Quantum Deformations of certain Prehomogeneous Vector Spaces

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This thesis consits of two parts. In Part I, the general theory is treated. This part has been published in Hiroshima Mathematical Journal, Vol. 28, No. 3 (1998) as a joint paper with A. Kamita and T. Tanisaki. In Part II, some special cases are treated in detail. This part has been accepted for publication in Osaka Journal of Mathematics.

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Quantum deformations of certain prehomogeneous vector spaces I

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Abstract. We shall construct a quantum analogue of the prehomogeneous vector space associated to a parabolic subgroup with commutative unipotent radical.

0. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbf{C}$, and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}^+$ be a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{l}$ is a maximal reductive subalgebra of $\mathfrak{p}$ and $\mathfrak{m}^+$ is the nilpotent part. We denote by $\mathfrak{m}^-$ the nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{l} \oplus \mathfrak{m}^-$ is a parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. Take an algebraic group L with Lie algebra $\mathfrak{l}$.

In this paper we shall deal with the case where $\mathfrak{m}^{\pm}$ is nonzero and commutative. Then $\mathfrak{m}^+$ consists of finitely many L-orbits.

Our aim is to give a quantum analogue of the prehomogeneous vector space $(L, \mathfrak{m}^+)$. More precisely, we shall construct a quantum analogue A_q of the ring $A = \mathbf{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$ as a noncommutative $\mathbf{C}(q)$-algebra endowed with the action of the quantized enveloping algebra $U_q(\mathfrak{l})$ of $\mathfrak{l}$, and show that for each L-orbit C on $\mathfrak{m}^+$ there exists a two-sided ideal $J_{C,q}$ of A_q which can be regarded as a quantum analogue of the defining ideal J_C of the closure $\bar{C}$ of C. Such an object was intensively studied in the cases $\mathfrak{g} = \mathfrak{sl}_n$ (see Hashimoto-Hayashi [3], Noumi-Yamada-Mimachi [10]) and $\mathfrak{g} = \mathfrak{so}_{2n}$ (see Strickland [13]).

Our method is as follows. Since $\mathfrak{m}^-$ is identified with the dual space of $\mathfrak{m}^+$ via the Killing form, A is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By the commutativity of $\mathfrak{m}^-$ the enveloping algebra $U(\mathfrak{m}^-)$ is naturally identified with the symmetric algebra $S(\mathfrak{m}^-)$. Hence we have an identification $A = U(\mathfrak{m}^-)$. Then using the Poincaré-Birkhoff-Witt type basis of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Lusztig [9]) we obtain a natural quantization A_q of A as a subalgebra of $U_q(\mathfrak{g})$. The algebra A_q has a canonical generator system satisfying quadratic fundamental relations. In particular, it is a graded algebra. The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined using the Hopf

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algebra structure, and we can show that A_q is preserved under the adjoint action of $U_q(\mathfrak{l})$. As a $U_q(\mathfrak{l})$-module A_q is a direct sum of finite dimensional irreducible submodules.

Let C be a non-open L-orbit on $\mathfrak{m}^+$. It is known that J_C is an $\mathfrak{l}$-stable homogeneous ideal generated by the lowest degree part J_C^0. Since A is a multiplicity free $\mathfrak{l}$-module, there exist unique $U_q(\mathfrak{l})$-submodules $J_{C,q}$ and $J_{C,q}^0$ of A_q satisfying $J_{C,q}|_{q=1} = J_C$ and $J_{C,q}^0|_{q=1} = J_C^0$. We can show that $J_{C,q}$ is a two-sided ideal of A_q and that $J_{C,q}$ is generated by $J_{C,q}^0$ both as a left ideal and a right ideal. The proof uses the quantum counterpart of the results on a generalized Verma module of $\mathfrak{g}$ whose maximal proper submodule is explicitly described in terms of J_C (see Enright-Joseph [2], Tanisaki [14]).

Explicit descriptions of A_q and $J_{C,q}$ in each individual case will be given in our subsequent papers.

1. Quantized enveloping algebras

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We fix an ordering on Δ, and denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where I_0 is an index set. We set

$$\mathfrak{n}^+ = \bigoplus\nolimits_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus\nolimits_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to i respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ satisfying $[e_i, f_i] = h_i$. Let $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α. Set

$$d_i = (\alpha_i, \alpha_i)/2 \quad (i \in I_0), \qquad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0).$$

For a subset I of I_0 we set

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, \qquad W_I = \langle s_i \mid i \in I \rangle,$$

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus\nolimits_{\alpha \in \Delta_I} \mathfrak{g}_\alpha\right), \quad \mathfrak{n}_I^+ = \bigoplus\nolimits_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad \mathfrak{n}_I^- = \bigoplus\nolimits_{\alpha \in -\Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha.$$

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$.

Let us recall the definition of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [7]). It is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i \in I_0}$ satisfying the following fundamental relations:

$$K_iK_j = K_jK_i,$$

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j,$$

$$K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}E_i^{1-a_{ij}-k}E_jE_i^k = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}F_i^{1-a_{ij}-k}F_jF_i^k = 0 \quad (i \neq j),$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^{m}[k]_t \quad \begin{bmatrix}m\\ n\end{bmatrix}_t = \frac{[m]_t!}{[n]_t![m-n]_t!} \quad (m \geq n \geq 0).$$

For $i \in I_0$ and $n \in \mathbb{Z}_{\geq 0}$ we set

$$E_i^{(n)} = \frac{1}{[n]_{q_i}!}E_i^n, \qquad F_i^{(n)} = \frac{1}{[n]_{q_i}!}F_i^n.$$

The algebra $U_q(\mathfrak{g})$ is endowed with a Hopf algebra structure via the following formula:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i,$$

where $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $\varepsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ are the algebra homomorphisms giving the comultiplication and the counit respectively, and $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra anti-automorphism giving the antipode.

We define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For x, $y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\operatorname{ad} x)(y) = \sum_k x_k^1 y S(x_k^2)$. Then

$$\operatorname{ad} : U_q(\mathfrak{g}) \to \operatorname{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$$

is a homomorphism of algebras.

Define subalgebras $U_q(\mathfrak{n}^\pm)$, $U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^+) = \langle E_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{h}) = \langle K_i^{\pm 1} \mid i \in I_0 \rangle,$$
$$U_q(\mathfrak{l}_I) = \langle K_i^{\pm 1}, E_j, F_j \mid i \in I_0, j \in I \rangle.$$

For $i \in I_0$ define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$T_i(K_j) = K_j K_i^{-a_{ij}},$$

$$T_i(E_j) = \begin{cases} -F_i K_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_i^{-1} E_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (i \neq j). \end{cases}$$

(see Lusztig [9]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that T_w does not depend on the choice of the reduced expression.

For $I \subset I_0$ let w_I be the longest element of W_I and define a subalgebra $U_q(\mathfrak{n}_I^-)$ by

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-).$$

Let w_0 be the longest element of W. Take a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_m}$ of $w_I w_0$ and set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}), \quad Y_{\beta_k}^{(n)} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}^{(n)})$$

for $k = 1, \dots, m$. Then it is known that $\{\beta_k \mid 1 \le k \le m\} = \Delta^+ \setminus \Delta_I$, and that $\{ Y_{\beta_1}^{(d_1)} \cdots Y_{\beta_m}^{(d_m)} \mid d_1, \dots, d_m \in \mathbb{Z}_{\ge 0} \}$ is a basis of $U_q(\mathfrak{n}_I^-)$. We note that this basis depends on the choice of the reduced expression of $w_I w_0$ in general.

Let $\tau : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the algebra anti-automorphism given by

$$\tau(K_i) = K_i^{-1}, \quad \tau(E_i) = E_i, \quad \tau(F_i) = F_i \quad (i \in I_0).$$

LEMMA 1.1. (i) $\tau T_{w_I}(U_q(\mathfrak{n}_I^-)) = U_q(\mathfrak{n}_I^-)$.

(ii) *Let $i, j \in I$ be such that $w_I(\alpha_i) = -\alpha_j$. Then we have*

$$(\mathrm{ad}\, F_i)(\tau T_{w_I}(x)) = \tau T_{w_I}((\mathrm{ad}\, E_j)(x)), \quad (\mathrm{ad}\, E_i)(\tau T_{w_I}(x)) = \tau T_{w_I}((\mathrm{ad}\, F_j)(x)),$$

$$(\mathrm{ad}\, K_i)(\tau T_{w_I}(x)) = \tau T_{w_I}((\mathrm{ad}(K_j^{-1}))(x))$$

for any $x \in U_q(\mathfrak{g})$.

PROOF. (i) We have $\tau T_k = T_k^{-1}\tau$ for any $k \in I_0$, and hence $\tau T_w = T_{w^{-1}}^{-1}\tau$ for any $w \in W$. Hence

$$\begin{aligned}\tau T_{w_I}(U_q(\mathfrak{n}_I^-)) &= \tau T_{w_I}(U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1}(U_q(\mathfrak{n}^-)) \\ &= T_{w_I}^{-1}(U_q(\mathfrak{n}^-)) \cap U_q(\mathfrak{n}^-) = U_q(\mathfrak{n}_I^-).\end{aligned}$$

(ii) We have

$$\tau T_{w_I}(E_j) = \tau T_{w_I s_j} T_{s_j}(E_j) = \tau T_{w_I s_j}(-F_j K_j) = -\tau(F_i K_i) = -K_i^{-1} F_i.$$

Here we have used the formula:

$$T_y(F_k) = F_\ell, \quad T_y(K_k) = K_\ell \quad (y \in W, k, \ell \in I_0, y(\alpha_k) = \alpha_\ell)$$

(see Lusztig [9]). Hence

$$\begin{aligned}\tau T_{w_I}((\operatorname{ad} E_j)(x)) &= \tau T_{w_I}((E_j x - x E_j)K_j) = K_i(z(-K_i^{-1}F_i) - (-K_i^{-1}F_i)z) \\ &= F_i z - (K_i z K_i^{-1})F_i = (\operatorname{ad} F_i)(z)\end{aligned}$$

with $z = \tau T_{w_I}(x)$. Other formulas are proved similarly. □

PROPOSITION 1.2. $(\operatorname{ad} U_q(\mathfrak{l}_I))(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$.

PROOF. We see easily that $(\operatorname{ad} U_q(\mathfrak{h}))(U_q(\mathfrak{n}_I^-)) = U_q(\mathfrak{n}_I^-)$. Hence it is sufficient to show that $U_q(\mathfrak{n}_I^-)$ is stable under $\operatorname{ad} E_i$, $\operatorname{ad} F_i$ for $i \in I$.

Let $i \in I$ and define $j \in I$ by $\alpha_j = -w_I(\alpha_i)$. By Lemma 1.1 we have

$$\begin{aligned}(\operatorname{ad} E_i)(U_q(\mathfrak{n}_I^-)) &= T_{w_I}^{-1}\tau^{-1}\tau T_{w_I}(\operatorname{ad} E_i)(U_q(\mathfrak{n}_I^-)) = T_{w_I}^{-1}\tau^{-1}(\operatorname{ad} F_j)(\tau T_{w_I} U_q(\mathfrak{n}_I^-)) \\ &\subset T_{w_I}^{-1}\tau^{-1}(\operatorname{ad} F_j)(U_q(\mathfrak{n}^-)) \subset T_{w_I}^{-1}(U_q(\mathfrak{n}^-)).\end{aligned}$$

Let us show $(\operatorname{ad} E_i)(U_q(\mathfrak{n}^-)) \subset U_q(\mathfrak{n}^-)$. For any $y \in U_q(\mathfrak{n}^-)$ we can write

$$[E_i, y] = K_i r_1(y) - r_2(y) K_i^{-1} \quad (r_1(y), r_2(y) \in U_q(\mathfrak{n}^-)),$$

and hence $(\operatorname{ad} E_i)(y) = K_i r_1(y) K_i - r_2(y)$. On the other hand by Jantzen [5] we have

$$\{y \in U_q(\mathfrak{n}^-) \mid r_1(y) = 0\} = U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-).$$

Hence we have to show $U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-)$. It is sufficient to show for any $y \in W$ and $k \in I_0$ satisfying $s_k y < y$ that $U_q(\mathfrak{n}^-) \cap T_{s_k y}^{-1} U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{n}^-) \cap T_y^{-1} U_q(\mathfrak{n}^-)$. This follows from Lusztig [9]. Therefore we have $(\operatorname{ad} E_i)(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$. Then we see from Lemma 1.1 that $(\operatorname{ad} F_\ell)(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$. □

Let $U_q^0(\mathfrak{n}^-)$ be the $\mathbf{C}[q^{\pm 1}]$-subalgebra of $U_q(\mathfrak{n}^-)$ generated by $\{F_i^{(n)} \mid i \in I_0, n \in \mathbf{Z}_{\geq 0}\}$. We have a natural $\mathbf{C}$-algebra homomorphism $\varphi : U_q^0(\mathfrak{n}^-) \to U(\mathfrak{n}^-)$ given by $F_i^{(n)} \to f_i^n/n!$, and it induces the isomorphism $\mathbf{C} \otimes_{\mathbf{C}[q^{\pm 1}]} U_q^0(\mathfrak{n}^-) \simeq U(\mathfrak{n}^-)$ where $\mathbf{C}[q^{\pm 1}] \to \mathbf{C}$ is given by $q \mapsto 1$. For $I \subset I_0$ the restriction of φ to $U_q^0(\mathfrak{n}_I^-) = U_q^0(\mathfrak{n}^-) \cap U_q(\mathfrak{n}_I^-)$ gives a surjective $\mathbf{C}$-algebra homomorphism $\varphi_I : U_q^0(\mathfrak{n}_I^-) \to U(\mathfrak{n}_I^-)$ inducing $\mathbf{C} \otimes_{\mathbf{C}[q^{\pm 1}]} U_q^0(\mathfrak{n}_I^-) \simeq U(n_I^-)$.

For $N \in \mathbf{Z}_{>0}$ set

$$U_{q,N}(\mathfrak{g}) = \mathbf{C}(q^{1/N}) \otimes_{\mathbf{C}(q)} U_q(\mathfrak{g}),$$

and let $U_{q,N}(\mathfrak{n}^{\pm})$, $U_{q,N}(\mathfrak{h})$, $U_{q,N}(\mathfrak{l}_I)$, $U_{q,N}(\mathfrak{n}_I^-)$ be the $\mathbf{C}(q^{1/N})$-subalgebras of $U_{q,N}(\mathfrak{g})$ generated by $U_q(\mathfrak{n}^{\pm})$, $U_q(\mathfrak{h})$, $U_q(\mathfrak{l}_I)$, $U_q(\mathfrak{n}_I^-)$ respectively.

2. Highest weight modules

For a $U(\mathfrak{h})$-module M and $\mu \in \mathfrak{h}^*$ we set

$$M_\mu = \{m \in M \mid hm = \mu(h)m \quad (h \in \mathfrak{h})\}.$$

It is called a weight space of M with weight μ. A $U(\mathfrak{h})$-module M satisfying $M = \bigoplus_\mu M_\mu$ and $\dim M_\mu < \infty$ for any μ is called a weight module. We define its character $\mathrm{ch}(M)$ as the formal infinite sum

$$\mathrm{ch}(M) = \sum_\mu \dim M_\mu \, e^\mu.$$

A $U(\mathfrak{g})$-module M is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $m \in M_\lambda \setminus \{0\}$ satisfying $M = U(\mathfrak{g})m$, $\mathfrak{n}^+ m = 0$. Such m is determined up to a nonzero constant multiple and is called the highest weight vector of M. For each $\lambda \in \mathfrak{h}^*$ there exists a unique (up to an isomorphism) irreducible highest weight module with highest weight λ, which we denote by $L(\lambda)$. Since highest weight modules are weight modules, their characters are defined. For $I \subset I_0$ set

$$\mathfrak{h}_I^* = \bigoplus_{i \in I_0 \setminus I} \mathbf{C}\varpi_i \subset \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g}) \Big/ \left(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})(\mathfrak{l}_I \cap \mathfrak{n}^-) \right).$$

It is a highest weight module with highest weight λ and the highest weight vector $m_{I,\lambda} = \bar{1}$, where $\bar{1}$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$. Moreover it is a rank one free $U(\mathfrak{n}_I^-)$-module generated by the highest weight vector $m_{I,\lambda}$, and hence we have

$$\mathrm{ch}(M_I(\lambda)) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+ \setminus \Delta_I}(1 - e^{-\alpha})}.$$

It contains a unique maximal proper submodule $K_I(\lambda)$, and we have $L(\lambda) = M_I(\lambda)/K_I(\lambda)$.

Now we define the corresponding notions for the quantized enveloping algebras. Set

$$\mathfrak{h}_{\mathbf{Z}}^* = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbf{Z} \ (i \in I_0)\} = \bigoplus_{i \in I_0} \mathbf{Z}\varpi_i \subset \mathfrak{h}^*.$$

For a $U_{q,N}(\mathfrak{h})$-module M the weight space M_μ with weight $\mu \in \mathfrak{h}_{\mathbf{Z}}^*/N$ is defined by

$$M_\mu = \{m \in M \mid K_i m = q_i^{\mu(h_i)} m \quad (i \in I_0)\}.$$

We call a $U_{q,N}(\mathfrak{h})$-module M a weight module if $M = \bigoplus_\mu M_\mu$ and $\dim M_\mu < \infty$ for any $\mu \in \mathfrak{h}_{\mathbf{Z}}^*/N$. Let M be a $U_{q,N}(\mathfrak{g})$-module. If there exists $m \in M_\lambda$ satisfying $U_{q,N}(\mathfrak{g})m = M$, $E_i m = 0$ $(i \in I_0)$, then M is called a highest weight module with highest weight λ and m is called its highest weight vector. There exists a unique irreducible highest weight module $L_{q,N}(\lambda)$ with highest weight λ. Highest weight modules are weight modules. For $I \subset I_0$ set

$$\mathfrak{h}_{I,\mathbf{Z}}^* = \bigoplus_{i \in I_0 \setminus I} \mathbf{Z}\varpi_i \subset \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}_{I,\mathbf{Z}}^*/N$ we define a highest weight module $M_{I,q,N}(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(\mathfrak{g}) \Big/ \left(\sum_{i \in I_0} U_{q,N}(\mathfrak{g})(K_i - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(\mathfrak{g})E_i + \sum_{j \in I} U_{q,N}(\mathfrak{g})F_j \right).$$

Its highest weight vector is given by $m_{I,\lambda,q,N} = \bar{1}$. Since $M_{I,q,N}(\lambda)$ is a rank one free module generated by $m_{I,\lambda,q,N}$, we have

$$\mathrm{ch}(M_{I,q,N}(\lambda)) = \mathrm{ch}(M_I(\lambda)).$$

We have a unique maximal proper submodule $K_{I,q,N}(\lambda)$ of $M_{I,q,N}(\lambda)$, and hence $L_{q,N}(\lambda) = M_{I,q,N}(\lambda)/K_{I,q,N}(\lambda)$.

Proposition 2.1. *Let $I \subset I_0$ and $\lambda \in \mathfrak{h}_{I,\mathbf{Z}}^*/N$. Let Y be a subset of $U_q^0(\mathfrak{n}_I^-)$ such that $Y m_{I,\lambda,q,N} \subset K_{I,q,N}(\lambda)$ and $U(\mathfrak{g})\varphi_I(Y)m_{I,\lambda} = K_I(\lambda)$. Then we have $U_{q,N}(\mathfrak{g})Y m_{I,\lambda,q,N} = K_{I,q,N}(\lambda)$ and $\mathrm{ch}(L_{q,N}(\lambda)) = \mathrm{ch}(L(\lambda))$.*

Proof. Let M be any highest weight $U_{q,N}(\mathfrak{g})$-module with highest weight λ. Take a highest weight vector $m \in M$ and set

$$M^0 = U_q^0(\mathfrak{n}^-)m, \quad \overline{M}^0 = M^0|_{q=1} = \mathbf{C} \otimes_{\mathbf{C}[q^{\pm 1/N}]} M^0.$$

Then we can show as in Lusztig [8] that M^0 is stable under the actions of E_i, F_i, $(K_i - K_i^{-1})/(q_i - q_i^{-1})$ $(i \in I_0)$ and that $\overline{M}^0$ becomes a highest weight $U(\mathfrak{g})$-module with highest weight λ via the operators

$$e_i = \bar{E}_i, \quad f_i = \bar{F}_i, \quad h_i = \overline{\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}} \quad (i \in I_0).$$

In particular we have

$$\dim M_\mu = \dim(\overline{M}^0)_\mu \geq \dim L(\lambda)_\mu.$$

Now we set

$$M = M_{I,q,N}(\lambda)/U_{q,N}(\mathfrak{g})Ym_{I,\lambda,q,N}, \quad m = \overline{m_{I,\lambda,q,N}} \in M.$$

By the above argument $\overline{M}^0$ is a highest weight $U(\mathfrak{g})$-module with highest weight λ and the highest weight vector $\bar{m}$. Moreover, since $Ym = 0$, we have $\varphi_I(Y)\bar{m} = 0$. Hence we have $\overline{M}^0 \simeq L(\lambda)$. It follows that

$$\dim L_{q,N}(\lambda)_\mu \leq \dim M_\mu = \dim(\overline{M}^0)_\mu = \dim L(\lambda)_\mu \leq \dim L_{q,N}(\lambda)_\mu.$$

Therefore we have $M \simeq L_{q,N}(\lambda)$ and $\mathrm{ch}(L_{q,N}(\lambda)) = \mathrm{ch}(L(\lambda))$. □

3. Parabolic subalgebras with commutative nilpotent radicals

In the rest of this paper we fix $I \subset I_0$ satisfying $\mathfrak{n}_I^+ \neq \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$ (see, for example, [14] for the list of $(\mathfrak{g}, I)$'s satisfying the condition). We have $I = I_0 \backslash \{i_0\}$ for some $i_0 \in I_0$.

We set $\mathfrak{l} = \mathfrak{l}_I$, $\mathfrak{m}^\pm = \mathfrak{n}_I^\pm$ for simplicity.

PROPOSITION 3.1. *The element $Y_\beta \in U_q(\mathfrak{m}^-)$ for $\beta \in \Delta^+ \backslash \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$.*

PROOF. For $i, j \in I_0$ set

$$r(i,j) = (\overbrace{i,j,i,j,\ldots}^{m_{ij}}),$$

where m_{ij} denotes the order of $s_i s_j \in W$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced expression of $w \in W$. Then $s_{j_1} \cdots s_{j_r}$ is a reduced expression of w if and only if $(j_1, \ldots, j_r)$ can be obtained from $(i_1, \ldots, i_r)$ by successively exchanging a subsequence of the form $r(i,j)$ to $r(j,i)$.

We first show that for any reduced expression $s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ the sequence $(i_1, \ldots, i_r)$ does not contain a subsequence of the form $r(i,j)$ with $m_{ij} \geq 3$. Assume that there exists a subsequence $r(i,j)$ with $m_{ij} = 3$ in $(i_1, \ldots, i_r)$. We have $(i_p, i_{p+1}, i_{p+2}) = (i,j,i)$ for some p. Set $y = s_{i_1} \cdots s_{i_{p-1}}$.

Then we have

$$\beta_p = y(\alpha_i), \quad \beta_{p+1} = ys_i(\alpha_j) = y(\alpha_i + \alpha_j), \quad \beta_{p+2} = ys_is_j(\alpha_i) = y(\alpha_j),$$

and hence $\beta_p + \beta_{p+2} = \beta_{p+1}$. This contradicts the commutativity of $\mathfrak{m}^-$. Thus the sequence $(i_1, \dots, i_r)$ does not contain a subsequence of the form $r(i,j)$ with $m_{ij} = 3$. Similarly we can show that there does not exist a subsequence of the form $r(i,j)$ with $m_{ij} = 4, 6$.

Therefore it is sufficient to show that for two reduced expressions

$$s_{i_1} \cdots s_{i_p} s_i s_j s_{j_1} \cdots s_{j_q}, \quad s_{i_1} \cdots s_{i_p} s_j s_i s_{j_1} \cdots s_{j_q}, \quad (s_i s_j = s_j s_i)$$

of $w_I w_0$ the resulting Y_β's are the same. This follows from $T_i(F_j) = F_j$, $T_j(F_i) = F_i$, and $T_i T_j = T_j T_i$. □

We fix a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ and set $\beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$. Set

$$Q^+ = \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i, \quad Q_I^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i,$$

$$U_q(\mathfrak{m}^-)^m = \sum_{p_1, \dots, p_m = 1}^{r} \mathbb{C}(q) Y_{\beta_{p_1}} \cdots Y_{\beta_{p_m}} \quad (m \geq 0).$$

LEMMA 3.2. *We have*

$$U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m.$$

$$U_q(\mathfrak{m}^-)^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_r}^{(m_r)} = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

Here $U_q(\mathfrak{m}^-)_{-\gamma}$ is the weight space with respect to the adjoint action of $U_q(\mathfrak{h})$ on $U_q(\mathfrak{m}^-)$.

PROOF. Set

$$V_0^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_r}^{(m_r)}, \quad V_1^m = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

By $\beta_p \in \alpha_{i_0} + Q_I^+$ we have $V_0^m \subset U_q(\mathfrak{m}^-)^m \subset V_1^m$. Since $U_q(\mathfrak{m}^-) = \bigoplus_m V_0^m$, we obtain $V_0^m = U_q(\mathfrak{m}^-)^m = V_1^m$ and $U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m$. □

By Lemma 3.2 we can write

$$(3.1) \qquad Y_{\beta_{p_1}} Y_{\beta_{p_2}} = \sum_{\substack{s_1 \leq s_2 \\ \beta_{p_1} + \beta_{p_2} = \beta_{s_1} + \beta_{s_2}}} a_{s_1, s_2}^{p_1, p_2} Y_{\beta_{s_1}} Y_{\beta_{s_2}} \quad (a_{s_1, s_2}^{p_1, p_2} \in \mathbb{C}(q))$$

for $p_1 > p_2$.

PROPOSITION 3.3. *The $\mathbf{C}(q)$-algebra $U_q(\mathfrak{m}^-)$ is generated by the elements $\{Y_{\beta_p} \mid 1 \leq p \leq r\}$ satisfying the fundamental relations* (3.1) *for $p_1 > p_2$.*

PROOF. It is sufficient to show that any element of the form $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}}$ $(1 \leq t_i \leq r)$ can be rewritten as a linear combination of the elements of the form $Y_{\beta_{s_1}} \cdots Y_{\beta_{s_n}}$ $(1 \leq s_1 \leq \cdots \leq s_n \leq r)$ by a successive use of the relations (3.1) for $p_1 > p_2$. For $1 \leq k \leq r$ let V_k be the subalgebra of $U_q(\mathfrak{m}^-)$ generated by $\{Y_{\beta_p} \mid 1 \leq p \leq k\}$. By Lusztig [9] we have

$$V_k = \bigoplus_{m_1, \dots, m_k} \mathbf{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_k}^{(m_k)}.$$

We shall show by the induction on k that any element of the form $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}}$ $(1 \leq t_i \leq k)$ can be rewritten as a linear combination of the elements of the form $Y_{\beta_{s_1}} \cdots Y_{\beta_{s_n}}$ $(1 \leq s_1 \leq \cdots \leq s_n \leq k)$ by a successive use of the relations (3.1) for $k \geq p_1 > p_2$. It is trivial for $k = 1$. Assume that $k \geq 2$ and the assertion is proved up to $k - 1$. We shall show the statement by induction on n. It is obvious for $n = 0$. Assume that $n > 0$ and the statement is already proved up to $n - 1$. Take j such that $t_1 = \cdots = t_j = k$, $t_{j+1} \neq k$. We use induction on j. Assume that $j = 0$. Then we have $t_1 \neq k$. By using the inductive hypothesis on n we may assume that $t_2 \leq \cdots \leq t_n \leq k$. If $t_n < k$, then we have $t_i \leq k - 1$ for any i, and hence the statement holds by the inductive hypothesis on k. If $t_n = k$, then we can apply the inductive hypothesis on n to $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_{n-1}}}$, and hence the statement also holds. Assume $0 < j < n$. Then we have

$$Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}} = Y_{\beta_k}^j Y_{\beta_{t_{j+1}}} \cdots Y_{\beta_{t_n}}$$

with $t_{j+1} \neq k$. Applying (3.1) for $(p_1, p_2) = (k, t_{j+1})$ we obtain

$$Y_{\beta_k} Y_{\beta_{t_{j+1}}} = \sum_{\substack{s_1 \leq s_2 \leq k \\ \beta_k + \beta_{t_{j+1}} = \beta_{s_1} + \beta_{s_2}}} a_{s_1, s_2}^{k, t_{j+1}} Y_{\beta_{s_1}} Y_{\beta_{s_2}}.$$

Since $s_1 < k$ by the condition $\beta_k + \beta_{t_{j+1}} = \beta_{s_1} + \beta_{s_2}$, we can apply the inductive hypothesis on j to $Y_{\beta_k}^{j-1} Y_{\beta_{s_1}} Y_{\beta_{s_2}} Y_{\beta_{t_{j+2}}} \cdots Y_{\beta_{t_n}}$, and the statement holds. If $j = n$, then we have $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}} = Y_{\beta_k}^n$, and the statement is obvious. □

Since $\mathfrak{m}^-$ is commutative, $U(\mathfrak{m}^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By identifying $\mathfrak{m}^-$ with $(\mathfrak{m}^+)^*$ via the Killing form of $\mathfrak{g}$, $S(\mathfrak{m}^-)$ is naturally identified with the algebra $\mathbf{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$. Hence we have an identification $U(\mathfrak{m}^-) = \mathbf{C}[\mathfrak{m}^+]$. We denote by $\mathbf{C}[\mathfrak{m}^+]^m$ $(m \in \mathbf{Z}_{\geq 0})$ the subspace of $\mathbf{C}[\mathfrak{m}^+]$ consisting of homogeneous polynomials with degree m.

Set

$$\mathfrak{h}^*_{\mathbf{Z}}(I,+) = \{\lambda \in \mathfrak{h}^*_{\mathbf{Z}} \mid \lambda(h_i) \geq 0 \ (i \in I)\}.$$

For $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}(I,+)$ we denote the finite dimensional irreducible $U(\mathfrak{l})$-module (resp. $U_q(\mathfrak{l})$-module) with highest weight λ by $V(\lambda)$ (resp. $V_q(\lambda)$). We can decompose the finite dimensional $\mathfrak{l}$-module $\mathbf{C}[\mathfrak{m}^+]^m$ into a direct sum of submodules isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}(I,+)$. Moreover, it is known that

$$\dim \mathrm{Hom}_{\mathfrak{l}}(V(\lambda), \mathbf{C}[\mathfrak{m}^+]) \geq 1 \quad (\lambda \in \mathfrak{h}^*_{\mathbf{Z}}(I,+)),$$

and hence we have

$$\mathbf{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma^m} V(\lambda)$$

for finite subsets Γ^m of $\mathfrak{h}^*_{\mathbf{Z}}(I,+)$ satisfying $\Gamma^m \cap \Gamma^{m'} = \varnothing$ for $m \neq m'$ (see Schmid [11], Takeuchi [12], Johnson [6] for the explicit description of Γ^m). On the other hand, since $U_q(\mathfrak{m}^-)^m$ is a finite dimensional $U_q(\mathfrak{l})$-module whose character is the same as that of $\mathbf{C}[\mathfrak{m}^+]^m$, we have

$$U_q(\mathfrak{m}^-)^m \simeq \bigoplus_{\lambda \in \Gamma^m} V_q(\lambda).$$

Let L be the algebraic group corresponding to $\mathfrak{l}$. It is known that the set of L-orbits on $\mathfrak{m}^+$ is a finite totally ordered set with respect to the closure relation. Hence we can label the orbits by

$$\{L\text{-orbits on } \mathfrak{m}^+\} = \{C_0, C_1, \dots, C_t\}, \quad \{0\} = C_0 \subset \bar{C}_1 \subset \cdots \subset \bar{C}_t = \mathfrak{m}^+.$$

Set

$$\mathscr{I}(\bar{C}_p) = \{f \in \mathbf{C}[\mathfrak{m}^+] \mid f(\bar{C}_p) = 0\}.$$

Since $\mathscr{I}(\bar{C}_p)$ is an $\mathfrak{l}$-submodule of $\mathbf{C}[\mathfrak{m}^+]$, we have

$$\mathscr{I}(\bar{C}_p) = \bigoplus_m \mathscr{I}^m(\bar{C}_p), \quad \mathscr{I}^m(\bar{C}_p) = \mathscr{I}(\bar{C}_p) \cap \mathbf{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma^m_p} V(\lambda)$$

for a subset Γ^m_p of Γ^m. Moreover the following fact is known (see, for example, [14]):

PROPOSITION 3.4. *Let $p = 0, \dots, t-1$.*

(i) *$\mathscr{I}^m(\bar{C}_p) = 0$ for $m \leq p$.*

(ii) *$\mathscr{I}^{p+1}(\bar{C}_p)$ is an irreducible $\mathfrak{l}$-module, i.e. Γ^{p+1}_p consists of a single element ν_p.*

(iii) *$\mathscr{I}(\bar{C}_p)$ is generated by $\mathscr{I}^{p+1}(\bar{C}_p)$ as an ideal of $\mathbf{C}[\mathfrak{m}^+]$.*

PROPOSITION 3.5. *For $p = 0, \dots, t-1$ there exists a unique $\lambda_p \in \mathfrak{h}^*_I$ such that $K_I(\lambda_p) = \mathscr{I}(\bar{C}_p) m_{I,\lambda_p}$. Moreover, we have $\lambda_p \in \mathfrak{h}^*_{I,\mathbf{Z}}/2$.*

Let v^p be the highest weight vector of the $\mathfrak{l}$-module $\mathscr{I}^{p+1}(\bar{C}_p)(\simeq V(\nu_p))$. Then we have

$$\begin{aligned} K_I(\lambda_p) &= \mathscr{I}(\bar{C}_p)m_{I,\lambda_p} = U(\mathfrak{m}^-)\mathscr{I}^{p+1}(\bar{C}_p)m_{I,\lambda_p} \\ &= U(\mathfrak{m}^-)((\mathrm{ad}\, U(\mathfrak{l}\cap\mathfrak{n}^-))(v^p))m_{I,\lambda_p} \\ &= U(\mathfrak{m}^-)(U(\mathfrak{l}\cap\mathfrak{n}^-))v^p m_{I,\lambda_p} = U(\mathfrak{n}^-)v^p m_{I,\lambda_p} \end{aligned}$$

and hence $K_I(\lambda_p)$ is a highest weight module with highest weight $\lambda_p+\nu_p$.

We set

$$\mathscr{I}_q^m(\bar{C}_p) = \bigoplus_{\lambda\in\Gamma_p^m} V_q(\lambda) \subset U_q(\mathfrak{m}^-)^m, \quad \mathscr{I}_q(\bar{C}_p) = \bigoplus_m \mathscr{I}_q^m(\bar{C}_p) \subset U_q(\mathfrak{m}^-),$$

$$\mathscr{I}_{q,N}^m(\bar{C}_p) = \mathbf{C}(q^{1/N})\otimes_{\mathbf{C}(q)}\mathscr{I}_q^m(\bar{C}_p) \subset U_{q,N}(\mathfrak{m}^-)^m,$$

$$\mathscr{I}_{q,N}(\bar{C}_p) = \bigoplus_m \mathscr{I}_{q,N}^m(\bar{C}_p) \subset U_{q,N}(\mathfrak{m}^-).$$

Here we identify $U_q(\mathfrak{m}^-)^m$ with $\bigoplus_{\lambda\in\Gamma^m} V_q(\lambda)$.

PROPOSITION 3.6. *For $p=0,\dots,t-1$ we have*

$$\mathrm{ch}(L_{q,2}(\lambda_p)) = \mathrm{ch}(L(\lambda_p)), \quad K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{m}^-)\mathscr{I}_{q,2}^{p+1}(\bar{C}_p)m_{I,\lambda_p,q,2}.$$

PROOF. We shall only give a sketch of the proof. We can prove a quantum analogue of the determinant formula for the contravariant forms on generalized Verma modules given by Jantzen [4]. It implies that $K_{I,q,N}(\lambda)_\mu = 0$ if and only if $K_I(\lambda)_\mu = 0$. In particular, we have $K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p} \neq 0$ and $K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p+\alpha_i} = 0$ for any $i\in I_0$. Let $vm_{I,\lambda_p,q,2}$ ($v\in U_{q,2}(\mathfrak{m}^-)_{\nu_p}$) be a nonzero element of $K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p}$. Then for $i\in I$ we have

$$\begin{aligned} ((\mathrm{ad}\, E_i)(v))m_{I,\lambda_p,q,2} &= (E_i v - vE_i)K_i m_{I,\lambda_p,q,2} \\ &\in \mathbf{C}(q^{1/2})E_i v m_{I,\lambda_p,q,2} \subset K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p+\alpha_i} = \{0\}. \end{aligned}$$

Hence $(\mathrm{ad}\, E_i)(v) = 0$ for any $i\in I$. It follows that v is a highest weight vector of the $U_{q,2}(\mathfrak{l})$-module $V_{q,2}(\nu_p)$. We may assume $v\in U_q^0(\mathfrak{m}^-)$ and $\varphi_I(v)\neq 0$. By Proposition 2.1 we conclude that $\mathrm{ch}(L_{q,2}(\lambda_p)) = \mathrm{ch}(L(\lambda_p))$ and $K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{g})vm_{I,\lambda_p,q,2}$. Then we have

$$\begin{aligned} K_{I,q,2}(\lambda_p) &= U_{q,2}(\mathfrak{g})vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)(U_{q,2}(\mathfrak{l})\cap U_{q,2}(\mathfrak{n}^-))U_{q,2}(\mathfrak{h})U_{q,2}(\mathfrak{n}^+)vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)(U_{q,2}(\mathfrak{l})\cap U_{q,2}(\mathfrak{n}^-))vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)((\mathrm{ad}(U_{q,2}(\mathfrak{l})\cap U_{q,2}(\mathfrak{n}^-))(v))m_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)\mathscr{I}_{q,2}^{p+1}(\bar{C}_p)m_{I,\lambda_p,q,2}. \end{aligned}$$

□

THEOREM 3.7. *We have*

$$\mathscr{I}_q(\bar{C}_p) = U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p) = \mathscr{I}_q^{p+1}(\bar{C}_p)U_q(\mathfrak{m}^-).$$

PROOF. By Proposition 3.6 we have

$$\mathrm{ch}(U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p)) = \mathrm{ch}(U_{q,2}(\mathfrak{m}^-)\mathscr{I}_{q,2}^{p+1}(\bar{C}_p)) = \mathrm{ch}(\mathscr{I}(\bar{C}_p)),$$

and hence $\mathscr{I}_q(\bar{C}_p) = U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p)$. Let us show $U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p) = \mathscr{I}_q^{p+1}(\bar{C}_p)U_q(\mathfrak{m}^-)$. Since τT_{w_I} is an anti-automorphism of the algebra $U_q(\mathfrak{m}^-)$ (see Lemma 1.1), it is sufficient to show that τT_{w_I} preserves $\mathscr{I}_q^{p+1}(\bar{C}_p)$. Since $U_q(\mathfrak{m}^-)$ is a multiplicity free $U_q(\mathfrak{l})$-module, we have only to show that $\tau T_{w_I}(V_q(\lambda))$ is a $U_q(\mathfrak{l})$-submodule isomorphic to $V_q(\lambda)$ for any $\lambda \in \bigcup_m \Gamma^m$. By Lemma 1.1 we see easily that $\tau T_{w_I}(V_q(\lambda))$ is an irreducible $U_q(\mathfrak{l})$-module with lowest weight $w_I(\lambda)$. Hence we have $\tau T_{w_I}(V_q(\lambda)) \simeq V_q(\lambda)$. □

References

[1] V. G. Drinfel'd, Hopf algebra and the Yang-Baxter equation, *Soviet Math. Dokl.* **32** (1985), 254–258.

[2] T. J. Enright, A. Joseph, An intrinsic analysis of unitarizable highest weight modules, *Math. Ann.* **288** (1990), 571–594.

[3] M. Hashimoto, T. Hayashi, Quantum multilinear algebra, *Tohoku Math. J.*, **44** (1992), 471–521.

[4] J. C. Jantzen, Kontravariante Formen auf indzierten Darstellungen halbeinfacher Lie-algebren, *Math. Ann.* **226** (1977), 53–65.

[5] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, **6**, American Mathematical Society, 1995.

[6] K. Johnson, On a ring of invariant polynomials on a hermitian symmetric spaces, *J. Alg.* **67** (1980), 72–81.

[7] M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.

[8] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, *Adv. in Math.* **70** (1988), 237–249.

[9] G. Lusztig, Quantum groups at roots of 1, *Geometriae Dedicata* **35** (1990), 89–114.

[10] M. Noumi, H. Yamada, K. Mimachi, Finite dimensional representations of the quantum group $GL_q(n;\mathbf{C})$ and the zonal spherical functions, *Japan. J. Math.* **19** (1993), 31–80.

[11] W. Schmid, Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, *Invent. Math.* **9** (1969), 61–80.

[12] M. Takeuchi, Polynomial representations associated with symmetric bounded domains, *Osaka J. Math.* **10** (1973), 441–475.

[13] E. Strickland, Classical invariant theory for the quantum symplectic group, *Adv. Math.* **123** (1996), 78–90.

[14] T. Tanisaki, Highest weight modules associated to parabolic subgroups with commutative unipotent radicals, to appear in *Algebraic groups and their representations*, Proceedings of the NATO ASI conference, Kluwer Academic Publishers, Dordrecht, 1998.

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Quantum Deformations of certain Prehomogeneous Vector Spaces. II

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Quantum deformations of certain prehomogeneous vector spaces. II

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Introduction

Let G be a reductive algebraic group over the complex number field $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. The quantized coordinate algebra $A_q(G)$ of G is constructed as a certain dual Hopf algebra of the quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. The Hopf algebras $U_q(\mathfrak{g})$ and $A_q(G)$ over $\mathbb{C}(q)$ tend to the ordinary enveloping algebra $U(\mathfrak{g})$ and the coordinate algebra $A(G)$ respectively when the parameter q tends to 1 in a certain sense (Drinfeld [1], Jimbo [3]).

Let us consider what object we should regard as a quantum deformation of an affine variety X with G-action.

An affine variety X is endowed with an action of G if and only if its coordinate algebra $A(X)$ is equipped with a right $A(G)$-comodule structure

$$\tau : A(X) \to A(X) \otimes A(G)$$

which is simultaneously an algebra homomorphism. By the duality between $U(\mathfrak{g})$ and $A(G)$ we obtain a locally finite left $U(\mathfrak{g})$-module structure

$$\gamma : U(\mathfrak{g}) \otimes A(X) \to A(X) \qquad (*)$$

given by

$$\tau(n) = \sum_i n_i \otimes f_i \quad \Rightarrow \quad \gamma(u \otimes n) = \sum_i \langle u, f_i \rangle n_i, \qquad (**)$$

where $\langle , \rangle : U(\mathfrak{g}) \times A(G) \to \mathbb{C}$ is the dual pairing. Since τ is an algebra homomorphism, we have

$$u \in U(\mathfrak{g}),\ m, n \in A(X),\ \Delta(u) = \sum_i u_i \otimes v_i \quad \Rightarrow \quad u(mn) = \sum_i (u_i m)(v_i n), \qquad (***)$$

where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the coproduct. Then the action of G on X is uniquely determined by the infinitesimal action γ. Moreover, for a locally finite left $U(\mathfrak{g})$-module structure $(*)$ on $A(X)$ satisfying $(***)$ and a certain condition on irreducible $U(\mathfrak{g})$-modules appearing as submodules of $A(X)$, there exists a unique action of G on X whose infinitesimal action is given by γ.

Now we define the notion of a quantum deformation of an affine variety X with G-action as follows. A (not necessarily commutative) $\mathbb{C}(q)$-algebra $A_q(X)$ endowed with a locally finite left $U_q(\mathfrak{g})$-module structure

$$\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$$

is called a quantum deformation of X if $A_q(X)$ and γ_q tend to $A(X)$ and $\gamma : U(\mathfrak{g}) \otimes A(X) \to A(X)$ respectively when q tends to 1 and if it satisfies

$$u \in U_q(\mathfrak{g}),\ m, n \in A_q(X),\ \Delta(u) = \sum_i u_i \otimes v_i \quad \Rightarrow \quad u(mn) = \sum_i (u_i m)(v_i n).$$

It seems to be an interesting problem to determine in which case X admits a quantum deformation. In this paper we consider the problem when X is a prehomogeneous vector space, that is, when X is a vector space with a linear G-action containing an open G-orbit. Such a quantum deformation was intensively studied in the case where $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ and $X = M_{mn}(\mathbb{C})$ (see Taft-Towber [10], Hashimoto-Hayashi [2] and Noumi-Yamada-Mimachi [7]), and also in the case where $G = GL_n(\mathbb{C})$ and X is the set of skew symmetric matrices of degree n (see Strickland [8]).

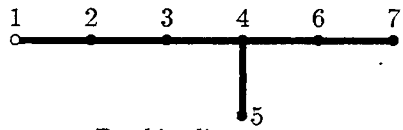
In our previous paper [4] we gave a general method to construct quantum deformations of prehomogeneous vector spaces of parabolic type. Moreover, for each non-open G-orbit C on X, we have shown that the defining ideal of the closure $\overline{C}$ and its canonical generators admit quantum deformations inside $A_q(X)$. It includes the existence of the quantum deformation of the irreducible relative invariant when X is a regular prehomogeneous vector space. Indeed, the canonical generator of the defining ideal of the closure of the one-codimensional orbit is nothing but the irreducible relative invariant.

Quantum deformations of prehomogeneous vector spaces of commutative parabolic type associated to classical simple Lie algebras are intensively studied in Kamita [5]. In this paper we shall deal with the remaining two cases

(I) $G = \mathbb{C}^\times \times Spin(10, \mathbb{C})$, $X = \mathbb{C}^{16}$, the scalar multiplication and the half-spin representation,

(II) $G = \mathbb{C}^\times \times E_6$, $X = \mathbb{C}^{27}$, the scalar multiplication and the 27-dimensional irreducible representation of E_6, which naturally arise from the exceptional simple Lie algebras of type E_6 and E_7 respectively using the method in our previous paper [4]. In Introduction we shall only state the results in case (II).

Let $\mathfrak{g}_{E_7}$ be a simple Lie algebra of type E_7 over $\mathbb{C}$ and let $\mathfrak{h}$ be its Cartan subalgebra. We shall use the labelling of the vertices of the Dynkin diagram 1.



Dynkin diagram 1.

Set $I_0 = \{1,2,\ldots,7\}$, $I = I_0 \setminus \{1\}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of type E_7. We denote the set of simple roots by $\{\alpha_i\}_{i\in I_0}$ and the set of positive roots by Δ^+. Let $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be a standard symmetric bilinear form. Set $D = \Delta^+ \setminus \sum_{i\in I} \mathbb{Z}\alpha_i$. Then we have $\sharp D = 27$. Set $\Lambda = \{1,2,\ldots,27\}$, and fix a bijection $\Lambda \ni j \mapsto \beta_j \in D$ such that $\beta_k - \beta_j \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\alpha_i$ implies $j \leq k$, where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Set $\delta = 3\alpha_1+4\alpha_2+5\alpha_3+6\alpha_4+3\alpha_5+4\alpha_6+2\alpha_7$. For each $n \in \Lambda$ there exist exactly five pairs $(i,j) \in \Lambda^2$ such that $\beta_i + \beta_j = \delta - \beta_n, i < j$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n$. Let $K_i^{\pm 1}, E_i, F_i$ $(i \in I_0)$ be the canonical generators of $U_q(\mathfrak{g}_{E_7})$, and set $U_q(\mathfrak{g}) = \langle K_1^{\pm 3},\ K_j^{\pm 1}, E_j, F_j \mid j \in I\rangle \subset U_q(\mathfrak{g}_{E_7})$. Then $U_q(\mathfrak{g})$ is isomorphic to the tensor product of $\mathbb{C}(q)[K, K^{-1}]$ and the quantized enveloping algebra of type E_6, where $K = K_1^3K_2^4K_3^5K_4^6K_5^3K_6^4K_7^2$.

Theorem 0.1 *A quantum deformation of the 27-dimensional irreducible prehomogeneous vector space X of $G = \mathbb{C}^\times \times E_6$ is given by the following.*

(a) *$A_q(X)$ is an associative $\mathbb{C}(q)$-algebra defined by the following generators and fundamental relations:*
Generators: Y_i with $i = 1,\ldots,27$.
Fundamental relations: For $i < j$

$$Y_iY_j = \begin{cases} qY_jY_i \\ \quad \text{if } \beta_i + \beta_j \text{ does not have another decomposition } \beta + \beta',\ \beta, \beta' \in D, \\ Y_jY_i + qY_bY_a - q^{-1}Y_aY_b \\ \quad \text{if there exist } k \in I,\ a,b \in \Lambda \text{ such that } \beta_a = \beta_i + \alpha_k,\ \beta_b = \beta_j - \alpha_k, \\ Y_jY_i \quad \text{otherwise.} \end{cases}$$

(b) *The action $\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$ is given by the following.*

For $2 \leq k \leq 7,\ 1 \leq m \leq 7$

$$\begin{aligned}
\gamma_q(F_k \otimes Y_i) &= \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i + \alpha_k, \\ 0 & \text{otherwise}, \end{cases} \\
\gamma_q(E_k \otimes Y_i) &= \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i - \alpha_k, \\ 0 & \text{otherwise}, \end{cases} \\
\gamma_q(K_m \otimes Y_i) &= q^{-(\alpha_m, \beta_i)} Y_i.
\end{aligned}$$

(c) *The quantum deformation of the irreducible relative invariant of* X *is given by*

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n| - 1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ $(\beta = \sum_{i \in I_0} m_i \alpha_i)$, $\psi_n = Y_{i_5^n} Y_{j_5^n} - q Y_{i_4^n} Y_{j_4^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_2^n} Y_{j_2^n} + q^4 Y_{i_1^n} Y_{j_1^n}$.

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1 Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra of type E_6 or E_7 over the complex number field $\mathbb{C}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system, and let $W \subset GL(\mathfrak{h})$ be the Weyl group. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where I_0 is an index set. For $i \in I_0$ we denote the simple reflection corresponding to α_i by $s_i \in W$. Let $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for any $\alpha \in \Delta$. Set $a_{ij} = (\alpha_i, \alpha_j)$. The matrix $(a_{ij})_{i,j \in I_0}$ is called the Cartan matrix of type E_6 or E_7. For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. Set $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. For a subset $I \subset I_0$ we define

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i \mid i \in I \rangle.$$

We set

$$\mathfrak{l}_I = \mathfrak{h} \oplus (\textstyle\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha), \quad \mathfrak{n}_I^+ = \textstyle\bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad \mathfrak{n}_I^- = \textstyle\bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha}.$$

Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. We denote by L_I the subgroup of G corresponding to $\mathfrak{l}_I$. Then L_I acts on $\mathfrak{n}_I^{\pm}$ via the adjoint action.

The quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [3]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements E_i, F_i, K_i, K_i^{-1} $(i \in I_0)$ satisfying the following fundamental relations:

$$\begin{array}{ll}
K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, & \\
K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{-a_{ij}}F_jK_i, & \\
E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}, & \\
E_iE_j = E_jE_i & (i \neq j,\ a_{ij} = 0), \\
E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 & (i \neq j,\ a_{ij} = -1), \\
F_iF_j = F_jF_i & (i \neq j,\ a_{ij} = 0), \\
F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0 & (i \neq j,\ a_{ij} = -1).
\end{array}$$

A Hopf algebra structure on $U_q(\mathfrak{g})$ is defined as follows. The comultiplication $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$

The counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i.$$

Using the Hopf algebra structure, we define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $\mathrm{ad}(x)y = \sum_k x_k^1 y S(x_k^2)$. Then $\mathrm{ad} : U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism. For $x, y, z \in U_q(\mathfrak{g})$ we have $\mathrm{ad}(x)(yz) = \sum_k(\mathrm{ad}(x_k^1)y)(\mathrm{ad}(x_k^2)z)$, where $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$.

We define subalgebras $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^-) = \langle F_i \,|\, i \in I_0\rangle, \quad U_q(\mathfrak{l}_I) = \langle E_i, F_i, K_j, K_j^{-1} \,|\, i \in I,\ j \in I_0\rangle.$$

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$\begin{aligned}
T_i(K_j) &= K_j K_i^{-a_{ij}}, \\
T_i(E_j) &= \begin{cases} -F_i K_i & (i = j) \\ E_j & (i \neq j,\ a_{ij} = 0) \\ E_i E_j - q^{-1} E_j E_i & (i \neq j,\ a_{ij} = -1), \end{cases} \\
T_i(F_j) &= \begin{cases} -K_i^{-1} E_i & (i = j) \\ F_j & (i \neq j,\ a_{ij} = 0) \\ F_j F_i - q F_i F_j & (i \neq j,\ a_{ij} = -1) \end{cases}
\end{aligned}$$

(see Lusztig [6]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$ and set $T_w = T_{i_1} \cdots T_{i_r}$. It is known that T_w does not depend on the choice of a reduced expression.

We shall use the following later (see Lusztig [6]).

Lemma 1.1 *If $w(\alpha_i) = \alpha_j$ for $w \in W$ and $i, j \in I_0$, then we have $T_w(F_i) = F_j$.*

For $I \subset I_0$ let w_I be the longest element of W_I and let w_0 be the longest element of W. Choose a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ and set

$$\beta_j = s_{i_1} s_{i_2} \dots s_{i_{j-1}}(\alpha_{i_j}), \quad Y_j = Y_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(F_{i_j})$$

for $1 \leq j \leq r$. Then it is known that $\{\beta_j \mid 1 \leq j \leq r\} = \Delta^+ \setminus \Delta_I$. Set

$$U_q(\mathfrak{n}_I^-) = \sum_{d_j \geq 0} \mathbb{C}(q) Y_1^{d_1} \dots Y_r^{d_r}.$$

Then $\{Y_1^{d_1} \dots Y_r^{d_r} \mid d_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq r\}$ is a basis of $U_q(\mathfrak{n}_I^-)$ and $U_q(\mathfrak{n}_I^-)$ is a subalgebra of $U_q(\mathfrak{n}^-)$. we have

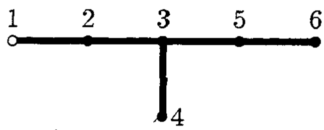
$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-)$$

and $U_q(\mathfrak{n}_I^-)$ does not depend on the choice of a reduced expression of $w_I w_0$ (see Lusztig [6]).

If $\mathfrak{n}_I^+ \neq \{0\}, [\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$, then Y_β for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$ (see [4]). In this case we denote the $\mathbb{C}(q)$-algebra $U_q(\mathfrak{n}_I^-)$ by A_q. We can regard it as a quantum deformation of the coordinate algebra $A = \mathbb{C}[\mathfrak{n}_I^+]$ of $\mathfrak{n}_I^+$ as explained in [4].

2 Case of type E_6

Let $\mathfrak{g}$ be a simple Lie algebra of type E_6. We shall use the labelling of the vertices of the Dynkin diagram 2.



Dynkin diagram 2.

Hence we have $I_0 = \{1,2,3,4,5,6\}$. Set $I = \{2,3,4,5,6\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}, [\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $\mathfrak{l}_I$ is isomorphic to $\mathbb{C} \oplus \mathfrak{o}(10, \mathbb{C})$ and $\mathfrak{n}_I^+$ is a 16-dimensional irreducible prehomogeneous vector space. There are three L_I-orbits $\{0\}, C_0, O$ on $\mathfrak{n}_I^+$ satisfying $\{0\} \subset \overline{C_0} \subset \overline{O}$. Let $J_{C_0} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_0, and let $J_{C_0}^0$ denote the subspace of J_{C_0} consisting of the polynomials in J_{C_0} with homogeneous degree 2. Then $J_{C_0}^0$ is a ten-dimensional irreducible $\mathfrak{l}_I$-module, and it generates the ideal J_{C_0}.

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ and define the elements Y_i ($i \in \Lambda = \{1, 2, \dots, 16\}$) as in Section 1.

Set $I_0' = \{1,2,3,4,5\}, I' = \{2,3,4,5\}, \Lambda' = \{1,2,\dots,8\}$. Then $\{\alpha_i\}_{i \in I_0'}$ is a set of simple roots of type D_5. Let $\mathfrak{g}'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to I_0'. We choose a reduced expression $w_{I'} w_{I_0'} = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1$ of $w_{I'} w_{I_0'}$. The elements Y_i ($i \in \Lambda'$) can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \dots, m_6^j)$ for $j \in \Lambda$. Then we have

$$\begin{array}{lll}
\mathbf{m}^1 = (1,0,0,0,0,0), & \mathbf{m}^2 = (1,1,0,0,0,0), & \mathbf{m}^3 = (1,1,1,0,0,0), \\
\mathbf{m}^4 = (1,1,1,1,0,0), & \mathbf{m}^5 = (1,1,1,0,1,0), & \mathbf{m}^6 = (1,1,1,1,1,0), \\
\mathbf{m}^7 = (1,1,2,1,1,0), & \mathbf{m}^8 = (1,2,2,1,1,0), & \mathbf{m}^9 = (1,1,1,0,1,1), \\
\mathbf{m}^{10} = (1,1,1,1,1,1), & \mathbf{m}^{11} = (1,1,2,1,1,1), & \mathbf{m}^{12} = (1,2,2,1,1,1), \\
\mathbf{m}^{13} = (1,1,2,1,2,1), & \mathbf{m}^{14} = (1,2,2,1,2,1), & \mathbf{m}^{15} = (1,2,3,1,2,1), \\
\mathbf{m}^{16} = (1,2,3,2,2,1). & &
\end{array}$$

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+$. Since $k \neq 1$ and $m_1^j = 1$, we have $\beta_j + \alpha_k \notin \Delta_I$. Therefore there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j + \alpha_k = \beta_l$ ($j, l \in \Lambda, k \in I$), then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

There exist 20 triplets $(k, j, l) \in I \times \Lambda \times \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. The triplets are the following: $(2,1,2)$, $(3,2,3)$, $(4,3,4)$, $(5,3,5)$, $(5,4,6)$, $(4,5,6)$, $(3,6,7)$, $(2,7,8)$, $(6,5,9)$, $(4,9,10)$, $(3,10,11)$, $(2,11,12)$, $(5,11,13)$, $(5,12,14)$, $(2,13,14)$, $(3,14,15)$, $(4,15,16)$, $(6,6,10)$, $(6,7,11)$, $(6,8,12)$.

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Lemma 2.1 *Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ $(k \in I)$. Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ and $p \in \Lambda$ satisfying*

$$\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}),\ \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}}),\ (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1,$$
$$\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}).$$

Proof. Among the 20 triplets (k, j, l) satisfying $\beta_j + \alpha_k = \beta_l$ $(k \in I,\ j, k \in \Lambda)$, the 12 triplets satisfy $l = j + 1$, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 8 cases. In the cases $(k, j, l) = (5,3,5)$, $(5,4,6)$, $(5,11,13)$, $(5,12,14)$, the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_3 s_4 s_2 s_3 s_5 s_6$$

of $w_I w_0$ with $p = 3, 5, 11, 13$ respectively satisfies the required properties. In the cases $(k, j, l) = (6,5,9)$, $(6,6,10)$, $(6,7,11)$, $(6,8,12)$, the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_3 s_5 s_2 s_3 s_1 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ with $p = 5, 7, 9, 11$ respectively satisfies the required properties. □

It is known that $U_q(\mathfrak{n}_I^+)^1 = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_I} \mathbb{C}(q) Y_\beta$ is an irreducible $U_q(\mathfrak{l}_I)$-module. (see [4])

Lemma 2.2 *For $k \in I$, $j \in \Lambda$, we have*

$$\mathrm{ad}(F_k) Y_j = \begin{cases} Y_l & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j + \alpha_k, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathrm{ad}(E_k) Y_j = \begin{cases} Y_l & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j - \alpha_k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\bigoplus_{j \in \Lambda} \mathbb{C}(q) Y_j$ is a $U_q(\mathfrak{l}_I)$-module, we have $\mathrm{ad}(F_k) Y_j = 0$ if $\beta_j + \alpha_k \notin \Delta^+ \setminus \Delta_I$, and we have $\mathrm{ad}(E_k) Y_j = 0$ if $\beta_j - \alpha_k \notin \Delta^+ \setminus \Delta_I$.

We shall show $\mathrm{ad}(F_k) Y_\beta = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta + \alpha_k$. By Lemma 2.1 we can choose a reduced expression of $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ satisfying $\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p})$, $\beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}})$,

$(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$. Then we can write $Y_\beta = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_p})$, $Y_{\beta'} = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}})$. Since $(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$, we have $T_{i_p}(F_{i_{p+1}}) = F_{i_{p+1}}F_{i_p} - qF_{i_p}F_{i_{p+1}}$. Moreover, since $\alpha_k = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_{p+1}})$, we have $T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_{p+1}}) = F_k$ by Lemma 1.1, and hence

$$\begin{aligned} Y_{\beta'} &= T_{i_1}T_{i_2}\cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}}) = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_{p+1}}F_{i_p} - qF_{i_p}F_{i_{p+1}}) \\ &= F_kY_\beta - qY_\beta F_k. \end{aligned}$$

Since $(\beta, \alpha_k) = -1$, we have $\mathrm{ad}(F_k)Y_\beta = F_kY_\beta - qY_\beta F_k$. Hence we have $\mathrm{ad}(F_k)Y_\beta = Y_{\beta'}$.

Let us show $\mathrm{ad}(E_k)Y_\beta = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta - \alpha_k$. By the above argument we have $Y_\beta = \mathrm{ad}(F_k)Y_{\beta'} = F_kY_{\beta'} - qY_{\beta'}F_k$. Since $\beta' - \alpha_k = \beta - 2\alpha_k \notin \Delta^+ \setminus \Delta_I$, we have $\mathrm{ad}(E_k)Y_{\beta'} = 0$, and hence $E_kY_{\beta'} = Y_{\beta'}E_k$. Since $(\beta', \alpha_k) = -1$, we have $K_kY_{\beta'} = qY_{\beta'}K_k$. Hence we have

$$\begin{aligned} \mathrm{ad}(E_k)Y_\beta &= (E_kY_\beta - Y_\beta E_k)K_k \\ &= (E_k(F_kY_{\beta'} - qY_{\beta'}F_k) - (F_kY_{\beta'} - qY_{\beta'}F_k)E_k)K_k \\ &= \left(\frac{K_k - K_k^{-1}}{q - q^{-1}}Y_{\beta'} - qY_{\beta'}\frac{K_k - K_k^{-1}}{q - q^{-1}}\right)K_k = (Y_{\beta'}K_k^{-1})K_k = Y_{\beta'}. \ \square \end{aligned}$$

Next we shall consider quadratic fundamental relations among the elements Y_i. Since we have

$$\sum_{i,j\in\Lambda} \mathbb{C}(q)Y_iY_j = \bigoplus_{s\leq t} \mathbb{C}(q)Y_sY_t,$$

we can write

$$Y_iY_j = \sum_{\substack{s\leq t \\ \beta_i+\beta_j=\beta_s+\beta_t}} a_{s,t}^{i,j}Y_sY_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for $i > j$ (see [4]). Hence if $\beta_i + \beta_j$ does not have another decomposition $\beta + \beta'$ ($\beta, \beta' \in \Delta^+ \setminus \Delta_I$, $\beta_i + \beta_j = \beta + \beta'$) then we have $Y_iY_j = a_{i,j}Y_jY_i$ for some $a_{i,j} \in \mathbb{C}(q)$. We denote the set of weights of the ten-dimensional irreducible highest weight $\mathfrak{l}_I$-module $J_{C_0}^0$ with highest weight $-\beta_1 - \beta_8$ by Γ. For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. We fix a bijection $\{1, 2, \ldots, 10\} \ni n \mapsto -\delta_n \in \Gamma$ such that if $\delta_m - \delta_n \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\alpha_i$, then $n \leq m$. For each n there exist exactly four pairs $(i,j) \in \Lambda^2$ such that $i < j, \beta_i + \beta_j = \delta_n$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n) \in \Lambda^2$ where $i_4^n < i_3^n < i_2^n < i_1^n$. Set $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n) \in \Lambda^8$ $(1 \le n \le 10)$. Then we have

$$
\begin{array}{ll}
\mathbf{A}(1) = (1,2,3,4,5,6,7,8), & \mathbf{A}(2) = (1,2,3,4,9,10,11,12), \\
\mathbf{A}(3) = (1,2,5,6,9,10,13,14), & \mathbf{A}(4) = (1,3,5,7,9,11,13,15), \\
\mathbf{A}(5) = (2,3,5,8,9,12,14,15), & \mathbf{A}(6) = (1,4,6,7,10,11,13,16), \\
\mathbf{A}(7) = (2,4,6,8,10,12,14,16), & \mathbf{A}(8) = (3,4,7,8,11,12,15,16), \\
\mathbf{A}(9) = (5,6,7,8,13,14,15,16), & \mathbf{A}(10) = (9,10,11,12,13,14,15,16).
\end{array}
$$

We denote the set $\{i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n\}$ by $|\mathbf{A}(n)|$ for $1 \le n \le 10$. For any $i, j \in \Lambda$ there exists n satisfying $i, j \in |\mathbf{A}(n)|$.

Set

$$\mathcal{A} = \{(k, n, n') \in I \times \Lambda \times \Lambda \mid \delta_n + \alpha_k = \delta_{n'}\}.$$

Then $\mathcal{A} = \{(6,1,2), (5,2,3), (3,3,4), (2,4,5), (4,4,6), (2,6,7), (4,5,7), (3,7,8), (5,8,9), (6,9,10)\}$. For any $n \in \{2,3,\ldots,10\}$ we can take a sequence $((k_1, n_1, n_1'), \ldots, (k_s, n_s, n_s'))$ of $\mathcal{A}$ satisfying $n_1 = 1$, $n_s' = n$, $n_j' = n_{j+1}$ $(1 \le j \le s-1)$.

For $(k, n, n') \in \mathcal{A}$ and $m \in \{1,2,3,4\}$, we have either

$$(\beta_{i_m^n}, \alpha_k) = 0,\ i_m^{n'} = i_m^n,\ (\beta_{j_m^n}, \alpha_k) = -1,\ \beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k \qquad (\mathrm{P}_m^+)$$

or

$$(\beta_{i_m^n}, \alpha_k) = -1,\ \beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k,\ (\beta_{j_m^n}, \alpha_k) = 0,\ j_m^{n'} = j_m^n. \qquad (\mathrm{P}_m^-)$$

Proposition 2.3 *For any $i, j \in \Lambda$ satisfying $i < j$, we have*

$$
(\mathrm{Q6}) \quad Y_i Y_j = \begin{cases}
Y_j Y_i & \text{if there exists } n \text{ such that } i = i_1^n, j = j_1^n, \\
Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \\
\quad \text{if there exists } n \text{ such that } i = i_2^n, j = j_2^n, & \\
Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & \\
\quad \text{if there exist } n,\ m \in \{3,4\} \text{ such that } i = i_m^n, j = j_m^n, & \\
q Y_j Y_i & \text{otherwise.}
\end{cases}
$$

Proof. Since there exists some n satisfying $i, j \in |\mathbf{A}(n)|$ for any $i, j \in \Lambda$, it is sufficient to show that for any $1 \le n \le 10$ the elements $Y_{i_m^n}, Y_{j_m^n}$ $(1 \le m \le 4)$ satisfy the following relations.

$$
(\mathrm{R}n) \begin{cases}
Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n} & (\mathrm{R}n, 1) \\
Y_{i_m^n} Y_{j_m^n} = Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} \quad (2 \le m \le 4) & (\mathrm{R}n, 2) \\
Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1} & \\
\quad (l_1, l_2 \in |\mathbf{A}(n)|, l_1 < l_2, (l_1, l_2) \ne (i_m^n, j_m^n)\ (1 \le m \le 4)) & (\mathrm{R}n, 3)
\end{cases}
$$

When $n = 1$, the elements Y_i $(1 \leq i \leq 8)$ satisfy the same relations as those for type D_5, hence the relations (R1) hold.

For any $m > 1$ there exists a sequence $((k_1, n_1, n_1'), \ldots, (k_s, n_s, n_s'))$ of $\mathcal{A}$ satisfying $n_1 = 1, n_s' = m$, $n_j' = n_{j+1}$ $(1 \leq j \leq s-1)$, and hence it is sufficient to show the relations (Rn') for $(k, n, n') \in \mathcal{A}$ assuming the relations (Rn).

Let $(k, n, n') \in \mathcal{A}$. Assume that the relations (Rn) hold.

We first show that the relation (Rn',1) holds. If the condition (P_1^+) is satisfied, then we have $Y_{i_1^{n'}} = Y_{i_1^n}, F_k Y_{i_1^n} = Y_{i_1^n} F_k, Y_{j_1^{n'}} = \mathrm{ad}(F_k) Y_{j_1^n} = F_k Y_{j_1^n} - qY_{j_1^n} F_k$. Since $Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n}$, we have

$$\begin{aligned} Y_{i_1^{n'}} Y_{j_1^{n'}} &= Y_{i_1^n}\, \mathrm{ad}(F_k) Y_{j_1^n} = Y_{i_1^n}(F_k Y_{j_1^n} - qY_{j_1^n} F_k) = (F_k Y_{j_1^n} - qY_{j_1^n} F_k) Y_{i_1^n} \\ &= Y_{j_1^{n'}} Y_{i_1^{n'}}. \end{aligned}$$

If the condition (P_1^-) is satisfied, then we can prove the formula (Rn',1) similarly.

Next we prove the formula (Rn',2). Assume the condition (P_m^+) is satisfied, then we have

$$\begin{aligned} Y_{i_m^{n'}} Y_{j_m^{n'}} &= Y_{i_m^n}(F_k Y_{j_m^n} - qY_{j_m^n} F_k) \\ &= F_k Y_{j_m^n} Y_{i_m^n} - qY_{j_m^n} F_k Y_{i_m^n} \\ &\quad + q(F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - qY_{j_{m-1}^n} Y_{i_{m-1}^n} F_k) \\ &\quad - q^{-1}(F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - qY_{i_{m-1}^n} Y_{j_{m-1}^n} F_k). \end{aligned}$$

If the condition (P_{m-1}^+) is satisfied, then we have

$$\begin{aligned} F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - qY_{j_{m-1}^n} Y_{i_{m-1}^n} F_k &= Y_{j_{m-1}^n}(F_k Y_{i_{m-1}^n} - qY_{i_{m-1}^n} F_k) = Y_{j_{m-1}^{n'}} Y_{i_{m-1}^{n'}}, \\ F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - qY_{i_{m-1}^n} Y_{j_{m-1}^n} F_k &= (F_k Y_{i_{m-1}^n} - qY_{i_{m-1}^n} F_k) Y_{j_{m-1}^n} = Y_{i_{m-1}^{n'}} Y_{j_{m-1}^{n'}}, \end{aligned}$$

and if the condition (P_{m-1}^-) is satisfied, then we have

$$\begin{aligned} F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - qY_{j_{m-1}^n} Y_{i_{m-1}^n} F_k &= (F_k Y_{j_{m-1}^n} - qY_{j_{m-1}^n} F_k) Y_{i_{m-1}^n} = Y_{j_{m-1}^{n'}} Y_{i_{m-1}^{n'}}, \\ F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - qY_{i_{m-1}^n} Y_{j_{m-1}^n} F_k &= Y_{i_{m-1}^n}(F_k Y_{j_{m-1}^n} - qY_{j_{m-1}^n} F_k) = Y_{i_{m-1}^{n'}} Y_{j_{m-1}^{n'}}. \end{aligned}$$

Hence we have $Y_{i_m^{n'}} Y_{j_m^{n'}} = Y_{j_m^{n'}} Y_{i_m^{n'}} + qY_{j_{m-1}^{n'}} Y_{i_{m-1}^{n'}} - q^{-1} Y_{i_{m-1}^{n'}} Y_{j_{m-1}^{n'}}$. The formula (R$n'$,2) is proved. When the condition (P_m^-) is satisfied, we can prove it similarly.

Finally we prove the formula (Rn',3). Let $l'_1, l'_2 \in |\mathbf{A}(n')|$ satisfying $l'_1 < l'_2$ and $(l'_1, l'_2) \neq (i^{n'}_m, j^{n'}_m)$ for $1 \leq m \leq 4$. When $l'_p = i^{n'}_m \in |\mathbf{A}(n')|$ (resp. $l'_p = j^{n'}_m$), we denote $i^n_m \in |\mathbf{A}(n)|$ (resp. j^n_m) by l_p for $p = 1, 2$. Since $l_1 < l_2$ and $(l_1, l_2) \neq (i^n_m, j^n_m)$ for $1 \leq m \leq 4$, we have $Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1}$. We have the following possibilities:

(1) $l'_1 = l_1$, $l'_2 = l_2$, $(\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = 0$,
(2) $l'_1 = l_1$, $(\beta_{l_1}, \alpha_k) = 0$, $\beta_{l'_2} = \beta_{l_2} + \alpha_k$, $(\beta_{l_2}, \alpha_k) = -1$,
(3) $\beta_{l'_1} = \beta_{l_1} + \alpha_k$, $(\beta_{l_1}, \alpha_k) = -1$, $l'_2 = l_2$, $(\beta_{l_2}, \alpha_k) = 0$,
(4) $\beta_{l'_1} = \beta_{l_1} + \alpha_k$, $\beta_{l'_2} = \beta_{l_2} + \alpha_k$, $(\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = -1$.

In the case (1) the formula (Rn',3) is obvious.

In the case (2) we have $F_kY_{l_1} = Y_{l_1}F_k$, $Y_{l'_2} = \mathrm{ad}(F_k)Y_{l_2} = F_kY_{l_2} - qY_{l_2}F_k$. Hence we have

$$Y_{l'_1}Y_{l'_2} = Y_{l_1}(F_kY_{l_2} - qY_{l_2}F_k) = q(F_kY_{l_2} - qY_{l_2}F_k)Y_{l_1} = qY_{l'_2}Y_{l'_1}.$$

In the case (3) we can prove it similarly to the case (2).

In the case (4) we have $Y_{l'_p} = F_kY_{l_p} - qY_{l_p}F_k$ for $p = 1, 2$. Since $\beta_{l'_p} + \alpha_k = \beta_{l_p} + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$ and $(\beta_{l'_p}, \alpha_k) = 1$, we have $\mathrm{ad}(F_k)Y_{l'_p} = F_kY_{l'_p} - q^{-1}Y_{l'_p}F_k = 0$ for $p = 1, 2$. Hence we have $F_kF_kY_{l_p} - (q + q^{-1})F_kY_{l_p}F_k + Y_{l_p}F_kF_k = 0, F_kY_{l_p}F_k = (q + q^{-1})^{-1}(F_kF_kY_{l_p} + Y_{l_p}F_kF_k)$ for $p = 1, 2$. By these formulas we have

$$\begin{aligned}
Y_{l'_1}Y_{l'_2} &= (F_kY_{l_1} - qY_{l_1}F_k)(F_kY_{l_2} - qY_{l_2}F_k) \\
&= F_kY_{l_1}F_kY_{l_2} - qF_kY_{l_1}Y_{l_2}F_k - qY_{l_1}F_kF_kY_{l_2} + q^2Y_{l_1}F_kY_{l_2}F_k \\
&= \frac{1}{q+q^{-1}}F_kF_kY_{l_1}Y_{l_2} + \frac{1}{q+q^{-1}}Y_{l_1}F_kF_kY_{l_2} \\
&\quad - qF_kY_{l_1}Y_{l_2}F_k - qY_{l_1}F_kF_kY_{l_2} \\
&\quad + \frac{q^2}{q+q^{-1}}Y_{l_1}F_kF_kY_{l_2} + \frac{q^2}{q+q^{-1}}Y_{l_1}Y_{l_2}F_kF_k \\
&= \frac{1}{q+q^{-1}}F_kF_kY_{l_1}Y_{l_2} - qF_kY_{l_1}Y_{l_2}F_k + \frac{q^2}{q+q^{-1}}Y_{l_1}Y_{l_2}F_kF_k.
\end{aligned}$$

Similarly we have

$$Y_{l'_2}Y_{l'_1} = \frac{1}{q+q^{-1}}F_kF_kY_{l_2}Y_{l_1} - qF_kY_{l_2}Y_{l_1}F_k + \frac{q^2}{q+q^{-1}}Y_{l_2}Y_{l_1}F_kF_k.$$

Since $Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1}$, we have $Y_{l'_1}Y_{l'_2} = qY_{l'_2}Y_{l'_1}$. □

By [4] and Proposition 2.3 we obtain the following:

Theorem 2.4 *The formulas* (Q6) *give fundamental relations for the generator system* $\{Y_i\}_{i\in\Lambda}$ *of the algebra* $A_q = U_q(\mathfrak{n}_I^-)$.

We shall construct a quantum deformation of the lowest degree part $J_{C_0}^0$ of the defining ideal J_{C_0} and we shall give canonical generators of a quantum analogue of J_{C_0}.

Set

$$\psi_n = Y_{i_4^n} Y_{j_4^n} - qY_{i_3^n} Y_{j_3^n} + q^2 Y_{i_2^n} Y_{j_2^n} - q^3 Y_{i_1^n} Y_{j_1^n},$$

for $1 \le n \le 10$. Recall that $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n)$. Using the formulas (R$n$,1), (R$n$,2), we can write $\psi_n = Y_{j_4^n} Y_{i_4^n} - q^{-1} Y_{j_3^n} Y_{i_3^n} + q^{-2} Y_{j_2^n} Y_{i_2^n} - q^{-3} Y_{j_1^n} Y_{i_1^n}$.

Lemma 2.5 *We have*

$$\mathrm{ad}(F_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n + \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{ad}(E_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n - \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise} \end{cases}$$

for $k \in I$, *and*

$$\mathrm{ad}(K_k)\psi_n = q^{-(\delta_n, \alpha_k)}\psi_n$$

for $k \in I_0$.

Proof. Let $(k, n, n') \in \mathcal{A}$. We shall show $\mathrm{ad}(F_k)\psi_n = \psi_{n'}$. If the condition ($\mathrm{P}_m^+$) is satisfied, then we have $\mathrm{ad}(F_k)Y_{i_m^n} = 0$, $Y_{i_m^{n'}} = Y_{i_m^n}$, $\mathrm{ad}(K_k)Y_{i_m^n} = Y_{i_m^n}$ $\mathrm{ad}(F_k)Y_{j_m^n} = Y_{j_m^{n'}}$. Hence

$$\mathrm{ad}(F_k)(Y_{i_m^n} Y_{j_m^n}) = (\mathrm{ad}(F_k)Y_{i_m^n})Y_{j_m^n} + (\mathrm{ad}(K_k)Y_{i_m^n})(\mathrm{ad}(F_k)Y_{j_m^n}) = Y_{i_m^{n'}} Y_{j_m^{n'}}.$$

If the condition (P_m^-) is satisfied, then we have $\mathrm{ad}(F_k)Y_{i_m^n} = Y_{i_m^{n'}}$, $\mathrm{ad}(F_k)Y_{j_m^n} = 0$. Hence $\mathrm{ad}(F_k)(Y_{i_m^n} Y_{j_m^n}) = Y_{i_m^{n'}} Y_{j_m^{n'}}$ similarly. Therefore we have $\mathrm{ad}(F_k)\psi_n = \psi_{n'}$.

Next we prove $\mathrm{ad}(E_k)\psi_{n'} = \psi_n$. We have $\mathrm{ad}(E_k)Y_{i_m^{n'}} = 0$, $\mathrm{ad}(E_k)Y_{j_m^{n'}} = Y_{j_m^n}$ if the condition (P_m^+) is satisfied, and we have $\mathrm{ad}(E_k)Y_{i_m^{n'}} = Y_{i_m^n}$, $\mathrm{ad}(K_k^{-1})Y_{j_m^{n'}} = Y_{j_m^{n'}}$, $j_m^{n'} = j_m^n$, $\mathrm{ad}(E_k)Y_{j_m^{n'}} = 0$ if the condition (P_m^-) is satisfied. Hence we have

$$\mathrm{ad}(E_k)(Y_{i_m^{n'}} Y_{j_m^{n'}}) = (\mathrm{ad}(E_k)Y_{i_m^{n'}})(\mathrm{ad}(K_k^{-1})Y_{j_m^{n'}}) + Y_{i_m^{n'}}(\mathrm{ad}(E_k)Y_{j_m^{n'}}) = Y_{i_m^n} Y_{j_m^n}$$

for $1 \le m \le 4$. Therefore we have $\mathrm{ad}(E_k)\psi_{n'} = \psi_n$.

In other 50 cases, where $\delta_n + \alpha_k \notin \{\delta_l \,|\, 1 \le l \le 10\}$, we can check $\mathrm{ad}(F_k)\psi_n = 0$ by a case-by-case consideration as follows.

In the 10 cases where there exists n' satisfying $\mathrm{ad}(F_k)\psi_{n'} = \psi_n$, ($(k,n) = (6,2), (5,3), (3,4), (2,5), (4,6), (2,7), (4,7), (3,8), (5,9), (6,10)$), we have $\mathrm{ad}(F_k)Y_{i_m^n} = \mathrm{ad}(F_k)Y_{j_m^n} = 0$ for $1 \le m \le 4$, and hence the assertion is obvious.

In the 8 cases $(k,n) = (5,1), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8), (5,10)$, we have $\mathrm{ad}(F_k)Y_{i_m^n} = \mathrm{ad}(F_k)Y_{j_m^n} = 0$ for $m = 3, 4$, $\mathrm{ad}(F_k)Y_{i_2^n} = Y_{j_1^n}$, $\mathrm{ad}(F_k)Y_{j_2^n} = 0$, $\mathrm{ad}(F_k)Y_{i_1^n} = Y_{j_2^n}$, $\mathrm{ad}(F_k)Y_{j_1^n} = 0$, and hence $\mathrm{ad}(F_k)(Y_{i_2^n}Y_{j_2^n}) = Y_{j_1^n}Y_{j_2^n}$, $\mathrm{ad}(F_k)(Y_{i_1^n}Y_{j_1^n}) = Y_{j_2^n}Y_{j_1^n}$. Thus we have $\mathrm{ad}(F_k)\psi_n = q^2(Y_{j_1^n}Y_{j_2^n} - qY_{j_2^n}Y_{j_1^n}) = 0$ by Proposition 2.3.

In the remaining 32 cases there exists $m' \in \{2,3,4\}$ such that $\mathrm{ad}(F_k)Y_{i_m^n} = 0$ $(m \ne m')$, $\mathrm{ad}(F_k)Y_{j_m^n} = 0$ $(m \ne m'-1)$, $\mathrm{ad}(F_k)Y_{i_{m'}^n} = Y_{i_{m'-1}^n}$, $\mathrm{ad}(F_k)Y_{j_{m'-1}^n} = Y_{j_{m'}^n}$, $\mathrm{ad}(K_k)Y_{i_{m'-1}^n} = q^{-1}Y_{i_{m'-1}^n}$. Then we have $\mathrm{ad}(F_k)(Y_{i_{m'}^n}Y_{j_{m'}^n}) = Y_{i_{m'-1}^n}Y_{j_{m'}^n}$, $\mathrm{ad}(F_k)(Y_{i_{m'-1}^n}Y_{j_{m'-1}^n}) = q^{-1}Y_{i_{m'-1}^n}Y_{j_{m'}^n}$, $\mathrm{ad}(F_k)\psi_n = q^{4-m'}(1 - qq^{-1})Y_{i_{m'-1}^n}Y_{j_{m'}^n} = 0$.

The weight $\beta_{i_m^n} + \beta_{j_m^n}$ does not depend on m. Hence we have $\mathrm{ad}(K_k)\psi_n = q^{-(\delta_n,\alpha_k)}\psi_n$ where $\delta_n = \beta_{i_m^n} + \beta_{j_m^n}$.

Finally we show $\mathrm{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \,|\, 1 \le l \le 10\}$. We can check $\mathrm{ad}(E_k)\psi_1 = 0$ for any $k = 2, 3, \ldots, 6$ directly. It follows that $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n = U_q(\mathfrak{l}_I)\psi_1$ and hence $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is an $\mathrm{ad}\, U_q(\mathfrak{l}_I)$-stable subspace with weights in $\{-\delta_l \,|\, 1 \le l \le 10\}$. Therefore we have $\mathrm{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \,|\, 1 \le l \le 10\}$. □

Proposition 2.6 *$\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(\mathfrak{l}_I)$-module with highest weight vector ψ_1.*

Proof. By Lemma 2.5 $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is a finite dimensional $U_q(\mathfrak{l}_I)$-submodule generated by a highest weight vector ψ_1 with highest weight $-\delta_1$. Thus it is irreducible. □

By [4] and Proposition 2.6 we obtain the following:

Theorem 2.7 *A quantum analogue of the defining ideal J_{C_0} of the closure of the non-trivial non-open orbit C_0 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid 1 \le n \le 10\}$.*

3 Case of type E_7

Let $\mathfrak{g}$ be a simple Lie algebra of type E_7. We shall use the labelling of the vertices of the Dynkin diagram 1. Hence we have $I_0 = \{1,2,3,4,5,6,7\}$. Set $I = \{2,3,4,5,6,7\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}, [\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $\mathfrak{l}_I$ is isomorphic to $\mathbb{C} \oplus \mathfrak{g}_{E_6}$, where $\mathfrak{g}_{E_6}$ is a Lie algebra of type E_6 over $\mathbb{C}$, and $\mathfrak{n}_I^+$ is a 27-dimensional irreducible prehomogeneous vector space. There are four L_I-orbits $\{0\}, C_1, C_2, O$ on $\mathfrak{n}_I^+$ satisfying $\{0\} \subset \overline{C_1} \subset \overline{C_2} \subset \overline{O}$. Let $J_{C_1} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_1, and let $J_{C_1}^0$ denote the subspace of J_{C_1} consisting of the polynomials in J_{C_1} with homogeneous degree 2. Then $J_{C_1}^0$ is a 27-dimensional irreducible $\mathfrak{l}_I$-module, and it generates the ideal J_{C_1}. Let $J_{C_2} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_2, and let $J_{C_2}^0$ denote the subspace of J_{C_2} consisting of the polynomials in J_{C_2} with homogeneous degree 3. Then $J_{C_2}^0$ is a one-dimensional irreducible $\mathfrak{l}_I$-module generated by the irreducible relative invariant, and it generates the ideal J_{C_2}.

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_3 s_5 s_4 s_6 s_7 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$$

of $w_I w_0$ and define the elements Y_i $(i \in \Lambda = \{1, 2, \ldots, 27\})$ as in Section 1.

Set $I_0' = \{1,2,3,4,5,6\}, I' = \{2,3,4,5,6\}, \Lambda' = \{1,2,\ldots,10\}$. Then $\{\alpha_i\}_{i \in I_0'}$ is a set of simple roots of type D_6. Let $\mathfrak{g}'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to I_0'. We choose a reduced expression $w_{I'} w_{I_0'} = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$ of $w_{I'} w_{I_0'}$. The elements Y_i $(i \in \Lambda')$ can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \ldots, m_7^j)$ for $j \in \Lambda$. Then we have

$$\begin{aligned}
&\mathbf{m}^1 = (1,0,0,0,0,0,0),\ \mathbf{m}^2 = (1,1,0,0,0,0,0),\ \mathbf{m}^3 = (1,1,1,0,0,0,0),\\
&\mathbf{m}^4 = (1,1,1,1,0,0,0),\ \mathbf{m}^5 = (1,1,1,1,1,0,0),\ \mathbf{m}^6 = (1,1,1,1,0,1,0),\\
&\mathbf{m}^7 = (1,1,1,1,1,1,0),\ \mathbf{m}^8 = (1,1,1,2,1,1,0),\ \mathbf{m}^9 = (1,1,2,2,1,1,0),\\
&\mathbf{m}^{10} = (1,2,2,2,1,1,0),\ \mathbf{m}^{11} = (1,1,1,1,0,1,1),\ \mathbf{m}^{12} = (1,1,1,1,1,1,1),\\
&\mathbf{m}^{13} = (1,1,1,2,1,1,1),\ \mathbf{m}^{14} = (1,1,2,2,1,1,1),\ \mathbf{m}^{15} = (1,1,1,2,1,2,1),\\
&\mathbf{m}^{16} = (1,1,2,2,1,2,1),\ \mathbf{m}^{17} = (1,1,2,3,1,2,1),\ \mathbf{m}^{18} = (1,1,2,3,2,2,1),\\
&\mathbf{m}^{19} = (1,2,2,2,1,1,1),\ \mathbf{m}^{20} = (1,2,2,2,1,2,1),\ \mathbf{m}^{21} = (1,2,2,3,1,2,1),\\
&\mathbf{m}^{22} = (1,2,2,3,2,2,1),\ \mathbf{m}^{23} = (1,2,3,3,1,2,1),\ \mathbf{m}^{24} = (1,2,3,3,2,2,1),\\
&\mathbf{m}^{25} = (1,2,3,4,2,2,1),\ \mathbf{m}^{26} = (1,2,3,4,2,3,1),\ \mathbf{m}^{27} = (1,2,3,4,2,3,2).
\end{aligned}$$

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+ \setminus \Delta_I$ and there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j, \beta_l \in \Delta^+ \setminus \Delta_I$ satisfying $\beta_l - \beta_j = \alpha_k$ $(k \in I)$, then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Set

$$\mathcal{B} = \{(k, j, l) \in I \times \Lambda \times \Lambda \mid \beta_j + \alpha_k = \beta_l\}.$$

We have
$\mathcal{B} = \{(2,1,2), (3,2,3), (4,3,4), (5,4,5), (6,4,6), (6,5,7), (5,6,7), (4,7,8), (3,8,9), (2,9,10), (7,6,11), (7,7,12), (7,8,13), (7,9,14), (7,10,19), (5,11,12), (4,12,13), (3,13,14), (6,13,15), (6,14,16), (3,15,16), (4,16,17), (5,17,18), (2,14,19), (2,16,20), (2,17,21), (2,18,22), (6,19,20), (4,20,21), (5,21,22), (3,21,23), (3,22,24), (5,23,24), (4,24,25), (6,25,26), (7,26,27)\}$.
In particular, we have $|\mathcal{B}| = 36$.

Lemma 3.1 *Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ $(k \in I)$. Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{27}}$ and $p \in \Lambda$ satisfying*

$$\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}),\ \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}}),\ (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1,$$
$$\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}).$$

Proof. The 21 triplets (k, j, l) in $\mathcal{B}$ satisfy $l = j + 1$, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 15 cases. In the cases $(k, j, l) = (6,4,6), (6,5,7), (6,13,15), (6,14,16), (3,21,23), (3,22,24)$, we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_5 s_3 s_4 s_6 s_7 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$$

with $p = 4, 6, 13, 15, 21, 23$, and in the cases $(k, j, l) = (7,6,11), (7,7,12), (7,8,13), (7,9,14), (7,10,19)$, we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_4 s_6 s_3 s_4 s_2 s_3 s_1 s_2 s_5 s_4 s_6 s_7 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$$

with $p = 6, 8, 10, 12, 14$, and in the cases $(k, j, l) = (2,14,19), (2,16,20), (2,17,21), (2,18,22)$, we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_5 s_3 s_2 s_4 s_3 s_6 s_4 s_7 s_6 s_5 s_4 s_3 s_2 s_1$$

with $p = 15, 17, 19, 21$. □

We can show the following similarly to the case E_6. We omit the details.

Lemma 3.2 *For $k \in I$, $j \in \Lambda$, we have*

$$\mathrm{ad}(F_k) Y_j = \begin{cases} Y_l & \text{if there exists } (k, j, l) \in \mathcal{B}, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathrm{ad}(E_k) Y_j = \begin{cases} Y_l & \text{if there exists } (k, l, j) \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

The $U_q(\mathfrak{l}_I)$-module $\bigoplus_{j\in\Lambda}\mathbb{C}(q)Y_j$ is an irreducible highest weight module with highest weight vector Y_1 and lowest weight vector Y_{27}. Hence, for any $1 \le m \le 26$, there exists a sequence $((k_1, n_1', n_1), \dots, (k_s, n_s', n_s))$ of $\mathcal{B}$ satisfying $n_1 = 27$, $n_s' = m$, $n_j' = n_{j+1}$ $(1 \le j \le s-1)$.

Next we shall consider relations among the elements Y_i. We can write

$$Y_iY_j = \sum_{\substack{s\le t\\ \beta_i+\beta_j=\beta_s+\beta_t}} a_{s,t}^{i,j} Y_s Y_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for $i > j$ (see [4]). Hence if $\beta_i + \beta_j$ does not have another decomposition $\beta + \beta'$ $(\beta, \beta' \in \Delta^+ \setminus \Delta_I,\ \beta_i + \beta_j = \beta + \beta')$ then we have $Y_iY_j = a_{i,j}Y_jY_i$ for some $a_{i,j} \in \mathbb{C}(q)$. Set $\delta = 2\varpi_1 = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$, where ϖ_1 is the fundamental weight corresponding to α_1. We denote a set of weights of the 27-dimensional irreducible highest weight $\mathfrak{l}_I$-module $J_{C_1}^0$ with highest weight $-\beta_1 - \beta_{10}$ by Γ. Set $\gamma_n = \delta - \beta_n$ $(n \in \Lambda)$, and we have $\Gamma = \{-\gamma_n \mid n \in \Lambda\}$. For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. For each $n \in \Lambda$ there exist exactly five pairs $(i,j) \in \Lambda^2$ such that $i < j, \beta_i + \beta_j = \gamma_n$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n, j_1^n < j_2^n < j_3^n < j_4^n < j_5^n$, and i_1^n, j_1^n satisfy the following condition (P_1^+) or (P_1^-). Set $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10}$ $(n \in \Lambda)$. Then we have

$\mathbf{B}(1) = (10, 19, 20, 21, 23, 22, 24, 25, 26, 27)$, $\mathbf{B}(2) = (9, 14, 16, 17, 23, 18, 24, 25, 26, 27)$,
$\mathbf{B}(3) = (8, 13, 15, 17, 21, 18, 22, 25, 26, 27)$, $\mathbf{B}(4) = (7, 12, 15, 16, 20, 18, 22, 24, 26, 27)$,
$\mathbf{B}(5) = (6, 11, 15, 16, 20, 17, 21, 23, 26, 27)$, $\mathbf{B}(6) = (5, 12, 13, 14, 19, 18, 22, 24, 25, 27)$,
$\mathbf{B}(7) = (4, 11, 13, 14, 19, 17, 21, 23, 25, 27)$, $\mathbf{B}(8) = (3, 11, 12, 14, 19, 16, 20, 23, 24, 27)$,
$\mathbf{B}(9) = (2, 11, 12, 13, 19, 15, 20, 21, 22, 27)$, $\mathbf{B}(10) = (1, 11, 12, 13, 14, 15, 16, 17, 18, 27)$,
$\mathbf{B}(11) = (5, 7, 8, 9, 10, 18, 22, 24, 25, 26)$, $\mathbf{B}(12) = (4, 6, 8, 9, 10, 17, 21, 23, 25, 26)$,
$\mathbf{B}(13) = (3, 6, 7, 9, 10, 16, 20, 23, 24, 26)$, $\mathbf{B}(14) = (2, 6, 7, 8, 10, 15, 20, 21, 22, 26)$,
$\mathbf{B}(15) = (3, 4, 5, 9, 10, 14, 19, 23, 24, 25)$, $\mathbf{B}(16) = (2, 4, 5, 8, 10, 13, 19, 21, 22, 25)$,
$\mathbf{B}(17) = (2, 3, 5, 7, 10, 12, 19, 20, 22, 24)$, $\mathbf{B}(18) = (2, 3, 4, 6, 10, 11, 19, 20, 21, 23)$,
$\mathbf{B}(19) = (1, 6, 7, 8, 9, 15, 16, 17, 18, 26)$, $\mathbf{B}(20) = (1, 4, 5, 8, 9, 13, 14, 17, 18, 25)$,
$\mathbf{B}(21) = (1, 3, 5, 7, 9, 12, 14, 16, 18, 24)$, $\mathbf{B}(22) = (1, 3, 4, 6, 9, 11, 14, 16, 17, 23)$,
$\mathbf{B}(23) = (1, 2, 5, 7, 8, 12, 13, 15, 18, 22)$, $\mathbf{B}(24) = (1, 2, 4, 6, 8, 11, 13, 15, 17, 21)$,
$\mathbf{B}(25) = (1, 2, 3, 6, 7, 11, 12, 15, 16, 20)$, $\mathbf{B}(26) = (1, 2, 3, 4, 5, 11, 12, 13, 14, 19)$,
$\mathbf{B}(27) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

For $n \in \Lambda$ we denote the set $\{i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n\}$ by $|\mathbf{B}(n)|$. For any $i, j \in \Lambda$ there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$.

For $(k,n',n)\in\mathcal{B}$ and $m\in\{1,2,3,4,5\}$, we have either

$$(\beta_{i^n_m},\alpha_k)=0,\ i^{n'}_m=i^n_m,\ (\beta_{j^n_m},\alpha_k)=-1,\ \beta_{j^{n'}_m}=\beta_{j^n_m}+\alpha_k \qquad (\mathrm{P}^+_m)$$

or

$$(\beta_{i^n_m},\alpha_k)=-1,\ \beta_{i^{n'}_m}=\beta_{i^n_m}+\alpha_k,\ (\beta_{j^n_m},\alpha_k)=0,\ j^{n'}_m=j^n_m. \qquad (\mathrm{P}^-_m)$$

Proposition 3.3 *For any $i,j\in\Lambda$ satisfying $i<j$, we have*

$$(\mathrm{Q7})\ Y_iY_j=\begin{cases} Y_jY_i & \text{if there exists } n\in\Lambda \text{ such that } \{i,j\}=\{i^n_1,j^n_1\},\\ Y_{j^n_2}Y_{i^n_2}+(q-q^{-1})Y_{i^n_1}Y_{j^n_1} & \\ \quad \text{if there exists } n\in\Lambda \text{ such that } i=i^n_2, j=j^n_2, & \\ Y_{j^n_m}Y_{i^n_m}+qY_{j^n_{m-1}}Y_{i^n_{m-1}}-q^{-1}Y_{i^n_{m-1}}Y_{j^n_{m-1}} & \\ \quad \text{if there exist } n\in\Lambda,\ m\in\{3,4,5\} \text{ such that } i=i^n_m, j=j^n_m, & \\ qY_jY_i & \text{otherwise.}\end{cases}$$

Proof. Since there exists $n\in\Lambda$ satisfying $i,j\in|\mathbf{B}(n)|$ for any $i,j\in\Lambda$, it is sufficient to show

$$(\mathrm{R}n)\begin{cases} Y_{i^n_1}Y_{j^n_1}=Y_{j^n_1}Y_{i^n_1} & (\mathrm{R}n,1)\\ Y_{i^n_m}Y_{j^n_m}=Y_{j^n_m}Y_{i^n_m}+qY_{j^n_{m-1}}Y_{i^n_{m-1}}-q^{-1}Y_{i^n_{m-1}}Y_{j^n_{m-1}} \quad (2\le m\le 5) & (\mathrm{R}n,2)\\ Y_{l_1}Y_{l_2}=qY_{l_2}Y_{l_1} & \\ \quad (l_1,l_2\in|\mathbf{B}(n)|, l_1<l_2, \{l_1,l_2\}\ne\{i^n_m,j^n_m\}\ (1\le m\le 5)) & (\mathrm{R}n,3)\end{cases}$$

for $n\in\Lambda$ and $1\le m\le 5$.

When $n=27$, the elements Y_i $(1\le i\le 10)$ satisfy the same relations as those for type D_6, and hence relations (R27) hold.

Since there exists a sequence $((k_1,n'_1,n_1),\dots,(k_s,n'_s,n_s))$ of $\mathcal{B}$ satisfying $n_1=27$, $n'_s=m$, $n'_j=n_{j+1}$ $(1\le j\le s-1)$ for any $1\le m\le 26$, it is sufficient to show $(\mathrm{R}n')$ for $(k,n',n)\in\mathcal{B}$ assuming $(\mathrm{R}n)$. This is proved similarly to Proposition 2.3. Details are omitted. □

By [4] and Proposition 3.3 we obtain the following:

Theorem 3.4 *The formulas* (Q7) *give fundamental relations for the generator system $\{Y_i\}_{i\in\Lambda}$ of the algebra $A_q=U_q(\mathfrak{n}^-_I)$.*

We shall construct a quantum deformation of the lowest degree part $J^0_{C_1}$ of the defining ideal J_{C_1} and we shall give canonical generators of a quantum deformation of J_{C_1}.

Set

$$\psi_n=Y_{i^n_5}Y_{j^n_5}-qY_{i^n_4}Y_{j^n_4}+q^2Y_{i^n_3}Y_{j^n_3}-q^3Y_{i^n_2}Y_{j^n_2}+q^4Y_{i^n_1}Y_{j^n_1},$$

for $n \in \Lambda$, where $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n)$. Using the formulas (R$n$,1), (R$n$,2), we can write $\psi_n = Y_{j_5^n} Y_{i_5^n} - q^{-1} Y_{j_4^n} Y_{i_4^n} + q^{-2} Y_{j_3^n} Y_{i_3^n} - q^{-3} Y_{j_2^n} Y_{i_2^n} + q^{-4} Y_{j_1^n} Y_{i_1^n}$.

Similarly to Lemma 2.5 and Proposition 2.6 we can show the following:

Lemma 3.5 *We have*

$$\mathrm{ad}(F_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } (k, n', n) \in \mathcal{B}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{ad}(E_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } (k, n, n') \in \mathcal{B}, \\ 0 & \text{otherwise} \end{cases}$$

for $k \in I$, and

$$\mathrm{ad}(K_k)\psi_n = q^{-(\gamma_n, \alpha_k)}\psi_n$$

for $k \in I_0$.

Proposition 3.6 $\sum_{n \in \Lambda} \mathbb{C}(q)\psi_n$ *is an irreducible highest weight $U_q(\mathfrak{l}_I)$-module with highest weight vector ψ_{27}.*

By [4] and Proposition 3.6 we obtain the following:

Theorem 3.7 *A quantum deformation of the defining ideal J_{C_1} of the closure of the non-open orbit C_1 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid n \in \Lambda\}$.*

Set

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n|-1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ $(\beta = \sum_{i \in I_0} m_i \alpha_i)$.

Proposition 3.8 $\mathbb{C}(q)\varphi$ *is a one-dimensional $U_q(\mathfrak{l}_I)$-module.*

Proof. By Proposition 3.3 we can check that the coefficient $a_{1,10,27}$ of $Y_1 Y_{10} Y_{27}$ in $\varphi = \sum_{i<j<k} a_{ijk} Y_i Y_j Y_k$ is $1 + q^8 + q^{16}$. Therefore we have $\varphi \neq 0$.

Let $(k, n, n') \in \mathcal{B}$. Then we have $|\beta_{n'}| = |\beta_n| + 1$, $\mathrm{ad}(F_k)Y_n = Y_{n'}$, $\mathrm{ad}(F_k)Y_{n'} = 0$, $\mathrm{ad}(F_k)\psi_{n'} = \psi_n$, $\mathrm{ad}(F_k)\psi_n = 0$, $(\beta_{n'}, \alpha_k) = 1$. Hence $\mathrm{ad}(F_k)(Y_n\psi_n - qY_{n'}\psi_{n'}) = Y_{n'}\psi_n - qq^{-1}Y_{n'}\psi_n = 0$. Therefore we have $\mathrm{ad}(F_k)\varphi = 0$ for any $k \in I$, and similarly we have $\mathrm{ad}(E_k)\varphi = 0$ for any $k \in I$. Since $\gamma_n + \beta_n = \delta$ for any $n \in \Lambda$, we have $\mathrm{ad}(K_k)\varphi = q^{-(\delta,\alpha_k)}\varphi$ for any $k \in I_0$. In particular, we have $\mathrm{ad}(K_k)\varphi = \varphi$ for any $k \in I$, and $\mathrm{ad}(K_1)\varphi = q^{-2}\varphi$. □

The element φ is a quantum deformation of the irreducible relative invariant on the prehomogeneous vector space.

Theorem 3.9 *A quantum deformation of the defining ideal J_{C_2} of the closure of the non-open orbit C_2 is given by the two-sided ideal of A_q generated by φ.*

References

[1] V.G. Drinfel'd: *Hopf algebra and the Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.

[2] M. Hashimoto, T. Hayashi: *Quantum multilinear algebra*, Tohoku Math. J. **44** (1992), 471–521.

[3] M. Jimbo: *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.

[4] A. Kamita, Y. Morita, T. Tanisaki: *Quantum deformations of certain prehomogeneous vector spaces I*, Hiroshima Math. J. **28** (1998), 527–540.

[5] A. Kamita: *Quantum deformations of certain prehomogeneous vector spaces. III*, preprint, 1999.

[6] G. Lusztig: *Quantum groups at roots of 1*, Geometriae Dedicata **35** (1990), 89–114.

[7] M. Noumi, H. Yamada, K. Mimachi: *Finite dimensional representations of the quantum group $GL_q(n;\mathbb{C})$ and the zonal spherical functions*, Japan. J. Math. **19** (1993), 31–80.

[8] E. Strickland: *Classical invariant theory for the quantum symplectic group*, Adv. Math. **123** (1996), 78–90.

[9] T. Tanisaki: *Highest weight modules associated to parabolic subgroups with commutative unipotent radicals*, in Algebraic Groups and their Representations, R. W. Carter and J. Saxt (eds.), Proceedings of the NATO ASI conference, Kluwer Academic Publishers, Dordrecht, 1998, 73–90.

[10] E. Taft, J. Towber: *Quantum deformation of flag schemes and Grassmann schemes. I. A q-deformation of the shape-algebra for $GL(n)$*, J. Algebra **142** (1991), 1–36.

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