{r}\sum_{i=1}^{n}\mathrm{Cov}(X_{[i]k},X_{[i]k})+\sum_{k=1}^{r}\sum\sum_{i\neq j}\mathrm{Cov}(X_{[i]k},X_{[j]k})\right.\\
&\qquad\left.+\sum\sum_{k\neq h}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_{[i]k}X_{[j]h})\right)\\
&= \frac{1}{n^2r^2}\left(r\sum_{i=1}^{n}\widetilde{\alpha}_{n:ii}+r\sum\sum_{i\neq j}\widetilde{\gamma}_{n:ij}+r(r-1)\sum_{i=1}^{n}\sum_{j=1}^{n}\widetilde{\gamma}_{n:ij}\right). \qquad (2.5.8)
\end{aligned}$$

Substituting (2.5.1) and (2.5.2) for the most right hand of (2.5.8), we obtain (2.5.7). The proof is complete.

Now,we can prove the following theorem.

Theorem 2.5.2 *Let $n^2r \leq N$. Then,*

$$RP \geq 1,$$

with equality holding if and only if one of the following conditions for populations is fulfilled:

(i) $\{x_1 < x_2 = x_3 = \cdots = x_N\}$;

(ii) $\{x_1 = x_2 = \cdots = x_{N-1} < x_N\}$;

(iii) $\{x_1 = x_2 = \cdots = x_N\}$.

Proof. By Theorem 2.4.1 and Theorem 2.4.2, we have

$$\sum\sum_{i\neq j}\widetilde{\alpha}_{n:ij}-\sum\sum_{i\neq j}\widetilde{\gamma}_{n:ij}\geq 0 \qquad (2.5.9)$$

From this and Theorem 2.5.1, we have

$$\mathrm{Var}(\overline{X}_{[n]r}) \leq \mathrm{Var}(\overline{X}_{nr}).$$

Therefore, $RP \geq 1$. The equality in (2.5.9) holds if and only if $\tilde{\alpha}_{n:ij} = 0$ and $\tilde{\gamma}_{n:ij} = 0$ for all $1 \leq i, j \leq n$ $(i \neq j)$. Now, we assume that $\tilde{\alpha}_{n:ij} = 0$ and $\tilde{\gamma}_{n:ij} = 0$ for all $1 \leq i, j \leq n$ $(i \neq j)$. Then, from Theorem 2.4.1 or Theorem 2.5.1, we must have $\mathcal{A}_{n:i} \cap \mathcal{B} = \emptyset$ or $\mathcal{A}_{n:j} \cap \mathcal{B} = \emptyset$ for any (i, j) such that $1 \leq i, j \leq n$ and $i \neq j$. Suppose that, for $1 < k < n$, $\mathcal{A}_{n:k} \cap \mathcal{B} \neq \emptyset$. From the assumption, $\mathcal{A}_{n:k-1} \cap \mathcal{B} = \emptyset$ and $\mathcal{A}_{n:k+1} \cap \mathcal{B} = \emptyset$. Recall that $\mathcal{A}_{n:k} = \{s \mid k \leq s < N - n + k\}$. Because $\mathcal{A}_{n:k} \subseteq \mathcal{A}_{n:k-1} \cup \mathcal{A}_{n:k+1}$, it follows that $\mathcal{A}_{n:k} \cap \mathcal{B} = \emptyset$. This is a contradiction. Therefore, under the assumption, one of the following cases must hold:

(a) $\mathcal{A}_{n:1} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{A}_{n:k} \cap \mathcal{B} = \emptyset$ for $2 \leq k \leq n$;

(b) $\mathcal{A}_{n:n} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{A}_{n:k} \cap \mathcal{B} = \emptyset$ for $1 \leq k \leq n - 1$;

(c) $\mathcal{A}_{n:k} \cap \mathcal{B} = \emptyset$ for $1 \leq k \leq n$.

It is obvious that (a) implies (i), (b) implies (ii) and (c) implies (iii). Conversely, it is clear that (i), (ii) or (iii) implies $RP = 1$. This completes the proof.

From Theorem 2.5.2, we can say that $RP > 1$, unless $N - 1$ elements of the finite population of size N have the same value.

Chapter 3

Extremal Finite Populations Maximizing Relative Precisions in Ranked Set Sampling

3.1 Introduction

Let $\Omega = \{x_1, x_2, \cdots, x_N\}$ be a finite population of size N. Let n, r, $X_{[i]k}$, $\overline{X}_{[n]r}$, RP and $\mathrm{Var}(\overline{X}_{nr})$ have the same meaning as in Section 2.5. In this chapter, we discuss the following problems: Given the population size N, the set-size n and the number of cycles r, what is the maximum RP, and what is the population that has the maximum RP ? In the case of the replacement ranked set sampling from an infinite population, Takahasi and Wakimoto (1968) have shown that for any distribution with finite variance the RP is bounded above by $(n+1)/2$ and that the upper bound is attained only by the uniform distribution. In section 3.3, we shall show by numerical examples that the maximum RP in the case of sampling without replacement from finite populations is larger than $(n+1)/2$ and that the discrete uniform distribution does not lead to the maximum RP.

3.2 Variance of RSS estimator

Let us denote by μ and σ^2 the mean and the variance of the finite population Ω. Rearranging the elements of Ω in the order of their magnitude, we write $x_{(1)} \le x_{(2)} \le \cdots \le x_{(N)}$. Now, consider a simple random sample of size $2n$, $\tilde{z} = \{Z_1, Z_2, \cdots, Z_{2n}\}$, without replacement from Ω. Let us denote by $Z_{(1)} \le Z_{(2)} \le \cdots \le Z_{(2n)}$ order statistics of this sample. Let $\{X_1, X_2, \cdots, X_n\}$ be the first n elements $\{Z_1, Z_2, \cdots, Z_n\}$ of $\tilde{z}$ and let $\{Y_1, Y_2, \cdots, Y_n\}$ be the last n elements $\{Z_{n+1}, Z_{n+2}, \cdots, Z_{2n}\}$ of $\tilde{z}$. Suppose that $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$, $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$are order statistics of these samples. Put

$$\mu_{n:i} = \mathrm{E}(X_{(i)}) = \mathrm{E}(Y_{(i)}),$$

$$\alpha_{n:ij} = \mathrm{E}(X_{(i)}X_{(j)}) = \mathrm{E}(Y_{(i)}Y_{(j)}),$$

$$\alpha_{2n:ab} = \mathrm{E}(Z_{(a)}Z_{(b)})$$

and

$$\gamma_{n:ij} = \mathrm{E}(X_{(i)}Y_{(j)}).$$

LEMMA 3.2.1 *If $n^2 \le N$, then*

$$\sum_{i=1}^{n} \alpha_{n:ii} = n(\sigma^2 + \mu^2), \tag{3.2.1}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} \gamma_{n:ij} = -n^2\left(\frac{\sigma^2}{N-1} - \mu^2\right) \tag{3.2.2}$$

and

$$\sum\sum_{1 \le a < b \le 2n} \alpha_{2n:ab} = -n(2n-1)\left(\frac{\sigma^2}{N-1} - \mu^2\right). \tag{3.2.3}$$

PROOF. Since $\sum_{i=1}^{n} \alpha_{n:ii} = \mathrm{E}(\sum_{i=1}^{n} X_{(i)}^2) = \mathrm{E}(\sum_{i=1}^{n} X_i^2) = n(\sigma^2 + \mu^2)$, (3.2.1) holds. (3.2.2) follows from the equations

$$-\frac{\sigma^2}{N-1} = \mathrm{Cov}(\overline{X},\overline{Y}) = \mathrm{E}(\overline{X}\cdot\overline{Y}) - \mathrm{E}(\overline{X})\cdot\mathrm{E}(\overline{Y})$$
$$= \frac{1}{n^2}\mathrm{E}((\sum_{i=1}^{n} X_i)\cdot(\sum_{j=1}^{n} Y_j)) - \mu^2$$
$$= \frac{1}{n^2}\mathrm{E}((\sum_{i=1}^{n} X_{(i)})\cdot(\sum_{j=1}^{n} Y_{(j)})) - \mu^2$$
$$= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\gamma_{n:ij} - \mu^2.$$

Using

$$\sum_{a=1}^{2n}\sum_{b=1}^{2n}\alpha_{2n:ab} = \mathrm{E}((\sum_{a=1}^{2n} Z_{(a)})^2) = \mathrm{E}((\sum_{a=1}^{2n} Z_a)^2)$$

$$= 4n^2\left(\frac{(N-2n)\sigma^2}{(N-1)2n} + \mu^2\right)$$

and $\sum_{a=1}^{2n}\alpha_{2n:aa} = 2n(\sigma^2+\mu^2)$, we have (3.2.3).

LEMMA 3.2.2 *For $rn^2 \le N$,*

$$\begin{aligned}
\mathrm{Var}(\overline{X}_{[n]r}) &= \mathrm{Var}(\overline{X}_{nr}) - \frac{1}{nr}\left(\frac{\sigma^2}{N-1} - \mu^2 + \frac{1}{n}\sum_{i=1}^{n}\gamma_{n:ii}\right) \\
&= \mathrm{Var}(\overline{X}_{nr}) - \frac{1}{nr}\left(\frac{\sigma^2}{N-1} + \frac{1}{n}\sum_{i=1}^{n}(\mu_{n:i}-\mu)^2\right. \\
&\qquad \left. + \frac{1}{n}\sum_{i=1}^{n}\mathrm{Cov}(X_{(i)},Y_{(i)})\right) \qquad (3.2.4) \\
&= \mathrm{Var}(\overline{X}_{nr}) - \frac{1}{n^2r}\left(-\frac{1}{2n-1}\sum\sum_{1\le a<b\le 2n}\alpha_{2n:ab} + \sum_{i=1}^{n}\gamma_{n:ii}\right). \quad (3.2.5)
\end{aligned}$$

PROOF. We see that

$$\mathrm{Var}(\overline{X}_{[n]r}) = \mathrm{E}\left(\left(\frac{1}{nr}\sum_{k=1}^{r}\sum_{i=1}^{n}X_{[i]k}\right)^2\right) - \mu^2$$

$$
\begin{aligned}
&= \frac{1}{n^2r^2}\sum_{k=1}^{r}\sum_{i=1}^{n}\sum_{h=1}^{r}\sum_{j=1}^{n}\mathrm{E}(X_{[i]k}X_{[j]h}) - \mu^2 \\
&= \frac{1}{n^2r^2}\left(\sum_{k=1}^{r}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}(X_{[i]k}X_{[j]k}) + \sum\sum_{k\neq h}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}(X_{[i]k}X_{[j]h})\right) - \mu^2 \\
&= \frac{1}{n^2r^2}\left(\sum_{k=1}^{r}\sum_{i=1}^{n}\mathrm{E}(X_{[i]k}^2) + \sum_{k=1}^{r}\sum\sum_{i\neq j}\mathrm{E}(X_{[i]k}X_{[j]k})\right. \\
&\qquad \left. + \sum\sum_{k\neq h}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}(X_{[i]k}X_{[j]h})\right) - \mu^2 \\
&= \frac{1}{n^2r^2}\left(r\sum_{i=1}^{n}\alpha_{n:ii} + r\sum\sum_{i\neq j}\gamma_{n:ij} + r(r-1)\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{n:ij}\right) - \mu^2 \\
&= \frac{1}{n^2r}\left(\sum_{i=1}^{n}\alpha_{n:ii} + r\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{n:ij} - \sum_{i=1}^{n}\gamma_{n:ii}\right) - \mu^2 \\
&= \frac{1}{n^2r}\left(n(\sigma^2+\mu^2) - rn^2\left(\frac{\sigma^2}{N-1} - \mu^2\right) - \sum_{i=1}^{n}\gamma_{n:ii}\right) - \mu^2 \\
&= \frac{(N-nr)\sigma^2}{(N-1)nr} - \frac{1}{nr}\left(\frac{\sigma^2}{N-1} - \mu^2 + \frac{1}{n}\sum_{i=1}^{n}\gamma_{n:ii}\right) \\
&= \mathrm{Var}(\overline{X}_{nr}) - \frac{1}{n^2r}\left(-\frac{1}{2n-1}\sum\sum_{1\leq a<b\leq 2n}\alpha_{2n:ab} + \sum_{i=1}^{n}\gamma_{n:ii}\right).
\end{aligned}
$$

The equality (3.2.4) follows from this statement and $\mu = (\sum_{i=1}^{n}\mu_{n:i})/n$.

Hereafter, we use the convention that $\binom{l}{m} = 0$ for $m < 0 \leq l$ and for $0 \leq l < m$, and that $\binom{0}{0} = 1$.

LEMMA 3.2.3 *The cross product moment $\gamma_{n:ij}$ is given in terms of the product moments $\alpha_{2n:ab}$, $1 \leq a < b \leq 2n$, by*

$$
\begin{aligned}
\gamma_{n:ij} = \sum\sum_{1\leq a<b\leq 2n}&\left\{\binom{a-1}{i-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-j}\right. \\
&\left. + \binom{a-1}{j-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-i}\right\}\alpha_{2n:ab}\Big/\binom{2n}{n}.
\end{aligned}
$$

PROOF. It is seen from combinatorial consideration that the conditional joint probability distribution of $X_{(i)}$ and $Y_{(j)}$ given $\tilde{z}$ is

$$\Pr\left(X_{(i)} = Z_{(a)}, Y_{(j)} = Z_{(b)} \mid \tilde{z}\right)$$

$$= \begin{cases} \binom{a-1}{i-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-j} \Big/ \binom{2n}{n} & (a < b) \\ \\ \binom{b-1}{j-1}\binom{a-1-b}{a-i-j}\binom{2n-a}{n-i} \Big/ \binom{2n}{n} & (a > b). \end{cases}$$

Thus,

$$\begin{aligned}
\gamma_{n:ij} &= \mathrm{E}(X_{(i)}Y_{(j)}) = \mathrm{E}'(\mathrm{E}(X_{(i)}Y_{(j)} \mid \tilde{z})) \\
&= \mathrm{E}'\left(\sum_{a=1}^{2n}\sum_{b=1}^{2n} Z_{(a)}Z_{(b)} \Pr\left(X_{(i)} = Z_{(a)}, Y_{(j)} = Z_{(b)} \mid \tilde{z}\right)\right) \\
&= \mathrm{E}'\left(\sum\sum_{1\le a<b\le 2n} Z_{(a)}Z_{(b)}\binom{a-1}{i-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-j} \Big/ \binom{2n}{n}\right. \\
&\qquad \left. + \sum_{1\le b<a\le 2n} Z_{(b)}Z_{(a)}\binom{b-1}{j-1}\binom{a-1-b}{a-i-j}\binom{2n-a}{n-i} \Big/ \binom{2n}{n}\right) \\
&= \mathrm{E}'\left(\sum\sum_{1\le a<b\le 2n}\left\{\binom{a-1}{i-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-j}\right.\right. \\
&\qquad \left.\left. +\binom{a-1}{j-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-i}\right\} Z_{(a)}Z_{(b)} \Big/ \binom{2n}{n}\right) \\
&= \sum\sum_{1\le a<b\le 2n}\left\{\binom{a-1}{i-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-j}\right. \\
&\qquad \left. +\binom{a-1}{j-1}\binom{b-1-a}{b-i-j}\binom{2n-b}{n-i}\right\} \mathrm{E}'(Z_{(a)}Z_{(b)}) \Big/ \binom{2n}{n},
\end{aligned}$$

where the symbol E' denotes the average over all possible samples.

THEOREM 3.2.1 *For $1 \le a < b \le 2n$ and $1 \le s < t \le N$, put*

$$A(2n,a,b) = \left(\sum_{i=1}^{n} 2\binom{a-1}{i-1}\binom{b-1-a}{b-2i}\binom{2n-b}{n-i} \Big/ \binom{2n}{n}\right) - \frac{1}{2n-1}$$

and

$$C(N,2n,s,t,a,b)=\binom{s-1}{a-1}\binom{t-1-s}{b-1-a}\binom{N-t}{2n-b}\Big/\binom{N}{2n}.$$

Further, put

$$D(N,n,s,t)=\sum\sum_{1\le a<b\le 2n} A(2n,a,b)\cdot C(N,2n,s,t,a,b)/2 \quad \textit{for} \ \ 1\le s<t\le N,$$

$$D(N,n,s,t)=0 \qquad \textit{for} \ \ 1\le s=t\le N$$

and

$$D(N,n,s,t)=D(N,n,t,s) \qquad \textit{for} \ \ 1\le t<s\le N.$$

Let Γ *denote the* $N\times N$ *matrix whose element in* i*-th row and* j*-th column is* $D(N,n,i,j)$ *and let* θ *denote the column vector* $(x_{(1)},x_{(2)},\cdots,x_{(N)})'$.
Then, from these notations, $\mathrm{Var}(\overline{X}_{[n]r})$ *can be written as*

$$\mathrm{Var}(\overline{X}_{[n]r})=\mathrm{Var}(\overline{X}_{nr})-\frac{1}{n^2r}(\theta'\Gamma\theta). \tag{3.2.6}$$

PROOF. Two equations

$$\gamma_{n:ii}=\sum\sum_{1\le a<b\le 2n} 2\binom{a-1}{i-1}\binom{b-1-a}{b-2i}\binom{2n-b}{n-i}\alpha_{2n:ab}\Big/\binom{2n}{n} \tag{3.2.7}$$

and

$$\begin{aligned}\alpha_{2n:ab} &= \sum_{t=1}^{N}\sum_{s=1}^{N} x_{(s)}x_{(t)}\Pr(Z_{(a)}=x_{(s)},Z_{(b)}=x_{(t)})\\ &= \sum\sum_{1\le s<t\le N} x_{(s)}x_{(t)}\binom{s-1}{a-1}\binom{t-1-s}{b-1-a}\binom{N-t}{2n-b}\Big/\binom{N}{2n}\end{aligned} \tag{3.2.8}$$

follow from lemma 3.2.3 and Wilks (1962), respectively. Therefore, inserting (3.2.7) and (3.2.8) into (3.2.5) and changing the order of the summations, we obtain (3.2.6).

3.3 Extremal finite populations maximizing relative precisions

THEOREM 3.3.1 *For any fixed N and n, suppose that $\delta' = (y_1, y_2, \cdots, y_N)$ is the eigenvector corresponding to the largest eigenvalue of the matrix Γ in Theorem 3.2.1. If the order condition $y_1 \le y_2 \le \cdots \le y_N$ are satisfied, then $\{y_1, y_2, \cdots, y_N\}$ is the population that leads to the maximum RP for every r.*

PROOF. By theorem 3.2.1, the relative precision is

$$RP = \frac{1}{1 - \frac{(N-1)}{n(N-nr)}\left(\frac{\theta'\Gamma\theta}{\sigma^2}\right)}. \tag{3.3.1}$$

There is no loss of generality in assuming that $\mu = 0$ and $\sigma^2 = 1$, because $\theta'\Gamma\theta/\sigma^2$ does not depend on μ and σ^2. Suppose

$$\widetilde{R^N} = \left\{(t_1, t_2, \cdots, t_N) \in R^N \mid \sum_{i=1}^N t_i = 0, \sum_{i=1}^N t_i^2/N = 1\right\}$$

and

$$T^N = \left\{(t_1, t_2, \cdots, t_N) \in \widetilde{R^N} \mid t_1 \le t_2 \le \cdots \le t_N\right\}.$$

If we can find $\hat{\delta}' = (\hat{t_1}, \hat{t_2}, \cdots, \hat{t_N}) \in T^N$ such that $\hat{\delta}'\Gamma\hat{\delta} = \sup\{\delta'\Gamma\delta \mid \delta' \in \widetilde{R^N}\}$, then, from (3.3.1), the population $\{\hat{t_1}, \hat{t_2}, \cdots, \hat{t_N}\}$ has the maximum RP. By the method of Lagrange multipliers, the vector δ that maximizes $\delta'\Gamma\delta$ subject to the constraint $\delta' \in \widetilde{R^N}$ is given by the eigenvector corresponding to the largest eigenvalue of Γ. Further, since the elements of Γ are not functions of r, the vector at which this constrained maximum occurs does not depend on r. This completes the proof of the theorem.

We computed the eigenvector $(y_1, y_2, \cdots, y_N)$ corresponding to the largest eigenvalue of the matrix Γ for $4 \le N \le 100$ and $2 \le n \le 5$. As a result, all these computed $(y_1, y_2, \cdots, y_N)$ satisfied the order condition $y_1 \le y_2 \le \cdots \le y_N$. We have, however, not proved theoretically that, for every n and N, the eigenvector corresponding to the largest eigenvalue of the matrix Γ satisfies the order condition $y_1 \leq y_2 \leq \cdots \leq y_N$. Table 3.3.1 shows the elements of the extremal finite populations for $N = 25$ and Figure 3.3.1 illustrates the extremal populations for some n, N. Table 3.3.2 shows the maximum values of $e_{[n]}^{(1)}$ and, for comparison, the values of $e_{[n]}^{(1)}$ of the discrete uniform distributions. Table 3.3.3 gives the algebraic representations of RP for the discrete uniform distributions. These algebraic calculations were done on a Micro-VAX using the MACSYMA.

Table 3.3.1 Extremal Finite Populations Maximizing Relative Precisions for $N = 25$

n	$x_{(25)}$	$x_{(24)}$	$x_{(23)}$	$x_{(22)}$	$x_{(21)}$	$x_{(20)}$	$x_{(19)}$	$x_{(18)}$	$x_{(17)}$	$x_{(16)}$	$x_{(15)}$	$x_{(14)}$
2	1.617	1.501	1.380	1.256	1.127	0.995	0.859	0.721	0.580	0.437	0.292	0.146
3	1.623	1.509	1.388	1.260	1.126	0.989	0.850	0.709	0.567	0.425	0.284	0.142
4	1.625	1.516	1.393	1.261	1.124	0.984	0.844	0.703	0.563	0.422	0.281	0.141
5	1.627	1.521	1.395	1.260	1.121	0.981	0.841	0.701	0.561	0.421	0.281	0.140
DU	1.664	1.525	1.387	1.248	1.109	0.971	0.832	0.693	0.555	0.416	0.277	0.139

DU : Discrete uniform distribution of size $N = 25$

Figure 3.3.1: Extremal Finite Populations Maximizing Relative Precisions

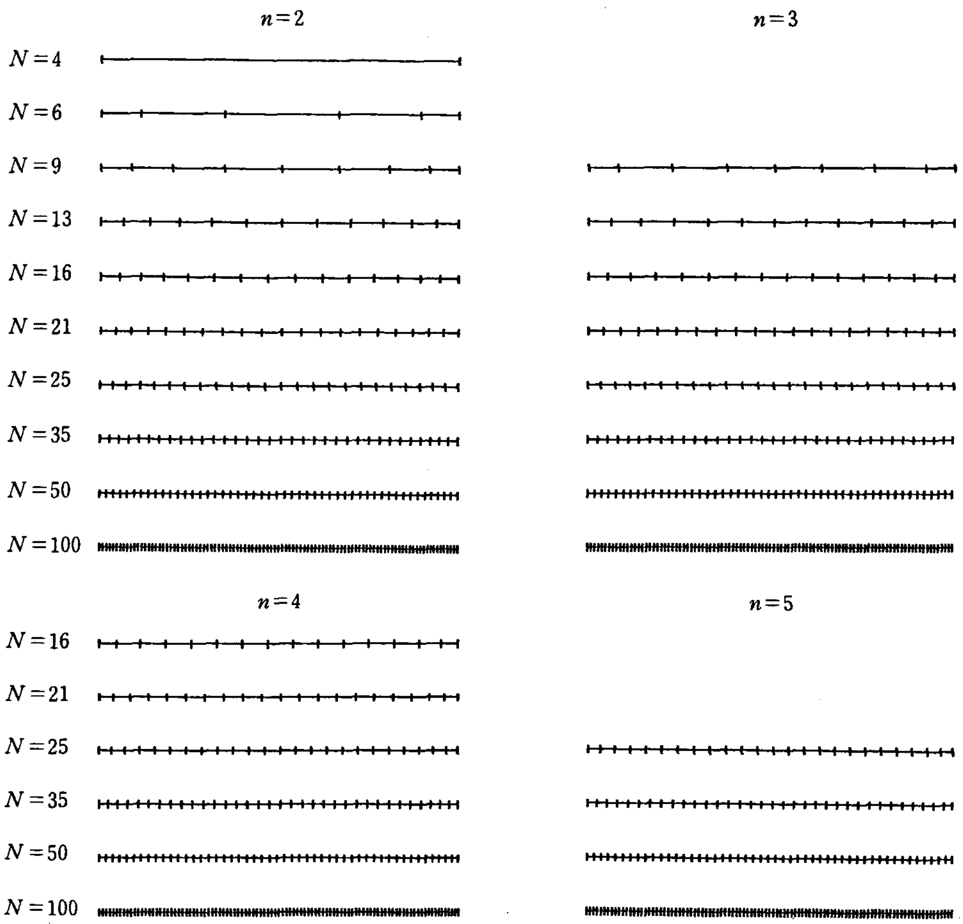


Table 3.3.2: Maximum Relative Precisions and Relative Precisions for Discrete Uniform Distributions

N		n = 2	3	4	5
4	DU	5.00000			
	max	∞			
6	DU	2.30769			
	max	2.39019			
9	DU	1.87500	5.31646		
	max	1.88695	5.42134		
13	DU	1.71875	3.19635		
	max	1.72210	3.20728		
16	DU	1.66667	2.80864	6.05769	
	max	1.66847	2.81335	6.07896	
21	DU	1.61932	2.52505	4.27034	
	max	1.62017	2.52693	4.27525	
25	DU	1.59722	2.41002	3.76280	6.91617
	max	1.59777	2.41113	3.76523	6.92366
35	DU	1.56646	2.26491	3.23559	4.81919
	max	1.56670	2.26536	3.23638	4.82073
50	DU	1.54506	2.17305	2.95232	4.00887
	max	1.54517	2.17324	2.95261	4.00933
100	DU	1.52174	2.08027	2.69807	3.40602
	max	1.52176	2.08031	2.69813	3.40609

Table 3.3.3: Relative Precisions for Discrete Uniform Distributions

	n			
	2	3	4	5
RP	$\frac{15N - 30r}{10N - 4 - 30r}$	$\frac{70N - 210r}{35N - 26 - 210r}$	$\frac{105N - 420r}{42N - 44 - 420r}$	$\frac{231N - 1155r}{77N - 102 - 1155r}$

Chapter 4

Comparative Monte-Carlo Studies on Confidence Intervals for the Mean of a Normal Distribution from Censored Samples

4.1 Introduction

Consider a life testing experiment in which a sample of identical units is put on test. We assume that the lifetimes of these units are independent random variables, having a common normal distribution (possibly after transformation of the data) with mean μ and standard deviation σ. Suppose the sample is censored on the right at a fixed point or at a pre-specified sample percentage point. These two kinds of censored sample are called Type I censored sample and Type II censored sample, respectively. Methods and tables for calculating the maximum likelihood estimates (MLE) of μ and σ from Type I and Type II censored samples have been given by Hald (1949), Cohen (1959), (1961) and David (1981). These papers have also given large-sample confidence intervals with confidence level α for μ and σ. Since these confidence intervals have been based on asymptotic properties of related statistics, we can not necessarily guarantee the correct confidence levels for censored samples of small or moderate size.

In this chapter, we consider five kinds of confidence intervals for μ from Type I and Type II censored samples, where sample sizes are small or moderate. In Section 4.2, we explain how to calculate these confidence intervals. We perform a Monte-Carlo simulation study of the confidence intervals for μ in Section 4.3. In Section 4.4, we discuss the results of the Monte-Carlo simulation study.

4.2 Constructions of Confidence Intervals

In the first subsection we construct confidence intervals based on the MLEs. In the second subsection we construct confidence intervals based on the best linear unbiased estimates (BLUE). In the third subsection we present the confidence intervals proposed by Halperin (1961).

4.2.1 Confidence intervals based on the MLEs

First we consider the case of Type I censoring. Suppose a sample of size n from $N(\mu, \sigma^2)$ is censored at a fixed point x_0. Let k be the number of non-censored observations. If $k = 0$, then we define, for convenience, the confidence interval of μ by $(-\infty, \infty)$. Now, we assume that $k \geq 1$. Let $x_1, x_2, \cdots, x_k$ denote the non-censored observations and $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(k)}$ the corresponding ordered observations. The likelihood function in the Type I censored sample can be written as

$$L = \frac{n!}{(n-k)!}\left(1 - \Phi\left(\frac{x_0 - \mu}{\sigma}\right)\right)^{n-k} \frac{1}{(2\pi\sigma^2)^{k/2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{k}(x_{(i)} - \mu)^2\}$$

where $\Phi(x)$ is the unit-normal cumulative distribution function. Let $\hat{\mu}$ and $\hat{\sigma}$ be the MLEs for μ and σ, respectively. Hald(1949), Cohen(1959),(1961) and David(1981) have also given the asymptotic variance-covariance matrix of $(\hat{\mu}, \hat{\sigma})$. We invert the matrix whose elements are negatives of expected values of second order derivatives of logarithms of the likelihood function. That is, by putting $\xi = \frac{x_0 - \mu}{\sigma}$ and $\phi(x) = \frac{d\Phi(x)}{dx}$, we calculate

$$\begin{pmatrix} \mu_{11}(\xi) & \mu_{12}(\xi) \\ \mu_{12}(\xi) & \mu_{22}(\xi) \end{pmatrix} = \begin{pmatrix} \varphi_{11}(\xi) & \varphi_{12}(\xi) \\ \varphi_{12}(\xi) & \varphi_{22}(\xi) \end{pmatrix}^{-1},$$

where

$Z(\xi) = \phi(\xi)/(1 - \Phi(\xi))$,

$\varphi_{11}(\xi) = \Phi(\xi) + \phi(\xi)(Z(\xi) - \xi)$,

$\varphi_{12}(\xi) = -\phi(\xi) + \xi\phi(\xi)(Z(\xi) - \xi)$,

and

$\varphi_{22}(\xi) = 2\Phi(\xi) + \xi\varphi_{12}(\xi)$.

The asymptotic variance of $\hat{\mu}$ is given by $\mu_{11}(\xi)\sigma^2/n$. Put $c(\alpha) = \Phi^{-1}((1+\alpha)/2)$ for $0 < \alpha < 1$ and $\hat{\xi} = (x_0 - \hat{\mu})/\hat{\sigma}$ ($\hat{\xi} = \infty$ for $k = n$). Then, a large-sample 100α percent confidence interval for μ is

$$\left(\hat{\mu} - c(\alpha)\sqrt{\frac{\mu_{11}(\hat{\xi})}{n}}\,\hat{\sigma},\quad \hat{\mu} + c(\alpha)\sqrt{\frac{\mu_{11}(\hat{\xi})}{n}}\,\hat{\sigma}\right). \tag{4.2.1}$$

Now, we consider the case of Type II censoring. Suppose a sample of size n is censored at the occurrence of the k-th ordered observation. We assume that $k \geq 2$. Denote the ordered non-censored observations by $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(k)}$. The likelihood function in the Type II censored sample can be written as

$$L = \frac{n!}{(n-k)!}(1 - \Phi(\frac{x_{(k)} - \mu}{\sigma}))^{n-k}(2\pi\sigma^2)^{-k/2}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{k}(x_{(i)} - \mu)^2\}.$$

Methods and tables for calculating the MLEs of the parameters of a normal distribution and for calculating the asymptotic variances of those MLEs have also been given by Gupta(1952), Cohen(1959),(1961) and David(1981). These papers suggest that if $k/n \to p$ as $n \to \infty$, then the asymptotic variance for $\hat{\mu}$ is given by $\sigma^2\mu_{11}(\Phi^{-1}(p))/n$. Thus, replacing $\mu_{11}(\hat{\xi})$ by $\mu_{11}(\Phi^{-1}(k/n))$ throughout (4.2.1), we get the large-sample 100α percent confidence interval for μ. That is,

$$\left(\hat{\mu} - c(\alpha)\sqrt{\frac{\mu_{11}(\Phi^{-1}(k/n))}{n}}\,\hat{\sigma},\quad \hat{\mu} + c(\alpha)\sqrt{\frac{\mu_{11}(\Phi^{-1}(k/n))}{n}}\,\hat{\sigma}\right). \tag{4.2.2}$$

For convenience, we call the intervals (4.2.1) and (4.2.2) ML intervals.

Let $c_{k-1}(\alpha)$ be the $(1+\alpha)/2$-th quantile of Student's t distribution with $k-1$ degrees of freedom. Replacing $c(\alpha)$ by $c_{k-1}(\alpha)$ ($c_1(\alpha)$ for $k = 1$), and $\hat{\sigma}$ by $\sqrt{k/(k-1)}\hat{\sigma}$ ($\sqrt{2}\hat{\sigma}$ for $k=1$) throughout (4.2.1), we obtain, in Type I censoring, a confidence interval for μ

$$\left(\hat{\mu} - c_{k-1}(\alpha)\sqrt{\frac{\mu_{11}(\hat{\xi})k}{n(k-1)}}\,\hat{\sigma},\quad \hat{\mu} + c_{k-1}(\alpha)\sqrt{\frac{\mu_{11}(\hat{\xi})k}{n(k-1)}}\,\hat{\sigma}\right). \tag{4.2.3}$$

Similarly, by replacing $c(\alpha)$ by $c_{k-1}(\alpha)$, and $\hat{\sigma}$ by $\sqrt{k/(k-1)}\hat{\sigma}$ throughout (4.2.2), we obtain, in the Type II censoring, a confidence interval for μ

$$\left(\hat{\mu} - c_{k-1}(\alpha)\sqrt{\frac{\mu_{11}(\Phi^{-1}(k/n))k}{n(k-1)}}\,\hat{\sigma},\quad \hat{\mu} + c_{k-1}(\alpha)\sqrt{\frac{\mu_{11}(\Phi^{-1}(k/n))k}{n(k-1)}}\,\hat{\sigma}\right). \tag{4.2.4}$$

We call these modified ML intervals (4.2.3) and (4.2.4) ML-T intervals.

4.2.2 Confidence intervals based on the BLUEs

Methods and tables for calculating the BLUEs of the parameters of a normal distribution from Type II censored samples have been given by Sarhan and Greenberg (1962). Let μ^* and σ^* be the BLUEs of μ and σ, and let $c(n,k)\sigma^2$ be the variances of μ^*. Now, we can obtain a confidence interval for μ

$$\left(\mu^* - c(\alpha)\sqrt{c(n,k)}\,\sigma^*,\quad \mu^* + c(\alpha)\sqrt{c(n,k)}\,\sigma^*\right). \tag{4.2.5}$$

We may regard a Type I censored sample as a Type II censored sample. Then, for Type I censored sample, we can obtain a confidence interval for μ from (4.2.5). Unlike the case of the ML interval in Type I censoring, the truncation point x_0 does not appear in the calculation of this confidence interval.

For convenience, we call the confidence interval (4.2.5) and the corresponding interval in Type I censoring LU intervals.

Replacing $\hat{\mu}$ by μ^*, $\hat{\sigma}$ by σ^*, and $\mu_{11}(\hat{\xi})/n$ by $c(n,k)$ throughout (4.2.3), we obtain a confidence interval for μ from a Type I censored sample.

Similarly, replacing $\hat{\mu}$ by μ^*, $\hat{\sigma}$ by σ^*, and $\mu_{11}(\Phi^{-1}(k/n))/n$ by $c(n,k)$ throughout (4.2.4), we obtain a confidence interval for μ from a Type II censored sample. For convenience, we call these modified LU intervals LU-T intervals.

4.2.3 Confidence intervals by Halperin

In Type I censoring, suppose that $X_1, X_2, \cdots, X_k$ and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(k)}$ are the random variables corresponding to the non-censored observations $x_1, x_2, \cdots, x_k$ and $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(k)}$, respectively. Set $\Phi_{x_0} = \Phi((x_0 - \mu)/\sigma)$. First, for $k(\geq 1)$ and given α $(0 < \alpha < 1)$, we determine Φ_L, Φ_U, r, s and δ_0 in order. Since the random variable K corresponding to the number of the non-censored observations k has the binomial distribution with two parameters n and Φ_{x_0}, we can get a usual $100\alpha^{1/2}$ percent confidence interval for Φ_{x_0}. We denote this interval by (Φ_L, Φ_U). Further, $X_1, X_2, \cdots, X_k$ given $K = k$ can be considered as a random sample of size k from the truncated cumulative distribution function $F(x) = \Phi[(x-\mu)/\sigma]/\Phi_{x_0}$. Therefore, $U_{(1)} = F(X_{(1)}), U_{(2)} = F(X_{(2)}), \cdots, U_{(k)} = F(X_{(k)})$ are the order statistics from the uniform distribution over the interval $[0,1]$. For $0 \leq \delta \leq 1$, $k \geq 1$ and $0 \leq s' \leq r' \leq k+1$, we consider the following inequality:

$$\Pr(U_{(s')} \leq \delta \leq U_{(r')} \mid K = k) \geq \alpha^{1/2} \tag{4.2.6}$$

where $U_{(0)} = 0$ and $U_{(k+1)} = 1$. The left hand side (l.h.s.) of (4.2.6) is equal to $\sum_{j=s'}^{r'-1} \binom{k}{j} \delta^j (1-\delta)^{k-j}$. To determine s and r, we substitute $\delta = 1/2$ and $s' = k - r' + 1$ into (4.2.6) and we get the following ineqaulity:

$$\sum_{j=k-r'+1}^{r'-1} \binom{k}{j} \frac{1}{2}^k \geq \alpha^{1/2}. \tag{4.2.7}$$

Let u denote the smallest number of all integers r' which satisfy the inequalty (4.2.7). For u, we determine s and r as follows:

(a) If $u \neq k+1$, then $s = k - u + 1$ and $r = u$;

(b) If $u = k+1$ and $\sum_{j=1}^{k} \binom{k}{j} \frac{1}{2}^k \geq \alpha^{1/2}$, then $s = 1$ and $r = k+1$;

(c) If $u = k+1$ and $\sum_{j=1}^{k} \binom{k}{j} \frac{1}{2}^k < \alpha^{1/2}$, then $s = 0$ and $r = k$.

Now, using such s and r, we can find a δ $(0 \leq \delta \leq 1/2)$ such that, for $s' = s$ and $r' = r$, the inequality (4.2.6) holds equality. We define δ_0 by such a δ.

Put $x_{(0)} = -\infty$, $x_{(k+1)} = x_0$,

$$R_U = \frac{\Phi^{-1}(\Phi_U)}{\Phi^{-1}(\Phi_U) - \Phi^{-1}(\delta_0 \Phi_U)}$$

and

$$R_L = \frac{\Phi^{-1}(\Phi_L)}{\Phi^{-1}(\Phi_L) - \Phi^{-1}(\delta_0 \Phi_L)},$$

where $R_U = 1$ for $\Phi_U = 1$. Using Φ_L, Φ_U, r, s, δ_0, R_L, and R_U determined above, we make a confidence interval for μ as follows:

If $\Phi_L \geq 1/2$, then,

$$\left(x_0 - (x_0 - x_{(s)})R_U, \;\; x_0 - (x_0 - x_{(r)})R_L\right).$$

If $\Phi_U < 1/2$, then,

$$\left(x_0 - (x_0 - x_{(r)})R_U, \;\; x_0 - (x_0 - x_{(s)})R_L\right).$$

If $\Phi_U \geq 1/2$ and if $\Phi_L < 1/2$, then,

$$\left(x_0 - (x_0 - x_{(s)})R_U, \;\; x_0 - (x_0 - x_{(s)})R_L\right).$$

For convenience, we call this interval H or H(α) interval in Type I censoring.

In Type II censoring, replacing k by $k-1$ and x_0 by $x_{(k)}$ in the H interval in Type I censoring, we obtain a confidence interval for μ. For convenience, we call this confidence interval H or H(α) interval in Type II censoring.

4.3 Monte-Carlo study

The ML, ML-T, LU and LU-T intervals described in the previous section are based on asymptotic properties of related statistics. Therefore, they might not work satisfactorily for a sample of small or moderate size. On the other hand, the H interval by Halparin (1961) is based on a joint confidence region for μ and σ. Such a interval might be too long for single parameter.

In this section, the confidence coefficients and expected lengths of these intervals are studied by Monte-Carlo simulation.

4.3.1 Type I censoring

Let $X = (X_1, X_2, \cdots, X_k)$ be a random vector corresponding to non-censored observations in Type I censoring. Let $(l(X), u(X))$ a confidence interval for μ from X with confidence coefficient $\geq \beta$. It means that

$$\Pr{}^{x_0}_{\mu,\sigma}\{l(X) \leq \mu \leq u(X)\} \geq \beta \tag{4.3.1}$$

for all μ and σ $(-\infty < \mu < \infty, \quad 0 < \sigma < \infty)$, where the potential sample of size n from $N(\mu, \sigma^2)$ is censored at x_0 on the right. Put $Y_i = (X_i - \mu)/\sigma$ and $y_0 = (x_0 - \mu)/\sigma$. It can be easily seen that (4.3.1) is equivalent to

$$\Pr{}^{y_0}_{0,1}\{l(Y) \leq 0 \leq u(Y)\} \geq \beta. \tag{4.3.2}$$

For the ML, ML-T, LU, LU-T and H intervals, the evaluations of the l.h.s. of (4.3.2) are too hard to derive mathematically. Instead, we estimate them by Monte-Carlo simulation.

Let us define the coverage of the interval $(l(Y), u(Y))$ by $p = \Pr^{y_0}_{0,1}\{-\infty < l(Y) \leq 0 \leq u(Y) < \infty\} \times 100$. We estimate the coverage of the confidence intervals for $\alpha = 0.95$ and 0.99, and $n = 10, 20, 30, 50, 100$ (10, 20 for the LU and LU-T intervals) and $y_0 = \Phi^{-1}(\gamma)$ with $\gamma = 0.10, 0.25, 0.50, 0.75, 0.90$. For this purpose, we generated 2000 samples for each size $n = 10, 20, 30, 50, 100$, independently. For each of these samples, we calculated the ML, LT, ML-T, LU-T and H intervals for each pair of α and y_0. For each combination of n, α and γ, we then counted the number N_p of times that the calculated intervals contained 0 and both the ends were finite. The estimate of coverage p was $\hat{p} = (N_p/2000) \times 100$. Tables 4.3.1, 4.3.2, 4.3.3, 4.3.4 and 4.3.5 present $\hat{p}$ for the ML, LU, H, ML-T and LU-T intervals, respectively. In these tables, the entries in the first row correspond to $\hat{p}$ with the nominal confidence coefficient $\alpha = 0.95$, while those in the second row correspond to $\hat{p}$ with $\alpha = 0.99$. The combinations (α, γ, n) such that the probability of infinite intervals is greater than $1 - \alpha$ are shown by the parenthesis ( ).

Since the Monte-Carlo simulation results are based on 2000 samples, the 95% confidence limits for the coverage $p\%$ are $95 \pm 1.8(\%)$ for $\hat{p} = 95$ and $99 \pm 0.4(\%)$ for $\hat{p} = 99$.

Secondly, by Monte-Carlo simulation, we examined the expected lengths of the intervals. Tables 4.3.6, 4.3.7 and 4.3.8 give the average lengths of calculated finite intervals. The number in the bracket [ ] of these tables denotes m such that the expected length of the usual confidence interval, based on a complete normal sample of size m, is numerically closest to that average length. Further, the blank in these tables means the pair of (γ, n) in which $\hat{p} < 94$ for $\alpha = 0.95$ or $\hat{p} < 98$ for $\alpha = 0.99$.

4.3.2 Type II censoring

In Type II censoring, in order to show that our confidence intervals have confidence coefficients $\geq \beta$, it is sufficient to show the fact for the standard normal parent. Thus, we consider the confidence intervals for the standard normal parent. For given n (potential sample size) and γ (non-censored proportion), we choose $k = [n\gamma + 0.5]$ (number of non-censored observations), where $[n\gamma + 0.5]$ denotes the integer part of $n\gamma + 0.5$. For each (n, γ, α), 2000 samples were used also in Type II censoring. Tables 4.3.9, 4.3.10, 4.3.11, 4.3.12 and 4.3.13 present the empirical coverages $\hat{p}$ for the ML, LU , H, ML-T and LU-T intervals, respectively. Here, the blank in these tables indicates the combination of (α, γ, n) that either end of the intervals is infinite.

Tables 4.3.14, 4.3.15 and 4.3.16 give the average lengths of the ML-T, LU-T and H intervals, respectively.

Table 4.3.1: Empirical coverages for the ML interval in Type I censoring

γ		n = 10	20	30	50	100
0.1	$\alpha = .95$	(42.3)	(66.6)	76.8	84.9	90.1
	$\alpha = .99$	(46.5)	(71.8)	(81.9)	89.7	94.8
0.25	$\alpha = .95$	(74.7)	85.8	90.6	91.8	94.0
	$\alpha = .99$	(81.2)	91.5	94.7	96.2	97.6
0.50	$\alpha = .95$	93.4	94.1	94.2	94.2	95.3
	$\alpha = .99$	97.7	98.2	97.9	98.4	98.8
0.75	$\alpha = .95$	93.0	93.8	94.0	94.3	94.9
	$\alpha = .99$	97.6	98.2	98.3	98.7	98.8
0.90	$\alpha = .95$	91.9	93.0	93.5	94.2	94.6
	$\alpha = .99$	97.0	97.8	97.9	98.6	98.9

Table 4.3.2: Empirical coverages for the LU interval in Type I censoring

γ		n = 10	20
0.1	$\alpha = .95$	(13.5)	(41.3)
	$\alpha = .99$	(15.5)	(45.2)
0.25	$\alpha = .95$	(52.7)	78.5
	$\alpha = .99$	(58.7)	(84.9)
0.50	$\alpha = .95$	86.8	90.7
	$\alpha = .99$	(92.7)	95.2
0.75	$\alpha = .95$	91.9	93.1
	$\alpha = .99$	97.2	97.7
0.90	$\alpha = .95$	92.9	93.2
	$\alpha = .99$	97.6	98.0

Table 4.3.3: Empirical coverages for the H interval in Type I censoring

γ		n				
		10	20	30	50	100
0.1	$\alpha = .95$	(0.0)	(0.9)	(7.3)	(36.1)	(93.5)
	$\alpha = .99$	(0.0)	(0.0)	(0.8)	(11.7)	(79.1)
0.25	$\alpha = .95$	(2.1)	(38.9)	(80.5)	99.1	99.9
	$\alpha = .99$	(0.0)	(11.3)	(48.6)	(95.3)	100.0
0.50	$\alpha = .95$	(37.5)	97.8	98.7	98.1	98.1
	$\alpha = .99$	(4.8)	(87.7)	99.8	99.5	99.8
0.75	$\alpha = .95$	(92.2)	99.9	99.8	99.8	99.8
	$\alpha = .99$	(54.0)	100.0	100.0	100.0	99.9
0.90	$\alpha = .95$	99.5	99.7	99.5	99.3	99.6
	$\alpha = .99$	(93.2)	100.0	99.9	99.9	100.0

Table 4.3.4: Empirical coverages for the ML-T interval in Type I censoring

γ		n				
		10	20	30	50	100
0.1	$\alpha = .95$	(62.9)	(81.4)	90.4	92.2	93.9
	$\alpha = .99$	(65.7)	(85.7)	(94.7)	96.8	97.9
0.25	$\alpha = .95$	(89.9)	92.3	93.6	94.0	95.0
	$\alpha = .99$	(93.5)	96.8	97.4	98.2	98.1
0.50	$\alpha = .95$	97.3	96.4	95.7	96.0	96.1
	$\alpha = .99$	99.4	99.0	99.2	99.2	99.2
0.75	$\alpha = .95$	97.3	96.4	96.0	95.7	95.7
	$\alpha = .99$	99.3	99.4	99.1	99.4	99.4
0.90	$\alpha = .95$	96.3	95.6	95.5	95.6	95.5
	$\alpha = .99$	99.1	99.2	98.8	99.2	99.3

Table 4.3.5: Empirical coverages for the LU-T interval in Type I censoring

γ		n = 10	n = 20
0.1	$\alpha = .95$	(22.9)	(55.1)
	$\alpha = .99$	(25.8)	(60.0)
0.25	$\alpha = .95$	(69.5)	86.4
	$\alpha = .99$	(74.1)	(93.0)
0.50	$\alpha = .95$	94.0	93.0
	$\alpha = .99$	(97.9)	98.0
0.75	$\alpha = .95$	96.6	95.9
	$\alpha = .99$	99.6	99.1
0.90	$\alpha = .95$	96.8	95.5
	$\alpha = .99$	99.5	99.4

Table 4.3.6: Average lengths for the ML-T interval in Type I censoring

γ		n				
		10	20	30	50	100
0.25	$\alpha = .95$				1.40	0.86
	$\alpha = .99$				2.02	1.17
					[10]	[23]
0.50	$\alpha = .95$	4.24	1.38	1.03	0.74	0.50
	$\alpha = .99$	13.27	2.02	1.43	1.01	0.67
		[3]	[10]	[16]	[30]	[63]
0.75	$\alpha = .95$	1.76	1.03	0.81	0.61	0.42
	$\alpha = .99$	2.81	1.44	1.10	0.81	0.56
		[7]	[16]	[25]	[44]	[90]
0.90	$\alpha = .95$	1.50	0.96	0.76	0.58	0.40
	$\alpha = .99$	2.20	1.32	1.03	0.77	0.53
		[7]	[16]	[25]	[44]	[90]

Table 4.3.7: Average lengths for the LU-T interval in Type I censoring

γ		n	
		10	20
0.75	$\alpha = .95$	1.78	1.02
	$\alpha = .99$	2.78	1.41
		[7]	[16]
0.90	$\alpha = .95$	1.54	0.96
	$\alpha = .99$	2.26	1.32
		[9]	[19]

Table 4.3.8: Average lengths for the H interval in Type I censoring

γ		n				
		10	20	30	50	100
0.10	$\alpha = .95$					4.29
						[3]
	$\alpha = .99$					
0.25	$\alpha = .95$				2.82	1.76
					[4]	[7]
	$\alpha = .99$					2.39
						[8]
0.50	$\alpha = .95$		2.78	2.00	1.33	0.81
			[4]	[6]	[11]	[25]
	$\alpha = .99$			2.95	1.88	1.10
				[6]	[11]	[25]
0.75	$\alpha = .95$		1.77	1.34	1.01	0.72
			[7]	[11]	[17]	[31]
	$\alpha = .99$		2.49	1.77	1.27	0.89
			[8]	[12]	[20]	[37]
0.90	$\alpha = .95$	2.64	1.71	1.37	1.06	0.74
		[4]	[7]	[10]	[16]	[30]
	$\alpha = .99$		2.12	1.71	1.31	0.92
			[9]	[12]	[19]	[34]

Table 4.3.9: Empirical coverages for the ML interval in Type II censoring

γ		n				
		10	20	30	50	100
0.1	$\alpha = .95$		43.7	60.8	75.0	83.7
	$\alpha = .99$		50.5	68.6	81.0	90.7
0.25	$\alpha = .95$	67.0	76.2	83.6	87.5	90.7
	$\alpha = .99$	73.3	82.5	90.2	93.1	96.1
0.50	$\alpha = .95$	80.7	88.7	89.4	91.9	93.6
	$\alpha = .99$	88.0	93.7	95.1	97.2	98.6
0.75	$\alpha = .95$	89.9	90.8	92.6	93.7	94.8
	$\alpha = .99$	95.4	96.8	97.5	98.5	98.7
0.90	$\alpha = .95$	90.0	92.3	93.1	93.5	94.4
	$\alpha = .99$	96.2	97.4	97.8	98.4	98.8

Table 4.3.10: Empirical coverages for the LU interval in Type II censoring

γ		n	
		10	20
0.1	$\alpha = .95$		73.8
	$\alpha = .99$		78.2
0.25	$\alpha = .95$	83.4	87.5
	$\alpha = .99$	88.0	91.6
0.50	$\alpha = .95$	88.9	91.5
	$\alpha = .99$	93.8	96.0
0.75	$\alpha = .95$	92.5	92.6
	$\alpha = .99$	97.2	97.7
0.90	$\alpha = .95$	93.0	93.4
	$\alpha = .99$	97.4	98.1

Table 4.3.11: Empirical coverages for the H interval in Type II censoring

γ		n				
		10	20	30	50	100
0.1	$\alpha = .95$					99.6
	$\alpha = .99$					100.0
0.25	$\alpha = .95$			99.5	99.6	99.9
	$\alpha = .99$				99.9	100.0
0.50	$\alpha = .95$		99.6	99.7	99.7	99.6
	$\alpha = .99$		100.0	100.0	100.0	99.9
0.75	$\alpha = .95$	99.5	98.9	99.1	99.5	99.8
	$\alpha = .99$		100.0	99.7	99.9	100.0
0.90	$\alpha = .95$	98.6	99.5	99.4	99.4	99.8
	$\alpha = .99$	99.8	99.9	99.9	99.9	99.9

Table 4.3.12: Empirical coverages for the ML-T interval in Type II censoring

γ		n				
		10	20	30	50	100
0.1	$\alpha = .95$		87.0	84.2	86.3	88.5
	$\alpha = .99$		96.9	96.0	94.5	95.6
0.25	$\alpha = .95$	87.8	87.1	89.0	90.6	92.7
	$\alpha = .99$	96.8	94.1	96.0	95.6	97.4
0.50	$\alpha = .95$	92.2	92.2	93.0	94.2	95.1
	$\alpha = .99$	96.8	97.1	97.6	98.1	98.8
0.75	$\alpha = .95$	94.6	94.2	94.9	95.0	95.4
	$\alpha = .99$	99.2	98.5	98.6	99.1	99.2
0.90	$\alpha = .95$	94.7	94.8	95.2	95.0	96.0
	$\alpha = .99$	98.8	99.1	98.7	99.1	99.2

Table 4.3.13: Empirical coverage for the LU-T interval in Type II censoring

γ		n: 10	n: 20
0.1	$\alpha = .95$		95.2
	$\alpha = .99$		99.0
0.25	$\alpha = .95$	95.1	93.0
	$\alpha = .99$	98.9	97.9
0.50	$\alpha = .95$	95.8	94.8
	$\alpha = .99$	99.0	98.2
0.75	$\alpha = .95$	96.8	95.3
	$\alpha = .99$	99.6	99.1
0.90	$\alpha = .95$	96.5	95.6
	$\alpha = .99$	99.4	99.3

Table 4.3.14: Average lengths for the ML-T interval in Type II censoring

γ		n: 10	n: 20	n: 30	n: 50	n: 100
0.50	$\alpha = .95$			0.95	0.71	0.49
	$\alpha = .99$			1.31	0.96	0.66
				[19]	[32]	[65]
0.75	$\alpha = .95$	1.48	0.98	0.78	0.59	0.42
	$\alpha = .99$	2.19	1.36	1.06	0.79	0.55
		[9]	[18]	[27]	[46]	[90]
0.90	$\alpha = .95$	1.42	0.94	0.75	0.57	0.40
	$\alpha = .99$	2.07	1.29	1.02	0.76	0.53
		[10]	[19]	[29]	[49]	[100]

Table 4.3.15: Average lengths for the LU-T interval

γ		n	
		10	20
0.10	$\alpha = .95$		48.23
	$\alpha = .99$		241.63
			[*]
0.25	$\alpha = .95$	6.71	3.05
	$\alpha = .99$	15.48	5.06
		[3]	[4]
0.50	$\alpha = .95$	2.50	1.34
	$\alpha = .99$	4.15	1.92
		[5]	[11]
0.75	$\alpha = .95$	1.66	1.04
	$\alpha = .99$	2.46	1.45
		[8]	[16]
0.90	$\alpha = .95$	1.57	0.98
	$\alpha = .99$	2.28	1.35
		[8]	[18]

Table 4.3.16: Average lengths for the H interval in Type II censoring

γ		n: 10	20	30	50	100
0.10	$\alpha = .95$					4.28
						[3]
	$\alpha = .99$					6.47
						[4]
0.25	$\alpha = .95$			3.67	2.47	1.73
				[3]	[5]	[7]
	$\alpha = .99$				3.61	2.24
					[5]	[9]
0.50	$\alpha = .95$		2.54	1.85	1.27	0.78
			[5]	[7]	[12]	[27]
	$\alpha = .99$		4.25	2.74	1.75	1.07
			[5]	[7]	[12]	[27]
0.75	$\alpha = .95$	2.66	1.52	1.23	0.97	0.71
		[4]	[9]	[12]	[18]	[32]
	$\alpha = .99$		2.28	1.54	1.21	0.87
			[8]	[15]	[21]	[36]
0.90	$\alpha = .95$	2.17	1.60	1.36	1.01	0.73
		[5]	[8]	[10]	[17]	[31]
	$\alpha = .99$	3.32	1.98	1.64	1.27	0.92
		[6]	[10]	[13]	[20]	[43]

4.4 Discussion

4.4.1 Type I censoring

Tables 4.3.1 and 4.3.2 indicate that the empirical coverages for the ML and LU intervals are always less than the nominal confidence coefficient α and that these two confidence intervals will not satisfy the confidence levels. Thus, the ML and LU intervals will not be available for a sample of small or moderate size. On the other hand, Table 4.3.3 indicates that the H intervals become infinite too often for small sample size and the empirical coverages are fairly more than the nominal confidence coefficient. Halperin (1966) has suggested that, by numerical conclusion, H(α) interval really had the $\sqrt{\alpha}$ confidence coefficient. In view of such points, H(0.95) interval will have the confidence coefficient $\sqrt{0.95} = 0.975$. Since the empirical coverages at $\alpha = 0.95$ in Table 4.3.3 satisfy $\hat{p} > 97.5$ except for the effect of infinite intervals, the result of our simulation supports the suggestion by Halperin (1966). Nevertheless, the empirical coverages are fairly more than the confidence coefficient modified above. Additionally, Table 4.3.8 shows that the average lengths of the H intervals are too long as compared with those of the ML-T intervals. Thus, the H intervals will not be available for a sample of small or moderate size.

Table 4.3.4 shows that the empirical coverages $\hat{p}$ for the ML-T intervals are greater than $(100\alpha - 1)\%$ except for $\gamma = 0.1$ or $\gamma = 0.25$ and $n \leq 30$. Table 4.3.5 shows that the empirical coverages $\hat{p}$ for the LU-T intervals are greater than $(100\alpha - 1)\%$ except for $\gamma \geq 0.5$. Thus, the ML-T and LU-T intervals may be available, in some cases, for a sample of small or moderate size. From the comparisons between Table 4.3.4 and Table 4.3.5 and between Table 4.3.6 and Table 4.3.7, we can say that the ML-T interval will be superior to the LU-T interval. Furthermore, it can be shown, by Table 4.3.6, that the average length of the ML-T interval from a censored sample of size k is smaller than the expected length of the confidence interval based on a complete sample of size k.

For Type I censoring, our conclusion is that the ML-T interval will be useful as the confidence interval for μ for a sample of small or moderate size except for $\gamma \leq 0.25$.

4.4.2 Type II censoring

Table 4.3.12 shows that the confidence coefficients of the ML-T interval for $\gamma \leq 0.25$ or $\gamma = 0.5$ and $n \leq 30$ will be less than the nominal confidence coefficient α. Table 4.3.13 shows that the empirical coverages $\hat{p}$ for the LU-T interval are greater than $(100\alpha - 1)\%$ except for $\gamma = 0.1$ and $n = 10$. Tables 4.3.14 and 4.3.15 show that the expected lengths of the ML-T interval and LU-T interval are not so different.

Thus, our conclusion in Type II censoring is that the LU-T interval will be useful as the confidence interval for μ for a sample of small or moderate size. The table for calculating the BLUE given by Sarhan and Greenberg (1962) does not cover figures for $n > 20$. It will be helpful to make the table for $n > 20$.

In our Monte-Carlo studies, we also counted the number of times l that the calculated interval could not contain $\mu(= 0)$ and the number of times m that $\mu(= 0)$ exceeded the upper end point of that interval. We set $Q = m/l \times 100(\%)$. For the ML , ML-T, LU and LU-T intervals, we obtained the following fact for $\alpha = 0.95$: $Q = 100\%$ for $\gamma \leq 0.25$; $85 \leq Q \leq 100\%$ for $\gamma = 0.5$; Q was about 70% for $\gamma = 0.75$; Q was about 60% for $\gamma = 0.9$. Such tendency may be explained easily from the fact that the asymptotic correlation coefficient between $\hat{\mu}$ and $\hat{\sigma}$ given by Cohen(1961) or the correlation coefficient between μ^* and σ^* given by Sarhan and Greenberg (1962) are monotonically close to 1 as $\gamma \to 0$. That is, if γ is small, then, as $\hat{\mu}$ becomes smaller, $\hat{\sigma}$ becomes smaller. Since $\hat{\sigma}$ is closely related to the width of the interval, we have many intervals such that the upper end point is less than μ. From this fact it will be suggested that, for further improvement of confidence intervals for μ, some asymmetric intervals should be considered.

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