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Author(s)	Hiramine, Yutaka
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Osaka University

平 峰 豐

ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS
IN WHICH $\text{socle}(G_\alpha)$ IS NONSOLVABLE

Yutaka Hiramine

1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and $\alpha \in \Omega$. In [8], O'Nan has proved that $\text{socle}(G_\alpha) = A \times N$, where A is an abelian group and N is 1 or a nonabelian simple group. Here $\text{socle}(G_\alpha)$ is the product of all minimal normal subgroups of G_α .

In the previous paper [4], we have studied doubly transitive permutation groups in which N is isomorphic to $\text{PSL}(2, q)$, $\text{Sz}(q)$ or $\text{PSU}(3, q)$ with q even. In this paper we shall prove the following :

Theorem. Let G be a doubly transitive permutation group on a finite set Ω with $|\Omega|$ even and let $\alpha \in \Omega$. If G_α has a normal simple subgroup N^α isomorphic to $\text{PSL}(2, q)$, where q is odd, then one of the following holds.

- (i) G^Ω has a regular normal subgroup.
- (ii) $G^\Omega \cong A_6$ or S_6 , $N^\alpha \cong \text{PSL}(2, 5)$ and $|\Omega| = 6$.
- (iii) $G^\Omega \cong M_{11}$, $N^\alpha \cong \text{PSL}(2, 11)$ and $|\Omega| = 12$.

In the case that G^Ω has a regular normal subgroup, by a result of Hering [3] we have $(|\Omega|, q) = (16, 9)$, $(16, 5)$ or $(8, 7)$.

We introduce some notations :

$F(X)$: the set of fixed points of a nonempty subset X of G

$X(\Delta)$: the global stabilizer of a subset $\Delta (\subseteq \Omega)$ in X

X_Δ : the pointwise stabilizer of Δ in X

X^Δ : the restriction of X on Δ

$m|n$: an integer m divides an integer n

X^H : the set of H -conjugates of X

$|X|_p$: maximal power of p dividing the order of X

$I(X)$: the set of involutions in X

D_m : dihedral group of order m

In this paper all sets and groups are finite.

2. Preliminaries.

Lemma 2.1. Let G be a transitive permutation group on Ω , $\alpha \in \Omega$ and N^α a normal subgroup of G_α such that $F(N^\alpha) = \{\alpha\}$. Let the subgroup $X \leq N^\alpha$ be conjugate in G_α to every group Y which lies in N^α and which is conjugate to X in G . Then $N_G(X)$ is transitive on $\Delta = \{\gamma \in \Omega \mid X \leq N^\gamma\}$.

Proof. Let $\beta \in \Delta$ and let $g \in G$ such that $\beta^g = \alpha$. Then, as $X \leq N^\beta$, $X^g \leq N^{\beta^g} = N^\alpha$. By assumption, $(X^g)^h = X$ for some $h \in G_\alpha$. Hence $gh \in N_G(X)$ and $\alpha^{(gh)^{-1}} = \alpha^{g^{-1}} = \beta$. Obviously $N_G(X)$ stabilizes Δ . Thus Lemma 2.1 holds.

Lemma 2.2. Let G be a doubly transitive permutation group on Ω of even degree and N^α a nonabelian simple normal subgroup of G_α with $\alpha \in \Omega$. If $C_G(N^\alpha) \neq 1$, then $N_\beta^\alpha = N^\alpha \cap N^\beta$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semi-regular on $\Omega - \{\alpha\}$.

Proof. See Lemma 2.1 of [4].

Lemma 2.3. Let G be a transitive permutation group on Ω , H a stabilizer of a point of Ω and M a nonempty subset of G . Then

$$|F(M)| = |N_G(M)| \cdot |M^G \cap H| / |H|.$$

Here $M^G \cap H = \{g^{-1}Mg \mid g^{-1}Mg \subseteq H, g \in G\}$.

Proof. See Lemma 2.2 of [4].

Lemma 2.4. Let G be a doubly transitive permutation group on Ω and N^α a normal subgroup of G_α with $\alpha \in \Omega$. Assume that a subgroup X of N^α satisfies $X^{G_\alpha} = X^{N^\alpha}$. Then the following hold.

$$(i) |F(X) \cap \beta^{N^\alpha}| = |F(X) \cap \gamma^{N^\alpha}| \text{ for } \beta, \gamma \in \Omega - \{\alpha\}.$$

(ii) $|F(X)| = 1 + |F(X) \cap \beta^N| \cdot r$, where r is the number of N^α -orbits on $\Omega - \{\alpha\}$.

Proof. Let $\Gamma = \{\Delta_1, \Delta_2, \dots, \Delta_r\}$ be the set of N^α -orbits on $\Omega - \{\alpha\}$. Since G_α is transitive on $\Omega - \{\alpha\}$ and $G_\alpha \cong N^\alpha$, we have $|\Delta_i| = |\Delta_j|$ for $1 \leq i, j \leq r$. By assumption, $G_\alpha = N_{G_\alpha}(X)N^\alpha$ and so $N_{G_\alpha}(X)$ is transitive on Γ . Hence for each i with $1 \leq i \leq r$ there exists $g \in N_{G_\alpha}(X)$ such that $(\Delta_1)^g = \Delta_i$. Therefore $|F(X) \cap \Delta_i| = |F(X^g) \cap (\Delta_1)^g| = |F(X) \cap \Delta_1|$. Thus (i) holds and (ii) follows immediately from (i).

Lemma 2.5. (Huppert [5]) Let G be a doubly transitive permutation group on Ω . Suppose that $O_2(G) \neq 1$ and G_α is solvable. Then for any involution z in G_α , $|F(z)|^2 = |\Omega|$.

We list now some properties of $\text{PSL}(2, q)$ with q odd which will be required in the proof of our theorem.

Lemma 2.6. ([2], [6], [10]) Set $N = \text{PSL}(2, q)$ and $G = \text{Aut}(N)$, where $q = p^n$ and p is an odd prime. Let z be an involution in N . Then the following hold.

$$(i) |N| = (q-1)q(q+1)/2, I(N) = z^N \text{ and } C_N(z) \simeq D_{q-\varepsilon}, \text{ where } q \equiv \varepsilon \pmod{4}.$$

(ii) If $q \neq 3$, N is a nonabelian simple group and a Sylow r -subgroup of N is cyclic when $r \neq 2, p$.

(iii) If X and Y are cyclic subgroups of N and $|X| = |Y| \neq 2, p$, then X is conjugate to Y in $\langle X, Y \rangle$ and $N_N(X) \simeq D_{q \pm \varepsilon}$.

(iv) If $X \leq N$ and $X \simeq Z_2 \times Z_2$, $N_N(X)$ is isomorphic to A_4 or S_4 .

(v) If $|N|_2 \geq 8$, N has two conjugate classes of four-groups in N .

(vi) There exist a field automorphism f of N of order n and a diagonal automorphism d of N of order 2 and if we identify N with its inner automorphism group, $\langle d \rangle N \simeq \text{PGL}(2, q)$, $\langle f \rangle \langle d \rangle N = G$ and $G/N \simeq Z_2 \times Z_n$.

(vii) $C_N(d) \simeq D_{q+\varepsilon}$ and $C_{\langle d \rangle N}(z) \simeq D_{2(q-\varepsilon)}$.

(viii) Suppose $n = mk$ for some positive integers m, k . Then $C_N(f^m) \simeq \text{PSL}(2, p^m)$ if k is odd and $C_N(f^m) \simeq \text{PGL}(2, p^m)$ if k is even.

(ix) Assume n is even and let u be a field automorphism of order 2. Then $I(G) = I(N) \cup d^N \cup u^{\langle d \rangle N}$. If n is odd, $I(G) = I(N) \cup d^N$.

(x) If H is a subgroup of N of odd index, then one of the following holds :

(1) H is a subgroup of $C_N(z)$ of odd index for some involution $z \in N$.

$z \in N$.

(2) $H \simeq \text{PGL}(2, p^m)$, where $n = 2mk$ and k is odd.

(3) $H \simeq \text{PSL}(2, p^m)$, where $n = mk$ and k is odd.

(4) $H \simeq A_4$ and $q \equiv 3, 5 \pmod{8}$.

(5) $H \simeq S_4$ and $q \equiv 7, 9 \pmod{16}$.

(6) $H \simeq A_5$, $q \equiv 3, 5 \pmod{8}$ and $5 \mid (q-1)q(q+1)$.

Lemma 2.7. Let G, N, d and f be as defined in Lemma 2.6 and H an $\langle f, d \rangle$ -invariant subgroup of N isomorphic to $D_{q-\varepsilon}$. Let W be a cyclic subgroup of $\langle d \rangle H$ of index 2 (cf. (vii) of Lemma 2.6) and set $Y = O_2(W \cap H)$. Then $C_G(Y) = W \cdot C_{\langle f \rangle}(Y)$.

Proof. By (viii) of Lemma 2.6, we can take an involution t satisfying $\langle d \rangle H = \langle t \rangle W$ and $[f, t] = 1$. Since $N_G(Y) = \langle f, d \rangle N_N(Y) = \langle f, d \rangle H$, $C_G(Y) = C_{\langle f \rangle \langle d \rangle H}(Y) = W \cdot C_{\langle f \rangle \times \langle t \rangle}(Y)$. Suppose $ht \in C(Y)$ for some $h \in \langle f \rangle$. Since t inverts Y , h also inverts Y and so h^2 centralizes Y . Hence some nontrivial 2-element $g \in \langle h \rangle$ inverts Y , so that $C_H(g)$ contains no

element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, G^Ω will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume G^Ω has no regular normal subgroup.

Notation. $C^\alpha = C_G(N^\alpha)$, which is semi-regular on $\Omega - \{\alpha\}$ by Lemma 2.2.

Let r be the number of N^α -orbits on $\Omega - \{\alpha\}$.

Since $G_\alpha \supseteq N^\alpha$, $|\beta^{N^\alpha}| = |\gamma^{N^\alpha}|$ for $\beta, \gamma \in \Omega - \{\alpha\}$ and so $|\Omega| = 1 + r \cdot |\beta^{N^\alpha}|$. Hence r is odd and N_β^α is a subgroup of N^α of odd index.

Therefore N_β^α is isomorphic to one of the groups listed in (x) of Lemma 2.6.

Accordingly the proof of our theorem will be divided in six cases.

Lemma 2.8. Let Z be a cyclic subgroup of N_β^α with $|Z| \neq 1, p$. Then

(i) If $|Z| = 2$, $|F(Z)| = 1 + (q-\varepsilon)|I(N_\beta^\alpha)|r/|N_\beta^\alpha|$.

(ii) If $|Z| \neq 2$, $|F(Z)| = 1 + |N_{N^\alpha}(Z)|r/|N_\beta^\alpha(Z)|$.

Proof. It follows from Lemmas 2.3, 2.4 and 2.6(i), (iii).

Lemma 2.9. If $N_\beta^\alpha \not\cong D_{q-\varepsilon}$ and Z is a cyclic subgroup of N_β^α with $|Z| \neq 1, p$ and $N_G(Z)^{F(Z)}$ is doubly transitive. Then $C^\alpha = 1$ and one of the following holds.

(i) $N_G(Z)^{F(Z)} \leq \text{AFL}(1, q_1)$ for some q_1 .

(ii) $C_G(Z)^{F(Z)} \cong \text{PSL}(2, p_1)$, $r = 1$ and $|F(Z)| - 1 = |N_{N^\alpha}(Z) : N_{N_\beta^\alpha}(Z)| = p_1$, where $p_1 (\geq 5)$ is a prime.

(iii) $N_G(Z)^{F(Z)} = \text{R}(3)$, the smallest Ree group, $|F(Z)| = 28$.

Proof. Set $N_G(Z) = L$ and $F(Z) = \Delta$. By Lemma 2.6(iii), $L \cap N^\alpha \cong D_{q \pm \varepsilon}$ and $L \cap N^\alpha = \langle t \rangle Y \supseteq Y \supseteq Z$, where $o(t) = 2$, $Y \cong Z_{(q \pm \varepsilon)/2}$.

If $(L \cap N^\alpha)^\Delta = 1$, then $L \cap N^\alpha = N_\beta^\alpha$ because $L \cap N^\alpha$ is a maximal subgroup of N^α . Since $|N^\alpha : N_\beta^\alpha|$ is odd, $L \cap N^\alpha = N_\beta^\alpha \cong D_{q-\varepsilon}$, contrary to the assumption. Hence $(L \cap N^\alpha)^\Delta \neq 1$ and as $L_\alpha \supseteq L_\alpha \cap N^\alpha$ and $L_\alpha \supseteq Y$,

$(L_\alpha)^\Delta$ has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs :

- (a) L^Δ has a regular normal subgroup
- (b) $L^\Delta \cong \text{PSL}(2, p_1)$, $|\Delta| = p_1 + 1$, where $p_1 (\geq 5)$ is a prime
- (c) $L^\Delta \cong \text{PSU}(3, p_1)$, $p_1 \geq 3$, $|\Delta| = (p_1)^3 + 1$
- (d) $L^\Delta = R(3)$, $|\Delta| = 28$.

Suppose $C^\alpha \neq 1$. Then there exists a subgroup D of C^α of prime order such that $(L_\alpha)^\Delta \triangleright D^\Delta$. Since $[L_\alpha, D] \leq D \cdot L_\Delta \cap C^\alpha = D(L_\Delta \cap C^\alpha) = D$, D is a normal subgroup of L_α . By (i) and (iii) of Lemma 2.6, $G_\alpha = L_\alpha \cdot N^\alpha$ and so D is a normal subgroup of G_α . By Theorem 3 of [1], G_α^Δ has a regular normal subgroup, contrary to the hypothesis. Thus $C^\alpha = 1$.

If (a) occurs, L^Δ is solvable because $L_\alpha/L \cap N^\alpha \cong L_\alpha N^\alpha/N^\alpha \leq \text{Out}(N^\alpha)$ and $L \cap N^\alpha \cong D_{q \pm \varepsilon}$. Hence by [5], (i) holds in this case.

If (b) occurs, we have $Y^\Delta \neq 1$, for otherwise $(L \cap N^\alpha)^\Delta = 1$ and so $N_B^\alpha = L \cap N^\alpha \cong D_{q-\varepsilon}$, a contradiction. Hence $1 \neq C_G(Z)^\Delta \leq L^\Delta$ and so $C_G(Z)^\Delta \cong \text{PSL}(2, p_1)$ and $Y^\Delta \cong Z_{p_1}$. Therefore $|\Delta \cap \beta^{N^\alpha}| = p_1$ and $r = 1$ by Lemma 2.4(ii). Since $|\beta^Y| = p_1$, we have $|\beta^{L \cap N^\alpha}| = p_1$, so that $|L \cap N^\alpha : L \cap N_B^\alpha| = p_1$. Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of $\text{PSU}(3, p_1)$, a Sylow p_1 -subgroup of $(L_\alpha)^\Delta$ is not cyclic, while $(L_\alpha)^\Delta \leq L \cap N^\alpha \cong D_{q \pm \varepsilon}$, a contradiction.

3. Case (I)

In this section we assume that $N_B^\alpha \leq D_{q-\varepsilon}$, where $B \neq \alpha$, $q = p^n$.

(3.1) (i) If $N_B^\alpha \not\cong Z_2 \times Z_2$, $N_{N^\alpha}(N_B^\alpha) = N_B^\alpha$ and $|F(N_B^\alpha)| = r + 1$.

(ii) If $N_B^\alpha \cong Z_2 \times Z_2$, $N_{N^\alpha}(N_B^\alpha) \cong A_4$ and $|F(N_B^\alpha)| = 3r + 1$.

Proof. Put $X = N_{N^\alpha}(N_\beta^\alpha)$. Let S be a Sylow 2-subgroup of N_β^α and Y a cyclic subgroup of N_β^α of index 2.

If $N_\beta^\alpha \not\cong Z_2 \times Z_2$, then $|Y| > 2$ and so Y is characteristic in N_β^α . Hence $X \leq N_{N^\alpha}(Y) \simeq D_{q-\varepsilon}$. From this $[N_X(S), S \cap Y] \leq S \cap Y$ and $O^2(N_X(S))$ stabilizes a normal series $S \supseteq S \cap Y \supseteq 1$, so that $O^2(N_X(S)) \leq C_{N^\alpha}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^\alpha}(S) \leq S$ and hence $N_X(S) = S$. On the other hand by a Frattini argument, $X = N_X(S)N_\beta^\alpha$ and so $X = N_\beta^\alpha$. By Lemma 2.6(i), $(N_\beta^\alpha)^{G_\alpha} = (N_\beta^\alpha)^{N^\alpha}$ and so, by Lemmas 2.3 and 2.4(ii), $|F(N_\beta^\alpha)| = 1 + |F(N_\beta^\alpha) \cap \beta^{N^\alpha}| \cdot r = 1 + |N_\beta^\alpha| \cdot r / |N_\beta^\alpha| = r + 1$. Thus (i) holds.

If $N_\beta^\alpha \cong Z_2 \times Z_2$, $N_{N^\alpha}(N_\beta^\alpha) \simeq A_4$ by Lemma 2.6(iv). Similarly as in the case $N_\beta^\alpha \not\cong Z_2 \times Z_2$, we have $|F(N_\beta^\alpha)| = 3r + 1$.

$$(3.2) \quad N_\beta^\alpha / N^\alpha \cap N^\beta \leq Z_2 \times Z_2.$$

Proof. By Lemma 2.2, it suffices to consider the case $C^\alpha = 1$. Suppose $C^\alpha = 1$. Then $N_\beta^\alpha / N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta / N^\beta \leq \text{Out}(N^\alpha) \simeq Z_2 \times Z_n$ by Lemma 2.6(vi) and hence $(N_\beta^\alpha)' \leq N^\alpha \cap N^\beta$. Since N_β^α is dihedral, $N_\beta^\alpha / (N_\beta^\alpha)' \simeq Z_2 \times Z_2$, so that $N_\beta^\alpha / N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

(3.3) Suppose $N_\beta^\alpha = N^\alpha \cap N^\beta$ and let U be a subgroup of N_β^α isomorphic to $Z_2 \times Z_2$. Then $|F(U)| = 3r + 1$ and $N_G(U)^{F(U)}$ is doubly transitive.

Proof. Set $X = N_G(N_\beta^\alpha)$, $\Delta = F(N_\beta^\alpha)$ and let $\{\Delta_1, \Delta_2, \dots, \Delta_r\}$ be the set of N^α -orbits on $\Omega - \{\alpha\}$. If $g^{-1}N_\beta^\alpha g \leq G_{\alpha\beta}$, then $g^{-1}N_\beta^\alpha g \leq N_\alpha^\gamma \cap N_\beta^\gamma = N_\gamma^\alpha \cap N_\gamma^\beta \leq N_\beta^\alpha$, where $\gamma = \alpha^g$. By a Witt's theorem, X^Δ is doubly transitive.

If U is a Sylow 2-subgroup of N_β^α , by a Witt's theorem, $N_G(U)^{F(U)}$ is doubly transitive. Moreover $N_{N^\alpha}(U) \simeq A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)| = 1 + |A_4| \cdot |N_\beta^\alpha : N_{N^\alpha}(U)| \cdot r / |N_\beta^\alpha| = 3r + 1$.

If $|N_\beta^\alpha|_2 > 4$, by Lemma 2.6(iv) and (v), $N_{N^\alpha}(U) \simeq S_4$ and N_β^α has two conjugate classes of four-groups, say $\pi = \{K_1, K_2\}$. Set $X_\pi = M$. Then

$M \cong N_B^\alpha$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_i \neq \emptyset$ for each i and so $|F(U) \cap \Delta_i| = 3$ by Lemma 2.3. Hence $|F(U)| = 3r + 1$. Since $N_{N^\beta}(U) \cong S_4$, we may assume $r > 1$. Hence by (3.1)(i) $|\Delta| = r + 1 \geq 4$, so that M^Δ is doubly transitive. Since $M = N_B^\alpha N_M(U)$, $N_M(U)^\Delta$ is also doubly transitive and so $N_{M^\alpha}(U)$ is transitive on $\Delta - \{\alpha\}$. As $|\Delta \cap \Delta_i| = 1$, $\Delta \cap \Delta_i \subseteq F(U)$ and $N_{N^\alpha}(U)$ is transitive on $F(U) \cap \Delta_i$ for each i , $N_G(U)^{F(U)}$ is doubly transitive.

(3.4) (i) $C^\alpha = 1$.

(ii) Let U be a subgroup of N_B^α isomorphic to $Z_2 \times Z_2$. If $N_B^\alpha = N^\alpha \cap N^\beta$, then $N_G(U)^{F(U)}$ has a regular normal 2-subgroup. In particular $|F(U)| = 3r + 1 = 2^b$ for some positive integer b .

Proof. Since $N_{G_\alpha}(U)^{F(U)} \cong N_{N^\alpha}(U)^{F(U)} \cong S_3$ or Z_3 , by (3.3) and Theorem 3 of [1], $N_G(U)^{F(U)}$ has a regular normal subgroup, $N_G(U)^{F(U)} \cong \text{PSU}(3,3)$ or $N_G(U)^{F(U)} = R(3)$.

Suppose $C^\alpha \neq 1$. Let D be a minimal characteristic subgroup of C^α . Clearly $G_\alpha \triangleright D$. If $N_G(U)^{F(U)} \neq R(3)$, D is cyclic. By Theorem 3 of [1], G^Ω has a regular normal subgroup, contrary to the hypothesis. Hence $N_G(U)^{F(U)} = R(3)$. Therefore $(N_{G_\alpha}(U)^{F(U)})'$ contains an element of order 9. Since $N_{G_\alpha}(U)/C^\alpha N_{N^\alpha}(U) \cong N_{G_\alpha}(U)C^\alpha N^\alpha / C^\alpha N^\alpha \leq \text{Out}(N^\alpha)$, by (vi) of Lemma 2.6 we have $(N_{G_\alpha}(U))' \leq C^\alpha \times N_{N^\alpha}(U)$. From this, C^α contains an element of order 9 and so $C^\alpha \cong Z_9$ or $M_3(3)$. In both cases, C^α contains a characteristic subgroup of order 3. Since $G_\alpha \triangleright D$, by Theorem 3 of [1] G^Ω has a regular normal subgroup, a contradiction. Thus $C^\alpha = 1$.

Let R be a Sylow 3-subgroup of $N_{G_\alpha}(U)$. Since $N_{G_\alpha}(U)/N_{N^\alpha}(U) \cong N_{G_\alpha}(U)N^\alpha/N^\alpha \leq \text{Out}(N^\alpha) \cong Z_2 \times Z_n$, $R/R \cap N_{N^\alpha}(U)$ is cyclic. Clearly $R \cap N_{N^\alpha}(U) \cong Z_3$. Therefore $N_G(U)^{F(U)} \not\cong \text{PSU}(3,3), R(3)$. Thus (3.4) holds.

Since N_B^α is dihedral, we set $N_B^\alpha = \langle t \rangle W$ and $Y = W \cap N^\alpha \cap N^\beta$, where W

is a cyclic subgroup of N_β^α of index 2 and t is an involution in N_β^α which inverts W .

(3.5) (i) If $|Y| \geq 3$, $N_G(Y)^{F(Y)}$ is doubly transitive.

(ii) If $|Y| < 3$, $N_\beta^\alpha \simeq Z_2 \times Z_2$ or $N_\beta^\alpha \simeq D_8$ and $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

Proof. Suppose $|Y| \geq 3$. If $Y^g \leq G_{\alpha\beta}$, $Y^g \leq N^\gamma \cap G_{\alpha\beta} \leq N_\alpha^\gamma$, where $\gamma = \alpha^g$. If $\gamma = \alpha$, obviously $Y^g \leq N^\alpha$. If $\gamma \neq \alpha$, $N_\alpha^\gamma \simeq N_\beta^\alpha$. Therefore, as $|Y| \geq 3$, N_α^γ has a unique cyclic subgroup of order $|Y|$. Hence $Y^g \leq N^\gamma \cap N^\alpha \leq N^\alpha$, so that $Y^g \leq N^\alpha$. Similarly $Y^g \leq N^\beta$. Thus $Y^g \leq N^\alpha \cap N^\beta$ and so $Y^g = Y$.

By a Witt's theorem, $N_G(Y)$ is doubly transitive on $F(Y)$.

Suppose $|Y| < 3$. Since $|N^\alpha \cap N^\beta : Y| \leq 2$, we have $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

On the other hand, as N_β^α is dihedral, $(N_\beta^\alpha)'$ is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set $\Delta = F(N_\beta^\alpha)$, $L = G(\Delta)$, $K = G_\Delta$ and suppose $N_\beta^\alpha \not\leq Z_2 \times Z_2$. Then $L_\alpha \triangleq N_\beta^\alpha$, $(L_\alpha)' \leq N_\beta^\alpha$, $K' \leq N^\alpha \cap N^\beta$ and $(L_\alpha)^\Delta \simeq Z_r$. If $r \neq 1$, L^Δ is a doubly transitive Frobenius group of degree $r + 1$.

Proof. By Corollary B1 of [7] and (i) of (3.1), L^Δ is doubly transitive and $|\Delta| = r + 1$. Since $N^\alpha \cap L \triangleq N^\alpha \cap K = N_\beta^\alpha$, by (i) of (3.1), we have $N^\alpha \cap L = N_\beta^\alpha$. Hence $L_\alpha \triangleq N_\beta^\alpha$. By (i) of (3.4), $L_\alpha/N_\beta^\alpha \simeq L_\alpha N^\alpha/N^\alpha \leq \text{Out}(N^\alpha) \simeq Z_2 \times Z_n$ and so $(L_\alpha)' \leq N_\beta^\alpha$ and $(L_\alpha)^\Delta \simeq Z_r$. If $r \neq 1$, then $(L_\alpha)^\Delta \neq 1$. On the other hand $(L_{\alpha\beta})^\Delta = 1$ as $(L_\alpha)^\Delta$ is abelian. Hence L^Δ is a Frobenius group.

(3.7) Suppose $|Y| \geq 3$. Then there exists an involution z in $N_\beta^\alpha \cap Y$ such that $Z(N_\beta^\alpha) = \langle z \rangle$.

Proof. Since $N_\beta^\alpha \not\leq Z_2 \times Z_2$, $|N_\beta^\alpha|_2 \geq 2^2$ and N_β^α is dihedral, we have $\langle I(W) \rangle = Z(N_\beta^\alpha) \simeq Z_2$ and $N_\beta^\alpha / (N_\beta^\alpha)' \simeq Z_2 \times Z_2$. Let $Z(N_\beta^\alpha) = \langle z \rangle$ and suppose that z is not contained in Y . By (3.2), $(N_\beta^\alpha)' \leq N^\alpha \cap N^\beta \cap W = Y$ and so $|(N_\beta^\alpha)'|$

is odd. Hence $|N_\beta^\alpha|_2 = 4$ and $q \equiv p^n = 3$ or $5 \pmod{8}$, so that n is odd. By (3.2) and (i) of (3.4), $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta/N^\beta \simeq 1$ or Z_2 . If $N_\beta^\alpha = N^\alpha \cap N^\beta$, then $W = Y$ and so $z \in Y$, contrary to the assumption. Therefore we have $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq Z_2$ and $N_\beta^\alpha = \langle z \rangle \times (N^\alpha \cap N^\beta)$. Since n is odd and $z \in N_\beta^\alpha N^\beta - N^\beta$, by Lemma 2.6(vi), (vii) and (ix), $N_\beta^\alpha N^\beta \simeq \text{PGL}(2, q)$ and $C_{N^\beta}(z) \simeq D_{q+\varepsilon}$. But $N^\alpha \cap N^\beta \leq C_{N^\beta}(z)$ and besides it is isomorphic to a subgroup of $D_{q-\varepsilon}$. Hence $N^\alpha \cap N^\beta \simeq Z_2$ and $N_\beta^\alpha \simeq Z_2 \times Z_2$, a contradiction.

(3.8) Suppose $|Y| \geq 3$. Then $N_\beta^\alpha = N^\alpha \cap N^\beta$.

Proof. Suppose $N_\beta^\alpha \neq N^\alpha \cap N^\beta$ and let Δ, L, K be as defined in (3.6) and $x \in L_\alpha$ such that its order is odd and $\langle x \rangle$ is transitive on $\Delta - \{\alpha\}$. As $|Y| \geq 3$, W is characteristic in N_β^α and hence by (3.6), x stabilizes a normal series $L_\alpha \supseteq N_\beta^\alpha \supseteq W \supseteq (N_\beta^\alpha)'$. By Theorem 5.3.2 of [2], $[x, O_2(L_\alpha/(N_\beta^\alpha)')] = 1$. Since $L_\alpha/(N_\beta^\alpha)'$ has a normal Sylow 2-subgroup and $(N_\beta^\alpha)' \leq K'$, we have $[x, O_2(L_\alpha/K')] = 1$, so that $[x, N_\beta^\alpha] \leq K' \leq N^\alpha \cap N^\beta$ by (3.6). If $r \neq 1$, then $\beta^x \neq \beta$ and $\beta^x \in \Delta$, hence $N_\alpha^\beta = x^{-1}N_\alpha^\beta x = N_\alpha^\gamma$, where $\gamma = \beta^x$. Since $\gamma \in \Delta$ and $\Delta = F(N_\alpha^\beta)$, $N_\alpha^\beta \leq N^\beta \cap G_\gamma = N_\gamma^\beta$ and so $N_\alpha^\beta = N_\gamma^\beta$. Similarly $N_\alpha^\gamma = N_\beta^\gamma$. Hence $N_\beta^\alpha = N_\beta^\gamma$, which implies $N_\beta^\alpha = N^\beta \cap N^\gamma$. By the doubly transitivity of G , we have $N_\beta^\alpha = N^\alpha \cap N^\beta$, contrary to the assumption. Therefore we obtain $r = 1$.

Let z be as defined in (3.7) and put $k = (q-\varepsilon)/|N_\beta^\alpha|$. By Lemma 2.8(i) we have $|F(z)| = 1 + (q-\varepsilon)(|N_\beta^\alpha|/2 + 1)/|N_\beta^\alpha| = (q-\varepsilon)/2 + k + 1$. Similarly $|F(Y)| = k + 1$. As $N_\beta^\alpha \neq N^\alpha \cap N^\beta$, there is an involution t in N_β^α which is not contained in N^β . By Lemma 2.6(i), $t^y = z$ for some $y \in N^\alpha$. Set $\gamma = \beta^y$. Then $\gamma \in F(z)$ and $z \notin N^\gamma$. By Lemma 2.6(vii), (viii) and (ix), $C_{N^\gamma}(z) \simeq D_{q+\varepsilon}$ or $\text{PGL}(2, \sqrt{q})$. Assume $C_{N^\gamma}(z) \simeq D_{q+\varepsilon}$ and let R be a cyclic subgroup of $C_{N^\gamma}(z)$ of index 2. We note that R is semi-regular on $\Omega - \{\alpha\}$. Set $X = C_G(z)$. Since $2 \leq k + 1 \leq (q-\varepsilon)/|q-\varepsilon|_2 + 1$, we have $(q+\varepsilon)/2 \nmid k + 1$ and so $|\alpha^X| > k + 1$. By (i) of (3.5) and (3.7), $N_G(Y) \leq C_G(z) = X$ and

$\alpha^X \geq F(Y)$. It follows from Lemma 2.1 that $\alpha^X = \{\mu \mid z \in N^\mu\} \not\cong \gamma$. Hence $|F(z)| > |\alpha^X| \geq |F(Y)| + (q+\varepsilon)/2 = k + 1 + (q-\varepsilon)/2 + \varepsilon = |F(z)| + \varepsilon$.

Therefore $\varepsilon = -1$ and $\gamma^X = \{\gamma\}$, so that $\gamma \in F(Y)$, a contradiction. Thus

$C_{N\delta}(z) \simeq \text{PGL}(2, \sqrt{q})$, $\varepsilon = 1$, $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq Z_2$ and $|\langle z^G \cap G_\alpha \rangle : N^\alpha| = 2$.

Set $\Delta_1 = \alpha^X$ and $\Delta_2 = F(z) - \Delta_1$. Let $\delta \in \Delta_2$ and g an element of G satisfying $\delta^g = \gamma$. Then $z \in N_\delta^\alpha N^\delta - N^\delta$ and so $z^g \in N_\delta^\gamma N^\delta - N^\delta$, where $\gamma = \alpha^g$. Since $|\langle z^G \cap G_\gamma \rangle : N^\delta| = 2$ and $z \in G_\gamma - N^\delta$, it follows from Lemma 2.6(ix) that $(z^g)^h = z$ for some $h \in G_\gamma$. Hence $gh \in X$ and $\delta^{gh} = \gamma$. Thus $\Delta_2 = \gamma^X$.

Let $\delta \in \Delta_2$. Then $z \in N_\delta^\alpha$ and $z \notin Z(N_\delta^\alpha)$ by (3.7) and so $X \cap N_\delta^\alpha \simeq Z_2 \times Z_2$, which implies $|C_{N^\alpha}(z)| = (q-1)/4$. Hence $(|\Delta_1|, |\Delta_2|) = ((q-1)/4 + k + 1,$

$(q-1)/4)$ or $(k + 1, (q-1)/2)$. Let P be a subgroup of $C_{N\delta}(z)$ of order \sqrt{q} . Then $F(P) = \{\delta\}$ and P is semi-regular on $\Omega - \{\delta\}$. If $|\Delta_2| = (q-1)/4$,

then $\sqrt{q} \mid (q-1)/4 - 1 = (q-5)/4$ and $\sqrt{q} \mid (q-1)/4 + k + 1$. From this, $q = 5^2$, $k = 3$, $|\Delta_1| = 10$ and $|\Delta_2| = 6$. Since $(C_{N\delta}(z))^{\Delta_2} \simeq S_5$, $X^{\Delta_2} \simeq S_6$ and so

$|X|_3 \geq 3^2$. As X acts on Δ_1 and $|\Delta_1| \equiv 1 \pmod{3}$, $|G_\alpha|_3 \geq |X_\alpha|_3 \geq 3^2$, contrary to $N^\alpha \simeq \text{PSL}(2, 25)$. If $|\Delta_2| = (q-1)/2$, $\sqrt{q} \mid (q-1)/2 - 1 = (q-3)/2$, so

$q = 3^2$, $k = 1$, $N_\beta^\alpha \simeq D_8$ and $\Delta_1 = \{\alpha, \beta\}$. Hence $C_{N\delta}(z)$ fixes α and β , so that $\text{PGL}(2, 3) \simeq C_{N\delta}(z) \leq N_\alpha^\delta \simeq N_\beta^\alpha \simeq D_8$, a contradiction.

(3.9) Suppose $|Y| \geq 3$. Then $r = 1$.

Proof. By (3.6), $r + 1 = 2^c$ for some integer $c \geq 0$. On the other hand $3r + 1 = 2^b$ by (3.8) and (ii) of (3.4). Hence $2r = 2^c(2^{b-c} - 1)$ and so $c = 1$ as r is odd. Thus $r = 1$.

(3.10) Put $k = (q-\varepsilon)/|N_\beta^\alpha|$. If $N_\beta^\alpha = N^\alpha \cap N^\beta$ and $r = 1$, then

$$q-\varepsilon + 2k + 2 \mid 2((2k + 2 - \varepsilon)(k + 1 - \varepsilon)k + 1)(2k + 2 - \varepsilon)(k + 1 - \varepsilon).$$

Proof. Set $S = \{(\gamma, u) \mid \gamma \in F(u), u \in z^G\}$, where z is an involution in N_β^α . We now count the number of elements of S in two ways. Since

$N_B^\alpha = N^\alpha \cap N^\beta$, $F(z) = \{\gamma \mid z \in N^\gamma\}$ and hence $C_G(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S| = |\Delta| |z^{G_\alpha}| = |z^G| |F(z)|$. Since $r = 1$, $|\Delta| = 1 + |N^\alpha : N_B^\alpha| = kq(q+\varepsilon)/2 + 1$ and by Lemma 2.8 $|F(z)| = (q-\varepsilon)/2 + k + 1$. Since $G_\alpha \supseteq N^\alpha$, z^{G_α} is contained in N^α and so $|G_\alpha : C_{G_\alpha}(z)| = |N^\alpha : C_{N^\alpha}(z)| = q(q+\varepsilon)/2$. Hence $(q-\varepsilon)/2 + k + 1 \mid (kq(q+\varepsilon)/2 + 1)q(q+\varepsilon)/2$. On the other hand $|F(z)|_2 = |C_G(z)|_2 / |C_{G_\alpha}(z)|_2 \leq |G|_2 / |C_{G_\alpha}(z)|_2 = |G|_2 / |G_\alpha|_2 = |\Delta|_2$ because $|G_\alpha : C_{G_\alpha}(z)| = q(q+\varepsilon)/2 \equiv 1 \pmod{2}$. Hence $|q-\varepsilon + 2k + 2|_2 \leq |kq(q+\varepsilon) + 2|_2$. Since $kq(q+\varepsilon) + 2 = (kq + 2k(\varepsilon - k - 1))(q-\varepsilon + 2k + 2) + 2((2k + 2 - \varepsilon)(k + 1 - \varepsilon)k + 1)$ and $q(q+\varepsilon) = (q+2\varepsilon - 2k - 2)(q-\varepsilon + 2k + 2) + 2(2k + 2 - \varepsilon)(k + 1 - \varepsilon)$, we have (3.10).

(3.11) Suppose $|Y| \geq 3$. Then one of the following holds.

(i) $N_B^\alpha = N^\alpha \cap N^\beta \simeq D_{q-\varepsilon}$.

(ii) $N_B^\alpha = N^\alpha \cap N^\beta \not\cong D_{q-\varepsilon}$ and $N_G(Y)^{F(Y)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(Y)} = R(3)$ or there exists a prime $p_1 \geq 5$ such that $C_G(Y)^{F(Y)} \supseteq \text{PSL}(2, p_1)$ and $V/Y \simeq Z_{p_1}$, where $V = C_{N^\alpha}(Y)$. By (i) of (3.1) and (3.9), $F(N_B^\alpha) = \{\alpha, \beta\}$. On the other hand, $(N_B^\alpha)^{F(Y)} \simeq N_B^\alpha/Y \simeq Z_2$. Hence $N_G(Y)^{F(Y)} \neq R(3)$ and $C_G(Y)^{F(Y)} \supseteq \text{PSL}(2, p_1)$.

By (i) of (3.4) and Lemma 2.7, we have $C_{G_\alpha}(Y) = V \langle f_1 \rangle$, where f_1 is a field automorphism of N^α . Let t be the order of f_1 , $n = tm$ and let $p^m \equiv \varepsilon_1 \in \{\pm 1\} \pmod{4}$. Clearly $C_{G_\alpha}(Y)^{F(Y)} \supseteq V^{F(Y)} \simeq Z_{p_1}$ and $|C_{G_\alpha}(Y)^{F(Y)}| \mid t$, so that $(p_1-1)/2 \mid t$.

First we assume that t is even and set $t = 2t_1$. Then $Y \leq C_{N^\alpha}(f_1) \simeq \text{PGL}(2, p^m)$ by Lemma 2.6(viii). As $|V/Y| = p_1$ and p_1 is a prime, Y is a cyclic subgroup of $C_{N^\alpha}(f_1)$ of order $p^m - \varepsilon_1$ and $(p^n-1)/2(p^m - \varepsilon_1) = p_1$.

Put $s = \sum_{i=0}^{t_1-1} (p^{2m})^i$. Then $(p^m + \varepsilon_1)s/2 = p_1$, so that we have either

(i) $t_1 = 1$ and $p_1 = (p^m + \varepsilon_1)/2$ or (ii) $t_1 \geq 2$, $p^m = 3$ and $p_1 = s$.

In the case (i), $2 \leq (p_1 - 1)/2 = (p^m + \varepsilon_1 - 2)/4 \mid 2t_1 = 2$. Hence $(p_1, q) = (5, 3^4)$ or $(5, 11^2)$. Let z be as in (3.7). As mentioned in the proof of

(3.10), $|F(z)| = (q-1)/2 + k + 1$, $|\Delta| = kq(q+1)/2 + 1$ and $C_G(z)$ is transitive on $F(z)$. If $q = 3^4$, then $|F(z)| = 46$ and $|\Delta| = 2 \cdot 19^2 \cdot 23$.

Hence $|C_G(z)| = |F(z)| |C_{G_\alpha}(z)| = |F(z)| |C_{G_\alpha}(z)^{N^\alpha/N^\alpha}| |C_{N^\alpha}(z)| = 46 \cdot 2^i \cdot 80 = 2^{5+i} \cdot 5 \cdot 23$ with $0 \leq i \leq 3$. Let P be a Sylow 23-subgroup of $C_G(z)$ and Q a Sylow 5-subgroup of $C_G(z)$. It follows from a Sylow's theorem

that P is a normal subgroup of $C_G(z)$ and so $[P, Q] = 1$. Therefore $|F(Q)| \geq 23$, contrary to $5 \nmid |N_\beta^\alpha|$.

If $q = 11^2$, then $|F(z)| = 66$ and $|\Delta| = 2 \cdot 3 \cdot 6151$. Let P be a Sylow 11-subgroup of $C_G(z)$. Since $11 \nmid |\Delta|$, P is a subgroup of N^γ for some $\gamma \in \Delta$ and $F(P) = \{\gamma\}$. Hence $\gamma \in F(z)$, so that

$z \in N^\gamma$, contrary to $C_{N^\gamma}(z) \simeq D_{120}$. In the case(ii), we have

$(p_1 - 1)/2 = (\sum_{i=1}^{t_1-1} 9^i)/2 \mid t = 2t_1$. From this, $9^{t_1-1} \leq 4t_1$, hence $t_1 = 1$, a contradiction.

Assume t is odd. Then $Y \leq C_{N^\alpha}(f_1) \simeq \text{PSL}(2, p^m)$ by Lemma 2.6(viii).

As $|V/Y| = p_1$ and p_1 is a prime, $Y \simeq Z_{(p^m - \varepsilon_1)/2}$ and $(q - \varepsilon)/(p^m - \varepsilon_1) = p_1$.

Hence $\sum_{i=0}^{t-1} (p^m)^i (\varepsilon_1)^{t-1-i} = p_1$ and $(p_1 - 1)/2 = ((\sum_{i=1}^{t-1} (p^m)^i (\varepsilon_1)^{t-1-i}) - 1)/2 \mid t$.

In particular $2t \geq (p^m)^{t-1} - (p^m)^{t-2} = (p^m - 1)(p^m)^{t-2} \geq 2(p^m)^{t-2}$. From this

$t = 3$, $m = 1$, $p_1 = 7$ and $q = 3^3$, so that $N_\beta^\alpha \simeq Z_2 \times Z_2$, a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let G^Δ be a minimal counterexample to (3.12) and M a minimal normal subgroup of G . By the hypothesis, G has no regular normal subgroup and hence $M_\alpha \neq 1$. As M_α is a normal subgroup of G_α , by (i) of (3.4),

M_α contains N^α . By (3.9), $r = 1$, hence M is doubly transitive on Δ .

Therefore $G = M$ and G is a nonabelian simple group.

Since $N_B^\alpha \cong D_{q-\varepsilon}$, $k = 1$ and so $q-\varepsilon+4 \mid 2((4-\varepsilon)(2-\varepsilon)+1)(4-\varepsilon)(2-\varepsilon)$

by (3.10). Hence we have $q = 7, 9, 11, 19, 27$ or 43 .

Let x be an element of N_B^α . If $|x| > 2$, by Lemma 2.8, $|F(x)| = 1 +$

$|N_B^\alpha| \cdot 1/|N_B^\alpha| = 2$ and if $|x| = 2$, similarly we have $|F(x)| = (q-\varepsilon)/2 + 2$.

Assume $q \neq 9$ and let d be an involution in $G_\alpha - N^\alpha$ such that $\langle d \rangle N^\alpha$ is isomorphic to $PGL(2, q)$. We may assume $d \in G_{\alpha\beta}$. Since $\langle d \rangle N^\alpha$ is transitive on $\Omega - \{\alpha\}$, by Lemmas 2.3 and 2.6(vii), (ix), $|F(d)| = 2(q-1)(q+1/2)/2(q+1) + 1 = (q+1)/2$, while $|F(x)| = (q+1)/2 + 2$ for $x \in I(N^\alpha)$. Hence d is an odd permutation, contrary to the simplicity of G . Thus $G_\alpha = N^\alpha$ if $q \neq 9, 27$ and $|G_\alpha/N^\alpha| = 1, 3$ if $q = 27$.

If $q = 9$, $|\Omega| = 1 + |N^\alpha : N_B^\alpha| = 1 + 9 \cdot 10/2 = 2 \cdot 23$ and $|G_\alpha| = 2^i |PSL(2, 9)| = 2^{3+i} \cdot 3^2 \cdot 5$ with $0 \leq i \leq 2$. Let P be a Sylow 23-subgroup of G . Since $\text{Aut}(Z_{23}) \cong Z_2 \times Z_{11}$, $3 \nmid |N_G(P)|$, for otherwise P centralizes a nontrivial 3-element x and so $F(P) \supseteq F(x)$ because $|F(x)| = 1$, contrary to $|F(P)| = 0$. Similarly $5 \nmid |N_G(P)|$. Hence $|G : N_G(P)| = 2^a \cdot 3^b \cdot 5$ for some a with $0 \leq a \leq 6$. By a Sylow's theorem, $2^a \cdot 3^2 \cdot 5 \equiv -2^a \equiv 1 \pmod{23}$, a contradiction.

If $q = 27$, $|\Omega| = 1 + 27 \cdot 26/2 = 2^5 \cdot 11$ and $|G_\alpha| = 2^2 \cdot 3^{3+i} \cdot 7 \cdot 13$ with $0 \leq i \leq 1$. Let P be a Sylow 11-subgroup of G . Since $P \cong Z_{11}$ and $\text{Aut}(Z_{11}) \cong Z_2 \times Z_5$, $3^{1+i}, 7, 13 \nmid |N_G(P)|$ by the similar argument as above. Hence $|G : N_G(P)| = 2^a \cdot 3^b \cdot 7 \cdot 13$ with $0 \leq a \leq 7$ and $3 \leq b \leq 3+i$. By a Sylow's theorem, $2^a \cdot 3^b \cdot 7 \cdot 13 = 2^a \cdot 3^{b-3} \cdot 3^3 \cdot 7 \cdot 13 \equiv 2^a \cdot 3^{b-3} \cdot 4 \equiv 1 \pmod{11}$. Hence $a = 0, b = 4$. Therefore $N_G(P)$ contains a Sylow 2-subgroup S of G . Let T be a Sylow 2-subgroup of N_B^α and g an element such that $T^g \leq S$. Then $T^g \cap C_G(P) \neq 1$ as $N_S(P)/C_S(P) \leq Z_2$. Let u be an involution in $T^g \cap C_G(P)$. Then $|F(u)| = (27+1)/2 + 2 = 16$, while $11 \nmid |F(u)|$ because $[P, u] = 1$ and $|F(P)| = 0$, a contradiction.

If $q = 7, 11, 19$ or 43 , then $G_\alpha = N^\alpha$ and $\varepsilon = -1$. Set $\Gamma = \{\{\gamma, \delta\} \mid$

$\gamma, \delta \in \Omega, \gamma \neq \delta$. We consider the action of G on Γ . Since G^Ω is doubly transitive, G^Γ is transitive and $G_\Gamma = 1$. Let z be an involution of $Z(N_\beta^\alpha)$. There exists an involution t such that $t \in z^G$ and $\alpha^t = \beta$. Since $G_{\alpha\beta} = N_\beta^\alpha$ and $F(N_\beta^\alpha) = \{\alpha, \beta\}$ we have $G_{\{\alpha, \beta\}} = \langle t \rangle N_\beta^\alpha$. By Lemma 2.3, $|F(z^\Gamma)| = |C_G(z)| \times |\langle t \rangle N_\beta^\alpha \cap z^G|/2|N_\beta^\alpha| = |F(z)| \times |C_{G_\alpha}(z)| \times |\langle t \rangle N_\beta^\alpha \cap z^G|/2|N_\beta^\alpha| = |F(z)| \times |\langle t \rangle N_\beta^\alpha \cap z^G|/2$. As $|F(z^\Gamma)| = |F(z)|(|F(z)| - 1)/2 + (|\Omega| - |F(z)|)/2$, $|\langle t \rangle N_\beta^\alpha \cap z^G| = |F(z)| + |\Omega|/|F(z)| - 2$. In particular $|F(z)| \mid |\Omega|$. Since $|F(z)| = (q+1)/2 + 2 = (q+5)/2$ and $|\Omega| = 1 + q(q-1)/2 = (q^2 - q + 2)/2$, we have $q = 11$ and $|\langle t \rangle N_\beta^\alpha \cap z^G| = 13$. Moreover $|\Omega| = 56$, $|G_\alpha| = |\text{PSL}(2, 11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$ and $|G| = 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.

We now argue that $\langle t \rangle N_\beta^\alpha \simeq D_{24}$. Let R be the Sylow 3-subgroup of N_β^α . If t centralizes R , R acts on $F(t)$ and so $F(R) \subseteq F(t)$ as $|F(t)| = 8$ and $|F(R)| = 2$. Hence $\alpha^t = \alpha$, contrary to the choice of t . Therefore t inverts R and $\langle t \rangle N_\beta^\alpha$ is isomorphic to $Z_2 \times D_{12}$ or D_{24} . Suppose $\langle t \rangle N_\beta^\alpha \simeq Z_2 \times D_{12}$. Then $\langle t \rangle N_\beta^\alpha$ contains fifteen involutions and so we can take $u \in I(\langle t \rangle N_\beta^\alpha)$ satisfying $|F(u)| = 0$ and $\langle t \rangle N_\beta^\alpha = \langle u \rangle \times N_\beta^\alpha$. As $|F(u)| = 0$, $|F(u^\Gamma)| = |\Omega|/2 = 28$. By Lemma 2.3, $28 = |C_G(u)| \times |\langle u \rangle N_\beta^\alpha \cap u^G|/24$ and hence $|C_G(u)| = 2^4 \cdot 3 \cdot 7$ or $2^5 \cdot 3 \cdot 7$. Since $\langle u \rangle N_\beta^\alpha = N_G(R)$, we have $|C_G(u) : C_G(u) \cap N_G(R)| = 2 \cdot 7$ or $2^2 \cdot 7$. By a Sylow's theorem, $|C_G(u) : C_G(u) \cap N_G(R)| = 2^2 \cdot 7$, so that $|C_G(u)| = 2^5 \cdot 3 \cdot 7$. Let Q be a Sylow 7-subgroup of $C_G(u)$. Then $|C_G(u) \cap N_G(Q)| = 2^5 \cdot 3 \cdot 7$ or $2^2 \cdot 3 \cdot 7$ by a Sylow's theorem. Hence $2^2 \cdot 3 \cdot 7 \mid |N_G(Q)|$. Since $\text{Aut}(Z_7) \simeq Z_2 \times Z_3$, $5 \nmid |N_G(Q)|$ and $11 \nmid |N_G(Q)|$ by the similar argument as in the case $q = 9$. Therefore $|G : N_G(Q)| = 2^a \cdot 5 \cdot 11$ for some a with $0 \leq a \leq 3$. Hence $|G : N_G(Q)| \not\equiv 1 \pmod{7}$, a contradiction. Thus $\langle t \rangle N_\beta^\alpha \simeq D_{24}$.

Let U be a Sylow 2-subgroup of N_β^α and set $L = N_G(U)$. It follows from (3.3) and Lemma 2.6(iv) that $L \cap N^\alpha \simeq A_4$, $L^{F(U)} \simeq A_4$ and $|L| = 2^4 \cdot 3$. Let $T, \langle x \rangle$ be Sylow 2- and 3-subgroup of L , respectively. Obviously $L \triangleright T$ and $C_T(x) = 1$. On the other hand $T \trianglelefteq L \cap \langle t \rangle N_\beta^\alpha \simeq D_8$ and so $T' \simeq Z_2 \times Z_2$

because $C_T(x) = 1$. By Theorem 5.4.5 of [2], T is dihedral or semi-dihedral. Hence $N_G(T)/C_G(T) (\leq \text{Aut}(T))$ is a 2-group, so that $C_T(x) = T$, a contradiction.

(3.13) (ii) of (3.11) does not occur.

Proof. Let G^Ω be a doubly transitive permutation group satisfying (ii) of (3.11). Let x be an involution in N_β^α with $x \notin Y$. Then $F(x^{F(Y)}) = F(\langle x \rangle Y) = F(N_\beta^\alpha) = \{\alpha, \beta\}$ by (i) of (3.1) and (3.9). Since $|F(Y)| = 1 + (q-\varepsilon)/|N_\beta^\alpha| = 1 + k \geq 4$, $x^{F(Y)}$ is an involution. By Lemma 2.5, $1 + k = 2^2$ and so $k = 3$. By (3.11), $q-\varepsilon+8 \mid 2((8-\varepsilon)(4-\varepsilon) \times 3 + 1)(8-\varepsilon)(4-\varepsilon)$. Hence $q+7 \mid 2^7 \cdot 3 \cdot 7$ if $\varepsilon = 1$ and $q+9 \mid 2^4 \cdot 3^2 \cdot 5 \cdot 17$ if $\varepsilon = -1$. Since $k = 3 \mid q-\varepsilon$, $3 \nmid q-\varepsilon+8$. From this $q+7 \mid 2^7 \cdot 7$ if $\varepsilon = 1$ and $q+9 \mid 2^4 \cdot 5 \cdot 17$ if $\varepsilon = -1$. Therefore $q = 5^2, 7^2, 11^2, 59$ or 71 .

Let p_1 be an odd prime such that $p_1 \mid |\Omega|$ and $p_1 \nmid |G_\alpha|$ and let P be a Sylow p_1 -subgroup of G . Clearly P is semi-regular on Ω and so any element in $C_{G_\alpha}(P)$ has at least p_1 fixed points. If x is an element of N_β^α and its order is at least three, $|F(x)| = |F(Y)| = 4$ by Lemma 2.8. Since $|N_\beta^\alpha| = (q-\varepsilon)/3$, we have $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 1 + 3q(q+\varepsilon)/2$.

If $q = 5^2$, then $|\Omega| = 2^4 \cdot 61$ and $|G_\alpha| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 13$ ($0 \leq i \leq 2$). Let P be a Sylow 61-subgroup of G . Then $P \cong Z_{61}$. As mentioned above, $5, 13 \nmid |C_G(P)|$ and so $5^2, 13 \nmid |N_G(P)|$. Hence $|G : N_G(P)| = 2^a \cdot 3^b \cdot 5^{c+1} \cdot 13$, where $0 \leq a \leq 10$ and $0 \leq b, c \leq 1$. But we can easily verify $|G : N_G(P)| \not\equiv 1 \pmod{61}$, contrary to a Sylow's theorem.

If $q = 7^2$, then $|\Omega| = 2^2 \cdot 919$ and $|G_\alpha| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 7^2$ ($0 \leq i \leq 2$). Let P be a Sylow 919-subgroup of G . By the similar argument as above, we obtain $5, 7 \nmid |N_G(P)|$ and so $|G : N_G(P)| = 2^a \cdot 3^b \cdot 5^2 \cdot 7^2 \equiv 2^a \cdot 306$ or $-2^a \pmod{919}$, where $0 \leq a \leq 8$ and $0 \leq b \leq 1$. Hence $|G : N_G(P)| \not\equiv 1$, a contradiction.

If $q = 11^2$, then $|\Delta| = 2^7 \cdot 173$ and $|G_\alpha| = 2^{3+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$ ($0 \leq i \leq 2$).

Let P be a Sylow 173-subgroup of G . Similarly we have $3, 5, 11, 61 \nmid |N_G(P)|$ and so $|G : N_G(P)| = 2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -5 \cdot 2^a \pmod{173}$, where $0 \leq a \leq 12$. Hence $|G : N_G(P)| \neq 1$, a contradiction.

If $q = 59$, then $|\Delta| = 2 \cdot 17 \cdot 151$ and $|G_\alpha| = 2^{2+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59$ ($0 \leq i \leq 1$).

Let P be a Sylow 17-subgroup of G . Similarly we have $3, 5, 29, 59 \nmid |N_G(P)|$ and so $|G : N_G(P)| = 2^a \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151^b \equiv 10 \cdot 2^a$ or $12 \cdot 2^a \pmod{17}$, where $0 \leq a \leq 4$ and $0 \leq b \leq 1$. From this, we have a contradiction.

If $q = 71$, then $|\Delta| = 2^5 \cdot 233$ and $|G_\alpha| = 2^{3+i} \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$ ($0 \leq i \leq 1$).

Let P be a Sylow 233-subgroup of G . Since $3, 5, 7, 71 \nmid |N_G(P)|$, $|G : N_G(P)| = 2^a \cdot 3^2 \cdot 5 \cdot 7 \cdot 71 \equiv -3 \cdot 2^a \pmod{233}$, where $0 \leq a \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N_\beta^\alpha \simeq Z_2 \times Z_2$ or $N_\beta^\alpha \simeq D_8$ and $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$.

(3.14) The case that $N_\beta^\alpha \simeq Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N_\beta^\alpha)$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N_\beta^\alpha N_\alpha^\beta)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^\alpha|_2 = 4$, we have $q = p^n \equiv 3, 5 \pmod{8}$ and so n is odd. Hence $|G_\alpha/N^\alpha|_2 \leq 2$ and $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta/N^\beta \simeq 1$ or Z_2 by (3.2). Suppose $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq Z_2$. Then $N_\beta^\alpha N_\alpha^\beta$ is a Sylow 2-subgroup of G_α , hence $N_G(N_\beta^\alpha N_\alpha^\beta)^\Delta$ is doubly transitive by a Witt's theorem. Since $N_\beta^\alpha N_\alpha^\beta \simeq D_8$ and $|\Delta|$ is even, $C_G(N_\beta^\alpha N_\alpha^\beta)^\Delta$ is also doubly transitive. Let g be an element of $C_G(N_\beta^\alpha N_\alpha^\beta)$ such that $\alpha^g = \beta$ and $\beta^g = \alpha$. Then $N_\beta^\alpha = g^{-1} N_\beta^\alpha g = N_\alpha^\beta$ and hence $N_\beta^\alpha = N^\alpha \cap N^\beta$, a contradiction. Thus $N_\beta^\alpha = N^\alpha \cap N^\beta \simeq Z_2 \times Z_2$.

Let z be an involution in N_β^α and $t \in Z^G$ an involution such that $\alpha^t = \beta$. Set $\Gamma = \{ \{\gamma, \delta\} \mid \gamma, \delta \in \Delta, \gamma \neq \delta \}$. We consider the action of the element z on Γ . By the similar argument as in the proof of (3.12), $|F(z)| (|F(z)| - 1) / 2 + (|\Delta| - |F(z)|) / 2 = |F(z^\Gamma)| = |C_G(z)| |z^G \cap \langle t \rangle G_{\alpha\beta}| / |\langle t \rangle G_{\alpha\beta}|$.

Since $N_{\beta}^{\alpha} = N^{\alpha} \cap N^{\beta}$, by Lemma 2.6(i), $z^G \cap G_{\alpha} = z^{G_{\alpha}}$ and so $|C_G(z)| = |F(z)| \times |C_{G_{\alpha}}(z)|$. Hence $|G_{\alpha\beta}| (|F(z)| (|F(z)| - 1) + |\Delta| - |F(z)|) = |F(z)| |C_{G_{\alpha}}(z)| |z^G \cap \langle t \rangle G_{\alpha\beta}|$, so that $|G_{\alpha\beta}| |\Delta| \equiv 0 \pmod{|F(z)|}$. Since $|G_{\alpha\beta}/N_{\beta}^{\alpha}| = |G_{\alpha\beta} N^{\alpha}/N^{\alpha}| \mid 2n$, we have $|G_{\alpha\beta}| \mid 8n$. Clearly $|\Delta| = 1 + q(q-\varepsilon)(q+\varepsilon)r/8$ and by Lemma 2.8(i), $|F(z)| = 1 + 3(q-\varepsilon)r/4$. Hence $1 + 3(q-\varepsilon)r/4 \mid 8n(1 + q(q-\varepsilon)(q+\varepsilon)r/8)$. Put $n = rs$. Then $3qr-3\varepsilon r+4 \mid 4rs(8+q(q-\varepsilon)(q+\varepsilon)r)3^3r = 864r^2s + 4s(3qr)(3qr-3\varepsilon r)(3qr+3\varepsilon r)$. Hence $3qr-3\varepsilon r+4 \mid 864r^2s + 4s(3\varepsilon r-4)(3\varepsilon r-4-3\varepsilon r)(3\varepsilon r-4+3\varepsilon r) = 864r^2s - 32s(3\varepsilon r-4)(3\varepsilon r-2)$. (*)

We argue that $r = 1$. Suppose false. Then $32s(3\varepsilon r-4)(3\varepsilon r-2) > 0$ and so $3r(q-\varepsilon) < 864r^2s$. Therefore $288n + \varepsilon > q = p^n \geq 3^n$ and so $288n > 3^n$. Hence $(n, r, p, \varepsilon) = (5, 5, 3, -1)$, $(3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (*). Thus $r = 1$.

Hence $3q-3\varepsilon+4 \mid 64(5+9\varepsilon)n$ and $|F(z)| = 1 + 3(q-\varepsilon)/4$, $|\Delta| = 1 + q(q-\varepsilon)(q+\varepsilon)/8$. If $\varepsilon = -1$, then $3 \cdot 3^n < 3q+7 \mid 256n$. Hence $n = 1$ or $(n, p) = (5, 3)$, $(3, 3)$. Since $3 \cdot 3^5 + 7 \nmid 256 \cdot 5$ and $3 \cdot 3^3 + 7 \mid 256 \cdot 3$, $n = 1$ and $3q + 7 \mid 256$. From this, $q = 19$ or 83 . If $\varepsilon = 1$, then $3 \cdot 5^n < 3q + 1 \mid 896n$ and so $n = 1$ or $(n, p) = (3, 5)$. Since $3 \cdot 5^3 + 1 \nmid 896 \cdot 3$, we have $n = 1$ and $3q + 1 \mid 896$. From this, $q = 5, 37$ or 149 . As $\text{PSL}(2, 5) \simeq \text{PSL}(2, 4)$, $q \neq 5$ by [4]. Thus $q = 19, 37, 83$ or 149 .

Set $m = |z^G \cap \langle t \rangle G_{\alpha\beta}|$. As we mentioned above, $|G_{\alpha\beta}| (|F(z)| (|F(z)| - 1) + |\Delta| - |F(z)|) = |F(z)| |C_{G_{\alpha}}(z)| m$. Since $|G_{\alpha}/N^{\alpha}| = 1$ or 2 , $|C_{G_{\alpha}}(z)| / |G_{\alpha\beta}| = (q-\varepsilon)/4$. Therefore $m = (2q^2 + (2\varepsilon+9)q - 9\varepsilon)/(3q-3\varepsilon+4)$. It follows that $(q, m) = (19, 27/2)$, $(37, 28)$, $(83, 449/8)$ or $(149, 411/4)$. Since m is an integer, we have $(q, m) = (37, 28)$. But $m \leq |\langle t \rangle G_{\alpha\beta}| \leq 16$, a contradiction. Thus (3.14) holds.

(3.15) The case that $N_\beta^\alpha \simeq D_8$ and $N^\alpha \cap N^\beta \leq Z_2 \times Z_2$ does not occur.

Proof. Let Δ , L and K be as defined in (3.6). By (3.6), there exists an element x in L_α such that its order is odd and $\langle x^\Delta \rangle$ is regular on $\Delta - \{\alpha\}$. Since $(L_\alpha)' \leq N_\beta^\alpha$ by (3.6) and $N_\beta^\alpha \simeq D_8$, x stabilizes a normal series $N_\alpha^\beta N_\beta^\alpha \geq N_\beta^\alpha \geq 1$. Hence x centralizes $N_\alpha^\beta N_\beta^\alpha$ by Theorem 5.3.2 of [2] and so $x^{-1} N_\alpha^\beta x = N_\alpha^\beta$. Put $\gamma = \beta^x$. If $r \neq 1$, then $\beta \neq \gamma$, so that $N_\alpha^\gamma = N_\alpha^\beta$. From this, $N_\beta^\gamma = N_\beta^\beta$. By the doubly transitivity of G , $N_\beta^\alpha = N_\alpha^\beta$, hence $N_\beta^\alpha = N^\alpha \cap N^\beta$, a contradiction. Therefore $r = 1$ and $\Delta = \{\alpha, \beta\}$.

Set $\langle z \rangle = Z(N_\beta^\alpha)$, $\Delta_1 = \alpha^{C_G(z)}$ and let $\{\Delta_1, \Delta_2, \dots, \Delta_k\}$ be the set of $C_G(z)$ -orbits on $F(z)$. Since $L \geq N^\alpha \cap N^\beta$ and by (3.2), $N^\alpha \cap N^\beta \neq 1$, z is contained in $N^\alpha \cap N^\beta$. Hence, by Lemma 2.1, $\beta \in \Delta_1$ and k is at least two. By Lemma 2.8, $|F(z)| = 1 + (q-\varepsilon)5/|N_\beta^\alpha| = 1 + 5(q-\varepsilon)/8$. Clearly $|C_{N^\alpha}(z) : N_\beta^\alpha| = (q-\varepsilon)/8$ and so $|\Delta_1| \geq 1 + (q-\varepsilon)/8$. If $\gamma \in F(z) - \Delta_1$, then $C_{N_\gamma^\alpha}(z) \simeq Z_2 \times Z_2$, for otherwise $\langle z \rangle = Z(N_\gamma^\alpha) \leq N^\alpha \cap N^\gamma$ and by Lemma 2.1 $\gamma \in \Delta_1$, a contradiction. Hence one of the following holds.

(i) $k = 3$ and $|\Delta_1| = 1 + (q-\varepsilon)/8$, $|\Delta_2| = |\Delta_3| = (q-\varepsilon)/4$.

(ii) $k = 2$ and $|\Delta_1| = 1 + (q-\varepsilon)/8$, $|\Delta_2| = (q-\varepsilon)/2$.

(iii) $k = 2$ and $|\Delta_1| = 1 + 3(q-\varepsilon)/8$, $|\Delta_2| = (q-\varepsilon)/4$.

Let $\gamma \in F(z) - \Delta_1$. Then, $z \in G_\gamma - N^\gamma$ and so $C_{N^\gamma}(z) \simeq D_{q+\varepsilon}$ or $\text{PGL}(2, \sqrt{q})$ by Lemma 2.6(vii), (viii), (ix). If $C_{N^\gamma}(z) \simeq D_{q+\varepsilon}$, then $(q+\varepsilon)/2 \mid |\Delta_1|$ and so $q = 7$ and (iii) occurs. But $(q+\varepsilon)/2 = 3 \mid |\Delta_2| - 1 = 1$, a contradiction. If $C_{N^\gamma}(z) \simeq \text{PGL}(2, \sqrt{q})$, then (i) does not occur because $\sqrt{q} \nmid q-\varepsilon$. Hence $\sqrt{q} \mid |\Delta_1|$ and $\sqrt{q} \mid |\Delta_2| - 1$. From this, $q = 25$ and (iii) occurs. In this case, we have $|\Delta_1| = 10$, so that an element of $C_{N^\gamma}(z)$ of order 3 is contained in N_δ^γ for some $\delta \in \Delta_1$, contrary to $N_\delta^\gamma \simeq N_\beta^\alpha \simeq D_8$.

4. Case (II)

In this section we assume that $N_B^\alpha \simeq \text{PGL}(2, p^m)$, where $n = 2mk$ and

k is odd. Since n is even, $q = p^n \equiv 1 \pmod{4}$. We set $p^m \equiv \varepsilon \in \{\pm 1\} \pmod{4}$.

In section 7 we shall consider the case that $N_B^\alpha \simeq S_4$. Therefore we assume $(p, m) \neq (3, 1)$ in this section.

(4.1) The following hold.

(i) $N_B^\alpha / N^\alpha \cap N^\beta \simeq 1$ or Z_2 and $N^\alpha \cap N^\beta \geq (N_B^\alpha)' \simeq \text{PSL}(2, p^m)$.

(ii) If $(p, m) \neq (5, 1)$, there exists a cyclic subgroup Y of $(N_B^\alpha)'$ such that $N_{N^\alpha}(Y) \simeq D_{q-\varepsilon}$ and $N_G(Y)^{F(Y)}$ is doubly transitive.

Proof. As $N_B^\alpha \geq N^\alpha \cap N^\beta$, either $N_B^\alpha / N^\alpha \cap N^\beta \leq Z_2$ or $N^\alpha \cap N^\beta = 1$.

If $N^\alpha \cap N^\beta = 1$, by Lemmas 2.2 and 2.6(vi), $N_B^\alpha \simeq N_B^\alpha / N^\alpha \cap N^\beta \simeq N_B^\alpha N^\beta / N^\beta \simeq Z_2 \times Z_n$, a contradiction. Therefore $N_B^\alpha / N^\alpha \cap N^\beta \simeq 1$ or Z_2 and $N^\alpha \cap N^\beta \geq (N_B^\alpha)' \simeq \text{PSL}(2, p^m)$.

Now we assume that $(p, m) \neq (5, 1)$ and let z be an involution in $(N_B^\alpha)'$. Then $C_{N_B^\alpha}(z) \simeq D_{2(p^m - \varepsilon)}$ by Lemma 2.6(vii). Suppose $C_{N_B^\alpha}(z)$ is not a 2-subgroup and put $Y = O(C_{N_B^\alpha}(z))$. Then, if $Y^g \leq G_{\alpha\beta}$ for some $g \in G$, we have $Y^g \leq N_\alpha^\gamma$ and $Y^g \leq N_\beta^\delta$, where $\gamma = \alpha^g$ and $\delta = \beta^g$. By (i) $Y^g \leq N^\alpha \cap N^\beta$ and so $Y^g = Y^h$ for some $h \in N^\alpha \cap N^\beta$. Thus $N_G(Y)^{F(Y)}$ is doubly transitive. Assume that $C_{N_B^\alpha}(z)$ is a 2-subgroup and set $C_{N_B^\alpha}(z) = \langle u, v \mid u^v = u^{-1}, v^2 = 1 \rangle$. We may assume that $v \in (N_B^\alpha)'$ and $\langle u^2, v \rangle$ is a Sylow 2-subgroup of $(N_B^\alpha)'$. Since $p^m \neq 3, 5$, the order of u^2 is at least four. On the other hand there is no element of order $|u^2|$ in $\langle u, v \rangle - \langle u^2, v \rangle$. Hence any element of order $|u^2|$ which is contained in N_B^α is necessarily an element of $N^\alpha \cap N^\beta$. By the similar argument as above, $N_G(Y)^{F(Y)}$ is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose $(p, m) \neq (3, 1), (5, 1)$

and set $\Delta = F(Y)$ and $X = N_G(Y)$. Then $|\Delta| = rs(p^m + \varepsilon)/2 + 1$, where

$s = \sum_{i=0}^{k-1} p^{2mi}$, $C_G(N^\alpha) = 1$ and one of the following holds.

(i) $X^\Delta \leq \text{AFL}(1, 2^c)$ for some integer c .

(ii) $X^\Delta \simeq \text{PSL}(2, p_1)$ or $\text{PGL}(2, p_1)$, $r = 1$, $k = 1$ and $2p_1 = p^m + \varepsilon$.

Proof. By Lemma 2.8(ii), $|\Delta| = 1 + |N^\alpha \cap X| r / |N_\beta^\alpha \cap X| = 1 + (p^{2mk} - 1)r / 2(p^m - \varepsilon) = rs(p^m + \varepsilon) / 2 + 1$. By (4.1) and Lemma 2.9, we have (i), (ii) or $X^\Delta = R(3)$.

Assume that $X^\Delta = R(3)$. Then $rs(p^m + \varepsilon) / 2 + 1 = 28$, hence $k = 1$ and $r(p^m + \varepsilon) / 2 = 27$. Since r is odd and $r \mid 2m = n$, we have $r = m = 1$ and $q = 53^2$. But a Sylow 3-subgroup of X_α is cyclic because $N^\alpha \cap X \simeq D_{q-\varepsilon}$ and $X_\alpha / X \cap N^\alpha \simeq X_\alpha N^\alpha / N^\alpha \leq Z_2 \times Z_2$, a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose $X^\Delta \leq \text{AFL}(1, 2^c)$ and put $W = C_{N_\beta^\alpha}(Y)$. Then $Y \leq W \simeq Z_{p^m - \varepsilon}$. Since $C_{N_\beta^\alpha}(Y)$ is cyclic, W is a characteristic subgroup of $C_{N_\beta^\alpha}(Y)$ and so W is a normal subgroup of X_α . Hence $W \leq X_\Delta$ and $(X \cap N_\beta^\alpha)^\Delta \simeq 1$ or Z_2 . By Lemmas 2.4 and 2.6, $F(X \cap N_\beta^\alpha) = 1 + |X \cap N_\beta^\alpha| |N_\beta^\alpha : X \cap N_\beta^\alpha| r / |N_\beta^\alpha| = 1 + r$. Since $1 + r < |\Delta|$, $(X \cap N_\beta^\alpha)^\Delta \simeq Z_2$ and hence $(1+r)^2 = rs(p^m + \varepsilon) / 2 + 1$ by Lemma 2.5. From this, $r = s(p^m + \varepsilon) / 2 - 2 \mid mk$ and so $p^{2m(k-1)} + mk \leq 2$. Hence $m = k = r = 1$ and $q = 7^2$.

Let R be a Sylow 3-subgroup of N_β^α . Since $N_\beta^\alpha \simeq \text{PGL}(2, 7)$, we have $R \simeq Z_3$. By Lemmas 2.4 and 2.6, $|F(R)| = 1 + (7^2 - 1) |N_\beta^\alpha : N_{N_\beta^\alpha}(R)| / |N_\beta^\alpha| = 4$. Hence $N_G(R)^{F(R)} \simeq A_4$ or S_4 . But R is a Sylow 3-subgroup of $N_{G_\alpha}(R)$ because $N^\alpha \simeq \text{PSL}(2, 7^2)$, contrary to $N_{G_\alpha}(R)^{F(R)} \simeq A_3$ or S_3 .

(4.4) (ii) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose $X^\Delta \cong \text{PSL}(2, p_1)$. By the similar argument as in (4.3), $C_{N_\beta^\alpha}(Y) \leq X_\Delta$ and so $C_{N_\beta^\alpha}(Y)^\Delta \cong Z_{p_1}$, and $N_{N_\beta^\alpha}(Y)^\Delta \cong D_{2p_1}$. Hence $|X_\Delta| \mid 2p_1 \cdot 2n$. Since $X^\Delta \cong \text{PSL}(2, p_1)$, $p_1(p_1-1)/2 \mid |X_\Delta|$, hence $p_1-1 \mid 8n$. As $k=1$ and $2p_1 = p^m + \varepsilon$, we have $p^{m+\varepsilon-2} \mid 32n$. From this, $(p, m, p_1) = (11, 1, 5)$, $(3, 2, 5)$ or $(3, 3, 13)$.

Let R be a cyclic subgroup of N_β^α such that $R \cong Z_{(p^m+\varepsilon)/2}$.

By Lemma 2.6, $N_G(R)^{F(R)}$ is doubly transitive and by Lemma 2.8(ii), $|F(R)| = 1 + |N_{N_\beta^\alpha}(R)|/|N_{N_\beta^\alpha}(R)| = 1 + (p^{2m}-1)/2(p^m+\varepsilon) = (p^m-\varepsilon)/2 + 1$.

If $(p, m, p_1) = (11, 1, 5)$, $|F(R)| = 7$ and so by [9] $|N_G(R)^{F(R)}| = 42$ and $N_{G_\alpha}(R)^{F(R)} \cong Z_6$. Since $|N_{N_\beta^\alpha}(R) : N_{N_\beta^\alpha}(R)| = 6$, $N_{N_\beta^\alpha}(R)^{F(R)} = N_{G_\alpha}(R)^{F(R)}$. Hence $N_{N_\beta^\alpha}(R)/K \cong Z_6$, where $K = (N_{N_\beta^\alpha}(R))_{F(R)}$. But $N_{N_\beta^\alpha}(R)/(N_{N_\beta^\alpha}(R))' \cong Z_2 \times Z_2$, a contradiction.

If $(p, m, p_1) = (3, 2, 5)$, $|F(R)| = 5$ and so by [9], $|N_G(R)^{F(R)}| = 20$ and $N_{G_\alpha}(R)^{F(R)} \cong Z_4$. Since $|N_{N_\beta^\alpha}(R) : N_{N_\beta^\alpha}(R)| = 4$, $N_{N_\beta^\alpha}(R)^\Delta \cong Z_4$, contrary to $N_{N_\beta^\alpha}(R)/(N_{N_\beta^\alpha}(R))' \cong Z_2 \times Z_2$.

If $(p, m, p_1) = (3, 3, 13)$, $|F(R)| = 15$. By [9], $N_{G_\alpha}(R)^{F(R)}$ is not solvable, a contradiction.

(4.5) $p^m \neq 5$.

Proof. Assume that $p^m = 5$. Then $n = 2k$ with k odd and $N_\beta^\alpha \cong \text{PGL}(2, 5) \cong S_5$. First we argue that $N_\beta^\alpha = N^\alpha \cap N^\beta$. Suppose false. Then $C_G(N^\alpha) = 1$ by Lemma 2.2, and $N_\beta^\alpha/N^\alpha \cap N^\beta \cong Z_2$ by (4.1). Since $N_\beta^\alpha/N^\alpha \cong N_\beta^\alpha/N^\alpha \cap N^\beta \cong Z_2$ and the outer automorphism group of S_5 is trivial, we have $Z(N_\beta^\alpha/N^\alpha) \cong Z_2$.

Let w_1 be the involution of $Z(N_\beta^\alpha/N^\alpha)$ and let $w \in I(N_\beta^\alpha) - I(N^\alpha)$. Since $C_{N_\beta^\alpha}(w_1) \geq N_\beta^\alpha$, by Lemma 2.6(viii) and (ix), w acts on N^α as a field automorphism of order 2 and $C_{N_\beta^\alpha}(w) \cong \text{PGL}(2, 5^k)$.

By Lemma 2.8 $|F(w)| = 1 + r(q-\varepsilon)|I(N_B^\alpha)|/|N_B^\alpha| = 1 + 5r(5^{2k}-1)/24$. Let P be a Sylow 5-subgroup of $C_{N^\alpha}(w)$. Then $|P| = 5^k$ and $|\gamma^P| = 5^{k-1}$ or 5^k for $\gamma \in \Omega - \{\alpha\}$. Since P acts on $F(w) - \{\alpha\}$, we have $5^{k-1} \mid 5r(5^{2k}-1)/24$, so that $k = 1$ and $|F(w)| = 6$ as $r \mid k$. Hence $C_{N^\alpha}(w)^{F(w)} \simeq S_5$ and so $C_G(w)^{F(w)} \simeq S_6$. But clearly $w \in N^\alpha \cap N^\beta$ by Lemma 2.1, a contradiction. Thus $N_B^\alpha = N^\alpha \cap N^\beta$.

Let V be a cyclic subgroup of N_B^α of order 4. Since $N_B^\alpha = N^\alpha \cap N^\beta \simeq S_5$, $N_G(V)^{F(V)}$ is doubly transitive and by Lemma 2.8, $|F(V)| = 1 + |N_{N^\alpha}(V)|r/|N_{N_B^\alpha}(V)| = 1 + (5^{2k}-1)r/8 = 3rs + 1$, where $s = \sum_{i=0}^{k-1} 25^i$. By Lemma 2.9, $C_G(N^\alpha) = 1$ and (a) $N_G(V)^{F(V)} \leq \text{AGL}(1, 2^c)$ or (b) $N_G(V)^{F(V)} = R(3)$.

Put $P = N_{N_B^\alpha}(V)$. Then $P \simeq D_8$, $|F(P)| = 1 + |N_{N^\alpha}(P)| |N_B^\alpha : N_{N_B^\alpha}(P)|r/|N_B^\alpha| = r + 1$ and $P^{F(V)} \simeq Z_2$. If (b) occurs, $k = 1$ and $r = 9$, hence $|F(P)| = 10$, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r+1)^2 = 3rs + 1$ and so $r = 3s - 2 \mid k$. Hence $k = r = 1$ and $G_\alpha/N^\alpha \leq Z_2 \times Z_2$. Let z be an involution in N_B^α . Then $|F(z)| = 1 + 24 \cdot 25/120 = 6$ by Lemma 2.8 and $|\Omega| = 1 + |N^\alpha : N_B^\alpha| = 66$ as $r = 1$. By the similar argument as in the proof of (3.12), $|F(z)|(|F(z)|-1)/2 + (|\Omega| - |F(z)|)/2 = |C_G(z)| |z^G \cap \langle t \rangle G_{\alpha\beta}| / |\langle t \rangle G_{\alpha\beta}|$, where t is an involution such that $\alpha^t = \beta$. Hence $|z^G \cap \langle t \rangle G_{\alpha\beta}| = 15 |G_{\alpha\beta}| / |C_{G_\alpha}(z)|$. Set $H = \langle t \rangle G_{\alpha\beta}$ and let R be a Sylow 3-subgroup of N_B^α . By Lemma 2.8, $|F(R)| = 1 + 24 \cdot 10/120 = 3$. Set $F(R) = \{\alpha, \beta, \gamma\}$. On the other hand, as $N_B^\alpha \simeq S_5$ and $\text{Out}(S_5) = 1$, we have $H = Z(H) \times N_B^\alpha$ and $|Z(H)| = 2, 4$ or $H = C_H(N_B^\alpha) \times N_B^\alpha$ and $C_H(N_B^\alpha) \simeq D_8$. In the latter case $G_{\alpha\beta} = Z(G_{\alpha\beta}) \times N_B^\alpha$ and $Z(G_{\alpha\beta}) \simeq Z_2 \times Z_2$, contrary to Lemma 2.6(ix). In the former case, we have $|Z(H)| = 2$. For otherwise $Z(H) \leq G_\gamma$ and $Z(H) \cap z^G \neq \emptyset$ and so letting $u \in Z(H) \cap z^G$, we have $|R| = 3 \mid |F(u)| - 1 = 5$, a contradiction. Therefore $Z(H) \simeq Z_2$ and so $|z^G \cap H| \leq 25 + 25 = 50$, while $|z^G \cap H| = 15 |G_{\alpha\beta}| / |C_{G_\alpha}(z)| = 15 \cdot 120/24 = 75$, a contradiction.

5. Case (III)

In this section we assume that $N_\beta^\alpha \simeq \text{PSL}(2, p^m)$, where $n = mk$ and k is odd. Set $p^m \equiv \varepsilon \in \{\pm 1\} \pmod{4}$. Then $q \equiv \varepsilon \pmod{4}$ as k is odd. In section 6 we shall consider the case that $N_\beta^\alpha \simeq A_4$, so we assume $(p, m) \neq (3, 1)$ in this section. From this N_β^α is a nonabelian simple group and so $N_\beta^\alpha = N^\alpha \cap N^\beta$ or $N^\alpha \cap N^\beta = 1$. If $N^\alpha \cap N^\beta = 1$, then $C_G(N^\alpha) = 1$ by Lemma 2.2 and $N_\beta^\alpha \simeq N_\beta^\alpha / N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta / N^\beta \simeq Z_2 \times Z_n$, a contradiction. Hence $N_\beta^\alpha = N^\alpha \cap N^\beta$.

Let z be an involution of N_β^α . Suppose $z^g \in G_{\alpha\beta}$ for some $g \in G$ and set $\gamma = \alpha^g$, $\delta = \beta^g$. Then $z^g \in N_\delta^\gamma \cap G_{\alpha\beta} \leq N_\alpha^\gamma \cap N_\beta^\delta \leq N^\alpha \cap N^\beta$ and so $z^g \in Z^{N_\beta^\alpha}$. Hence $C_G(z)^{F(z)}$ is doubly transitive and by Lemma 2.8(i), $|F(z)| = (q - \varepsilon)r / (p^m - \varepsilon) + 1$. In particular $|F(z)| > 3r + 1$ as $(p^n - \varepsilon) / (p^m - \varepsilon) \geq p^{2m} + \varepsilon p^m + 1 > 3$.

By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

(a) $C_G(z)^{F(z)} \leq \text{AFL}(1, 2^c)$.

(b) $C_G(z)^{F(z)} \cong \text{PSL}(2, p_1)$ ($p_1 \geq 5$), $r = 1$ and $|C_{N^\alpha}(z) : C_{N_\beta^\alpha}(z)|$

$= p_1$.

(c) $C_G(z)^{F(z)} = R(3)$.

Let Y be a cyclic subgroup of $C_{N_\beta^\alpha}(z) \simeq D_{p^m - \varepsilon}$ of index 2. Since $C_{G_\alpha}(z) \supseteq Y$, $z \in Y$ and $C_G(z)^{F(z)}$ is doubly transitive, we have $F(Y) = F(z)$. By the

similar argument as in (3.1), $N^\alpha \cap N(C_{N_\beta^\alpha}(z)) = C_{N_\beta^\alpha}(z)$ or $N^\alpha \cap N(C_{N_\beta^\alpha}(z))$

$\simeq A_4$. Hence by Lemmas 2.3 and 2.4 $|F(C_{N_\beta^\alpha}(z))| = 1 +$

$|C_{N_\beta^\alpha}(z)| |N_\beta^\alpha : C_{N_\beta^\alpha}(z)| r / |N_\beta^\alpha|$ or $1 + |A_4| |N_\beta^\alpha : C_{N_\beta^\alpha}(z)| r / |N_\beta^\alpha|$. Therefore

$|F(C_{N_\beta^\alpha}(z))| = r + 1$ or $3r + 1$. From this $C_{N_\beta^\alpha}(z)^{F(z)} \simeq Z_2$.

In the case (a), $(r+1)^2 = 1 + (p^n - \varepsilon)r / (p^m - \varepsilon)$ by Lemma 2.5 and hence

$r = (p^n - \varepsilon) / (p^m - \varepsilon) - 2 \mid mk$. Since $(p^n - \varepsilon) / (p^m - \varepsilon) \geq ((p^m)^k + 1) / (p^m + 1)$

$= \sum_{i=0}^{k-1} (-p^m)^i$ and $k \geq 3$, we have $p^{m(k-1)} (p^{2m} - p^m + 1) \leq mk$, hence $(p^m)^{k-3} / k$

$m/(p^{2m}-p^m+1) < 1$. Thus $k = 3$, $m = 1$ and $p = 3$, contrary to $(p, m) \neq (3, 1)$.

In the case (b), $r = 1$, $p_1 = (p^n - \varepsilon)/(p^m - \varepsilon)$, $p_1(p_1 - 1)/2 \mid s$ and $s \mid 4m p_1$, where s is the order of $C_{G_\alpha}(z)^{F(z)}$. Hence $p_1 - 1 \mid 8mk$. Since $p_1 - 1 = (p^n - \varepsilon)/(p^m - \varepsilon) - 1 \geq (p^n + 1)/(p^m + 1) - 1 = \sum_{i=0}^{k-1} (-p^m)^i \geq p^{m(k-2)}(p^m - 1)$, we have $p^{m(k-2)}/2k \leq 4m/(p^m - 1) \leq 1$ because $p^m \neq 3$. Hence $k = 3$ and $p^m = 5$, so that $p_1 - 1 = 30 \nmid 8mk = 24$, a contradiction.

In the case (c), $r + 1 = 4$ and $1 + (p^n - \varepsilon)r/(p^m - \varepsilon) = 28$ and so $r = 3$ and $(p^n - \varepsilon)/(p^m - \varepsilon) = 9$. Hence $9 \geq (p^{mk} + 1)/(p^m + 1) \geq p^{2m} - p^m + 1$, so that $p^m = 3$, a contradiction.

6. Case (IV)

In this section we assume that $N_B^\alpha \simeq A_4$ and $q = 3, 5 \pmod{8}$. If $N^\alpha \cap N^\beta = 1$, by Lemma 2.2, $C_G(N^\alpha) = 1$ and so $N_B^\alpha/N^\alpha \cap N^\beta \simeq N_B^\alpha N^\beta/N^\beta \leq Z_2 \times Z_n$. Hence $N_B^\alpha/N^\alpha \cap N^\beta \simeq 1$ or Z_3 , so that $z^G \cap G_{\alpha\beta} = z^G \cap N_B^\alpha = z^{N_B^\alpha}$ for an involution $z \in N_B^\alpha$. Therefore $C_G(z)^{F(z)}$ is doubly transitive.

By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

- (a) $C_G(z)^{F(z)} \leq \text{AFL}(1, 2^c)$ for some integer $c \geq 1$.
- (b) $C_G(z)^{F(z)} \cong \text{PSL}(2, p_1)$ ($p_1 \geq 5$), $r = 1$ and $|C_{N^\alpha}(z) : C_{N_B^\alpha}(z)| = p_1$.
- (c) $C_G(z)^{F(z)} = R(3)$.

Let T be a Sylow 2-subgroup of N_B^α . Then $z \in T$ and by Lemmas 2.3 and 2.4, $|F(T)| = 1 + |N_{N^\alpha}(T)|r/|N_B^\alpha| = r + 1$. By Lemma 2.8(i), $|F(z)| = (q - \varepsilon)r/4 + 1$. Hence $T^{F(z)} \simeq Z_2$ if $q \neq 5$. If $q = 5$, as $\text{PSL}(2, 5) \simeq \text{PSL}(2, 4)$, (ii) of our theorem holds by [4]. Therefore we may assume $q \neq 5$.

In the case (a), $(r+1)^2 = 1 + (q - \varepsilon)r/4$ by Lemma 2.5. Hence $r = (q - \varepsilon - 8)/4$ and $r \mid n$, so that $q = 11$ or 13 and $r = 1$. Let R be a Sylow

3-subgroup of $G_{\alpha\beta}$. Then $R \cong Z_3$ and $R \leq N_{\beta}^{\alpha}$ because $G_{\alpha\beta}/N_{\beta}^{\alpha} \cong G_{\alpha\beta}N^{\alpha}/N^{\alpha} \cong 1$ or Z_2 and $N_{\beta}^{\alpha} \cong A_4$. By Lemma 2.8(ii), $|F(R)| = 1 + 12/3 = 5$ and $N_G(R)^{F(R)}$ is doubly transitive. Since $N_{G_{\alpha}}(R) \cong D_{12}$ or D_{24} and $|F(R)| = 5$, we have $|N_G(R)|_5 = 5$. Let S be a Sylow 5-subgroup of $N_G(R)$. Then $[S, R] = 1$ as $N_G(R)/C_G(R) \leq Z_2$. Since $5 \nmid |G_{\alpha\beta}|$, $|F(S)| = 0$ or 1 . If $|F(S)| = 1$, $F(S) \subseteq F(R)$ and so $5 \mid |F(R)| - 1 = 4$, a contradiction. Therefore S is semi-regular on Ω . But $|\Omega| = 1 + |N^{\alpha} : N_{\beta}^{\alpha}| = 56$ or 92 . This is a contradiction.

In the case (b), $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/2 = 4np_1$, where s is the order of $C_{G_{\alpha}}(z)^{F(z)}$. Hence $p_1-1 \mid 8n$. Since $p_1 = (q-\varepsilon)/4$, $p_1^{n-\varepsilon-4} \mid 32n$ and so we have $q = 11, 13, 19, 27$ or 37 . If $q \neq 27$, by Lemma 2.6, $C_{G_{\alpha}}(z) \cong D_{q-\varepsilon}$ or $D_{2(q-\varepsilon)}$ and so $C_{G_{\alpha\beta}}(z)^{F(z)} \cong Z_2$. Hence $(p_1-1)/2 = 2$. From this $q = 19$. Let R be a Sylow 3-subgroup of $G_{\alpha\beta}$. By the similar argument as in the case (a), $N_G(R)^{F(R)}$ is doubly transitive and $|F(R)| = 1 + 18/3 = 7$. Hence $7 \mid |G|$. On the other hand $|G| = |\Omega||G_{\alpha}| = (1 + |N^{\alpha} : N_{\beta}^{\alpha}|)|G_{\alpha}| = (1 + 18 \cdot 19 \cdot 20/2 \cdot 12) \cdot 2^i \cdot 18 \cdot 19 \cdot 20/2 = 2^{3+i} \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 19$ with $0 \leq i \leq 1$, a contradiction. If $q = 27$, then $|C_G(z)|_2 = |F(z)|_2 \times |C_{G_{\alpha}}(z)|_2 = 8 \times |G_{\alpha}|_2$, while $|\Omega| = 1 + |N^{\alpha} : N_{\beta}^{\alpha}| = 1 + 26 \cdot 27 \cdot 28/2 \cdot 12 = 820 = 2^2 \cdot 5 \cdot 41$ and so $|G|_2 = 4|G_{\alpha}|_2$. Therefore $|C_G(z)| \nmid |G|$, a contradiction.

In the case (c), $r + 1 = 4$ and $1 + (q-\varepsilon)r/4 = 28$. Hence $r = 3$ and $q = 37$, contrary to $r \mid n$.

7. Case (V)

In this section we assume that $N_{\beta}^{\alpha} \cong S_4$ and $q \equiv 7, 9 \pmod{16}$. We note that $4 \nmid n$.

First we argue that $N_{\beta}^{\alpha} = N^{\alpha} \cap N^{\beta}$. Suppose $N_{\beta}^{\alpha} \neq N^{\alpha} \cap N^{\beta}$. Then $C_G(N^{\alpha}) = 1$ by Lemma 2.2. Since $N_{\beta}^{\alpha}/N^{\alpha} \cap N^{\beta} \cong N_{\beta}^{\alpha}N^{\beta}/N^{\beta} \leq Z_2 \times Z_n$, we have $N^{\alpha} \cap N^{\beta} \cong A_4$ and $N_{\beta}^{\alpha}/N^{\alpha} \cap N^{\beta} \cong Z_2$, so that $N_{\alpha}^{\beta}N_{\beta}^{\alpha}/N_{\beta}^{\alpha} \cong N_{\alpha}^{\beta}/N^{\alpha} \cap N^{\beta} \cong Z_2$. Hence as

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Out $(S_4) = 1$, $Z(N_B^\alpha N_\alpha^\beta) \simeq Z_2$. Set $\langle t_1 \rangle = Z(N_B^\alpha N_\alpha^\beta)$ and let $t \in I(N_\alpha^\beta) - I(N^\alpha)$. Since $C_{N^\alpha}(t_1) \geq N_B^\alpha \simeq S_4$ and $\langle t \rangle N^\alpha = N_\alpha^\beta N^\alpha$, by Lemma 2.6, we have $C_{N^\alpha}(t) \simeq \text{PGL}(2, \sqrt{q})$ and $|F(t)| = 1 + 3(q-\varepsilon)r/8$ by Lemma 2.8.

Let P be a Sylow p -subgroup of $C_{N^\alpha}(t)$. Then $|P| = \sqrt{q}$. If $p \neq 3$, P acts semi-regularly on $F(t) - \{\alpha\}$ and so $\sqrt{q} \mid 3(q-\varepsilon)r/8$. Therefore $\sqrt{q} \mid r$ and so $5^n \leq n^2$ as $p \geq 5$ and $r \mid n$. But obviously $5^n > n^2$ for any positive integer n . This is a contradiction. If $p = 3$, $|P : P_\gamma| = \sqrt{q}/3$ or \sqrt{q} for each $\gamma \in \Omega - \{\alpha\}$. Hence $\sqrt{q}/3 \mid 3(q-\varepsilon)r/8$ and so $q \mid 81r^2$. In particular, $3^n = q \mid 81n^2$. From this, $n \leq 7$. Since $q = 3^n \equiv 7$ or $9 \pmod{16}$, we have $q = 3^2$ or 3^6 . If $q = 3^2$, $|\Omega| = 1 + |N^\alpha : N_B^\alpha| = 1 + 8 \cdot 9 \cdot 10 / 2 \cdot 24 = 16$, a contradiction by [9].

If $q = 3^6$, $|F(t)| = 1 + 273r$ and $|F(t) - \{\alpha\}| \geq |C_{N^\alpha}(t) : C_{N_B^\alpha}(t)| \geq |\text{PGL}(2, 3^3)|/8 = 2457$, contrary to $r \mid 3$. Thus $N_B^\alpha = N^\alpha \cap N^\beta$.

Let V be a cyclic subgroup of N_B^α of order 4 and let U be a Sylow 2-subgroup of N_B^α containing V . Then $U = N_{N_B^\alpha}(V)$, $|F(V)| = 1 + (q-\varepsilon)r/8$ by Lemma 2.8 and $|F(U)| = 1 + 8 \cdot 3r/24 = r + 1$ by Lemmas 2.3 and 2.4. If $q \neq 7, 9$, then $|F(U)| < |F(V)|$ and hence $U^{F(V)} \simeq Z_2$. Suppose $q = 7$ or 9 . Then $r = 1$ as $r \mid n$. Hence $|\Omega| = 1 + |N^\alpha : N_B^\alpha| = 8$ or 16 . By [10], we have a contradiction. Therefore $U^{F(V)} \simeq Z_2$.

Suppose $V^g \leq G_{\alpha\beta}$ for some $g \in G$ and set $\gamma = \alpha^g$. Then $V^g \leq g^{-1}N^\alpha g \cap G_{\alpha\beta} \leq N^\gamma \cap G_{\alpha\beta} \leq N_\alpha^\gamma \cap N_\beta^\gamma \leq N^\alpha \cap N^\beta = N_B^\alpha$. As $N_B^\alpha \simeq S_4$, $V^g = V^h$ for some $h \in N_B^\alpha$. Hence $N_G(V)^{F(V)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

$$(a) N_G(V)^{F(V)} \leq \text{AGL}(1, 2^c).$$

$$(b) N_G(V)^{F(V)} \cong \text{PSL}(2, p_1), p_1 = (q-\varepsilon)/8 \geq 5.$$

$$(c) N_G(V)^{F(V)} = R(3).$$

In the case (a), $(r+1)^2 = 1 + (q-\varepsilon)r/8$ by Lemma 2.5 and so $r = (q-\varepsilon-16)/8$ and $r \mid n$. From this $q = 23$ or 25 and $r = 1$. Since $|\Delta_1| = 1 + |N^\alpha : N_\beta^\alpha| = 2 \cdot 127$ or $2 \cdot 163$, we have $|G|_2 = 2|G_\alpha|_2$, while $|N_G(V)|_2 = |F(V)|_2 |N_{G_\alpha}(V)|_2 = 4|G_\alpha|_2$, contrary to $|N_G(V)| \mid |G|$.

In the case (b), $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/4 = 4np_1$, where s is the order of $N_{G_\alpha}(V)^{F(V)}$. Hence $p_1-1 \mid 8n$. From this, $p_1^{n-\varepsilon-8} \mid 64n$ and so $q = 23, 41, 71$ or 73 . Since p_1 is a prime and $p_1 = (q-\varepsilon)/8 \geq 5$, $q \neq 23, 71, 73$. Therefore $q = 41$ and $|\Delta_1| = 1 + |N^\alpha : N_\beta^\alpha| = 1 + 40 \cdot 41 \cdot 42 / 2 \cdot 24 = 2^2 \cdot 359$, so that $|G|_2 = 4|G_\alpha|_2$. Since $N_\beta^\alpha = N^\alpha \cap N^\beta$, $C_G(z)^{F(z)}$ is transitive by Lemma 2.1. On the other hand $|F(z)| = 1 + 40 \cdot 9 / 24 = 16$ by Lemma 2.8(i) and so $|C_G(z)|_2 = 16|C_{G_\alpha}(z)|_2 = 16|G_\alpha|_2$, contrary to $|C_G(z)| \mid |G|$.

In the case (c), $r+1 = 4$ and $1 + (q-\varepsilon)r/8 = 28$. Hence $r = 3$ and $q = 71$ or 73 , contrary to $r \mid n$.

8. Case (VI)

In this section we assume that $N_\beta^\alpha \cong A_5$ and $q \equiv 3, 5 \pmod{8}$. In particular, n is odd. If $N_\beta^\alpha \neq N^\alpha \cap N^\beta$, then $N^\alpha \cap N^\beta = 1$, $C_G(N^\alpha) = 1$ and so $N_\beta^\alpha \cong N_\beta^\alpha N^\beta / N^\beta \leq \text{Out}(N^\beta) \cong Z_2 \times Z_n$, a contradiction. Hence $N_\beta^\alpha = N^\alpha \cap N^\beta$. Let z be an involution in N_β^α and T a Sylow 2-subgroup of N_β^α containing z . Then, by Lemma 2.8 $|F(z)| = 1 + (q-\varepsilon)15r/60 = 1 + (q-\varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)| = 1 + 12 \cdot 5r/60 = 1 + r$. Since $N_\beta^\alpha = N^\alpha \cap N^\beta$, $z^G \cap G_{\alpha\beta} = z^G \cap N_\beta^\alpha = z^{N_\beta^\alpha}$ and so $C_G(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha) = 1$ and one of the following holds.

$$(a) C_G(z)^{F(z)} \leq \text{AFL}(1, 2^C).$$

$$(b) C_G(z)^{F(z)} \cong \text{PSL}(2, p_1), p_1 = (q-\varepsilon)/4 \geq 5.$$

$$(c) C_G(z)^{F(z)} = R(3).$$

• In the case (a), by Lemma 2.5, $(q-\varepsilon)/4 = 1$ or $(r+1)^2 = 1 + (q-\varepsilon)r/4$. Hence $q = 5$ or $r = (q-\varepsilon-8)/4 \mid n$. If $q = 5$, then $N_{\mathcal{B}}^{\alpha} = N^{\alpha}$, a contradiction. Therefore $p^{n-\varepsilon-8} \mid 4n$ and so $n = 1$ and $q = 11$ or 13 . If $q = 13$, we have $5 \nmid |G_{\alpha}|$, a contradiction. Hence $q = 11$ and $|\Delta\Omega| = 1 + |N^{\alpha} : N_{\mathcal{B}}^{\alpha}| = 1 + 10 \cdot 11 \cdot 12 / 2 \cdot 60 = 12$. By [9], $G^{\Delta\Omega} \cong M_{11}$, $|\Delta\Omega| = 12$ and so (iii) of our theorem holds.

• In the case (b), we have $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/2 = 4np_1$, where s is the order of $C_{G_{\alpha}}(z)^{F(z)}$. Hence $p_1-1 \mid 8n$ and so $p^{n-\varepsilon-4} \mid 32n$. From this $q = 19, 27$ or 37 . Since $5 \mid |G_{\alpha}|$, $q \neq 27, 37$. Hence $q = 19$ and $|\Delta\Omega| = 1 + |N^{\alpha} : N_{\mathcal{B}}^{\alpha}| = 1 + 18 \cdot 19 \cdot 20 / 2 \cdot 60 = 2 \cdot 29$. Since $G_{\alpha} \cong \text{PSL}(2, 19)$ or $\text{PGL}(2, 19)$, $|G| = |\Delta\Omega| |G_{\alpha}| = 2 \cdot 29 \cdot 2^i \cdot 18 \cdot 19 \cdot 20 / 2 = 2^{3+i} \cdot 3^2 \cdot 5 \cdot 19 \cdot 29$ with $0 \leq i \leq 1$. Let P be a Sylow 29-subgroup of G . Then P is semi-regular on Ω and $3, 5, 19 \nmid |N_G(P)|$ because $N_G(P)/C_G(P) \leq Z_4 \times Z_7$. Hence $|G : N_G(P)| = 2^j \cdot 3^2 \cdot 5 \cdot 19$ with $0 \leq j \leq 4$, while $2^j \cdot 3^2 \cdot 5 \cdot 19 \not\equiv 1 \pmod{29}$ for any j with $0 \leq j \leq 4$, contrary to a Sylow's theorem.

• If $C_G(z)^{F(z)} = R(3)$, $r+1 = 4$ and $1 + (q-\varepsilon)r/4 = 28$ and hence $r = 3$, $q = 37$, contrary to $r \mid n$.

Osaka Kyoiku University

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