Title	Generalized Date Envelopment Analysis and Its Applications
Author(s)	尹, 禮分
Citation	大阪大学, 2000, 博士論文
Version Type	VoR
URL	https://doi.org/10.11501/3169413
rights	
Note	

Generalized Data Envelopment Analysis and Its Applications

(一般化包絡分析法とその応用)

by

Ye Boon YUN

A dissertation submitted in partial fulfillment
of the requirements for the degree of
DOCTOR of PHILOSOPHY in ENGINEERING
at
Graduate School of Engineering of Osaka University
Osaka, Japan
January 2000

Acknowledgement

I wish to express my appreciation to Professor Tetsuzo Tanino of Osaka University for supervising this thesis, encouraging me during the course of this work and providing me with the ideal environment for my research activity.

My sincere thanks are due to Professor Kiichiro Tsuji and Professor Sadatoshi Kumagai for serving as members on my dissertation committee. I also wish to acknowledge Professor Kenji Taniguchi, Professor Ken-ichi Kitayama and Professor Fumio Kishino for their useful comments.

Especially, I wish to express my sincere gratitude to Professor Hirotaka Nakayama of Konan University for his continuous guidance and numerous valuable discussion throughout this work. In addition, I am grateful to Professor Masao Arakawa of Kagawa University for his advise throughout this work.

I am thankful to Professor Masahiro Inuiguchi and Mr. Keiji Tatsumi. Also, I would like to thank the members of Tanino Laboratory at Osaka University for their kind cooperation.

I would like to appreciate Professor Do-Sang Kim of Pukyung National University for his encouragement.

Finally, I warmly thank my mother and brother, who always support and encourage me.

Abstract

DEA, which was originally suggested by Charnes, Cooper and Rhodes, is a method to calculate measuring a relative efficiency of decision making unit (DMU) performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs. So far, many DEA models have been developed: The CCR model (1978), the BCC model (1984) and the FDH model (1993) are well known as basic DEA models. These models based on the domination structure in primal form are characterized by how to determine the production possibility set in a viewpoint of dual form; a convex cone, a convex hull and a free disposable hull for the observed data, respectively.

In this thesis, we suggest a model called GDEA (generalized DEA) model, which can treat the above stated DEA models in a unified way. In addition, by establishing the theoretical properties on relationships among the GDEA model and those DEA models, we prove that the GDEA model makes it possible to calculate the efficiency of decision making unit incorporating various preference structures of decision makers. Furthermore, we propose a dual approach to GDEA, GDEA$_D$ and also show that GDEA$_D$ can reveal domination relations among all DMUs.

On the other hand, in many practical problems such as engineering de-

sign problems, criteria functions can not be given explicitly in terms of design variables. Under this circumstance, values of criteria functions for given value of design variables are usually obtained by some analyses such as structural analysis, thermodynamical analysis or fluid mechanical analysis. These analyses require considerably much computation time. Therefore, it is not unrealistic to apply existing interactive optimization methods to those problems. Recently, multi-objective optimization methods using genetic algorithms (GA) have been studied actively by many authors.

We suggest, in this thesis, the method employing GDEA as the fitness in GA in order to generate Pareto optimal solutions in multi-objective optimization problems. Consequently, we prove that the method using GDEA can remove dominated individuals faster than methods based on only GA, and can overcome the shortcomings of existing methods. Furthermore, through several numerical examples, we show that the method using GDEA can yield desirable efficient frontiers even in non-convex problems as well as convex problems.

Through the study on GDEA in the thesis, it will be expected that GDEA makes it helpful to evaluate an efficiency of complex management systems such as banks, chain stores, communications enterprise, hospitals, etc. Moreover, GDEA is promising to be very useful method to construct decision support systems such as administrative reforms of (local) goverments, engineering design, schools, courts, and so on.

Contents

Chapter 1

Introduction

There are two fundamental approaches used for estimating efficient frontiers in economics. They are called the parametric and nonparametric approaches. The parametric approach requires the imposition of a specific functional form (for example, a regression equation, a production function, and so on) for the technology [3, 21]. The selected functional form also requires specific assumptions about the distribution of the inefficiency terms and many other restrictions. In contrast to the parametric approach, the non-parametric approach initiated as data envelopment analysis (DEA) by Charnes, Cooper and Rhodes does not require any assumption about the functional form. DEA calculates the relative efficiency for each decision making unit (DMU) to all other DMUs with the only requirement that all DMUs lie on or 'below' the so-called efficient frontier based on the observed data.

We use an example to explain those approaches. The example consists of eight DMUs with consuming one input to produce one output. (See Figure 1.1.) As shown in Figure 1.1, the parametric approach is to estimate the degree how far the performance of each DMU is on the above or below from the single regression

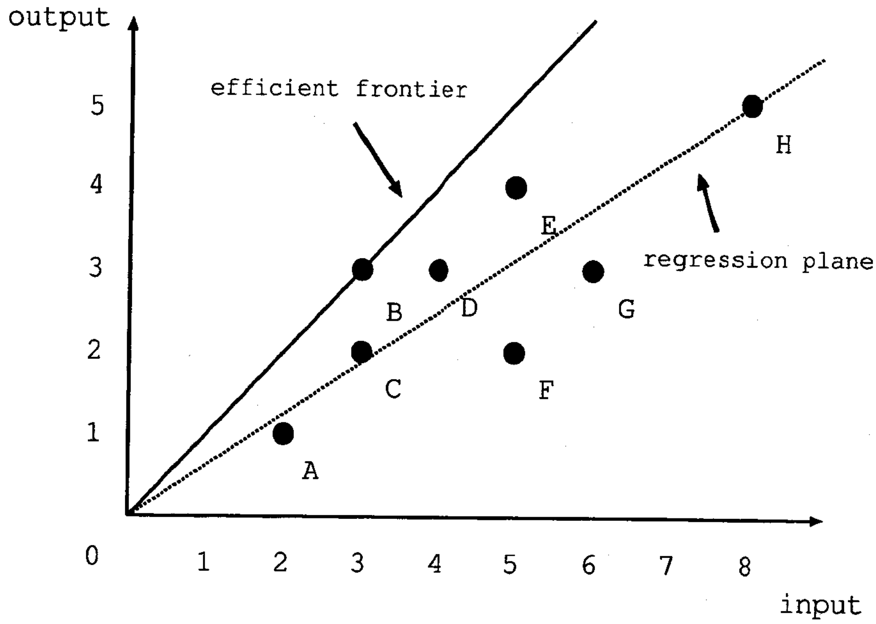


Figure 1.1: Comparison of DEA and Regression.

plane through the observed data. In contrast, DEA optimizes the performance of each DMU with respect to a solid line, which represents the efficient frontier derived by DEA from the observed data. It is distinguished from the parametric approach by the fact that DEA calculations produce the only relative efficiency because they are obtained from actual observed data for each DMU.

The initial DEA model was presented by Charnes, Cooper and Rhodes (CCR), and built on the idea of Farrell [13] which is concerned with the estimation of technical efficiency and efficient frontiers. The CCR model [8, 9] generalized the single output/single input ratio efficiency measure for each DMU to multiple outputs/multiple inputs situations by forming the ratio of a weighted sum of outputs to a weighted sum of inputs. DEA is a method for measuring the relative efficiency of DMUs performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs. The main characteristics of DEA are that (i) it can be applied to analyze multiple outputs and multiple inputs without preassigned weights, (ii) it can be used for measuring a relative efficiency based on the observed data without knowing information on the production function and (iii) decision makers' preferences can be incorporated in DEA models. Later, Banker, Charnes and Cooper (BCC) suggested a model for estimating technical efficiency and scale inefficiency in DEA. The BCC model [6] relaxed the constant returns to scale assumption of the CCR model and made it possible to investigate whether the performance of each DMU was conducted in region of increasing, constant or decreasing returns to scale in multiple outputs and multiple inputs situations. In addition, Tulkens [29] introduced a relative efficiency to non-convex free disposable hull (FDH) of the observed data defined by Deprins *et al.* [12], and formulated a mixed integer programming to calculate the relative efficiency for each DMU. Besides basic models as mentioned in the above, a number of extended models have been studied, for example, cone ratio model [11], polyhedral cone ratio model [10], Seiford and Thrall's model [25], Wei and Yu's model [32], and so on.

On the other hand, relationships between DEA and multiple criteria decision analysis (MCDA) have been studied from several viewpoints by many authors. Belton [4], and Belton and Vickers [5] measured an efficiency as a weighted sum of input and output. Stewart [26] showed the equivalence between the CCR model and some linear value function model for multiple outputs and multiple inputs. Joro *et al.* [18] proved structural correspondences between DEA models and multiple objective linear programming (MOLP) using an achievement scalarizing function proposed by Wierzbicki [31]. Especially, various ways of introducing preference information into DEA formulations have been developed. Golany [15] suggested a so-called target setting model, which allows decision makers to select the preferred set of output levels given the input levels of a DMU. Thanassoulis and Dyson [28] introduced models that can be used to estimate alternative output and input levels, in order to render relatively inefficient DMUs efficient. Zhu [38] proposed a model that calculates efficiency scores incorporating the decision makers' preference informations, whereas Korhonen [20] applied an interactive technique to progressively reveal preferences. Hamel *et al.* [17] evaluated an efficiency of DMU in terms of pseudo-concave value function, by considering a tangent cone of the feasible set at the most preferred solution of decision maker. Agrell and Tind [1] showed correspondences among the CCR model [8], the BCC model [6] and the FDH model [29] and MCDA model according to the property of a partial Lagrangean relaxation. Yun, Nakayama and Tanino [33, 34] suggested a concept of "value free efficiency" in the observed data and proposed a new model called GDEA (generalized DEA) model which can treat basic DEA models, specifically, the CCR model, the BCC model and the FDH model in a unified way. They showed theoretical properties on relationships among the GDEA model and those DEA models and, GDEA model made it possible to calculate the efficiency of DMUs incorporating various preference structures of decision makers. Furthermore, they [35] proposed a dual approach GDEA_D to GDEA and showed also that GDEA_D can reveal domination relations among all DMUs. In addition, as an application of GDEA, they [36, 37] suggested a method combining GDEA and genetic algorithms for generating efficient frontiers in multi-objective optimization problems.

The rest of this thesis is organized as follows. Chapter 2 introduces notations used in this thesis and presents brief explanations on basic DEA models. In Chapter 3, we propose the GDEA model based on a parametric domination. Chapter 4 presents a dual approach to GDEA, that is, the GDEA_D model based on a production possibility set. In Chapter 5, we compare the efficiency of GDEA and several DEAs for each DMU through illustrative examples. As an application of GDEA, Chapter 6 explains a method combining GDEA and genetic algorithms (GA) for generating efficient frontiers in multi-objective optimization problems. Finally, Chapter 7 concludes this thesis.

Chapter 2

Basic DEA Models

In the following discussion, we assume that there exist n DMUs to be evaluated. Each DMU consumes varying amounts of m different inputs to produce p different outputs. Specifically, DMUj consumes amounts $\boldsymbol{x}_j := (x_{ij})$ of inputs $(i = 1, \cdots, m)$ and produces amounts $\boldsymbol{y}_j := (y_{kj})$ of outputs $(k = 1, \cdots, p)$. For these constants, which generally take the form of observed data, we assume $x_{ij} > 0$ for each $i = 1, \cdots, m$ and $y_{kj} > 0$ for each $k = 1, \cdots, p$. Further, we assume that there are no duplicated units in the observed data. The $p \times n$ output matrix for the n DMUs is denoted by $\boldsymbol{Y}$, and the $m \times n$ input matrix for the n DMUs is denoted by $\boldsymbol{X}$. $\boldsymbol{x}_o := (x_{1o}, \cdots, x_{mo})$ and $\boldsymbol{y}_o := (y_{1o}, \cdots, y_{po})$ are amounts of inputs and outputs of DMUo, which is evaluated. In addition, ε is a small positive number ("non-Archimedean[1]") and $\mathbf{1} = (1, \cdots, 1)$ is a unit vector.

For convenience, the following notations for vectors in $\mathbb{R}^{p+m}$ will be used:

$$\begin{aligned}
\boldsymbol{z}_o > \boldsymbol{z}_j &\iff z_{io} > z_{ij}, \ i = 1, \cdots, p+m, \\
\boldsymbol{z}_o \geqq \boldsymbol{z}_j &\iff z_{io} \geqq z_{ij}, \ i = 1, \cdots, p+m, \\
\boldsymbol{z}_o \geq \boldsymbol{z}_j &\iff z_{io} \geqq z_{ij}, \ i = 1, \cdots, p+m \text{ but } \boldsymbol{z}_o \neq \boldsymbol{z}_j.
\end{aligned}$$

[1]Archimedean property : If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$. ε is a small positive number not satisfying Archimedean property.

So far, a number of DEA models have been developed. Among them, the CCR model [8, 9], the BCC model [6] and the FDH model [29] are well known as basic DEA models. These models are based on the domination structure in primal form and moreover, these are characterized by how to determine the production possibility set in a viewpoint of dual form; a convex cone, a convex hull and a free disposable hull [2] for the observed data, respectively. The development of the models in this chapter is as follows. Section 1 starts with the CCR model. Sections 2 and 3 discuss the BCC model and the FDH model, respectively.

2.1 The CCR Model

The CCR model, which was suggested by Charnes *et al.* [8], is a fractional linear programming problem and can be solved by being transformed into an equivalent linear programming one. Therefore, the primal problem (CCR) with an input oriented model [3] can be formulated as the following:

$$
\begin{array}{lll}
\text{(CCR)} & \underset{\mu_k,\,\nu_i}{\text{maximize}} & \displaystyle\sum_{k=1}^{p} \mu_k y_{ko} \\
& \text{subject to} & \displaystyle\sum_{i=1}^{m} \nu_i x_{io} = 1, \\
& & \displaystyle\sum_{k=1}^{p} \mu_k y_{kj} - \sum_{i=1}^{m} \nu_i x_{ij} \leqq 0, \quad j = 1, \cdots, n, \\
& & \mu_k \geqq \varepsilon, \quad \nu_i \geqq \varepsilon, \quad k = 1, \cdots, p\,;\; i = 1, \cdots, m.
\end{array}
$$

[2]The free disposable hull (FDH) by Deprins *et al.* [12] is a non-convex hull consisting of any points that perform less output with the same amount of input as the observed data, and/or those that perform more input with the same amount of output.

[3]The CCR model, the BCC model and the FDH model are dependent on the orientation. For instance, in an input orientation, one focuses on maximal movement toward the efficient frontier through proportional reduction of inputs, whereas in an output orientation one focuses on maximal movement via proportional augmentation of outputs. In this thesis, to condense the text, we deal with only the input oriented model for simplicity.

The dual problem (CCR_D) to the problem (CCR) is given by

$$(\mathrm{CCR}_D) \qquad \begin{array}{ll} \underset{\theta,\,\boldsymbol{\lambda},\,\boldsymbol{s}_x,\,\boldsymbol{s}_y}{\text{minimize}} & \theta - \varepsilon(\mathbf{1}^T\boldsymbol{s}_x + \mathbf{1}^T\boldsymbol{s}_y) \\ \text{subject to} & \boldsymbol{X\lambda} - \theta\boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\ & \boldsymbol{Y\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\ & \boldsymbol{\lambda} \geqq \mathbf{0},\ \ \boldsymbol{s}_x \geqq \mathbf{0},\ \ \boldsymbol{s}_y \geqq \mathbf{0}, \\ & \theta \in \mathbb{R},\ \ \boldsymbol{\lambda} \in \mathbb{R}^n,\ \ \boldsymbol{s}_x \in \mathbb{R}^m, \boldsymbol{s}_y \in \mathbb{R}^p. \end{array}$$

Well, we introduce the 'efficiency' in the CCR model.

Definition 2.1.1. (CCR-efficiency) A DMUo is *CCR-efficient* if and only if the optimal value $\sum_{k=1}^{p} \mu_k^* y_{ko}$ to the problem (CCR) equals one. Otherwise, the DMUo is said to be *CCR-inefficient.*

Definition 2.1.2. (CCR_D-efficiency) A DMUo is *CCR_D-efficient* if and only if for the optimal solution (θ^*, $\boldsymbol{\lambda}^*$, $\boldsymbol{s}_x^*$, $\boldsymbol{s}_y^*$) to the problem (CCR_D), the following two conditions are satisfied:

(i) θ^* is equal to one;

(ii) the slack variables $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are all zero.

Otherwise, the DMUo is *CCR_D-inefficient.*

Note that evidently, the above two definitions are equivalent.

It is worthy of notice the optimal solution (θ^*, $\boldsymbol{\lambda}^*$, $\boldsymbol{s}_x^*$, $\boldsymbol{s}_y^*$) to the dual problem (CCR_D). A DMUo is CCR_D-inefficient if θ^* is less than one, which represents the efficiency degree for a DMUo in the CCR model. A CCR_D-inefficient DMUo can be made more CCR_D-efficient by projection onto the CCR efficient

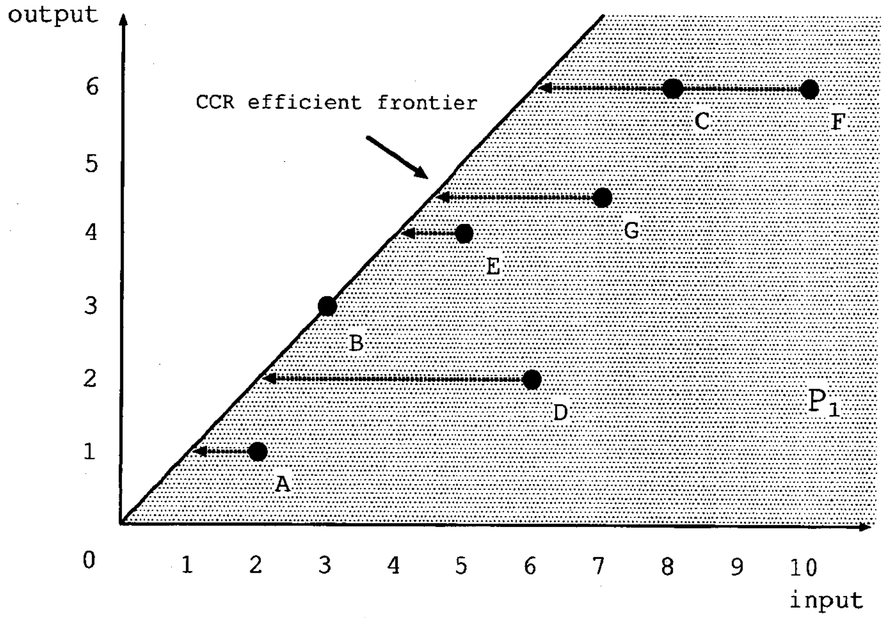


Figure 2.1: CCR efficient frontier and production possibility set generated by the CCR model from the observed data.

frontier [4], i.e. one improves the efficiency degree in the CCR model through the proportional reduction of all inputs. The projection of $\boldsymbol{x}_o$ and $\boldsymbol{y}_o$ yields $\theta^*\boldsymbol{x}_o$ and $\boldsymbol{y}_o$, respectively, and we obtain

$$\theta^*\boldsymbol{x}_o = \boldsymbol{X}\boldsymbol{\lambda}^* + \boldsymbol{s}_x^* \quad \text{and} \quad \boldsymbol{y}_o = \boldsymbol{Y}\boldsymbol{\lambda}^* - \boldsymbol{s}_y^*. \tag{2.1}$$

As seen in the above expression (2.1), $\boldsymbol{s}_x^*$ represents surplus of input and $\boldsymbol{s}_y^*$ does slack of output. $\theta^*\boldsymbol{x}_o$ and $\boldsymbol{y}_o$ can be expressed by a linear combination of the other $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$, and thus, DMUo in terms of the CCR model is dominated by DMU$\hat{\jmath}$, when a $\hat{\jmath}$th-component of $\boldsymbol{\lambda}^*$ is positive.

Consequently, the production possibility set P_1 in the CCR model is the

[4]We call the CCR efficient frontier generated by the CCR model in order to avoid confusing with an efficient frontier, i.e. the set of Pareto optimal values in usual multi-objective programming problems. As the same, the BCC efficient frontier and the FDH efficient frontier are obtained by the BCC model and the FDH model, respectively.

convex cone (or conical hull) generated by the observed data, since one takes a viewpoint of the fact that the scale efficiency of a DMU is constant, that is to say, constant returns to scale. Therefore, P_1 can be denoted by

$$P_1 = \left\{ (\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y},\ X\boldsymbol{\lambda} \leqq \boldsymbol{x},\ \boldsymbol{\lambda} \geqq \boldsymbol{0} \right\}.$$

and the definition of CCR-efficiency (or CCR$_D$-efficiency) can be transformed into the following:

Definition 2.1.3. DMUo is said to be *Pareto efficient in* P_1 if and only if there does not exist $(\boldsymbol{y}, \boldsymbol{x}) \in P_1$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

2.2 The BCC Model

The BCC model of Banker *et al.* [6] is formulated similarly to that for the CCR model. The dual problem for the BCC model is obtained by adding the convexity constraint $\boldsymbol{1}^T\boldsymbol{\lambda} = 1$ to the dual problem (CCR$_D$) and thus, the variable u_o appears in the primal problem. The efficiency degree of a DMUo with respect to the BCC model can be measured by solving the problem

$$
\begin{array}{llll}
\text{(BCC)} & \underset{\mu_k,\,\nu_i,\,u_o}{\text{maximize}} & \displaystyle\sum_{k=1}^{p} \mu_k y_{ko} - u_o & \\
& \text{subject to} & \displaystyle\sum_{i=1}^{m} \nu_i x_{io} = 1, & \\
& & \displaystyle\sum_{k=1}^{p} \mu_k y_{kj} - \sum_{i=1}^{m} \nu_i x_{ij} - u_o \leqq 0, \quad j = 1, \cdots, n, & \\
& & \mu_k \geqq \varepsilon, \ \ \nu_i \geqq \varepsilon, \ \ k = 1, \cdots, p\,;\ i = 1, \cdots, m. &
\end{array}
$$

The dual problem (BCC_D) to the problem (BCC) is formulated as follows:

$$
(\mathrm{BCC}_D) \qquad \begin{array}{ll}
\underset{\theta,\, \boldsymbol{\lambda},\, \boldsymbol{s}_x,\, \boldsymbol{s}_y}{\text{minimize}} & \theta - \varepsilon(\mathbf{1}^T \boldsymbol{s}_x + \mathbf{1}^T \boldsymbol{s}_y) \\
\text{subject to} & \boldsymbol{X}\boldsymbol{\lambda} - \theta \boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\
& \boldsymbol{Y}\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\
& \mathbf{1}^T \boldsymbol{\lambda} = 1, \\
& \boldsymbol{\lambda} \geqq \mathbf{0}, \;\; \boldsymbol{s}_x \geqq \mathbf{0}, \;\; \boldsymbol{s}_y \geqq \mathbf{0}, \\
& \theta \in \mathbb{R}, \;\; \boldsymbol{\lambda} \in \mathbb{R}^n, \;\; \boldsymbol{s}_x \in \mathbb{R}^m, \boldsymbol{s}_y \in \mathbb{R}^p.
\end{array}
$$

The definition of 'efficiency' in the BCC model is given as follows, and the two definitions are equivalent.

Definition 2.2.1. (BCC-efficiency) A DMUo is *BCC-efficient* if and only if the optimal value $(\sum_{k=1}^{p} \mu_k^* y_{ko} - u_o^*)$ to the problem (BCC) equals one. Otherwise, the DMUo is said to be *BCC-inefficient.*

Definition 2.2.2. (BCC_D-efficiency) A DMUo is *BCC_D-efficient* if and only if for an optimal solution $(\theta^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$ to the problem (BCC_D), the following two conditions are satisfied:

(i) θ^* is equal to one;

(ii) the slack variables $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are all zero.

Otherwise, the DMUo is said to be *BCC_D-inefficient.*

The meanings of an optimal solution $(\theta^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$ to the problem (BCC_D) are similar to those in the CCR model. In particular, we see that the presence of the constraint $\mathbf{1}^T\boldsymbol{\lambda} = 1$ in the dual problem (BCC_D) yields that

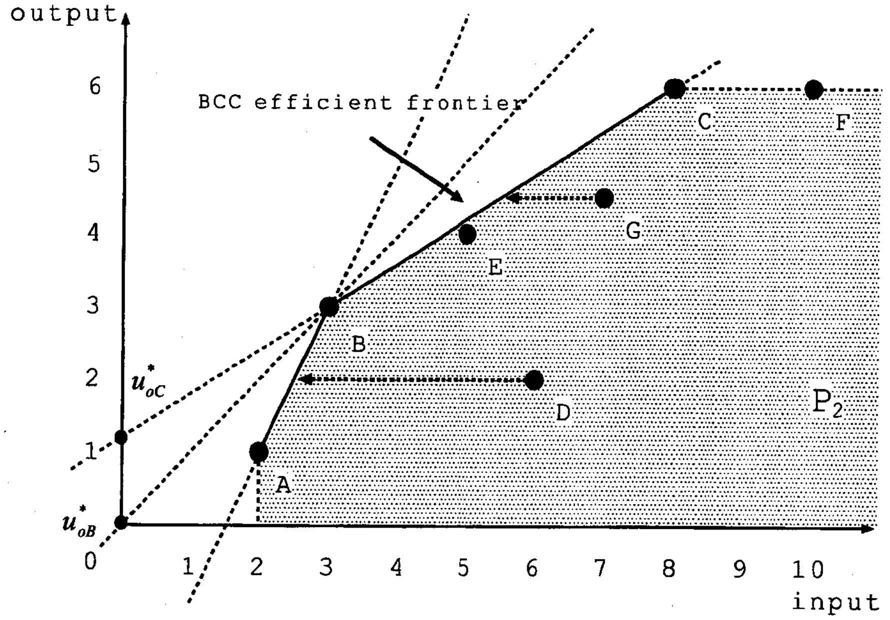


Figure 2.2: BCC efficient frontier and production possibility set generated by the BCC model from the observed data.

the production possibility set P_2 in the BCC model is the *convex hull* generated by the observed data. In addition, the constraint $\mathbf{1}^T\boldsymbol{\lambda} = 1$ makes it possible to consider the scale efficiency (i.e. decreasing, constant and increasing returns to scale) under multiple outputs/multiple inputs situations. The one of a DMU*o* is judged by the optimal solution u_o^* to the problem (BCC). Here, u_o^* means an intercept of output-axis. For example, as seen in Figure 2.2, u_o^* for DMU A is negative, u_o^* for DMU B is equal to zero, and u_o^* for DMU C is positive. Then, the following can be established:

(i) increasing returns to scale $\Leftrightarrow u_o^* < 0$,

(ii) constant returns to scale $\Leftrightarrow u_o^* = 0$,

(iii) decreasing returns to scale $\Leftrightarrow u_o^* > 0$.

Therefore, P_2 can be obtained as

$$P_2 = \left\{ (\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y},\ X\boldsymbol{\lambda} \leqq \boldsymbol{x},\ \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geqq \mathbf{0} \right\}.$$

and the definition of BCC-efficiency (or BCC$_D$-efficiency) can be transformed into the following:

Definition 2.2.3. DMUo is said to be *Pareto efficient in* P_2 if and only if there does not exist $(\boldsymbol{y}, \boldsymbol{x}) \in P_2$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

2.3 The FDH Model

The FDH model by Tulkens [29] is formulated as follows:

$$(\mathrm{FDH}_D) \qquad \begin{array}{ll} \underset{\theta,\, \boldsymbol{\lambda},\, \boldsymbol{s}_x,\, \boldsymbol{s}_y}{\text{minimize}} & \theta - \varepsilon(\mathbf{1}^T\boldsymbol{s}_x + \mathbf{1}^T\boldsymbol{s}_y) \\ \text{subject to} & \boldsymbol{X}\boldsymbol{\lambda} - \theta\boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\ & \boldsymbol{Y}\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\ & \mathbf{1}^T\boldsymbol{\lambda} = 1;\ \ \lambda_j \in \{0,\ 1\} \text{ for each } j = 1, \cdots, n, \\ & \boldsymbol{\lambda} \geqq \mathbf{0},\ \ \boldsymbol{s}_x \geqq \mathbf{0},\ \ \boldsymbol{s}_y \geqq \mathbf{0}, \\ & \theta \in \mathbb{R},\ \ \boldsymbol{\lambda} \in \mathbb{R}^n,\ \ \boldsymbol{s}_x \in \mathbb{R}^m, \boldsymbol{s}_y \in \mathbb{R}^p. \end{array}$$

However, here, it is seen that the problem (FDH$_D$) is a mixed integer programming problem and hence, the traditional linear optimization methods cannot apply to it. An optimal solution is obtained by means of a simple vector comparision procedure, to the end.

For a DMUo, the optimal solution θ^* to the problem (FDH$_D$) is equal to the value R_o^* defined by

$$R_o^* = \min_{j \in D(o)} \ \max_{i=1,\cdots m} \left\{ \frac{x_{ij}}{x_{io}} \right\}, \tag{2.2}$$

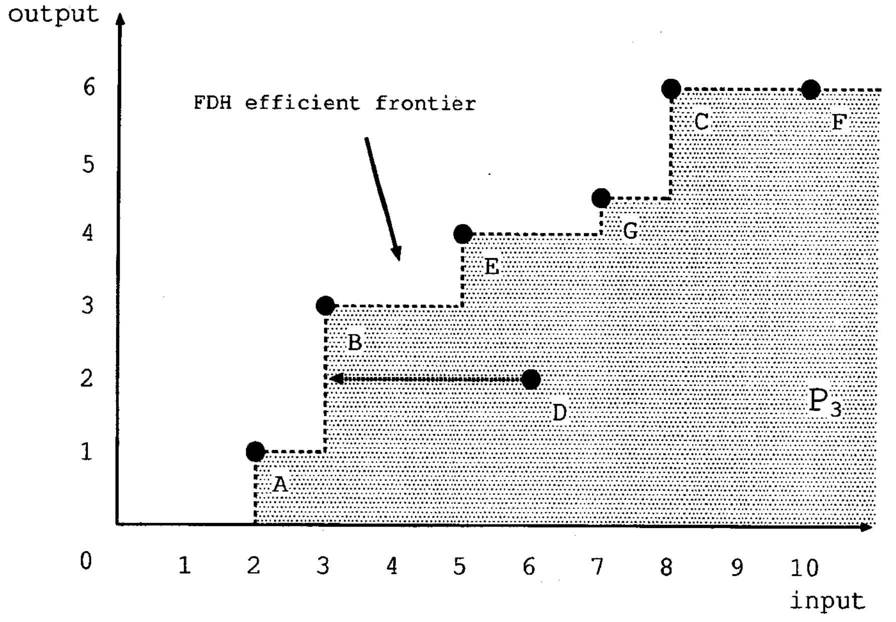


Figure 2.3: FDH efficient frontier and production possibility set generated by the FDH model from the observed data.

where $D(o) = \{\, j \mid \boldsymbol{x}_j \leq \boldsymbol{x}_o \text{ and } \boldsymbol{y}_j \geqq \boldsymbol{y}_o,\ j = 1, \cdots, n \,\}$.

R_o^* is subutituted for θ^* as the efficiency degree for DMUo in the FDH model. Also, the 'efficiency' in the FDH model is given in the following.

Definition 2.3.1. (FDH-efficiency) A DMUo is *FDH-efficient* if and only if R_o^* equals to one. If $R_o^* < 1$, the DMUo is said to be *FDH-inefficient.*

Definition 2.3.2. (FDH$_D$-efficiency) A DMUo is *FDH$_D$-efficient* if and only if for an optimal solution (θ^*, $\boldsymbol{\lambda}^*$, $\boldsymbol{s}_x^*$, $\boldsymbol{s}_y^*$) to the problem (FDH$_D$), the following two conditions are satisfied:

(i) θ^* is equal to one;

(ii) the slack variables $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are all zero.

Otherwise, the DMUo is said to be *FDH$_D$-inefficient.*

The above two definitions are equivalent forms, and the production possibility set P_3, which is a free disposable hull, is given by

$$(2.3)\quad P_3 = \Big\{(\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y},\ X\boldsymbol{\lambda} \leqq \boldsymbol{x},\ \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \lambda_j \in \{0,1\},\ j = 1, \cdots, n\Big\}.$$

Besides, the definition of FDH-efficiency (or FDH_D-efficiency) can be transformed into the following:

Definition 2.3.3. DMUo is said to be *Pareto efficient in* P_3 if and only if there does not exist $(\boldsymbol{y}, \boldsymbol{x}) \in P_3$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

Chapter 3

GDEA Based on Parametric Domination Structure

In this chapter, we formulate a GDEA model based on a domination structure and define a new 'efficiency' in the GDEA model. Next, we establish relationships between the GDEA model and basic DEA models mentioned in chapter 2.

3.1 The GDEA Model

We formulate a generalized DEA model by employing the augemented Tchebyshev scalarizing function [23]. The GDEA model, which can evaluate the efficiency in several basic models as special cases, is the following:

$$
\begin{aligned}
\text{(GDEA)} \quad & \underset{\Delta,\,\mu_k,\,\nu_i}{\text{maximize}} \quad \Delta \\
& \text{subject to} \quad \Delta \leqq \tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \mu_k (y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i (-x_{io} + x_{ij}) \right), \\
& \qquad\qquad j = 1, \cdots, n, \\
& \qquad \sum_{k=1}^{p} \mu_k + \sum_{i=1}^{m} \nu_i = 1, \\
& \qquad \mu_k,\ \nu_i \geqq \varepsilon,\ k = 1, \cdots, p;\ i = 1, \cdots, m,
\end{aligned}
$$

where $\tilde{d}_j = \max_{\substack{k=1,\cdots,p \\ i=1,\cdots,m}} \{\mu_k(y_{ko} - y_{kj}),\ \nu_i(-x_{io} + x_{ij})\}$ and α is a positive number.

Note that when $j = o$, the right-hand side of the inequality constraint in the problem (GDEA) is zero, and hence its optimal value is not greater than zero. We define 'efficiency' in the GDEA model as follows.

Definition 3.1.1. (α-efficiency) For a given positive number, α, DMUo is defined to be *α-efficient* if and ony if the optimal value to the problem (GDEA) is equal to zero. Otherwise, DMUo is said to be *α-inefficient.*

3.2 Relationships between GDEA and DEA

In this section, we establish theoretical properties on relationships among efficiency in the basic DEA models and that in the GDEA model.

Theorem 3.2.1. *DMUo is FDH-efficient if and only if DMUo is α-efficient for some sufficiently small positive number α.*

Proof. (*only if part*) Let Δ^*, $(\mu_1^*, \cdots, \mu_p^*)$ and $(\nu_1^*, \cdots, \nu_m^*)$ be the optimal solution for the DMUo. Negate that DMUo is α-efficient for some sufficiently small positive α. Then for any sufficiently small positive α, $\Delta^* < 0$, that is,

$$\tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \mu_k^*(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^*(-x_{io} + x_{ij}) \right) < 0 \text{ for some } j \neq o. \tag{3.1}$$

The necessary and sufficient condition so that the above inequality (3.1) holds for any sufficiently small positive α is that

$$\tilde{d}_j = \max_{\substack{k=1,\cdots,p \\ i=1,\cdots,m}} \{\mu_k^*(y_{ko} - y_{kj}),\ \nu_i^*(-x_{io} + x_{ij})\} < 0 \tag{3.2}$$

and since $(\mu_1^*, \cdots, \mu_p^*)$ and $(\nu_1^*, \cdots, \nu_m^*)$ are strictly positive, the inequality (3.2) implies that $(\boldsymbol{y}_j, -\boldsymbol{x}_j) > (\boldsymbol{y}_o, -\boldsymbol{x}_o)$ for some $j \neq o$. Thus, $j \in D(o)$ and $\max\limits_{i=1,\cdots m} \{x_{ij}/x_{io}\} < 1$, which means that $R_o^* = \min\limits_{j \in D(o)} \max\limits_{i=1,\cdots m} \{x_{ij}/x_{io}\} < 1$. This contradicts the assumption that DMUo is FDH-efficient and therefore, DMUo is α-efficient for some sufficiently small positive α.

(*if part*) Suppose that DMUo is FDH-inefficient. Then $R_o^* < 1$, which yields that there exists some $j \in D(o) = \{j \mid \boldsymbol{x}_j \leq \boldsymbol{x}_o \text{ and } \boldsymbol{y}_j \geqq \boldsymbol{y}_o, \ j = 1, \cdots, n\}$ such that $\max\limits_{i=1,\cdots m} \{x_{ij}/x_{io}\} < 1$. Thus, $\boldsymbol{y}_j \geqq \boldsymbol{y}_o$ and $\boldsymbol{x}_j < \boldsymbol{x}_o$ for such a j. For any positive $(\mu_1, \cdots, \mu_p)$ and $(\nu_1, \cdots, \nu_m)$, we have

$$(3.3) \quad \mu_k(y_{ko} - y_{kj}) \leqq 0, \ \ k = 1, \cdots, p \ \text{ and } \ \nu_i(-x_{io} + x_{ij}) < 0, \ \ i = 1, \cdots, m.$$

From inequalities of the above (3.3), the following inequalities hold:

$$(3.4) \qquad \tilde{d}_j = \max_{\substack{k=1,\cdots,p \\ i=1,\cdots,m}} \{\mu_k(y_{ko} - y_{kj}), \ \nu_i(-x_{io} + x_{ij})\} \leqq 0$$

and

$$(3.5) \qquad \left(\sum_{k=1}^{p} \mu_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i(-x_{io} + x_{ij}) \right) < 0.$$

Multiplying (3.5) by any positive α and adding to (3.4) yields that

$$\tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \mu_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i(-x_{io} + x_{ij}) \right) < 0 \text{ for some } j,$$

which is a contradiction to the α-efficiency for some sufficiently small positive α. Hence, it has been shown that the DMUo is FDH-efficient. □

Theorem 3.2.2. *DMUo is BCC-efficient if and only if DMUo is α-efficient for some sufficiently large positive number α.*

Proof. (*only if part*) Let Δ^*, $(\mu_1^*, \cdots, \mu_p^*)$ and $(\nu_1^*, \cdots, \nu_m^*)$ be the optimal solution for the DMUo. Negate that DMUo is α-efficient for some sufficiently large positive α. Then for any sufficiently large positive α, $\Delta^* < 0$, that is, we have

$$\tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \mu_k^*(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^*(-x_{io} + x_{ij}) \right) < 0 \text{ for some } j \neq o. \tag{3.6}$$

The necessary and sufficient condition so that the above inequality (3.6) holds for any sufficiently large positive α is that

$$\left(\sum_{k=1}^{p} \mu_k^*(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^*(-x_{io} + x_{ij}) \right) < 0. \tag{3.7}$$

The inequality (3.7) is rewritten as

$$\sum_{k=1}^{p} \mu_k^* y_{ko} - \sum_{i=1}^{m} \nu_i^* x_{io} < \sum_{k=1}^{p} \mu_k^* y_{kj} - \sum_{i=1}^{m} \nu_i^* x_{ij}$$

and hence

$$\sum_{k=1}^{p} \mu_k^* y_{ko} - \sum_{i=1}^{m} \nu_i^* x_{io} - u_o^* < \sum_{k=1}^{p} \mu_k^* y_{kj} - \sum_{i=1}^{m} \nu_i^* x_{ij} - u_o^* = 0. \tag{3.8}$$

Let $\hat{\mu}_k := \mu_k^* / \sum_{i=1}^{m} \nu_i^* x_{io}$, $\hat{\nu}_i := \nu_i^* / \sum_{i=1}^{m} \nu_i^* x_{io}$ and $\hat{u}_o := u_o^* / \sum_{i=1}^{m} \nu_i^* x_{io}$. Then, $\sum_{i=1}^{m} \hat{\nu}_i x_{io} = 1$ and from the expression (3.8),

$$\sum_{k=1}^{p} \hat{\mu}_k y_{ko} - \sum_{i=1}^{m} \hat{\nu}_i x_{io} - \hat{u}_o < \sum_{k=1}^{p} \hat{\mu}_k y_{kj} - \sum_{i=1}^{m} \hat{\nu}_i x_{ij} - \hat{u}_o = 0.$$

Therefore, $(\hat{\mu}_1, \cdots, \hat{\mu}_p)$ and $(\hat{\nu}_1, \cdots, \hat{\nu}_m)$ is a feasible solution of the problem (BCC), $\sum_{k=1}^{p} \hat{\mu}_k y_{ko} - \hat{u}_o = \max_{\mu_k\, \nu_i,\, u_o} \{\sum_{k=1}^{p} \mu_k y_{ko} - u_o\} < 1$. This contradicts the assumption that DMUo is BCC-efficient, and hence DMUo is α-efficient for some sufficiently large positive α.

(*if part*) Assume that DMUo is α-efficient for some sufficiently large positive α. Then, there exist $(\mu_1^*, \cdots, \mu_p^*)$ and $(\nu_1^*, \cdots, \nu_m^*)$ such that

$$\tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \mu_k^* (y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^* (-x_{io} + x_{ij}) \right) \geqq 0, \;\; j = 1, \cdots, n. \tag{3.9}$$

In the inequality (3.9), the equality holds only when $j = o$. Since α is a sufficiently large positive, we obtain from the inequality (3.9) that for each $j = 1, \cdots, n$, $\sum_{k=1}^{p} \mu_k^* (y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^* (-x_{io} + x_{ij}) \geqq 0$, that is,

$$\sum_{k=1}^{p} \mu_k^* y_{ko} - \sum_{i=1}^{m} \nu_i^* x_{io} \geqq \sum_{k=1}^{p} \mu_k^* y_{kj} - \sum_{i=1}^{m} \nu_i^* x_{ij}, \;\; j = 1, \cdots, n. \tag{3.10}$$

Let $\hat{\mu}_k := \mu_k^* / \sum\limits_{i=1}^{m} \nu_i^* x_{io}$, $\hat{\nu}_i := \nu_i^* / \sum\limits_{i=1}^{m} \nu_i^* x_{io}$ and $\hat{u}_o := (\sum_{k=1}^{p} \hat{\mu}_k y_{ko} - \sum_{i=1}^{m} \hat{\nu}_i x_{io})$. Then from the inequality (3.10), we get $\sum_{i=1}^{m} \hat{\nu}_i x_{io} = 1$ and

$$\hat{u}_o = \sum_{k=1}^{p} \hat{\mu}_k y_{ko} - \sum_{i=1}^{m} \hat{\nu}_i x_{io} \geqq \sum_{k=1}^{p} \hat{\mu}_k y_{kj} - \sum_{i=1}^{m} \hat{\nu}_i x_{ij}, \;\; j = 1, \cdots, n.$$

This implies that $(\hat{\mu}_1, \cdots, \hat{\mu}_p)$ and $(\hat{\nu}_1, \cdots, \hat{\nu}_m)$ is a feasible solution of the problem (BCC) and $\sum_{k=1}^{p} \hat{\mu}_k y_{ko} - \hat{u}_o = \max\limits_{\mu, \nu, u_o} \{ \sum_{k=1}^{p} \mu_k y_{ko} - u_o \} = 1$. Consequently, we established that the DMUo is BCC-efficient. $\square$

Consider the problem (GDEA$'$) in which the constraint $\sum_{k=1}^{p} \mu_k y_{ko} = \sum_{i=1}^{m} \nu_i x_{io}$ is added to the problem (GDEA):

$$\begin{array}{lll}
(\text{GDEA}') & \underset{\Delta,\, \mu_k,\, \nu_i}{\text{maximize}} & \Delta \\
& \text{subject to} & \Delta \leqq \tilde{d}_j + \alpha \left(\sum\limits_{k=1}^{p} \mu_k (y_{ko} - y_{kj}) + \sum\limits_{i=1}^{m} \nu_i (-x_{io} + x_{ij}) \right), \\
& & \qquad\qquad\qquad\qquad\qquad\qquad j = 1, \cdots, n, \\
& & \sum\limits_{k=1}^{p} \mu_k y_{ko} - \sum\limits_{i=1}^{m} \nu_i x_{io} = 0, \\
& & \sum\limits_{k=1}^{p} \mu_k + \sum\limits_{i=1}^{m} \nu_i = 1, \\
& & \mu_k, \;\; \nu_i \geqq \varepsilon, \;\; k = 1, \cdots, p; \; i = 1, \cdots, m,
\end{array}$$

where $\tilde{d}_j = \max\limits_{\substack{k=1,\cdots,p \\ i=1,\cdots,m}} \{\mu_k(y_{ko} - y_{kj}),\ \nu_i(-x_{io} + x_{ij})\}$ and α is a given positive number.

Theorem 3.2.3. *DMUo is CCR-efficient if and only if DMUo is α-efficient for some sufficiently large positive α when regarding the problem (GDEA) as the problem (GDEA$'$).*

Proof. (*only if part*) Assume that DMUo is CCR-efficient. Then, from the definition of CCR-efficiency and constraints of the problem (CCR), it follows that $\sum_{k=1}^{p} \mu_k^* y_{ko} = \sum_{i=1}^{m} \nu_i^* x_{io}$ and $\sum_{k=1}^{p} \mu_k^* y_{kj} \leqq \sum_{i=1}^{m} \nu_i^* x_{ij},\ j = 1, \cdots, n$, where $(\mu_1^*, \cdots, \mu_p^*)$ and $(\nu_1^*, \cdots, \nu_m^*)$ is an optimal solution to the problem (CCR). That is,

$$\sum_{k=1}^{p} \mu_k^*(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^*(-x_{io} + x_{ij}) \geqq 0, \;\; j = 1, \cdots, n,$$

which implies that at least one of $\mu_k^*(y_{ko} - y_{kj}),\ k = 1, \cdots, p$ and $\nu_i^*(-x_{io} + x_{ij})$, $i = 1, \cdots, m$ is non-negative.

Let $\hat{\mu}_k := \mu_k^*/(\sum_{k=1}^{p} \mu_k^* + \sum_{i=1}^{m} \nu_i^*)$ and $\hat{\nu}_i := \nu_i^*/(\sum_{k=1}^{p} \mu_k^* + \sum_{i=1}^{m} \nu_i^*)$, then $\sum_{k=1}^{p} \hat{\mu}_k + \sum_{i=1}^{m} \hat{\nu}_i = 1$.
Furthermore, $\tilde{d}_j = \max\limits_{\substack{k=1,\cdots,p \\ i=1,\cdot,m}} \{\hat{\mu}_k(y_{ko} - y_{kj}),\ \hat{\nu}_i(-x_{io} + x_{ij})\}$, $j = 1, \cdots, n$ are non-negative and for some sufficiently large positive α, we have

$$\tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \hat{\mu}_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \hat{\nu}_i(-x_{io} + x_{ij}) \right) \geqq 0, \;\; j = 1, \cdots, n. \tag{3.11}$$

Hence, $(\hat{\mu}_1, \cdots, \hat{\mu}_p)$ and $(\hat{\nu}_1, \cdots, \hat{\nu}_m)$ are feasible in the problem (GDEA$'$) and an optimal value Δ^* is equal to zero, from the inequality (3.11). Thus, we showed that the DMUo is α-efficient for some sufficiently large positive α in the problem (GDEA$'$).

(*if part*) Assume that DMUo is α-efficient for some sufficiently large positive α when regarding the problem (GDEA) as the problem (GDEA$'$). Then, by α-efficiency of the DMUo, we obtain

$$
\begin{aligned}
0 = \Delta^* &\leqq \tilde{d}_j + \alpha\left(\sum_{k=1}^{p} \mu_k^*(y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i^*(-x_{io} + x_{ij})\right) \\
&= \tilde{d}_j - \alpha\left(\sum_{k=1}^{p} \mu_k^* y_{kj} - \sum_{i=1}^{m} \nu_i^* x_{ij}\right), \quad j = 1, \cdots, n, \qquad (3.12)
\end{aligned}
$$

where Δ^*, $(\mu_1^*, \cdots, \mu_p^*)$ and $(\nu_1^*, \cdots, \nu_m^*)$ is an optimal solution to the problem (GDEA$'$).

From the fact that the above inequality (3.12) holds for sufficiently large positive α, the following inequality is obtained.

$$
\sum_{k=1}^{p} \mu_k^* y_{kj} - \sum_{i=1}^{m} \nu_i^* x_{ij} \leqq 0, \; j = 1, \cdots, n. \qquad (3.13)
$$

Let $\hat{\mu}_k := \mu_k^* / \sum_{i=1}^{m} \nu_i^* x_{io}$ and $\hat{\nu}_i := \nu_i^* / \sum_{i=1}^{m} \nu_i^* x_{io}$. Then, in this equality (3.13), $\sum_{k=1}^{p} \hat{\mu}_k y_{kj} - \sum_{i=1}^{m} \hat{\nu}_i x_{ij} \leqq 0$, for all $j = 1, \cdots, n$ and by the second constraint of the problem GDEA$'$, $\sum_{k=1}^{p} \hat{\mu}_k y_{ko} = \sum_{i=1}^{m} \hat{\nu}_i x_{io} = 1$ holds.

Therefore, $(\hat{\mu}_1, \cdots, \hat{\mu}_p)$ and $(\hat{\nu}_1, \cdots, \hat{\nu}_m)$ is a feasible solution of the problem (CCR) and an optimal value $\sum_{k=1}^{p} \hat{\mu}_k y_{ko}$ is equal to zero. Thus, we showed that the DMUo is CCR-efficient. □

From the stated theorems, it is seen that the CCR-efficiency, BCC-efficiency and FDH-efficiency for each DMU can be evaluated by varying the parameter α in the problem (GDEA).

3.3 An Illustrative Example

In this section, we explain the α-efficiency in the GDEA model with a simple illustrative example and reveal domination relations among all DMUs by GDEA.

Assume that there are six DMUs which consume one input to produce one output, as seen in Table 3.1.

Table 3.1: An Example of 1-input and 1-output.

DMU	A	B	C	D	E	F
input	2	3	4.5	4	6	5.5
output	1	3	3.5	2	5	4

Table 3.2 shows the results of efficiency in the basic DEA models and α-efficiency in the GDEA model. In the upper half part of Table 3.2, we see that a DMU is efficient if the optimal value is equal to one in the CCR model, the BCC model and the FDH models, respectively. The lower half part of Table 3.2 shows the α-efficiency by changing a parameter α. It can be seen that if $\alpha = 0.1$, the α-efficiency of each DMU is the same as the FDH-efficiency. If $\alpha = 10$, the α-efficiency of each DMU is the same as the BCC-efficiency, and moreover if $\alpha = 10$ in the problem (GDEA$'$), then the α-efficiency is equivalent to the CCR-efficiency. Furthermore, Figure 3.1-Figure 3.3 represent the efficient frontier generated by varying α in the GDEA model.

Through this example, it was shown that by varying the value of parameter α, various efficiency of the basic DEA models can be measured in a unified way on the basis of this GDEA model, and furthermore the relationships among efficiency for these models become transparent.

Table 3.2: The Optimal values in basic DEA models and GDEA model.

DMU	A	B	C	D	E	F
CCR model	0.50	**1.00**	0.78	0.50	0.83	0.73
BCC model	**1.00**	**1.00**	0.83	0.63	**1.00**	0.75
FDH model	**1.00**	**1.00**	**1.00**	0.75	**1.00**	**1.00**
(i) $\alpha = 10$ (GDEA′)	-9.33	**0.00**	-3.25	-11.33	-0.73	-3.74
(ii) $\alpha = 10$	**0.00**	**0.00**	-2.10	-11.00	**0.00**	-3.35
(iii) $\alpha = 3$	**0.00**	**0.00**	**0.00**	-4.00	**0.00**	-0.55
(iv) $\alpha = 1$	**0.00**	**0.00**	**0.00**	-2.00	**0.00**	**0.00**
(v) $\alpha = 0.1$	**0.00**	**0.00**	**0.00**	-1.10	**0.00**	**0.00**

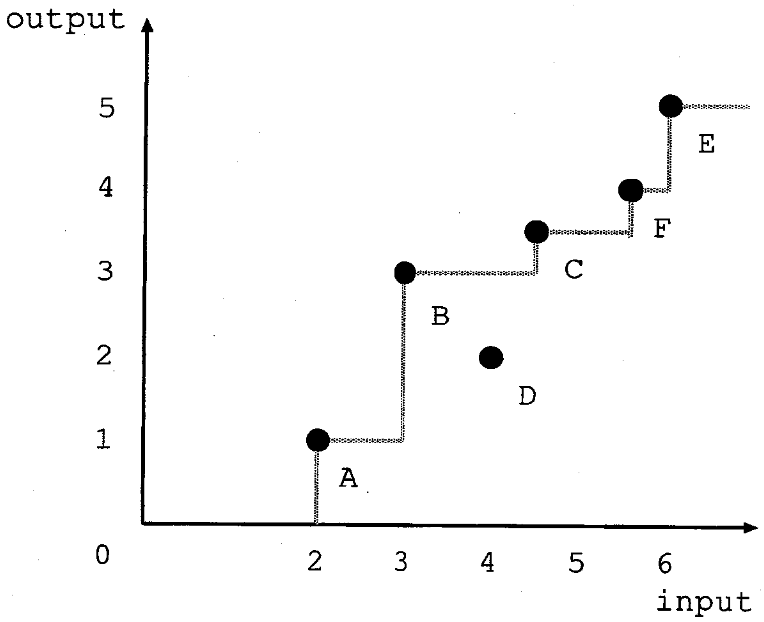


Figure 3.1: Efficient frontier generated by GDEA model with $\alpha \risingdotseq 0$.

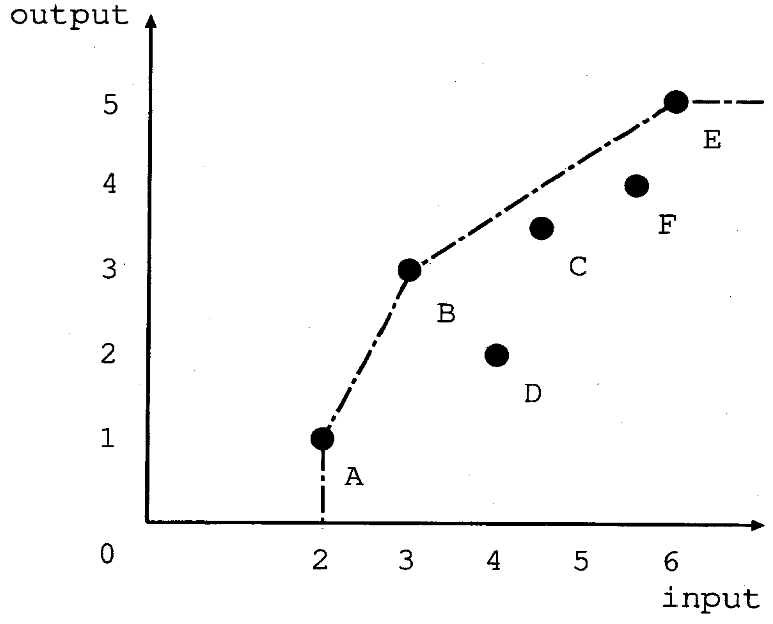


Figure 3.2: Efficient frontier generated by GDEA model with $\alpha = 10$.

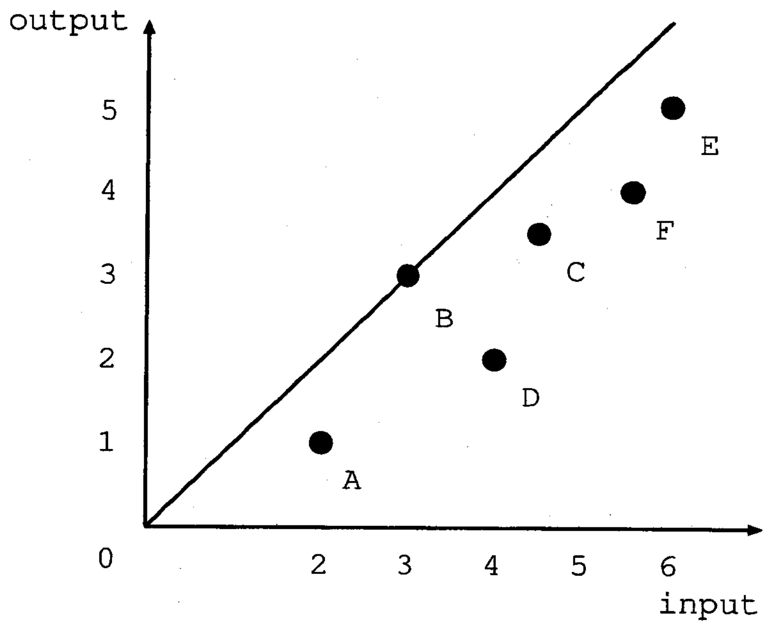


Figure 3.3: Efficient frontier generated by GDEA$'$ model with $\alpha = 10$.

Chapter 4

GDEA Based on Production Possibility

In this chapter, we consider a dual approach to GDEA introduced in chapter 3. We formulate a GDEA$_D$ model based on the production possibility set and define 'efficiency' in the GDEA$_D$ model. Next, we establish relationships between the GDEA$_D$ model and dual models to basic DEA models introduced in chapter 2.

4.1 The GDEA$_D$ Model

To begin with, an output-input vector $\boldsymbol{z}_j$ of a DMUj, $j = 1, \cdots, n$ and output-input matrix Z of all DMUs respectively, denoted by

$$\boldsymbol{z}_j := \begin{pmatrix} \boldsymbol{y}_j \\ -\boldsymbol{x}_j \end{pmatrix}, \; j = 1, \cdots, n \text{ and } Z := \begin{pmatrix} Y \\ -X \end{pmatrix}.$$

In addition, we denote a $(p+m) \times n$ matrix Z_o by $Z_o := (\boldsymbol{z}_o, \cdots, \boldsymbol{z}_o)$, where o is the index of DMU to be evaluated.

The production possibility sets in the CCR model, the BCC model and the

FDH model in chapter 2 are reformulated as follws:

$$P'_1 = \{z \mid Z\boldsymbol{\lambda} \geqq z,\ \boldsymbol{\lambda} \geqq \mathbf{0}\}$$

$$P'_2 = \left\{z \mid Z\boldsymbol{\lambda} \geqq z,\ \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geqq \mathbf{0}\right\}$$

$$P'_3 = \left\{z \mid Z\boldsymbol{\lambda} \geqq z,\ \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \lambda_j \in \{0,1\},\ j = 1,\cdots,n\right\}$$

and the 'efficiency' in these models are redefined.

Definition 4.1.1. DMUo is said to be *Pareto efficient in* P'_1 if and only if there does not exist $(\boldsymbol{y},\boldsymbol{x}) \in P'_1$ such that $(\boldsymbol{y},-\boldsymbol{x}) \geq (\boldsymbol{y}_o,-\boldsymbol{x}_o)$.

Definition 4.1.2. DMUo is said to be *Pareto efficient in* P'_2 if and only if there does not exist $(\boldsymbol{y},\boldsymbol{x}) \in P'_2$ such that $(\boldsymbol{y},-\boldsymbol{x}) \geq (\boldsymbol{y}_o,-\boldsymbol{x}_o)$.

Definition 4.1.3. DMUo is said to be *Pareto efficient in* P'_3 if and only if there does not exist $(\boldsymbol{y},\boldsymbol{x}) \in P'_3$ such that $(\boldsymbol{y},-\boldsymbol{x}) \geq (\boldsymbol{y}_o,-\boldsymbol{x}_o)$.

The definitions 4.1.1-4.1.3 are corresponding to the CCR-efficiency (or CCR$_D$-efficiency), BCC-efficiency (or BCC$_D$-efficiency) and the FDH-efficiency (or FDH$_D$-efficiency), respectively.

The dual problem to the problem (GDEA$'$) introduced in chapter 3 is formulated as follows:

$$(\text{GDEA}_D) \qquad \begin{aligned} \underset{\omega,\,\kappa,\,\boldsymbol{\lambda},\,\boldsymbol{s_z}}{\text{minimize}} \quad & \omega - \varepsilon\mathbf{1}^T\boldsymbol{s_z} \\ \text{subject to} \quad & \{\alpha(Z_o - Z) + D_z\}\,\boldsymbol{\lambda} - \boldsymbol{\omega} + \boldsymbol{s_z} + \kappa\boldsymbol{z}_o = \mathbf{0} \\ & \mathbf{1}^T\boldsymbol{\lambda} = 1, \\ & \boldsymbol{\lambda} \geqq \mathbf{0},\ \ \boldsymbol{s_z} \geqq \mathbf{0}, \end{aligned}$$

where $\boldsymbol{\omega} = (\omega, \cdots, \omega)$ and α is a given positive number. A $(p+m) \times n$ matrix $D_{\boldsymbol{z}} := (\boldsymbol{d}_1, \cdots, \boldsymbol{d}_n)$ is a matrix $(Z - Z_o)$ is replaced by 0, except for the maximal component (if there exist plural maximal components, only one is chosen from among them) in each row.

Especially, it is seen that when κ is fixed at 0, a problem (GDEA$_D$) becomes the dual problem to the problem (GDEA), since κ is dual variable to the second constraint in the problem (GDEA$'$).

We define an 'efficiency' for a DMUo in the GDEA$_D$ model.

Definition 4.1.4. (α_D-efficiency) For a given positive α, DMUo is said to be *α_D-efficient* if and ony if the optimal solution (ω^*, κ^*, $\boldsymbol{\lambda}^*$, $\boldsymbol{s}_z^*$) to the problem (GDEA$_D$) satisfies the following two conditions:

(i) ω^* is equal to zero;

(ii) the slack variable $\boldsymbol{s}_z^*$ is zero.

Otherwise, DMUo is said to be *α_D-inefficient.*

We, particularly, note that for an optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem GDEA$_D$, ω^* is not greater than zero because of the strong duality of (GDEA) and (GDEA$_D$) (in linear programming problem), and 'non-Archimedean' property of ε.

4.2 Relationships between GDEA$_D$ and DEA

In this section, we establish theoretical properties on relationships among efficiencies in basic DEA models and the GDEA$_D$ model.

Theorem 4.2.1. *Let κ be fixed at 0 in the problem ($GDEA_D$). DMUo is Pareto efficient in P_3' if and only if DMUo is α_D-efficient for some sufficiently small positive number α.*

Proof. (*only if part*) Assume that DMUo is Pareto efficient in P_3'. Then, there does not exist $\boldsymbol{\lambda}$ such that

$$\boldsymbol{z}_j = Z\boldsymbol{\lambda} \geq Z_o\boldsymbol{\lambda} = \boldsymbol{z}_o. \tag{4.1}$$

where $\boldsymbol{\lambda} \in \{\boldsymbol{\lambda} \mid \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \lambda_j \in \{0,1\},\ j = 1, \cdots, n\}$.

Negate that DMUo is α_D-efficient for some sufficiently small positive α. Then, for an optimal solution $(\omega^*, \boldsymbol{s}_z^*, \boldsymbol{\lambda}^*)$ to the problem (GDEA$_D$), $\omega^* < 0$ or $\boldsymbol{s}_z^* \geq 0$. In other words, for any sufficiently small positive α, the following inequality holds.

$$\{\alpha(Z_o - Z) + D_z\}\boldsymbol{\lambda}^* = \boldsymbol{\omega}^* - \boldsymbol{s}_z^* \leq \mathbf{0}. \tag{4.2}$$

The necessary condition that the inequality (4.2) holds for any sufficiently small positive α is that $D_z\boldsymbol{\lambda}^* \leq \mathbf{0}$, This implies $\boldsymbol{d}_j \leq 0$, for some j, since $\boldsymbol{\lambda}^* \geq 0$. Besides, from the definition of $\boldsymbol{d}_j$, we have $\boldsymbol{z}_o - \boldsymbol{z}_j \leq 0$. This is a contradiction to the inequality (4.1).

(*if part*) If DMUo is α_D-efficient for some sufficiently small positive α, then from the first constraint of the problem (GDEA$_D$), the following equality is obtained.

$$\{\alpha(Z_o - Z) + D_z\}\boldsymbol{\lambda}^* = \boldsymbol{\omega}^* - \boldsymbol{s}_z^* = \mathbf{0}, \tag{4.3}$$

where $(\omega^*, \boldsymbol{s}_z^*, \boldsymbol{\lambda}^*)$ is an optimal solution to the problem (GDEA$_D$).

Suppose that DMUo is not Pareto efficient in P_3'. Then there exists $\boldsymbol{z} \in P_3'$ such that $\boldsymbol{z} \geq \boldsymbol{z}_o$. this means that

$$\boldsymbol{z}_j = Z\hat{\boldsymbol{\lambda}} \geq Z_o\hat{\boldsymbol{\lambda}} = \boldsymbol{z}_o, \quad \hat{\boldsymbol{\lambda}} \in \{\mathbf{1}^T\hat{\boldsymbol{\lambda}} = 1, \ \hat{\lambda}_j \in \{0,1\}, \ j = 1, \cdots, n\}. \tag{4.4}$$

From the expression (4.4), $\boldsymbol{d}_j \leq 0$ for $j : \hat{\lambda}_j = 1$ and $D_{\boldsymbol{z}}\hat{\boldsymbol{\lambda}} \leq 0$. Multiplying the inequality of (4.4) by an arbitrary positive α and adding it to $D_{\boldsymbol{z}}\hat{\boldsymbol{\lambda}}$ yields

$$\{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\}\,\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\omega}} - \hat{\boldsymbol{s}}_{\boldsymbol{z}} \leq 0$$

and thus, $(\hat{\omega}, \hat{\boldsymbol{s}}_{\boldsymbol{z}}, \hat{\boldsymbol{\lambda}})$ is a feasible solution of the problem (GDEA$_D$). However, this contradicts the fact that the expression (4.3) holds for $(\omega^*, \boldsymbol{s}_{\boldsymbol{z}}^*, \boldsymbol{\lambda}^*)$. □

Theorem 4.2.2. *Let κ be fixed at 0 in the problem (GDEA$_D$). DMUo is Pareto efficient in P_2' if and only if DMUo is α_D-efficient for some sufficiently large positive number α.*

Proof. (*only if part*) Assume that DMUo is Pareto efficient in P_2'. Then, there does not exist $\boldsymbol{\lambda}$ such that

$$\boldsymbol{z} = Z\boldsymbol{\lambda} \geq Z_o\boldsymbol{\lambda} = \boldsymbol{z}_o, \quad \boldsymbol{\lambda} \in \left\{\boldsymbol{\lambda} \mid \mathbf{1}^T\boldsymbol{\lambda} = 1, \ \boldsymbol{\lambda} \geqq 0\right\}. \tag{4.5}$$

Negate that DMUo is α_D-efficient for some sufficiently large positive α. Then, from the α_D-efficiency od DMUo, it is seen that $\omega^* < 0$ or $\boldsymbol{s}_{\boldsymbol{z}}^* \geq 0$ for an optimal solution $(\omega^*, \boldsymbol{s}_{\boldsymbol{z}}^*, \boldsymbol{\lambda}^*)$ to the problem (GDEA$_D$). This means that for any sufficiently large positive α,

$$\{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\}\,\boldsymbol{\lambda}^* = \boldsymbol{\omega}^* - \boldsymbol{s}_{\boldsymbol{z}}^* \leq \mathbf{0}. \tag{4.6}$$

The necessary condition that the inequality (4.6) holds for any sufficiently large positive α is that

$$(Z_o - Z)\boldsymbol{\lambda}^* \leq 0, \;\; \boldsymbol{\lambda}^* \in \left\{\boldsymbol{\lambda} \mid \mathbf{1}^T\boldsymbol{\lambda} = 1, \;\; \boldsymbol{\lambda} \geqq 0\right\}.$$

This is a contradiction to the expression (4.5).

(*if part*) If DMUo is α_D-efficient for some sufficiently large positive α, then from the first constraint of the problem (GDEA$_D$), the following equality is obtained.

$$\{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\}\,\boldsymbol{\lambda}^* = \boldsymbol{\omega}^* - \boldsymbol{s}_{\boldsymbol{z}}^* = \mathbf{0}, \tag{4.7}$$

where $(\omega^*, \boldsymbol{s}_{\boldsymbol{z}}^*, \boldsymbol{\lambda}^*)$ is an optimal solution to the problem (GDEA$_D$).
Suppose that DMUo is not Pareto efficient in P_2'. Then there exists $\boldsymbol{z} \in P_2'$ such that $\boldsymbol{z} \geq \boldsymbol{z}_o$. This means that

$$\boldsymbol{z}_j = Z\hat{\boldsymbol{\lambda}} \geq Z_o\hat{\boldsymbol{\lambda}} = \boldsymbol{z}_o, \;\; \hat{\boldsymbol{\lambda}} \in \left\{\mathbf{1}^T\boldsymbol{\lambda} = 1, \;\; \boldsymbol{\lambda} \geqq 0\right\}. \tag{4.8}$$

Hence, for a sufficiently large positive α,

$$\{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\}\,\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\omega}} - \hat{\boldsymbol{s}}_{\boldsymbol{z}} \leq 0$$

and thus, $(\hat{\omega}, \hat{\boldsymbol{s}}_{\boldsymbol{z}}, \hat{\boldsymbol{\lambda}})$ is a feasible solution of the problem (GDEA$_D$). However, this contradicts the fact that the expression (4.7) holds for $(\omega^*, \boldsymbol{s}_{\boldsymbol{z}}^*, \boldsymbol{\lambda}^*)$. □

Theorem 4.2.3. *DMUo is Pareto efficient in P_1' if and only if DMUo is α_D-efficient for some sufficiently large positive number α.*

Proof. (*only if part*) Assume that DMUo is Pareto efficient in P_3'. Then, there does not exist $\boldsymbol{\lambda} \geqq 0$ such that

$$\boldsymbol{z} = Z\boldsymbol{\lambda} \geq Z_o\boldsymbol{\lambda} = \boldsymbol{z}_o. \tag{4.9}$$

Negate that DMUo is α_D-efficient for some sufficiently large positive α. Then, from the α_D-efficiency, it is seen that $\omega^* < 0$ or $\boldsymbol{s}_{\boldsymbol{z}}^* \geq 0$ for an optimal solution $(\omega^*, \kappa^*, \boldsymbol{s}_{\boldsymbol{z}}^*, \boldsymbol{\lambda}^*)$ to the problem (GDEA$_D$). This means that for any sufficiently large positive α,

$$\{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\} \boldsymbol{\lambda}^* + \kappa^* \boldsymbol{z}_o = \boldsymbol{\omega}^* - \boldsymbol{s}_{\boldsymbol{z}}^* \leq \mathbf{0}. \tag{4.10}$$

The necessary condition that the inequality (4.10) holds for any sufficiently large positive α is that

$$(Z_o - Z)\boldsymbol{\lambda}^* \leq 0, \ \ \boldsymbol{\lambda}^* \in \left\{\boldsymbol{\lambda} \mid \mathbf{1}^T\boldsymbol{\lambda} = 1, \ \ \boldsymbol{\lambda} \geqq 0\right\}.$$

This is a contradiction to the expression (4.9).

(*if part*) If DMUo is α_D-efficient for some sufficiently large positive α, then from the first constraint of the problem (GDEA$_D$), the following equality is obtained.

$$\{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\} \boldsymbol{\lambda}^* + \kappa^* \boldsymbol{z}_o = \boldsymbol{\omega}^* - \boldsymbol{s}_{\boldsymbol{z}}^* = \mathbf{0}. \tag{4.11}$$

where $(\omega^*, \kappa^*, \boldsymbol{s}_{\boldsymbol{z}}^*, \boldsymbol{\lambda}^*)$ is an optimal solution to the problem (GDEA$_D$).

Suppose that DMUo is not Pareto efficient in P_1'. Then there exists $\boldsymbol{z} \in P_1'$ such that $\boldsymbol{z} \geq \boldsymbol{z}_o$. This implies that there exists $\hat{\boldsymbol{\lambda}}$ such that

$$\boldsymbol{z}_o - Z\hat{\boldsymbol{\lambda}} \leq \mathbf{0}, \ \hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \cdots, \hat{\lambda}_n) \geqq 0. \tag{4.12}$$

Let $\overline{\boldsymbol{\lambda}} := \hat{\boldsymbol{\lambda}}/\sum_{j=1}^n \hat{\lambda}_j$. Then we have $\mathbf{1}^T\overline{\boldsymbol{\lambda}} = 1$ and $\overline{\boldsymbol{\lambda}} \geq 0$. (If $\overline{\boldsymbol{\lambda}} = \mathbf{0}$, then from the inequality (4.12), $\boldsymbol{z}_o \leq \mathbf{0}$ and this contradicts to the positiveness of inputs and outputs.) The inequality (4.12) becomes

$$\begin{aligned} 0 &\geq \frac{1}{\sum_{j=1}^{n}\hat{\lambda}_j}\boldsymbol{z}_o - Z\overline{\boldsymbol{\lambda}} = \boldsymbol{z}_o - Z\overline{\boldsymbol{\lambda}} + \left(\frac{1}{\sum_{j=1}^{n}\hat{\lambda}_j} - 1\right)\boldsymbol{z}_o \\ &= (Z_o - Z)\overline{\boldsymbol{\lambda}} + \left(\frac{1}{\sum_{j=1}^{n}\hat{\lambda}_j} - 1\right)\boldsymbol{z}_o. \end{aligned}$$

and by multiplying the inequality (4.13) by a sufficiently large positive α, which renders the inequality sign of it to remain though adding $D_z\overline{\boldsymbol{\lambda}}$ to the right hand side of it, we have

$$\begin{aligned} 0 &\geq \{\alpha(Z_o - Z) + D_z\}\overline{\boldsymbol{\lambda}} + \alpha\left(\frac{1}{\sum_{j=1}^{n}\hat{\lambda}_j} - 1\right)\boldsymbol{z}_o \\ &= \overline{\boldsymbol{\omega}} - \overline{\boldsymbol{s}}_z. \end{aligned}$$

Define $\overline{\kappa}$ by $\overline{\kappa} := \alpha\left(1/\sum_{j=1}^{n}\hat{\lambda}_j - 1\right)$, thus, $(\overline{\omega}, \overline{\kappa}, \overline{\boldsymbol{s}}_z, \overline{\boldsymbol{\lambda}})$ is a feasible solution of the problem (GDEA$_D$). However, this contradicts the fact that the expression (4.11) holds for $(\omega^*, \kappa^*, \boldsymbol{s}_z^*, \boldsymbol{\lambda}^*)$. □

4.3 Optimal Solutions to (GDEA$_D$)

In this section, we explain the meaning of optimal solutions ω^*, $\boldsymbol{\lambda}^*$, $\boldsymbol{s}_z^*$ to the problem (GDEA$_D$).

ω^* gives a measurement of relative efficiency for DMUo. In other words, it represents the degree how inefficient DMUo is, that is, how far DMUo is from the efficient frontier generated with the given α. $\boldsymbol{\lambda}^* := (\lambda_1^*, \cdots, \lambda_n^*)$ represents a domination relation between DMUo and another DMUs. That is, it means that the DMUo is dominated by DMUj if λ_j for some $j \neq 0$ is positive. $\boldsymbol{s}_x^*$ represents the slack of inputs and $\boldsymbol{s}_y^*$ does the surplus of outputs for performance of the DMUo.

Consider an illustrative example as shown in Table 4.1. Table 4.2 shows the results of the CCR-efficiency, BCC-efficiency and FDH-efficiency, respectively, in the example.

Table 4.1: An Example of 1-input and 1-output.

DMU	A	B	C	D	E	F	G
input	2	3	8	6	5	10	7
output	1	3	6	2	4	6	4

Table 4.2: Optimal value in the problems (CCR), (BCC) and (FDH).

DMU	CCR model	BCC model	FDH model
A	0.5	1	1
B	1	1	1
C	0.75	1	1
D	0.333	0.417	0.5
E	0.8	0.933	1
F	0.6	-2×10^{-6}	0.8
G	0.571	0.667	0.714

Table 4.3 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_D$) ($\varepsilon = 10^{-6}$) when α is given as 10^{-6} and κ is fixed at 0. Table 4.4 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_D$) ($\varepsilon = 10^{-6}$) when α is given by 10 and κ is fixed at 0. Finally, Table 4.5 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_D$) ($\varepsilon = 10^{-6}$) when α is given as 10.

Here, we can see that the FDH-efficiency, BCC-efficiency and CCR-efficiency are equivalent to the α-efficiency, respectively, from the result of Tables 4.3-

Tables 4.5 and Figure 4.1-Figure 4.3. In other words, the FDH-efficiency, BCC-efficiency and CCR-efficiency can be obtained by changing the paremter α in the GDEA$_D$ model.

Now, we interpret a meaning of optimal solutions $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_D$). Note that ω^* gives a measurement of relative efficiency for DMUo. In other words, it represents the degree how inefficient DMUo is, that is, how far DMUo is from the efficient frontier generated with the given α.

$\boldsymbol{\lambda}^* := (\lambda_1^*, \cdots, \lambda_n^*)$ represents a domination relation between DMUo and another DMUs. That is, it means that the DMUo is dominated by DMUj if λ_j for some $j \neq o$ is positive. For example, as seen in Table 4.3, the optimal solution for the DMU D is $\lambda_B^* = 0.5$ and $\lambda_E^* = 0.5$, and hence DMU D is dominated by DMU B and DMU E. (See Figure 4.1.) In addition, in Table 4.4, the optimal solution for the DMU E is $\lambda_B^* = 0.631$ and $\lambda_C^* = 0.369$, and hence DMU E is dominated by linear combination of DMU B and DMU C. (See Figure 4.2.) As seen in Table 4.5, the optimal solution for the DMU C is $\lambda_B^* = 1$, and hence DMU D is dominated by a point on line through DMU B and original point. (See Figure 4.3.)

$\boldsymbol{s}_x^*$ represents the slack of inputs and $\boldsymbol{s}_y^*$ does the surplus of outputs for performance of the DMUo. For instance, DMU G has the optimal solution $\omega^* = 0$, $\lambda_E^* = 1$ and $\boldsymbol{s}_x^* = 0$, and it is α-inefficient because $\boldsymbol{s}_x^*$ is not equal to zero although $\omega^* = 0$. It implies that DMU G has the suplus amount of input than DMU E with the same output.

Table 4.3: Optimal solution to (GDEA$_D$) with $\alpha = 10^{-6}$ and fixed $\kappa = 0$.

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$
A	0	$\lambda_A^* = 1$	(0, 0)
B	0	$\lambda_B^* = 1$	(0, 0)
C	0	$\lambda_C^* = 1$	(0, 0)
D	-0.5	$\lambda_B^* = \lambda_E^* = 0.5$	(0, 0)
E	0	$\lambda_E^* = 1$	(0, 0)
F	0	$\lambda_C^* = 1$	(2, 0)
G	0	$\lambda_E^* = 1$	(2, 0)

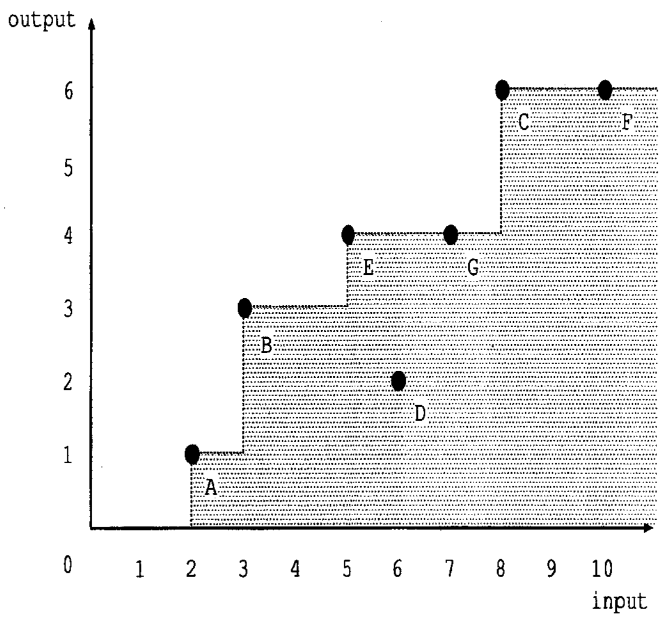


Figure 4.1: Efficient frontier generated by GDEA$_D$ model with $\alpha = 10^{-6}$ and fixed $\kappa = 0$.

Table 4.4: Optimal solution to (GDEA$_D$) with $\alpha = 10$ and fixed $\kappa = 0$.

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$
A	0	$\lambda_A^* = 1$	(0, 0)
B	0	$\lambda_B^* = 1$	(0, 0)
C	0	$\lambda_C^* = 1$	(0, 0)
D	-7.803	$\lambda_B^* = 0.765,\ \lambda_C^* = 0.235$	(0, 0)
E	-0.441	$\lambda_B^* = 0.631,\ \lambda_C^* = 0.369$	(0, 0)
F	0	$\lambda_C^* = 1$	(20, 0)
G	-8.281	$\lambda_B^* = 0.378, \lambda_C^* = 0.622$	(0, 0)

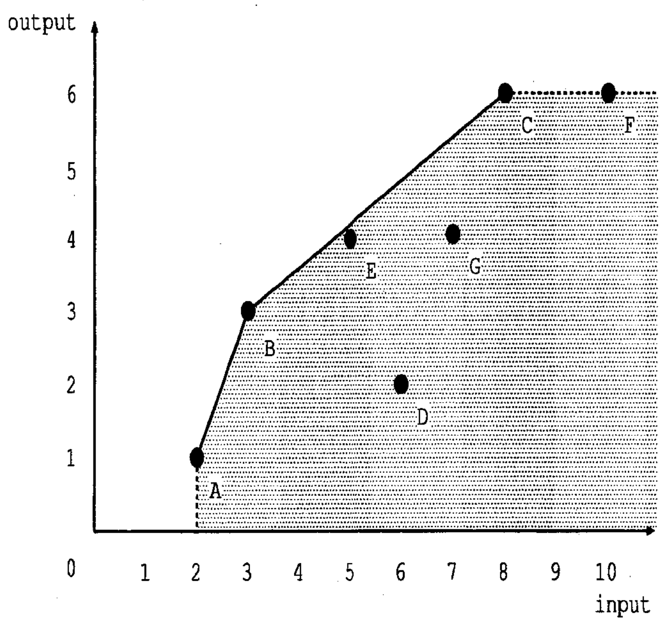


Figure 4.2: Efficient frontier generated by GDEA$_D$ model with $\alpha = 10$ and fixed $\kappa = 0$.

Table 4.5: Optimal solution to (GDEA_D) with $\alpha = 10$ and non-fixed κ.

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$	κ^*
A	-11.333	$\lambda_C^* = 1$	(0, 0)	38.667
B	0	$\lambda_B^* = 1$	(0, 0)	0
C	-2.571	$\lambda_B^* = 1$	(0, 0)	-5.929
D	-24.500	$\lambda_C^* = 1$	(0, 0)	7.750
E	-2.778	$\lambda_B^* = 1$	(0, 0)	-3.444
F	-7.500	$\lambda_C^* = 1$	(0, 0)	-1.250
G	-8.727	$\lambda_C^* = 1$	(0, 0)	2.818

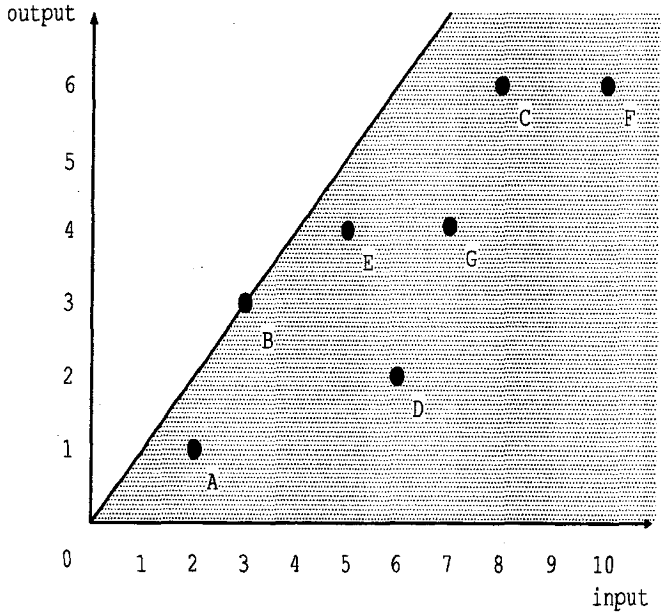


Figure 4.3: Efficient frontier generated by GDEA_D model with $\alpha = 10$ and non-fixed κ.

Chapter 5

Comparision between GDEA and DEA Models with Real Data

In this chapter, we compare the efficiency in basic DEA models and GDEA model with a real data.

The data for thirteen Mexican commercial banks in two years (1990-1991) is from Taylor *et al.* [30]. As shown in Table 5.1, each bank has total income as the single output. Total income is the sum of a bank's interest and non-interest income. Total deposits and total non-interest expense are the two inputs used to generate the output. Interest income includes interest earned from loan activities. Total non-interest income includes dividends, fees, and other non-interest revenue. The total deposits input variable includes the bank's interest paying deposit liabilities. Total non-interest expense includes personnel and administrative costs, commissions paid, banking support fund contributions and other non-interest operating costs.

Thus, we evaluate the efficiency for each bank with the annual data, that is, consider α-efficiency corresponding to several values $\alpha = 0.1,\ 0.5,\ 1,\ 10,\ 15$ (only 1991) and 10^3. Therefore, Table 5.2 and Table 5.3 represent the results of analyses by the basic DEA models and the GDEA model. (In this example, FDH-efficiency is not mentioned, since it can be evaluated by vector comparision.)

As shown there, GDEA model with $\alpha = 0.1$ provides FDH efficiency. It means that there is no change in α-efficient DMUs for smaller α than 0.1. In addition, GDEA model with $\alpha = 10$ yields BCC efficiency in Table 5.3, while $\alpha = 15$ does in Table 5.4. Also, there is no change in α-efficiency of DMUs, even if taking greater α than 10 or 15. Moreover, CCR-efficiency can be conducted by taking α sufficiently large in the GDEA model adding the constraint $\boldsymbol{x}_o^T \nu = \boldsymbol{y}_o^T \mu$. From this fact, we see that the number of efficient DMUs decreases as a parameter α increases in general.

Particularly, note the α-efficiency for $\alpha = 0.5$ and $\alpha = 1$: This represents an intermediate efficiency between FDH-efficiency and BCC-efficiency. Among decision making problems, there exist the cases that it is impossible to correspond to a special value judgement of decision makers such as "ratio value efficiency[1]" in the CCR model, "sum value efficiency[2]" in the BCC model, and so on. In contrast to the existing DEA models, GDEA model can incorporate his/her various value judgement by changing a parameter α, and then several kinds of efficiency of the basic DEA models can be measured in a unified way on the basis of the GDEA model. Furthermore, the relationships among efficiency for these models become transparent by considering GDEA.

[1] We named the CCR-efficiency *ratio value efficiency*, because the ratio of the weighted sum of outputs to the weighted sum of inputs is maximized by the CCR model. See [33].

[2] We named the BCC-efficiency *sum value efficiency*, because the difference of the weighted sum of outputs and the weighted sum of inputs is maximized by the BCC model. See [33].

Table 5.1: Input and output values for 13 Mexican banks, 1990–1991 (billions of nominal pesos)

Bank	1990			1991		
	deposits	non-int. expense	int. income plus non-int. income	deposits	non-int. expense	int. income plus non-int. income
(1) Banamex	35313.90	2500.88	14247.10	57510.90	3670.33	15764.60
(2) Bancomer	34504.60	2994.70	12682.10	59965.00	3872.40	15877.00
(3) Serfin	30558.20	1746.50	11766.40	46987.20	2709.20	12694.10
(4)Intermac	7603.53	1011.40	3422.40	13458.00	1165.20	4212.20
(5) Cremi	1977.18	1628.80	2889.10	5108.97	760.60	2102.70
(6) Bancreser	2405.00	140.70	1050.50	3314.32	190.80	1681.10
(7) MercNort	2146.06	338.30	1320.10	3714.72	463.30	1377.40
(8) BCH	2944.00	260.8	1410.00	3728.00	402.90	1794.10
(9) Confia	1962.34	266.60	1568.00	3324.43	364.90	1944.40
(10) Bancen	1815.73	196.70	946.20	2544.96	242.70	848.80
(11) Promex	1908.23	251.30	1162.80	3080.00	320.40	1251.40
(12) Banoro	1372.78	169.60	598.20	2799.00	224.40	810.50
(13) Banorie	488.17	71.90	340.80	680.88	86.80	373.00

Table 5.2: DEA Mexican bank analysis, 13 banks, 1990. Output is total interest and non-interest income; inputs are total deposits and non-interest expense

Bank	1990								
	CCR		BCC		GDEA				
	θ	class	θ	RTS	$\alpha = 10^3$ $(\boldsymbol{x}_o^T\nu = \boldsymbol{y}_o^T\mu)$	$\alpha = 10$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.1$
(1) Banamex	0.816	NE	1.000	D	-123.46	0.00	0.00	0.00	0.00
(2) Bancomer	0.646	NE	0.890	–	-744.67	-7282.88	-358.41	0.00	0.00
(3) Serfin	0.902	NE	1.000	D	-11.88	0.00	0.00	0.00	0.00
(4)Intermac	0.573	NE	0.809	–	-285.50	-1648.99	0.00	0.00	0.00
(5) Cremi	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00
(6) Bancreser	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00
(7) MercNort	0.750	NE	0.757	–	-126.73	-1078.91	-149.92	-102.55	-19.69
(8) BCH	0.829	NE	0.837	–	-70.89	-390.60	-11.27	-0.08	0.00
(9) Confia	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00
(10) Bancen	0.778	NE	0.803	–	-94.29	-390.09	-8.06	0.00	0.00
(11) Promex	0.782	NE	0.797	–	-79.50	-506.79	-29.08	-6.76	0.00
(12) Banoro	0.588	NE	0.644	–	-299.20	-606.52	-12.81	0.00	0.00
(13) Banorie	0.862	NE	1.000	I	-58.55	0.00	0.00	0.00	0.00

E: Efficient
NE: Not Efficient
D: Decreasing Returns to Scale(RTS)
C: Constant Returns to Scale
I: Increasing Returns to Scale

Table 5.3: DEA Mexican bank analysis, 13 banks, 1991. Output is total interest and non-interest income; inputs are total deposits and non-interest expense

Bank	1991									
	CCR		BCC		GDEA					
	θ	class	θ	RTS	$\alpha = 10^3$ $(\boldsymbol{x}_o^T \nu = \boldsymbol{y}_o^T \mu)$	$\alpha = 15$	$\alpha = 10$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.1$
(1) Banamex	0.531	NE	1.000	D	-181.32	0.00	0.00	0.00	0.00	0.00
(2) Bancomer	0.511	NE	1.000	D	-281.95	0.00	0.00	0.00	0.00	0.00
(3) Serfin	0.532	NE	1.000	D	-136.52	0.00	0.00	0.00	0.00	0.00
(4)Intermac	0.569	NE	0.908	–	-257.11	-717.26	0.00	0.00	0.00	0.00
(5) Cremi	0.704	NE	0.772	–	-282.58	-3134.25	-1957.76	0.00	0.00	0.00
(6) Bancreser	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00	0.00
(7) MercNort	0.634	NE	0.638	–	-284.80	-4371.50	-2999.54	-385.14	-212.60	-42.31
(8) BCH	0.826	NE	0.828	–	-112.88	-1481.79	-982.50	-99.34	-60.03	-15.61
(9) Confia	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00	0.00
(10) Bancen	0.592	NE	0.612	–	-253.70	-1621.77	-1075.07	-50.54	0.00	0.00
(11) Promex	0.705	NE	0.715	–	-191.64	-2262.34	-1504.08	-74.49	0.00	0.00
(12) Banoro	0.535	NE	0.554	–	-295.19	-1410.08	-934.00	-80.67	-5.37	0.00
(13) Banorie	0.937	NE	1.000	I	-73.42	0.00	0.00	0.00	0.00	0.00

E: Efficient
D: Decreasing Returns to Scale(RTS)
I: Increasing Returns to Scale
NE: Not Efficient
C: Constant Returns to Scale

Chapter 6

Application of GDEA to Multi-objective Optimization

In multi-objective optimization problems, there does not necessarily exist the solution that optimizes all objective functions simultaneously, and then the concept which is called Pareto optimal solution (or efficient solution) is introduced [23]. Usually, there exist a number of Pareto optimal solutions, which are considered as candidates of final decision making solution [19]. It is an issue how decision makers decide one from the set of Pareto optimal solutions as the final solution. Consequently, interactive multi-objective optimization methods have been developed to this end. In many practical problems such as engineering design problems, however, criteria functions can not be given explicitly in terms of design variables. Under this circumstance, values of criteria functions for given value of design variables are usually obtained by some analyses such as structural analysis, thermodynamical analysis or fluid mechanical analysis. These analyses require considerably much computation time. Therefore, it is not unrealistic to apply existing interactive optimization methods to those problems.

Recently, multi-objective optimization methods using genetic algorithm (GA) have been studied actively by many authors [2, 7, 14, 16, 26, 27]. GAs are useful for generating efficient frontiers with two or three objective functions. Decision making can be easily performed on the basis of visualized efficient frontiers. However, these methods have the following shortcomings; there is a tendency for VEGA (Vector Evaluated Genetic Algorithms) [24] to generate such solutions that one of the objective functions is extremely good. It is difficult to generate smooth efficient frontier by ranking methods [14, 16]. Moreover, many non-dominated individuals generated at intermediate generation in these methods are not necessarily exact Pareto optimal solutions. By the method using DEA, which was proposed by Arakawa *et al.* [2], almost of all non-dominated individuals obtained at intermediate generations become Pareto optimal solutions. However, the DEA method cannot produce the sunken part of efficient frontier, because CCR model [8] or BCC model [6] is used of DEA there.

In this chapter, we employ GDEA as the fitness of GA [36, 37] in order to generate Pareto optimal solutions in multi-objective optimization problems. Consequently, we prove that GDEA can remove dominated individuals faster than methods based on only GA, and overcomes the shortcomings of existing methods. The proposed method can yield desirable efficient frontiers even in non-convex problems as well as convex problems. Finally, the effectiveness of the proposed method will be shown through several numerical examples.

6.1 Multi-Objective Optimization Methods Using GAs

Consider a multi-objective optimization problem (MOP):

$$\text{(MOP)} \qquad \underset{\boldsymbol{x}}{\text{minimize}} \quad \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \cdots, f_m(\boldsymbol{x}))^T$$
$$\text{subject to} \quad \boldsymbol{x} \in S = \{ \, \boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \leqq 0, \; j = 1, \cdots, l \, \},$$

where $\boldsymbol{x}$ $(= (x_1, \cdots, x_n)^T)$ is a design variable and S is the set of all feasible solutions.

In general, unlike traditional optimization problems with a single objective function, an optimal solution which minimizes all objective functions $f_i(\boldsymbol{x})$ $(i = 1, \cdots, m)$ simultaneously does not necessarily exist in the problem (MOP). Hence, the concept of Pareto optimal solution is introduced as follows [23]:

Definition 6.1.1. (Pareto optimal solution) A point $\hat{\boldsymbol{x}} \in S$ is said to be a Pareto optimal solution to the problem (MOP) if there exists no $x \in S$ such that $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\hat{\boldsymbol{x}})$.

A final decision making solution to the problem (MOP) may be found from the set of Pareto optimal solutions by existing methods, for example, aspiration level techniques [23], if the value of objective function can be obtained easily. In cases in which it takes much computation time to evaluate objective functions, however, interactive methods become unsuitable due to the time limitation in decision making. In the problem (MOP) with two or three objective functions, under this circumstance, figuring out efficient frontiers helps decision makers decide the final solution.

In order to generate efficient frontiers, Schaffer [26] proposed the vector evaluated genetic algorithms (VEGA), in which sub-populations of the next generation are reproduced from the current population according to each of the

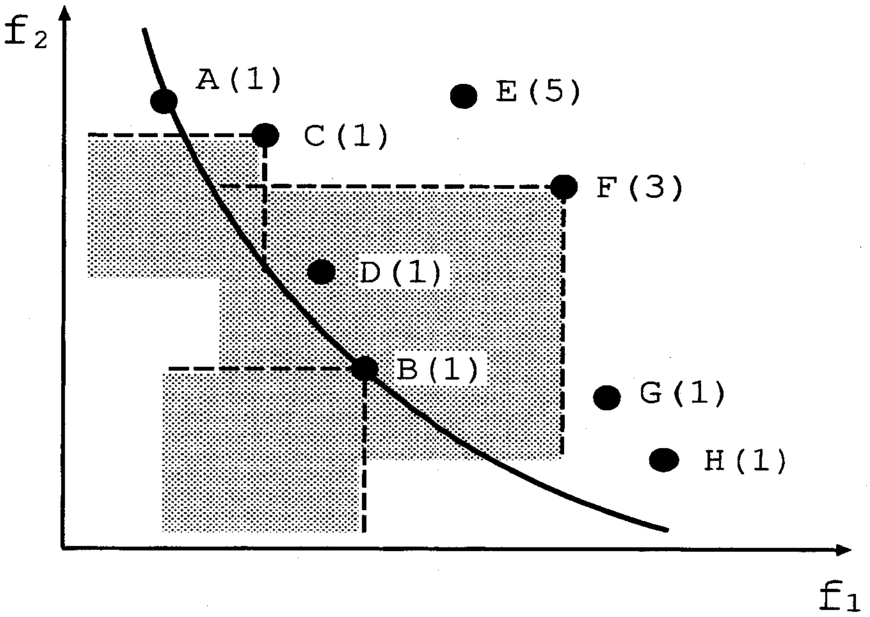


Figure 6.1: Ranking method

objective functions, separately. Since then, several approaches have been studied by many authors [2, 14, 16, 29]. To begin with, we give a brief explanation on the ranking method given by Fonseca *et al.* [14]. Consider an individual $\boldsymbol{x}^o$ at a generation which is dominated by n individuals in the current population. Then its rank is given by $(1 + n)$: From this, we can see that all non-dominated individuals are assigned rank 1. In Figure 6.1, each number in parentheses represents the rank of each individual and the curve represents the exact efficient frontier. The ranking method based on the Pareto domination among individuals has a merit to be computationally simple. However, the ranking method has a shortcoming to need to repeat GA process until a large number of generations, since non-dominated individuals in the current generation such as C and G in Figure 6.1 are often kept alive long, even though they are not Pareto optimal solutions in the final generation.

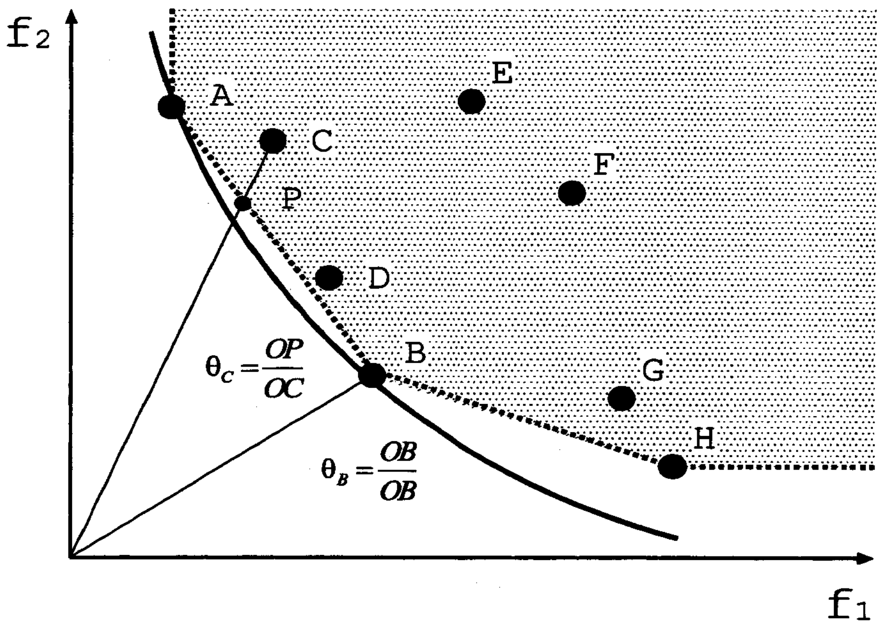


Figure 6.2: GA with DEA method

Arakawa *et al.* [2] suggested a method using DEA in order to overcome the shortcomings of the methods stated above. In the method using DEA, the efficiency θ of an individual $\boldsymbol{x}^o$ $(o = 1, \cdots, p)$ is given by solving the following linear programming problem:

$$\begin{aligned} \underset{\theta,\boldsymbol{\lambda}}{\text{minimize}} \quad & \theta \\ \text{subject to} \quad & \left[\boldsymbol{f}(\boldsymbol{x}^1), \cdots, \boldsymbol{f}(\boldsymbol{x}^p)\right] \boldsymbol{\lambda} - \theta \boldsymbol{f}(\boldsymbol{x}^o) \leqq 0, \\ & \boldsymbol{\lambda} \geqq 0, \;\; \boldsymbol{\lambda} \in \mathbb{R}^p. \end{aligned}$$

The degree of efficiency θ represents how far $\boldsymbol{f}(\boldsymbol{x}^o)$ is from the DEA-efficient frontier. And only when θ is equal to one, $\boldsymbol{f}(\boldsymbol{x}^o)$ is located on the DEA-efficient frontier. Selection in GA is performed by taking the degree of efficiency θ for fitness. In other words, this method investigates the relation of domination among individuals with respect to the shaded region in Figure 6.2. In this figure, the solid curve represents the exact efficient frontier and the dotted line represents DEA-efficient frontier at a generation. As shown in there, individuals C and G are removed fast, and then a good approximation of the exact efficient frontier can be obtained efficiently. Therefore, when the efficient frontier is convex [1], non-Pareto solutions can be removed at a young generation. However, when the efficient frontier is non-convex, the sunken part of it can not be generated by the method using DEA.

6.2 Multi-Objective Optimization Combining GDEA and GA

In this section, we propose a method combining GDEA and GA to overcome the shortcomings of the ranking methods and the method using DEA. In applying GA to problems with constraints, we introduce an augmented objective function using penalty functions imposed on constraints. Here, an augmented objective function of f_i $(i = 1, \cdots, m)$ in the problem (MOP) is given by

$$F_i(\boldsymbol{x}) = f_i(\boldsymbol{x}) + \sum_{j=1}^{l} p_j \times \left\{[g_j(\boldsymbol{x})]_+\right\}^a,$$

where p_j is a penalty coefficient, a is a penalty exponent and $[y]_+ = \max\{y,\ 0\}$.

Thus, the initial problem (MOP) can be converted into a problem to minimize the augmented objective function $(F_1(\boldsymbol{x}), \cdots, F_m(\boldsymbol{x}))$. Here, we need to prepare the data set in order to evaluate the α-efficiency of an individual $\boldsymbol{x}^o$ in the current population. Let inputs and outputs in GDEA model be substituted

[1]Let E be an efficient frontier set in $\mathbb{R}^n$ and let $\mathbb{R}^n_+$ be the non-negative orthant in the objective space. Then we say *the efficient frontier to be convex* if $(E + \mathbb{R}^n_+)$ is a convex set. Otherwise, the efficient frontier is non-convex.

by a value $F_i(\boldsymbol{x}^o)$ and the unit, respectively. Then the problem (GDEA) reduces to the following problem (P).

$$
\begin{aligned}
\text{(P)} \qquad & \underset{\Delta,\nu}{\text{maximize}} \quad \Delta \\
& \text{subject to} \quad \Delta \leqq \tilde{d}_j - \alpha \sum_{i=1}^{m} \nu_i (F_i(\boldsymbol{x}^o) - F_i(\boldsymbol{x}^j)), \;\; j = 1, \cdots, p, \\
& \qquad\qquad\quad \sum_{i=1}^{m} \nu_i = 1, \\
& \qquad\qquad\quad \nu_i \geqq \varepsilon, \;\; i = 1, \cdots, m,
\end{aligned}
$$

where $\tilde{d}_j = \max_{i=1,\cdots,m} \{\nu_i \left(-F_i(\boldsymbol{x}^o) + F_i(\boldsymbol{x}^j)\right)\}$ and α is the value of a monotonically decreasing function with respect to the number of generations.

Practically, α is given by

$$\alpha(t) := \omega \cdot \exp(-\beta \cdot t), \;\; t = 0, 1, \cdots, N,$$

where ω, β and N are positive fixed numbers. ω $(= \alpha(0))$ is determined to be sufficiently large as $10, 10^2$ and 10^3. N (the number of generations until the termination of computation) is given by the time limitation for decision making. For given ω and N, β is decided by solving the equation $\alpha(N) = \omega \cdot \exp(-\beta \cdot N) \fallingdotseq 0$.

The degree of α-efficiency of an individual $\boldsymbol{x}^o$ in the current population is given by the optimal value Δ^* to the problem (P), and is considered as the fitness in GA. Therefore, the selection of an individual is determined by the degree of α-efficiency, i.e. if Δ^* equals to zero, the individual remains at the next generation. With making the best use of the properties of GDEA, it is possible to keep merits of ranking methods and the method using DEA, and at the same

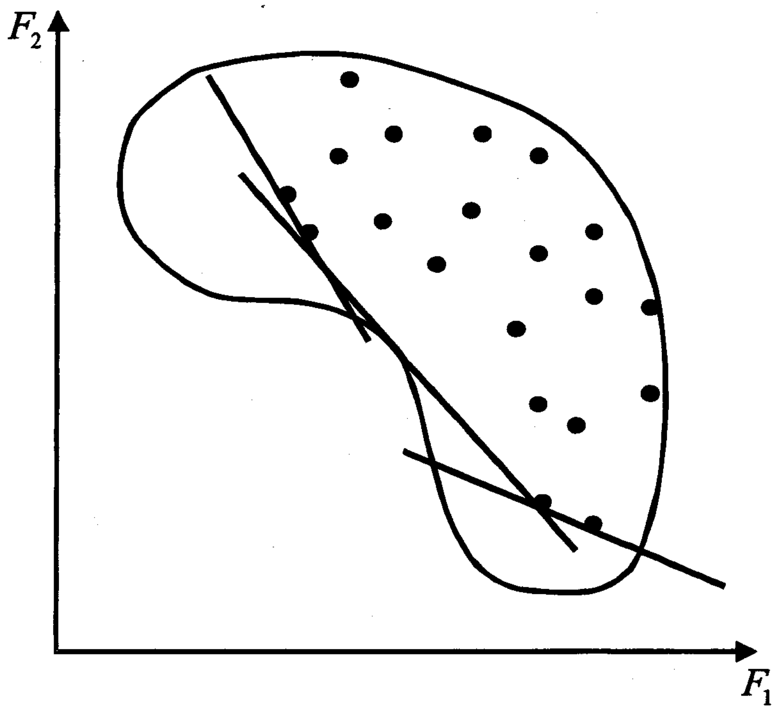


Figure 6.3: When α is a sufficiently large positive number

time, to overcome the shortcomings of existing methods. Namely, taking a large α can remove individuals which are located far from the efficient frontier, and taking a small α can generate non-convex efficient frontiers. (See Figure 6.3 and 6.4.)

Finally, the proposed method is summarized as follows:

Step 1. (Initialization)

Generate p-individuals randomly. Here, the number of p is given a prior.

Step 2. (Crossover · Mutation)

Make $p/2$-pairs randomly among the population. Making crossover each pair generates a new population. Mutate them according to the given

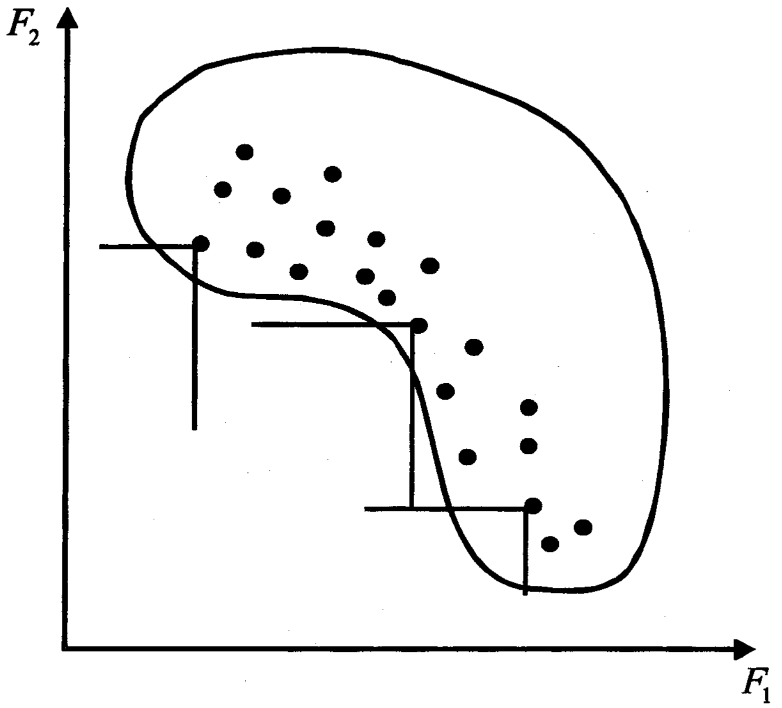


Figure 6.4: When α is a sufficiently small positive number

probability of mutation, if necessary.

Step 3. (Evaluation of Fitness by GDEA)

Evaluate the GDEA-efficiency by solving the problem (P)

Step 4. (Selection)

Select p-individuals from current population on the basis of the fitness given by GDEA-efficiency.

The process Step 2-Step 4 is continued until the number of generations attains a given number.

6.3 Illustrative Examples

We consider the following examples with two objective functions.

Example 1

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})) = (x_1,\ x_2) \\ \text{subject to} \quad & (x_1 - 2)^2 + (x_2 - 2)^2 - 4 \leqq 0, \\ & x_1 \geqq 0,\ \ x_2 \geqq 0. \end{aligned}$$

Example 2

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})) = (2x_1 - x_2,\ \ -x_1) \\ \text{subject to} \quad & (x_1 - 1)^3 + x_2 \leqq 0, \\ & x_1 \geqq 0,\ \ x_2 \geqq 0. \end{aligned}$$

Example 3

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})) = (x_1,\ x_2) \\ \text{subject to} \quad & x_1^3 - 3x_1 - x_2 \leqq 0, \\ & x_1 \geqq -1,\ \ x_2 \leqq 2. \end{aligned}$$

The efficient frontier in Example 1 is convex, and non-convex in both Example 2 and Example 3. In order to show the effectiveness of GDEA method, we compare the results by (a) ranking method, (b) DEA method and (c) GDEA method. Parameters in GA and the problem (P) are set as follows:

(i) the number of generations : 10, 20, 30 (examples 1–3)

(ii) the size of population : 80,

(iii) the representation of chromosome : 10 bits

(iv) the probability of crossover : 1,

(v) the probability of mutation : 0.05

(vi) $\alpha(t) =$

$$\begin{cases} 10 \times \exp(-0.6 \times t), & t = 0, \cdots, 10 \\ & \text{(when the maximal number of generation is 10)} \\ 10 \times \exp(-0.3 \times t), & t = 0, \cdots, 20 \\ & \text{(when the maximal number of generation is 20)} \\ 10 \times \exp(-0.2 \times t), & t = 0, \cdots, 30 \\ & \text{(when the maximal number of generation is 30)} \end{cases}$$

(vii) $\varepsilon = 10^{-6}$.

The elitist preserving selection [16] is adopted. The results are shown in Figure 6.5-6.7. The horizontal axis and the vertical axis indicate the values of objective functions f_1 and f_2, respectively. The symbol • represents a Pareto optimal solution among the whole generations, and ◦ does a non-dominated individuals at some generation but not Pareto optimal among the whole generations. Note here that non-dominated individual depends on the domination structure of each method: For example, individuals with rank 1 are non-dominated in ranking method, the ones with $\theta^* = 1$ are non-dominated in DEA method. In GDEA method, non-dominated individuals are identical with α-efficient ones.

(a) Ranking method

We obtained relatively many Pareto optimal solutions. However, there are also many non-Pareto optimal solutions among non-dominated individuals at each generation. Moreover, it is usually difficult to generate smooth efficient frontiers as shown in (a) of Figure 6.5-6.7.

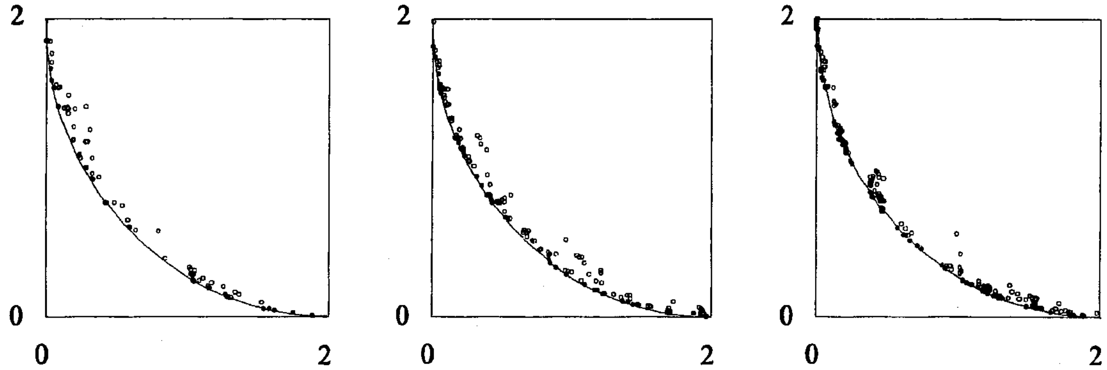
(b) DEA method

Many non-dominated individuals at each generation become finally Pareto optimal among the whole generation in (a) of Figure 6.5, nevertheless the obtained Pareto optimal solutions are fewer than by the ranking methods. On the other hand, for non-convex efficient frontiers in (b) of Figure 6.6 and Figure 6.7, the sunken part of it can not be generated by this method. Therefore, DEA method cannot be applied to multi-objective optimization problems with non-convex functions.

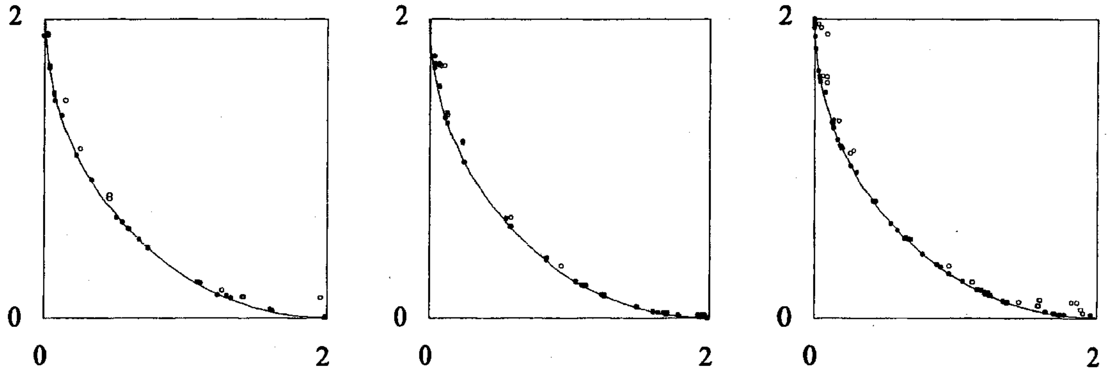
(c) GDEA method

In (c) of Figure 6.5-6.7, the largest number of Pareto optimal solutions are obtained among the stated methods. Moreover, efficient frontiers generated by the proposed method are smooth, even if they are non-convex. In addition, it is seen that almost all of non-dominated individuals at each generation become the final Pareto optimal solutions.

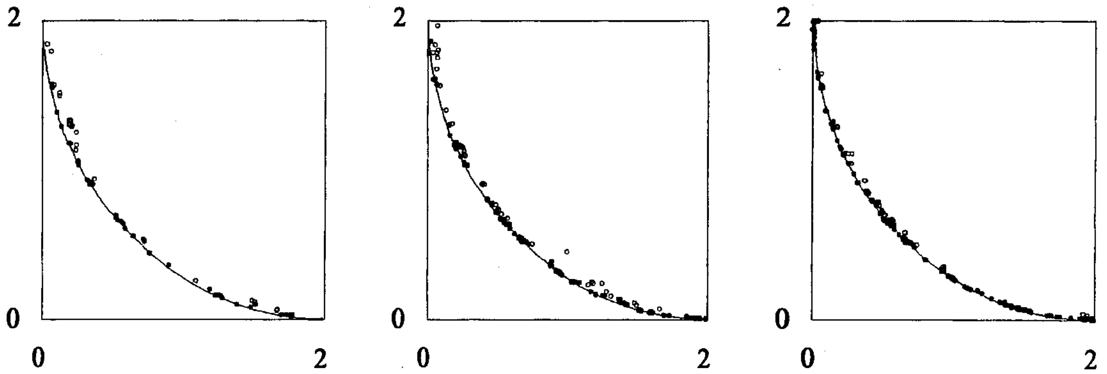
In particular, it should be noted in the ranking method that non-dominated individuals obtained at intermediate generations are often not Pareto optimal solutions. In practical problems, we do not know when to stop the computation in advance. Usually, the computation is terminated at a relatively early generation due to the time limitation. It is an important requirement, therefore, that non-dominated individuals at intermediate generations are finally Pareto optimal solutions. GDEA method has a desirable performance from this point of view.



(a) Ranking method (Fonseca *et al.*)

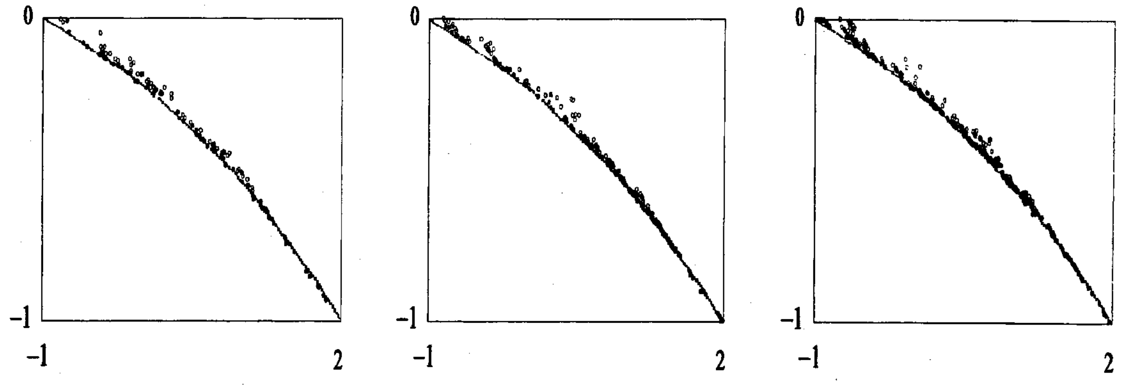


(b) DEA method

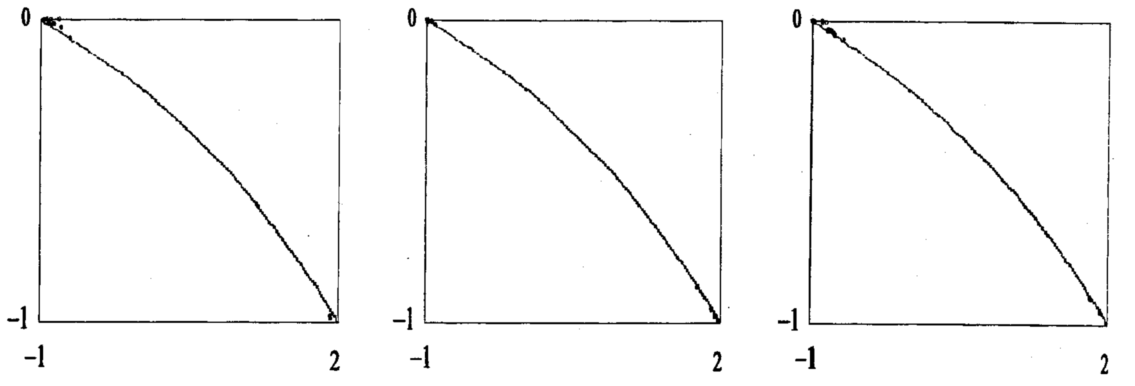


(c) GDEA method

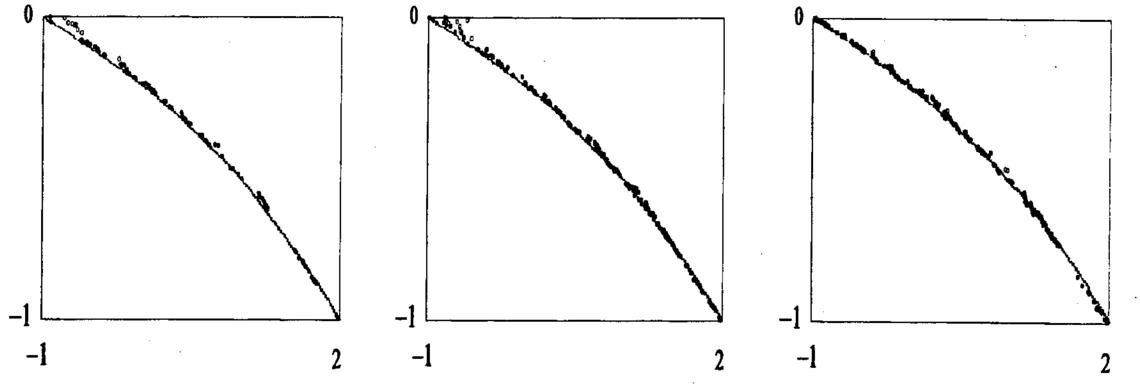
Figure 6.5: Results to the Example 1.
(from left to right, 10, 20, 30 generations, respectively)



(a) Ranking method (Fonseca *et al.*)

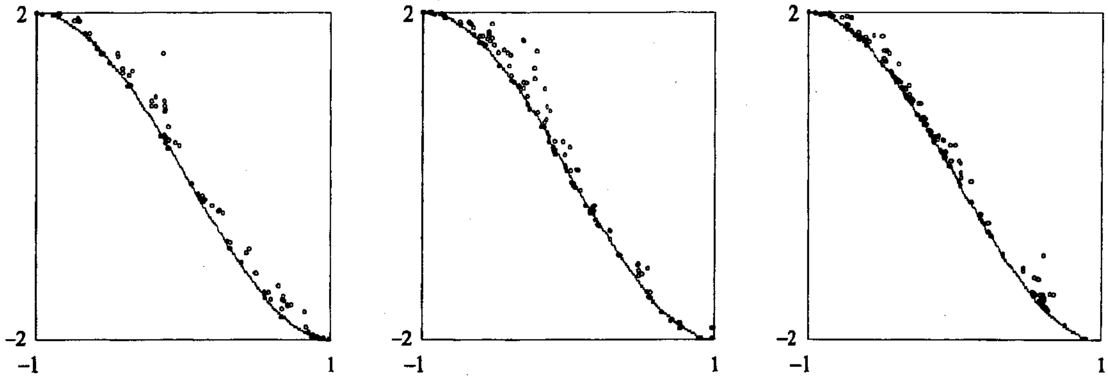


(b) DEA method

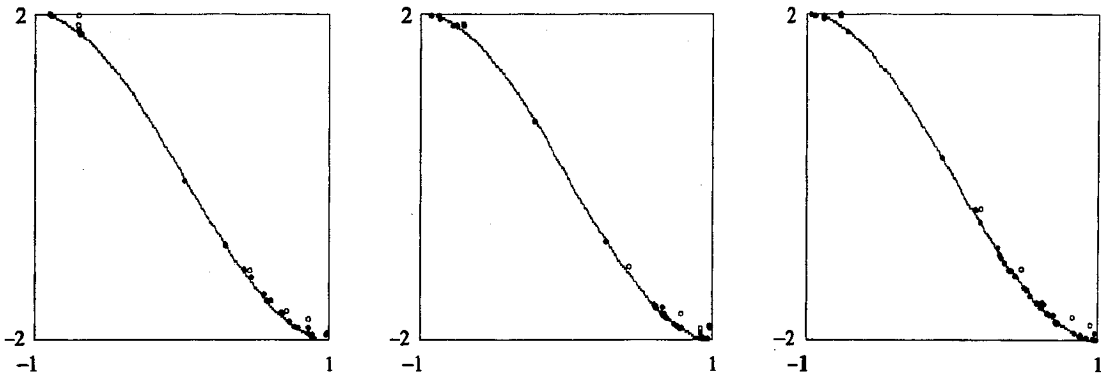


(c) GDEA method

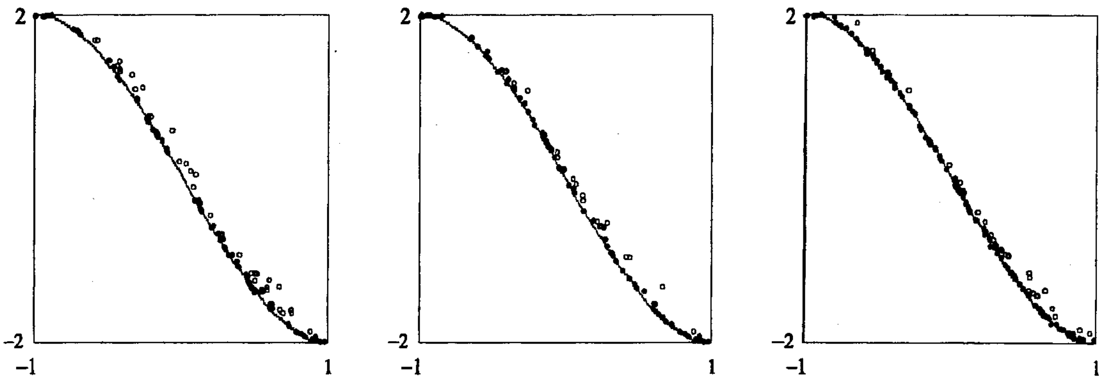
Figure 6.6: Results to the Example 2.
(from left to right, 10, 20, 30 generations, respectively)



(a) Ranking method (Fonseca *et al.*)



(b) DEA method



(c) GDEA method

Figure 6.7: Results to the Example 3.
(from left to right, 10, 20, 30 generations, respectively)

Chapter 7

Conclusions

This thesis has presented several results related to the generalization of DEA model and its applications. The following results have been obtained.

First, the GDEA model based on parametric domination structure and the concept of α-efficiency in the GDEA model have been suggested. In addition, the relationships between the GDEA model and existing DEA models, specifically, the CCR model, the BCC model and the FDH model have been investigated. It has been proved that the GDEA model can evaluate the efficiency in several DEA models, and moreover the efficiency of decision making unit incorporating various preference structures of decision makers. Through a numerical example with real data, it has been shown that the mutual relations among all decision making units corresponding to a parametric domination structure by varying α in the GDEA can be grasped.

Secondly, the GDEA$_D$ model based on production possibility as a dual approach to GDEA and the concept of α_D-efficiency in the GDEA$_D$ model have been proposed. The relations between the GDEA$_D$ model and existing DEA dual models have been established, and the meaning of an optimal value to the problem (GDEA_D) has been interpreted. Therefore, it is possible to evaluate the efficiency for each decision making unit by considering surplus of inputs/slack of outputs as well as the technical efficiency. Moreover, through an illustrative example, it has been shown that GDEA_D can reveal domination relations among all decision making units.

Thirdly, the method combining GDEA and GA for generating efficient frontiers in multi-objective optimization problems has been proposed. The method using GDEA can overcome the shortcomings of existing methods: it provides a lot of Pareto optimal solutions in a relatively small number of generations, and can be applied to multi-objective optimization problems with non-convex functions as well as convex functions. It requires a certain amount of time to solve the problem (P) in order to evaluate the α-efficiency. However, since the time required for analyses such as structural analysis, thermodynamical analysis and fluid mechanical analysis in engineering design problems is extremely large, the computational time for solving the problem (P) is not so serious. It can be considered that the method using GDEA is effective especially to the problems requiring analysis such as engineering design problems from a view point of obtaining a good approximation of efficient frontiers in a small number of generations.

Finally, through the study on GDEA in the thesis, it will be expected that GDEA makes it helpful to evaluate an efficiency of complex management systems such as banks, chain stores, communications enterprise, hospitals, etc. Moreover, GDEA is promising to be very useful method to construct decision support systems such as administrative reforms of (local) goverments, engineering design, schools, courts, and so on.

Bibliography

[1] P.J. Agrell and J. Tind, An Extension of the DEA-MCDM Liaison for the Free Disposable Hull Model, *University of Copenhagen, Department of Operations Research, Working Paper* **3**, 1–18 (1998).

[2] M. Arakawa, H. Nakayama, I. Hagiwara and H. Yamakawa, Multiobjective Optimization Using Adaptive Range Genetic Algorithms with Data Envelopment Analysis, *A Collection of Technical papers on 7th Symposium on Multidisciplinary Analysis and Optimization (TP98-4970), AIAA* **3**, 2074–2082 (1998).

[3] P.W. Bauer, Recent Developments in the Econometric Estimation of Frontiers, *Journal of Econometrics* **46**, 39–56 (1990).

[4] V. Belton, An Integrating Data Envelopment Analysis with Multiple Criteria Decision Analysis, *Proceedings of the Ninth International Conference on Multiple Criteria Decision Making:Theory and Applications in Business, Industry and Commerce*, eds. A. Goicoechea, L. Duckstein and S. Zoints (Springer-Verlag, Berlin), 71–79 (1992).

[5] V. Belton and S.P. Vickers, Demystifying DEA-A Visual Interactive Approach Based on Multiple Criteria Analysis, *Journal of Operational Research*

Society **44**, 883–896 (1993).

[6] R.D. Banker, A. Charnes and W.W. Cooper, Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis, *Management Science* **30**, 1078–1092 (1984).

[7] C.A. Coello Coello, An Updated Survey of Evolutionary Multiobjective Optimization Techniques: State of Art and Future Trends, *Proceedings of the 1999 Congress on Evolutionary Computation*, 3–13 (1999).

[8] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the Efficiency of Decision Making Units, *European Journal of Operational Research* **2**, 429–444 (1978).

[9] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the Efficiency of Decision Making Units, *European Journal of Operational Research* **3**, 339 (1979).

[10] A. Charnes, W.W. Cooper, Z.M. Huang and D.B. Sun, Polyhedral Cone Ratio DEA Models with an Illustrative Application to Large Commercial Banks, *Journal of Econometrics* **46**, 73–91 (1990).

[11] A. Charnes, W.W. Cooper, Q.L. Wei and Z.M. Huang, Cone Ratio Data Envelopment Analysis and Multi-objective Programming, *International of Systems Science* **20**, 1099–1118 (1989).

[12] D. Deprins, L. Simar and H. Tulkens, Measuring Labor-Efficiency in Post Offices, *The Performance of Public Enterprises:Concepts and Measurements*, eds. M. Marchand, P. Pestieu and H. Tulkens (North Holland, Amsterdam), 247–263 (1984).

[13] M.J. Farrell, The Measurement of Productive Efficiency, *Journal of the Royal Statistical Society A* **120**, 253–281 (1957).

[14] C.M. Fonseca and P.J. Fleming, Genetic Algorithms for Multi-Objective Optimization: Formulation, Discussion and Generalization, *Proceedings of the Fifth International Conference on Genetic Algorithms*, 416–426 (1993).

[15] B. Golany, An Interactive MOLP Procedure for the Extension of DEA to Effectiveness Analysis, *Journal of Operational Research Society* **39**, 725–734 (1988).

[16] D.E. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning* (Addison-Wesley, Massachusetts, 1989).

[17] M. Halme, T. Joro, P. Korhonen, A. Salo and J. Wallenius, A Value Efficiency Approach to Incorporating Preference Information in Data Envelopment Analysis, *Management Science* **45**, 103–115 (1999).

[18] T. Joro, P. Korhonen and J. Wallenius, Structural Comparison of Data Envelopment Analysis and Multiple Objective Linear Programming, *Management Science* **44**, 962–970 (1998).

[19] T.C. Koopmans, Analysis of Production as an Efficient Combination of Activities, *Analysis of Production and Allocation* (John Wiley), 33–97 (1997).

[20] P. Korhonen, Searching the Efficient Frontier in Data Envelopment Analysis, *Interim Report IR-97-79/October* (International Institute for Applied Systems Analysis, Austria).

[21] C.A.L. Lovell and P. Schmit, A Comparision of Alternative Approaches to the Measurement of Productive Efficiency, *Applications of Modern Production Theory: Efficiency and Productivity*, eds. A. Dogramaci and R. Färe (Kluwer Academic Publishers, Boston, MA, 1988).

[22] H. Nakayama, Aspiration level approach to interactive multi-objective programming and its applications, *Advances in Multicriteria Analysis*, ed. P.M. Pardolas *et al.* (Kluwer), 147–174 (1995).

[23] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization* (Academic Press, 1985).

[24] J.D. Schaffer, Multiple Objective Optimization with Vector Evaluated Genetic Algorithms, *Proceedings of the First International Conference on Genetic Algorithms*, 93–100 (1985).

[25] S.M. Seiford and R.M. Thrall, Recent Developments in DEA: The Mathematical Programming Approach to Frontier Analysis, *Journal of Econometrics* **46**, 7–38 (1990).

[26] T.J. Stewart, Relationships Between Data Envelopment Analysis and Multiple Criteria Decision Analysis, *Journal of Operational Research Society* **47**, 654–665 (1996).

[27] H. Tamaki, H. Kita and S. Kobayashi, Multi-Objective Optimization by Genetic Algorithms: A Review, *Proceedings of the ICEC'96*, 517–522 (1996).

[28] E. Thanassoulis and R.G. Dyson, Estimating Preffered Target Input-Output Levels Using Data Envelopment Analysis, *European Journal of Operational Research* **56**, 80–97 (1992).

[29] H. Tulkens, On FDH efficiency: Some Methodological Issues and Applications to Retail Banking, Courts, and Urban Transit, *Journal of Productivity Analysis* **4**, 183–210 (1993).

[30] W.M. Taylor, R.G. Thompson, R.M. Thrall and P.S. Dharmapala, DEA/AR Efficiency and Profitability of Mexican Banks: A Total Income Model, *European Journal of Operational Research* **98**, 346–363 (1997).

[31] A. Wierzbicki, The use of Reference Objectives in Multiobjective Optimization, *Multiple Objective Decision Making, Theory and Application* eds. G. Fandel and T. Gal (Springer-Verlag, New York, 1980).

[32] Q.L. Wei and G.Yu, Analyzing properties of K-Cones in the Generalized Data Envelopment Analysis Model, *Journal of Econometrics* **80**, 63–84 (1997).

[33] Y.B. Yun, H. Nakayama and T. Tanino, On efficiency of Data Envelopment Analysis, *Forthcoming in Proceedings of the 14th International Conference on Multiple Criteria Decision Making* (Charlottesville, Virginia, USA).

[34] Y.B. Yun, H. Nakayama and T. Tanino, A Generalization of DEA Model, *Journal of the Society of Instrument and Control Engineers (SICE)* **35**, 1813–1818 (1999). (in Japanese)

[35] Y.B. Yun, H. Nakayama and T. Tanino, Dual Approach to Generalized Data Envelopment Analysis, *Submitting to Journal of the Society of Instrument and Control Engineers (SICE).* (in Japanese)

[36] Y.B. Yun, H. Nakayama, T. Tanino and M. Arakawa, A Multi-Objective Optimization Method Combining Generalized Data Envelopment Analysis and Genetic Algorithms, *Forthcoming in Transactions of the Institute of System, Control and Information Engineers (ISCIE).* (in Japanese)

[37] Y.B. Yun, H. Nakayama, T. Tanino and M. Arakawa, Generation of Efficient Frontiers in Multi-Objective Optimization Problems by Generalized Data Envelopment Analysis, *Forthcoming in European Journal of Operational Research.*

[38] J. Zhu, Data Envelopment Analysis with Preference Structure, *Journal of Operational Research Society* **47**, 136–150 (1996).

List of Publications by the Author

I. Transactions

(a) Do Sang Kim, Ye Boon Yun and Hun Kuk, "Second-Order Symmetric and Self Duality in Multi Objective Programming", *Applied Mathematics Letters*, Vol.10 (2), pp. 17-22, 1997.

(b) Do Sang Kim, Ye Boon Yun and Won Jung Lee, "Multi Objective Symmetric Duality with Cone Constraints", *European Journal of Operational Research*, Vol.107 (3), pp. 686–691, 1998.

(c) Ye Boon Yun, Hirotaka Nakayama and Tetsuzo Tanino, "A Generalization of DEA Model", *Journal of the Society of Instrument and Control Engineers (SICE)*, Vol.35(8), pp. 1813–1818, 1999. (in Japanese)

(d) Ye Boon Yun, Hirotaka Nakayama, Tetsuzo Tanino and Masao Arakawa, "A Multi-Objective Optimization Method Combining Generalized Data Envelopment Analysis and Genetic Algorithms", *Forthcoming in Transactions of the Institute of System, Control and Information Engineers (ISCIE)*. (in Japanese)

(e) Ye Boon Yun, Hirotaka Nakayama, Tetsuzo Tanino and Masao Arakawa, "Generation of Efficient Frontiers in Multi-Objective Optimization Problems by Generalized Data Envelopment Analysis", *Forthcoming in European Journal of Operational Research.*

(f) Ye Boon Yun, Hirotaka Nakayama and Tetsuzo Tanino, "Dual Approach to Generalized Data Envelopment Analysis", *Submitting to Journal of the Society of Instrument and Control Engineers (SICE).* (in Japanese)

II. International Conferences

(a) Ye Boon Yun, Hirotaka Nakayama and Tetsuzo Tanino, "On efficiency of Data Envelopment Analysis", *Forthcoming in Proceedings of the 14th International Conference on Multiple Criteria Decision Making* (Charlottesville, Virginia, USA).

(b) Ye Boon Yun, Hirotaka Nakayama, Tetsuzo Tanino and Masao Arakawa, "A Multi-Objective Optimization Method Combining Generalized Data Envelopment Analysis and Genetic Algorithms", *Proceedings of 1999 IEEE, Systems, Man, Cybernetics Conference*, pp. 671–677, 1999.

(c) Ye Boon Yun, Hirotaka Nakayama, Tetsuzo Tanino and Masao Arakawa, "Generalized Data Envelopment Analysis and Its Application to Multi-Objective Optimization", *Proceedings of First China-Japan-Korea Joint Symposium on Optimization of Structural and Mechanical System*, pp. 563–569, 1999.

III. Books

(a) Hirotaka Nakayama, Tetsuzo Tanino and Ye Boon Yun, "Generalized Data Envelopment Analysis and Its Applications", *Forthcoming in New Frontiers of Decision Making for the Information Technology Era.*.