ON p-RADICAL DESCENT OF HIGHER EXPONENT

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§ 0. Introduction

In the paper [8], P. Samuel has developed the theory of p-radical descent of exponent one by making use of logarithmic derivatives. In this article we shall give a generalization of his theory to the case of p-radical descent of higher exponent with the aid of a finite set of higher derivations of finite rank.

In the first section some preparatory results are collected. Let A be a Krull domain of characteristic $p > 0$ and K be its quotient field. Let $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations $\underline{D}^{(i)}$'s of rank m_i on K which leave A invariant. For simplicity we shall abuse the notation $\underline{D}^{(i)}$ to denote the ring homomorphism of K into a truncated polynomial ring of order m_i over K, i.e., $K[t_i : m_i] := K[T_i]/T_i^{m_i+1}$ associated to the higher derivation $\underline{D}^{(i)}$. Let K' be the intersection of the fields of $\underline{D}^{(i)}$-constants $(1 \leq i \leq r)$ and let $A' := A \cap K'$. Let $\mathbb{T} = (T_1, \ldots, T_r)$ be an r-tuple of indeterminates and let t_i be the residue class of T_i modulo $T_i^{m_i+1}$ in $K[T_i]/T_i^{m_i+1}$. We shall set $\mathbf{t} := (t_1, \ldots, t_r)$ and $\mathbb{m} := (m_1, \ldots, m_r)$. We shall denote $\prod_{i=1}^{r} K[t_i : m_i]$ by $K[\mathbf{t} : \mathbb{m}]$. Similarly we denote

$\prod_{i=1}^{r} A[t_i : m_i]$ by $A[t : m]$ where $A[t_i : m_i]$ is a truncated polynomial ring of order m_i over A. Furthermore we shall define a ring homomorphism $\mathbb{D}$ of K into $K[t : m]$ by $\mathbb{D}(z) = (\underline{D}^{(1)}(z), \ldots, \underline{D}^{(r)}(z))$ $(z \in K)$. Let $\mathcal{L}_A$ and $\mathcal{L}'_A$ be the sets of elements defined respectively by

$$\mathcal{L}_A = \left\{ \mathbb{D}(z)/z \in K[t : m] \,\middle|\, z \in K^*,\ \mathbb{D}(z)/z \in A[t : m] \right\},$$
$$\mathcal{L}'_A = \left\{ \mathbb{D}(u)/u \,\middle|\, u \in A^* \right\}.$$

Let $\mathbb{j} : \mathrm{Div}(A') \longrightarrow \mathrm{Div}(A)$ be the homomorphism defined by $\mathbb{j}(\mathfrak{p}) = e(\wp)\wp$ where, $\mathfrak{p}$ is a prime ideal of height one in A', $\wp$ is the unique prime ideal of height one in A with $\wp \cap A' = \mathfrak{p}$ and $e(\wp)$ is the ramification index of $\wp$ over $\mathfrak{p}$. Then we can define the homomorphism $\bar{\mathbb{j}} : \mathrm{Cl}(A') \longrightarrow \mathrm{Cl}(A)$ induced by $\mathbb{j}$ (cf. [8]). Let $\mathcal{D}$ be the subgroup of $\mathrm{Div}(A')$ consisting of divisors E's such that $\mathbb{j}(E)$ is principal and let $\Phi_0 : \mathcal{D} \longrightarrow \mathcal{L}_A/\mathcal{L}'_A$ be the homomorphism defined by $\Phi_0(E) = \mathbb{D}(x)/x$ modulo $\mathcal{L}'_A$, where $E \in \mathcal{D}$ and $\mathbb{j}(E) = \mathrm{div}_A(x)$. Let $\Phi : \mathrm{Ker}(\bar{\mathbb{j}}) = \mathcal{D}/F(A') \longrightarrow \mathcal{L}_A/\mathcal{L}'_A$ be the homomorphism induced by Φ_0 where $F(A')$ denotes the subgroup of $\mathrm{Div}(A')$ generated by principal divisors. Furthermore we put $\mu_i = \min\left\{ j \,\middle|\, D_j^{(i)} \neq 0,\ 1 \leq j \leq m_i \right\}$ and, $n_i = \min \left\{ n \,\middle|\, m_i < \mu_i p^n \right\}$ where $\underline{D}^{(i)} = \left\{ D_j^{(i)} \,\middle|\, 0 \leq j \leq m_i \right\}$ $(1 \leq i \leq r)$. We denote the Jacobian $\det(D_{\mu_i}^{(i)}(\alpha_k))_{s \leq i,\ k \leq r}$ by $J(\mathbb{D} : \alpha;\ s,\ r)$ for $\alpha = (\alpha_1, \ldots, \alpha_r) \in A^r$ and $1 \leq s \leq r$. We shall use the notation $J(\mathbb{D}:\alpha)$ instead of $J(\mathbb{D}:\alpha;\ 1,\ r)$. Our main result in § 1 is the following:

Theorem (cf. 1.6) Assume that the following two conditions hold:

(1) $[K : K'] = p^{n_1+\cdots+n_r}$.

(2) For each prime ideal $\wp$ of height one in A, there exists α in A^r such that the Jacobian $J(D:\alpha)$ is not contained in $\wp$.

Then the homomorphism $\Phi : \mathrm{Ker}(\bar{\mathbb{J}}) \longrightarrow \mathcal{L}_A/\mathcal{L}'_A$ is an isomorphism.

The property (2) in the above theorem will be referred to as "the height one property". When the height one property is not satisfied, Φ is not necessarily surjective. Even if Φ is not surjective, we can determine, in some cases, the cokernel of Φ (§ 2). As a byproduct we get the following:

Theorem (cf. 2.7). Assume that A is a unique factorization domain with $J(\mathbb{D}: A) := \{ J(\mathbb{D} : A) \mid \alpha \in A^r \} \neq \{0\}$ and $[K : K'] = p^{n_1+\cdots+n_r}$. Let $\wp = cA$ be a principal prime ideal of height one in A and let $s^{(i)}(\wp) := \min \{ s \in \mathbb{N} \mid (\underline{D}^{(i)}(c)/c)^s \in A[t_i : m_i] \}$ for $1 \leq i \leq r$, and $s(\wp) := \max \{ s^{(i)}(\wp) \mid 1 \leq i \leq r \}$. Then the followings are equivalent to each other:

(i) $\Phi : \mathrm{Ker}(\bar{J}) \longrightarrow \mathcal{L}_A/\mathcal{L}'_A$ is an isomorphism.

(ii) For each prime ideal $\wp$ of height one in A, either $J(D : A) \not\subset \wp$ or $e(\wp) = s(\wp)$ occurs.

If A is a unique factorization domain, it turns out that $\mathrm{Ker}(\bar{J})$ is isomorphic to $\mathrm{Cl}(A')$. Therefore, in order to determine $\mathrm{Cl}(A')$, it suffices to know $\mathrm{Ker}(\bar{J})$. In the final section some examples of rings are presented whose divisor class groups are calculated by applying Theorem 1.6.

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Each ring appeared in this paper is commutative with identity. Our terminology and notation are as follows:

Let A be a Krull domain.

$P(A)$: the set of prime ideals of height one in A.

$\mathrm{Div}(A)$: the free abelian group generated by elements of $P(A)$. An element of $\mathrm{Div}(A)$ is called a divisor.

We shall define the divisor $\mathrm{div}_A(a)$ $(a \in A - \{0\})$ by $\mathrm{div}_A(a) = \sum v_\wp(a)\wp$ where the sum is taken over all prime ideals $\wp$'s in $P(A)$ and $v_\wp$ is the normalized valuation associated to the prime ideal $\wp$. Let K be the quotient field of A and x be an element of K^*. We define $\mathrm{div}_A(x) := \mathrm{div}_A(a) - \mathrm{div}_A(b)$ where $x = a/b$ $(a, b \in A, b \neq 0)$.

$F(A)$: the subgroup of $\mathrm{Div}(A)$ generated by $\{\mathrm{div}_A(x) \mid x \in K^*\}$. We call an element of $F(A)$ a principal divisor.

$Cl(A) := Div(A)/F(A)$: the divisor class group of A.

$cl(E)$: the divisor class of a divisor E.

$Supp(E)$: the support of a divisor E, i.e., the set of all prime ideals $\wp$'s in $P(A)$ such that $E = \sum n_\wp \wp$ and $n_\wp \neq 0$.

§ 1. Fundamental theorem

Let A and B be commutative rings with common identity such that $A \subset B$. A higher derivation $\underline{D} = \{ D_j \mid 0 \leq j \leq m \}$ of rank m of A into B is a collection of additive homomorphisms of A into B satisfying the following conditions:

(1) $D_0(a) = a$ for all a in A.

(2) $D_n(ab) = \sum_{j=0}^{n} D_j(a)D_{n-j}(b)$

for $0 \leq n \leq m$ and $a, b \in A$ (cf. [5], [6]).

Let $B[t : m]$ be a truncated polynomial ring of order m over B, i.e., $B[t : m] = B[T]/T^{m+1}$. We can define the ring homomorphism $\phi_{\underline{D}}$ of A into $B[t : m]$ associated to a higher derivation $\underline{D}$ by the following manner:

$$\phi_{\underline{D}}(a) = \sum_{j=0}^{m} D_j(a)t^j \quad \text{for } a \in A.$$

For simplicity we shall abuse the notation $\underline{D}$ to denote the ring homomorphism $\phi_{\underline{D}}$ when there is no fear of confusion. If $\underline{D}(a) = a$, a is called a $\underline{D}$-constant. We say that $\underline{D}$ is non-trivial if there exists an element in A which is not a

$\underline{D}$-constant. For a non-trivial higher derivation $\underline{D}$, the smallest integer among those j such that $D_j \neq 0$ for $1 \leq j \leq m$ is denoted by $\mu(\underline{D})$. Let C be a subset of A. We say that $\underline{D}$ leaves C invariant if $D_j(C) \subset C$ for $1 \leq j \leq m$. Let $\underline{D}^{(i)}$ be a higher derivation of rank m_i of A into B for $1 \leq i \leq r$. Let $T = (T_1, \ldots, T_r)$ be an r-tuple of indeterminates $T_1, \ldots, T_r$ and let $t := (t_1, \ldots, t_r)$ where t_i is the residue class of T_i modulo $T_i^{m_i+1}$ in $B[T_i]/T_i^{m_i+1}$. We shall denote $\prod_{i=1}^{r} B[t_i : m_i]$ by $B[t : m]$ where $m := (m_1, \ldots, m_r)$. Then $B[t : m]$ is a B-algebra in the usual way. Let $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r)})$ be an r-tuple of higher derivations of rank m of A into B. A ring homomorphism $\mathbb{D}$ of A into $B[t : m]$ is defined by $\mathbb{D}(a) = (\underline{D}^{(1)}(a), \ldots, \underline{D}^{(r)}(a))$ $(a \in A)$. The intersection of $\underline{D}^{(i)}$-constants for $1 \leq i \leq r$ is called the ring of $\mathbb{D}$-constants. First we shall prove two lemmas:

Lemma 1.1. Let $A \subset B$ be integral domains of characteristic $p > 0$ and let $\underline{D} = \{ D_j \mid 0 \leq j \leq m \}$ be a non-trivial higher derivation of rank m of A into B. Set $\mu := \mu(\underline{D})$ and $d_i := D_{\mu p^i}$. Then $d_s(\alpha^{p^k}) = 0$ if $s < k$ and $d_s(\alpha^{p^k}) = d_{s-k}(\alpha)^{p^k}$ if $s \geq k$ $(\alpha \in A,\ \mu p^s \leq m)$.

Proof. The proof is easy, hence we omit it. Q.E.D.

Lemma 1.2. Let $M = (a_{ij})_{1\le i,\ j\le r}$ be a non-singular matrix. Then after a suitable change of columns we can bring M into the one such that every $M^{(k)}$ $(1 \le k \le r)$ is a non-singular matrix where

$$M^{(k)} = \begin{pmatrix} a_{kk} & \cdots & a_{kr} \\ & \cdots & \\ a_{rk} & \cdots & a_{rr} \end{pmatrix}.$$

Proof. Let α_{ij} be the cofactor of a_{ij}. Then $\det M = a_{11}\alpha_{11} + a_{12}\alpha_{12} + \cdots + a_{1r}\alpha_{1r}$. Since $\det M$ does not vanish, $\alpha_{1j'} \neq 0$ for some j'. Interchanging the first column with the j'-th column, we may assume $\alpha_{11} \neq 0$, i.e., $\det M^{(2)} \neq 0$. Continuing this process we will arrive at the desired situation. Q.E.D.

Let $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations of rank $m = (m_1, \ldots, m_r)$. We shall set $\mu_i := \mu(\underline{D}^{(i)})$ and $n_i := \min\{n \in \mathbb{N} \mid m_i < \mu_i p^n\}$ where $\mathbb{N}$ denotes the set of positive integers. Furthermore we shall set $n(\mathbb{D}) = n_1 + \cdots + n_r$. Then $D^{(i)}_{\mu_i}$ is a derivation. We denote the Jacobian $\det(D^{(i)}_{\mu_i}(\alpha_k))$ by $J(\mathbb{D} : \boldsymbol{\alpha})$ for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in A^r$. Let $T = (T_1, \ldots, T_r)$ be an r-tuple of indeterminates $T_1, \ldots, T_r$. We shall denote $(T_1^{\mu_1 p^j}, \ldots, T_r^{\mu_r p^j})$

by $T^{p^j\mu}$ where $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$.

Proposition 1.3. Let $L \subset F$ be fields of characteristic $p > 0$ and let $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r)})$ be an r-tuple of higher derivations of rank $m = (m_1, \ldots, m_r)$ of L into F. Let L' be the field of $\mathbb{D}$-constants. Suppose that there exists an element $\alpha = (\alpha_1, \ldots, \alpha_r)$ in L^r such that the Jacobian $J(\mathbb{D} : \alpha)$ does not vanish. Then we have $[L : L'] \geq p^{n(\mathbb{D})}$. Furthermore if the equality holds, then $L = L'[\alpha_1, \ldots, \alpha_r]$.

Proof. (I) First we shall prove the Proposition in the case $n := n_1 = \ldots = n_r$. Let L_j be a subfield of L defined by $\{ z \in L \mid \mathbb{D}(z) = (z, \ldots, z) \mod T^{p^j\mu} \}$ for $1 \leq j \leq n$. Then we have $L_0 \supset L_1 \supset \ldots \supset L_n$ where we put $L_0 := L$ and $L_n := L'$. It suffices to show that $[L_{j-1} : L_j] \geq p^r$ for $1 \leq j \leq n$. For simplicity we shall set $d_j^{(i)} := D^{(i)}_{\mu_i p^j}$. From the definition of L_{j-1}, the restriction of $d_{j-1}^{(i)}$ to L_{j-1} is a derivation of L_{j-1} for $1 \leq i \leq r$. Let $\widetilde{L_{j-1}}$ be the intersection of the kernels of these derivations. Then we have $L_{j-1} \supset \widetilde{L_{j-1}} \supset L_j$. By Lemma 1.1 we know $J(\mathbb{D}|L_{j-1} : \alpha^{p^{j-1}}) = J(\mathbb{D} : \alpha)^{p^{j-1}} \neq 0$ and $\alpha^{p^{j-1}} \in L_{j-1}^r$. Hence these derivations are linearly independent over F. This implies that $[L_{j-1} : \widetilde{L_{j-1}}] \geq p^r$, hence $[L_{j-1} : L_j] \geq p^r$. From our argument we get the following sequence:

$$L_{j-1} \supset L_j^{\#} := L_j[\alpha_1^{p^{j-1}}, \ldots, \alpha_r^{p^{j-1}}] \supset L_j$$

for $1 \leq j \leq n$. To prove the latter half, assume that $[L : L'] = p^{nr}$. Then we have $[L_{j-1} : L_j] = p^r$. Since $d_{j-1}^{(i)}|L_j^{\#}$ $(1 \leq i \leq r)$ are linearly independent over F, $[L_j^{\#} : L_j] \geq p^r$. Therefore we see that $L_{j-1} = L_j^{\#}$ for $1 \leq j \leq n$, hence $L = L'[\alpha_1, \ldots, \alpha_r]$.

(II) Next we shall prove the general case. Without loss of generality we may assume that $n_1 \leq n_2 \leq \ldots \leq n_r$. Moreover by Lemma 1.2 we may assume that $J(\mathbb{D} : \alpha ; k, r) \neq 0$ for $1 \leq k \leq r$. This implies that for every k there exists an integer k' such that $d_0^{(k)}(\alpha_{k'}) \neq 0$ and $k \leq k' \leq r$. Let $\bar{n}_1 < \ldots < \bar{n}_\rho$ be integers with the property $\{n_1, \ldots, n_r\} = \{\bar{n}_1, \ldots, \bar{n}_\rho\}$ and let $r_\lambda := \#\{i \mid n_i = \bar{n}_\lambda, 1 \leq i \leq r\}$ for $1 \leq \lambda \leq \rho$. Then we know

$$r_1 + r_2 + \ldots + r_\rho = r,$$

$$r_1\bar{n}_1 + r_2\bar{n}_2 + \ldots + r_\rho\bar{n}_\rho = n_1 + n_2 + \ldots + n_r.$$

For convenience sake we put $r_0 := 0$, $\bar{n}_0 := 0$ and $\delta_\lambda := r_0 + \ldots + r_\lambda$. Let K_λ be the subfield of L defined by

$$\Big\{ z \in L \,\Big|\, \underline{D}^{(h)}(z) \equiv z \mod T_h^{m_h+1} \quad (1 \leq h \leq \delta_\lambda),$$

$$\underline{D}^{(\chi)}(z) \equiv z \mod T_\chi^{w_\chi} \quad (w_\chi = \mu_\chi p^{\bar{n}_\lambda}, \quad \delta_\lambda < \chi \leq r) \Big\}$$

for $1 \leq \lambda \leq \rho - 1$ (note that $n_\chi \geq \bar{n}_{\lambda+1} > \bar{n}_\lambda$). Then we have

$$K_0 := L \supset K_1 \supset \dots \supset K_{\rho-1} \supset K_\rho := L'.$$

We shall claim the following inequality for $1 \leq \lambda \leq \rho$:

$$[K_{\lambda-1} : K_\lambda] \geq p^{\varepsilon_\lambda}$$

where $\varepsilon_\lambda := (r - \delta_{\lambda-1})(\bar{n}_\lambda - \bar{n}_{\lambda-1})$. Let $\underline{\Delta}^{(i)}$ be the restriction of $\underline{D}^{(i)}$ to $K_{\lambda-1}$. Then for $1 \leq \lambda < \rho$, $\underline{\Delta}^{(i)}$ is a higher derivation of $K_{\lambda-1}$ into F of rank m_i for $\delta_{\lambda-1} < i \leq \delta_\lambda$ and of rank $w_i - 1$ for $\delta_\lambda < i \leq r$ respectively. For $\lambda = \rho$, $\underline{\Delta}^{(i)}$ is a higher derivation of $K_{\lambda-1}$ into F of rank m_i for $\delta_{\rho-1} < i \leq r$. The following five assertions are easily verified:

(1) $K_\lambda = \bigcap_{i=\delta_{\lambda-1}+1}^{r}$ (the field of $\underline{\Delta}^{(i)}$-constants).

(2) $\mu(\underline{\Delta}^{(i)}) = \mu_i p^{\bar{n}_{\lambda-1}}$ $(\delta_{\lambda-1} < i \leq r)$.

(3) For $1 \leq \lambda \leq \rho$,

$$\min\left\{ s \in \mathbb{N} \mid m_u < \mu_u p^{\bar{n}_{\lambda-1}+s} \right\} = \bar{n}_\lambda - \bar{n}_{\lambda-1} \quad (\delta_{\lambda-1} < u \leq \delta_\lambda).$$

For $1 \leq \lambda < \rho$,

$$\min\left\{ s \in \mathbb{N} \mid \mu_v p^{\bar{n}_\lambda} \leq \mu_v p^{\bar{n}_{\lambda-1}+s} \right\} = \bar{n}_\lambda - \bar{n}_{\lambda-1} \quad (\delta_\lambda < v \leq r)$$

where $\mathbb{N}$ denotes the set of positive integers.

(4) $\alpha_i^q \in K_{\lambda-1}$ where $q := p^{\bar{n}_{\lambda-1}}$ $(\delta_{\lambda-1} < i \leq r)$.

(5) $J(\Delta : \alpha^q ; \delta_{\lambda-1} + 1, r) = J(\mathbb{D} : \alpha ; \delta_{\lambda-1} + 1, r)^q$ $\neq 0$ where $\Delta = (\underline{\Delta}^{(1)}, \dots, \underline{\Delta}^{(r)})$. Therefore we get $[K_{\lambda-1} : K]$

$\geq p^{\varepsilon_\lambda}$. Furthermore $\sum_{\lambda=1}^{\rho} \varepsilon_\lambda = n_1 + \dots + n_r = n(\mathbb{D})$. Hence we have $[L : L'] \geq p^{n(\mathbb{D})}$. In order to prove the latter half, it suffices to prove the following: $K_{\lambda-1} = \widetilde{K}_\lambda$ where $\widetilde{K}_\lambda := K_\lambda[\alpha_i^q;\ \delta_{\lambda-1} < i \leq r]$ for $1 \leq \lambda \leq \rho$. Since $[L : L'] = p^{n(\mathbb{D})}$, we have $[K_{\lambda-1} : K_\lambda] = p^{\varepsilon_\lambda}$. Applying the step (I) to $\widetilde{K}_\lambda$ and $\underline{\Delta}^{(i)}|\widetilde{K}_\lambda$ ($\delta_{\lambda-1} < i \leq r$), it is seen that $[\widetilde{K}_\lambda : K_\lambda] \geq p^{\varepsilon_\lambda}$. Since $K_{\lambda-1} \supset \widetilde{K}_\lambda \supset K_\lambda$, we have $K_{\lambda-1} = \widetilde{K}_\lambda$.

Q.E.D.

Remark 1.4. The converse of the latter half of the Proposition 1.3 does not hold. Let k be a field of characteristic $p > 0$. Let x, y be indeterminates over k and let $L := k(x, y)$. Let $\underline{D}^{(i)}$ $(i = 1, 2)$ be higher derivations on L over k of rank $p - 1$ and $p^2 - 1$ defined respectively by

$$\underline{D}^{(1)}(x) = x(1 + t_1),\ \underline{D}^{(1)}(y) = y + t_1,$$

$$\underline{D}^{(2)}(x) = x + t_2,\ \underline{D}^{(2)}(y) = y(1 + t_2).$$

Then $n_1 = 1$, $n_2 = 2$ and $J(\mathbb{D} : (x, y)) = xy - 1 \neq 0$. By a simple calculation we see that $L' = k(x^{p^2}, y^{p^2})$. Therefore $L = L'[x, y]$, while $[L : L'] = p^4 > p^{n_1+n_2}$.

(1.5) Let A be a Krull domain of characteristic $p > 0$ with the quotient field K. Let $\mathbb{D} = (\underline{D}^{(1)}, \dots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations of rank $m = (m_1, \ldots, m_r)$ on K which leave A invariant. Let K' be the field of $\mathbb{D}$-constants and $A' := A \cap K'$. Then A' is also a Krull domain. Since any element of K is of the form a/b with $a \in A$, $b \in A'$, K' is the quotient field of A'. For any prime ideal $\mathfrak{q}$ in $P(A')$, there exists only one prime ideal $\wp$ in $P(A)$ such that $\wp \cap A' = \mathfrak{q}$. From this fact we define the homomorphism $\mathbb{j} : \mathrm{Div}(A') \longrightarrow \mathrm{Div}(A)$ by $\mathbb{j}(\mathfrak{q}) = e(\wp)\wp$ where $e(\wp)$ stands for the ramification index of $\wp$ over $\mathfrak{q}$. Since A is integral over A', we can define the canonical homomorphism $\bar{\mathbb{j}} : \mathrm{Cl}(A') \longrightarrow \mathrm{Cl}(A)$ induced by the homomorphism $\mathbb{j}$ (cf. [8]).

Let $\mathcal{L}_A$ and $\mathcal{L}'_A$ be sets of elements defined respectively by

$$\mathcal{L}_A := \{ \mathbb{D}(z)/z \in K[t : m] \mid z \in K^*, \mathbb{D}(z)/z \in A[t : m] \},$$

$$\mathcal{L}'_A := \{ \mathbb{D}(u)/u \mid u \in A^* \}$$

where $*$ denotes the set of invertible elements. Since we have

$$(\mathbb{D}(z_1)/z_1)(\mathbb{D}(z_2)/z_2) = \mathbb{D}(z_1 z_2)/z_1 z_2$$

and

$$(\mathbb{D}(z)/z)^{-1} = \mathbb{D}(z^{-1})/z^{-1} \quad (z \neq 0),$$

$\mathcal{L}_A$ is an abelian group and $\mathcal{L}'_A$ is its subgroup.

Let $\mathcal{D}$ be the subgroup of $\mathrm{Div}(A')$ consisting of divisors E's such that $\mathbb{j}(E)$ is principal. Then we get $\mathrm{Ker}(\bar{\mathbb{j}}) = \mathcal{D}/F(A')$. We shall define the homomorphism Φ_0 of $\mathcal{D}$ into $\mathcal{L}_A/\mathcal{L}'_A$ by the following manner: Let E be a divisor of $\mathcal{D}$ and x be an element of K^* satisfying $\mathbb{j}(E) = \mathrm{div}_A(x)$. Then we set $\Phi_0(E) := \mathbb{D}(x)/x$ modulo $\mathcal{L}'_A$. It is easily seen that Φ_0 is well-defined. Moreover if x' is in K', $\Phi_0(\mathrm{div}_{A'}(x')) = \mathbb{D}(x')/x' = \mathbb{1}$ where $\mathbb{1} = (1, \ldots, 1) \in A^r$, hence the homomorphism Φ of $\mathrm{Ker}(\bar{\mathbb{j}})$ into $\mathcal{L}_A/\mathcal{L}'_A$ induced by the homomorphism Φ_0 is also well-defined. On the other hand, the relation $\mathbb{D}(x)/x = \mathbb{D}(u)/u$ $(x \in K^*, u \in A^*)$ implies $\mathbb{D}(xu^{-1})/xu^{-1} = \mathbb{1}$, i.e., $xu^{-1} \in K'$ and $E = \mathrm{div}_{A'}(xu^{-1})$. This implies that Φ is injective (cf. [8], p.86). Set $\mu := (\mu_1, \ldots, \mu_r)$ and $n(\mathbb{D}) := n_1 + \cdots + n_r$ where $\mu_i := \mu(\underline{D}^{(i)})$ and $n_i := \min\{ n \in \mathbb{N} \mid m_i < \mu_i p^n \}$ $(1 \leq i \leq r)$.

Theorem 1.6. Let A, K, K', $\mathbb{D}$ and $n(\mathbb{D})$ have the same meaning as in 1.5. Assume the following two conditions hold:

(1) $[K : K'] = p^{n(\mathbb{D})}$.

(2) For each prime ideal $\wp$ in $P(A)$, there exists an element α in A^r such that the Jacobian $J(\mathbb{D} : \alpha)$ is not contained in $\wp$.

Then the homomorphism $\Phi : \mathrm{Ker}(\bar{\mathbb{j}}) \longrightarrow \mathcal{L}_A/\mathcal{L}'_A$ is an isomorphsm.

Proof. Since Φ is injective, it suffices to prove the following: If $\mathbb{D}(x)/x$ is in $\mathcal{L}_A$ $(x \in K^*)$, then there exists a divisor E in $\mathcal{D}$ such that $\mathbb{j}(E) = \mathrm{div}_A(x)$. Set

$n := \max\{n_1, \ldots, n_r\}$. Note that for each prime ideal in $P(A')$ there exists a unique prime ideal in $P(A)$ which contracts to $\mathfrak{p}$ because $A^{p^n} \subset A'$. Therefore the surjectivity of Φ is seen by showing that if $\mathbb{D}(x)/x$ is in $\mathcal{L}_A$ $(x \in K^*)$, then $e(\mathfrak{p})$ divides $v_{\mathfrak{p}}(x)$ for every prime ideal $\mathfrak{p}$ in $P(A)$ where $v_{\mathfrak{p}}(x)$ denotes the normalized valuation of K associated to the prime ideal $\mathfrak{p}$. Hence by localizing, we may assume that A is a discrete valuation ring with the maximal ideal $\mathfrak{p}$. Thus we have only to show the following:

Proposition 1.7. Let A be a discrete valuation ring with the maximal ideal $\mathfrak{p}$ and let K, K', $\mathbb{D}$ and $n(\mathbb{D})$ have the same meaning as in 1.5. Assume that the following two conditions hold:

(1) $[K : K'] = p^{n(\mathbb{D})}$.

(2) There exists an element α in A^r such that the Jacobian $J(\mathbb{D} : \alpha)$ is not contained in $\mathfrak{p}$.

If $\mathbb{D}(x)/x$ is in $\mathcal{L}_A$ $(x \in K^*)$, then e divides $v(x)$ where we put $e := e(\mathfrak{p})$ and v is the normalized valuation of K associated to A.

Proof. Our proof consists of several steps:

(I) First we shall consider the case $m_i = 1$ (hence $\mu_i = n_i = 1$) for $1 \leq i \leq r$. We shall set $\underline{D}^{(i)} = \{id, D^{(i)}\}$.

Then $D^{(i)}$'s are derivations. We shall define the higher derivation $\underline{\Delta}^{(i)} = \{ \mathrm{id}, \Delta^{(i)} \}$ of rank 1 on K in the following way:

$$\Delta^{(i)}(z) = J^{-1}\det\begin{pmatrix} D^{(1)}(\alpha_1), & \ldots, & \overset{i}{\overbrace{D^{(1)}(z)}}, & \ldots, & D^{(1)}(\alpha_r) \\ & \ldots & & \ldots & \\ D^{(r)}(\alpha_1), & \ldots, & D^{(r)}(z), & \ldots, & D^{(r)}(\alpha_r) \end{pmatrix}$$

for $z \in K$ $(1 \leq i \leq r)$ where $J := J(\mathbb{D} : \alpha)$. Then we have $\Delta^{(i)}(\alpha_k) = \delta_{ik}$ where δ_{ik} denotes the Kronecker's delta $(1 \leq i, k \leq r)$. Since J is not in $\wp$, J is a unit of A, hence $\Delta^{(i)}(A) \subset A$ for $1 \leq i \leq r$. Set $\mathbb{\Delta} := (\underline{\Delta}^{(1)}, \ldots, \underline{\Delta}^{(r)})$. Since $\Delta^{(i)}$ is an A-linear combination of $D^{(k)}$'s and $D^{(k)}$ is also an A-linear combination of $\Delta^{(k)}$'s, we have the following three assertions:

(1) K' is the field of $\mathbb{\Delta}$-constants.

(2) $J(\mathbb{\Delta} : \alpha) = 1$.

(3) $\mathbb{\Delta}(x)/x \in \mathcal{L}_A$.

Hence it suffices to prove the Proposition with respect to $\mathbb{\Delta}$ instead of $\mathbb{D}$. We shall prove that e divides $v(x)$ by induction on r. As is well known e takes no other value than some power of p. Hence in the case $r = 1$, it suffices to prove the following: If p does not divide $v(x)$, then $e = 1$.

Let π be a uniformisant of the discrete valuation ring A. Then we can write $x = u\pi^{v(x)}$ for some $u \in A^*$. Since

$$\Delta^{(1)}(u)/u + v(x)\Delta^{(1)}(\pi)/\pi = \Delta^{(1)}(x)/x \in A$$

and since p does not divide $v(x)$, we have $\Delta^{(1)}(\pi)/\pi \in A$. This implies that we can define the derivation $\tilde{\Delta}^{(1)}$ of $A/\wp$ induced by $\Delta^{(1)}$. Set $\mathfrak{k} := A/\wp$ and $\mathfrak{k}' := A'/\mathfrak{g}$ where $\mathfrak{g} := \wp \cap A'$. Since $\Delta^{(1)}(\alpha_1) = 1$ implies $\tilde{\Delta}^{(1)} \neq 0$, we have $[\mathfrak{k} : \mathfrak{k}'] > 1$. Therefore from the inequality $e[\mathfrak{k} : \mathfrak{k}'] \leq [K : K'] = p$, it follows that $e = 1$.

Suppose $r > 1$ and the assertion holds for $r - 1$. Set $\overline{K}$:= the field of $\underline{\Delta}^{(1)}$-constants and $\overline{A} := A \cap \overline{K}$. Since $[K : K'] = p^r$ and $J(\Delta|\overline{K} : \alpha ; 2, r) = 1$, Proposition 1.3 implies that $[K : \overline{K}] = p$ and $[\overline{K} : K'] = p^{r-1}$. Furthermore we have $K = \overline{K}[\alpha_1]$ and $\overline{K} = K'[\alpha_2, \ldots, \alpha_r]$. Then the restriction of $\Delta^{(i)}$ to $\overline{K}$ is a derivation on $\overline{K}$ such that $\Delta^{(i)}(\overline{A}) \subset \overline{A}$ for $2 \leq i \leq r$. Let e_1 be the ramification index of $\wp$ over $\wp \cap \overline{A}$. Since $[K : \overline{K}] = p$ and $\Delta^{(1)}(\alpha_1) = 1$, e_1 divides $v(x)$ from the argument in the case $r = 1$. Therefore we can write $x = uy$ for some u in A^* and y in $\overline{K}^*$. It follows from $\Delta(x)/x = (\Delta(u)/u)(\Delta(y)/y)$ that $\Delta(y)/y \in (A \cap \overline{K})[t : m] = \overline{A}[t : m]$. Furthermore $J(\Delta|\overline{K} : \alpha ; 2, r) = 1 \in \overline{A}^*$ and $\alpha_2, \ldots, \alpha_r \in \overline{A}$. Let e_2 be the ramification index of $\overline{\mathfrak{g}} := \wp \cap A$ over $\mathfrak{g}' := \wp \cap A'$ and $\overline{v}$ be the normalized valuation of K associated to the prime ideal $\overline{\mathfrak{g}}$. Apply the induction assumption to $\Delta|\overline{K}$, then we see that e_2 divides $\overline{v}(y)$. On the other hand $v(x) = v(y)$

$= e_1\bar{v}(y)$ and $e = e_1e_2$. Hence e divides $v(x)$.

(II) Suppose that $n := n_1 = \dots = n_r$. We shall prove the Proposition by induction on n. For the case $n = 1$, let $\widetilde{K} = \{ z \in K \mid \mathbb{D}(z) \equiv (z, \dots, z) \mod T^{\mu+\mathbb{1}} \}$. Then $K \supset \widetilde{K} \supset K'$ and Proposition 1.3 implies that $[K : \widetilde{K}] \geq p^r$. Since $[K : K'] = p^r$, we get $\widetilde{K} = K'$ and e divides $v(x)$ by the previous argument. Suppose that $n > 1$ and the Proposition is proved for $n - 1$. Let $L_1 = \{ z \in K \mid \mathbb{D}(z) \equiv (z, \dots, z) \mod T^{p\mu} \}$ and $A_1' := A \cap L_1$. It is easily seen that

(1) $\mu(\underline{D}^{(i)} | L_1) = \mu_i p$.

(2) $\min\{ s \in \mathbb{N} \mid m_i < \mu_i p^{1+s} \} = n_i - 1 = n - 1$ $(1 \leq i \leq r)$.

(3) $J(\mathbb{D}|L_1 : \alpha^p) = J(\mathbb{D} : \alpha)^p \notin \mathfrak{P}_1 := \mathfrak{P} \cap A_1'$.

(4) $\alpha^p \in A_1^r$.

Hence Proposition 1.3 implies that $[K : L_1] = p^r$ and $[L_1 : K'] = p^{(n-1)r}$ because $[K : K'] = p^{nr}$. We shall prove that the restriction of $\mathbb{D}$ to L_1 is an r-tuple of non-trivial higher derivations of rank m on L_1 which leave A_1' invariant. We know $L_1 = K'[\alpha_1^p, \dots, \alpha_r^p]$ by Proposition 1.3. For any element z in L_1, z is of the form

$$z = \sum_{i_1, \dots, i_r \in \mathbb{Z}_+} c_{i_1 \dots i_r} (\alpha_1^p)^{i_1} \dots (\alpha_r^p)^{i_r} \quad (c_{i_1 \dots i_r} \in K')$$

where $\mathbb{Z}_+$ denotes the set of non-negative integers. Therefore we get

$$\mathbb{D}(z) = \sum c_{i_1\ldots i_r}\mathbb{D}(\alpha_1^p)^{i_1}\cdots\mathbb{D}(\alpha_r^p)^{i_r}.$$

From Lemma 1.1 and the definition of L_1, it follows that $\mathbb{D}(\alpha_k^p) \in L_1[t : m]$. This implies that $\mathbb{D}(L_1) \subset L_1[t : m]$. Since $A_1' = A \cap L_1$, $\mathbb{D}|L_1$ becomes an r-tuple of non-trivial higher derivations of rank m on L_1 with the desired property. Let e_1 be the ramification index of $\wp$ over $\mathfrak{P}_1$. Let $\widetilde{K}$ be a subfield of K defined by $\{ z \in K \mid \mathbb{D}(z) \equiv (z, \ldots, z) \bmod T^{\mu+\mathbb{1}} \}$ where $\mathbb{1} = (1, \ldots, 1)$. Then we have $K \supset \widetilde{K} \supset L_1$ and Proposition 1.3 implies $[K : \widetilde{K}] \geq p^r$. Since $[K : L_1] = p^r$, we get $\widetilde{K} = L_1$ and e_1 divides $v(x)$ by the argument in (I). Hence we can write $x = uy$ for some u in A^* and y in L_1^*. Therefore $\mathbb{D}(y)/y \in A_1[t : m]$. Let e_2 be the ramification index of $\mathfrak{P}_1$ over $\wp \cap A'$ and v' be the normalized valuation of L_1 associated to the prime ideal $\mathfrak{P}_1$. By induction hypothesis, we know that e_2 divides $v'(y)$ and therefore e divides $v(x)$.

(III) We shall prove the general case. Without loss of generality we may assume the following:

(1) $n_1 \leq n_2 \leq \ldots \leq n_r$.

(2) $J(\mathbb{D} : \alpha ; k, r) \notin \wp$ for $1 \leq k \leq r$.

Let $\bar{n}_1, \ldots, \bar{n}_\rho$ and K_λ have the same meaning as in the step (II) of the proof of Proposition 1.3. We shall use the induction on ρ. The case $\rho = 1$ is treated in (II). Suppose that $\rho > 1$ and the Proposition is proved for $\rho - 1$. Proposition 1.3 and its proof shows $[K_{\lambda-1} : K_\lambda] \geq p^{\varepsilon_\lambda}$. Since $[K : K']$ $= p^{n(\mathbb{D})}$, we have $[K : K_1] = p^{r\bar{n}_1}$ and $[K_1 : K'] = p^{n(\mathbb{D})-r\bar{n}_1}$. Let $A_1 := A \cap K_1$ and e_1 be the ramification index of $\wp$ over $\mathfrak{g}_1 := \wp \cap A_1$. Then the step (II) implies that e_1 divides $v(x)$. Hence we can write $x = uy$ for some u in A^* and y in K_1^*. Then $\mathbb{D}(y)/y \in A_1[t : m]$. For $r_1 < i \leq r$, we have the followings:

(1) $\mu(\underline{D}^{(i)}|K_1) = \mu_i p^{\bar{n}_1}$.

(2) $\min\left\{ s \in \mathbb{N} \,\middle|\, m_i < \mu_i p^{\bar{n}_1+s} \right\} = n_i - \bar{n}_1$.

(3) $J(\mathbb{D}|K_1 : \alpha^q ; r_1 + 1, r) = J(\mathbb{D} : \alpha ; r_1 + 1, r)^q$ $\in A_1^*$ where $q := p^{\bar{n}_1}$.

(4) $\#\left\{ n_i - \bar{n}_1 \,\middle|\, r_1 < i \leq r \right\} < \rho$.

Let e_2 be the ramification index of $\mathfrak{g}_1$ over $\wp \cap A'$ and v' be the normalized valuation of K_1 associated to the prime ideal $\mathfrak{g}_1$. Then induction hypothesis implies that e_2 divides $v'(y)$, hence e divides $v(x)$. Q.E.D.

§ 2. Cokernel of Φ

We shall retain the same notations and assumptions used in § 1, (1.5).

Proposition 2.1. Let S be a multiplicatively closed subset of A' consisting of prime elements in A. Let H be the subgroup of $\mathrm{Div}(A')$ generated by $\mathfrak{p} \in P(A')$ such that $\mathfrak{p} \cap S \neq \emptyset$, and L be the subgroup of $\mathcal{L}_A$ generated by the set $\{D(a)/a \in \mathcal{L}_A \mid a \in A \cap A_S^*\}$. Let $L \vee \mathcal{L}'_A$ denote the subgroup of $\mathcal{L}_A$ generated by L and $\mathcal{L}'_A$. Let f be the restriction of Φ to $(H + F(A')/F(A')) \cap \mathrm{Ker}(\mathbb{J})$. Let the homomorphisms $\mathbb{J}_S : \mathrm{Cl}(A'_S) \longrightarrow \mathrm{Cl}(A_S)$, $\Phi_S : \mathrm{Ker}(\mathbb{J}_S) \longrightarrow \mathcal{L}_{A_S}/\mathcal{L}'_{A_S}$ be defined in a similar way as $\mathbb{J}$ and Φ respectively. Then we have the following commutative diagram of exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{Coker}(f) \arrow[r] \arrow[u] & \mathrm{Coker}(\Phi) \arrow[r] \arrow[u] & \mathrm{Coker}(\Phi_S) \arrow[u] & \\
0 \arrow[r] & L \vee \mathcal{L}'_A/\mathcal{L}'_A \arrow[r] \arrow[u] & \mathcal{L}_A/\mathcal{L}'_A \arrow[r] \arrow[u] & \mathcal{L}_{A_S}/\mathcal{L}'_{A_S} \arrow[u] & \\
0 \arrow[r] & (H + F(A')/F(A')) \cap \mathrm{Ker}(\mathbb{J}) \arrow[r] \arrow[u, "f"] & \mathrm{Ker}(\mathbb{J}) \arrow[r] \arrow[u, "\Phi"] & \mathrm{Ker}(\mathbb{J}_S) \arrow[r] \arrow[u, "\Phi_S"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $\mathcal{L}_A/\mathcal{L}'_A \longrightarrow \mathcal{L}_{A_S}/\mathcal{L}'_{A_S}$ is the homomorphism induced by the inclusion $\mathcal{L}_A \longrightarrow \mathcal{L}_{A_S}$ and $\mathrm{Ker}(\bar{j}) \longrightarrow \mathrm{Ker}(\bar{j}_S)$ is the natural homomorphism $\mathrm{Cl}(A') \longrightarrow \mathrm{Cl}(A'_S)$.

Proof. The homomorphism $\mathrm{Ker}(\bar{j}) \longrightarrow \mathrm{Ker}(\bar{j}_S)$ is well-defined since we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Cl}(A) \arrow[r] & \mathrm{Cl}(A_S) \\ \mathrm{Cl}(A') \arrow[r] \arrow[u, "\bar{j}"] & \mathrm{Cl}(A'_S) \arrow[u, "\bar{j}_S"'] \end{tikzcd}$$

The middle sequence forms evidently a complex. For any element $D(x)/x \in \mathcal{L}_A \cap \mathcal{L}'_{A_S}$ $(x \in K^*)$, we can write

$$D(x)/x = D(a/s)/(a/s) = D(a)/a$$

for some $a/s \in A_S^*$ $(a \in A,\ s \in S)$. Since a/s is a unit of A_S, a is in A_S^*. Hence $D(a)/a$ is in $L \vee \mathcal{L}'_A$ and the middle row is exact. The exactness of the third row is seen as follows:

$$\begin{tikzcd} 0 \arrow[r] & H + F(A')/F(A') \arrow[r] & \mathrm{Cl}(A') \arrow[r] & \mathrm{Cl}(A'_S) \arrow[r] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[u] & \mathrm{Ker}(\bar{j}) \arrow[r] \arrow[u] & \mathrm{Ker}(\bar{j}_S) \arrow[u] & \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

is commutative where $G = (H + F(A')/F(A')) \cap \mathrm{Ker}(\bar{j})$. Since S is generated by prime elements of A, we have $\mathrm{Cl}(A) \cong \mathrm{Cl}(A_S)$ ([4], Cor. 7.3, [7]). Therefore $\mathrm{Ker}(\bar{j}) \longrightarrow \mathrm{Ker}(\bar{j}_S)$ is surjective. Furthermore $\mathrm{Im}(f) \subset L \vee \mathcal{L}'_A/\mathcal{L}'_A$. The rest is immediate from the Snake lemma ([2], Chap. 1, §1. Prop. 2).

Q.E.D.

Proposition 2.2. Let $\underline{D} = \{ D_i \mid 0 \leq j \leq m \}$ be a higher derivation of rank m on A and let $\wp$ be a principal prime ideal in P(A), say, $\wp = cA$. Let

$$s_0 := \min \{ s \in \mathbb{N} \mid (\underline{D}(c)/c)^s \in A[t : m] \}$$

and

$$r_0 := \min \{ \gamma \in \mathbb{N} \mid D_\gamma(c) \notin \wp \}$$

(if $D_\gamma(c) \in \wp$ for all $1 \leq \gamma \leq m$, we put $r_0 := m + 1$).

Then the following three assertions hold:

(1) s_0 is a power of p.

(2) Write $s_0 = p^{\alpha_0}$, then $\alpha_0 = \min \{ \alpha \in \mathbb{Z}_+ \mid r_0 p^\alpha \geq m + 1 \}$

where $\mathbb{Z}_+$ denotes the set of non-negative integers.

(3) $(\underline{D}(c)/c)^h \in A[t : m]$ if and only if s_0 divides h.

Proof. (1) Write $s_0 = s' p^\alpha$, $p \nmid s'$. Then it suffices to prove that $s' = 1$. In the relation

$$(\underline{D}(c)/c)^{s_0} = (1 + \dots + (D_{r_0}(c)/c)^{p^\alpha} t^{r_0 p^\alpha} + \dots)^{s'},$$

the coefficient of $t^{r_0 p^\alpha}$ is of the form $s'(D_{r_0}(c)/c)^{p^\alpha} + a$ $(a \in A)$. If $r_0 p^\alpha > m$, then $(\underline{D}(c)/c)^{p^\alpha} \in A[t : m]$, i.e., $s' = 1$ because of the minimality of s_0. Hence if $s' > 1$, we must have $r_0 p^\alpha \leq m$. Then the coefficient of $t^{r_0 p^\alpha}$ is in A and $D_{r_0}(c)^{p^\alpha}$ is in $c^{p^\alpha}A$. This implies that $D_{r_0}(c)$ is in $cA = \wp$, which contradicts to the definition of r_0 (note that $r_0 \leq m$).

(2) Set $\alpha' := \min\{\alpha \in \mathbb{Z}_+ \mid r_0 p^\alpha \geq m+1\}$. Then we have $(\underline{D}(c)/c)^{p^{\alpha'}} \in A[t : m]$, hence by the minimality of s_0 we have $s_0 \leq p^{\alpha'}$. On the other hand $r_0 p^{\alpha_0} \geq m + 1$ because otherwise $(\underline{D}(c)/c)^{p^{\alpha_0}} \notin A[t : m]$. Hence $\alpha_0 \geq \alpha'$. Combining these, $\alpha_0 = \alpha'$.

(3) It suffices to prove the "only if" part. Write $h = s_0 q + h'$, $0 \leq h' < s_0$. Suppose that $(\underline{D}(c)/c)^h \in A[t : m]$. Since $(\underline{D}(c)/c)^{s_0} \in A[t : m]$ and $(\underline{D}(c)/c)^{s_0}$ is a unit of $A[t : m]$, we see that $(\underline{D}(c)/c)^{-s_0 q} \in A[t : m]$. Hence $(\underline{D}(c)/c)^{h'} \in A[t : m]$ and $h' = 0$ by the minimality of s_0. Q.E.D.

Corollary 2.3. In the above notations, s_0 divides e where $e := e(\wp)$.

Proof. Notice that e is a power of p because $\wp^{p^n} \subset \wp \cap A'$ for some n. Hence it remains only to prove that $(\underline{D}(c)/c)^e \in A[t : m]$. For every prime ideal $\mathfrak{q}$ in $P(A)$, we can write $c^e = ux$ for some $u \in A_{\mathfrak{q}}^*$ and $x \in K'$. Then we know that $(\underline{D}(c)/c)^e = \underline{D}(u)/u \in A_{\mathfrak{q}}[t : m]$. Since $A = \bigcap_{\mathfrak{q}} A_{\mathfrak{q}}$, we have $(\underline{D}(c)/c)^e \in A[t : m]$. Q.E.D.

Lemma 2.4. Let A be a Krull domain and let $a_1, \ldots, a_\nu$ $(\nu \geq 2)$ be elements of A such that $\mathrm{Supp}(\mathrm{div}_A(a_k)) \cap \mathrm{Supp}(\mathrm{div}_A(a_\chi)) = \emptyset$ for $1 \leq k, \chi \leq \nu$, $k \neq \chi$. Let $f_k(X)$ $(1 \leq k \leq \nu)$ be polynomials in one variable X over the quotient field of A defined by

$$f_k(X) = 1 + (\alpha_1^{(k)}X + \ldots + \alpha_m^{(k)}X^m)/a_k$$

with $\alpha_1^{(k)}, \ldots, \alpha_m^{(k)} \in A$. If the product $f_1(t)\cdots f_\nu(t)$ is in $A[t : m]$, then all $f_k(t)$'s are in $A[t : m]$ $(1 \leq k \leq \nu)$.

Proof. We shall use the induction on ν. Let γ_k be the smallest integer among those j such that $\alpha_j^{(k)}/a_k \notin A$ (if $\alpha_j^{(k)}/a_k \in A$ for all $1 \leq j \leq m$, we put $\gamma_k = m + 1$). In the case $\nu = 2$, we may assume that $\gamma_1 \leq \gamma_2$. If $\gamma_1 = m + 1$, then $\gamma_2 = m + 1$ and $f_1(t)$, $f_2(t)$ are already in $A[t : m]$, hence the Lemma is proved. Suppose that $\gamma_1 \leq m$. The coefficient of t^{γ_1} of $f_1(t)f_2(t)$ is

$$(\alpha_{\gamma_1}^{(1)}/a_1) + (\alpha_{\gamma_1-1}^{(1)}/a_1)(\alpha_1^{(2)}/a_2) + \ldots + (\alpha_{\gamma_1}^{(2)}/a_2).$$

Hence $(\alpha_{\gamma_1}^{(1)}/a_1) + (\alpha_{\gamma_1}^{(2)}/a_2)$ is in A. This means that $a_2\alpha_{\gamma_1}^{(1)} + a_1\alpha_{\gamma_1}^{(2)}$ is in a_1a_2A, hence $a_2\alpha_{\gamma_1}^{(1)}$ is in a_1A.

Since $\mathrm{Supp}(\mathrm{div}_A(a_1)) \cap \mathrm{Supp}(\mathrm{div}_A(a_2)) = \phi$, $\alpha_{\gamma_1}^{(1)}$ is in a_1A. This is absurd. Suppose that $\nu > 2$ and the assertion holds for $\nu - 1$. Notice that $\mathrm{Supp}(\mathrm{div}_A(a_1)) \cap \mathrm{Supp}(\mathrm{div}_A(a_2 \cdots a_\nu)) = \phi$. By our argument in the case $\nu = 2$, $f_1(t)$ is in $A[t : m]$ and $f_2(t) \cdots f_\nu(t)$ is in $A[t : m]$. From the induction hypothesis, it follows that $f_2(t), \ldots, f_\nu(t)$ is in $A[t : m]$. Q.E.D.

Proposition 2.5. Let $\underline{D}$ be a higher derivation of rank m on A and let $a = uc_1^{j_1} \cdots c_\nu^{j_\nu}$ ($u \in A^*$, $j_1, \ldots, j_\nu \in \mathbb{Z}$ and $c_1, \ldots, c_\nu$ are distinct prime elements of A). Let

$$s_k := \min \left\{ s \in \mathbb{N} \mid (\underline{D}(c_k)/c_k)^s \in A[t : m] \right\}.$$

Then $\underline{D}(a)/a \in A[t : m]$ if and only if s_k divides j_k for $1 \leq k \leq \nu$.

Proof. The "if" part of the Proposition is obvious. We shall prove the "only if" part. Assume that $\underline{D}(a)/a$ is in $A[t : m]$. Then we have $(\underline{D}(c_1)/c_1)^{j_1} \cdots (\underline{D}(c_\nu)/c_\nu)^{j_\nu}$ is in $A[t : m]$. Since $c_1, \ldots, c_\nu$ are distinct prime elements of A, the assumptions of Lemma 2.4 are satisfied. Hence by Lemma 2.4, $(\underline{D}(c_k)/c_k)^{j_k}$ is in $A[t : m]$ for $1 \leq k \leq \nu$. Therefore Proposition 2.2, (3) implies that s_k divides j_k

for $1 \leq k \leq \nu$. Q.E.D.

Let $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations of rank $m = (m_1, \ldots, m_r)$ on A. Let c be a prime element of A. Set

$$s^{(i)} := \min \left\{ s \in \mathbb{N} \;\middle|\; (\underline{D}^{(i)}(c)/c)^s \in A[t_i : m_i] \right\} \quad (1 \leq i \leq r)$$

and

$$s_0 := \max \left\{ s^{(i)} \;\middle|\; 1 \leq i \leq r \right\}.$$

Then s_0 is a power of p by Proposition 2.2, (1) and s_0 divides the ramification index of cA over $cA \cap A'$ by Corollary 2.3.

Let $J(\mathbb{D} : A) := \left\{ J(\mathbb{D} : \alpha) \;\middle|\; \alpha = (\alpha_1, \ldots, \alpha_r) \in A^r \right\}$. If $J(\mathbb{D} : A) \neq \{0\}$, $\left\{ \wp \in P(A) \;\middle|\; J(\mathbb{D} : A) \subset \wp \right\}$ is a finite set because A is a Krull domain.

Theorem 2.6. Let A, A', K, K', $\mathbb{D}$ and $n(\mathbb{D})$ be as before. Assume that $J(\mathbb{D} : A) \neq \{0\}$ and let $\wp_1, \ldots, \wp_\nu$ be all of $\wp$'s in P(A) such that $J(\mathbb{D} : A) \subset \wp$. Furthermore assume that $[K : K'] = p^{n(\mathbb{D})}$ and $\wp_k$'s $(1 \leq k \leq \nu)$ are principal. Set $\wp_k = c_k A$,

$$s_k^{(i)} := \min \left\{ s \in \mathbb{N} \;\middle|\; (\underline{D}^{(i)}(c_k)/c_k)^s \in A[t_i : m_i] \right\} \quad (1 \leq i \leq r)$$

and,

$$s_k := \max \left\{ s_k^{(i)} \;\middle|\; 1 \leq i \leq r \right\}.$$

Let e_k be the ramification index of $\wp_k$ over $\wp_k \cap A'$ for $1 \leq k \leq \nu$. Then we get the following exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\mathbb{j}) \xrightarrow{\Phi} \mathcal{L}_A/\mathcal{L}_A' \longrightarrow \prod_{k=1}^{\nu} \mathbb{Z}/(e_k/s_k)\mathbb{Z} \longrightarrow 0.$$

Proof. Let $n := \max\{n_1, \ldots, n_r\}$ and S be the multiplicatively closed subset of A' generated by $c_1^{p^n}$, $\ldots$, $c_\nu^{p^n}$. Then we get an isomorphism $\Phi_S : \mathrm{Ker}(\mathbb{j}_S) \longrightarrow \mathcal{L}_{A_S}/\mathcal{L}_{A_S}'$ from Theorem 1.6. Therefore Propotion 2.1 implies that $\mathrm{Coker}(f) \cong \mathrm{Coker}(\Phi)$. Hence it suffices to prove $\mathrm{Coker}(f) \cong \prod_{k=1}^{\nu} \mathbb{Z}/(e_k/s_k)\mathbb{Z}$. Set $\mathfrak{p}_k := \wp_k \cap A'$ $(1 \leq k \leq \nu)$. Then $\mathfrak{p}_1, \ldots, \mathfrak{p}_\nu$ are all prime ideals in $P(A')$ with $\mathfrak{p}_k \cap S \neq \emptyset$. For each k $(1 \leq k \leq \nu)$, we have $\mathbb{j}(\mathfrak{p}_k) = e_k \wp_k = \mathrm{div}_A(c_k^{e_k})$ by the definition. Hence $f(\mathrm{cl}(\mathfrak{p}_k)) = (\mathbb{D}(c_k)/c_k)^{e_k}$ and

$$\mathrm{Im}(f) = \langle (\mathbb{D}(c_k)/c_k)^{e_k} \mid 1 \leq k \leq \nu \rangle \vee \mathcal{L}_A'/\mathcal{L}_A'.$$

Next we shall prove the following:

$$L \vee \mathcal{L}_A'/\mathcal{L}_A' = \langle (\mathbb{D}(c_k)/c_k)^{s_k} \mid 1 \leq k \leq \nu \rangle \vee \mathcal{L}_A'/\mathcal{L}_A'.$$

Suppose that $\mathbb{D}(a)/a \in L$ $(a \in A \cap A_S^*)$, then it is seen that

$$\mathrm{Supp}(\mathrm{div}_A(a)) \subset \{\wp_1, \ldots, \wp_\nu\}.$$

Hence we can write $a = uc_1^{j_1} \cdots c_\nu^{j_\nu}$ for some $u \in A^*$ and

$j_1, \ldots, j_\nu \in \mathbb{Z}$. Notice that $\underline{D}^{(i)}(a)/a \in A[t_i : m_i]$ for $1 \leq i \leq r$. Then Proposition 2.5 implies that $s_k^{(i)}$ divides j_k for $1 \leq i \leq r$ and $1 \leq k \leq \nu$. Therefore s_k divides j_k for $1 \leq k \leq \nu$. Conversely, it is easily seen that $(D(c_k)/c_k)^{s_k}$ is in L $(1 \leq k \leq \nu)$. So we have the required result.

Conseqently we know

$$\mathrm{Coker}(f) \cong \frac{\left\langle (\mathbb{D}(c_k)/c_k)^{s_k} \,\middle|\, 1 \leq k \leq \nu \right\rangle \vee \mathcal{L}'_A}{\left\langle (\mathbb{D}(c_k)/c_k)^{e_k} \,\middle|\, 1 \leq k \leq \nu \right\rangle \vee \mathcal{L}'_A}.$$

We shall define the homomorphism θ by the following manner:

$$\theta : \prod_{k=1}^{\nu} \mathbb{Z}/(e_k/s_k)\mathbb{Z} \longrightarrow \mathrm{Coker}(f),$$

$$\theta(\text{the residue class of } (j_1, \ldots, j_\nu))$$

$$= \text{the residue class of } \prod_{k=1}^{\nu} (\mathbb{D}(c_k)/c_k)^{s_k j_k}.$$

Then it is easily seen that θ is well-defined and surjective. We shall show that θ is injective. Suppose that

$$\theta(\text{the residue class of } (j_1, \ldots, j_\nu)) = \mathbb{1}.$$

Then there exist elements $i_1, \ldots, i_\nu \in \mathbb{Z}$ and $\alpha \in A^*$ such that

$$(\mathbb{D}(\alpha)/\alpha) \prod_{k=1}^{\nu} (\mathbb{D}(c_k)/c_k)^{s_k j_k} = \prod_{k=1}^{\nu} (\mathbb{D}(c_k)/c_k)^{e_k i_k}.$$

Put $x := \prod_{k=1}^{\nu} \alpha c_k^{d_k}$ where $d_k := s_k j_k - e_k i_k$. Then $\mathbb{D}(x)/x$

$= \mathbb{1}$ and $x \in K'$. Let v_k be the normalized valuation of K associated to the prime ideal $\wp_k$ and A'_k be the localization of A' with respect to $\wp_k$. Let u_k be a uniformisant of A'_k for $1 \leq k \leq \nu$. Since x is in K', there exist elements $\alpha_k \in A_k'^{*}$ and $f_k \in \mathbb{Z}$ such that $x = \alpha_k u_k^{f_k}$ for $1 \leq k \leq \nu$. Then we have $d_k = v_k(x) = v_k(\alpha_k u_k^{f_k}) = v_k(u_k^{f_k}) = f_k e_k$. Hence e_k divides $s_k j_k$, i.e., e_k/s_k divides j_k for $1 \leq k \leq \nu$. This implies that θ is injective. Q.E.D.

Let $\wp = cA$ be a principal prime ideal in $P(A)$ and let $s^{(i)}(\wp) := \min\left\{ s \in \mathbb{N} \mid (\underline{D}^{(i)}(c)/c)^s \in A[t_i : m_i] \right\}$ $(1 \leq i \leq r)$, and $s(\wp) := \max\left\{ s^{(i)}(\wp) \mid 1 \leq i \leq r \right\}$.

Theorem 2.7. Assume that A is a unique factorization domain and let $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations on A satisfying the conditions $J(\mathbb{D} : A) \neq \{0\}$ and $[K : K'] = p^{n(\mathbb{D})}$. Then the followings are equivalent to each other:

(i) $\Phi : \mathrm{Ker}(\mathbb{J}) \longrightarrow \mathcal{L}_A/\mathcal{L}'_A$ is an isomorphism.

(ii) For each prime ideal $\wp$ in $P(A)$, either $J(\mathbb{D} : A) \not\subset \wp$ or $e(\wp) = s(\wp)$ occurs where $e(\wp)$ stands for the ramification index of $\wp$ over $\wp \cap A'$.

Proof. Immediate from Theorem 2.6. Q.E.D.

§ 3. Calculus of divisor class groups

In this section we shall determine divisor class groups of certain rings as applications of the preceding results. As before k will be a field of characteristic $p > 0$ unless otherwise specified.

Proposition 3.1. Let $A = k[x, y]$ be a two-dimensional polynomial ring over k with the quotient field K. Let α, β be integers such that $0 < \alpha, \beta < p^n$. Let $\underline{D}$ be the higher derivation of rank $p^n - 1$ on K over k defined by

$$\underline{D}(x) = x(1 + t)^{\alpha}, \ \underline{D}(y) = y(1 + t)^{\beta}$$

and let K' be the field of $\underline{D}$-constants. Let p^{γ} be the maximal p-th power which divides $\mathrm{GCD}(\alpha, \beta)$. Set $\alpha = \alpha' p^{\gamma}$, and $\beta = \beta' p^{\gamma}$. Then we have the following assertions:

(1) $[K : K'] = p^{n-\gamma}$.

(2) $\mathcal{L}_A / \mathcal{L}'_A = \mathbb{Z}/p^{n-\gamma}\mathbb{Z}$.

(3) Assume that p does not divide either α or β. Then $\mathrm{Cl}(A') \cong \mathbb{Z}/p^n\mathbb{Z}$ where $A' := A \cap K'$, and A' is the normalization of $k[x^{p^n}, y^{p^n}, x^{p^n - \beta'} y^{\alpha'}]$.

Proof. (I) We may assume that p does not divide α'. Set $F_s := k(x^{p^s}, y^{p^s}, x^{-\beta'} y^{\alpha'})$ for $0 \leq s \leq n$. Then we have

$$K = F_0 \supset F_1 \supset \cdots \supset F_{n-1} \supset F_n.$$

Hence $GCD(\alpha', p^s) = 1$ implies that $F_{s-1} = F_s(x^{p^{s-1}})$ and $x^{p^{s-1}} \in F_{s-1} - F_s$. Therefore $[F_{s-1} : F_s] = p$ for $1 \leq s \leq n$. Set $s_0 := \min\left\{ s \mid x^{p^s} \in K',\ 1 \leq s \leq n \right\}$. We shall show that $s_0 = n - \gamma$. From $\underline{D}(x^{p^{n-\gamma}}) = x^{p^{n-\gamma}}$, it follows that $x^{p^{n-\gamma}} \in K'$ and $s_0 \leq n - \gamma$. On the other hand $\underline{D}(x^{p^{n-\gamma-1}}) \neq x^{p^{n-\gamma-1}}$ because p does not divide α'. This implies that $s_0 = n - \gamma$. Since $\mu(\underline{D}) = p^\gamma$, we know that $[K : K'] \geq p^{n-\gamma}$ by Proposition 1.3. Then we get $K' = F_{s_0}$ because $F_{s_0} \subset K' \subset K = F_0$ and $[F_0 : F_{s_0}] = p^{s_0} = p^{n-\gamma}$. Hence $[K : K'] = p^{s_0} = p^{n-\gamma}$.

(2) Since $A^* = k^*$, we have $\mathcal{L}'_A = \{1\}$. We shall show that $\mathcal{L}_A = \left\{ (1+t)^{ds} \in k[t : m] \mid s \in \mathbb{Z} \right\}$ where $d := GCD(\alpha, \beta)$ and $m := p^n - 1$. Notice that

$$\mathcal{L}_A = \left\{ \underline{D}(f)/f \in K[t : m] \mid f \in A - \{0\},\ \underline{D}(f)/f \in A[t : m] \right\}$$

because $\underline{D}(f_1/f_2)/(f_1/f_2) = \underline{D}(f_1 f_2^m)/f_1 f_2^m$ $(f_1, f_2 (\neq 0) \in A)$. For every polynomial $f \in A - \{0\}$, the total degree of the coefficient of t^j in $\underline{D}(f)$ is not more than that of f for $0 \leq j \leq m$ by the definition of $\underline{D}$. Hence $\underline{D}(f)/f \in A[t : m]$ implies that $\underline{D}(f)/f \in k[t : m]$. Set

$f := \sum a_{ij}x^iy^j$ $(a_{ij} \in k^*)$ and $\underline{D}(f)/f = h(t)$ where $h(T) \in k[T]$. Then we see

$$\sum a_{ij}x^iy^j(1 + t)^{i\alpha + j\beta} = \sum a_{ij}x^iy^jh(t).$$

Since x, y and T are algebraically independent over k, we get $(1 + t)^{i\alpha + j\beta} = h(t)$. Hence $i\alpha + j\beta$ is constant modulo p^n for any i, j with $a_{ij} \neq 0$. On the other hand $i\alpha + j\beta$ is a multiple of $d = GCD(\alpha, \beta)$. Therefore we know $\underline{D}(f)/f = (1 + t)^{ds'}$ where $s' = (i\alpha + j\beta)/d$. This means that $\mathcal{L}_A$ is contained in $\{(1 + t)^{ds} \in k[t : m] | s \in \mathbb{Z}\}$. Since $GCD(\alpha, \beta) = d$, there exist integers a, b such that $a\alpha + b\beta = d$. Then we have $\underline{D}(x^ay^b)/x^ay^b = (1 + t)^d$. This implies that $(1 + t)^d$ is in $\mathcal{L}_A$. Hence $\mathcal{L}_A = \{(1 + t)^{ds} \in k[t : m] \mid s \in \mathbb{Z}\}$. Let $\theta : \mathbb{Z}/p^{n-\gamma}\mathbb{Z} \longrightarrow \mathcal{L}_A$ be the homomorphism defined by θ(the residue class of s) $= (1 + t)^{ds}$. Then we see easily that θ is well-defined and surjective. We shall prove the injectivity of θ. Assume that θ(the residue class of s) $= 1$. Then $(1 + t)^{ds} = 1$ in a truncated polynomial ring $k[t : m]$. Write $d = d'p^\gamma$ and $s = s'p^\delta$ $(p \nmid d'$ and $p \nmid s')$. Since $(1 + t)^{ds} = (1 + t^{p^{\gamma+\delta}})^{d's'}$ and $p \nmid d's'$, the coefficient of $t^{p^{\gamma+\delta}}$ does not vanish.

Hence $p^{\gamma+\delta} \geq p^n$ and $\delta \geq n - \gamma$. This implies that $s \in p^{n-\gamma}\mathbb{Z}$ and θ is injective. Finally we have $\mathcal{L}_A/\mathcal{L}'_A \cong \mathcal{L}_A \cong \mathbb{Z}/p^{n-\gamma}\mathbb{Z}$.

(3) Since p does not divide either α or β, we see that the height one property for $\underline{D}$ is satisfied. It follws from (1) that $[K : K'] = p^n$ (note that $\gamma = 0$). Therefore Theorem 1.6 implies that $\mathrm{Ker}(\bar{j}) \cong \mathcal{L}_A/\mathcal{L}'_A$. Since A is a unique factorization domain, we have $\mathrm{Cl}(A') = \mathrm{Ker}(\bar{j})$, hence $\mathrm{Cl}(A') \cong \mathbb{Z}/p^n\mathbb{Z}$. The rest is obvious from the fact A' is normal and integral over $k[x^{p^n}, y^{p^n}, x^{p^n-\beta'}y^{\alpha'}]$ (note that $K' = F_n$). Q.E.D.

By making use of Proposition 3.1 we get the following:

Proposition 3.2. The divisor class group of a surface $S : Z^{p^n} = XY$ is a cyclic group of order p^n.

Proof. Let x, y be independent variables over k. Then the coordinate ring of the surface S is isomorphic to $A'_1 := k[x^{p^n}, y^{p^n}, xy]$. Set $\alpha := 1$ and $\beta := p^n - 1$ in Proposition 3.1, then we have $\mathrm{Cl}(A') \cong \mathbb{Z}/p^n\mathbb{Z}$ where $A' = A \cap K'$ is a Krull domain in Proposition 3.1. We shall show that $A'_1 = A'$. We see that A'_1 is normal because the surface

S has only isolated singular point (cf. [4], Th. 4.1). Since A' is the normalization of $k[x^{p^n}, y^{p^n}, xy]$ by Proposition 3.1, (3), we get $A_1' = A'$. Q.E.D.

Remark 3.3. Let $\mathfrak{P}$ be a prime ideal in $P(A')$ generated by x^{p^n} and xy. Since $\mathfrak{j}(\mathfrak{P}) = \mathrm{div}_A(x)$ and since $\Phi(\mathrm{cl}(\mathfrak{P})) = \underline{D}(x)/x$, $\mathrm{cl}(\mathfrak{P})$ generates $\mathrm{Cl}(A') \cong \mathbb{Z}/p^n\mathbb{Z}$.

In order to generalize Proposition 3.2, we shall prove $\mathrm{Cl}(R_1 \otimes_k \cdots \otimes_k R_r) \cong \prod_{i=1}^{r} \mathrm{Cl}(R_i)$ in a certain restricted case as an application of Theorem 1.6.

Proposition 3.4. Let A_i be a polynomial ring in a finite set of variables over k and set $K_i := Q(A_i)$ $(1 \leq i \leq r)$. Let $\underline{D}^{(i)}$ be a non-trivial higher derivation of rank m_i on K_i over k leaving A_i invariant. Let K_i' be the field of $\underline{D}^{(i)}$-constants and set $A_i' := A_i \cap K_i'$ $(1 \leq i \leq r)$. Assume that the height one property holds for $\underline{D}^{(i)}$ and $[K_i : K_i'] = p^{n_i}$ where $n_i := n(\underline{D}^{(i)})$ for $1 \leq i \leq r$. Set $A := A_1 \otimes_k \cdots \otimes_k A_r$ and $A' := A_1' \otimes_k \cdots \otimes_k A_r'$ with $L := Q(A)$ and $L' := Q(A')$. Then we have $\mathrm{Cl}(A') \cong \prod_{i=1}^{r} \mathrm{Cl}(A_i')$.

Proof. We have only to prove the Proposition in the case $r = 2$ because we can get the general case by induction on r. Set $A_1 = k[x_1, \ldots, x_d]$ and $A_2 = k[y_1, \ldots, y_e]$ where $x_1, \ldots, x_d$ and $y_1, \ldots, y_e$ are independent variables over k. Then $A \cong k[x_1, \ldots, x_d, y_1, \ldots, y_e]$. We shall extend $\underline{D}^{(1)}$ to L by the following way:

$$\underline{D}^{(1)}(y_1) = y_1, \ldots, \underline{D}^{(1)}(y_e) = y_e.$$

Similarly we shall extend $\underline{D}^{(2)}$ to L. Then $\mathbb{D} := (\underline{D}^{(1)}, \underline{D}^{(2)})$ is a 2-tuple of non-trivial higher derivations of rank $\mathbf{m} := (m_1, m_2)$ on L over k leaving A invariant.

We shall show that $A' = A \cap L'$. Since K_i $(i = 1, 2)$ are regular extensions of k, K_i' $(i = 1, 2)$ are also regular extensions of k. Besides, A_i' $(i = 1, 2)$ are integrally closed integral domains. Therefore $A' = A_1' \otimes_k A_2'$ is an integrally closed integral domain ([2], Chap. 5, §1, Cor. of Prop. 19). Furthermore $A \cap L'$ is an integral extension of A' with the same quotient field $L' = Q(A \cap L') = Q(A')$. Hence we have $A' = A \cap L'$.

Next we shall prove that L' is the field of $\mathbb{D}$-constants. It is easily seen that $A_1' \otimes_k A_2 = A_1'[y_1, \ldots, y_e]$ is the ring of $\underline{D}^{(1)}$-constants in A. Similarly $A_1 \otimes_k A_2'$ is the ring of

$\underline{D}^{(2)}$-constants in A. We know that $A_1' \otimes_k A_2' = (A_1' \otimes_k A_2) \cap (A_1 \otimes_k A_2')$ ([2], Chapter 1, §2, Proposition 7). Therefore $A' = A_1' \otimes_k A_2'$ is the ring of $\mathbb{D}$-constants in A. It is clear that $L' = Q(A')$ is contained in the field of $\mathbb{D}$-constants. Since A is the integral closure of A' in L, any element of L is of the form a/b $(a \in A,\ b \in A')$. Suppose that $\mathbb{D}(a/b) = a/b$ $(a \in A,$ $b \in A')$. Then we have $\mathbb{D}(a) = \mathbb{D}((a/b)b) = \mathbb{D}(a/b)\mathbb{D}(b) = (a/b)b$ $= a$, hence a is in A'. This implies that a/b is in L'. Finally L' is the field of $\mathbb{D}$-constants.

We shall show that the height one property holds for $\mathbb{D}$. Since A is A_i-flat, we know that $\mathrm{ht}(\wp \cap A_i) \leq 1$ $(i = 1, 2)$ for all $\wp \in P(A)$ ([4], Proposition 6.4). Set $\wp_i := \wp \cap A_i$. Then there exists an element α_i in A_i such that the Jacobian $J(\underline{D}^{(i)} : \alpha_i)$ is not contained in $\wp_i$ because the height one property holds for $\underline{D}^{(i)}$. On the other hand we have $J(\mathbb{D} : (\alpha_1, \alpha_2)) = J(\underline{D}^{(1)} : \alpha_1)J(\underline{D}^{(2)} : \alpha_2)$. Suppose that $J(\mathbb{D} : (\alpha_1, \alpha_2))$ $\in \wp$, then either $J(\underline{D}^{(1)} : \alpha_1)$ or $J(\underline{D}^{(2)} : \alpha_2)$ is in $\wp$, say, $J(\underline{D}^{(1)} : \alpha_1) \in \wp$. This means that $J(\underline{D}^{(1)} : \alpha_1)$ $\in \wp \cap A_1 = \wp_1$, which contradicts to the height one property for $\underline{D}^{(1)}$.

We shall show that $[L : L'] = p^{n(\mathbb{D})}$. Set $L_1 = Q(A_1' \otimes_k A_2)$, then we have $L \supset L_1 \supset L'$. We know that $[L : L'] \geq p^{n(\mathbb{D})}$ because of Proposition 1.3. Since $[L : L'] = [L : L_1][L_1 : L']$, it suffices to prove that $[L : L_1] \leq p^{n_1}$ and $[L_1 : L'] \leq p^{n_2}$. We shall prove that $[L : L_1] \leq p^{n_1}$. It is easily verified that $L = Q(K_1 \otimes_k K_2)$, $L_1 = Q(K_1' \otimes_k K_2)$ and $K_1' \otimes_k K_2 = L_1 \cap (K_1 \otimes_k K_2)$. Therefore any element of L is of the form α/β with $\alpha \in K_1 \otimes_k K_2$ and $\beta \in K_1' \otimes K_2$. Let $a_1, \ldots, a_\nu$ ($\nu := p^{n_1}$) be K_1'-basis of K_1. Then $K_1 \otimes_k K_2$ is generated by $a_1 \otimes 1, \ldots, a_\nu \otimes 1$ over $K_1' \otimes_k K_2$. Since any element of L is of the form α/β ($\alpha \in K_1 \otimes_k K_2$, $\beta \in K_1' \otimes_k K_2$), L is generated by $a_1 \otimes 1, \ldots, a_\nu \otimes 1$ over L_1, hence $[L : L_1] \leq \nu = p^{n_1}$. Similarly we have $[L_1 : L'] \leq p^{n_2}$.

Let

$$\mathcal{L}_i = \left\{ \underline{D}^{(i)}(z_i)/z_i \;\middle|\; z_i \in K_i^*,\ \underline{D}^{(i)}(z_i)/z_i \in A_i[t_i : m_i] \right\},$$

$$\mathcal{L}_i' = \left\{ \underline{D}^{(i)}(u_i)/u_i \;\middle|\; u_i \in A_i^* \right\} \quad \text{for } i = 1, 2,$$

$$\mathcal{L} = \left\{ \mathbb{D}(z)/z \;\middle|\; z \in L^*,\ \mathbb{D}(z)/z \in A[\mathbf{t} : \mathbf{m}] \right\}$$

and,

$$\mathcal{L}' = \left\{ \mathbb{D}(u)/u \;\middle|\; u \in A^* \right\}$$

where $\mathbf{t} = (t_1, t_2)$. Since we know that $Cl(A_i') \cong \mathcal{L}_i/\mathcal{L}_i'$ ($i = 1, 2$), $Cl(A') \cong \mathcal{L}/\mathcal{L}'$ and $\mathcal{L}_i' = \mathcal{L}' = \{1\}$, it remains only to prove that $\mathcal{L}_1 \times \mathcal{L}_2 \cong \mathcal{L}$. Let θ be the homomorphism of $\mathcal{L}_1 \times \mathcal{L}_2$ into $\mathcal{L}$ defined by

$$(\underline{D}^{(1)}(a_1)/a_1,\ \underline{D}^{(2)}(a_2)/a_2) = \mathbb{D}(a_1a_2)/a_1a_2,\ (a_i \in K_i^*).$$

It is easily seen that θ is injective. We shall show that θ is surjective. Suppose that $\mathbb{D}(f)/f \in \mathcal{L}$ $(f \in A - \{0\})$. Then there exist polynomials $g_i(T_i)$ in $A[T_i]$ $(i = 1, 2)$ such that $\mathbb{D}(f)/f = (g_1(t_1), g_2(t_2))$. Comparing the total degree with respect to $y_1, \ldots, y_e$ of $\underline{D}^{(1)}(f)$ with that of $fg_1(t_1)$, we see that $g_1(t_1)$ is in $A_1[t_1 : m_1]$. Write $f = \sum_{\gamma} a_\gamma b_\gamma$ $(a_\gamma \in A_1,\ b_\gamma \in A_2$ and $\{b_\gamma\}$ is linearly independent over $k)$, then we have

$$\sum_{\gamma}(\underline{D}^{(1)}(a_\gamma) - g_1(t_1)a_\gamma)b_\gamma = 0.$$

This implies that $\underline{D}^{(1)}(a_\gamma) = g_1(t_1)a_\gamma$ for all γ. Therefore $\underline{D}^{(1)}(a)/a = g_1(t_1)$ for some $a \in A_1$. Similarly $\underline{D}^{(2)}(b)/b = g_2(t_2)$ for some $b \in A_2$. Hence $\theta(\underline{D}^{(1)}(a)/a, \underline{D}^{(2)}(b)/b) = \mathbb{D}(f)/f$. Furthermore we know that $\mathcal{L} = \{\mathbb{D}(f)/f \mid f \in A - \{0\}, \mathbb{D}(f)/f \in A[t : m]\}$. Therefore θ is surjective and we get the desired result. Q.E.D.

Remark 3.5. By the similar method as the proof of Proposition 3.4, we can get the following fact using units theorem ([10], Corollary 1.8). But the proof is more complicated, so we omit it:

"Let $A_i := \bigoplus_{s \in \mathbb{Z}_+} (A_i)_s$ $(1 \leq i \leq r)$ be graded unique factorization domains with $(A_i)_0 = k$ and let K_i be its quotient field. Assume that K_i $(1 \leq i \leq r)$ are regular extensions of k. Let $\underline{D}^{(i)}$ be a non-trivial higher derivation of rank m_i on K_i over k leaving A_i invariant for $1 \leq i \leq r$. Let K_i' be the field of $\underline{D}^{(i)}$-constants and set $A_i' := A_i \cap K_i'$ $(1 \leq i \leq r)$. Assume that the height one property holds for $\underline{D}^{(i)}$ and $[K_i : K_i'] = p^{n_i}$ where $n_i := n(\underline{D}^{(i)})$ for $1 \leq i \leq r$. Set $A := A_1 \otimes_k \cdots \otimes_k A_r$ and $A' := A_1' \otimes_k \cdots \otimes_k A_r'$ with $L := Q(A)$ and $L' := Q(A')$. Furthermore assume that $A_1 \otimes_k \cdots \otimes_k A_i$ $(1 \leq i \leq r)$ are unique factorization domains.

Then we have $\mathrm{Cl}(A') \cong \prod_{i=1}^{r} \mathrm{Cl}(A_i')$."

The following Proposition is immediate from Proposition 3.4.

Proposition 3.6. The divisor class group of an affine variety in A^{3r} defined by the equations $Z_i^{q_i} = X_iY_i$ $(1 \leq i \leq r)$ is isomorphic to $\prod_{i=1}^{r} \mathbb{Z}/q_i\mathbb{Z}$ where $q_i := p^{n_i}$.

Remark 3.7. The coordinate ring of this variety is isomorphic to $A' := k[x_1^{q_1}, y_1^{q_1}, x_1y_1, \ldots, x_r^{q_r}, y_r^{q_r}, x_ry_r]$. And if we denote by $\mathfrak{P}_i$ a prime ideal in $P(A')$ generated by $x_i^{q_i}$, x_iy_i for $1 \leq i \leq r$, then $\mathrm{cl}(\mathfrak{P}_i)$ $(1 \leq i \leq r)$ generate $\mathrm{Cl}(A')$.

As another generalization of Proposition 3.2 we have the following:

Proposition 3.8. The divisor class group of a hypersurface $S : Z^{p^n} = X_1X_2\cdots X_r$ $(r \geq 2)$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^{r-1}$. The coordinate ring of this hypersurface S is isomorphic to $A' := k[x_1^{p^n}, x_2^{p^n}, \ldots, x_r^{p^n}, x_1x_2\cdots x_r]$ where $x_1, x_2, \ldots, x_r$ are independent variables over k. If we denote by $\mathfrak{P}_i$ a prime ideal in $P(A')$ generated by $x_i^{p^n}$ and $x_1x_2\cdots x_r$ for $1 \leq i \leq r-1$, then $\mathrm{cl}(\mathfrak{P}_i)$ $(1 \leq i \leq r-1)$ generate $\mathrm{Cl}(A')$.

Proof. We see easily that A' is the coordinate ring of the hypersurface S. We shall set $A = k[x_1, x_2, \ldots, x_r]$ and $K := Q(A)$. Let $\underline{D}^{(i)}$ be the higher derivation of rank $p^n - 1$ on K over k satisfying

$$\underline{D}^{(i)}(x_i) = x_i(1 + t_i),$$

$$\underline{D}^{(i)}(x_j) = x_j \quad (1 \leq j \leq r-1, \quad j \neq i),$$

$$\underline{D}^{(i)}(x_r) = x_r(1 + t_i)^{-1}$$

for $1 \leq i \leq r-1$. Then we have

$$J(\mathbb{D} : (x_1, \ldots, \hat{x}_s, \ldots, x_r)) = (-1)^{r+s} x_1 \ldots \hat{x}_s \ldots x_r$$

for $1 \leq s \leq r$ where $\mathbb{D} = (\underline{D}^{(1)}, \ldots, \underline{D}^{(r-1)})$ and the symbol $\wedge$ over a letter means that the letter is missing. Let K' be the field of $\mathbb{D}$-constants. Then Proposition 1.3 implies that $[K : K'] \geq p^{n(r-1)}$. We shall set

$$K_i := k(x_1^{p^n}, x_2^{p^n}, \ldots, x_i^{p^n}, x_{i+1}, \ldots, x_r, x_1 \ldots x_r)$$

for $1 \leq i \leq r-1$ and $K_r := k(x_1^{p^n}, \ldots, x_r^{p^n}, x_1 \ldots x_r)$. Then $K = K_1$, $K_i = K_{i+1}(x_{i+1})$ and $x_{i+1}^{p^n} \in K_{i+1}$ for $1 \leq i \leq r-1$. Besides, $K \supset K' \supset K_r$. This implies that $[K : K'] \leq p^{n(r-1)}$, hence $[K : K'] = p^{n(r-1)}$. Since the hypersurface S has no singularity of codimension one, we see that A' is normal. Then we get $A' = A \cap K'$. Therefore we have $Cl(A') \cong \mathcal{L}_A/\mathcal{L}'_A$ by Theorem 1.6. Let θ be the homomorphism of $(\mathbb{Z}/p^n\mathbb{Z})^{r-1}$ into $\mathcal{L}_A$ defined by

$$\theta(\text{the residue class of } (j_1, \ldots, j_{r-1}))$$

$$= \mathbb{D}(a)/a$$

$$= ((1 + t_1)^{j_1}, \ldots, (1 + t_{r-1})^{j_{r-1}})$$

where $a := x_1^{j_1} \ldots x_{r-1}^{j_{r-1}}$. Then θ is well-defined and bijective by the similar device to the proof of Proposition 3.1. Consequently $Cl(A') \cong \mathcal{L}_A/\mathcal{L}'_A \cong \mathcal{L}_A \cong (\mathbb{Z}/p^n\mathbb{Z})^{r-1}$. Since $\mathbb{D}(x_i)/x_i$ $(1 \leq i \leq r-1)$ generate $\mathcal{L}_A$, $cl(\mathfrak{p}_i)$ $(1 \leq i \leq r-1)$ generate $Cl(A')$.

Q.E.D.

For future reference we shall recollect the known results concerning Galois descent and semigroup rings. Let G be a finite group of automorphisms of a Krull domain A and let A' be the invariant subring of A with respect to G. Since A is integral over A', we can define the homomorphism $\mathbb{J} : Cl(A') \longrightarrow Cl(A)$ by $\mathbb{J}(cl(\mathfrak{p})) = cl(\sum e(\wp)\wp)$ where the sum is taken over all prime ideal $\wp$ in $P(A)$ such that $\wp \cap A' = \mathfrak{p}$. If every prime ideal $\wp$ in $P(A)$ is unramified over $\wp \cap A'$, A is called divisorially unramified over A'.

Lemma 3.9. If A is divisorially unramified over A', there is an isomorphism $Ker(\mathbb{J}) \cong H^1(G, A^*)$ (cf. [4], Theorem 16.1).

Lemma 3.10. Let $\mathcal{D}(A/A')$ be the Dedekind different of A over A'. Then we have the following; a prime ideal $\wp$ in $P(A)$ is unramified over $\wp \cap A'$ if and only if

$\mathfrak{D}(A/A') \not\subset \mathfrak{P}$ ([4], Proposition 16.3).

Let $f(X)$ be the minimal polynomial for a primitive element α of $Q(A)$ over $Q(A')$. Let $f'(X)$ denote the derivative of $f(X)$ with respect to X. Then we have $f'(\alpha) \in \mathfrak{D}(A/A')$. Hence each prime ideal $\mathfrak{P}$ in $P(A)$ such that $f'(\alpha) \notin \mathfrak{P}$ is unramified over $\mathfrak{P} \cap A'$ by Lemma 3.10.

Furthermore we need the following fact concerning semigroup rings.

Lemma 3.11. Let $K_i[\Gamma]$ be a semigroup ring over a field K_i generated by a semigroup $\Gamma \subset \mathbb{Z}_+^n$ $(i = 1, 2)$. Assume that $K_i[\Gamma]$ $(i = 1, 2)$ are Krull domains. Then we have $\mathrm{Cl}(K_1[\Gamma]) = \mathrm{Cl}(K_2[\Gamma])$ (cf. [1], Proposition 7.3).

By making use of Proposition 3.8 and Galois descent we get the following:

Proposition 3.12. Let k be a field of arbitrary characteristic. Then the divisor class group of a hypersurface $S : Z^d = X_1X_2\cdots X_r$ $(r \geq 2)$ over k is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{r-1}$.

Proof. It is easily seen that the coordinate ring of the hypersurface S is isomorphic to $A' := k[x_1^d, \ldots, x_r^d, x_1\cdots x_r]$ where $x_1, \ldots, x_r$ are independent variables over k. Since A' is generated by monomials, we may assume that k is algebraically closed by Lemma 3.11. Let p denote the characteristic of k. In the case $p = 0$, we can conclude the result simply through Galois descent. So we omit the proof. Assume that $p > 0$ and write $d = ap^n$, $p \nmid a$. We shall set $B = k[x_1^{p^n}, \ldots, x_r^{p^n}, x_1 \cdots x_r]$, then we have $B \supset A'$. Let ω be a primitive a-th root of unity and σ_i be the automorphism of B defined by the following manner:

$$\sigma_i(x_i^{p^n}) = \omega x_i^{p^n}, \quad \sigma_i(x_j^{p^n}) = x_j^{p^n} \ (1 \leq j \leq r-1, \ j \neq i),$$
$$\sigma_i(x_r^{p^n}) = \omega^{-1} x_r^{p^n} \text{ and, } \sigma_i(x_1 \cdots x_r) = x_1 \cdots x_r$$

for $1 \leq i \leq r-1$. Then σ_i is well-defined. Let G be the subgroup of $\mathrm{Aut}\, B$ generated by σ_i $(1 \leq i \leq r-1)$. Then we get $B^G = A'$. In order to use Galois descent, we must prove that B is divisorially unramified over A'. We shall set

$$K_i := k(x_1^d, \ldots, x_i^d, x_{i+1}^{p^n}, \ldots, x_r^{p^n}, x_1 \cdots x_r)$$

for $1 \leq i \leq r-1$. Then $F_s(T) = T^a - x_s^d$ is the minimal polynomial for a primitive element $x_s^{p^n}$ of K_{s-1} over K_s and $F_s'(x_s^{p^n}) = a(x_s^{p^n})^{a-1}$ for $1 \leq s \leq r$ where $K_0 := Q(B)$ and $K_r := Q(A')$. Therefore every prime ideal $\mathfrak{P}$ in $P(B)$ except $\mathfrak{P}_s = (x_s^{p^n}, x_1 \cdots x_r)$ $(1 \leq s \leq r)$ is unramified over $\mathfrak{P} \cap A'$. By a direct calculation the ramification index of $\mathfrak{P}_s$ over $\mathfrak{P}_s \cap A'$ is one. Hence B is divisorially unramified over A'. By Galois descent we get the following exact sequence:

$$0 \longrightarrow H^1(G, B^*) \longrightarrow Cl(B^G) \longrightarrow Cl(B).$$

Since G acts trivially on $B^* = k^*$, we know that $H^1(G, B^*) \cong \mathrm{Hom}_{\mathbb{Z}}(G, k^*)$. Furthermore it is easily verified that $\mathrm{Hom}_{\mathbb{Z}}(G, k^*) \cong (\mathbb{Z}/a\mathbb{Z})^{r-1}$ because ω is in k. On the other hand, Proposition 3.8 shows that $\mathrm{Cl}(B) \cong (\mathbb{Z}/p^n\mathbb{Z})^{r-1}$. Let $\mathfrak{P}_i$ be a prime ideal in $P(A')$ generated by x_i^d and $x_1 \cdots x_r$ for $1 \leq i \leq r-1$. Then we have $\wp_i \cap A' = \mathfrak{P}_i$ and $\mathfrak{j}(\mathfrak{P}_i) = \wp_i$ where $\mathfrak{j} : \mathrm{Div}(A') \longrightarrow \mathrm{Div}(B)$. Besides, $\mathrm{cl}(\wp_i)$ $(1 \leq i \leq r-1)$ generate $\mathrm{Cl}(B) \cong (\mathbb{Z}/p^n\mathbb{Z})^{r-1}$. Finally we get the following exact sequence:

$$0 \longrightarrow (\mathbb{Z}/a\mathbb{Z})^{r-1} \longrightarrow \mathrm{Cl}(A') \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})^{r-1} \longrightarrow 0.$$

Since a and p^n are relatively prime, $\mathrm{Ext}^1_{\mathbb{Z}}((\mathbb{Z}/p^n\mathbb{Z})^{r-1}, (\mathbb{Z}/a\mathbb{Z})^{r-1})$ vanishes and the above sequence splits ([3], p. 290, Theorem 1.1). This implies that $\mathrm{Cl}(A') \cong (\mathbb{Z}/d\mathbb{Z})^{r-1}$. Q.E.D.

Remark 3.13. In the notations of the proof of Proposition 3.12, $p^n \mathrm{cl}(\mathfrak{P}_i)$ $(1 \leq i \leq r-1)$ generate $\mathrm{Ker}(\bar{\mathfrak{j}})$ because $j(p^n \mathfrak{P}_i) = \mathrm{div}_B(x_i^{p^n})$ and $\mathrm{Ker}(\bar{\mathfrak{j}}) \cong \mathrm{Hom}_{\mathbb{Z}}(G, k^*) \cong (\mathbb{Z}/a\mathbb{Z})^{r-1}$. Furthermore it follows from Proposition 3.8 that $\mathrm{cl}(\mathfrak{P}_i)$ $(1 \leq i \leq r-1)$ generate $\mathrm{Cl}(A')$ modulo $\mathrm{Ker}(\bar{\mathfrak{j}})$. Hence $\mathrm{cl}(\mathfrak{P}_i)$ $(1 \leq i \leq r-1)$ generate $\mathrm{Cl}(A')$.

Proposition 3.14. Let k be a field of arbitrary characteristic. Then the divisor class group of the homogeneous coordinate ring of a Veronese transform $v_d(\mathbb{P}^r)$ of a projective space $\mathbb{P}^r$ over k $(d \geq 2)$ is a cyclic group of order d.

Proof. Let $x_0, x_1, \ldots, x_r$ be independent variables over k. We shall set $A := k[x_0, x_1, \ldots, x_r]$. Let A' be the subring of A generated by monomials with degree d. Then A' is isomorphic to the homogeneous coordinate ring of $v_d(\mathbb{P}^r)$. We may assume that k is algebraically closed by Lemma 3.11. Let p denote the characteristic of k. In the case $p = 0$, we have $\mathrm{Cl}(A') \cong \mathbb{Z}/d\mathbb{Z}$ by [8], p. 85, (1). Assume that $p > 0$ and d is a power of p, say, $d = p^n$. Let $\underline{D}$ be the higher derivation on $Q(A)$ over k of rank $d - 1$ defined by $\underline{D}(x_i) = x_i(1 + t)$ $(0 \leq i \leq r)$. Then we see easily that A' is the ring of $\underline{D}$-constants and $[K : K'] = d$ where $K := Q(A)$ and $K' := Q(A')$. Since $J(\underline{D} : x_i) = x_i$ $(0 \leq i \leq r)$, the height one property is satisfied. Hence by Theorem 1.6, $\mathrm{Cl}(A') \cong \mathrm{Ker}(\bar{\mathrm{j}}) \cong \mathcal{L}_A/\mathcal{L}_A' \cong \mathcal{L}_A$. Let θ be the homomorphism of $\mathbb{Z}/d\mathbb{Z}$ into $\mathcal{L}_A$ satisfying θ(the residue class of j) $= (\underline{D}(x_0)/x_0)^j$. It is easily seen that θ is well-defined and bijective. Hence we have $\mathrm{Cl}(A') \cong \mathbb{Z}/d\mathbb{Z}$. If d is not a power of p, write $d = ap^n$, $p \nmid a$ and let B be the subring of A generated by monomials with degree p^n. Let ω be a primitive a-th root of unity and let σ be the automorphism of B defined by $\sigma(M) = \omega M$ for every monomial M with degree p^n. Let G be the subgroup of $\mathrm{Aut}\, B$ generated by σ. Then we have $A' = B^G$. Since $x_i^{p^n}$ is a primitive element of $Q(B)$ over $Q(A')$ for $0 \leq i \leq r$, it is easily seen that B is divisorially unramified over A'. By the similar device to the proof of Proposition

3.12, we get $Cl(A') \cong \mathbb{Z}/d\mathbb{Z}$. Q.E.D.

All rings appeared in the above Propositions are generated by monomials. The coordinate ring of the following surface is not generated by monomials:

Proposition 3.15. Let n be a positive integer and s be a non-negative integer with $0 \leq s \leq n$. Then the divisor class group of a surface $S : Z^{p^n} = X^{p^s}Y^{p^n} - Y$ is isomorphic to $\mathbb{Z}/p^{n-s}\mathbb{Z}$.

Proof. Let x, y be independent variables over k. Then it is easily seen that the affine coordinate ring of the surface S is given by $A' := k[x^{p^n}, y^{p^n}, x^{p^s}y^{p^n} - y]$. Set $A := k[x, y]$ and let $\underline{D}$ be the higher derivation of rank $m := p^n - 1$ on $Q(A)$ over k defined by $\underline{D}(x) = x + t$, $\underline{D}(y) = y + y^{p^n}t^{p^s}$. Then it is easily checked that the assumptions in Theorem 1.6 are satisfied. Define the homomorphism of $\mathbb{Z}/p^{n-s}\mathbb{Z}$ into $\mathcal{L}_A$ by θ(the residue class of i) $= (\underline{D}(y)/y)^i$. Then θ is well-defined and injective. We shall show that θ is surjective. Suppose that $\underline{D}(f)/f \in A[t : m]$ ($f \in A - \{0\}$), then there exists an element $g(T)$ of $A[T]$ such that $\underline{D}(f)/f = g(t)$. Since the degree with respect to x of the coefficient of t^j in $\underline{D}(f)$ is not more than that of f for $0 \leq j \leq m$, we have $g(t) \in k[y][t : m]$. Write

$$f = a_0(y) + a_1(y)x + \dots + a_h(y)x^h,$$
$$a_\nu(y) \in k[y] \ (0 \le \nu \le h) \quad \text{and} \quad a_h(y) \ne 0.$$

From $\underline{D}(f) = fg(t)$, we get

$$\underline{D}(a_0(y)) + \underline{D}(a_1(y))(x + t) + \dots + \underline{D}(a_h(y))(x + t)^h$$
$$= a_0(y)g(t) + a_1(y)g(t)x + \dots + a_h(y)g(t)x^h.$$

Comparing the coefficients of x^h on both sides, we have $\underline{D}(a_h(y)) = a_h(y)g(t)$ because x, y and T are algebraically independent over k. By Lemma 3.17, there exists an integer i such that $g(t) = (\underline{D}(y)/y)^i$. Hence θ is surjective and $Cl(A') \cong \mathbb{Z}/p^{n-s}\mathbb{Z}$. Q.E.D.

Remark 3.16. Let $\mathfrak{P}$ be the prime ideal in $P(A')$ generated by y^{p^n} and $x^{p^s}y^{p^n} - y$. Then $cl(\mathfrak{P})$ generates $Cl(A')$. The q-th symbolic power $\mathfrak{P}^{(q)}$ of $\mathfrak{P}$ is a principal ideal generated by $y^{p^{n-s}}$ where $q := p^{n-s}$.

Lemma 3.17. Let $A = k[y]$ be a one-dimensional polynomial ring over k. Let n be a positive integer and s be a non-negative integer with $0 \le s \le n$. Let $\underline{D}$ be the higher derivation of rank $m := p^n - 1$ on $Q(A)$ over k defined by $\underline{D}(y) = y + y^{p^n}t^{p^s}$. If $\underline{D}(f)/f$ $(f \in A - \{0\})$ is in $A[t : m]$, there exists an integer i such that $\underline{D}(f)/f = (\underline{D}(y)/y)^i$.

Proof. Set $A' := k[y^{p^{n-s}}]$, then we have $A' = A \cap K'$

where K' is the field of $\underline{D}$-constants. Notice that $\mathfrak{p} := yA$ is the only prime ideal in $P(A)$ such that $D_q(A) \subset \mathfrak{p}$ $(q := p^s)$. Then we have $e(\mathfrak{p}) = p^{n-s}$ and $s(\mathfrak{p}) = 1$. Hence we get the following exact sequence by Theorem 2.6.

$$0 \longrightarrow \mathrm{Ker}(\bar{\mathfrak{j}}) \longrightarrow \mathcal{L}_A/\mathcal{L}'_A \xrightarrow{\eta} \mathbb{Z}/p^{n-s}\mathbb{Z} \longrightarrow 0.$$

Notice that η(the residue class of $(\underline{D}(y)/y)^j$) = the residue class of j. Furthermore $\mathrm{Ker}(\bar{\mathfrak{j}}) \cong Cl(A') = 0$ and $\mathcal{L}'_A = \{1\}$. So we have the desired result. Q.E.D.

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