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ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS IN WHICH SOCLE(G_α) IS NONSOLVABLE

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1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and $\alpha \in \Omega$. In [8], O'Nan has proved that socle(G_α) = $A \times N$, where A is an abelian group and N is 1 or a nonabelian simple group. Here socle(G_α) is the product of all minimal normal subgroups of G_α.

In the previous paper [4], we have studied doubly transitive permutation groups in which N is isomorphic to PSL(2,q), Sz(q) or PSU(3,q) with q even. In this paper we shall prove the following :

Theorem. Let G be a doubly transitive permutation group on a finite set Ω with $|\Omega|$ even and let $\alpha \in \Omega$. If G_α has a normal simple subgroup N^α isomorphic to PSL(2,q), where q is odd, then one of the following holds.

(i) G^Ω has a regular normal subgroup.

(ii) $G^\Omega \simeq A_6$ or S_6, $N^\alpha \simeq$ PSL(2,5) and $|\Omega| = 6$.

(iii) $G^\Omega \simeq M_{11}$, $N^\alpha \simeq$ PSL(2,11) and $|\Omega| = 12$.

In the case that G^Ω has a regular normal subgroup, by a result of Hering [3] we have $(|\Omega|, q) = (16,9)$, $(16,5)$ or $(8,7)$.

We introduce some notations :

F(X) : the set of fixed points of a nonempty subset X of G

X(Δ) : the global stabilizer of a subset Δ ($\subseteq \Omega$) in X

X_Δ : the pointwise stabilizer of Δ in X

X^Δ : the restriction of X on Δ

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$m|n$: an integer m divides an integer n

X^H : the set of H-conjugates of X

$|X|_p$: maximal power of p dividing the order of X

$I(X)$: the set of involutions in X

D_m : dihedral group of order m

In this paper all sets and groups are finite.

2. Preliminaries

Lemma 2.1. Let G be a transitive permutation group on Ω, $\alpha \in \Omega$ and N^α a normal subgroup of G_α such that $F(N^\alpha) = \{\alpha\}$. Let the subgroup $X \leq N^\alpha$ be conjugate in G_α to every group Y which lies in N^α and which is conjugate to X in G. Then $N_G(X)$ is transitive on $\Delta = \{\gamma \in \Omega \mid X \leq N^\gamma\}$.

Proof. Let $\beta \in \Delta$ and let $g \in G$ such that $\beta^g = \alpha$. Then, as $X \leq N^\beta$, $X^g \leq N^{\beta^g} = N^\alpha$. By assumption, $(X^g)^h = X$ for some $h \in G_\alpha$. Hence $gh \in N_G(X)$ and $\alpha^{(gh)^{-1}} = \alpha^{g^{-1}} = \beta$. Obviously $N_G(X)$ stabilizes Δ. Thus Lemma 2.1 holds.

Lemma 2.2. Let G be a doubly transitive permutation group on Ω of even degree and N^α a nonabelian simple normal subgroup of G_α with $\alpha \in \Omega$. If $C_G(N^\alpha) \neq 1$, then $N^\alpha_\beta = N^\alpha \cap N^\beta$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semi-regular on $\Omega - \{\alpha\}$.

Proof. See Lemma 2.1 of [4].

Lemma 2.3. Let G be a transitive permutation group on Ω, H a stabilizer of a point of Ω and M a nonempty subset of G. Then

$$|F(M)| = |N_G(M)| \times |M^G \cap H| / |H|.$$

Here $M^G \cap H = \{g^{-1}Mg \mid g^{-1}Mg \subseteq H,\ g \in G\}$.

Proof. See Lemma 2.2 of [4].

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Lemma 2.4. Let G be a doubly transitive permutation group on Ω and N^α a normal subgroup of G_α with $\alpha \in \Omega$. Assume that a subgroup X of N^α satisfies $X^{G_\alpha} = X^{N^\alpha}$. Then the following hold.

(i) $|F(X) \cap \beta^{N^\alpha}| = |F(X) \cap \gamma^{N^\alpha}|$ for $\beta, \gamma \in \Omega - \{\alpha\}$.

(ii) $|F(X)| = 1 + |F(X) \cap \beta^N| \times r$, where r is the number of N^α-orbits on $\Omega - \{\alpha\}$.

Proof. Let $\Gamma = \{\Delta_1, \Delta_2, \cdots \Delta_r\}$ be the set of N^α-orbits on $\Omega - \{\alpha\}$. Since G_α is transitive on $\Omega - \{\alpha\}$ and $G_\alpha \trianglerighteq N^\alpha$, we have $|\Delta_i| = |\Delta_j|$ for $1 \leq i, j \leq r$. By assumption, $G_\alpha = N_{G_\alpha}(X)N^\alpha$ and so $N_{G_\alpha}(X)$ is transitive on Γ. Hence for each i with $1 \leq i \leq r$ there exists $g \in N_{G_\alpha}(X)$ such that $(\Delta_1)^g = \Delta_i$. Therefore $|F(X) \cap \Delta_i| = |F(X^g) \cap (\Delta_1)^g| = |F(X) \cap \Delta_1|$. Thus (i) holds and (ii) follows immediately from (i).

Lemma 2.5. (Huppert [5]) Let G be a doubly transitive permutation group on Ω. Suppose that $O_2(G) \neq 1$ and G_α is solvable. Then for any involution z in G_α, $|F(z)|^2 = |\Omega|$.

We list now some properties of PSL(2,q) with q odd which will be required in the proof of our theorem.

Lemma 2.6. ([2], [6], [10]) Set N = PSL(2,q) and G = Aut(N), where $q = p^n$ and p is an odd prime. Let z be an involution in N. Then the following hold.

(i) $|N| = (q-1)q(q+1)/2$, $I(N) = z^N$ and $C_N(z) \simeq D_{q-\varepsilon}$, where $q \equiv \varepsilon \in \{\pm 1\}$ (mod 4).

(ii) If $q \neq 3$, N is a nonabelian simple group and a Sylow r-subgroup of N is cyclic when $r \neq 2, p$.

(iii) If X and Y are cyclic subgroups of N and $|X| = |Y| \neq 2, p$, then X is conjugate to Y in $\langle X, Y \rangle$ and $N_N(X) \simeq D_{q\pm\varepsilon}$.

(iv) If $X \leq N$ and $X \simeq Z_2 \times Z_2$, $N_N(X)$ is isomorphic to A_4 or S_4.

(v) If $|N|_2 \geq 8$, N has two conjugate classes of four-groups in N.

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(vi) There exist a field automorphism f of N of order n and a diagonal automorphism d of N of order 2 and if we identify N with its inner automorphism group, $\langle d\rangle N \simeq PGL(2,q)$, $\langle f\rangle\langle d\rangle N = G$ and $G/N \simeq Z_2 \times Z_n$.

(vii) $C_N(d) \simeq D_{q+\varepsilon}$ and $C_{\langle d\rangle N}(z) \simeq D_{2(q-\varepsilon)}$.

(viii) Suppose n = mk for some positive integers m, k. Then $C_N(f^m) \simeq PSL(2,p^m)$ if k is odd and $C_N(f^m) \simeq PGL(2,p^m)$ if k is even.

(ix) Assume n is even and let u be a field automorphism of order 2. Then $I(G) = I(N) \cup d^N \cup u^{\langle d\rangle N}$. If n is odd, $I(G) = I(N) \cup d^N$.

(x) If H is a subgroup of N of odd index, then one of the following holds :

(1) H is a subgroup of $C_N(z)$ of odd index for some involution $z \in N$.

(2) $H \simeq PGL(2,p^m)$, where n = 2mk and k is odd.

(3) $H \simeq PSL(2,p^m)$, where n = mk and k is odd.

(4) $H \simeq A_4$ and $q \equiv 3, 5 \pmod 8$.

(5) $H \simeq S_4$ and $q \equiv 7, 9 \pmod{16}$.

(6) $H \simeq A_5$, $q \equiv 3, 5 \pmod 8$ and $5 \mid (q-1)q(q+1)$.

Lemma 2.7. Let G, N, d and f be as defined in Lemma 2.6 and H an $\langle f,d\rangle$-invariant subgroup of N isomorphic to $D_{q-\varepsilon}$. Let W be a cyclic subgroup of $\langle d\rangle H$ of index 2 (cf. (vii) of Lemma 2.6) and set $Y = O_2(W \cap H)$. Then $C_G(Y) = W\cdot C_{\langle f\rangle}(Y)$.

Proof. By (viii) of Lemma 2.6, we can take an involution t satisfying $\langle d\rangle H = \langle t\rangle W$ and $[f,t] = 1$. Since $N_G(Y) = \langle f, d\rangle N_N(Y) = \langle f, d\rangle H$, $C_G(Y) = C_{\langle f\rangle\langle d\rangle H}(Y) = W\cdot C_{\langle f\rangle \times \langle t\rangle}(Y)$. Suppose $ht \in C(Y)$ for some $h \in \langle f\rangle$. Since t inverts Y, h also inverts Y and so h^2 centralizes Y. Hence some nontrivial 2-element $g \in \langle h\rangle$ inverts Y, so that $C_H(g)$ contains no

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element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, G^{Ω} will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume G^{Ω} has no regular normal subgroup.

Notation. $C^{\alpha} = C_G(N^{\alpha})$, which is semi-regular on $\Omega - \{\alpha\}$ by Lemma 2.2. Let r be the number of N^{α}-orbits on $\Omega - \{\alpha\}$.

Since $G_{\alpha} \trianglerighteq N^{\alpha}$, $|\beta^{N^{\alpha}}| = |\gamma^{N^{\alpha}}|$ for $\beta, \gamma \in \Omega - \{\alpha\}$ and so $|\Omega| = 1 + r \times |\beta^{N^{\alpha}}|$. Hence r is odd and N^{α}_{β} is a subgroup of N^{α} of odd index. Therefore N^{α}_{β} is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

Lemma 2.8. Let Z be a cyclic subgroup of N^{α}_{β} with $|Z| \neq 1, p$. Then

(i) If $|Z| = 2$, $|F(Z)| = 1 + (q-\varepsilon)|I(N^{\alpha}_{\beta})| r/|N^{\alpha}_{\beta}|$.

(ii) If $|Z| \neq 2$, $|F(Z)| = 1 + |N_{N^{\alpha}}(Z)| r/|N_{N^{\alpha}_{\beta}}(Z)|$.

Proof. It follows from Lemmas 2.3,2.4 and 2.6(i), (iii).

Lemma 2.9. If $N^{\alpha}_{\beta} \not\simeq D_{q-\varepsilon}$ and Z is a cyclic subgroup of N^{α}_{β} with $|Z| \neq 1, p$ and $N_G(Z)^{F(Z)}$ is doubly transitive. Then $C^{\alpha} = 1$ and one of the following holds.

(i) $N_G(Z)^{F(Z)} \leq A\Gamma L(1,q_1)$ for some q_1.

(ii) $C_G(Z)^{F(Z)} \trianglerighteq PSL(2,p_1)$, $r = 1$ and $|F(Z)| - 1 = |N_{N^{\alpha}}(Z) : N_{N^{\alpha}_{\beta}}(Z)| = p_1$, where p_1 (≥ 5) is a prime.

(iii) $N_G(Z)^{F(Z)} = R(3)$, the smallest Ree group, $|F(Z)| = 28$.

Proof. Set $N_G(Z) = L$ and $F(Z) = \Delta$. By Lemma 2.6(iii), $L \cap N^{\alpha} \simeq D_{q\pm\varepsilon}$ and $L \cap N^{\alpha} = \langle t \rangle Y \trianglerighteq Y \geq Z$, where $o(t) = 2$, $Y \simeq Z_{(q\pm\varepsilon)/2}$.

If $(L \cap N^{\alpha})^{\Delta} = 1$, then $L \cap N^{\alpha} = N^{\alpha}_{\beta}$ because $L \cap N^{\alpha}$ is a maximal subgroup of N^{α}. Since $|N^{\alpha} : N^{\alpha}_{\beta}|$ is odd, $L \cap N^{\alpha} = N^{\alpha}_{\beta} \simeq D_{q-\varepsilon}$, contrary to the assumption. Hence $(L \cap N^{\alpha})^{\Delta} \neq 1$ and as $L_{\alpha} \trianglerighteq L_{\alpha} \cap N^{\alpha}$ and $L_{\alpha} \trianglerighteq Y$,

$(L_\alpha)^\Delta$ has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs :

(a) L^Δ has a regular normal subgroup

(b) $L^\Delta \trianglerighteq PSL(2,p_1)$, $|\Delta| = p_1 + 1$, where $p_1 (\geq 5)$ is a prime

(c) $L^\Delta \trianglerighteq PSU(3,p_1)$, $p_1 \geq 3$, $|\Delta| = (p_1)^3 + 1$

(d) $L^\Delta = R(3)$, $|\Delta| = 28$.

Suppose $C^\alpha \neq 1$. Then there exists a subgroup D of C^α of prime order such that $(L_\alpha)^\Delta \triangleright D^\Delta$. Since $[L_\alpha, D] \leq D\cdot L_\Delta \cap C^\alpha = D(L_\Delta \cap C^\alpha) = D$, D is a normal subgroup of L_α. By (i) and (iii) of Lemma 2.6, $G_\alpha = L_\alpha \cdot N^\alpha$ and so D is a normal subgroup of G_α. By Theorem 3 of [1], G^Ω has a regular normal subgroup, contrary to the hypothesis. Thus $C^\alpha = 1$.

If (a) occurs, L^Δ is solvable because $L_\alpha/L\cap N^\alpha \simeq L_\alpha N^\alpha/N^\alpha \leq Out(N^\alpha)$ and $L\cap N^\alpha \simeq D_{q\pm\varepsilon}$. Hence by [5], (i) holds in this case.

If (b) occurs, we have $Y^\Delta \neq 1$, for otherwise $(L\cap N^\alpha)^\Delta = 1$ and so $N^\alpha_\beta = L\cap N^\alpha \simeq D_{q-\varepsilon}$, a contradiction. Hence $1 \neq C_G(Z)^\Delta \trianglelefteq L^\Delta$ and so $C_G(Z)^\Delta \trianglerighteq PSL(2,p_1)$ and $Y^\Delta \simeq Z_{p_1}$. Therefore $|\Delta \cap \beta^{N^\alpha}| = p_1$ and $r = 1$ by Lemma 2.4(ii). Since $|\beta^Y| = p_1$, we have $|\beta^{L\cap N^\alpha}| = p_1$, so that $|L\cap N^\alpha : L\cap N^\alpha_\beta| = p_1$. Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of $PSU(3,p_1)$, a Sylow p_1-subgroup of $(L^\Delta_\alpha)'$ is not cyclic, while $(L_\alpha)' \leq L\cap N^\alpha \simeq D_{q\pm\varepsilon}$, a contradiction.

3. Case (I)

In this section we assume that $N^\alpha_\beta \leq D_{q-\varepsilon}$, where $\beta \neq \alpha$, $q = p^n$.

(3.1) (i) If $N^\alpha_\beta \not\simeq Z_2 \times Z_2$, $N_{N^\alpha}(N^\alpha_\beta) = N^\alpha_\beta$ and $|F(N^\alpha_\beta)| = r + 1$.

(ii) If $N^\alpha_\beta \simeq Z_2 \times Z_2$, $N_{N^\alpha}(N^\alpha_\beta) \simeq A_4$ and $|F(N^\alpha_\beta)| = 3r + 1$.

Proof. Put $X = N_{N^\alpha}(N^\alpha_\beta)$. Let S be a Sylow 2-subgroup of N^α_β and Y a cyclic subgroup of N^α_β of index 2.

If $N^\alpha_\beta \not\simeq Z_2 \times Z_2$, then $|Y| > 2$ and so Y is characteristic in N^α_β. Hence $X \leq N_{N^\alpha}(Y) \simeq D_{q-\varepsilon}$. From this $[N_X(S), S \cap Y] \leq S \cap Y$ and $O^2(N_X(S))$ stabilizes a normal series $S \trianglerighteq S \cap Y \trianglerighteq 1$, so that $O^2(N_X(S)) \leq C_{N^\alpha}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^\alpha}(S) \leq S$ and hence $N_X(S) = S$. On the other hand by a Frattini argument, $X = N_X(S)N^\alpha_\beta$ and so $X = N^\alpha_\beta$. By Lemma 2.6(i), $(N^\alpha_\beta)^{G_\alpha} = (N^\alpha_\beta)^{N^\alpha}$ and so, by Lemmas 2.3 and 2.4(ii), $|F(N^\alpha_\beta)| = 1 + |F(N^\alpha_\beta) \cap \beta^{N^\alpha}| \times r = 1 + |N^\alpha_\beta| r / |N^\alpha_\beta| = r + 1$. Thus (i) holds.

If $N^\alpha_\beta \simeq Z_2 \times Z_2$, $N_{N^\alpha}(N^\alpha_\beta) \simeq A_4$ by Lemma 2.6(iv). Similarly as in the case $N^\alpha_\beta \not\simeq Z_2 \times Z_2$, we have $|F(N^\alpha_\beta)| = 3r + 1$.

(3.2) $N^\alpha_\beta / N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^\alpha = 1$. Suppose $C^\alpha = 1$. Then $N^\alpha_\beta / N^\alpha \cap N^\beta \simeq N^\alpha_\beta N^\beta / N^\beta \leq \mathrm{Out}(N^\alpha) \simeq Z_2 \times Z_n$ by Lemma 2.6(vi) and hence $(N^\alpha_\beta)' \leq N^\alpha \cap N^\beta$. Since N^α_β is dihedral, $N^\alpha_\beta/(N^\alpha_\beta)' \simeq Z_2 \times Z_2$, so that $N^\alpha_\beta / N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

(3.3) Suppose $N^\alpha_\beta = N^\alpha \cap N^\beta$ and let U be a subgroup of N^α_β isomorphic to $Z_2 \times Z_2$. Then $|F(U)| = 3r + 1$ and $N_G(U)^{F(U)}$ is doubly transitive.

Proof. Set $X = N_G(N^\alpha_\beta)$, $\Delta = F(N^\alpha_\beta)$ and let $\{\Delta_1, \Delta_2, \cdots, \Delta_r\}$ be the set of N^α-orbits on $\Omega - \{\alpha\}$. If $g^{-1} N^\alpha_\beta g \leq G_{\alpha\beta}$, then $g^{-1} N^\alpha_\beta g \leq N^\gamma_\alpha \cap N^\gamma_\beta = N^\alpha_\gamma \cap N^\beta_\gamma \leq N^\alpha_\beta$, where $\gamma = \alpha^g$. By a Witt's theorem, X^Δ is doubly transitive.

If U is a Sylow 2-subgroup of N^α_β, by a Witt's theorem, $N_G(U)^{F(U)}$ is doubly transitive. Moreover $N_{N^\alpha}(U) \simeq A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)| = 1 + |A_4| \times |N^\alpha_\beta : N_{N^\alpha_\beta}(U)| \times r / |N^\alpha_\beta| = 3r + 1$.

If $|N^\alpha_\beta|_2 > 4$, by Lemma 2.6(iv) and (v), $N_{N^\alpha}(U) \simeq S_4$ and N^α_β has two conjugate classes of four-groups, say $\pi = \{K_1, K_2\}$. Set $X_\pi = M$. Then

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$M \trianglerighteq N_\beta^\alpha$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_i \neq \phi$ for each i and so $|F(U) \cap \Delta_i| = 3$ by Lemma 2.3. Hence $|F(U)| = 3r + 1$. Since $N_{N^\beta}(U) \simeq S_4$, we may assume $r > 1$. Hence by (3.1)(i) $|\Delta| = r + 1 \geq 4$, so that M^Δ is doubly transitive. Since $M = N_\beta^\alpha N_M(U)$, $N_M(U)^\Delta$ is also doubly transitive and so $N_{M_\alpha}(U)$ is transitive on $\Delta - \{\alpha\}$. As $|\Delta \cap \Delta_i| = 1$, $\Delta \cap \Delta_i \subseteq F(U)$ and $N_{N^\alpha}(U)$ is transitive on $F(U) \cap \Delta_i$ for each i, $N_G(U)^{F(U)}$ is doubly transitive.

(3.4) (i) $C^\alpha = 1$.

(ii) Let U be a subgroup of N_β^α isomorphic to $Z_2 \times Z_2$. If $N_\beta^\alpha = N^\alpha \cap N^\beta$, then $N_G(U)^{F(U)}$ has a regular normal 2-subgroup. In particular $|F(U)| = 3r + 1 = 2^b$ for some positive integer b.

Proof. Since $N_{G_\alpha}(U)^{F(U)} \trianglerighteq N_{N^\alpha}(U)^{F(U)} \simeq S_3$ or Z_3, by (3.3) and Theorem 3 of [1], $N_G(U)^{F(U)}$ has a regular normal subgroup, $N_G(U)^{F(U)} \trianglerighteq PSU(3,3)$ or $N_G(U)^{F(U)} = R(3)$.

Suppose $C^\alpha \neq 1$. Let D be a minimal characteristic subgroup of C^α. Clearly $G_\alpha \triangleright D$. If $N_G(U)^{F(U)} \neq R(3)$, D is cyclic. By Theorem 3 of [1], G^Ω has a regular normal subgroup, contrary to the hypothesis. Hence $N_G(U)^{F(U)} = R(3)$. Therefore $(N_{G_\alpha}(U)^{F(U)})'$ contains an element of order 9. Since $N_{G_\alpha}(U)/C^\alpha N_{N^\alpha}(U) \simeq N_{G_\alpha}(U)C^\alpha N^\alpha/C^\alpha N^\alpha \leq Out(N^\alpha)$, by (vi) of Lemma 2.6 we have $(N_{G_\alpha}(U))' \leq C^\alpha \times N_{N^\alpha}(U)$. From this, C^α contains an element of order 9 and so $C^\alpha \simeq Z_9$ or $M_3(3)$. In both cases, C^α contains a caracteristic subgroup of order 3. Since $G_\alpha \triangleright D$, by Theorem 3 of [1] G^Ω has a regular normal subgroup, a contradiction. Thus $C^\alpha = 1$.

Let R be a Sylow 3-subgroup of $N_{G_\alpha}(U)$. Since $N_{G_\alpha}(U)/N_{N^\alpha}(U) \simeq N_{G_\alpha}(U)N^\alpha/N^\alpha \leq Out(N^\alpha) \simeq Z_2 \times Z_n$, $R/R \cap N_{N^\alpha}(U)$ is cyclic. Clearly $R \cap N_{N^\alpha}(U) \simeq Z_3$. Therefore $N_G(U)^{F(U)} \not\trianglerighteq PSU(3,3)$, $R(3)$. Thus (3.4) holds.

Since N_β^α is dihedral, we set $N_\beta^\alpha = \langle t \rangle W$ and $Y = W \cap N^\alpha \cap N^\beta$, where W

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is a cyclic subgroup of N_β^α of index 2 and t is an involution in N_β^α which inverts W.

(3.5) (i) If $|Y| \geq 3$, $N_G(Y)^{F(Y)}$ is doubly transitive.

(ii) If $|Y| < 3$, $N_\beta^\alpha \simeq Z_2 \times Z_2$ or $N_\beta^\alpha \simeq D_8$ and $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

Proof. Suppose $|Y| \geq 3$. If $Y^g \leq G_{\alpha\beta}$, $Y^g \leq N^\gamma \cap G_{\alpha\beta} \leq N_\alpha^\gamma$, where $\gamma = \alpha^g$. If $\gamma = \alpha$, obviously $Y^g \leq N^\alpha$. If $\gamma \neq \alpha$, $N_\alpha^\gamma \simeq N_\beta^\alpha$. Therefore, as $|Y| \geq 3$, N_α^γ has a unique cyclic subgroup of order $|Y|$. Hence $Y^g \leq N^\gamma \cap N^\alpha \leq N^\alpha$, so that $Y^g \leq N^\alpha$. Similarly $Y^g \leq N^\beta$. Thus $Y^g \leq N^\alpha \cap N^\beta$ and so $Y^g = Y$. By a Witt's theorem, $N_G(Y)$ is doubly transitive on $F(Y)$.

Suppose $|Y| < 3$. Since $|N^\alpha \cap N^\beta : Y| \leq 2$, we have $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$. On the other hand, as N_β^α is dihedral, $(N_\beta^\alpha)'$ is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set $\Delta = F(N_\beta^\alpha)$, $L = G(\Delta)$, $K = G_\Delta$ and suppose $N_\beta^\alpha \not\simeq Z_2 \times Z_2$. Then $L_\alpha \trianglerighteq N_\beta^\alpha$, $(L_\alpha)' \leq N_\beta^\alpha$, $K' \leq N^\alpha \cap N^\beta$ and $(L_\alpha)^\Delta \simeq Z_r$. If $r \neq 1$, L^Δ is a doubly transitive Frobenius group of degree $r + 1$.

Proof. By Corollary B1 of [7] and (i) of (3.1), L^Δ is doubly transitive and $|\Delta| = r + 1$. Since $N^\alpha \cap L \trianglerighteq N^\alpha \cap K = N_\beta^\alpha$, by (i) of (3.1), we have $N^\alpha \cap L = N_\beta^\alpha$. Hence $L_\alpha \trianglerighteq N_\beta^\alpha$. By (i) of (3.4), $L_\alpha / N_\beta^\alpha \simeq L_\alpha N^\alpha / N^\alpha \leq \mathrm{Out}(N^\alpha) \simeq Z_2 \times Z_n$ and so $(L_\alpha)' \leq N_\beta^\alpha$ and $(L_\alpha)^\Delta \simeq Z_r$. If $r \neq 1$, then $(L_\alpha)^\Delta \neq 1$. On the other hand $(L_{\alpha\beta})^\Delta = 1$ as $(L_\alpha)^\Delta$ is abelian. Hence L^Δ is a Frobenius group.

(3.7) Suppose $|Y| \geq 3$. Then there exists an involution z in $N_\beta^\alpha \cap Y$ such that $Z(N_\beta^\alpha) = \langle z \rangle$.

Proof. Since $N_\beta^\alpha \not\simeq Z_2 \times Z_2$, $|N_\beta^\alpha|_2 \geq 2^2$ and N_β^α is dihedral, we have $\langle I(W) \rangle = Z(N_\beta^\alpha) \simeq Z_2$ and $N_\beta^\alpha / (N_\beta^\alpha)' \simeq Z_2 \times Z_2$. Let $Z(N_\beta^\alpha) = \langle z \rangle$ and suppose that z is not contained in Y. By (3.2), $(N_\beta^\alpha)' \leq N^\alpha \cap N^\beta \cap W = Y$ and so $|(N_\beta^\alpha)'|$

is odd. Hence $|N_\beta^\alpha|_2 = 4$ and $q \equiv p^n = 3$ or $5 \pmod 8$, so that n is odd. By (3.2) and (i) of (3.4), $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta/N^\beta \simeq 1$ or Z_2. If $N_\beta^\alpha = N^\alpha \cap N^\beta$, then $W = Y$ and so $z \in Y$, contrary to the assumption. Therefore we have $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq Z_2$ and $N_\beta^\alpha = \langle z\rangle \times (N^\alpha \cap N^\beta)$. Since n is odd and $z \in N_\beta^\alpha N^\beta - N^\beta$, by Lemma 2.6(vi), (vii) and (ix), $N_\beta^\alpha N^\beta \simeq PGL(2,q)$ and $C_{N^\beta}(z) \simeq D_{q+\varepsilon}$. But $N^\alpha \cap N^\beta \leq C_{N^\beta}(z)$ and besides it is isomorphic to a subgroup of $D_{q-\varepsilon}$. Hence $N^\alpha \cap N^\beta \simeq Z_2$ and $N_\beta^\alpha \simeq Z_2 \times Z_2$, a contradiction.

(3.8) Suppose $|Y| \geq 3$. Then $N_\beta^\alpha = N^\alpha \cap N^\beta$.

Proof. Suppose $N_\beta^\alpha \neq N^\alpha \cap N^\beta$ and let Δ, L, K be as defined in (3.6) and $x \in L_\alpha$ such that its order is odd and $\langle x\rangle$ is transitive on $\Delta - \{\alpha\}$. As $|Y| \geq 3$, W is characteristic in N_β^α and hence by (3.6), x stabilizes a normal series $L_\alpha \trianglerighteq N_\beta^\alpha \trianglerighteq W \trianglerighteq (N_\beta^\alpha)'$. By Theorem 5.3.2 of [2], $[x, O_2(L_\alpha/(N_\beta^\alpha)')] = 1$. Since $L_\alpha/(N_\beta^\alpha)'$ has a normal Sylow 2-subgroup and $(N_\beta^\alpha)' \leq K'$, we have $[x, O_2(L_\alpha/K')] = 1$, so that $[x, N_\alpha^\beta] \leq K' \leq N^\alpha \cap N^\beta$ by (3.6). If $r \neq 1$, then $\beta^x \neq \beta$ and $\beta^x \in \Delta$, hence $N_\alpha^\beta = x^{-1}N_\alpha^\beta x = N_\alpha^\gamma$, where $\gamma = \beta^x$. Since $\gamma \in \Delta$ and $\Delta = F(N_\alpha^\beta)$, $N_\alpha^\beta \leq N^\beta \cap G_\gamma = N_\gamma^\beta$ and so $N_\alpha^\beta = N_\gamma^\beta$. Similarly $N_\alpha^\gamma = N_\beta^\gamma$. Hence $N_\gamma^\beta = N_\beta^\gamma$, which implies $N_\gamma^\beta = N^\beta \cap N^\gamma$. By the doubly transitivity of G, we have $N_\beta^\alpha = N^\alpha \cap N^\beta$, contrary to the assumption. Therefore we obtain $r = 1$.

Let z be as defined in (3.7) and put $k = (q-\varepsilon)/|N_\beta^\alpha|$. By Lemma 2.8(i) we have $|F(z)| = 1 + (q-\varepsilon)(|N_\beta^\alpha|/2 + 1)/|N_\beta^\alpha| = (q-\varepsilon)/2 + k + 1$. Similarly $|F(Y)| = k + 1$. As $N_\beta^\alpha \neq N^\alpha \cap N^\beta$, there is an involution t in N_β^α which is not contained in N^β. By Lemma 2.6(i), $t^y = z$ for some $y \in N^\alpha$. Set $\gamma = \beta^y$. Then $\gamma \in F(z)$ and $z \notin N^\gamma$. By Lemma 2.6(vii), (viii) and (ix), $C_{N^\gamma}(z) \simeq D_{q+\varepsilon}$ or $PGL(2,\sqrt{q})$. Assume $C_{N^\gamma}(z) \simeq D_{q+\varepsilon}$ and let R be a cyclic subgroup of $C_{N^\gamma}(z)$ of index 2. We note that R is semi-regular on $\Omega - \{\alpha\}$. Set $X = C_G(z)$. Since $2 \leq k + 1 \leq (q-\varepsilon)/|q-\varepsilon|_2 + 1$, we have $(q+\varepsilon)/2 \nmid k + 1$ and so $|\alpha^X| > k + 1$. By (i) of (3.5) and (3.7), $N_G(Y) \leq C_G(z) = X$ and

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$\alpha^X \geq F(Y)$. It follows from Lemma 2.1 that $\alpha^X = \{\mu \mid z \in N^\mu\} \not\ni \gamma$. Hence $|F(z)| > |\alpha^X| \geq |F(Y)| + (q+\varepsilon)/2 = k + 1 + (q-\varepsilon)/2 + \varepsilon = |F(z)| + \varepsilon$. Therefore $\varepsilon = -1$ and $\gamma^X = \{\gamma\}$, so that $\gamma \in F(Y)$, a contradiction. Thus $C_{N^\gamma}(z) \simeq PGL(2,\sqrt{q})$, $\varepsilon = 1$, $N^\alpha_\beta/N^\alpha \cap N^\beta \simeq Z_2$ and $|\langle z^G \cap G_\alpha\rangle : N^\alpha| = 2$.

Set $\Delta_1 = \alpha^X$ and $\Delta_2 = F(z) - \Delta_1$. Let $\delta \in \Delta_2$ and g an element of G satisfying $\delta^g = \gamma$. Then $z \in N^\alpha_\delta N^\delta - N^\delta$ and so $z^g \in N^\nu_\gamma N^\gamma - N^\gamma$, where $\nu = \alpha^g$. Since $|\langle z^G \cap G_\gamma\rangle : N^\gamma| = 2$ and $z \in G_\gamma - N^\gamma$, it follows from Lemma 2.6(ix) that $(z^g)^h = z$ for some $h \in G_\gamma$. Hence $gh \in X$ and $\delta^{gh} = \gamma$. Thus $\Delta_2 = \gamma^X$. Let $\delta \in \Delta_2$. Then $z \in N^\alpha_\delta$ and $z \notin Z(N^\alpha_\delta)$ by (3.7) and so $X \cap N^\alpha_\delta \simeq Z_2 \times Z_2$, which implies $|\delta^{(C_{N^\alpha}(z))}| = (q-1)/4$. Hence $(|\Delta_1|, |\Delta_2|) = ((q-1)/4 + k + 1, (q-1)/4\ )$ or $(k + 1, (q-1)/2\ )$. Let P be a subgroup of $C_{N^\gamma}(z)$ of order $\sqrt{q}$. Then $F(P) = \{\gamma\}$ and P is semi-regular on $\Omega - \{\gamma\}$. If $|\Delta_2| = (q-1)/4$, then $\sqrt{q} \mid (q-1)/4 - 1 = (q-5)/4$ and $\sqrt{q} \mid (q-1)/4 + k + 1$. From this, $q = 5^2$, $k = 3$, $|\Delta_1| = 10$ and $|\Delta_2| = 6$. Since $(C_{N^\gamma}(z))^{\Delta_2} \simeq S_5$, $X^{\Delta_2} \simeq S_6$ and so $|X|_3 \geq 3^2$. As X acts on Δ_1 and $|\Delta_1| \equiv 1 \pmod 3$, $|G_\alpha|_3 \geq |X_\alpha|_3 \geq 3^2$, contrary to $N^\alpha \simeq PSL(2,25)$. If $|\Delta_2| = (q-1)/2$, $\sqrt{q} \mid (q-1)/2 - 1 = (q-3)/2$, so $q = 3^2$, $k = 1$, $N^\alpha_\beta \simeq D_8$ and $\Delta_1 = \{\alpha, \beta\}$. Hence $C_{N^\gamma}(z)$ fixes α and β, so that $PGL(2,3) \simeq C_{N^\gamma}(z) \leq N^\gamma_\alpha \simeq N^\alpha_\beta \simeq D_8$, a contradiction.

(3.9) Suppose $|Y| \geq 3$. Then $r = 1$.

Proof. By (3.6), $r + 1 = 2^c$ for some integer $c \geq 0$. On the other hand $3r + 1 = 2^b$ by (3.8) and (ii) of (3.4). Hence $2r = 2^c(2^{b-c} - 1)$ and so $c = 1$ as r is odd. Thus $r = 1$.

(3.10) Put $k = (q-\varepsilon)/|N^\alpha_\beta|$. If $N^\alpha_\beta = N^\alpha \cap N^\beta$ and $r = 1$, then

$$q-\varepsilon + 2k + 2 \;\Big|\; 2((2k + 2 - \varepsilon)(k + 1 - \varepsilon)k + 1)(2k + 2 - \varepsilon)(k + 1 - \varepsilon).$$

Proof. Set $S = \{(\gamma, u) \mid \gamma \in F(u),\ u \in z^G\}$, where z is an involution in N^α_β . We now count the number of elements of S in two ways. Since

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$N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$, $F(z) = \{\gamma \mid z \in N^{\gamma}\}$ and hence $C_G(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S| = |\Omega||z^{G\alpha}| = |z^G||F(z)|$. Since $r = 1$, $|\Omega| = 1 + |N^{\alpha} : N^{\alpha}_{\beta}| = kq(q+\varepsilon)/2 + 1$ and by Lemma 2.8 $|F(z)| = (q-\varepsilon)/2 + k + 1$. Since $G\alpha \trianglerighteq N^{\alpha}$, $z^{G\alpha}$ is contained in N^{α} and so $|G_{\alpha} : C_{G_{\alpha}}(z)| = |N^{\alpha} : C_{N^{\alpha}}(z)| = q(q+\varepsilon)/2$. Hence $(q-\varepsilon)/2 + k + 1 \,\Big|\, (kq(q+\varepsilon)/2 + 1)q(q+\varepsilon)/2$. On the other hand $|F(z)|_2 = |C_G(z)|_2 / |C_{G_{\alpha}}(z)|_2 \leq |G|_2 / |C_{G_{\alpha}}(z)|_2 = |G|_2 / |G_{\alpha}|_2 = |\Omega|_2$ because $|G_{\alpha} : C_{G_{\alpha}}(z)| = q(q+\varepsilon)/2 \equiv 1 \pmod 2$. Hence $|q-\varepsilon + 2k + 2|_2 \leq |kq(q+\varepsilon) + 2|_2$. Since $kq(q+\varepsilon) + 2 =$
$(kq + 2k(\varepsilon - k - 1))(q-\varepsilon + 2k + 2) + 2((2k + 2 - \varepsilon)(k + 1 - \varepsilon)k + 1)$ and
$q(q+\varepsilon) = (q+2\varepsilon - 2k - 2)(q-\varepsilon + 2k + 2) + 2(2k + 2 - \varepsilon)(k + 1 - \varepsilon)$,
we have (3.10).

(3.11) Suppose $|Y| \geq 3$. Then one of the following holds.

(i) $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta} \simeq D_{q-\varepsilon}$.

(ii) $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta} \not\simeq D_{q-\varepsilon}$ and $N_G(Y)^{F(Y)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(Y)} = R(3)$ or there exists a prime $p_1 \geq 5$ such that $C_G(Y)^{F(Y)} \trianglerighteq PSL(2,p_1)$ and $V/Y \simeq Z_{p_1}$, where $V = C_{N^{\alpha}}(Y)$. By (i) of (3.1) and (3.9), $F(N^{\alpha}_{\beta}) = \{\alpha, \beta\}$. On the other hand, $(N^{\alpha}_{\beta})^{F(Y)} \simeq N^{\alpha}_{\beta}/Y \simeq Z_2$. Hence $N_G(Y)^{F(Y)} \neq R(3)$ and $C_G(Y)^{F(Y)} \trianglerighteq PSL(2,p_1)$.

By (i) of (3.4) and Lemma 2.7, we have $C_{G_{\alpha}}(Y) = V\langle f_1 \rangle$, where f_1 is a field automorphism of N^{α}. Let t be the order of f_1, $n = tm$ and let $p^m \equiv \varepsilon_1 \in \{\pm 1\} \pmod 4$. Clearly $C_{G_{\alpha}}(Y)^{F(Y)} \trianglerighteq V^{F(Y)} \simeq Z_{p_1}$ and $|C_{G_{\alpha\beta}}(Y)^{F(Y)}| \,\Big|\, t$, so that $(p_1-1)/2 \,\Big|\, t$.

First we assume that t is even and set $t = 2t_1$. Then $Y \leq C_{N^{\alpha}}(f_1) \simeq PGL(2,p^m)$ by Lemma 2.6(viii). As $|V/Y| = p_1$ and p_1 is a prime, Y is a cyclic subgroup of $C_{N^{\alpha}}(f_1)$ of order $p^m - \varepsilon_1$ and $(p^n-1)/2(p^m - \varepsilon_1) = p_1$.

Put $s = \sum_{i=0}^{t_1-1} (p^{2m})^i$. Then $(p^m + \varepsilon_1)s/2 = p_1$, so that we have either (i) $t_1 = 1$ and $p_1 = (p^m+\varepsilon_1)/2$ or (ii) $t_1 \geq 2$, $p^m = 3$ and $p_1 = s$. In the case (i), $2 \leq (p_1-1)/2 = (p^m+\varepsilon_1-2)/4 \mid 2t_1 = 2$. Hence $(p_1, q) = (5,3^4)$ or $(5,11^2)$. Let z be as in (3.7). As mentioned in the proof of (3.10), $|F(z)| = (q-1)/2 + k + 1$, $|\Omega| = kq(q+1)/2 + 1$ and $C_G(z)$ is transitive on F(z). If $q = 3^4$, then $|F(z)| = 46$ and $|\Omega| = 2\cdot 19^2\cdot 23$. Hence $|C_G(z)| = |F(z)|\,|C_{G\alpha}(z)| = |F(z)|\,|C_{G\alpha}(z)N^\alpha/N^\alpha|\,|C_{N^\alpha}(z)|$ $= 46\cdot 2^i\cdot 80 = 2^{5+i}\cdot 5\cdot 23$ with $0 \leq i \leq 3$. Let P be a Sylow 23-subgroup of $C_G(z)$ and Q a Sylow 5-subgroup of $C_G(z)$. It follows from a Sylow's theorem that P is a normal subgroup of $C_G(z)$ and so $[P, Q] = 1$. Therefore $|F(Q)| \geq 23$, contrary to $5 \nmid |N^\alpha_\beta|$. If $q = 11^2$, then $|F(z)| = 66$ and $|\Omega| = 2\cdot 3\cdot 6151$. Let P be a Sylow 11-subgroup of $C_G(z)$. Since $11 \nmid |\Omega|$, P is a subgroup of N^γ for some $\gamma \in \Omega$ and $F(P) = \{\gamma\}$. Hence $\gamma \in F(z)$, so that $z \in N^\gamma$, contrary to $C_{N^\gamma}(z) \simeq D_{120}$. In the case(ii), we have $(p_1-1)/2 = (\sum_{i=1}^{t_1-1} 9^i)/2 \mid t = 2t_1$. From this, $9^{t_1-1} \leq 4t_1$, hence $t_1 = 1$, a contradiction.

Assume t is odd. Then $Y \leq C_{N^\alpha}(f_1) \simeq PSL(2,p^m)$ by Lemma 2.6(viii). As $|V/Y| = p_1$ and p_1 is a prime, $Y \simeq Z_{(p^m-\varepsilon_1)/2}$ and $(q-\varepsilon)/(p^m-\varepsilon_1) = p_1$. Hence $\sum_{i=0}^{t-1} (p^m)^i(\varepsilon_1)^{t-1-i} = p_1$ and $(p_1-1)/2 = ((\sum_{i=1}^{t-1}(p^m)^i(\varepsilon_1)^{t-1-i})-1)/2 \mid t$. In particular $2t \geq (p^m)^{t-1} - (p^m)^{t-2} = (p^m-1)(p^m)^{t-2} \geq 2(p^m)^{t-2}$. From this $t = 3$, $m = 1$, $p_1 = 7$ and $q = 3^3$, so that $N^\alpha_\beta \simeq Z_2 \times Z_2$, a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let G^Ω be a minimal counterexample to (3.12) and M a minimal normal subgroup of G. By the hypothesis, G has no regular normal subgroup and hence $M_\alpha \neq 1$. As M_α is a normal subgroup of $G\alpha$, by (i) of (3.4), M_α contains N^α. By (3.9), $r = 1$, hence M is doubly transitive on Ω. Therefore $G = M$ and G is a nonabelian simple group.

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Since $N_\beta^\alpha \simeq D_{q-\varepsilon}$, $k = 1$ and so $q-\varepsilon+4 \mid 2((4-\varepsilon)(2-\varepsilon)+1)(4-\varepsilon)(2-\varepsilon)$ by (3.10). Hence we have $q = 7, 9, 11, 19, 27$ or 43.

Let x be an element of N_β^α. If $|x| > 2$, by Lemma 2.8, $|F(x)| = 1 + |N_\beta^\alpha|\times 1/|N_\beta^\alpha| = 2$ and if $|x| = 2$, similarly we have $|F(x)| = (q-\varepsilon)/2 + 2$. Assume $q \neq 9$ and let d be an involution in $G_\alpha - N^\alpha$ such that $\langle d\rangle N^\alpha$ is isomorphic to $PGL(2,q)$. We may assume $d \in G_{\alpha\beta}$. Since $\langle d\rangle N^\alpha$ is transitive on $\Omega - \{\alpha\}$, by Lemmas 2.3 and 2.6(vii),(ix), $|F(d)| = 2(q-1)(q+1/2)/2(q+1) + 1 = (q+1)/2$, while $|F(x)| = (q+1)/2 + 2$ for $x \in I(N^\alpha)$. Hence d is an odd permutation, contrary to the simplicity of G. Thus $G_\alpha = N^\alpha$ if $q \neq 9, 27$ and $|G_\alpha/N^\alpha| = 1, 3$ if $q = 27$.

If $q = 9$, $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 1 + 9\cdot 10/2 = 2\cdot 23$ and $|G_\alpha| = 2^i |PSL(2,9)| = 2^{3+i}\cdot 3^2\cdot 5$ with $0 \leq i \leq 2$. Let P be a Sylow 23-subgroup of G. Since $\mathrm{Aut}(Z_{23}) \simeq Z_2\times Z_{11}$, $3 \nmid |N_G(P)|$, for otherwise P centralizes a nontrivial 3-element x and so $F(P) \supseteq F(x)$ because $|F(x)| = 1$, contrary to $|F(P)| = 0$. Similarly $5 \nmid |N_G(P)|$. Hence $|G : N_G(P)| = 2^a\cdot 3^b\cdot 5$ for some a with $0 \leq a \leq 6$. By a Sylow's theorem, $2^a\cdot 3^2\cdot 5 \equiv -2^a \equiv 1 \pmod{23}$, a contradiction.

If $q = 27$, $|\Omega| = 1 + 27\cdot 26/2 = 2^5\cdot 11$ and $|G_\alpha| = 2^2\cdot 3^{3+i}\cdot 7\cdot 13$ with $0 \leq i \leq 1$. Let P be a Sylow 11-subgroup of G. Since $P \simeq Z_{11}$ and $\mathrm{Aut}(Z_{11}) \simeq Z_2\times Z_5$, $3^{1+i}, 7, 13 \nmid |N_G(P)|$ by the similar argument as above. Hence $|G : N_G(P)| = 2^a\cdot 3^b\cdot 7\cdot 13$ with $0 \leq a \leq 7$ and $3 \leq b \leq 3+i$. By a Sylow's theorem, $2^a\cdot 3^b\cdot 7\cdot 13 = 2^a\cdot 3^{b-3}\cdot 3^3\cdot 7\cdot 13 \equiv 2^a\cdot 3^{b-3}\cdot 4 \equiv 1 \pmod{11}$. Hence $a = 0$, $b = 4$. Therefore $N_G(P)$ contains a Sylow 2-subgroup S of G. Let T be a Sylow 2-subgroup of N_β^α and g an element such that $T^g \leq S$. Then $T^g \cap C_G(P) \neq 1$ as $N_S(P)/C_S(P) \leq Z_2$. Let u be an involution in $T^g \cap C_G(P)$. Then $|F(u)| = (27+1)/2 + 2 = 16$, while $11 \mid |F(u)|$ because $[P, u] = 1$ and $|F(P)| = 0$, a contradiction.

If $q = 7, 11, 19$ or 43, then $G_\alpha = N^\alpha$ and $\varepsilon = -1$. Set $\Gamma = \{\{\gamma, \delta\} \mid$

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$\gamma, \delta \in \Omega,\ \gamma \neq \delta \}$. We consider the action of G on Γ. Since G^{Ω} is doubly transitive, G^{Γ} is transitive and $G_{\Gamma} = 1$. Let z be an involution of $Z(N_\beta^\alpha)$. There exists an involution t such that $t \in z^G$ and $\alpha^t = \beta$. Since $G_{\alpha\beta} = N_\beta^\alpha$ and $F(N_\beta^\alpha) = \{\alpha, \beta\}$ we have $G_{\{\alpha,\beta\}} = \langle t\rangle N_\beta^\alpha$. By Lemma 2.3, $|F(z^\Gamma)| = |C_G(z)| \times |\langle t\rangle N_\beta^\alpha \cap z^G|/2|N_\beta^\alpha| = |F(z)| \times |C_{G_\alpha}(z)| \times |\langle t\rangle N_\beta^\alpha \cap z^G|/2|N_\beta^\alpha| = |F(z)| \times |\langle t\rangle N_\beta^\alpha \cap z^G|/2$. As $|F(z^\Gamma)| = |F(z)|(|F(z)| - 1)/2 + (|\Omega| - |F(z)|)/2$, $|\langle t\rangle N_\beta^\alpha \cap z^G| = |F(z)| + |\Omega|/|F(z)| - 2$. In particular $|F(z)| \,\big|\, |\Omega|$. Since $|F(z)| = (q+1)/2 + 2 = (q+5)/2$ and $|\Omega| = 1 + q(q-1)/2 = (q^2-q+2)/2$, we have $q = 11$ and $|\langle t\rangle N_\beta^\alpha \cap z^G| = 13$. Moreover $|\Omega| = 56$, $|G_\alpha| = |PSL(2,11)| = 2^2\cdot 3\cdot 5\cdot 11$ and $|G| = 2^5\cdot 3\cdot 5\cdot 7\cdot 11$.

We now argue that $\langle t\rangle N_\beta^\alpha \simeq D_{24}$. Let R be the Sylow 3-subgroup of N_β^α. If t centralizes R, R acts on $F(t)$ and so $F(R) \subseteq F(t)$ as $|F(t)| = 8$ and $|F(R)| = 2$. Hence $\alpha^t = \alpha$, contrary to the choice of t. Therefore t inverts R and $\langle t\rangle N_\beta^\alpha$ is isomorphic to $Z_2\times D_{12}$ or D_{24}. Suppose $\langle t\rangle N_\beta^\alpha \simeq Z_2\times D_{12}$. Then $\langle t\rangle N_\beta^\alpha$ contains fifteen involutions and so we can take $u \in I(\langle t\rangle N_\beta^\alpha)$ satisfying $|F(u)| = 0$ and $\langle t\rangle N_\beta^\alpha = \langle u\rangle\times N_\beta^\alpha$. As $|F(u)| = 0$, $|F(u^\Gamma)| = |\Omega|/2 = 28$. By Lemma 2.3, $28 = |C_G(u)|\times|\langle u\rangle N_\beta^\alpha \cap u^G|/24$ and hence $|C_G(u)| = 2^4\cdot 3\cdot 7$ or $2^5\cdot 3\cdot 7$. Since $\langle u\rangle N_\beta^\alpha = N_G(R)$, we have $|C_G(u) : C_G(u)\cap N_G(R)| = 2\cdot 7$ or $2^2\cdot 7$. By a Sylow's theorem, $|C_G(u) : C_G(u)\cap N_G(R)| = 2^2\cdot 7$, so that $|C_G(u)| = 2^5\cdot 3\cdot 7$. Let Q be a Sylow 7-subgroup of $C_G(u)$. Then $|C_G(u)\cap N_G(Q)| = = 2^5\cdot 3\cdot 7$ or $2^2\cdot 3\cdot 7$ by a Sylow's theorem. Hence $2^2\cdot 3\cdot 7 \,\big|\, |N_G(Q)|$. Since $Aut(Z_7) \simeq Z_2\times Z_3$, $5 \nmid |N_G(Q)|$ and $11 \nmid |N_G(Q)|$ by the similar argument as in the case $q = 9$. Therefore $|G : N_G(Q)| = 2^a\cdot 5\cdot 11$ for some a with $0 \leq a \leq 3$. Hence $|G : N_G(Q)| \not\equiv 1 \pmod 7$, a contradiction. Thus $\langle t\rangle N_\beta^\alpha \simeq D_{24}$.

Let U be a Sylow 2-subgroup of N_β^α and set $L = N_G(U)$. It follows from (3.3) and Lemma 2.6(iv) that $L\cap N^\alpha \simeq A_4$, $L^{F(U)} \simeq A_4$ and $|L| = 2^4\cdot 3$. Let T, $\langle x\rangle$ be Sylow 2- and 3-subgroup of L, respectively. Obviously $L \trianglerighteq T$ and $C_T(x) = 1$. On the other hand $T \trianglerighteq L \cap\langle t\rangle N_\beta^\alpha \simeq D_8$ and so $T' \simeq Z_2\times Z_2$

because $C_T(x) = 1$. By Theorem 5.4.5 of [2], T is dihedral or semi-dihedral. Hence $N_G(T)/C_G(T)$ ($\leq$ Aut(T)) is a 2-group, so that $C_T(x) = T$, a contradiction.

(3.13) (ii) of (3.11) does not occur.

Proof. Let G^{Ω} be a doubly transitive permutation group satisfying (ii) of (3.11). Let x be an involution in N_β^α with $x \notin Y$. Then $F(x^{F(Y)}) = F(\langle x\rangle Y) = F(N_\beta^\alpha) = \{\alpha, \beta\}$ by (i) of (3.1) and (3.9). Since $|F(Y)| = 1 + (q-\varepsilon)/|N_\beta^\alpha| = 1 + k \geq 4$, $x^{F(Y)}$ is an involution. By Lemma 2.5, $1 + k = 2^2$ and so $k = 3$. By (3.11), $q-\varepsilon+8 \mid 2((8-\varepsilon)(4-\varepsilon)\times 3 + 1)(8-\varepsilon)(4-\varepsilon)$. Hence $q + 7 \mid 2^7\cdot 3\cdot 7$ if $\varepsilon = 1$ and $q + 9 \mid 2^4\cdot 3^2\cdot 5\cdot 17$ if $\varepsilon = -1$. Since $k = 3 \mid q-\varepsilon$, $3 \nmid q-\varepsilon+8$. From this $q + 7 \mid 2^7\cdot 7$ if $\varepsilon = 1$ and $q + 9 \mid 2^4\cdot 5\cdot 17$ if $\varepsilon = -1$. Therefore $q = 5^2$, 7^2, 11^2, 59 or 71.

Let p_1 be an odd prime such that $p_1 \mid |\Omega|$ and $p_1 \nmid |G_\alpha|$ and let P be a Sylow p_1-subgroup of G. Clearly P is semi-regular on Ω and so any element in $C_{G_\alpha}(P)$ has at least p_1 fixed points. If x is an element of N_β^α and its order is at least three, $|F(x)| = |F(Y)| = 4$ by Lemma 2.8. Since $|N_\beta^\alpha| = (q-\varepsilon)/3$, we have $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 1 + 3q(q+\varepsilon)/2$.

If $q = 5^2$, then $|\Omega| = 2^4\cdot 61$ and $|G_\alpha| = 2^{4+i}\cdot 3\cdot 5^2\cdot 13$ $(0 \leq i \leq 2)$. Let P be a Sylow 61-subgroup of G. Then $P \simeq Z_{61}$. As mentioned above, $5, 13 \nmid |C_G(P)|$ and so $5^2, 13 \nmid |N_G(P)|$. Hence $|G : N_G(P)| = 2^a\cdot 3^b\cdot 5^{c+1}\cdot 13$, where $0 \leq a \leq 10$ and $0 \leq b, c \leq 1$. But we can easily verify $|G : N_G(P)| \not\equiv 1 \pmod{61}$, contrary to a Sylow's theorem.

If $q = 7^2$, then $|\Omega| = 2^2\cdot 919$ and $|G_\alpha| = 2^{4+i}\cdot 3\cdot 5^2\cdot 7^2$ $(0 \leq i \leq 2)$. Let P be a Sylow 919-subgroup of G. By the similar argument as above, we obtain $5, 7 \nmid |N_G(P)|$ and so $|G : N_G(P)| = 2^a\cdot 3^b\cdot 5^2\cdot 7^2 \equiv 2^a\cdot 306$ or $-2^a \pmod{919}$, where $0 \leq a \leq 8$ and $0 \leq b \leq 1$. Hence $|G : N_G(P)| \not\equiv 1$, a contradiction.

If $q = 11^2$, then $|\Omega| = 2^7 \cdot 173$ and $|G_\alpha| = 2^{3+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$ $(0 \leq i \leq 2)$. Let P be a Sylow 173-subgroup of G. Similarly we have $3,5,11,61 \nmid |N_G(P)|$ and so $|G : N_G(P)| = 2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -5 \cdot 2^a \pmod{173}$, where $0 \leq a \leq 12$. Hence $|G : N_G(P)| \neq 1$, a contradiction.

If $q = 59$, then $|\Omega| = 2 \cdot 17 \cdot 151$ and $|G_\alpha| = 2^{2+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59$ $(0 \leq i \leq 1)$. Let P be a Sylow 17-subgroup of G. Similarly we have $3,5,29,59 \nmid |N_G(P)|$ and so $|G : N_G(P)| = 2^a \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151^b \equiv 10 \cdot 2^a$ or $12 \cdot 2^a \pmod{17}$, where $0 \leq a \leq 4$ and $0 \leq b \leq 1$. From this, we have a contradiction.

If $q = 71$, then $|\Omega| = 2^5 \cdot 233$ and $|G_\alpha| = 2^{3+i} \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$ $(0 \leq i \leq 1)$. Let P be a Sylow 233-subgroup of G. Since $3,5,7,71 \nmid |N_G(P)|$, $|G : N_G(P)| = 2^a \cdot 3^2 \cdot 5 \cdot 7 \cdot 71 \equiv -3 \cdot 2^a \pmod{233}$, where $0 \leq a \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N^\alpha_\beta \simeq Z_2 \times Z_2$ or $N^\alpha_\beta \simeq D_8$ and $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

(3.14) The case that $N^\alpha_\beta \simeq Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N^\alpha_\beta)$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N^\alpha_\beta N^\beta_\alpha)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^\alpha|_2 = 4$, we have $q = p^n \equiv 3, 5 \pmod 8$ and so n is odd. Hence $|G_\alpha/N^\alpha|_2 \leq 2$ and $N^\alpha_\beta/N^\alpha \cap N^\beta \simeq N^\alpha_\beta N^\beta/N^\beta \simeq 1$ or Z_2 by (3.2). Suppose $N^\alpha_\beta/N^\alpha \cap N^\beta \simeq Z_2$. Then $N^\alpha_\beta N^\beta_\alpha$ is a Sylow 2-subgroup of G_α , hence $N_G(N^\alpha_\beta N^\beta_\alpha)^\Delta$ is doubly transitive by a Witt's theorem. Since $N^\alpha_\beta N^\beta_\alpha \simeq D_8$ and $|\Delta|$ is even, $C_G(N^\alpha_\beta N^\beta_\alpha)^\Delta$ is also doubly transitive. Let g be an element of $C_G(N^\alpha_\beta N^\beta_\alpha)$ such that $\alpha^g = \beta$ and $\beta^g = \alpha$. Then $N^\alpha_\beta = g^{-1} N^\alpha_\beta g = N^\beta_\alpha$ and hence $N^\alpha_\beta = N^\alpha \cap N^\beta$, a contradiction. Thus $N^\alpha_\beta = N^\alpha \cap N^\beta \simeq Z_2 \times Z_2$.

Let z be an involution in N^α_β and $t \in z^G$ an involution such that $\alpha^t = \beta$. Set $\Gamma = \{\{\gamma, \delta\} \mid \gamma, \delta \in \Omega, \gamma \neq \delta\}$. We consider the action of the element z on Γ. By the similar argument as in the proof of (3.12),

$$|F(z)|(|F(z)|-1)/2 + (|\Omega| - |F(z)|)/2 = |F(z^\Gamma)| = |C_G(z)||z^G \cap \langle t \rangle G_{\alpha\beta}|/|\langle t \rangle G_{\alpha\beta}| .$$

Since $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$, by Lemma 2.6(i), $z^G \cap G_{\alpha} = z^{G_{\alpha}}$ and so $|C_G(z)| = |F(z)| \times |C_{G_{\alpha}}(z)|$. Hence $|G_{\alpha\beta}|(|F(z)|(|F(z)| - 1) + |\Omega| - |F(z)|)$ $= |F(z)|\,|C_{G_{\alpha}}(z)|\,|z^G \cap \langle t \rangle G_{\alpha\beta}|$, so that $|G_{\alpha\beta}||\Omega| \equiv 0 \pmod{|F(z)|}$. Since $|G_{\alpha\beta}/N^{\alpha}_{\beta}| = |G_{\alpha\beta}N^{\alpha}/N^{\alpha}| \mid 2n$, we have $|G_{\alpha\beta}| \mid 8n$. Clearly $|\Omega| = 1 + q(q-\varepsilon)(q+\varepsilon)r/8$ and by Lemma 2.8(i), $|F(z)| = 1 + 3(q-\varepsilon)r/4$. Hence $1 + 3(q-\varepsilon)r/4 \mid 8n(1 + q(q-\varepsilon)(q+\varepsilon)r/8)$. Put $n = rs$. Then $3qr-3\varepsilon r+4 \mid 4rs(8+q(q-\varepsilon)(q+\varepsilon)r)3^3r = 864\, r^2s + 4s(3qr)(3qr-3\varepsilon r)(3qr+3\varepsilon r)$. Hence $3qr-3\varepsilon r+4 \mid 864r^2s + 4s(3\varepsilon r-4)(3\varepsilon r-4-3\varepsilon r)(3\varepsilon r-4+3\varepsilon r)$ $= 864r^2s - 32s(3\varepsilon r-4)(3\varepsilon r-2)$. $(*)$

We argue that $r = 1$. Suppose false. Then $32s(3\varepsilon r-4)(3\varepsilon r-2) > 0$ and so $3r(q-\varepsilon) < 864r^2s$. Therefore $288n + \varepsilon > q = p^n \geq 3^n$ and so $288n > 3^n$. Hence $(n,r,p,\varepsilon) = (5,5,3,-1)$, $(3,3,3,-1)$ or $(3,3,5,1)$, while none of these satisfy $(*)$. Thus $r = 1$.

Hence $3q-3\varepsilon+4 \mid 64(5+9\varepsilon)n$ and $|F(z)| = 1 + 3(q-\varepsilon)/4$, $|\Omega| = 1 + q(q-\varepsilon)(q+\varepsilon)/8$. If $\varepsilon = -1$, then $3\cdot 3^n < 3q+7 \mid 256n$. Hence $n = 1$ or $(n,p) = (5,3)$, $(3,3)$. Since $3\cdot 3^5 + 7 \nmid 256\cdot 5$ and $3\cdot 3^3 + 7 \mid 256\cdot 3$, $n = 1$ and $3q + 7 \mid 256$. From this, $q = 19$ or 83. If $\varepsilon = 1$, then $3\cdot 5^n < 3q + 1 \mid 896n$ and so $n = 1$ or $(n,p) = (3,5)$. Since $3\cdot 5^3 + 1 \nmid 896\cdot 3$, we have $n = 1$ and $3q + 1 \mid 896$. From this, $q = 5$, 37 or 149. As $PSL(2,5) \simeq PSL(2,4)$, $q \neq 5$ by [4]. Thus $q = 19$, 37, 83 or 149.

Set $m = |z^G \cap \langle t \rangle G_{\alpha\beta}|$. As we mentioned above, $|G_{\alpha\beta}|(|F(z)|(|F(z)|-1) + |\Omega| - |F(z)|) = |F(z)|\,|C_{G_{\alpha}}(z)|m$. Since $|G_{\alpha}/N^{\alpha}| = 1$ or 2, $|C_{G_{\alpha}}(z)|/|G_{\alpha\beta}| = (q-\varepsilon)/4$. Therefore $m = (2q^2 + (2\varepsilon+9)q - 9\varepsilon)/(3q-3\varepsilon+4)$. It follows that $(q,m) = (19, 27/2)$, $(37, 28)$, $(83, 449/8)$ or $(149, 411/4)$. Since m is an integer, we have $(q,m) = (37, 28)$. But $m \leq |\langle t \rangle G_{\alpha\beta}| \leq 16$, a contradiction. Thus (3.14) holds.

(3.15) The case that $N^{\alpha}_{\beta} \simeq D_8$ and $N^{\alpha} \cap N^{\beta} \le Z_2 \times Z_2$ does not occur.

Proof. Let Δ, L and K be as defined in (3.6). By (3.6), there exists an element x in L_{α} such that its order is odd and $\langle x^{\Delta} \rangle$ is regular on $\Delta - \{\alpha\}$. Since $(L_{\alpha})' \le N^{\alpha}_{\beta}$ by (3.6) and $N^{\alpha}_{\beta} \simeq D_8$, x stabilizes a normal series $N^{\beta}_{\alpha} N^{\alpha}_{\beta} \trianglerighteq N^{\alpha}_{\beta} \trianglerighteq 1$. Hence x centralizes $N^{\beta}_{\alpha} N^{\alpha}_{\beta}$ by Theorem 5.3.2 of [2] and so $x^{-1} N^{\beta}_{\alpha} x = N^{\beta}_{\alpha}$. Put $\gamma = \beta^{x}$. If $r \ne 1$, then $\beta \ne \gamma$, so that $N^{\gamma}_{\alpha} = N^{\beta}_{\alpha}$. From this, $N^{\gamma}_{\beta} = N^{\beta}_{\gamma}$. By the doubly transitivity of G, $N^{\alpha}_{\beta} = N^{\beta}_{\alpha}$, hence $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$, a contradiction. Therefore $r = 1$ and $\Delta = \{\alpha, \beta\}$.

Set $\langle z \rangle = Z(N^{\alpha}_{\beta})$, $\Delta_1 = \alpha^{C_G(z)}$ and let $\{\Delta_1, \Delta_2 \cdots \Delta_k\}$ be the set of $C_G(z)$-orbits on $F(z)$. Since $L \trianglerighteq N^{\alpha} \cap N^{\beta}$ and by (3.2), $N^{\alpha} \cap N^{\beta} \ne 1$, z is contained in $N^{\alpha} \cap N^{\beta}$. Hence, by Lemma 2.1, $\beta \in \Delta_1$ and k is at least two. By Lemma 2.8, $|F(z)| = 1 + (q-\varepsilon)5/|N^{\alpha}_{\beta}| = 1 + 5(q-\varepsilon)/8$. Clearly $|C_{N^{\alpha}}(z) : N^{\alpha}_{\beta}| = (q-\varepsilon)/8$ and so $|\Delta_1| \ge 1 + (q-\varepsilon)/8$. If $\gamma \in F(z) - \Delta_1$, then $C_{N^{\alpha}_{\gamma}}(z) \simeq Z_2 \times Z_2$, for otherwise $\langle z \rangle = Z(N^{\alpha}_{\gamma}) \le N^{\alpha} \cap N^{\gamma}$ and by Lemma 2.1 $\gamma \in \Delta_1$, a contradiction. Hence one of the following holds.

(i) $k = 3$ and $|\Delta_1| = 1 + (q-\varepsilon)/8$, $|\Delta_2| = |\Delta_3| = (q-\varepsilon)/4$.

(ii) $k = 2$ and $|\Delta_1| = 1 + (q-\varepsilon)/8$, $|\Delta_2| = (q-\varepsilon)/2$.

(iii) $k = 2$ and $|\Delta_1| = 1 + 3(q-\varepsilon)/8$, $|\Delta_2| = (q-\varepsilon)/4$.

Let $\gamma \in F(z) - \Delta_1$. Then, $z \in G_{\gamma} - N^{\gamma}$ and so $C_{N^{\gamma}}(z) \simeq D_{q+\varepsilon}$ or $PGL(2, \sqrt{q})$ by Lemma 2.6(vii),(viii),(ix). If $C_{N^{\gamma}}(z) \simeq D_{q+\varepsilon}$, then $(q+\varepsilon)/2 \mid |\Delta_1|$ and so $q = 7$ and (iii) occurs. But $(q+\varepsilon)/2 = 3 \mid |\Delta_2| - 1 = 1$, a contradiction. If $C_{N^{\gamma}}(z) \simeq PGL(2, \sqrt{q})$, then (i) does not occur because $\sqrt{q} \nmid q-\varepsilon$. Hence $\sqrt{q} \mid |\Delta_1|$ and $\sqrt{q} \mid |\Delta_2| - 1$. From this, $q = 25$ and (iii) occurs. In this case, we have $|\Delta_1| = 10$, so that an element of $C_{N^{\gamma}}(z)$ of order 3 is contained in N^{γ}_{δ} for some $\delta \in \Delta_1$, contrary to $N^{\gamma}_{\delta} \simeq N^{\alpha}_{\beta} \simeq D_8$.

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4. Case (II)

In this section we assume that $N^{\alpha}_{\beta} \simeq PGL(2,p^m)$, where $n = 2mk$ and k is odd. Since n is even, $q = p^n \equiv 1 \pmod 4$. We set $p^m \equiv \varepsilon \in \{\pm 1\} \pmod 4$. In section 7 we shall consider the case that $N^{\alpha}_{\beta} \simeq S_4$. Therefore we assume $(p,m) \neq (3,1)$ in this section.

(4.1) The following hold.

(i) $N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq 1$ or Z_2 and $N^{\alpha} \cap N^{\beta} \geq (N^{\alpha}_{\beta})' \simeq PSL(2,p^m)$.

(ii) If $(p,m) \neq (5,1)$, there exists a cyclic subgroup Y of $(N^{\alpha}_{\beta})'$ such that $N_{N^{\alpha}}(Y) \simeq D_{q-\varepsilon}$ and $N_G(Y)^{F(Y)}$ is doubly transitive.

Proof. As $N^{\alpha}_{\beta} \trianglerighteq N^{\alpha} \cap N^{\beta}$, either $N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \leq Z_2$ or $N^{\alpha} \cap N^{\beta} = 1$. If $N^{\alpha} \cap N^{\beta} = 1$, by Lemmas 2.2 and 2.6(vi), $N^{\alpha}_{\beta} \simeq N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq N^{\alpha}_{\beta}N^{\beta}/N^{\beta} \simeq Z_2 \times Z_n$, a contradiction. Therefore $N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq 1$ or Z_2 and $N^{\alpha} \cap N^{\beta} \geq (N^{\alpha}_{\beta})' \simeq PSL(2,p^m)$.

Now we assume that $(p,m) \neq (5,1)$ and let z be an involution in $(N^{\alpha}_{\beta})'$. Then $C_{N^{\alpha}_{\beta}}(z) \simeq D_{2(p^m-\varepsilon)}$ by Lemma 2.6(vii). Suppose $C_{N^{\alpha}_{\beta}}(z)$ is not a 2-subgroup and put $Y = O(C_{N^{\alpha}_{\beta}}(z))$. Then, if $Y^g \leq G_{\alpha\beta}$ for some $g \in G$, we have $Y^g \leq N^{\gamma}_{\alpha}$ and $Y^g \leq N^{\delta}_{\beta}$, where $\gamma = \alpha^g$ and $\delta = \beta^g$. By (i) $Y^g \leq N^{\alpha} \cap N^{\beta}$ and so $Y^g = Y^h$ for some $h \in N^{\alpha} \cap N^{\beta}$. Thus $N_G(Y)^{F(Y)}$ is doubly transitive. Assume that $C_{N^{\alpha}_{\beta}}(z)$ is a 2-subgroup and set $C_{N^{\alpha}_{\beta}}(z) = \langle u,v \mid u^v = u^{-1}, v^2 = 1\rangle$. We may assume that $v \in (N^{\alpha}_{\beta})'$ and $\langle u^2, v\rangle$ is a Sylow 2-subgroup of $(N^{\alpha}_{\beta})'$. Since $p^m \neq 3,5$, the order of u^2 is at least four. On the other hand there is no element of order $|u^2|$ in $\langle u,v\rangle - \langle u^2, v\rangle$. Hence any element of order $|u^2|$ which is contained in N^{α}_{β} is necessarily an element of $N^{\alpha} \cap N^{\beta}$. By the similar argument as above, $N_G(Y)^{F(Y)}$ is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose $(p,m) \neq (3,1), (5,1)$ and set $\Delta = F(Y)$ and $X = N_G(Y)$. Then $|\Delta| = rs(p^m+\varepsilon)/2 + 1$, where $s = \sum_{i=0}^{k-1} p^{2mi}$, $C_G(N^{\alpha}) = 1$ and one of the following holds.

(i) $X^{\Delta} \leq A\Gamma L(1,2^c)$ for some integer c.

(ii) $X^{\Delta} \simeq PSL(2,p_1)$ or $PGL(2,p_1)$, $r = 1$, $k = 1$ and $2p_1 = p^m + \varepsilon$.

Proof. By Lemma 2.8(ii), $|\Delta| = 1 + |N^{\alpha} \cap X|\, r/|N^{\alpha}_{\beta} \cap X| = 1 + (p^{2mk}-1)r/2(p^m-\varepsilon) = rs(p^m+\varepsilon)/2 + 1$. By (4.1) and Lemma 2.9, we have (i), (ii) or $X^{\Delta} = R(3)$.

Assume that $X^{\Delta} = R(3)$. Then $rs(p^m+\varepsilon)/2 + 1 = 28$, hence $k = 1$ and $r(p^m+\varepsilon)/2 = 27$. Since r is odd and $r \mid 2m = n$, we have $r = m = 1$ and $q = 53^2$. But a Sylow 3-subgroup of X_{α} is cyclic because $N^{\alpha} \cap X \simeq D_{q-\varepsilon}$ and $X_{\alpha}/X \cap N^{\alpha} \simeq X_{\alpha}N^{\alpha}/N^{\alpha} \leq Z_2 \times Z_2$, a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose $X^{\Delta} \leq A\Gamma L(1,2^c)$ and put $W = C_{N^{\alpha}_{\beta}}(Y)$. Then $Y \leq W \simeq Z_{p^m-\varepsilon}$. Since $C_{N^{\alpha}}(Y)$ is cyclic, W is a characteristic subgroup of $C_{N^{\alpha}}(Y)$ and so W is a normal subgroup of X_{α}. Hence $W \leq X_{\Delta}$ and $(X \cap N^{\alpha}_{\beta})^{\Delta} \simeq 1$ or Z_2. By Lemmas 2.4 and 2.6, $F(X \cap N^{\alpha}_{\beta}) = 1 + |X \cap N^{\alpha}_{\beta}|\,|N^{\alpha}_{\beta} : X \cap N^{\alpha}_{\beta}| \times r/|N^{\alpha}_{\beta}| = 1 + r$. Since $1 + r < |\Delta|$, $(X \cap N^{\alpha}_{\beta})^{\Delta} \simeq Z_2$ and hence $(1+r)^2 = rs(p^m+\varepsilon)/2 + 1$ by Lemma 2.5. From this, $r = s(p^m+\varepsilon)/2 - 2 \mid mk$ and so $p^{2m(k-1)} + mk \leq 2$. Hence $m = k = r = 1$ and $q = 7^2$.

Let R be a Sylow 3-subgroup of N^{α}_{β}. Since $N^{\alpha}_{\beta} \simeq PGL(2,7)$, we have $R \simeq Z_3$. By Lemmas 2.4 and 2.6, $|F(R)| = 1 + (7^2-1)|N^{\alpha}_{\beta} : N_{N^{\alpha}_{\beta}}(R)|/|N^{\alpha}_{\beta}| = 4$. Hence $N_G(R)^{F(R)} \simeq A_4$ or S_4. But R is a Sylow 3-subgroup of $N_{G_{\alpha}}(R)$ because $N^{\alpha} \simeq PSL(2,7^2)$, contrary to $N_{G_{\alpha}}(R)^{F(R)} \simeq A_3$ or S_3.

(4.4) (ii) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose $X^{\Delta} \trianglerighteq PSL(2,p_1)$. By the similar argument as in (4.3), $C_{N^{\alpha}_{\beta}}(Y) \leq X_{\Delta}$ and so $C_{N^{\alpha}}(Y)^{\Delta} \simeq Z_{p_1}$, and $N_{N^{\alpha}}(Y)^{\Delta} \simeq D_{2p_1}$. Hence $|(X_{\alpha})^{\Delta}| \mid 2p_1 \cdot 2n$. Since $X^{\Delta} \trianglerighteq PSL(2,p_1)$, $p_1(p_1-1)/2 \mid |(X_{\alpha})^{\Delta}|$, hence $p_1-1 \mid 8n$. As $k = 1$ and $2p_1 = p^m + \varepsilon$, we have $p^m+\varepsilon-2 \mid 32m$. From this, $(p,m,p_1) = (11,1,5)$, $(3,2,5)$ or $(3,3,13)$.

Let R be a cyclic subgroup of N^{α}_{β} such that $R \simeq Z_{(p^m+\varepsilon)/2}$. By Lemma 2.6, $N_G(R)^{F(R)}$ is doubly transitive and by Lemma 2.8(ii), $|F(R)| = 1 + |N_{N^{\alpha}}(R)|/|N_{N^{\alpha}_{\beta}}(R)| = 1 + (p^{2m}-1)/2(p^m+\varepsilon) = (p^m-\varepsilon)/2 + 1$.

If $(p,m,p_1) = (11,1,5)$, $|F(R)| = 7$ and so by [9] $|N_G(R)^{F(R)}| = 42$ and $N_{G_{\alpha}}(R)^{F(R)} \simeq Z_6$. Since $|N_{N^{\alpha}}(R) : N_{N^{\alpha}_{\beta}}(R)| = 6$, $N_{N^{\alpha}}(R)^{F(R)} = N_{G_{\alpha}}(R)^{F(R)}$. Hence $N_{N^{\alpha}}(R)/K \simeq Z_6$, where $K = (N_{N^{\alpha}}(R))_{F(R)}$. But $N_{N^{\alpha}}(R)/(N_{N^{\alpha}}(R))' \simeq Z_2 \times Z_2$, a contradiction.

If $(p,m,p_1) = (3,2,5)$, $|F(R)| = 5$ and so by [9], $|N_G(R)^{F(R)}| = 20$ and $N_{G_{\alpha}}(R)^{F(R)} \simeq Z_4$. Since $|N_{N^{\alpha}}(R) : N_{N^{\alpha}_{\beta}}(R)| = 4$, $N_{N^{\alpha}}(R)^{\Delta} \simeq Z_4$, contrary to $N_{N^{\alpha}}(R)/(N_{N^{\alpha}}(R))' \simeq Z_2 \times Z_2$.

If $(p,m,p_1) = (3,3,13)$, $|F(R)| = 15$. By [9], $N_{G_{\alpha}}(R)^{F(R)}$ is not solvable, a contradiction.

(4.5) $p^m \neq 5$.

Proof. Assume that $p^m = 5$. Then $n = 2k$ with k odd and $N^{\alpha}_{\beta} \simeq PGL(2,5) \simeq S_5$. First we argue that $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$. Suppose false. Then $C_G(N^{\alpha}) = 1$ by Lemma 2.2, and $N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq Z_2$ by (4.1). Since $N^{\beta}_{\alpha}N^{\alpha}_{\beta}/N^{\alpha}_{\beta} \simeq N^{\beta}_{\alpha}/N^{\alpha} \cap N^{\beta} \simeq Z_2$ and the outer automorphism group of S_5 is trivial, we have $Z(N^{\alpha}_{\beta}N^{\beta}_{\alpha}) \simeq Z_2$. Let w_1 be the involution of $Z(N^{\alpha}_{\beta}N^{\beta}_{\alpha})$ and let $w \in I(N^{\beta}_{\alpha}) - I(N^{\alpha})$. Since $C_{N^{\alpha}}(w_1) \geq N^{\alpha}_{\beta}$, by Lemma 2.6(viii) and (ix), w acts on N^{α} as a field automorphism of order 2 and $C_{N^{\alpha}}(w) \simeq PGL(2,5^k)$.

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By Lemma 2.8 $|F(w)| = 1 + r(q-\varepsilon)|I(N_\beta^\alpha)|/|N_\beta^\alpha| = 1 + 5r(5^{2k}-1)/24$. Let P be a Sylow 5-subgroup of $C_{N^\alpha}(w)$. Then $|P| = 5^k$ and $|\gamma^P| = 5^{k-1}$ or 5^k for $\gamma \in \Omega - \{\alpha\}$. Since P acts on $F(w)-\{\alpha\}$, we have $5^{k-1} \mid 5r(5^{2k}-1)/24$, so that $k = 1$ and $|F(w)| = 6$ as $r \mid k$. Hence $C_{N^\alpha}(w)^{F(w)} \simeq S_5$ and so $C_G(w)^{F(w)} \simeq S_6$. But clearly $w \in N^\alpha \cap N^\beta$ by Lemma 2.1, a contradiction. Thus $N_\beta^\alpha = N^\alpha \cap N^\beta$.

Let V be a cyclic subgroup of N_β^α of order 4. Since $N_\beta^\alpha = N^\alpha \cap N^\beta \simeq S_5$, $N_G(V)^{F(V)}$ is doubly transitive and by Lemma 2.8, $|F(V)| = 1 + |N_{N^\alpha}(V)|r/|N_{N_\beta^\alpha}(V)| = 1 + (5^{2k}-1)r/8 = 3rs + 1$, where $s = \sum_{i=0}^{k-1} 25^i$. By Lemma 2.9, $C_G(N^\alpha) = 1$ and (a) $N_G(V)^{F(V)} \le A\Gamma L(1,2^c)$ or (b) $N_G(V)^{F(V)} = R(3)$.

Put $P = N_{N_\beta^\alpha}(V)$. Then $P \simeq D_8$, $|F(P)| = 1 + |N_{N^\alpha}(P)||N_\beta^\alpha : N_{N_\beta^\alpha}(P)|r/|N_\beta^\alpha| = r + 1$ and $P^{F(V)} \simeq Z_2$. If (b) occurs, $k = 1$ and $r = 9$, hence $|F(P)| = 10$, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r+1)^2 = 3rs + 1$ and so $r = 3s - 2 \mid k$. Hence $k = r = 1$ and $G_\alpha/N^\alpha \le Z_2 \times Z_2$. Let z be an involution in N_β^α. Then $|F(z)| = 1 + 24\cdot 25/120 = 6$ by Lemma 2.8 and $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 66$ as $r = 1$. By the similar argument as in the proof of (3.12), $|F(z)|(|F(z)|-1)/2 + (|\Omega| - |F(z)|)/2 = |C_G(z)||z^G \cap \langle t\rangle G_{\alpha\beta}|/|\langle t\rangle G_{\alpha\beta}|$, where t is an involution such that $\alpha^t = \beta$. Hence $|z^G \cap \langle t\rangle G_{\alpha\beta}| = 15|G_{\alpha\beta}|/|C_{G_\alpha}(z)|$. Set $H = \langle t\rangle G_{\alpha\beta}$ and let R be a Sylow 3-subgroup of N_β^α. By Lemma 2.8, $|F(R)| = 1 + 24\cdot 10/120 = 3$. Set $F(R) = \{\alpha, \beta, \gamma\}$. On the other hand, as $N_\beta^\alpha \simeq S_5$ and $Out(S_5) = 1$, we have $H = Z(H)\times N_\beta^\alpha$ and $|Z(H)| = 2$, 4 or $H = C_H(N_\beta^\alpha)\times N_\beta^\alpha$ and $C_H(N_\beta^\alpha) \simeq D_8$. In the latter case $G_{\alpha\beta} = Z(G_{\alpha\beta})\times N_\beta^\alpha$ and $Z(G_{\alpha\beta}) \simeq Z_2\times Z_2$, contrary to Lemma 2.6(ix). In the former case, we have $|Z(H)| = 2$. For otherwise $Z(H) \le G_\gamma$ and $Z(H) \cap z^G \ne \emptyset$ and so letting $u \in Z(H) \cap z^G$, we have $|R| = 3 \mid |F(u)| - 1 = 5$, a contradiction. Therefore $Z(H) \simeq Z_2$ and so $|z^G \cap H| \le 25 + 25 = 50$, while $|z^G \cap H| = 15|G_{\alpha\beta}|/C_{G_\alpha}(z) = 15\cdot 120/24 = 75$, a contradiction.

5. Case (III)

In this section we assume that $N^{\alpha}_{\beta} \simeq PSL(2,p^m)$, where $n = mk$ and k is odd. Set $p^m \equiv \varepsilon \in \{\pm 1\}$ (mod 4). Then $q \equiv \varepsilon$ (mod 4) as k is odd. In section 6 we shall consider the case that $N^{\alpha}_{\beta} \simeq A_4$, so we assume $(p,m) \neq (3,1)$ in this section. From this N^{α}_{β} is a nonabelian simple group and so $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$ or $N^{\alpha} \cap N^{\beta} = 1$. If $N^{\alpha} \cap N^{\beta} = 1$, then $C_G(N^{\alpha}) = 1$ by Lemma 2.2 and $N^{\alpha}_{\beta} \simeq N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq N^{\alpha}_{\beta}N^{\beta}/N^{\beta} \simeq Z_2 \times Z_n$, a contradiction. Hence $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$.

Let z be an involution of N^{α}_{β}. Suppose $z^g \in G_{\alpha\beta}$ for some $g \in G$ and set $\gamma = \alpha^g$, $\delta = \beta^g$. Then $z^g \in N^{\gamma}_{\delta} \cap G_{\alpha\beta} \leq N^{\gamma}_{\alpha} \cap N^{\delta}_{\beta} \leq N^{\alpha} \cap N^{\beta}$ and so $z^g \in z^{N^{\alpha}_{\beta}}$. Hence $C_G(z)^{F(z)}$ is doubly transitive and by Lemma 2.8(i), $|F(z)| = (q-\varepsilon)r/(p^m-\varepsilon) + 1$. In particular $|F(z)| > 3r + 1$ as $(p^n-\varepsilon)/(p^m-\varepsilon) \geq p^{2m}+\varepsilon p^m+1 > 3$.

By Lemma 2.9, $C_G(N^{\alpha}) = 1$ and one of the following holds.

(a) $C_G(z)^{F(z)} \leq A\Gamma L(1,2^c)$.

(b) $C_G(z)^{F(z)} \trianglerighteq PSL(2,p_1)$ $(p_1 \geq 5)$, $r = 1$ and $|C_{N^{\alpha}}(z) : C_{N^{\alpha}_{\beta}}(z)| = p_1$.

(c) $C_G(z)^{F(z)} = R(3)$.

Let Y be a cyclic subgroup of $C_{N^{\alpha}_{\beta}}(z) \simeq D_{p^m-\varepsilon}$ of index 2. Since $C_{G_{\alpha}}(z) \trianglerighteq Y$, $z \in Y$ and $C_G(z)^{F(z)}$ is doubly transitive, we have $F(Y) = F(z)$. By the similar argument as in (3.1), $N^{\alpha} \cap N(C_{N^{\alpha}_{\beta}}(z)) = C_{N^{\alpha}_{\beta}}(z)$ or $N^{\alpha} \cap N(C_{N^{\alpha}_{\beta}}(z)) \simeq A_4$. Hence by Lemmas 2.3 and 2.4 $|F(C_{N^{\alpha}_{\beta}}(z))| = 1 + |C_{N^{\alpha}_{\beta}}(z)||N^{\alpha}_{\beta} : C_{N^{\alpha}_{\beta}}(z)|r/|N^{\alpha}_{\beta}|$ or $1 + |A_4||N^{\alpha}_{\beta} : C_{N^{\alpha}_{\beta}}(z)|r/|N^{\alpha}_{\beta}|$. Therefore $|F(C_{N^{\alpha}_{\beta}}(z))| = r + 1$ or $3r + 1$. From this $C_{N^{\alpha}_{\beta}}(z)^{F(z)} \simeq Z_2$.

In the case (a), $(r+1)^2 = 1 + (p^n-\varepsilon)r/(p^m-\varepsilon)$ by Lemma 2.5 and hence $r = (p^n-\varepsilon)/(p^m-\varepsilon) - 2 \mid mk$. Since $(p^n-\varepsilon)/(p^m-\varepsilon) \geq ((p^m)^k + 1)/(p^m+1) = \sum_{i=0}^{k-1}(-p^m)^i$ and $k \geq 3$, we have $p^{m(k-1)}(p^{2m}-p^m+1) \leq mk$, hence $(p^m)^{k-3}/k$

$m/(p^{2m}-p^m+1) < 1$. Thus $k = 3$, $m = 1$ and $p = 3$, cotrary to $(p,m) \neq (3,1)$.

In the case (b), $r = 1$, $p_1 = (p^n-\varepsilon)/(p^m-\varepsilon)$, $p_1(p_1-1)/2 \mid s$ and $s \mid 4mkp_1$, where s is the order of $C_{G_\alpha}(z)^{F(z)}$. Hence $p_1-1 \mid 8mk$. Since $p_1-1 = (p^n-\varepsilon)/(p^m-\varepsilon) - 1 \geq (p^n+1)/(p^m+1) - 1 = \sum_{i=0}^{k-1}(-p^m)^i \geq p^{m(k-2)}(p^m-1)$, we have $p^{m(k-2)}/2k \leq 4m/(p^m-1) \leq 1$ because $p^m \neq 3$. Hence $k = 3$ and $p^m = 5$, so that $p_1-1 = 30 \nmid 8mk = 24$, a contradiction.

In the case (c), $r + 1 = 4$ and $1 + (p^n-\varepsilon)r/(p^m-\varepsilon) = 28$ and so $r = 3$ and $(p^n-\varepsilon)/(p^m-\varepsilon) = 9$. Hence $9 \geq (p^{mk}+1)/(p^m+1) \geq p^{2m}-p^m+1$, so that $p^m = 3$, a contradiction.

6. Case (IV)

In this section we assume that $N^\alpha_\beta \simeq A_4$ and $q = 3,5 \pmod 8$. If $N^\alpha \cap N^\beta = 1$, by Lemma 2.2, $C_G(N^\alpha) = 1$ and so $N^\alpha_\beta/N^\alpha \cap N^\beta \simeq N^\alpha_\beta N^\beta/N^\beta \leq Z_2 \times Z_n$. Hence $N^\alpha_\beta/N^\alpha \cap N^\beta \simeq 1$ or Z_3, so that $z^G \cap G_{\alpha\beta} = z^G \cap N^\alpha_\beta = z^{N^\alpha_\beta}$ for an involution $z \in N^\alpha_\beta$. Therefore $C_G(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

(a) $C_G(z)^{F(z)} \leq A\Gamma L(1,2^c)$ for some integer $c \geq 1$.

(b) $C_G(z)^{F(z)} \trianglerighteq PSL(2,p_1)$ $(p_1 \geq 5)$, $r = 1$ and $|C_{N^\alpha}(z) : C_{N^\alpha_\beta}(z)| = p_1$.

(c) $C_G(z)^{F(z)} = R(3)$.

Let T be a Sylow 2-subgroup of N^α_β. Then $z \in T$ and by Lemmas 2.3 and 2.4, $|F(T)| = 1 + |N_{N^\alpha}(T)|r/|N^\alpha_\beta| = r + 1$. By Lemma 2.8(i), $|F(z)| = (q-\varepsilon)r/4 + 1$. Hence $T^{F(z)} \simeq Z_2$ if $q \neq 5$. If $q = 5$, as $PSL(2,5) \simeq PSL(2,4)$, (ii) of our theorem holds by [4]. Therefore we may assume $q \neq 5$.

In the case (a), $(r+1)^2 = 1 + (q-\varepsilon)r/4$ by Lemma 2.5. Hence $r = (q-\varepsilon-8)/4$ and $r \mid n$, so that $q = 11$ or 13 and $r = 1$. Let R be a Sylow

3-subgroup of $G_{\alpha\beta}$. Then $R \simeq Z_3$ and $R \leq N^{\alpha}_{\beta}$ because $G_{\alpha\beta}/N^{\alpha}_{\beta} \simeq G_{\alpha\beta}N^{\alpha}/N^{\alpha} \simeq 1$ or Z_2 and $N^{\alpha}_{\beta} \simeq A_4$. By Lemma 2.8(ii), $|F(R)| = 1 + 12/3 = 5$ and $N_G(R)^{F(R)}$ is doubly transitive. Since $N_{G_\alpha}(R) \simeq D_{12}$ or D_{24} and $|F(R)| = 5$, we have $|N_G(R)|_5 = 5$. Let S be a Sylow 5-subgroup of $N_G(R)$. Then $[S, R] = 1$ as $N_G(R)/C_G(R) \leq Z_2$. Since $5 \nmid |G_{\alpha\beta}|$, $|F(S)| = 0$ or 1. If $|F(S)| = 1$, $F(S) \subseteq F(R)$ and so $5 \mid |F(R)| - 1 = 4$, a contradiction. Therefore S is semi-regular on Ω. But $|\Omega| = 1 + |N^{\alpha} : N^{\alpha}_{\beta}| = 56$ or 92. This is a contradiction.

In the case (b), $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/2 = 4np_1$, where s is the order of $C_{G_\alpha}(z)^{F(z)}$. Hence $p_1 - 1 \mid 8n$. Since $p_1 = (q-\varepsilon)/4$, $p^n - \varepsilon - 4 \mid 32n$ and so we have q = 11,13,19,27 or 37. If $q \neq 27$, by Lemma 2.6, $C_{G_\alpha}(z) \simeq D_{q-\varepsilon}$ or $D_{2(q-\varepsilon)}$ and so $C_{G_{\alpha\beta}}(z)^{F(z)} \simeq Z_2$. Hence $(p_1-1)/2 = 2$. From this q = 19. Let R be a Sylow 3-subgroup of $G_{\alpha\beta}$. By the similar argument as in the case (a), $N_G(R)^{F(R)}$ is doubly transitive and $|F(R)| = 1 + 18/3 = 7$. Hence $7 \mid |G|$. On the other hand $|G| = |\Omega||G_\alpha| = (1 + |N^{\alpha} : N^{\alpha}_{\beta}|)|G_\alpha| = (1 + 18\cdot 19\cdot 20/2\cdot 12)\cdot 2^{i}\cdot 18\cdot 19\cdot 20/2 = 2^{3+i}\cdot 3^{2}\cdot 5\cdot 11\cdot 13\cdot 19$ with $0 \leq i \leq 1$, a contradiction. If q = 27, then $|C_G(z)|_2 = |F(z)|_2 \times |C_{G_\alpha}(z)|_2 = 8 \times |G_\alpha|_2$, while $|\Omega| = 1 + |N^{\alpha} : N^{\alpha}_{\beta}| = 1 + 26\cdot 27\cdot 28/2\cdot 12 = 820 = 2^{2}\cdot 5\cdot 41$ and so $|G|_2 = 4|G_\alpha|_2$. Therefore $|C_G(z)| \nmid |G|$, a contradiction.

In the case (c), $r + 1 = 4$ and $1 + (q-\varepsilon)r/4 = 28$. Hence $r = 3$ and $q = 37$, contrary to $r \mid n$.

7. Case (V)

In this section we assume that $N^{\alpha}_{\beta} \simeq S_4$ and $q \equiv 7,9 \pmod{16}$. We note that $4 \nmid n$.

First we argue that $N^{\alpha}_{\beta} = N^{\alpha} \cap N^{\beta}$. Suppose $N^{\alpha}_{\beta} \neq N^{\alpha} \cap N^{\beta}$. Then $C_G(N^{\alpha}) = 1$ by Lemma 2.2. Since $N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq N^{\alpha}_{\beta}N^{\beta}/N^{\beta} \leq Z_2 \times Z_n$, we have $N^{\alpha} \cap N^{\beta} \simeq A_4$ and $N^{\alpha}_{\beta}/N^{\alpha} \cap N^{\beta} \simeq Z_2$, so that $N^{\beta}_{\alpha}N^{\alpha}_{\beta}/N^{\alpha}_{\beta} \simeq N^{\beta}_{\alpha}/N^{\alpha} \cap N^{\beta} \simeq Z_2$. Hence as

$\mathrm{Out}(S_4) = 1$, $Z(N_\beta^\alpha N_\alpha^\beta) \simeq Z_2$. Set $\langle t_1 \rangle = Z(N_\beta^\alpha N_\alpha^\beta)$ and let $t \in I(N_\alpha^\beta) - I(N^\alpha)$. Since $C_{N^\alpha}(t_1) \geq N_\beta^\alpha \simeq S_4$ and $\langle t \rangle N^\alpha = N_\alpha^\beta N^\alpha$, by Lemma 2.6, we have $C_{N^\alpha}(t) \simeq PGL(2, \sqrt{q})$ and $|F(t)| = 1 + 3(q-\varepsilon)r/8$ by Lemma 2.8.

Let P be a Sylow p-subgroup of $C_{N^\alpha}(t)$. Then $|P| = \sqrt{q}$. If $p \neq 3$, P acts semi-regularly on $F(t) - \{\alpha\}$ and so $\sqrt{q} \mid 3(q-\varepsilon)r/8$. Therefore $\sqrt{q} \mid r$ and so $5^n \leq n^2$ as $p \geq 5$ and $r \mid n$. But obviously $5^n > n^2$ for any positive integer n. This is a contradiction. If $p = 3$, $|P : P_\gamma| = \sqrt{q}/3$ or $\sqrt{q}$ for each $\gamma \in \Omega - \{\alpha\}$. Hence $\sqrt{q}/3 \mid 3(q-\varepsilon)r/8$ and so $q \mid 81r^2$. In particular, $3^n = q \mid 81n^2$. From this, $n \leq 7$. Since $q = 3^n \equiv 7$ or $9 \pmod{16}$, we have $q = 3^2$ or 3^6. If $q = 3^2$, $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 1 + 8\cdot 9 \cdot 10/2\cdot 24 = 16$, a contradiction by [9].
If $q = 3^6$, $|F(t)| = 1 + 273r$ and $|F(t) - \{\alpha\}| \geq |C_{N^\alpha}(t) : C_{N_\beta^\alpha}(t)| \geq |PGL(2,3^3)|/8 = 2457$, contrary to $r \mid 3$. Thus $N_\beta^\alpha = N^\alpha \cap N^\beta$.

Let V be a cyclic subgroup of N_β^α of order 4 and let U be a Sylow 2- subgroup of N_β^α containing V. Then $U = N_{N_\beta^\alpha}(V)$, $|F(V)| = 1 + (q-\varepsilon)r/8$ by Lemma 2.8 and $|F(U)| = 1 + 8\cdot 3r/24 = r + 1$ by Lemmas 2.3 and 2.4. If $q \neq 7, 9$, then $|F(U)| < |F(V)|$ and hence $U^{F(V)} \simeq Z_2$. Suppose $q = 7$ or 9. Then $r = 1$ as $r \mid n$. Hence $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 8$ or 16. By [10], we have a contradiction. Therefore $U^{F(V)} \simeq Z_2$.

Suppose $V^g \leq G_{\alpha\beta}$ for some $g \in G$ and set $\gamma = \alpha^g$. Then $V^g \leq g^{-1}N^\alpha g \cap G_{\alpha\beta} \leq N^\gamma \cap G_{\alpha\beta} \leq N_\alpha^\gamma \cap N_\beta^\gamma \leq N^\alpha \cap N^\beta = N_\beta^\alpha$. As $N_\beta^\alpha \simeq S_4$, $V^g = V^h$ for some $h \in N_\beta^\alpha$. Hence $N_G(V)^{F(V)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

(a) $N_G(V)^{F(V)} \leq A\Gamma L(1,2^c)$.

(b) $N_G(V)^{F(V)} \trianglerighteq PSL(2,p_1)$, $p_1 = (q-\varepsilon)/8 \geq 5$.

(c) $N_G(V)^{F(V)} = R(3)$.

In the case (a), $(r+1)^2 = 1 + (q-\varepsilon)r/8$ by Lemma 2.5 and so $r = (q-\varepsilon-16)/8$ and $r \mid n$. From this $q = 23$ or 25 and $r = 1$. Since $|\Omega| = 1 + |N^\alpha : N^\alpha_\beta| = 2\cdot 127$ or $2\cdot 163$, we have $|G|_2 = 2|G_\alpha|_2$, while $|N_G(V)|_2 = |F(V)|_2\,|N_{G_\alpha}(V)|_2 = 4|G_\alpha|_2$, contrary to $|N_G(V)| \mid |G|$.

In the case (b), $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/4 = 4np_1$, where s is the order of $N_{G_\alpha}(V)^{F(V)}$. Hence $p_1-1 \mid 8n$. From this, $p^n-\varepsilon-8 \mid 64n$ and so $q = 23, 41, 71$ or 73. Since p_1 is a prime and $p_1 = (q-\varepsilon)/8 \geq 5$, $q \neq 23, 71, 73$. Therefore $q = 41$ and $|\Omega| = 1 + |N^\alpha : N^\alpha_\beta| = 1 + 40\cdot 41\cdot 42/2\cdot 24 = 2^2\cdot 359$, so that $|G|_2 = 4|G_\alpha|_2$. Since $N^\alpha_\beta = N^\alpha \cap N^\beta$, $C_G(z)^{F(z)}$ is transitive by Lemma 2.1. On the other hand $|F(z)| = 1 + 40\cdot 9/24 = 16$ by Lemma 2.8(i) and so $|C_G(z)|_2 = 16|C_{G_\alpha}(z)|_2 = 16|G_\alpha|_2$, contrary to $|C_G(z)| \mid |G|$.

In the case (c), $r + 1 = 4$ and $1 + (q-\varepsilon)r/8 = 28$. Hence $r = 3$ and $q = 71$ or 73, contrary to $r \mid n$.

8. Case (VI)

In this section we assume that $N^\alpha_\beta \simeq A_5$ and $q \equiv 3,5 \pmod 8$. In particular, n is odd. If $N^\alpha_\beta \neq N^\alpha \cap N^\beta$, then $N^\alpha \cap N^\beta = 1$, $C_G(N^\alpha) = 1$ and so $N^\alpha_\beta \simeq N^\alpha_\beta N^\beta/N^\beta \leq \mathrm{Out}(N^\beta) \simeq Z_2\times Z_n$, a contradiction. Hence $N^\alpha_\beta = N^\alpha \cap N^\beta$. Let z be an involution in N^α_β and T a Sylow 2-subgroup of N^α_β containing z. Then, by Lemma 2.8 $|F(z)| = 1 + (q-\varepsilon)\,15r/60 = 1 + (q-\varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)| = 1 + 12\cdot 5r/60 = 1 + r$. Since $N^\alpha_\beta = N^\alpha \cap N^\beta$, $z^G \cap G_{\alpha\beta} = z^G \cap N^\alpha_\beta = z^{N^\alpha_\beta}$ and so $C_G(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

(a) $C_G(z)^{F(z)} \leq A\Gamma L(1,2^c)$.

(b) $C_G(z)^{F(z)} \trianglerighteq PSL(2,p_1)$, $p_1 = (q-\varepsilon)/4 \geq 5$.

(c) $C_G(z)^{F(z)} = R(3)$.

In the case (a), by Lemma 2.5, $(q-\varepsilon)/4 = 1$ or $(r+1)^2 = 1 + (q-\varepsilon)r/4$. Hence $q = 5$ or $r = (q-\varepsilon-8)/4 \mid n$. If $q = 5$, then $N^{\alpha}_{\beta} = N^{\alpha}$, a contradiction. Therefore $p^n-\varepsilon-8 \mid 4n$ and so $n = 1$ and $q = 11$ or 13. If $q = 13$, we have $5 \nmid |G_{\alpha}|$, a contradiction. Hence $q = 11$ and $|\Omega| = 1 + |N^{\alpha} : N^{\alpha}_{\beta}| = 1 + 10\cdot 11\cdot 12/2\cdot 60 = 12$. By [9], $G^{\Omega} \simeq M_{11}$, $|\Omega| = 12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/2 = 4np_1$, where s is the order of $C_{G_{\alpha}}(z)^{F(z)}$. Hence $p_1-1 \mid 8n$ and so $p^n-\varepsilon-4 \mid 32n$. From this $q = 19, 27$ or 37. Since $5 \mid |G_{\alpha}|$, $q \neq 27, 37$. Hence $q = 19$ and $|\Omega| = 1 + |N^{\alpha} : N^{\alpha}_{\beta}| = 1 + 18\cdot 19\cdot 20/2\cdot 60 = 2\cdot 29$. Since $G_{\alpha} \simeq PSL(2,19)$ or $PGL(2,19)$, $|G| = |\Omega|\,|G_{\alpha}| = 2\cdot 29\cdot 2^i\cdot 18\cdot 19\cdot 20/2 = 2^{3+i}\cdot 3^2\cdot 5\cdot 19\cdot 29$ with $0 \leq i \leq 1$. Let P be a Sylow 29-subgroup of G. Then P is semi-regular on Ω and $3,5,19 \nmid |N_G(P)|$ because $N_G(P)/C_G(P) \leq Z_4\times Z_7$. Hence $|G : N_G(P)| = 2^j\cdot 3^2\cdot 5\cdot 19$ with $0 \leq j \leq 4$, while $2^j\cdot 3^2\cdot 5\cdot 19 \not\equiv 1 \pmod{29}$ for any j with $0 \leq j \leq 4$, contrary to a Sylow's theorem.

If $C_G(z)^{F(z)} = R(3)$, $r + 1 = 4$ and $1 + (q-\varepsilon)r/4 = 28$ and hence $r = 3$, $q = 37$, contrary to $r \mid n$.

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