Title	QUASI-CO-PRIME AUTOMORPHISMS AND THE GLAUBERMAN CORRESPONDENCE OF FINITE GROUPS
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Citation	
Issue Date	
Text Version	ETD
URL	http://hdl.handle.net/11094/27758
DOI	
rights	
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QUASI-CO-PRIME AUTOMORPHISMS AND THE GLAUBERMAN CORRESPONDENCE OF FINITE GROUPS

HIROSHI MATSUYAMA

1. Introduction and notation

In this paper, G denotes a finite group and σ denotes an automorphism of G of order n. Put $H=C_G(\sigma)$. Let $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$ be the set of irreducible complex characters of G and H respectively.

If χ is a character of G, then we define the character χ^σ of G by $\chi^\sigma(g^\sigma)=\chi(g)$ for each element g of G. Set $\mathrm{Irr}_\sigma(G)=\{\chi \mid \chi\in \mathrm{Irr}(G)$ and $\chi^\sigma=\chi\}$. Assuming σ to be co-prime (that is, n and $|G|$ are relatively prime), Glauberman [1] showed the existence of a natural bijection of $\mathrm{Irr}_\sigma(G)$ onto $\mathrm{Irr}(H)$ and extended the result to finite groups admitting solvable operator groups with relatively prime order. Since then, generalizations and variations of Glauberman's result were obtained by several authors, see for example H.Nagao [6]. Furthermore, more general character correspondences (through `norm mappings´) were considered by N.Kawanaka [4], [5], T.Shintani [7] and others.

The purpose of this paper is to study quasi-co-prime automorphisms (see Definition (1.2)) and to extend Glauberman's character correspondence to a finite group admitting a quasi-co-prime automorphism. (Indeed we shall characterize an automorphism which yields the Glauberman correspondence.)

Before stating our Theorems, we introduce the following notation:

$\Gamma=G\langle\sigma\rangle$; the semi-direct product of G by $\langle\sigma\rangle$.

$t=|G:H|$.

$h_1=1, h_2, \cdots, h_\alpha$; the representatives of the conjugacy classes of H.

$X(h_i)=\{ g^{-1}h_i g^\sigma \mid g \in G \}$, $i=1,2,\cdots,\alpha$.

$X=X(h_1)$.

$\mathrm{Irr}(G)=\{ \chi_1=1, \chi_2, \cdots\cdots, \chi_\gamma \}$.

$\mathrm{Irr}_\sigma(G)=\{ \chi_1=1, \chi_2, \cdots, \chi_\beta \}$.

$\mathrm{Irr}(H)=\{ \theta_1=1, \theta_2, \cdots, \theta_\alpha \}$.

$\mathrm{Hom}(\Gamma/G, C^\times)=\{\mu_1=1, \mu_2, \cdots, \mu_n\}$= the set of linear characters of Γ whose kernels contain G.

$n_i=\chi_i(1)$, $i=1,2,\cdots,\gamma$.

$e_i=n_i/|G| \sum_{g \in G} \chi_i(g^{-1})g$, $i=1,2,\cdots,\gamma$.

$C[G], C[\Gamma]$; the group algebras of G and Γ respectively over the field of complex numbers C.

$\hat{S}=\sum_{x \in S} x$ ($\in C[\Gamma]$) for any subset S of Γ.

$[\Lambda_1, \Lambda_2]_G= 1/_{|G|} \sum_{g \in G} \Lambda_1(g)\overline{\Lambda_2(g)}$, where Λ_1 and Λ_2 are class functions on G.

Remark (1.1). Let χ be a character of G and let R be a representation of G. Then, by linear extension, we may assume that both χ and R are defined on $C[G]$. Similarly, we may assume that characters and representations of Γ are defined on $C[\Gamma]$.

Definition (1.2). If $G=\bigcup_{i=1}^{\alpha} X(h_i)$, then σ is called a quasi-co-prime automorphism of G.

Remark (1.3). Some (but not all) of properties of co-prime automorphisms hold for quasi-co-prime automorphisms. Especially we can show $\alpha=\beta$ for a finite group G admitting a quasi-co-prime automorphism (see (2.8)).

Definition (1.4). Let $\chi\in \mathrm{Irr}_\sigma(G)$ and let χ^* be an extension of χ to Γ (see(2.1)). Let σ^m be a generator of $\langle\sigma\rangle$. Then we define a class function $\psi(\chi^*,\sigma^m)$ on H as follows;

$$\psi(\chi^*,\sigma^m)(h)=\chi^*(h\sigma^m) \quad \text{for each element h of H.}$$

Note that the definition of $\psi(\chi^*,\sigma^m)$ depends on the choice of χ^* and σ^m.

Definition (1.5). Suppose $\alpha=\beta$. A bijection π of $\mathrm{Irr}_\sigma(G)$ onto Irr(H) is called the Glauberman correspondence with respect to σ if the following condition is satisfied;

Let $\chi\in \mathrm{Irr}_\sigma(G)$. Then for any extension χ^* of χ to Γ, $\psi(\chi^*,\sigma)$ is a non-zero scalar multiple of $\pi(\chi)$.

Remark (1.6). If there exists the Glauberman correspondence π with respect to σ. Then there exists a non-zero complex number λ such that $\psi(\chi^*,\sigma)=\lambda\pi(\chi)$. Indeed λ is a 2n-th root of unity if n is odd and a n-th root of unity if n is even (see (4.1)(ii)). Furthermore $\psi(\chi^*,\sigma^m)$ is also a non-zero scalar multiple of $\pi(\chi)$ for any generator σ^m of $\langle\sigma\rangle$ (see (5.4)).

Now our Theorems are the following:

Theorem A. An automorphism σ is a quasi-co-prime automorphism if and only if there exists the Glauberman correspondence with respect to σ.

Theorem B. Let σ be a quasi-co-prime automorphism of G. Let π be the Glauberman correspondence with respect to σ and let $\pi(\chi_i)=\Theta_i$, $i=1,\cdots,\alpha$. Then the following hold;

(i) There exists a unique extension χ_i^* of χ_i to Γ such that $\chi_i^*(\sigma^m)=\varepsilon_i\Theta_i(1)$ for any generator σ^m of $\langle\sigma\rangle$, where $\varepsilon_i\in\{\pm 1\}$ if n is odd and $\varepsilon_i=1$ if n is even, $i=1,\cdots,\alpha$.

(In the sequel, the uniquely determined extension χ_i^* is called the canonical extension of χ_i, $i=1,\cdots,\alpha$. Furthermore, let ε_i be as is determined above.)

(ii) π is independent of the choice of a generator of $\langle\sigma\rangle$.

(iii) If n is a power of a prime p, then

$$[\chi_i|_H,\Theta_j]_H\equiv\delta_{ij}\varepsilon_i \pmod p,\quad 1\leq i,j\leq\alpha,$$

where δ_{ij} is the Kronecker's symbol.

(iv) n_i divides $t\theta_i(1)$, $i=1,2,\cdots,\alpha$.

(v) Let χ_i^* be the canonical extension of χ_i to Γ and let R_i^* be the representation of Γ which affords the character χ_i^*, $i=1,2,\cdots,\alpha$. Put $R_i=R_i^*|_G$. Then:

$R_i(\hat{X}) = \varepsilon_i t\theta_i(1)/n_i \; R_i^*(\sigma)$, consequently $\chi_i(\hat{X})$ is a positive rational integer, $i=1,2,\cdots,\alpha$.

Remark (1.7). Let σ be a co-prime automorphism of G. Then σ is a quasi-co-prime automorphism (see (3.3)). If n is odd, then the canonical extensions defined in Theorem B (i) coincide with the canonical extensions in the sense of Glauberman [1] (see (5.9)). If n is even, then they do not always coincide with each other (see Remark (5.10)).

Corollary C. Let σ be a quasi-co-prime automorphism of G. Then $\chi_i(\hat{X})\neq 0$ if and only if $\chi_i \in \mathrm{Irr}_\sigma(G)$.

The author is much indebted to Prof. H.Nagao for his kind guidance and helpful suggestions. Also he would like to thank Prof. N.Kawanaka who acquainted him with the result of [7].

Other notation is standard , see [2], [3] and [8].

2. Preliminaries

(2.1). Let $\chi_i \in \mathrm{Irr}_\sigma(G)$. Then we have the following.

(i) There exists an extension χ_i^* of χ_i to Γ. Furthermore $\{\mu_j\chi_i^* | j=1,\cdots,n\}$ is the full set of the extensions of χ_i.

(ii) $\sum_{g \in G\sigma} \chi_i^*(g)\overline{\chi_i^*(g)} = |G|$.

(iii) Let $j \neq i$ and let χ_j^* be an extension of $\chi_j \in \mathrm{Irr}_\sigma(G)$.

Then $\sum_{g \in G\sigma} \chi_i^*(g)\overline{\chi_j^*(g)} = 0$.

Proof. See [3] (6.17),(8.14),(11.22).

(2.2). The number of σ-invariant conjugacy classes of G coincides with β.

Proof. See [3] (6.32).

(2.3). Let G×G be the direct product of G and G. Defining the action of an element (x,y) of G×G on the set G by

$$g^{(x,y)} = x^{-1}gy \quad \text{for } g \in G$$

we can regard the set G as a G×G-set (see [8] Chap.1,§7).

(2.4). Let $\phi(g)=(g,g^\sigma)$ for each element $g \in G$. Then ϕ is a monomorphism of G into G×G. Through the homomorphism ϕ, we can regard the G×G-set G as a G-set.

(2.5). Regard $C[G]$ as a (right) $G\times G$-module (through the action defined in (2.3)). Then $C[G]e_i$ is an irreducible $G\times G$-submodule which affords the character $\overline{\chi_i}\times \chi_i$, $i=1,2,\cdots,\gamma$.

Proof. Since $C[G]e_i$ is a minimal two-sided ideal of $C[G]$, $C[G]e_i$ is an irreducible $G\times G$-submodule. Thus $C[G]e_i$ affords the character $\chi_m\times \chi_s$, $1\leqq m,s\leqq \gamma$. Let $G_1=\{(1,g)\mid g\in G\}$ and let $G_2=\{(g,1)\mid g\in G\}$. Regarding $C[G]e_i$ as a G_1-module, $C[G]e_i$ affords the character $\chi_i(1)\chi_i$ of G_1. Hence we have $\chi_s=\chi_i$. On the other hand, regarding $C[G]e_i$ as a (right) G_2-module, $C[G]e_i$ affords the character $\chi_i(1)\overline{\chi_i}$ of G_2. Hence we have $\chi_m=\overline{\chi_i}$. This proves (2.5).

(2.6). Regard $C[G]$ as a (right) G-module (through the action defined in (2.4)). Then $C[G]e_i$ is a G-submodule which affords the character $\overline{\chi_i}\chi_i^{\sigma^{-1}}$ of G, $i=1,\cdots,\gamma$.

Proof. By (2.5), $C[G]e_i$ affords the character $\overline{\chi_i}\times \chi_i$ of $G\times G$. Hence (2.6) is immediate.

(2.7). $R_i(\hat{X})=0$ for $i>\beta$, where R_i is a representation of G which affords the character χ_i.

Proof. It suffices to show that $\hat{X}e_i=0$. Suppose $\hat{X}e_i\neq 0$. Since $\hat{X}e_i$ is a G-invariant element of $C[G]e_i$, $\overline{\chi_i}\chi_i^{\sigma^{-1}}$ contains a principal character. Therefore $[\overline{\chi_i}\chi_i^{\sigma^{-1}},1]_G=[\chi_i^{\sigma^{-1}},\chi_i]_G\neq 0$. This contradicts the choice of χ_i. Thus (2.7) is proved.

(2.8). The number of orbits of the G-set G coincides with β.

Proof. Let χ be the character afforded by G-module $C[G]$. By (2.6), $\chi=\sum_{i=1}^{\gamma} \overline{\chi_i}\chi_i^{\sigma^{-1}}$. Hence we have $[\chi,1]_G=\beta$. On the other hand, $[\chi,1]_G$ coincides with the number of orbits of the G-set G (see [3] (5.15)). This proved (2.8).

In the remainder of this section, let Q be the field of rational numbers and let $E=Q(\zeta)$, where ζ is a primitive m-th root of unity.

(2.9). Let λ be an algebraic integer in E. Suppose $\lambda\overline{\lambda}=1$. Then the following hold.

(i) $\lambda^{2m}=1$, if m is odd.

(ii) $\lambda^{m}=1$, if m is even.

Proof. Considering the order of the Galois group of E over Q, (2.9) is immediate from Problem (3.2) of [3].

(2.10). Let $\lambda_1,\cdots,\lambda_s$ be algebraic integers in E. Suppose $\sum_{i=1}^{s} \lambda_i\overline{\lambda_i}=1$. Then there exists some j such that $\lambda_j\overline{\lambda_j}=1$ and $\lambda_i=0$ for all i distinct from j.

Proof. See [7], 405-406.

(2.11). Let r_1 be a rational integer relatively prime to m and let s be any integer. Then there exists an integer r_2 relatively prime to s with $r_1\equiv r_2 \pmod m$. (This is well-known and the proof is omitted.)

3. On quasi-co-prime automorphisms

(3.1). Let $\sigma_1=\sigma,\sigma_2,\cdots,\sigma_t$ be the conjugates of σ in Γ. Then the following conditions are equivalent:

(i) σ is a quasi-co-prime automorphism of G.

(ii) $G\sigma=\bigcup_{i=1}^{t} C_G(\sigma_i)\sigma_i$.

(iii) $\bigcup_{i=1}^{t} C_G(\sigma_i)\sigma_i$ is a disjoint sum.

Proof. ((i)→(ii)) Take an element g of G. Then $g^{-1}=x^{\sigma^{-1}}h_i x^{-1}$ for some h_i and some element x of G. It follows that $g\sigma= xh_i^{-1}(x^{-1})^{\sigma^{-1}}\sigma = xh_i^{-1}\sigma x^{-1}$, which implies $g\sigma\in \dot{\bigcup}_{i=1}^{t} C_G(\sigma_i)\sigma_i$. Since the converse inclusion is obvious, we get (ii).

((ii)→(iii)) Since $|G\sigma|=|G|$ and $|C_G(\sigma_i)\sigma_i|=|H|$, $\bigcup_{i=1}^{t} C_G(\sigma_i)\sigma_i$ is forced to be a disjoint sum. Hence we have (iii).

((iii)→(ii)) By the assumption, $|\bigcup_{i=1}^{t} C_G(\sigma_i)\sigma_i|=|G\sigma|$, thus (ii) is obvious.

((ii)→(i)) Take an element g of G. Then, by (ii), $g^{-1}\sigma= x(h_j^{-1}\sigma)x^{-1}$ for some h_j and some element x of G. But then $g= (xh_j^{-1}(x^{-1})^{\sigma^{-1}})^{-1}= x^{\sigma^{-1}}h_j x^{-1}\in X(h_j)$, which yields $G\subsetneq \bigcup_{i=1}^{\alpha} X(h_i)$. Hence we have (i). This completes the proof of (3.1).

(3.2). Let σ be a quasi-co-prime automorphism of G.

Then we have the following.

(i) $G=\bigcup_{i=1}^{\alpha} X(h_i)$ is a disjoint sum.

(ii) For any generator σ^m of $\langle\sigma\rangle$, where m is an integer relatively prime to n, σ^m is also a quasi-co-prime automorphism of G.

(iii) H controls fusion in H with respect to G (that is for any two elements x and y of H, they are conjugate in G if and only if they are conjugate in H).

(iv) Let x be in the center of H. Then σx is also a quasi-co-prime automorphism of G (where the automorphism σx is the restriction of the inner automorphism of Γ by σx to G).

Proof. (i) Let $K_i=\{g\in G|g^{-1}h_i g^{\sigma}=h_i\}$, $i=1,\cdots,\alpha$. Then K_i is a subgroup of G which contains $C_H(h_i)$. Hence $|K_i|\geqq|C_H(h_i)|$, and it follows $|X(h_i)|\leqq t|H:C_H(h_i)|$. But then $|G|\leqq\sum_{i=1}^{\alpha}|X(h_i)|\leqq t\sum_{i=1}^{\alpha}|H:C_H(h_i)|$ $=|G|$. This implies $|X(h_i)|=t|H:C_H(h_i)|$ and the sum $\bigcup_{i=1}^{\alpha} X(h_i)$ is disjoint.

(ii) Since m is relatively prime to n, there exists an integer r such that $rm\equiv1 \pmod n$. Then by (2.11), there exists an integer s relatively prime to $|H|$ such that $s\equiv r \pmod n$. Define a mapping f of $G\sigma^m$ into $G\sigma$ by $f(x)=x^s$ for each element x of $G\sigma^m$. Let $h\sigma^m$ be an element of $H\sigma^m$. Then $f(h\sigma^m)=h^s\sigma^{ms}=h^s\sigma$. Hence we conclude that f maps $H\sigma^m$ onto $H\sigma$. Similarly we have $f(C_G(\sigma_i)\sigma_i^m)=C_G(\sigma_i)\sigma_i, i=1,\cdots,t$ Therefore $C_G(\sigma_1)\sigma_1^m$, $C_G(\sigma_2)\sigma_2^m,\cdots,C_G(\sigma_t)\sigma_t^m$ are mutually disjoint. Thus (ii) follows from (3.1).

(iii) By (2.2) and (2.8), the number of σ-invariant conjugacy classes of G is α. Hence it suffices to show that $K\cap H\neq\phi$ for any σ-invariant conjugacy class K of G. Let K be a σ-invariant conjugacy class of G. Then there exists an element g of K with $C_\Gamma(g)\ni h\sigma$ for some $h\in H$. Hence $h\sigma=h^g\sigma^g$. But then, since $G\sigma=\bigcup_{i=1}^{t}C_G(\sigma_i)\sigma_i$ is a disjoint sum, we conclude $h=h^g$ and $\sigma=\sigma^g$. Therefore $g\in H$. Hence we have $K\cap H\neq\phi$. Thus (iii) is verified.

(iv) Let x be in the center of H. Put $\sigma'=\sigma x$. Let $\sigma_1=\sigma,\sigma_2,\cdots,\sigma_t$ be the conjugates of σ in Γ and let $\sigma_1'=\sigma',\sigma_2',\cdots,\sigma_t'$ be the conjugates of σ' in Γ. Rearranging the subindices if necessary, we may assume that $\sigma_i'\sigma_i^{-1}$ is contained in the center of $C_G(\sigma_i)$, $i=1,2,\cdots,t$. Then it is easily checked that $C_G(\sigma_i')\sigma_i'=C_G(\sigma_i)\sigma_i$, $i=1,2,\cdots,t$. Hence we conclude that $G\sigma'=G\sigma=\bigcup_{i=1}^{t}C_G(\sigma_i)\sigma_i=\bigcup_{i=1}^{t}C_G(\sigma_i')\sigma_i'$.

Thus (iv) is obtained from (3.1). This completes the proof of (3.2).

(3.3). Suppose n and $|H|$ are relatively prime. Then σ is a quasi-co-prime automorphism of G. Consequently, a co-prime automorphism is a quasi-co-prime automorphism.

Proof. Let $\sigma_1,\sigma_2,\cdots,\sigma_t$ be the conjugates of σ in Γ. Suppose $C_G(\sigma_i)\sigma_i\cap C_G(\sigma_j)\sigma_j\neq\phi$. Take an element g of $C_G(\sigma_i)\sigma_i\cap C_G(\sigma_j)\sigma_j$. Then $g=h\sigma_i=h'\sigma_j$ for some $h\in C_G(\sigma_i)$ and some $h'\in C_G(\sigma_j)$. By the assumption, n and the order of h are relatively prime, the same holds for n and the order of h'. Since such a decomposition of g is unique, we conclude $\sigma_i=\sigma_j$. Hence, $\bigcup_{k=1}^{t}C_G(\sigma_k)\sigma_k$ is a disjoint sum.

Therefore (3.3) follows from (3.1).

(3.4). Let σ be a quasi-co-prime automorphism of G. Suppose σ is an inner automorphism of G. Then $\sigma=1$. Consequently, a non-trivial quasi-co-prime automorphism is an outer automorphism.

Proof. Let $K_i=\{g \in G \mid g^{-1}h_i g^{\sigma}=h_i\}$, $i=1,2,\cdots,\alpha$. Then by the argument in the proof of (3.2)(i), $K_i=C_H(h_i)$, $i=1,2,\cdots,\alpha$. Suppose σ coincides with the inner automorphism by an element x of G. Since x is contained in the center of H, $x=h_j$ for some j. Then $g^{-1}h_j g^{\sigma}=g^{-1}h_j x^{-1}gx = x$ for each element g of G. Hence we have $X(h_j)=\{x\}$. Therefore $G=K_j=C_H(x)$. Thus (3.4) is proved.

Remark (3.5). (ii) of (3.2) is not valid for all integers. Let σ_1 be a co-prime automorphism of G. Suppose G has a trivial center and $C_G(\sigma_1)$ is nilpotent (indeed there exists such an example). Let x be an element of the center of $C_G(\sigma_1)$ distinct from the identity. Set $\sigma=\sigma_1 x$. Then by (3.2)(iv) and (3.3), σ is a quasi-co-prime automorphism of G. Take an integer m such that $\sigma^m=x$. By (3.4), σ^m is not a quasi-co-prime automorphism of G.

4. Proof of Theorem A

(4.1). Suppose σ is a quasi-co-prime automorphism of G. Then we have the following.

(i) There exists the Glauberman correspondence π with respect to σ.

(ii) Let χ^* be an extension of $\chi \in \mathrm{Irr}_\sigma(G)$ to Γ. Set $\psi(\chi^*,\sigma)=\lambda\pi(\chi)$. Then λ is a 2n-th root of unity if n is odd and a n-th root of unity if n is even.

Proof. (i) Let $\chi \in \mathrm{Irr}_\sigma(G)$ and let χ^* be an extension of χ to Γ. Set $\chi^*|_{H\times<\sigma>} = \sum_{\Theta \in \mathrm{Irr}(H)} \Theta\times\Lambda_\Theta$, where Λ_Θ is a character of $<\sigma>$ for each $\Theta \in \mathrm{Irr}(H)$. Then $\psi(\chi^*,\sigma) = \sum_{\Theta\in \mathrm{Irr}(H)} \Lambda_\Theta(\sigma)\Theta$. By (2.1)(ii) and (3.1), we have $[\psi(\chi^*,\sigma),\psi(\chi^*,\sigma)]_H=1$. Hence $\sum_{\Theta\in\mathrm{Irr}(H)} \Lambda_\Theta(\sigma)\overline{\Lambda_\Theta(\sigma)}=1$. Since $\Lambda_\Theta(\sigma)$ is an algebraic integer in $Q(\zeta)$ for each $\Theta \in \mathrm{Irr}(H)$, where ζ is a primitive n-th root of unity, we conclude from (2.10) that $\Lambda_\Theta(\sigma)\overline{\Lambda_\Theta(\sigma)}=1$ for some uniquely determined $\Theta \in \mathrm{Irr}(H)$ and $\Lambda_{\Theta'}(\sigma)=0$ for any $\Theta' \in \mathrm{Irr}(H)$ distinct from Θ. Take another extension $\mu_j\chi^*$ of χ. Then $\psi(\mu_j\chi^*,\sigma)=\mu_j(\sigma)\Lambda_\Theta(\sigma)\Theta$. Thus Θ is determined independently for the choice of an extension of χ to Γ. Therefore we can define a mapping π of $\mathrm{Irr}_\sigma(G)$ into $\mathrm{Irr}(H)$ by $\pi(\chi)=\Theta$. Now we shall show that π is the Glauberman correspondence with respect to σ. By the above argument, it suffices to prove that π is a bijection. Let $\chi_i\neq\chi_j$, $1\leqq i,j\leqq\alpha$. Set $\pi(\chi_i)=\Theta_i$ and $\pi(\chi_j)=\Theta''$.

Let χ_i^* and χ_j^* be extensions of χ_i and χ_j respectively. Then there exist non-zero scalars λ_i and λ_j such that $\psi(\chi_i^*,\sigma)=\lambda_i\Theta_i$ and $\psi(\chi_j^*,\sigma)=\lambda_j\Theta''$. By (2.1)(iii), $\sum_{g\in G\sigma}\chi_i^*(g)\overline{\chi_j^*(g)}=0$, it follows that $[\lambda_i\Theta_i,\lambda_j\Theta'']_H=\lambda_i\overline{\lambda_j}[\Theta_i,\Theta'']_H=0$. Hence we have $\Theta_i\neq\Theta''$. Since $\alpha=\beta$, we conclude π is a bijection of $\mathrm{Irr}_\sigma(G)$ onto $\mathrm{Irr}(H)$, which proves (i).

(ii) Set $\psi(\chi^*,\sigma)=\lambda\Theta$. Then by the above argument, λ is an algebraic integer in $Q(\zeta)$ with $\lambda\overline{\lambda}=1$. Hence by (2.9), (ii) is immediate. This completes the proof of (4.1).

(4.2). Suppose there exists the Glauberman correspondence π_1 with respect to σ^{-1}. Then σ is a quasi-co-prime automorphism.

Proof. By the assumption, it follows that $\alpha=\beta$. Then by (2.8), it suffices to show that $X(h_1), X(h_2), \cdots\cdots, X(h_\alpha)$ are mutually distinct subsets of G. Suppose to the contrary that $X(h_i)=X(h_j)$ for some distinct i and j. Then we can choose some $\Theta\in\mathrm{Irr}(H)$ such that $\Theta(h_i)\neq\Theta(h_j)$. Let $\chi\in\mathrm{Irr}_\sigma(G)$ be such that $\pi_1(\chi)=\Theta$. Let χ^* be an extension of χ and let R^* be a representation of Γ which affords χ^*. Set $\psi(\chi^*,\sigma^{-1})=\lambda\Theta$, where λ is a non-zero scalar. Put $|X(h_i)|=|X(h_j)|=s$. Considering $R^*(g^{-1}h_ig^\sigma\sigma^{-1})=R^*(g^{-1}h_i\sigma^{-1}g)$, we get $\chi^*(\widehat{X(h_i)}\sigma^{-1})=s\chi^*(h_i\sigma^{-1})=s\lambda\Theta(h_i)$. Similarly, $\chi^*(\widehat{X(h_j)}\sigma^{-1})=s\lambda\Theta(h_j)$. Since $\Theta(h_i)\neq\Theta(h_j)$, we obtain $\chi^*(\widehat{X(h_i)}\sigma^{-1})\neq\chi^*(\widehat{X(h_j)}\sigma^{-1})$. This contradicts the choice of $X(h_i)$ and $X(h_j)$. Hence we proved (4.2).

(4.3). Suppose there exists the Glauberman correspondence with respect to σ. Then σ is a quasi-co-prime automorphism.

Proof. By (4.2), σ^{-1} is a quasi-co-prime automorphism. Then by (3.2)(ii), σ is a quasi-co-prime automorphism, which proves (4.3).

By (4.1) and (4.3), the proof of Theorem A is completed.

5. Proof of Theorem B

In this section σ is a quasi-co-prime automorphism of G and π is the Glauberman correspondence with respect to σ. Arranging the subindices, we may assume $\pi(\chi_i)=\Theta_i$, $i=1,\cdots,\alpha$.

(5.1). There exists a unique extension χ_i^* of χ_i to Γ such that $\chi_i^*(\sigma)=\varepsilon_i\Theta_i(1)$, where $\varepsilon_i \in \{\pm 1\}$ if n is odd and $\varepsilon_i=1$ if n is even, $i=1,2,\cdots,\alpha$.

Proof. Let χ^* be an extension of χ_i. Then by (4.1)(ii), $\psi(\chi^*,\sigma)=\lambda\Theta_i$, where λ is a 2n-th root of unity if n is odd and a n-th root of unity if n is even. Assume n is odd, then there exists some $\mu_j \in \mathrm{Hom}(\Gamma/G, C^\times)$ with $\mu_j(\sigma)\lambda \in \{\pm 1\}$. Set $\chi_i^*=\mu_j\chi^*$. Then $\chi_i^*(\sigma) =\varepsilon_i\Theta_i(1)$, where $\varepsilon_i=\mu_j(\sigma)\lambda \in \{\pm 1\}$. Assume n is even, then there exists some $\mu_k \in \mathrm{Hom}(\Gamma/G,C^\times)$ with $\mu_k(\sigma)\lambda=1$. Set $\chi_i^*=\mu_k\chi^*$. Then $\chi_i^*(\sigma)=\Theta_i(1)$. Thus there exists an extension χ_i^* of χ_i which satisfies the condition of (5.1), $i=1,2,\cdots,\alpha$. Since $\mu_s\chi_i^*(\sigma)=\mu_s(\sigma)\chi_i^*(\sigma)= \mu_s(\sigma)\varepsilon_i\Theta_i(1)$, it can be easily checked that the above χ_i^* is uniquely determined. This proves (5.1).

In the rest of this section let χ_i^* be the canonical extension of χ_i to Γ (see Theorem B (i)) and set $\chi_i^*(\sigma)=\epsilon_i\Theta_i(1)$, $i=1,2,\cdots,\alpha$.

(5.2). Let m be a rational integer relatively prime to n. Then $\chi_i^*(\sigma^m)=\chi_i^*(\sigma)$, $i=1,2,\cdots,\alpha$.

Proof. Let ζ be a primitive n-th root of unity and let $E=Q(\zeta)$. Then there exists some $\tau\in Gal(E/Q)$ with $\zeta^\tau=\zeta^m$. Hence $\chi_i^*(\sigma)^\tau=\chi_i^*(\sigma^m)$. But then, since $\chi_i^*(\sigma)\in Q$, we have $\chi_i^*(\sigma^m)=\chi_i^*(\sigma)^\tau=\chi_i^*(\sigma)$. Thus (5.2) is proved.

Remark (5.3). Let m be a rational integer relatively prime to n. By (3.2)(ii), σ^m is a quasi-co-prime automorphism of G. Hence by Theorem A, there exists the Glauberman correspondence π' with respect to σ^m. By (5.2), the canonical extensions and ~~ϵ_i's are independent of the choice of a generator of $\langle\sigma\rangle$.~~ Furthermore, π' coincides with π (see the following assertion).

(5.4). Let π' be as above. Then $\pi'=\pi$.

Proof. Let $s=n|H|$. Let ζ_1 be a primitive s-th root of unity and let ζ_2 be a primitive n-th root of unity. Set $E=Q(\zeta_1)$ and $F=Q(\zeta_2)$. Let $\tau_2\in Gal(F/Q)$ be such that ${\zeta_2}^{\tau_2}={\zeta_2}^m$. Then there exists $\tau_1\in Gal(E/Q)$ whose restriction to F coincides with τ_2. Set ${\zeta_1}^{\tau_1}={\zeta_1}^r$. Then r and s are relatively prime and $r\equiv m \pmod n$.

Since $\varepsilon_i(\theta_i(h))^{\tau_1}=(\varepsilon_i\theta_i(h))^{\tau_1}=(\chi_i^*(h\sigma))^{\tau_1}=\chi_i^*(h^r\sigma^r)=\chi_i^*(h^r\sigma^m)$ $=\varepsilon_i\pi'(\chi_i)(h^r)=\varepsilon_i(\pi'(\chi_i)(h))^{\tau_1}$, we have $\theta_i(h)=\pi'(\chi_i)(h)$ for each element h of H. It follows that $\theta_i=\pi'(\chi_i)$, $i=1,2,\cdots,\alpha$. Hence we obtain $\pi'=\pi$, which proves (5.4).

Note that (i) and (ii) of Theorem B are proved by (5.1), (5.2) and (5.4).

(5.5). Suppose n is a power of a prime p. Then

$$[\chi_i|_H ,\theta_j]_H\equiv\delta_{ij}\varepsilon_i \pmod p, \quad 1\leqq i,j\leqq\alpha.$$

Proof. Let $\chi_i^*|_{H\times<\sigma>} = \sum_{k=1}^{\alpha} \theta_k\times\Lambda_{\theta_k}$, where Λ_{θ_k} is a character of $<\sigma>$. Assume $j\neq i$. Since $\Lambda_{\theta_j}(\sigma^m)=0$ for any $\sigma^m\in <\sigma>-<\sigma^p>$, we get $\Lambda_{\theta_j}(1)\equiv 0 \pmod p$ from Problem 2.16 of [3]. Considering $\chi_i^*|_H=\sum_{k=1}^{\alpha} \Lambda_{\theta_k}(1)\theta_k$, it follows that $[\chi_i^*|_H ,\theta_j]_H\equiv 0 \pmod p$ for any j distinct from i. Therefore, to prove (5.5), it suffices to show that $[\chi_i^*|_H ,\theta_i]_H\equiv\varepsilon_i \pmod p$, $i=1,2,\cdots,\alpha$. By (5.2), Λ_{θ_i} takes the constant value ε_i on $<\sigma>-<\sigma^p>$. Thus $\Lambda_{\theta_i}-\varepsilon_i 1_{<\sigma>}$ vanishes on $<\sigma>-<\sigma^p>$. Hence again from Problem 2.16 of [3], we have $\Lambda_{\theta_i}(1)-\varepsilon_i\equiv 0 \pmod p$, which implies $\Lambda_{\theta_i}(1)\equiv\varepsilon_i \pmod p$. This proved (5.5).

(The present form of the proof of (5.5) is due to H.Nagao, which is simplier than the original version.)

(5.6). The degree n_i divides $t\theta_i(1)$, $i=1,2,\cdots,\alpha$.

Proof. Since $|\Gamma:C_\Gamma(\sigma)|\chi_i^*(\sigma)/n_i$ is an algebraic integer and $\chi_i^*(\sigma)=\varepsilon_i\theta_i(1)$, $i=1,2,\cdots,\alpha$, we conclude (5.6).

(5.7). Let R_i^* be a representation of Γ which affords the character χ_i^*, $i=1,2,\cdots,\alpha$. Set $R_i^*|_G=R_i$. Then $R_i(\hat{X})=\varepsilon_i t\theta_i(1)/n_i\ R_i^*(\sigma)$. Consequently, $\chi_i(\hat{X})$ is a positive rational integer, $i=1,2,\cdots,\alpha$.

Proof. Let $G=Hg_1+Hg_2+\cdots\cdots+Hg_t$. Since $\hat{X}\sigma^{-1}=\sum_{j=1}^{t} g_j^{-1}g_j^{\sigma}\sigma^{-1}$ $=\sum_{j=1}^{t} g_j^{-1}\sigma^{-1}g_j$, $\hat{X}\sigma^{-1}$ is contained in the center of $C[\Gamma]$. Hence there exist complex numbers ν_i, $i=1,2,\cdots,\alpha$, such that $R_i^*(\hat{X}\sigma^{-1})$ $=\nu_iR_i^*(1)$. But then $t\chi_i^*(\sigma^{-1})=\nu_in_i$. It follows that $\nu_i=\varepsilon_it\theta_i(1)/n_i$. Therefore we have $R_i(\hat{X})=\nu_iR_i^*(\sigma)=\varepsilon_it\theta_i(1)/n_i\ R_i^*(\sigma)$, $i=1,2,\cdots,\alpha$. Moreover, $\chi_i(\hat{X})=\varepsilon_it\theta_i(1)/n_i\ \chi_i^*(\sigma)=t(\theta_i(1))^2/n_i$, which is a positive rational integer by (5.6). Thus (5.7) is verified.

By (5.5),(5.6) and (5.7), (iii),(iv) and (v) of Theorem B are proved. Thus the proof of Theorem B is completed.

Remark (5.8). $Q(\Theta_i)=Q(\chi_i)$, $i=1,\cdots,\alpha$, can be proved similarly as in Theorem 5 (c) of [1].

(5.9). Suppose one of the following conditions holds;

(i) n is relatively prime to $|H|$.

(ii) n is a prime.

Then $<\sigma^2>$ is contained in the kernel of det χ_i^*, $i=1,\cdots,\alpha$. Consequently, if n is odd, then det $\chi_i^*(\sigma)=1$, $i=1,\cdots,\alpha$.

Proof. First assume (i). Then it is easy to show that det $\chi_i^*(\sigma)=\pm 1$ by Remark (5.8), which implies (5.9). Next assume (ii). Let ζ be a primitive n-th root of unity. Set $\chi_i^*|_{<\sigma>} = m_1\lambda_1+\cdots+m_n\lambda_n$, where λ_i is an irreducible character with $\lambda_i(\sigma)=\zeta^{i-1}$, $i=1,\cdots,n$, and m_i's are non-negative rational integers. By (5.2), $\chi_i^*|_{<\sigma>}$ is rational valued. Hence we have $m_2=\cdots=m_n$. Then det $\chi_i^*(\sigma)=(\zeta\zeta^2\cdots\zeta^{n-1})^{m_2}$ $=\pm 1$ (-1 occurs only in the case n=2 and m_2 is odd), which implies (5.9). Furthermore, if n is odd, then $<\sigma^2>=<\sigma>$. This proved (5.9).

Remark (5.10). Let V be a two dimensional vector space over GF(5) and $N<\sigma>$ be a subgroup of GL(2,5) isomorphic to Σ_3 (the symmetric group on three letters), where N is a cyclic group of order 3 and $\sigma^2=1$. Set G=VN be a semi-direct product of V by N with respect to the natural action. Then $<\sigma>$ acts on G in the natural manner. Let $\chi \in \mathrm{Irr}_\sigma(G)$ be such that $\chi(1)=3$ and let χ^* be the canonical extension of χ. Since $\chi^*(\sigma)=1$, $\chi^*|_{<\sigma>} = 2\mu_1+\mu_2$, where $\mu_2(\sigma)=-1$. Hence we have det $\chi^*(\sigma)=-1$. In this case, the canonical extension χ^{**} of χ in the sense of Glauberman [1] satisfies $\chi^{**}(\sigma)=-1$.

Finally we shall prove Corollary C. If $i>\alpha$, then by (2.7), $\chi_i(\hat{X})=0$. If $i\leqq\alpha$, then by (5.7), $\chi_i(\hat{X})\neq 0$. Hence Corollary C is obtained.

Appendix. After finishing this work, the author was acquainted with T.Shintani's paper [7] by Prof. N.Kawanaka. Indeed the same argument as in the proof of (4.1) can be found in [7],405-406. Also the similar result to (2.8) was verified in the proof of Lemma 2-7 of [7].

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