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Quantum Deformations of certain
Prehomogeneous Vector Spaces

Yoshiyuki Morita

This thesis consists of two parts. In Part I, the general theory is treated. This part has been published in Hiroshima Mathematical Journal, Vol. 28, No. 3 (1998) as a joint paper with A. Kamita and T. Tanisaki. In Part II, some special cases are treated in detail. This part has been accepted for publication in Osaka Journal of Mathematics.

Y. Morita

Quantum deformations of certain prehomogeneous vector spaces I

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Quantum deformations of certain prehomogeneous vector spaces I

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ABSTRACT. We shall construct a quantum analogue of the prehomogeneous vector space associated to a parabolic subgroup with commutative unipotent radical.

0. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$, and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}^+$ be a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{l}$ is a maximal reductive subalgebra of $\mathfrak{p}$ and $\mathfrak{m}^+$ is the nilpotent part. We denote by $\mathfrak{m}^-$ the nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{l} \oplus \mathfrak{m}^-$ is a parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. Take an algebraic group L with Lie algebra $\mathfrak{l}$.

In this paper we shall deal with the case where $\mathfrak{m}^\pm$ is nonzero and commutative. Then $\mathfrak{m}^+$ consists of finitely many L -orbits.

Our aim is to give a quantum analogue of the prehomogeneous vector space $(L, \mathfrak{m}^+)$. More precisely, we shall construct a quantum analogue A_q of the ring $A = \mathbb{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$ as a noncommutative $\mathbb{C}(q)$ -algebra endowed with the action of the quantized enveloping algebra $U_q(\mathfrak{l})$ of $\mathfrak{l}$, and show that for each L -orbit C on $\mathfrak{m}^+$ there exists a two-sided ideal $J_{C,q}$ of A_q which can be regarded as a quantum analogue of the defining ideal J_C of the closure $\bar{C}$ of C . Such an object was intensively studied in the cases $\mathfrak{g} = \mathfrak{sl}_n$ (see Hashimoto-Hayashi [3], Noumi-Yamada-Mimachi [10]) and $\mathfrak{g} = \mathfrak{so}_{2n}$ (see Strickland [13]).

Our method is as follows. Since $\mathfrak{m}^-$ is identified with the dual space of $\mathfrak{m}^+$ via the Killing form, A is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By the commutativity of $\mathfrak{m}^-$ the enveloping algebra $U(\mathfrak{m}^-)$ is naturally identified with the symmetric algebra $S(\mathfrak{m}^-)$. Hence we have an identification $A = U(\mathfrak{m}^-)$. Then using the Poincaré-Birkhoff-Witt type basis of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Lusztig [9]) we obtain a natural quantization A_q of A as a subalgebra of $U_q(\mathfrak{g})$. The algebra A_q has a canonical generator system satisfying quadratic fundamental relations. In particular, it is a graded algebra. The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined using the Hopf

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algebra structure, and we can show that A_q is preserved under the adjoint action of $U_q(\mathfrak{l})$. As a $U_q(\mathfrak{l})$ -module A_q is a direct sum of finite dimensional irreducible submodules.

Let C be a non-open L -orbit on $\mathfrak{m}^+$. It is known that J_C is an $\mathfrak{l}$ -stable homogeneous ideal generated by the lowest degree part J_C^0 . Since A is a multiplicity free $\mathfrak{l}$ -module, there exist unique $U_q(\mathfrak{l})$ -submodules $J_{C,q}$ and $J_{C,q}^0$ of A_q satisfying $J_{C,q}|_{q=1} = J_C$ and $J_{C,q}^0|_{q=1} = J_C^0$. We can show that $J_{C,q}$ is a two-sided ideal of A_q and that $J_{C,q}$ is generated by $J_{C,q}^0$ both as a left ideal and a right ideal. The proof uses the quantum counterpart of the results on a generalized Verma module of $\mathfrak{g}$ whose maximal proper submodule is explicitly described in terms of J_C (see Enright-Joseph [2], Tanisaki [14]).

Explicit descriptions of A_q and $J_{C,q}$ in each individual case will be given in our subsequent papers.

1. Quantized enveloping algebras

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We fix an ordering on Δ , and denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where I_0 is an index set. We set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to i respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ satisfying $[e_i, f_i] = h_i$. Let $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α . Set

$$d_i = (\alpha_i, \alpha_i)/2 \quad (i \in I_0), \quad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0).$$

For a subset I of I_0 we set

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, \quad W_I = \langle s_i \mid i \in I \rangle,$$

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right), \quad \mathfrak{n}_I^+ = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad \mathfrak{n}_I^- = \bigoplus_{\alpha \in -\Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha.$$

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$.

Let us recall the definition of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [7]). It is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i \in I_0}$ satisfying the

following fundamental relations:

$$\begin{aligned}
 K_i K_j &= K_j K_i, \\
 K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\
 K_i E_j K_i^{-1} &= q_i^{a_{ij}} E_j, \\
 K_i F_j K_i^{-1} &= q_i^{-a_{ij}} F_j, \\
 E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
 \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k &= 0 \quad (i \neq j), \\
 \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k &= 0 \quad (i \neq j),
 \end{aligned}$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^m [k]_t, \quad \begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t!}{[n]_t! [m-n]_t!} \quad (m \geq n \geq 0).$$

For $i \in I_0$ and $n \in \mathbb{Z}_{\geq 0}$ we set

$$E_i^{(n)} = \frac{1}{[n]_{q_i}!} E_i^n, \quad F_i^{(n)} = \frac{1}{[n]_{q_i}!} F_i^n.$$

The algebra $U_q(\mathfrak{g})$ is endowed with a Hopf algebra structure via the following formula:

$$\begin{aligned}
 \Delta(K_i) &= K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i, \\
 \varepsilon(K_i) &= 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \\
 S(K_i) &= K_i^{-1}, \quad S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i,
 \end{aligned}$$

where $\Delta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $\varepsilon : U_q(\mathfrak{g}) \rightarrow \mathbb{C}(q)$ are the algebra homomorphisms giving the comultiplication and the counit respectively, and $S : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the algebra anti-automorphism giving the antipode.

We define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\text{ad } x)(y) = \sum_k x_k^1 y S(x_k^2)$. Then

$$\text{ad} : U_q(\mathfrak{g}) \rightarrow \text{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$$

is a homomorphism of algebras.

Define subalgebras $U_q(\mathfrak{n}^\pm)$, $U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$\begin{aligned} U_q(\mathfrak{n}^+) &= \langle E_i \mid i \in I_0 \rangle, & U_q(\mathfrak{n}^-) &= \langle F_i \mid i \in I_0 \rangle, & U_q(\mathfrak{h}) &= \langle K_i^{\pm 1} \mid i \in I_0 \rangle, \\ U_q(\mathfrak{l}_I) &= \langle K_i^{\pm 1}, E_j, F_j \mid i \in I_0, j \in I \rangle. \end{aligned}$$

For $i \in I_0$ define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$\begin{aligned} T_i(K_j) &= K_j K_i^{-a_{ij}}, \\ T_i(E_j) &= \begin{cases} -F_i K_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases} \\ T_i(F_j) &= \begin{cases} -K_i^{-1} E_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^k F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (i \neq j). \end{cases} \end{aligned}$$

(see Lusztig [9]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that T_w does not depend on the choice of the reduced expression.

For $I \subset I_0$ let w_I be the longest element of W_I and define a subalgebra $U_q(\mathfrak{n}_I^-)$ by

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-).$$

Let w_0 be the longest element of W . Take a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_m}$ of $w_I w_0$ and set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}), \quad Y_{\beta_k}^{(n)} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}^{(n)})$$

for $k = 1, \dots, m$. Then it is known that $\{\beta_k \mid 1 \leq k \leq m\} = \Delta^+ \setminus \Delta_I$, and that $\{Y_{\beta_1}^{(d_1)} \cdots Y_{\beta_m}^{(d_m)} \mid d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U_q(\mathfrak{n}_I^-)$. We note that this basis depends on the choice of the reduced expression of $w_I w_0$ in general.

Let $\tau: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ be the algebra anti-automorphism given by

$$\tau(K_i) = K_i^{-1}, \quad \tau(E_i) = E_i, \quad \tau(F_i) = F_i \quad (i \in I_0).$$

LEMMA 1.1. (i) $\tau T_{w_I}(U_q(\mathfrak{n}_I^-)) = U_q(\mathfrak{n}_I^-)$.

(ii) Let $i, j \in I$ be such that $w_I(\alpha_i) = -\alpha_j$. Then we have

$$\begin{aligned} (\text{ad } F_i)(\tau T_{w_I}(x)) &= \tau T_{w_I}((\text{ad } E_j)(x)), & (\text{ad } E_i)(\tau T_{w_I}(x)) &= \tau T_{w_I}((\text{ad } F_j)(x)), \\ (\text{ad } K_i)(\tau T_{w_I}(x)) &= \tau T_{w_I}((\text{ad } (K_j^{-1}))(x)) \end{aligned}$$

for any $x \in U_q(\mathfrak{g})$.

PROOF. (i) We have $\tau T_k = T_k^{-1} \tau$ for any $k \in I_0$, and hence $\tau T_w = T_w^{-1} \tau$ for any $w \in W$. Hence

$$\begin{aligned} \tau T_{w_I}(U_q(\mathfrak{n}_I^-)) &= \tau T_{w_I}(U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1}(U_q(\mathfrak{n}^-))) \\ &= T_{w_I}^{-1}(U_q(\mathfrak{n}^-)) \cap U_q(\mathfrak{n}^-) = U_q(\mathfrak{n}_I^-). \end{aligned}$$

(ii) We have

$$\tau T_{w_I}(E_j) = \tau T_{w_I s_j} T_{s_j}(E_j) = \tau T_{w_I s_j}(-F_j K_j) = -\tau(F_j K_j) = -K_i^{-1} F_i.$$

Here we have used the formula:

$$T_y(F_k) = F_\ell, \quad T_y(K_k) = K_\ell \quad (y \in W, k, \ell \in I_0, y(\alpha_k) = \alpha_\ell)$$

(see Lusztig [9]). Hence

$$\begin{aligned} \tau T_{w_I}((\text{ad } E_j)(x)) &= \tau T_{w_I}((E_j x - x E_j) K_j) = K_i(z(-K_i^{-1} F_i) - (-K_i^{-1} F_i)z) \\ &= F_i z - (K_i z K_i^{-1}) F_i = (\text{ad } F_i)(z) \end{aligned}$$

with $z = \tau T_{w_I}(x)$. Other formulas are proved similarly. $\square$

PROPOSITION 1.2. $(\text{ad } U_q(\mathfrak{l}_I))(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$.

PROOF. We see easily that $(\text{ad } U_q(\mathfrak{h}))(U_q(\mathfrak{n}_I^-)) = U_q(\mathfrak{n}_I^-)$. Hence it is sufficient to show that $U_q(\mathfrak{n}_I^-)$ is stable under $\text{ad } E_i$, $\text{ad } F_i$ for $i \in I$.

Let $i \in I$ and define $j \in I$ by $\alpha_j = -w_I(\alpha_i)$. By Lemma 1.1 we have

$$\begin{aligned} (\text{ad } E_i)(U_q(\mathfrak{n}_I^-)) &= T_{w_I}^{-1} \tau^{-1} \tau T_{w_I}(\text{ad } E_i)(U_q(\mathfrak{n}_I^-)) = T_{w_I}^{-1} \tau^{-1} (\text{ad } F_j)(\tau T_{w_I} U_q(\mathfrak{n}_I^-)) \\ &\subset T_{w_I}^{-1} \tau^{-1} (\text{ad } F_j)(U_q(\mathfrak{n}^-)) \subset T_{w_I}^{-1}(U_q(\mathfrak{n}^-)). \end{aligned}$$

Let us show $(\text{ad } E_i)(U_q(\mathfrak{n}^-)) \subset U_q(\mathfrak{n}^-)$. For any $y \in U_q(\mathfrak{n}^-)$ we can write

$$[E_i, y] = K_i r_1(y) - r_2(y) K_i^{-1} \quad (r_1(y), r_2(y) \in U_q(\mathfrak{n}^-)),$$

and hence $(\text{ad } E_i)(y) = K_i r_1(y) K_i - r_2(y)$. On the other hand by Jantzen [5] we have

$$\{y \in U_q(\mathfrak{n}^-) \mid r_1(y) = 0\} = U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-).$$

Hence we have to show $U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-)$. It is sufficient to show for any $y \in W$ and $k \in I_0$ satisfying $s_{ky} < y$ that $U_q(\mathfrak{n}^-) \cap T_{s_{ky}}^{-1} U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{n}^-) \cap T_y^{-1} U_q(\mathfrak{n}^-)$. This follows from Lusztig [9]. Therefore we have $(\text{ad } E_i)(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$. Then we see from Lemma 1.1 that $(\text{ad } F_\ell)(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$. $\square$

Let $U_q^0(\mathfrak{n}^-)$ be the $\mathbb{C}[q^{\pm 1}]$ -subalgebra of $U_q(\mathfrak{n}^-)$ generated by $\{F_i^{(n)} \mid i \in I_0, n \in \mathbb{Z}_{\geq 0}\}$. We have a natural $\mathbb{C}$ -algebra homomorphism $\varphi: U_q^0(\mathfrak{n}^-) \rightarrow U(\mathfrak{n}^-)$ given by $F_i^{(n)} \rightarrow f_i^n/n!$, and it induces the isomorphism $\mathbb{C} \otimes_{\mathbb{C}[q^{\pm 1}]} U_q^0(\mathfrak{n}^-) \simeq U(\mathfrak{n}^-)$ where $\mathbb{C}[q^{\pm 1}] \rightarrow \mathbb{C}$ is given by $q \mapsto 1$. For $I \subset I_0$ the restriction of φ to $U_q^0(\mathfrak{n}_I^-) = U_q^0(\mathfrak{n}^-) \cap U_q(\mathfrak{n}_I^-)$ gives a surjective $\mathbb{C}$ -algebra homomorphism $\varphi_I: U_q^0(\mathfrak{n}_I^-) \rightarrow U(\mathfrak{n}_I^-)$ inducing $\mathbb{C} \otimes_{\mathbb{C}[q^{\pm 1}]} U_q^0(\mathfrak{n}_I^-) \simeq U(\mathfrak{n}_I^-)$.

For $N \in \mathbb{Z}_{>0}$ set

$$U_{q,N}(\mathfrak{g}) = \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} U_q(\mathfrak{g}),$$

and let $U_{q,N}(\mathfrak{n}^{\pm})$, $U_{q,N}(\mathfrak{h})$, $U_{q,N}(\mathfrak{l}_I)$, $U_{q,N}(\mathfrak{n}_I^-)$ be the $\mathbb{C}(q^{1/N})$ -subalgebras of $U_{q,N}(\mathfrak{g})$ generated by $U_q(\mathfrak{n}^{\pm})$, $U_q(\mathfrak{h})$, $U_q(\mathfrak{l}_I)$, $U_q(\mathfrak{n}_I^-)$ respectively.

2. Highest weight modules

For a $U(\mathfrak{h})$ -module M and $\mu \in \mathfrak{h}^*$ we set

$$M_{\mu} = \{m \in M \mid hm = \mu(h)m \quad (h \in \mathfrak{h})\}.$$

It is called a weight space of M with weight μ . A $U(\mathfrak{h})$ -module M satisfying $M = \bigoplus_{\mu} M_{\mu}$ and $\dim M_{\mu} < \infty$ for any μ is called a weight module. We define its character $\text{ch}(M)$ as the formal infinite sum

$$\text{ch}(M) = \sum_{\mu} \dim M_{\mu} e^{\mu}.$$

A $U(\mathfrak{g})$ -module M is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $m \in M_{\lambda} \setminus \{0\}$ satisfying $M = U(\mathfrak{g})m$, $\mathfrak{n}^+m = 0$. Such m is determined up to a nonzero constant multiple and is called the highest weight vector of M . For each $\lambda \in \mathfrak{h}^*$ there exists a unique (up to an isomorphism) irreducible highest weight module with highest weight λ , which we denote by $L(\lambda)$. Since highest weight modules are weight modules, their characters are defined. For $I \subset I_0$ set

$$\mathfrak{h}_I^* = \bigoplus_{i \in I_0 \setminus I} \mathbb{C} \varpi_i \subset \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$ -module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g}) / \left(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})(\mathfrak{l}_I \cap \mathfrak{n}^-) \right).$$

It is a highest weight module with highest weight λ and the highest weight vector $m_{I,\lambda} = \bar{1}$, where $\bar{1}$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$. Moreover it is a rank one free $U(\mathfrak{n}_I^-)$ -module generated by the

highest weight vector $m_{I,\lambda}$, and hence we have

$$\text{ch}(M_I(\lambda)) = \frac{e^\lambda}{\prod_{\alpha \in \mathcal{A}^+ \setminus \mathcal{A}_I} (1 - e^{-\alpha})}.$$

It contains a unique maximal proper submodule $K_I(\lambda)$, and we have $L(\lambda) = M_I(\lambda)/K_I(\lambda)$.

Now we define the corresponding notions for the quantized enveloping algebras. Set

$$\mathfrak{h}_{\mathbf{Z}}^* = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \ (i \in I_0)\} = \bigoplus_{i \in I_0} \mathbb{Z} \varpi_i \subset \mathfrak{h}^*.$$

For a $U_{q,N}(\mathfrak{h})$ -module M the weight space M_μ with weight $\mu \in \mathfrak{h}_{\mathbf{Z}}^*/N$ is defined by

$$M_\mu = \{m \in M \mid K_i m = q_i^{\mu(h_i)} m \ (i \in I_0)\}.$$

We call a $U_{q,N}(\mathfrak{h})$ -module M a weight module if $M = \bigoplus_{\mu} M_\mu$ and $\dim M_\mu < \infty$ for any $\mu \in \mathfrak{h}_{\mathbf{Z}}^*/N$. Let M be a $U_{q,N}(\mathfrak{g})$ -module. If there exists $m \in M_\lambda$ satisfying $U_{q,N}(\mathfrak{g})m = M$, $E_i m = 0$ ($i \in I_0$), then M is called a highest weight module with highest weight λ and m is called its highest weight vector. There exists a unique irreducible highest weight module $L_{q,N}(\lambda)$ with highest weight λ . Highest weight modules are weight modules. For $I \subset I_0$ set

$$\mathfrak{h}_{I,\mathbf{Z}}^* = \bigoplus_{i \in I_0 \setminus I} \mathbb{Z} \varpi_i \subset \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}_{I,\mathbf{Z}}^*/N$ we define a highest weight module $M_{I,q,N}(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(\mathfrak{g}) / \left(\sum_{i \in I_0} U_{q,N}(\mathfrak{g})(K_i - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(\mathfrak{g})E_i + \sum_{j \in I} U_{q,N}(\mathfrak{g})F_j \right).$$

Its highest weight vector is given by $m_{I,\lambda,q,N} = \bar{1}$. Since $M_{I,q,N}(\lambda)$ is a rank one free module generated by $m_{I,\lambda,q,N}$, we have

$$\text{ch}(M_{I,q,N}(\lambda)) = \text{ch}(M_I(\lambda)).$$

We have a unique maximal proper submodule $K_{I,q,N}(\lambda)$ of $M_{I,q,N}(\lambda)$, and hence $L_{q,N}(\lambda) = M_{I,q,N}(\lambda)/K_{I,q,N}(\lambda)$.

PROPOSITION 2.1. *Let $I \subset I_0$ and $\lambda \in \mathfrak{h}_{I,\mathbf{Z}}^*/N$. Let Y be a subset of $U_q^0(\mathfrak{n}^-)$ such that $Ym_{I,\lambda,q,N} \subset K_{I,q,N}(\lambda)$ and $U(\mathfrak{g})\varphi_I(Y)m_{I,\lambda} = K_I(\lambda)$. Then we have $U_{q,N}(\mathfrak{g})Ym_{I,\lambda,q,N} = K_{I,q,N}(\lambda)$ and $\text{ch}(L_{q,N}(\lambda)) = \text{ch}(L(\lambda))$.*

PROOF. Let M be any highest weight $U_{q,N}(\mathfrak{g})$ -module with highest weight λ . Take a highest weight vector $m \in M$ and set

$$M^0 = U_q^0(\mathfrak{n}^-)m, \quad \bar{M}^0 = M^0|_{q=1} = \mathbb{C} \otimes_{\mathbb{C}[q^{\pm 1/N}]} M^0.$$

Then we can show as in Lusztig [8] that M^0 is stable under the actions of E_i , F_i , $(K_i - K_i^{-1})/(q_i - q_i^{-1})$ ($i \in I_0$) and that $\bar{M}^0$ becomes a highest weight $U(\mathfrak{g})$ -module with highest weight λ via the operators

$$e_i = \bar{E}_i, \quad f_i = \bar{F}_i, \quad h_i = \frac{\overline{K_i - K_i^{-1}}}{q_i - q_i^{-1}} \quad (i \in I_0).$$

In particular we have

$$\dim M_\mu = \dim(\bar{M}^0)_\mu \geq \dim L(\lambda)_\mu.$$

Now we set

$$M = M_{I,q,N}(\lambda)/U_{q,N}(\mathfrak{g})Ym_{I,\lambda,q,N}, \quad m = \overline{m_{I,\lambda,q,N}} \in M.$$

By the above argument $\bar{M}^0$ is a highest weight $U(\mathfrak{g})$ -module with highest weight λ and the highest weight vector $\bar{m}$. Moreover, since $Ym = 0$, we have $\varphi_I(Y)\bar{m} = 0$. Hence we have $\bar{M}^0 \simeq L(\lambda)$. It follows that

$$\dim L_{q,N}(\lambda)_\mu \leq \dim M_\mu = \dim(\bar{M}^0)_\mu = \dim L(\lambda)_\mu \leq \dim L_{q,N}(\lambda)_\mu.$$

Therefore we have $M \simeq L_{q,N}(\lambda)$ and $\text{ch}(L_{q,N}(\lambda)) = \text{ch}(L(\lambda))$. $\square$

3. Parabolic subalgebras with commutative nilpotent radicals

In the rest of this paper we fix $I \subset I_0$ satisfying $\mathfrak{n}_I^+ \neq \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$ (see, for example, [14] for the list of $(\mathfrak{g}, I)$'s satisfying the condition). We have $I = I_0 \setminus \{i_0\}$ for some $i_0 \in I_0$.

We set $\mathfrak{l} = \mathfrak{l}_I$, $\mathfrak{m}^\pm = \mathfrak{n}_I^\pm$ for simplicity.

PROPOSITION 3.1. *The element $Y_\beta \in U_q(\mathfrak{m}^-)$ for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$.*

PROOF. For $i, j \in I_0$ set

$$r(i, j) = \overbrace{(i, j, i, j, \dots)}^{m_{ij}},$$

where m_{ij} denotes the order of $s_i s_j \in W$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced expression of $w \in W$. Then $s_{j_1} \cdots s_{j_r}$ is a reduced expression of w if and only if $(j_1, \dots, j_r)$ can be obtained from $(i_1, \dots, i_r)$ by successively exchanging a subsequence of the form $r(i, j)$ to $r(j, i)$.

We first show that for any reduced expression $s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ the sequence $(i_1, \dots, i_r)$ does not contain a subsequence of the form $r(i, j)$ with $m_{ij} \geq 3$. Assume that there exists a subsequence $r(i, j)$ with $m_{ij} = 3$ in $(i_1, \dots, i_r)$. We have $(i_p, i_{p+1}, i_{p+2}) = (i, j, i)$ for some p . Set $y = s_{i_1} \cdots s_{i_{p-1}}$.

Then we have

$$\beta_p = y(\alpha_i), \quad \beta_{p+1} = y s_i(\alpha_j) = y(\alpha_i + \alpha_j), \quad \beta_{p+2} = y s_i s_j(\alpha_i) = y(\alpha_j),$$

and hence $\beta_p + \beta_{p+2} = \beta_{p+1}$. This contradicts the commutativity of $\mathfrak{m}^-$. Thus the sequence $(i_1, \dots, i_r)$ does not contain a subsequence of the form $r(i, j)$ with $m_{ij} = 3$. Similarly we can show that there does not exist a subsequence of the form $r(i, j)$ with $m_{ij} = 4, 6$.

Therefore it is sufficient to show that for two reduced expressions

$$s_{i_1} \cdots s_{i_p} s_i s_j s_{j_1} \cdots s_{j_q}, \quad s_{i_1} \cdots s_{i_p} s_j s_i s_{j_1} \cdots s_{j_q}, \quad (s_i s_j = s_j s_i)$$

of $w_I w_0$ the resulting Y_β 's are the same. This follows from $T_i(F_j) = F_j$, $T_j(F_i) = F_i$, and $T_i T_j = T_j T_i$. $\square$

We fix a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ and set $\beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$. Set

$$Q^+ = \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i, \quad Q_I^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i,$$

$$U_q(\mathfrak{m}^-)^m = \sum_{p_1, \dots, p_m=1}^r \mathbb{C}(q) Y_{\beta_{p_1}} \cdots Y_{\beta_{p_m}} \quad (m \geq 0).$$

LEMMA 3.2. *We have*

$$U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m.$$

$$U_q(\mathfrak{m}^-)^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_r}^{(m_r)} = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

Here $U_q(\mathfrak{m}^-)_{-\gamma}$ is the weight space with respect to the adjoint action of $U_q(\mathfrak{h})$ on $U_q(\mathfrak{m}^-)$.

PROOF. Set

$$V_0^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_r}^{(m_r)}, \quad V_1^m = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

By $\beta_p \in \alpha_{i_0} + Q_I^+$ we have $V_0^m \subset U_q(\mathfrak{m}^-)^m \subset V_1^m$. Since $U_q(\mathfrak{m}^-) = \bigoplus_m V_0^m$, we obtain $V_0^m = U_q(\mathfrak{m}^-)^m = V_1^m$ and $U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m$. $\square$

By Lemma 3.2 we can write

$$(3.1) \quad Y_{\beta_{p_1}} Y_{\beta_{p_2}} = \sum_{\substack{s_1 \leq s_2 \\ \beta_{p_1} + \beta_{p_2} = \beta_{s_1} + \beta_{s_2}}} a_{s_1, s_2}^{p_1, p_2} Y_{\beta_{s_1}} Y_{\beta_{s_2}} \quad (a_{s_1, s_2}^{p_1, p_2} \in \mathbb{C}(q))$$

for $p_1 > p_2$.

PROPOSITION 3.3. *The $\mathbb{C}(q)$ -algebra $U_q(\mathfrak{m}^-)$ is generated by the elements $\{Y_{\beta_p} \mid 1 \leq p \leq r\}$ satisfying the fundamental relations (3.1) for $p_1 > p_2$.*

PROOF. It is sufficient to show that any element of the form $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}}$ ($1 \leq t_i \leq r$) can be rewritten as a linear combination of the elements of the form $Y_{\beta_{s_1}} \cdots Y_{\beta_{s_n}}$ ($1 \leq s_1 \leq \cdots \leq s_n \leq r$) by a successive use of the relations (3.1) for $p_1 > p_2$. For $1 \leq k \leq r$ let V_k be the subalgebra of $U_q(\mathfrak{m}^-)$ generated by $\{Y_{\beta_p} \mid 1 \leq p \leq k\}$. By Lusztig [9] we have

$$V_k = \bigoplus_{m_1, \dots, m_k} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_k}^{(m_k)}.$$

We shall show by the induction on k that any element of the form $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}}$ ($1 \leq t_i \leq k$) can be rewritten as a linear combination of the elements of the form $Y_{\beta_{s_1}} \cdots Y_{\beta_{s_n}}$ ($1 \leq s_1 \leq \cdots \leq s_n \leq k$) by a successive use of the relations (3.1) for $k \geq p_1 > p_2$. It is trivial for $k = 1$. Assume that $k \geq 2$ and the assertion is proved up to $k - 1$. We shall show the statement by induction on n . It is obvious for $n = 0$. Assume that $n > 0$ and the statement is already proved up to $n - 1$. Take j such that $t_1 = \cdots = t_j = k$, $t_{j+1} \neq k$. We use induction on j . Assume that $j = 0$. Then we have $t_1 \neq k$. By using the inductive hypothesis on n we may assume that $t_2 \leq \cdots \leq t_n \leq k$. If $t_n < k$, then we have $t_i \leq k - 1$ for any i , and hence the statement holds by the inductive hypothesis on k . If $t_n = k$, then we can apply the inductive hypothesis on n to $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_{n-1}}}$, and hence the statement also holds. Assume $0 < j < n$. Then we have

$$Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}} = Y_{\beta_k}^j Y_{\beta_{t_{j+1}}} \cdots Y_{\beta_{t_n}}$$

with $t_{j+1} \neq k$. Applying (3.1) for $(p_1, p_2) = (k, t_{j+1})$ we obtain

$$Y_{\beta_k} Y_{\beta_{t_{j+1}}} = \sum_{\substack{s_1 \leq s_2 \leq k \\ \beta_k + \beta_{t_{j+1}} = \beta_{s_1} + \beta_{s_2}}} a_{s_1, s_2}^{k, t_{j+1}} Y_{\beta_{s_1}} Y_{\beta_{s_2}}.$$

Since $s_1 < k$ by the condition $\beta_k + \beta_{t_{j+1}} = \beta_{s_1} + \beta_{s_2}$, we can apply the inductive hypothesis on j to $Y_{\beta_k}^{j-1} Y_{\beta_{s_1}} Y_{\beta_{s_2}} Y_{\beta_{t_{j+2}}} \cdots Y_{\beta_{t_n}}$, and the statement holds. If $j = n$, then we have $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}} = Y_{\beta_k}^n$, and the statement is obvious. $\square$

Since $\mathfrak{m}^-$ is commutative, $U(\mathfrak{m}^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By identifying $\mathfrak{m}^-$ with $(\mathfrak{m}^+)^*$ via the Killing form of $\mathfrak{g}$, $S(\mathfrak{m}^-)$ is naturally identified with the algebra $\mathbb{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$. Hence we have an identification $U(\mathfrak{m}^-) = \mathbb{C}[\mathfrak{m}^+]$. We denote by $\mathbb{C}[\mathfrak{m}^+]^m$ ($m \in \mathbb{Z}_{\geq 0}$) the subspace of $\mathbb{C}[\mathfrak{m}^+]$ consisting of homogeneous polynomials with degree m .

Set

$$\mathfrak{h}_{\mathbf{Z}}^*(I, +) = \{\lambda \in \mathfrak{h}_{\mathbf{Z}}^* \mid \lambda(h_i) \geq 0 \ (i \in I)\}.$$

For $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*(I, +)$ we denote the finite dimensional irreducible $U(\mathfrak{l})$ -module (resp. $U_q(\mathfrak{l})$ -module) with highest weight λ by $V(\lambda)$ (resp. $V_q(\lambda)$). We can decompose the finite dimensional $\mathfrak{l}$ -module $\mathbb{C}[\mathfrak{m}^+]^m$ into a direct sum of submodules isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*(I, +)$. Moreover, it is known that

$$\dim \text{Hom}_{\mathfrak{l}}(V(\lambda), \mathbb{C}[\mathfrak{m}^+]) \geq 1 \quad (\lambda \in \mathfrak{h}_{\mathbf{Z}}^*(I, +)),$$

and hence we have

$$\mathbb{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma^m} V(\lambda)$$

for finite subsets Γ^m of $\mathfrak{h}_{\mathbf{Z}}^*(I, +)$ satisfying $\Gamma^m \cap \Gamma^{m'} = \emptyset$ for $m \neq m'$ (see Schmid [11], Takeuchi [12], Johnson [6] for the explicit description of Γ^m). On the other hand, since $U_q(\mathfrak{m}^-)^m$ is a finite dimensional $U_q(\mathfrak{l})$ -module whose character is the same as that of $\mathbb{C}[\mathfrak{m}^+]^m$, we have

$$U_q(\mathfrak{m}^-)^m \simeq \bigoplus_{\lambda \in \Gamma^m} V_q(\lambda).$$

Let L be the algebraic group corresponding to $\mathfrak{l}$. It is known that the set of L -orbits on $\mathfrak{m}^+$ is a finite totally ordered set with respect to the closure relation. Hence we can label the orbits by

$$\{L\text{-orbits on } \mathfrak{m}^+\} = \{C_0, C_1, \dots, C_t\}, \quad \{0\} = C_0 \subset \bar{C}_1 \subset \dots \subset \bar{C}_t = \mathfrak{m}^+.$$

Set

$$\mathcal{J}(\bar{C}_p) = \{f \in \mathbb{C}[\mathfrak{m}^+] \mid f(\bar{C}_p) = 0\}.$$

Since $\mathcal{J}(\bar{C}_p)$ is an $\mathfrak{l}$ -submodule of $\mathbb{C}[\mathfrak{m}^+]$, we have

$$\mathcal{J}(\bar{C}_p) = \bigoplus_m \mathcal{J}^m(\bar{C}_p), \quad \mathcal{J}^m(\bar{C}_p) = \mathcal{J}(\bar{C}_p) \cap \mathbb{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma_p^m} V(\lambda)$$

for a subset Γ_p^m of Γ^m . Moreover the following fact is known (see, for example, [14]):

PROPOSITION 3.4. *Let $p = 0, \dots, t-1$.*

- (i) $\mathcal{J}^m(\bar{C}_p) = 0$ for $m \leq p$.
- (ii) $\mathcal{J}^{p+1}(\bar{C}_p)$ is an irreducible $\mathfrak{l}$ -module, i.e. Γ_p^{p+1} consists of a single element v_p .
- (iii) $\mathcal{J}(\bar{C}_p)$ is generated by $\mathcal{J}^{p+1}(\bar{C}_p)$ as an ideal of $\mathbb{C}[\mathfrak{m}^+]$.

PROPOSITION 3.5. *For $p = 0, \dots, t-1$ there exists a unique $\lambda_p \in \mathfrak{h}_{\mathbf{I}}^*$ such that $K_I(\lambda_p) = \mathcal{J}(\bar{C}_p)m_{I, \lambda_p}$. Moreover, we have $\lambda_p \in \mathfrak{h}_{I, \mathbf{Z}}^*/2$.*

Let v^p be the highest weight vector of the $\mathfrak{l}$ -module $\mathcal{J}^{p+1}(\bar{C}_p)(\simeq V(v_p))$. Then we have

$$\begin{aligned} K_I(\lambda_p) &= \mathcal{J}(\bar{C}_p)m_{I,\lambda_p} = U(\mathfrak{m}^-)\mathcal{J}^{p+1}(\bar{C}_p)m_{I,\lambda_p} \\ &= U(\mathfrak{m}^-)((\text{ad } U(\mathfrak{l} \cap \mathfrak{n}^-))(v^p))m_{I,\lambda_p} \\ &= U(\mathfrak{m}^-)(U(\mathfrak{l} \cap \mathfrak{n}^-))v^p m_{I,\lambda_p} = U(\mathfrak{n}^-)v^p m_{I,\lambda_p} \end{aligned}$$

and hence $K_I(\lambda_p)$ is a highest weight module with highest weight $\lambda_p + v_p$.

We set

$$\begin{aligned} \mathcal{J}_q^m(\bar{C}_p) &= \bigoplus_{\lambda \in \Gamma_p^m} V_q(\lambda) \subset U_q(\mathfrak{m}^-)^m, \quad \mathcal{J}_q(\bar{C}_p) = \bigoplus_m \mathcal{J}_q^m(\bar{C}_p) \subset U_q(\mathfrak{m}^-), \\ \mathcal{J}_{q,N}^m(\bar{C}_p) &= \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} \mathcal{J}_q^m(\bar{C}_p) \subset U_{q,N}(\mathfrak{m}^-)^m, \\ \mathcal{J}_{q,N}(\bar{C}_p) &= \bigoplus_m \mathcal{J}_{q,N}^m(\bar{C}_p) \subset U_{q,N}(\mathfrak{m}^-). \end{aligned}$$

Here we identify $U_q(\mathfrak{m}^-)^m$ with $\bigoplus_{\lambda \in \Gamma^m} V_q(\lambda)$.

PROPOSITION 3.6. For $p = 0, \dots, t-1$ we have

$$\text{ch}(L_{q,2}(\lambda_p)) = \text{ch}(L(\lambda_p)), \quad K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{m}^-)\mathcal{J}_{q,2}^{p+1}(\bar{C}_p)m_{I,\lambda_p,q,2}.$$

PROOF. We shall only give a sketch of the proof. We can prove a quantum analogue of the determinant formula for the contravariant forms on generalized Verma modules given by Jantzen [4]. It implies that $K_{I,q,N}(\lambda)_\mu = 0$ if and only if $K_I(\lambda)_\mu = 0$. In particular, we have $K_{I,q,2}(\lambda_p)_{\lambda_p+v_p} \neq 0$ and $K_{I,q,2}(\lambda_p)_{\lambda_p+v_p+\alpha_i} = 0$ for any $i \in I_0$. Let $vm_{I,\lambda_p,q,2}$ ($v \in U_{q,2}(\mathfrak{m}^-)_{v_p}$) be a nonzero element of $K_{I,q,2}(\lambda_p)_{\lambda_p+v_p}$. Then for $i \in I$ we have

$$\begin{aligned} ((\text{ad } E_i)(v))m_{I,\lambda_p,q,2} &= (E_i v - v E_i)K_I m_{I,\lambda_p,q,2} \\ &\in \mathbb{C}(q^{1/2})E_i vm_{I,\lambda_p,q,2} \subset K_{I,q,2}(\lambda_p)_{\lambda_p+v_p+\alpha_i} = \{0\}. \end{aligned}$$

Hence $(\text{ad } E_i)(v) = 0$ for any $i \in I$. It follows that v is a highest weight vector of the $U_{q,2}(\mathfrak{l})$ -module $V_{q,2}(v_p)$. We may assume $v \in U_q^0(\mathfrak{m}^-)$ and $\varphi_I(v) \neq 0$. By Proposition 2.1 we conclude that $\text{ch}(L_{q,2}(\lambda_p)) = \text{ch}(L(\lambda_p))$ and $K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{g})vm_{I,\lambda_p,q,2}$. Then we have

$$\begin{aligned} K_{I,q,2}(\lambda_p) &= U_{q,2}(\mathfrak{g})vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)(U_{q,2}(\mathfrak{l}) \cap U_{q,2}(\mathfrak{n}^-))U_{q,2}(\mathfrak{h})U_{q,2}(\mathfrak{n}^+)vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)(U_{q,2}(\mathfrak{l}) \cap U_{q,2}(\mathfrak{n}^-))vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)((\text{ad}(U_{q,2}(\mathfrak{l}) \cap U_{q,2}(\mathfrak{n}^-)))(v))m_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)\mathcal{J}_{q,2}^{p+1}(\bar{C}_p)m_{I,\lambda_p,q,2}. \end{aligned}$$

□

THEOREM 3.7. *We have*

$$\mathcal{J}_q(\bar{C}_p) = U_q(\mathfrak{m}^-) \cdot \mathcal{J}_q^{p+1}(\bar{C}_p) = \mathcal{J}_q^{p+1}(\bar{C}_p) U_q(\mathfrak{m}^-).$$

PROOF. By Proposition 3.6 we have

$$\text{ch}(U_q(\mathfrak{m}^-) \cdot \mathcal{J}_q^{p+1}(\bar{C}_p)) = \text{ch}(U_{q,2}(\mathfrak{m}^-) \cdot \mathcal{J}_{q,2}^{p+1}(\bar{C}_p)) = \text{ch}(\mathcal{J}(\bar{C}_p)),$$

and hence $\mathcal{J}_q(\bar{C}_p) = U_q(\mathfrak{m}^-) \cdot \mathcal{J}_q^{p+1}(\bar{C}_p)$. Let us show $U_q(\mathfrak{m}^-) \cdot \mathcal{J}_q^{p+1}(\bar{C}_p) = \mathcal{J}_q^{p+1}(\bar{C}_p) U_q(\mathfrak{m}^-)$. Since τT_{w_I} is an anti-automorphism of the algebra $U_q(\mathfrak{m}^-)$ (see Lemma 1.1), it is sufficient to show that τT_{w_I} preserves $\mathcal{J}_q^{p+1}(\bar{C}_p)$. Since $U_q(\mathfrak{m}^-)$ is a multiplicity free $U_q(\mathfrak{l})$ -module, we have only to show that $\tau T_{w_I}(V_q(\lambda))$ is a $U_q(\mathfrak{l})$ -submodule isomorphic to $V_q(\lambda)$ for any $\lambda \in \bigcup_m \Gamma^m$. By Lemma 1.1 we see easily that $\tau T_{w_I}(V_q(\lambda))$ is an irreducible $U_q(\mathfrak{l})$ -module with lowest weight $w_I(\lambda)$. Hence we have $\tau T_{w_I}(V_q(\lambda)) \simeq V_q(\lambda)$. $\square$

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Quantum Deformations of certain
Prehomogeneous Vector Spaces. II

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Quantum deformations of certain prehomogeneous vector spaces. II

YOSHIYUKI MORITA

Introduction

Let G be a reductive algebraic group over the complex number field $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. The quantized coordinate algebra $A_q(G)$ of G is constructed as a certain dual Hopf algebra of the quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. The Hopf algebras $U_q(\mathfrak{g})$ and $A_q(G)$ over $\mathbb{C}(q)$ tend to the ordinary enveloping algebra $U(\mathfrak{g})$ and the coordinate algebra $A(G)$ respectively when the parameter q tends to 1 in a certain sense (Drinfeld [1], Jimbo [3]).

Let us consider what object we should regard as a quantum deformation of an affine variety X with G -action.

An affine variety X is endowed with an action of G if and only if its coordinate algebra $A(X)$ is equipped with a right $A(G)$ -comodule structure

$$\tau : A(X) \rightarrow A(X) \otimes A(G)$$

which is simultaneously an algebra homomorphism. By the duality between $U(\mathfrak{g})$ and $A(G)$ we obtain a locally finite left $U(\mathfrak{g})$ -module structure

$$\gamma : U(\mathfrak{g}) \otimes A(X) \rightarrow A(X) \tag{*}$$

given by

$$\tau(n) = \sum_i n_i \otimes f_i \quad \Rightarrow \quad \gamma(u \otimes n) = \sum_i \langle u, f_i \rangle n_i, \tag{**}$$

where $\langle, \rangle : U(\mathfrak{g}) \times A(G) \rightarrow \mathbb{C}$ is the dual pairing. Since τ is an algebra homomorphism, we have

$$u \in U(\mathfrak{g}), m, n \in A(X), \Delta(u) = \sum_i u_i \otimes v_i \quad \Rightarrow \quad u(mn) = \sum_i (u_i m)(v_i n), \tag{***}$$

where $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the coproduct. Then the action of G on X is uniquely determined by the infinitesimal action γ . Moreover, for a locally finite left $U(\mathfrak{g})$ -module structure $(*)$ on $A(X)$ satisfying (***) and a certain condition on irreducible $U(\mathfrak{g})$ -modules appearing as submodules of $A(X)$, there exists a unique action of G on X whose infinitesimal action is given by γ .

Now we define the notion of a quantum deformation of an affine variety X with G -action as follows. A (not necessarily commutative) $\mathbb{C}(q)$ -algebra $A_q(X)$ endowed with a locally finite left $U_q(\mathfrak{g})$ -module structure

$$\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \rightarrow A_q(X)$$

is called a quantum deformation of X if $A_q(X)$ and γ_q tend to $A(X)$ and $\gamma : U(\mathfrak{g}) \otimes A(X) \rightarrow A(X)$ respectively when q tends to 1 and if it satisfies

$$u \in U_q(\mathfrak{g}), m, n \in A_q(X), \Delta(u) = \sum_i u_i \otimes v_i \Rightarrow u(mn) = \sum_i (u_i m)(v_i n).$$

It seems to be an interesting problem to determine in which case X admits a quantum deformation. In this paper we consider the problem when X is a prehomogeneous vector space, that is, when X is a vector space with a linear G -action containing an open G -orbit. Such a quantum deformation was intensively studied in the case where $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ and $X = M_{mn}(\mathbb{C})$ (see Taft-Towber [10], Hashimoto-Hayashi [2] and Noumi-Yamada-Mimachi [7]), and also in the case where $G = GL_n(\mathbb{C})$ and X is the set of skew symmetric matrices of degree n (see Strickland [8]).

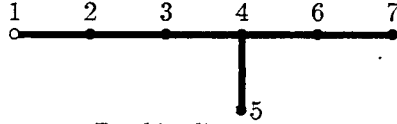
In our previous paper [4] we gave a general method to construct quantum deformations of prehomogeneous vector spaces of parabolic type. Moreover, for each non-open G -orbit C on X , we have shown that the defining ideal of the closure $\overline{C}$ and its canonical generators admit quantum deformations inside $A_q(X)$. It includes the existence of the quantum deformation of the irreducible relative invariant when X is a regular prehomogeneous vector space. Indeed, the canonical generator of the defining ideal of the closure of the one-codimensional orbit is nothing but the irreducible relative invariant.

Quantum deformations of prehomogeneous vector spaces of commutative parabolic type associated to classical simple Lie algebras are intensively studied in Kamita [5]. In this paper we shall deal with the remaining two cases

- (I) $G = \mathbb{C}^\times \times Spin(10, \mathbb{C})$, $X = \mathbb{C}^{16}$, the scalar multiplication and the half-spin representation,
- (II) $G = \mathbb{C}^\times \times E_6$, $X = \mathbb{C}^{27}$, the scalar multiplication and the 27-dimensional irreducible representation of E_6 ,

which naturally arise from the exceptional simple Lie algebras of type E_6 and E_7 respectively using the method in our previous paper [4]. In Introduction we shall only state the results in case (II).

Let $\mathfrak{g}_{E_7}$ be a simple Lie algebra of type E_7 over $\mathbb{C}$ and let $\mathfrak{h}$ be its Cartan subalgebra. We shall use the labelling of the vertices of the Dynkin diagram 1.



Dynkin diagram 1.

Set $I_0 = \{1, 2, \dots, 7\}$, $I = I_0 \setminus \{1\}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of type E_7 . We denote the set of simple roots by $\{\alpha_i\}_{i \in I_0}$ and the set of positive roots by Δ^+ . Let $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$ be a standard symmetric bilinear form. Set $D = \Delta^+ \setminus \sum_{i \in I} \mathbb{Z}\alpha_i$. Then we have $\#D = 27$. Set $\Lambda = \{1, 2, \dots, 27\}$, and fix a bijection $\Lambda \ni j \mapsto \beta_j \in D$ such that $\beta_k - \beta_j \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$ implies $j \leq k$, where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Set $\delta = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$. For each $n \in \Lambda$ there exist exactly five pairs $(i, j) \in \Lambda^2$ such that $\beta_i + \beta_j = \delta - \beta_n$, $i < j$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n$. Let $K_i^{\pm 1}, E_i, F_i$ ($i \in I_0$) be the canonical generators of $U_q(\mathfrak{g}_{E_7})$, and set $U_q(\mathfrak{g}) = \langle K_1^{\pm 3}, K_j^{\pm 1}, E_j, F_j \mid j \in I \rangle \subset U_q(\mathfrak{g}_{E_7})$. Then $U_q(\mathfrak{g})$ is isomorphic to the tensor product of $\mathbb{C}(q)[K, K^{-1}]$ and the quantized enveloping algebra of type E_6 , where $K = K_1^3 K_2^4 K_3^5 K_4^6 K_5^3 K_6^4 K_7^2$.

Theorem 0.1 *A quantum deformation of the 27-dimensional irreducible prehomogeneous vector space X of $G = \mathbb{C}^\times \times E_6$ is given by the following.*

(a) $A_q(X)$ is an associative $\mathbb{C}(q)$ -algebra defined by the following generators and fundamental relations:

Generators: Y_i with $i = 1, \dots, 27$.

Fundamental relations: For $i < j$

$$Y_i Y_j = \begin{cases} q Y_j Y_i & \text{if } \beta_i + \beta_j \text{ does not have another decomposition } \beta + \beta', \beta, \beta' \in D, \\ Y_j Y_i + q Y_b Y_a - q^{-1} Y_a Y_b & \text{if there exist } k \in I, a, b \in \Lambda \text{ such that } \beta_a = \beta_i + \alpha_k, \beta_b = \beta_j - \alpha_k, \\ Y_j Y_i & \text{otherwise.} \end{cases}$$

(b) The action $\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \rightarrow A_q(X)$ is given by the following.

For $2 \leq k \leq 7$, $1 \leq m \leq 7$

$$\begin{aligned}\gamma_q(F_k \otimes Y_i) &= \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i + \alpha_k, \\ 0 & \text{otherwise,} \end{cases} \\ \gamma_q(E_k \otimes Y_i) &= \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i - \alpha_k, \\ 0 & \text{otherwise,} \end{cases} \\ \gamma_q(K_m \otimes Y_i) &= q^{-(\alpha_m, \beta_i)} Y_i.\end{aligned}$$

(c) The quantum deformation of the irreducible relative invariant of X is given by

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n| - 1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ ($\beta = \sum_{i \in I_0} m_i \alpha_i$), $\psi_n = Y_{i_3^n} Y_{j_5^n} - q Y_{i_4^n} Y_{j_4^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_2^n} Y_{j_2^n} + q^4 Y_{i_1^n} Y_{j_1^n}$.

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1 Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra of type E_6 or E_7 over the complex number field $\mathbb{C}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system, and let $W \subset GL(\mathfrak{h})$ be the Weyl group. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where I_0 is an index set. For $i \in I_0$ we denote the simple reflection corresponding to α_i by $s_i \in W$. Let $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for any $\alpha \in \Delta$. Set $a_{ij} = (\alpha_i, \alpha_j)$. The matrix $(a_{ij})_{i, j \in I_0}$ is called the Cartan matrix of type E_6 or E_7 . For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. Set $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. For a subset $I \subset I_0$ we define

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i \mid i \in I \rangle.$$

We set

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right), \quad \mathfrak{n}_I^+ = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad \mathfrak{n}_I^- = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha}.$$

Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. We denote by L_I the subgroup of G corresponding to $\mathfrak{l}_I$. Then L_I acts on $\mathfrak{n}_I^\pm$ via the adjoint action.

The quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [3]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements E_i, F_i, K_i, K_i^{-1} ($i \in I_0$) satisfying the following fundamental relations:

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_i E_j &= q^{a_{ij}} E_j K_i, & K_i F_j &= q^{-a_{ij}} F_j K_i, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i E_j &= E_j E_i & (i \neq j, a_{ij} = 0), \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 & (i \neq j, a_{ij} = -1), \\ F_i F_j &= F_j F_i & (i \neq j, a_{ij} = 0), \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 & (i \neq j, a_{ij} = -1). \end{aligned}$$

A Hopf algebra structure on $U_q(\mathfrak{g})$ is defined as follows. The comultiplication $\Delta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$

The counit $\epsilon : U_q(\mathfrak{g}) \rightarrow \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The antipode $S : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i.$$

Using the Hopf algebra structure, we define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $\text{ad}(x)y = \sum_k x_k^1 y S(x_k^2)$. Then $\text{ad} : U_q(\mathfrak{g}) \rightarrow \text{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism. For $x, y, z \in U_q(\mathfrak{g})$ we have $\text{ad}(x)(yz) = \sum_k (\text{ad}(x_k^1)y)(\text{ad}(x_k^2)z)$, where $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$.

We define subalgebras $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{l}_I) = \langle E_i, F_i, K_j, K_j^{-1} \mid i \in I, j \in I_0 \rangle.$$

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$\begin{aligned} T_i(K_j) &= K_j K_i^{-a_{ij}}, \\ T_i(E_j) &= \begin{cases} -F_i K_i & (i = j) \\ E_j & (i \neq j, a_{ij} = 0) \\ E_i E_j - q^{-1} E_j E_i & (i \neq j, a_{ij} = -1), \end{cases} \\ T_i(F_j) &= \begin{cases} -K_i^{-1} E_i & (i = j) \\ F_j & (i \neq j, a_{ij} = 0) \\ F_j F_i - q F_i F_j & (i \neq j, a_{ij} = -1) \end{cases} \end{aligned}$$

(see Lusztig [6]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$ and set $T_w = T_{i_1} \cdots T_{i_r}$. It is known that T_w does not depend on the choice of a reduced expression.

We shall use the following later (see Lusztig [6]).

Lemma 1.1 *If $w(\alpha_i) = \alpha_j$ for $w \in W$ and $i, j \in I_0$, then we have $T_w(F_i) = F_j$.*

For $I \subset I_0$ let w_I be the longest element of W_I and let w_0 be the longest element of W . Choose a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ and set

$$\beta_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j}), \quad Y_j = Y_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(F_{i_j})$$

for $1 \leq j \leq r$. Then it is known that $\{\beta_j \mid 1 \leq j \leq r\} = \Delta^+ \setminus \Delta_I$. Set

$$U_q(\mathfrak{n}_I^-) = \sum_{d_j \geq 0} \mathbb{C}(q) Y_1^{d_1} \cdots Y_r^{d_r}.$$

Then $\{Y_1^{d_1} \cdots Y_r^{d_r} \mid d_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq r\}$ is a basis of $U_q(\mathfrak{n}_I^-)$ and $U_q(\mathfrak{n}_I^-)$ is a subalgebra of $U_q(\mathfrak{n}^-)$. we have

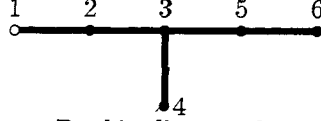
$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-)$$

and $U_q(\mathfrak{n}_I^-)$ does not depend on the choice of a reduced expression of $w_I w_0$ (see Lusztig [6]).

If $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$, then Y_β for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$ (see [4]). In this case we denote the $\mathbb{C}(q)$ -algebra $U_q(\mathfrak{n}_I^-)$ by A_q . We can regard it as a quantum deformation of the coordinate algebra $A = \mathbb{C}[\mathfrak{n}_I^+]$ of $\mathfrak{n}_I^+$ as explained in [4].

2 Case of type E_6

Let $\mathfrak{g}$ be a simple Lie algebra of type E_6 . We shall use the labelling of the vertices of the Dynkin diagram 2.



Dynkin diagram 2.

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6\}$. Set $I = \{2, 3, 4, 5, 6\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $\mathfrak{l}_I$ is isomorphic to $\mathbb{C} \oplus \mathfrak{o}(10, \mathbb{C})$ and $\mathfrak{n}_I^+$ is a 16-dimensional irreducible prehomogeneous vector space. There are three L_I -orbits $\{0\}, C_0, O$ on $\mathfrak{n}_I^+$ satisfying $\{0\} \subset \overline{C_0} \subset \overline{O}$. Let $J_{C_0} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_0 , and let $J_{C_0}^0$ denote the subspace of J_{C_0} consisting of the polynomials in J_{C_0} with homogeneous degree 2. Then $J_{C_0}^0$ is a ten-dimensional irreducible $\mathfrak{l}_I$ -module, and it generates the ideal J_{C_0} .

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ and define the elements Y_i ($i \in \Lambda = \{1, 2, \dots, 16\}$) as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5\}$, $I' = \{2, 3, 4, 5\}$, $\Lambda' = \{1, 2, \dots, 8\}$. Then $\{\alpha_i\}_{i \in I'_0}$ is a set of simple roots of type D_5 . Let $\mathfrak{g}'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to I'_0 . We choose a reduced expression $w_{I'} w_{I'_0} = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1$ of $w_{I'} w_{I'_0}$. The elements Y_i ($i \in \Lambda'$) can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \dots, m_6^j)$ for $j \in \Lambda$. Then we have

$$\begin{aligned}
 \mathbf{m}^1 &= (1, 0, 0, 0, 0, 0), & \mathbf{m}^2 &= (1, 1, 0, 0, 0, 0), & \mathbf{m}^3 &= (1, 1, 1, 0, 0, 0), \\
 \mathbf{m}^4 &= (1, 1, 1, 1, 0, 0), & \mathbf{m}^5 &= (1, 1, 1, 0, 1, 0), & \mathbf{m}^6 &= (1, 1, 1, 1, 1, 0), \\
 \mathbf{m}^7 &= (1, 1, 2, 1, 1, 0), & \mathbf{m}^8 &= (1, 2, 2, 1, 1, 0), & \mathbf{m}^9 &= (1, 1, 1, 0, 1, 1), \\
 \mathbf{m}^{10} &= (1, 1, 1, 1, 1, 1), & \mathbf{m}^{11} &= (1, 1, 2, 1, 1, 1), & \mathbf{m}^{12} &= (1, 2, 2, 1, 1, 1), \\
 \mathbf{m}^{13} &= (1, 1, 2, 1, 2, 1), & \mathbf{m}^{14} &= (1, 2, 2, 1, 2, 1), & \mathbf{m}^{15} &= (1, 2, 3, 1, 2, 1), \\
 \mathbf{m}^{16} &= (1, 2, 3, 2, 2, 1).
 \end{aligned}$$

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+$. Since $k \neq 1$ and $m_1^j = 1$, we have $\beta_j + \alpha_k \notin \Delta_I$. Therefore there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j + \alpha_k = \beta_l$ ($j, l \in \Lambda, k \in I$), then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

There exist 20 triplets $(k, j, l) \in I \times \Lambda \times \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. The triplets are the following: $(2, 1, 2), (3, 2, 3), (4, 3, 4), (5, 3, 5), (5, 4, 6), (4, 5, 6), (3, 6, 7), (2, 7, 8), (6, 5, 9), (4, 9, 10), (3, 10, 11), (2, 11, 12), (5, 11, 13), (5, 12, 14), (2, 13, 14), (3, 14, 15), (4, 15, 16), (6, 6, 10), (6, 7, 11), (6, 8, 12)$.

For $k \in I, j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Lemma 2.1 *Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ ($k \in I$). Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ and $p \in \Lambda$ satisfying*

$$\begin{aligned} \beta &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}), \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}}), (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1, \\ \alpha_k &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}). \end{aligned}$$

Proof. Among the 20 triplets (k, j, l) satisfying $\beta_j + \alpha_k = \beta_l$ ($k \in I, j, l \in \Lambda$), the 12 triplets satisfy $l = j + 1, (\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 8 cases. In the cases $(k, j, l) = (5, 3, 5), (5, 4, 6), (5, 11, 13), (5, 12, 14)$, the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_3 s_4 s_2 s_3 s_5 s_6$$

of $w_I w_0$ with $p = 3, 5, 11, 13$ respectively satisfies the required properties. In the cases $(k, j, l) = (6, 5, 9), (6, 6, 10), (6, 7, 11), (6, 8, 12)$, the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_3 s_5 s_2 s_3 s_1 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ with $p = 5, 7, 9, 11$ respectively satisfies the required properties. $\square$

It is known that $U_q(\mathfrak{n}_I^+)^1 = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_I} \mathbb{C}(q) Y_\beta$ is an irreducible $U_q(\mathfrak{l}_I)$ -module. (see [4])

Lemma 2.2 *For $k \in I, j \in \Lambda$, we have*

$$\begin{aligned} \text{ad}(F_k) Y_j &= \begin{cases} Y_l & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j + \alpha_k, \\ 0 & \text{otherwise,} \end{cases} \\ \text{ad}(E_k) Y_j &= \begin{cases} Y_l & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j - \alpha_k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Since $\bigoplus_{j \in \Lambda} \mathbb{C}(q) Y_j$ is a $U_q(\mathfrak{l}_I)$ -module, we have $\text{ad}(F_k) Y_j = 0$ if $\beta_j + \alpha_k \notin \Delta^+ \setminus \Delta_I$, and we have $\text{ad}(E_k) Y_j = 0$ if $\beta_j - \alpha_k \notin \Delta^+ \setminus \Delta_I$.

We shall show $\text{ad}(F_k) Y_\beta = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta + \alpha_k$. By Lemma 2.1 we can choose a reduced expression of $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ satisfying $\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}), \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}})$,

$(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$. Then we can write $Y_\beta = T_{i_1} T_{i_2} \cdots T_{i_{p-1}}(F_{i_p})$, $Y_{\beta'} = T_{i_1} T_{i_2} \cdots T_{i_{p-1}} T_{i_p}(F_{i_{p+1}})$. Since $(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$, we have $T_{i_p}(F_{i_{p+1}}) = F_{i_{p+1}} F_{i_p} - q F_{i_p} F_{i_{p+1}}$. Moreover, since $\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}})$, we have $T_{i_1} T_{i_2} \cdots T_{i_{p-1}}(F_{i_{p+1}}) = F_k$ by Lemma 1.1, and hence

$$\begin{aligned} Y_{\beta'} &= T_{i_1} T_{i_2} \cdots T_{i_{p-1}} T_{i_p}(F_{i_{p+1}}) = T_{i_1} T_{i_2} \cdots T_{i_{p-1}}(F_{i_{p+1}} F_{i_p} - q F_{i_p} F_{i_{p+1}}) \\ &= F_k Y_\beta - q Y_\beta F_k. \end{aligned}$$

Since $(\beta, \alpha_k) = -1$, we have $\text{ad}(F_k)Y_\beta = F_k Y_\beta - q Y_\beta F_k$. Hence we have $\text{ad}(F_k)Y_\beta = Y_{\beta'}$.

Let us show $\text{ad}(E_k)Y_\beta = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta - \alpha_k$. By the above argument we have $Y_\beta = \text{ad}(F_k)Y_{\beta'} = F_k Y_{\beta'} - q Y_{\beta'} F_k$. Since $\beta' - \alpha_k = \beta - 2\alpha_k \notin \Delta^+ \setminus \Delta_I$, we have $\text{ad}(E_k)Y_{\beta'} = 0$, and hence $E_k Y_{\beta'} = Y_{\beta'} E_k$. Since $(\beta', \alpha_k) = -1$, we have $K_k Y_{\beta'} = q Y_{\beta'} K_k$. Hence we have

$$\begin{aligned} \text{ad}(E_k)Y_\beta &= (E_k Y_\beta - Y_\beta E_k) K_k \\ &= (E_k (F_k Y_{\beta'} - q Y_{\beta'} F_k) - (F_k Y_{\beta'} - q Y_{\beta'} F_k) E_k) K_k \\ &= \left(\frac{K_k - K_k^{-1}}{q - q^{-1}} Y_{\beta'} - q Y_{\beta'} \frac{K_k - K_k^{-1}}{q - q^{-1}} \right) K_k = (Y_{\beta'} K_k^{-1}) K_k = Y_{\beta'}. \quad \square \end{aligned}$$

Next we shall consider quadratic fundamental relations among the elements Y_i . Since we have

$$\sum_{i,j \in \Lambda} \mathbb{C}(q) Y_i Y_j = \bigoplus_{s \leq t} \mathbb{C}(q) Y_s Y_t,$$

we can write

$$Y_i Y_j = \sum_{\substack{s \leq t \\ \beta_i + \beta_j = \beta_s + \beta_t}} a_{s,t}^{i,j} Y_s Y_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for $i > j$ (see [4]). Hence if $\beta_i + \beta_j$ does not have another decomposition $\beta + \beta'$ ($\beta, \beta' \in \Delta^+ \setminus \Delta_I$, $\beta_i + \beta_j = \beta + \beta'$) then we have $Y_i Y_j = a_{i,j} Y_j Y_i$ for some $a_{i,j} \in \mathbb{C}(q)$. We denote the set of weights of the ten-dimensional irreducible highest weight $\mathfrak{l}_I$ -module $J_{C_0}^0$ with highest weight $-\beta_1 - \beta_8$ by Γ . For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. We fix a bijection $\{1, 2, \dots, 10\} \ni n \mapsto -\delta_n \in \Gamma$ such that if $\delta_m - \delta_n \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$, then $n \leq m$. For each n there exist

exactly four pairs $(i, j) \in \Lambda^2$ such that $i < j, \beta_i + \beta_j = \delta_n$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n) \in \Lambda^2$ where $i_4^n < i_3^n < i_2^n < i_1^n$. Set $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n) \in \Lambda^8$ ($1 \leq n \leq 10$). Then we have

$$\begin{aligned} \mathbf{A}(1) &= (1, 2, 3, 4, 5, 6, 7, 8), & \mathbf{A}(2) &= (1, 2, 3, 4, 9, 10, 11, 12), \\ \mathbf{A}(3) &= (1, 2, 5, 6, 9, 10, 13, 14), & \mathbf{A}(4) &= (1, 3, 5, 7, 9, 11, 13, 15), \\ \mathbf{A}(5) &= (2, 3, 5, 8, 9, 12, 14, 15), & \mathbf{A}(6) &= (1, 4, 6, 7, 10, 11, 13, 16), \\ \mathbf{A}(7) &= (2, 4, 6, 8, 10, 12, 14, 16), & \mathbf{A}(8) &= (3, 4, 7, 8, 11, 12, 15, 16), \\ \mathbf{A}(9) &= (5, 6, 7, 8, 13, 14, 15, 16), & \mathbf{A}(10) &= (9, 10, 11, 12, 13, 14, 15, 16). \end{aligned}$$

We denote the set $\{i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n\}$ by $|\mathbf{A}(n)|$ for $1 \leq n \leq 10$. For any $i, j \in \Lambda$ there exists n satisfying $i, j \in |\mathbf{A}(n)|$.

Set

$$\mathcal{A} = \{(k, n, n') \in I \times \Lambda \times \Lambda \mid \delta_n + \alpha_k = \delta_{n'}\}.$$

Then $\mathcal{A} = \{(6, 1, 2), (5, 2, 3), (3, 3, 4), (2, 4, 5), (4, 4, 6), (2, 6, 7), (4, 5, 7), (3, 7, 8), (5, 8, 9), (6, 9, 10)\}$. For any $n \in \{2, 3, \dots, 10\}$ we can take a sequence $((k_1, n_1, n'_1), \dots, (k_s, n_s, n'_s))$ of $\mathcal{A}$ satisfying $n_1 = 1, n'_s = n, n'_j = n_{j+1}$ ($1 \leq j \leq s-1$).

For $(k, n, n') \in \mathcal{A}$ and $m \in \{1, 2, 3, 4\}$, we have either

$$(\beta_{i_m^n}, \alpha_k) = 0, i_m^{n'} = i_m^n, (\beta_{j_m^n}, \alpha_k) = -1, \beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k \quad (\mathbf{P}_m^+)$$

or

$$(\beta_{i_m^n}, \alpha_k) = -1, \beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k, (\beta_{j_m^n}, \alpha_k) = 0, j_m^{n'} = j_m^n. \quad (\mathbf{P}_m^-)$$

Proposition 2.3 For any $i, j \in \Lambda$ satisfying $i < j$, we have

$$(Q6) \quad Y_i Y_j = \begin{cases} Y_j Y_i & \text{if there exists } n \text{ such that } i = i_1^n, j = j_1^n, \\ Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \text{if there exists } n \text{ such that } i = i_2^n, j = j_2^n, \\ Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & \text{if there exist } n, m \in \{3, 4\} \text{ such that } i = i_m^n, j = j_m^n, \\ q Y_j Y_i & \text{otherwise.} \end{cases}$$

Proof. Since there exists some n satisfying $i, j \in |\mathbf{A}(n)|$ for any $i, j \in \Lambda$, it is sufficient to show that for any $1 \leq n \leq 10$ the elements $Y_{i_m^n}, Y_{j_m^n}$ ($1 \leq m \leq 4$) satisfy the following relations.

$$(Rn) \quad \begin{cases} Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n} & (Rn, 1) \\ Y_{i_m^n} Y_{j_m^n} = Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & (2 \leq m \leq 4) \quad (Rn, 2) \\ Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1} & (l_1, l_2 \in |\mathbf{A}(n)|, l_1 < l_2, (l_1, l_2) \neq (i_m^n, j_m^n) (1 \leq m \leq 4)) \quad (Rn, 3) \end{cases}$$

When $n = 1$, the elements Y_i ($1 \leq i \leq 8$) satisfy the same relations as those for type D_5 , hence the relations (R1) hold.

For any $m > 1$ there exists a sequence $((k_1, n_1, n'_1), \dots, (k_s, n_s, n'_s))$ of $\mathcal{A}$ satisfying $n_1 = 1, n'_s = m, n'_j = n_{j+1}$ ($1 \leq j \leq s-1$), and hence it is sufficient to show the relations (Rn') for $(k, n, n') \in \mathcal{A}$ assuming the relations (Rn).

Let $(k, n, n') \in \mathcal{A}$. Assume that the relations (Rn) hold.

We first show that the relation (Rn', 1) holds. If the condition (P_1^+) is satisfied, then we have $Y_{i_1^{n'}} = Y_{i_1^n}, F_k Y_{i_1^n} = Y_{i_1^n} F_k, Y_{j_1^{n'}} = \text{ad}(F_k) Y_{j_1^n} = F_k Y_{j_1^n} - q Y_{j_1^n} F_k$. Since $Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n}$, we have

$$\begin{aligned} Y_{i_1^{n'}} Y_{j_1^{n'}} &= Y_{i_1^n} \text{ad}(F_k) Y_{j_1^n} = Y_{i_1^n} (F_k Y_{j_1^n} - q Y_{j_1^n} F_k) = (F_k Y_{j_1^n} - q Y_{j_1^n} F_k) Y_{i_1^n} \\ &= Y_{j_1^{n'}} Y_{i_1^{n'}}. \end{aligned}$$

If the condition (P_1^-) is satisfied, then we can prove the formula (Rn', 1) similarly.

Next we prove the formula (Rn', 2). Assume the condition (P_m^+) is satisfied, then we have

$$\begin{aligned} Y_{i_m^{n'}} Y_{j_m^{n'}} &= Y_{i_m^n} (F_k Y_{j_m^n} - q Y_{j_m^n} F_k) \\ &= F_k Y_{j_m^n} Y_{i_m^n} - q Y_{j_m^n} F_k Y_{i_m^n} \\ &\quad + q (F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q Y_{j_{m-1}^n} Y_{i_{m-1}^n} F_k) \\ &\quad - q^{-1} (F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - q Y_{i_{m-1}^n} Y_{j_{m-1}^n} F_k). \end{aligned}$$

If the condition (P_{m-1}^+) is satisfied, then we have

$$\begin{aligned} F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q Y_{j_{m-1}^n} Y_{i_{m-1}^n} F_k &= Y_{j_{m-1}^n} (F_k Y_{i_{m-1}^n} - q Y_{i_{m-1}^n} F_k) = Y_{j_{m-1}^{n'}} Y_{i_{m-1}^{n'}}, \\ F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - q Y_{i_{m-1}^n} Y_{j_{m-1}^n} F_k &= (F_k Y_{i_{m-1}^n} - q Y_{i_{m-1}^n} F_k) Y_{j_{m-1}^n} = Y_{i_{m-1}^{n'}} Y_{j_{m-1}^{n'}}, \end{aligned}$$

and if the condition (P_{m-1}^-) is satisfied, then we have

$$\begin{aligned} F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q Y_{j_{m-1}^n} Y_{i_{m-1}^n} F_k &= (F_k Y_{j_{m-1}^n} - q Y_{j_{m-1}^n} F_k) Y_{i_{m-1}^n} = Y_{j_{m-1}^{n'}} Y_{i_{m-1}^{n'}}, \\ F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - q Y_{i_{m-1}^n} Y_{j_{m-1}^n} F_k &= Y_{i_{m-1}^n} (F_k Y_{j_{m-1}^n} - q Y_{j_{m-1}^n} F_k) = Y_{i_{m-1}^{n'}} Y_{j_{m-1}^{n'}}. \end{aligned}$$

Hence we have $Y_{i_m^{n'}} Y_{j_m^{n'}} = Y_{j_m^{n'}} Y_{i_m^{n'}} + q Y_{j_{m-1}^{n'}} Y_{i_{m-1}^{n'}} - q^{-1} Y_{i_{m-1}^{n'}} Y_{j_{m-1}^{n'}}$. The formula (Rn', 2) is proved. When the condition (P_m^-) is satisfied, we can prove it similarly.

Finally we prove the formula $(Rn', 3)$. Let $l'_1, l'_2 \in |\mathbf{A}(n')|$ satisfying $l'_1 < l'_2$ and $(l'_1, l'_2) \neq (i_m^{n'}, j_m^{n'})$ for $1 \leq m \leq 4$. When $l'_p = i_m^{n'} \in |\mathbf{A}(n')|$ (resp. $l'_p = j_m^{n'}$), we denote $i_m^{n'} \in |\mathbf{A}(n)|$ (resp. $j_m^{n'}$) by l_p for $p = 1, 2$. Since $l_1 < l_2$ and $(l_1, l_2) \neq (i_m^n, j_m^n)$ for $1 \leq m \leq 4$, we have $Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1}$. We have the following possibilities:

- (1) $l'_1 = l_1, l'_2 = l_2, (\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = 0$,
- (2) $l'_1 = l_1, (\beta_{l_1}, \alpha_k) = 0, \beta_{l'_2} = \beta_{l_2} + \alpha_k, (\beta_{l_2}, \alpha_k) = -1$,
- (3) $\beta_{l'_1} = \beta_{l_1} + \alpha_k, (\beta_{l_1}, \alpha_k) = -1, l'_2 = l_2, (\beta_{l_2}, \alpha_k) = 0$,
- (4) $\beta_{l'_1} = \beta_{l_1} + \alpha_k, \beta_{l'_2} = \beta_{l_2} + \alpha_k, (\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = -1$.

In the case (1) the formula $(Rn', 3)$ is obvious.

In the case (2) we have $F_k Y_{l_1} = Y_{l_1} F_k, Y_{l'_2} = \text{ad}(F_k) Y_{l_2} = F_k Y_{l_2} - q Y_{l_2} F_k$. Hence we have

$$Y_{l'_1} Y_{l'_2} = Y_{l_1} (F_k Y_{l_2} - q Y_{l_2} F_k) = q (F_k Y_{l_2} - q Y_{l_2} F_k) Y_{l_1} = q Y_{l_2} Y_{l'_1}.$$

In the case (3) we can prove it similarly to the case (2).

In the case (4) we have $Y_{l'_p} = F_k Y_{l_p} - q Y_{l_p} F_k$ for $p = 1, 2$. Since $\beta_{l'_p} + \alpha_k = \beta_{l_p} + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$ and $(\beta_{l'_p}, \alpha_k) = 1$, we have $\text{ad}(F_k) Y_{l'_p} = F_k Y_{l'_p} - q^{-1} Y_{l'_p} F_k = 0$ for $p = 1, 2$. Hence we have $F_k F_k Y_{l_p} - (q + q^{-1}) F_k Y_{l_p} F_k + Y_{l_p} F_k F_k = 0, F_k Y_{l_p} F_k = (q + q^{-1})^{-1} (F_k F_k Y_{l_p} + Y_{l_p} F_k F_k)$ for $p = 1, 2$. By these formulas we have

$$\begin{aligned} Y_{l'_1} Y_{l'_2} &= (F_k Y_{l_1} - q Y_{l_1} F_k)(F_k Y_{l_2} - q Y_{l_2} F_k) \\ &= F_k Y_{l_1} F_k Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k - q Y_{l_1} F_k F_k Y_{l_2} + q^2 Y_{l_1} F_k Y_{l_2} F_k \\ &= \frac{1}{q + q^{-1}} F_k F_k Y_{l_1} Y_{l_2} + \frac{1}{q + q^{-1}} Y_{l_1} F_k F_k Y_{l_2} \\ &\quad - q F_k Y_{l_1} Y_{l_2} F_k - q Y_{l_1} F_k F_k Y_{l_2} \\ &\quad + \frac{q^2}{q + q^{-1}} Y_{l_1} F_k F_k Y_{l_2} + \frac{q^2}{q + q^{-1}} Y_{l_1} Y_{l_2} F_k F_k \\ &= \frac{1}{q + q^{-1}} F_k F_k Y_{l_1} Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k + \frac{q^2}{q + q^{-1}} Y_{l_1} Y_{l_2} F_k F_k. \end{aligned}$$

Similarly we have

$$Y_{l'_2} Y_{l'_1} = \frac{1}{q + q^{-1}} F_k F_k Y_{l_2} Y_{l_1} - q F_k Y_{l_2} Y_{l_1} F_k + \frac{q^2}{q + q^{-1}} Y_{l_2} Y_{l_1} F_k F_k.$$

Since $Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1}$, we have $Y_{l'_1} Y_{l'_2} = q Y_{l'_2} Y_{l'_1}$. □

By [4] and Proposition 2.3 we obtain the following:

Theorem 2.4 *The formulas (Q6) give fundamental relations for the generator system $\{Y_i\}_{i \in \Lambda}$ of the algebra $A_q = U_q(\mathfrak{n}_I^-)$.*

We shall construct a quantum deformation of the lowest degree part $J_{C_0}^0$ of the defining ideal J_{C_0} and we shall give canonical generators of a quantum analogue of J_{C_0} .

Set

$$\psi_n = Y_{i_4^n} Y_{j_4^n} - q Y_{i_3^n} Y_{j_3^n} + q^2 Y_{i_2^n} Y_{j_2^n} - q^3 Y_{i_1^n} Y_{j_1^n},$$

for $1 \leq n \leq 10$. Recall that $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n)$. Using the formulas (Rn,1), (Rn,2), we can write $\psi_n = Y_{j_4^n} Y_{i_4^n} - q^{-1} Y_{j_3^n} Y_{i_3^n} + q^{-2} Y_{j_2^n} Y_{i_2^n} - q^{-3} Y_{j_1^n} Y_{i_1^n}$.

Lemma 2.5 *We have*

$$\begin{aligned} \text{ad}(F_k) \psi_n &= \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n + \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise,} \end{cases} \\ \text{ad}(E_k) \psi_n &= \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n - \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $k \in I$, and

$$\text{ad}(K_k) \psi_n = q^{-(\delta_n, \alpha_k)} \psi_n$$

for $k \in I_0$.

Proof. Let $(k, n, n') \in \mathcal{A}$. We shall show $\text{ad}(F_k) \psi_n = \psi_{n'}$. If the condition (P_m^+) is satisfied, then we have $\text{ad}(F_k) Y_{i_m^n} = 0$, $Y_{i_{m'}^{n'}} = Y_{i_m^n}$, $\text{ad}(K_k) Y_{i_m^n} = Y_{i_m^n}$, $\text{ad}(F_k) Y_{j_m^n} = Y_{j_{m'}^{n'}}$. Hence

$$\text{ad}(F_k)(Y_{i_m^n} Y_{j_m^n}) = (\text{ad}(F_k) Y_{i_m^n}) Y_{j_m^n} + (\text{ad}(K_k) Y_{i_m^n})(\text{ad}(F_k) Y_{j_m^n}) = Y_{i_{m'}^{n'}} Y_{j_{m'}^{n'}}.$$

If the condition (P_m^-) is satisfied, then we have $\text{ad}(F_k) Y_{i_m^n} = Y_{i_{m'}^{n'}}$, $\text{ad}(F_k) Y_{j_m^n} = 0$. Hence $\text{ad}(F_k)(Y_{i_m^n} Y_{j_m^n}) = Y_{i_{m'}^{n'}} Y_{j_{m'}^{n'}}$, similarly. Therefore we have $\text{ad}(F_k) \psi_n = \psi_{n'}$.

Next we prove $\text{ad}(E_k) \psi_{n'} = \psi_n$. We have $\text{ad}(E_k) Y_{i_{m'}^{n'}} = 0$, $\text{ad}(E_k) Y_{j_{m'}^{n'}} = Y_{j_m^n}$ if the condition (P_m^+) is satisfied, and we have $\text{ad}(E_k) Y_{i_{m'}^{n'}} = Y_{i_m^n}$, $\text{ad}(K_k^{-1}) Y_{j_{m'}^{n'}} = Y_{j_m^n}$, $j_m^{n'} = j_m^n$, $\text{ad}(E_k) Y_{j_m^n} = 0$ if the condition (P_m^-) is satisfied. Hence we have

$$\text{ad}(E_k)(Y_{i_{m'}^{n'}} Y_{j_{m'}^{n'}}) = (\text{ad}(E_k) Y_{i_{m'}^{n'}})(\text{ad}(K_k^{-1}) Y_{j_{m'}^{n'}}) + Y_{i_{m'}^{n'}}(\text{ad}(E_k) Y_{j_{m'}^{n'}}) = Y_{i_m^n} Y_{j_m^n}$$

for $1 \leq m \leq 4$. Therefore we have $\text{ad}(E_k)\psi_n = \psi_n$.

In other 50 cases, where $\delta_n + \alpha_k \notin \{\delta_l \mid 1 \leq l \leq 10\}$, we can check $\text{ad}(F_k)\psi_n = 0$ by a case-by-case consideration as follows.

In the 10 cases where there exists n' satisfying $\text{ad}(F_k)\psi_{n'} = \psi_n$, $((k, n) = (6, 2), (5, 3), (3, 4), (2, 5), (4, 6), (2, 7), (4, 7), (3, 8), (5, 9), (6, 10))$, we have $\text{ad}(F_k)Y_{i_m^n} = \text{ad}(F_k)Y_{j_m^n} = 0$ for $1 \leq m \leq 4$, and hence the assertion is obvious.

In the 8 cases $(k, n) = (5, 1), (6, 3), (6, 4), (6, 5), (6, 6), (6, 7), (6, 8), (5, 10)$, we have $\text{ad}(F_k)Y_{i_m^n} = \text{ad}(F_k)Y_{j_m^n} = 0$ for $m = 3, 4$, $\text{ad}(F_k)Y_{i_2^n} = Y_{j_1^n}$, $\text{ad}(F_k)Y_{j_2^n} = 0$, $\text{ad}(F_k)Y_{i_1^n} = Y_{j_2^n}$, $\text{ad}(F_k)Y_{j_1^n} = 0$, and hence $\text{ad}(F_k)(Y_{i_2^n}Y_{j_2^n}) = Y_{j_1^n}Y_{j_2^n}$, $\text{ad}(F_k)(Y_{i_1^n}Y_{j_1^n}) = Y_{j_2^n}Y_{j_1^n}$. Thus we have $\text{ad}(F_k)\psi_n = q^2(Y_{j_1^n}Y_{j_2^n} - qY_{j_2^n}Y_{j_1^n}) = 0$ by Proposition 2.3.

In the remaining 32 cases there exists $m' \in \{2, 3, 4\}$ such that $\text{ad}(F_k)Y_{i_m^n} = 0$ ($m \neq m'$), $\text{ad}(F_k)Y_{j_m^n} = 0$ ($m \neq m' - 1$), $\text{ad}(F_k)Y_{i_{m'}^n} = Y_{i_{m'-1}^n}$, $\text{ad}(F_k)Y_{j_{m'-1}^n} = Y_{j_{m'}^n}$, $\text{ad}(F_k)Y_{i_{m'-1}^n} = q^{-1}Y_{i_{m'}^n}$. Then we have $\text{ad}(F_k)(Y_{i_{m'}^n}Y_{j_{m'}^n}) = Y_{i_{m'-1}^n}Y_{j_{m'}^n}$, $\text{ad}(F_k)(Y_{i_{m'-1}^n}Y_{j_{m'-1}^n}) = q^{-1}Y_{i_{m'}^n}Y_{j_{m'}^n}$, $\text{ad}(F_k)\psi_n = q^{4-m'}(1 - qq^{-1})Y_{i_{m'-1}^n}Y_{j_{m'}^n} = 0$.

The weight $\beta_{i_m^n} + \beta_{j_m^n}$ does not depend on m . Hence we have $\text{ad}(K_k)\psi_n = q^{-(\delta_n, \alpha_k)}\psi_n$ where $\delta_n = \beta_{i_m^n} + \beta_{j_m^n}$.

Finally we show $\text{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \mid 1 \leq l \leq 10\}$. We can check $\text{ad}(E_k)\psi_1 = 0$ for any $k = 2, 3, \dots, 6$ directly. It follows that $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n = U_q(\mathfrak{l}_I)\psi_1$ and hence $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is an $\text{ad } U_q(\mathfrak{l}_I)$ -stable subspace with weights in $\{-\delta_l \mid 1 \leq l \leq 10\}$. Therefore we have $\text{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \mid 1 \leq l \leq 10\}$. $\square$

Proposition 2.6 $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(\mathfrak{l}_I)$ -module with highest weight vector ψ_1 .

Proof. By Lemma 2.5 $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is a finite dimensional $U_q(\mathfrak{l}_I)$ -submodule generated by a highest weight vector ψ_1 with highest weight $-\delta_1$. Thus it is irreducible. $\square$

By [4] and Proposition 2.6 we obtain the following:

Theorem 2.7 A quantum analogue of the defining ideal J_{C_0} of the closure of the non-trivial non-open orbit C_0 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid 1 \leq n \leq 10\}$.

3 Case of type E_7

Let $\mathfrak{g}$ be a simple Lie algebra of type E_7 . We shall use the labelling of the vertices of the Dynkin diagram 1. Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $I = \{2, 3, 4, 5, 6, 7\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $\mathfrak{l}_I$ is isomorphic to $\mathbb{C} \oplus \mathfrak{g}_{E_6}$, where $\mathfrak{g}_{E_6}$ is a Lie algebra of type E_6 over $\mathbb{C}$, and $\mathfrak{n}_I^+$ is a 27-dimensional irreducible prehomogeneous vector space. There are four $\mathfrak{l}_I$ -orbits $\{0\}, C_1, C_2, O$ on $\mathfrak{n}_I^+$ satisfying $\{0\} \subset \overline{C_1} \subset \overline{C_2} \subset \overline{O}$. Let $J_{C_1} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_1 , and let $\mathcal{J}_{C_1}^0$ denote the subspace of J_{C_1} consisting of the polynomials in J_{C_1} with homogeneous degree 2. Then $\mathcal{J}_{C_1}^0$ is a 27-dimensional irreducible $\mathfrak{l}_I$ -module, and it generates the ideal J_{C_1} . Let $J_{C_2} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_2 , and let $\mathcal{J}_{C_2}^0$ denote the subspace of J_{C_2} consisting of the polynomials in J_{C_2} with homogeneous degree 3. Then $\mathcal{J}_{C_2}^0$ is a one-dimensional irreducible $\mathfrak{l}_I$ -module generated by the irreducible relative invariant, and it generates the ideal J_{C_2} .

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_3 s_5 s_4 s_6 s_7 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$$

of $w_I w_0$ and define the elements Y_i ($i \in \Lambda = \{1, 2, \dots, 27\}$) as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5, 6\}$, $I' = \{2, 3, 4, 5, 6\}$, $\Lambda' = \{1, 2, \dots, 10\}$. Then $\{\alpha_i\}_{i \in I'_0}$ is a set of simple roots of type D_6 . Let $\mathfrak{g}'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to I'_0 . We choose a reduced expression $w_{I'} w_{I'_0} = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$ of $w_{I'} w_{I'_0}$. The elements Y_i ($i \in \Lambda'$) can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \dots, m_7^j)$ for $j \in \Lambda$. Then we have

$$\begin{aligned} \mathbf{m}^1 &= (1, 0, 0, 0, 0, 0, 0), & \mathbf{m}^2 &= (1, 1, 0, 0, 0, 0, 0), & \mathbf{m}^3 &= (1, 1, 1, 0, 0, 0, 0), \\ \mathbf{m}^4 &= (1, 1, 1, 1, 0, 0, 0), & \mathbf{m}^5 &= (1, 1, 1, 1, 1, 0, 0), & \mathbf{m}^6 &= (1, 1, 1, 1, 0, 1, 0), \\ \mathbf{m}^7 &= (1, 1, 1, 1, 1, 1, 0), & \mathbf{m}^8 &= (1, 1, 1, 2, 1, 1, 0), & \mathbf{m}^9 &= (1, 1, 2, 2, 1, 1, 0), \\ \mathbf{m}^{10} &= (1, 2, 2, 2, 1, 1, 0), & \mathbf{m}^{11} &= (1, 1, 1, 1, 0, 1, 1), & \mathbf{m}^{12} &= (1, 1, 1, 1, 1, 1, 1), \\ \mathbf{m}^{13} &= (1, 1, 1, 2, 1, 1, 1), & \mathbf{m}^{14} &= (1, 1, 2, 2, 1, 1, 1), & \mathbf{m}^{15} &= (1, 1, 1, 2, 1, 2, 1), \\ \mathbf{m}^{16} &= (1, 1, 2, 2, 1, 2, 1), & \mathbf{m}^{17} &= (1, 1, 2, 3, 1, 2, 1), & \mathbf{m}^{18} &= (1, 1, 2, 3, 2, 2, 1), \\ \mathbf{m}^{19} &= (1, 2, 2, 2, 1, 1, 1), & \mathbf{m}^{20} &= (1, 2, 2, 2, 1, 2, 1), & \mathbf{m}^{21} &= (1, 2, 2, 3, 1, 2, 1), \\ \mathbf{m}^{22} &= (1, 2, 2, 3, 2, 2, 1), & \mathbf{m}^{23} &= (1, 2, 3, 3, 1, 2, 1), & \mathbf{m}^{24} &= (1, 2, 3, 3, 2, 2, 1), \\ \mathbf{m}^{25} &= (1, 2, 3, 4, 2, 2, 1), & \mathbf{m}^{26} &= (1, 2, 3, 4, 2, 3, 1), & \mathbf{m}^{27} &= (1, 2, 3, 4, 2, 3, 2). \end{aligned}$$

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+ \setminus \Delta_I$ and there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j, \beta_l \in \Delta^+ \setminus \Delta_I$ satisfying $\beta_l - \beta_j = \alpha_k$ ($k \in I$), then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Set

$$\mathcal{B} = \{(k, j, l) \in I \times \Lambda \times \Lambda \mid \beta_j + \alpha_k = \beta_l\}.$$

We have

$$\begin{aligned} \mathcal{B} = & \{(2, 1, 2), (3, 2, 3), (4, 3, 4), (5, 4, 5), (6, 4, 6), (6, 5, 7), (5, 6, 7), (4, 7, 8), \\ & (3, 8, 9), (2, 9, 10), (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19), (5, 11, 12), \\ & (4, 12, 13), (3, 13, 14), (6, 13, 15), (6, 14, 16), (3, 15, 16), (4, 16, 17), (5, 17, 18), \\ & (2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22), (6, 19, 20), (4, 20, 21), (5, 21, 22), \\ & (3, 21, 23), (3, 22, 24), (5, 23, 24), (4, 24, 25), (6, 25, 26), (7, 26, 27)\}. \end{aligned}$$

In particular, we have $|\mathcal{B}| = 36$.

Lemma 3.1 *Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ ($k \in I$). Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{27}}$ and $p \in \Lambda$ satisfying*

$$\begin{aligned} \beta &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}), \quad \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}}), \quad (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1, \\ \alpha_k &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}). \end{aligned}$$

Proof. The 21 triplets (k, j, l) in $\mathcal{B}$ satisfy $l = j + 1$, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 15 cases. In the cases $(k, j, l) = (6, 4, 6), (6, 5, 7), (6, 13, 15), (6, 14, 16), (3, 21, 23), (3, 22, 24)$, we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_5 s_3 s_4 s_6 s_7 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$$

with $p = 4, 6, 13, 15, 21, 23$, and in the cases $(k, j, l) = (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19)$, we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_4 s_6 s_3 s_4 s_2 s_3 s_1 s_2 s_5 s_4 s_6 s_7 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$$

with $p = 6, 8, 10, 12, 14$, and in the cases $(k, j, l) = (2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22)$, we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_5 s_3 s_2 s_4 s_3 s_6 s_4 s_7 s_6 s_5 s_4 s_3 s_2 s_1$$

with $p = 15, 17, 19, 21$. □

We can show the following similarly to the case E_6 . We omit the details.

Lemma 3.2 *For $k \in I$, $j \in \Lambda$, we have*

$$\begin{aligned} \text{ad}(F_k)Y_j &= \begin{cases} Y_l & \text{if there exists } (k, j, l) \in \mathcal{B}, \\ 0 & \text{otherwise,} \end{cases} \\ \text{ad}(E_k)Y_j &= \begin{cases} Y_l & \text{if there exists } (k, l, j) \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The $U_q(\mathfrak{l}_I)$ -module $\bigoplus_{j \in \Lambda} \mathbb{C}(q)Y_j$ is an irreducible highest weight module with highest weight vector Y_1 and lowest weight vector Y_{27} . Hence, for any $1 \leq m \leq 26$, there exists a sequence $((k_1, n'_1, n_1), \dots, (k_s, n'_s, n_s))$ of $\mathcal{B}$ satisfying $n_1 = 27, n'_s = m, n'_j = n_{j+1} (1 \leq j \leq s-1)$.

Next we shall consider relations among the elements Y_i . We can write

$$Y_i Y_j = \sum_{\substack{\beta_i + \beta_j \stackrel{s \leq t}{=} \beta_s + \beta_t}} a_{s,t}^{i,j} Y_s Y_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for $i > j$ (see [4]). Hence if $\beta_i + \beta_j$ does not have another decomposition $\beta + \beta' (\beta, \beta' \in \Delta^+ \setminus \Delta_I, \beta_i + \beta_j = \beta + \beta')$ then we have $Y_i Y_j = a_{i,j} Y_j Y_i$ for some $a_{i,j} \in \mathbb{C}(q)$. Set $\delta = 2\varpi_1 = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$, where ϖ_1 is the fundamental weight corresponding to α_1 . We denote a set of weights of the 27-dimensional irreducible highest weight $\mathfrak{l}_I$ -module $J_{C_1}^0$ with highest weight $-\beta_1 - \beta_{10}$ by Γ . Set $\gamma_n = \delta - \beta_n (n \in \Lambda)$, and we have $\Gamma = \{-\gamma_n \mid n \in \Lambda\}$. For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. For each $n \in \Lambda$ there exist exactly five pairs $(i, j) \in \Lambda^2$ such that $i < j, \beta_i + \beta_j = \gamma_n$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n, j_1^n < j_2^n < j_3^n < j_4^n < j_5^n$, and i_1^n, j_1^n satisfy the following condition (P_1^+) or (P_1^-) : Set $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10} (n \in \Lambda)$. Then we have

$$\begin{aligned} \mathbf{B}(1) &= (10, 19, 20, 21, 23, 22, 24, 25, 26, 27), & \mathbf{B}(2) &= (9, 14, 16, 17, 23, 18, 24, 25, 26, 27), \\ \mathbf{B}(3) &= (8, 13, 15, 17, 21, 18, 22, 25, 26, 27), & \mathbf{B}(4) &= (7, 12, 15, 16, 20, 18, 22, 24, 26, 27), \\ \mathbf{B}(5) &= (6, 11, 15, 16, 20, 17, 21, 23, 26, 27), & \mathbf{B}(6) &= (5, 12, 13, 14, 19, 18, 22, 24, 25, 27), \\ \mathbf{B}(7) &= (4, 11, 13, 14, 19, 17, 21, 23, 25, 27), & \mathbf{B}(8) &= (3, 11, 12, 14, 19, 16, 20, 23, 24, 27), \\ \mathbf{B}(9) &= (2, 11, 12, 13, 19, 15, 20, 21, 22, 27), & \mathbf{B}(10) &= (1, 11, 12, 13, 14, 15, 16, 17, 18, 27), \\ \mathbf{B}(11) &= (5, 7, 8, 9, 10, 18, 22, 24, 25, 26), & \mathbf{B}(12) &= (4, 6, 8, 9, 10, 17, 21, 23, 25, 26), \\ \mathbf{B}(13) &= (3, 6, 7, 9, 10, 16, 20, 23, 24, 26), & \mathbf{B}(14) &= (2, 6, 7, 8, 10, 15, 20, 21, 22, 26), \\ \mathbf{B}(15) &= (3, 4, 5, 9, 10, 14, 19, 23, 24, 25), & \mathbf{B}(16) &= (2, 4, 5, 8, 10, 13, 19, 21, 22, 25), \\ \mathbf{B}(17) &= (2, 3, 5, 7, 10, 12, 19, 20, 22, 24), & \mathbf{B}(18) &= (2, 3, 4, 6, 10, 11, 19, 20, 21, 23), \\ \mathbf{B}(19) &= (1, 6, 7, 8, 9, 15, 16, 17, 18, 26), & \mathbf{B}(20) &= (1, 4, 5, 8, 9, 13, 14, 17, 18, 25), \\ \mathbf{B}(21) &= (1, 3, 5, 7, 9, 12, 14, 16, 18, 24), & \mathbf{B}(22) &= (1, 3, 4, 6, 9, 11, 14, 16, 17, 23), \\ \mathbf{B}(23) &= (1, 2, 5, 7, 8, 12, 13, 15, 18, 22), & \mathbf{B}(24) &= (1, 2, 4, 6, 8, 11, 13, 15, 17, 21), \\ \mathbf{B}(25) &= (1, 2, 3, 6, 7, 11, 12, 15, 16, 20), & \mathbf{B}(26) &= (1, 2, 3, 4, 5, 11, 12, 13, 14, 19), \\ \mathbf{B}(27) &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10). \end{aligned}$$

For $n \in \Lambda$ we denote the set $\{i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n\}$ by $|\mathbf{B}(n)|$. For any $i, j \in \Lambda$ there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$.

For $(k, n', n) \in \mathcal{B}$ and $m \in \{1, 2, 3, 4, 5\}$, we have either

$$(\beta_{i_m^n}, \alpha_k) = 0, i_m^{n'} = i_m^n, (\beta_{j_m^n}, \alpha_k) = -1, \beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k \quad (P_m^+)$$

or

$$(\beta_{i_m^n}, \alpha_k) = -1, \beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k, (\beta_{j_m^n}, \alpha_k) = 0, j_m^{n'} = j_m^n. \quad (P_m^-)$$

Proposition 3.3 For any $i, j \in \Lambda$ satisfying $i < j$, we have

$$(Q7) \quad Y_i Y_j = \begin{cases} Y_j Y_i & \text{if there exists } n \in \Lambda \text{ such that } \{i, j\} = \{i_1^n, j_1^n\}, \\ Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \text{if there exists } n \in \Lambda \text{ such that } i = i_2^n, j = j_2^n, \\ Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & \text{if there exist } n \in \Lambda, m \in \{3, 4, 5\} \text{ such that } i = i_m^n, j = j_m^n, \\ q Y_j Y_i & \text{otherwise.} \end{cases}$$

Proof. Since there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$ for any $i, j \in \Lambda$, it is sufficient to show

$$(Rn) \quad \begin{cases} Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n} & (Rn, 1) \\ Y_{i_m^n} Y_{j_m^n} = Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} \quad (2 \leq m \leq 5) & (Rn, 2) \\ Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1} & (Rn, 3) \end{cases} \quad \begin{cases} (l_1, l_2 \in |\mathbf{B}(n)|, l_1 < l_2, \{l_1, l_2\} \neq \{i_m^n, j_m^n\} \quad (1 \leq m \leq 5)) \end{cases}$$

for $n \in \Lambda$ and $1 \leq m \leq 5$.

When $n = 27$, the elements Y_i ($1 \leq i \leq 10$) satisfy the same relations as those for type D_6 , and hence relations (R27) hold.

Since there exists a sequence $((k_1, n'_1, n_1), \dots, (k_s, n'_s, n_s))$ of $\mathcal{B}$ satisfying $n_1 = 27, n'_s = m, n'_j = n_{j+1}$ ($1 \leq j \leq s-1$) for any $1 \leq m \leq 26$, it is sufficient to show (Rn') for $(k, n', n) \in \mathcal{B}$ assuming (Rn). This is proved similarly to Proposition 2.3. Details are omitted. $\square$

By [4] and Proposition 3.3 we obtain the following:

Theorem 3.4 The formulas (Q7) give fundamental relations for the generator system $\{Y_i\}_{i \in \Lambda}$ of the algebra $A_q = U_q(\mathfrak{n}_l^-)$.

We shall construct a quantum deformation of the lowest degree part $J_{C_1}^0$ of the defining ideal J_{C_1} and we shall give canonical generators of a quantum deformation of J_{C_1} .

Set

$$\psi_n = Y_{i_5^n} Y_{j_5^n} - q Y_{i_4^n} Y_{j_4^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_2^n} Y_{j_2^n} + q^4 Y_{i_1^n} Y_{j_1^n},$$

for $n \in \Lambda$, where $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n)$. Using the formulas (Rn,1), (Rn,2), we can write $\psi_n = Y_{j_5^n} Y_{i_5^n} - q^{-1} Y_{j_4^n} Y_{i_4^n} + q^{-2} Y_{j_3^n} Y_{i_3^n} - q^{-3} Y_{j_2^n} Y_{i_2^n} + q^{-4} Y_{j_1^n} Y_{i_1^n}$.

Similarly to Lemma 2.5 and Proposition 2.6 we can show the following:

Lemma 3.5 *We have*

$$\begin{aligned} \text{ad}(F_k)\psi_n &= \begin{cases} \psi_{n'} & \text{if there exists } (k, n', n) \in \mathcal{B}, \\ 0 & \text{otherwise,} \end{cases} \\ \text{ad}(E_k)\psi_n &= \begin{cases} \psi_{n'} & \text{if there exists } (k, n, n') \in \mathcal{B}, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $k \in I$, and

$$\text{ad}(K_k)\psi_n = q^{-(\gamma_n, \alpha_k)} \psi_n$$

for $k \in I_0$.

Proposition 3.6 $\sum_{n \in \Lambda} \mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(\mathfrak{l}_I)$ -module with highest weight vector ψ_{27} .

By [4] and Proposition 3.6 we obtain the following:

Theorem 3.7 A quantum deformation of the defining ideal J_{C_1} of the closure of the non-open orbit C_1 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid n \in \Lambda\}$.

Set

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n|-1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ ($\beta = \sum_{i \in I_0} m_i \alpha_i$).

Proposition 3.8 $\mathbb{C}(q)\varphi$ is a one-dimensional $U_q(\mathfrak{l}_I)$ -module.

Proof. By Proposition 3.3 we can check that the coefficient $a_{1,10,27}$ of $Y_1 Y_{10} Y_{27}$ in $\varphi = \sum_{i < j < k} a_{ijk} Y_i Y_j Y_k$ is $1 + q^8 + q^{16}$. Therefore we have $\varphi \neq 0$.

Let $(k, n, n') \in \mathcal{B}$. Then we have $|\beta_{n'}| = |\beta_n| + 1$, $\text{ad}(F_k)Y_n = Y_{n'}$, $\text{ad}(F_k)Y_{n'} = 0$, $\text{ad}(F_k)\psi_{n'} = \psi_n$, $\text{ad}(F_k)\psi_n = 0$, $(\beta_{n'}, \alpha_k) = 1$. Hence $\text{ad}(F_k)(Y_n \psi_n - q Y_{n'} \psi_{n'}) = Y_{n'} \psi_n - q q^{-1} Y_{n'} \psi_n = 0$. Therefore we have $\text{ad}(F_k)\varphi = 0$ for any $k \in I$, and similarly we have $\text{ad}(E_k)\varphi = 0$ for any $k \in I$. Since $\gamma_n + \beta_n = \delta$ for

any $n \in \Lambda$, we have $\text{ad}(K_k)\varphi = q^{-(\delta, \alpha_k)}\varphi$ for any $k \in I_0$. In particular, we have $\text{ad}(K_k)\varphi = \varphi$ for any $k \in I$, and $\text{ad}(K_1)\varphi = q^{-2}\varphi$. $\square$

The element φ is a quantum deformation of the irreducible relative invariant on the prehomogeneous vector space.

Theorem 3.9 *A quantum deformation of the defining ideal J_{C_2} of the closure of the non-open orbit C_2 is given by the two-sided ideal of A_q generated by φ .*

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