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QUASI-CO-PRIME AUTOMORPHISMS AND THE GLAUBERMAN CORRESPONDENCE  
OF FINITE GROUPS

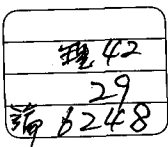
HIROSHI MATSUYAMA

1. Introduction and notation

In this paper,  $G$  denotes a finite group and  $\sigma$  denotes an automorphism of  $G$  of order  $n$ . Put  $H=C_G(\sigma)$ . Let  $\text{Irr}(G)$  and  $\text{Irr}(H)$  be the set of irreducible complex characters of  $G$  and  $H$  respectively.

If  $\chi$  is a character of  $G$ , then we define the character  $\chi^\sigma$  of  $G$  by  $\chi^\sigma(g^\sigma)=\chi(g)$  for each element  $g$  of  $G$ . Set  $\text{Irr}_\sigma(G)=\{\chi \mid \chi \in \text{Irr}(G) \text{ and } \chi^\sigma=\chi\}$ . Assuming  $\sigma$  to be co-prime (that is,  $n$  and  $|G|$  are relatively prime), Glauberman [1] showed the existence of a natural bijection of  $\text{Irr}_\sigma(G)$  onto  $\text{Irr}(H)$  and extended the result to finite groups admitting solvable operator groups with relatively prime order. Since then, generalizations and variations of Glauberman's result were obtained by several authors, see for example H.Nagao [6]. Furthermore, more general character correspondences (through 'norm mappings') were considered by N.Kawanaka [4], [5], T.Shintani [7] and others.

The purpose of this paper is to study quasi-co-prime automorphisms (see Definition (1.2)) and to extend Glauberman's character correspondence to a finite group admitting a quasi-co-prime automorphism. (Indeed we shall characterize an automorphism which yields the Glauberman correspondence.)



Before stating our Theorems, we introduce the following notation:

$\Gamma = G \langle \sigma \rangle$  ; the semi-direct product of  $G$  by  $\langle \sigma \rangle$ .

$t = |G:H|$ .

$h_1 = 1, h_2, \dots, h_\alpha$  ; the representatives of the conjugacy classes of  $H$ .

$X(h_i) = \{ g^{-1} h_i g^\sigma \mid g \in G \}$ ,  $i = 1, 2, \dots, \alpha$ .

$X = X(h_1)$ .

$\text{Irr}(G) = \{ \chi_1 = 1, \chi_2, \dots, \chi_\gamma \}$ .

$\text{Irr}_\sigma(G) = \{ \chi_1 = 1, \chi_2, \dots, \chi_\beta \}$ .

$\text{Irr}(H) = \{ \theta_1 = 1, \theta_2, \dots, \theta_\alpha \}$ .

$\text{Hom}(\Gamma/G, C^\times) = \{ \mu_1 = 1, \mu_2, \dots, \mu_n \}$  = the set of linear characters of  $\Gamma$  whose kernels contain  $G$ .

$n_i = \chi_i(1)$ ,  $i = 1, 2, \dots, \gamma$ .

$e_i = n_i / |G| \sum_{g \in G} \chi_i(g^{-1})g$ ,  $i = 1, 2, \dots, \gamma$ .

$C[G], C[\Gamma]$  ; the group algebras of  $G$  and  $\Gamma$  respectively over the field of complex numbers  $C$ .

$\hat{S} = \sum_{x \in S} x$  ( $\in C[\Gamma]$ ) for any subset  $S$  of  $\Gamma$ .

$[\Lambda_1, \Lambda_2]_G = 1/|G| \sum_{g \in G} \Lambda_1(g) \overline{\Lambda_2(g)}$ , where  $\Lambda_1$  and  $\Lambda_2$  are

class functions on  $G$ .

Remark (1.1). Let  $\chi$  be a character of  $G$  and let  $R$  be a representation of  $G$ . Then, by linear extension, we may assume that both  $\chi$  and  $R$  are defined on  $C[G]$ . Similarly, we may assume that characters and representations of  $\Gamma$  are defined on  $C[\Gamma]$ .

Definition (1.2). If  $G = \bigcup_{i=1}^{\alpha} X(h_i)$ , then  $\sigma$  is called a quasi-co-prime automorphism of  $G$ .

Remark (1.3). Some (but not all) of properties of co-prime automorphisms hold for quasi-co-prime automorphisms. Especially we can show  $\alpha = \beta$  for a finite group  $G$  admitting a quasi-co-prime automorphism (see (2.8)).

Definition (1.4). Let  $\chi \in \text{Irr}_{\sigma}(G)$  and let  $\chi^*$  be an extension of  $\chi$  to  $\Gamma$  (see(2.1)). Let  $\sigma^m$  be a generator of  $\langle \sigma \rangle$ . Then we define a class function  $\psi(\chi^*, \sigma^m)$  on  $H$  as follows;

$$\psi(\chi^*, \sigma^m)(h) = \chi^*(h\sigma^m) \quad \text{for each element } h \text{ of } H.$$

Note that the definition of  $\psi(\chi^*, \sigma^m)$  depends on the choice of  $\chi^*$  and  $\sigma^m$ .

Definition (1.5). Suppose  $\alpha = \beta$ . A bijection  $\pi$  of  $\text{Irr}_{\sigma}(G)$  onto  $\text{Irr}(H)$  is called the Glauberman correspondence with respect to  $\sigma$  if the following condition is satisfied;

Let  $\chi \in \text{Irr}_{\sigma}(G)$ . Then for any extension  $\chi^*$  of  $\chi$  to  $\Gamma$ ,  $\psi(\chi^*, \sigma)$  is a non-zero scalar multiple of  $\pi(\chi)$ .

Remark (1.6). If there exists the Glauberman correspondence  $\pi$  with respect to  $\sigma$ . Then there exists a non-zero complex number  $\lambda$  such that  $\psi(\chi^*, \sigma) = \lambda\pi(\chi)$ . Indeed  $\lambda$  is a  $2n$ -th root of unity if  $n$  is odd and a  $n$ -th root of unity if  $n$  is even (see (4.1)(ii)). Furthermore  $\psi(\chi^*, \sigma^m)$  is also a non-zero scalar multiple of  $\pi(\chi)$  for any generator  $\sigma^m$  of  $\langle \sigma \rangle$  (see (5.4)).

Now our Theorems are the following:

Theorem A. An automorphism  $\sigma$  is a quasi-co-prime automorphism if and only if there exists the Glauberman correspondence with respect to  $\sigma$ .

Theorem B. Let  $\sigma$  be a quasi-co-prime automorphism of  $G$ . Let  $\pi$  be the Glauberman correspondence with respect to  $\sigma$  and let  $\pi(\chi_i) = \theta_i$ ,  $i=1, \dots, \alpha$ . Then the following hold;

(i) There exists a unique extension  $\chi_i^*$  of  $\chi_i$  to  $\Gamma$  such that  $\chi_i^*(\sigma^m) = \epsilon_i \theta_i(1)$  for any generator  $\sigma^m$  of  $\langle \sigma \rangle$ , where  $\epsilon_i \in \{\pm 1\}$  if  $n$  is odd and  $\epsilon_i = 1$  if  $n$  is even,  $i=1, \dots, \alpha$ .

(In the sequel, the uniquely determined extension  $\chi_i^*$  is called the canonical extension of  $\chi_i$ ,  $i=1, \dots, \alpha$ . Furthermore, let  $\epsilon_i$  be as is determined above.)

(ii)  $\pi$  is independent of the choice of a generator of  $\langle \sigma \rangle$ .

(iii) If  $n$  is a power of a prime  $p$ , then

$$[\chi_i |_{\mathbb{H}}, \theta_j]_{\mathbb{H}} \equiv \delta_{ij} \epsilon_i \pmod{p}, \quad 1 \leq i, j \leq \alpha,$$

where  $\delta_{ij}$  is the Kronecker's symbol.

(iv)  $n_i$  divides  $t\theta_i(1)$ ,  $i=1,2,\dots,\alpha$ .

(v) Let  $\chi_i^*$  be the canonical extension of  $\chi_i$  to  $\Gamma$  and let  $R_i^*$  be the representation of  $\Gamma$  which affords the character  $\chi_i^*$ ,  $i=1,2,\dots,\alpha$ . Put  $R_i = R_i^*|_G$ . Then:

$R_i(\hat{X}) = \varepsilon_i t\theta_i(1)/n_i R_i^*(\sigma)$ , consequently  $\chi_i(\hat{X})$  is a positive rational integer,  $i=1,2,\dots,\alpha$ .

Remark (1.7). Let  $\sigma$  be a co-prime automorphism of  $G$ . Then  $\sigma$  is a quasi-co-prime automorphism (see (3.3)). If  $n$  is odd, then the canonical extensions defined in Theorem B (i) coincide with the canonical extensions in the sense of Glauberman [1] (see (5.9)). If  $n$  is even, then they do not always coincide with each other (see Remark (5.10)).

Corollary C. Let  $\sigma$  be a quasi-co-prime automorphism of  $G$ . Then  $\chi_i(\hat{X}) \neq 0$  if and only if  $\chi_i \in \text{Irr}_\sigma(G)$ .

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Other notation is standard, see [2], [3] and [8].

## 2. Preliminaries

(2.1). Let  $\chi_i \in \text{Irr}_\sigma(G)$ . Then we have the following.

(i) There exists an extension  $\chi_i^*$  of  $\chi_i$  to  $\Gamma$ . Furthermore  $\{\mu_j \chi_i^* \mid j=1, \dots, n\}$  is the full set of the extensions of  $\chi_i$ .

$$(ii) \quad \sum_{g \in G\sigma} \chi_i^*(g) \overline{\chi_i^*(g)} = |G|.$$

(iii) Let  $j \neq i$  and let  $\chi_j^*$  be an extension of  $\chi_j \in \text{Irr}_\sigma(G)$ .

$$\text{Then} \quad \sum_{g \in G\sigma} \chi_i^*(g) \overline{\chi_j^*(g)} = 0.$$

Proof. See [3] (6.17), (8.14), (11.22).

(2.2). The number of  $\sigma$ -invariant conjugacy classes of  $G$  coincides with  $\beta$ .

Proof. See [3] (6.32).

(2.3). Let  $G \times G$  be the direct product of  $G$  and  $G$ . Defining the action of an element  $(x, y)$  of  $G \times G$  on the set  $G$  by

$$g^{(x, y)} = x^{-1}gy \quad \text{for } g \in G$$

we can regard the set  $G$  as a  $G \times G$ -set (see [8] Chap.1, §7).

(2.4). Let  $\phi(g) = (g, g^\sigma)$  for each element  $g \in G$ . Then  $\phi$  is a monomorphism of  $G$  into  $G \times G$ . Through the homomorphism  $\phi$ , we can regard the  $G \times G$ -set  $G$  as a  $G$ -set.

(2.5). Regard  $C[G]$  as a (right)  $G \times G$ -module (through the action defined in (2.3)). Then  $C[G]e_i$  is an irreducible  $G \times G$ -submodule which affords the character  $\overline{\chi}_i \times \chi_i$ ,  $i=1,2,\dots,\gamma$ .

Proof. Since  $C[G]e_i$  is a minimal two-sided ideal of  $C[G]$ ,  $C[G]e_i$  is an irreducible  $G \times G$ -submodule. Thus  $C[G]e_i$  affords the character  $\chi_m \times \chi_s$ ,  $1 \leq m, s \leq \gamma$ . Let  $G_1 = \{(1, g) | g \in G\}$  and let  $G_2 = \{(g, 1) | g \in G\}$ . Regarding  $C[G]e_i$  as a  $G_1$ -module,  $C[G]e_i$  affords the character  $\chi_i(1)\chi_i$  of  $G_1$ . Hence we have  $\chi_s = \chi_i$ . On the other hand, regarding  $C[G]e_i$  as a (right)  $G_2$ -module,  $C[G]e_i$  affords the character  $\chi_i(1)\overline{\chi}_i$  of  $G_2$ . Hence we have  $\chi_m = \overline{\chi}_i$ . This proves (2.5).

(2.6). Regard  $C[G]$  as a (right)  $G$ -module (through the action defined in (2.4)). Then  $C[G]e_i$  is a  $G$ -submodule which affords the character  $\overline{\chi}_i \chi_i^{\sigma^{-1}}$  of  $G$ ,  $i=1,\dots,\gamma$ .

Proof. By (2.5),  $C[G]e_i$  affords the character  $\overline{\chi}_i \times \chi_i$  of  $G \times G$ . Hence (2.6) is immediate.

(2.7).  $R_i(\hat{X})=0$  for  $i > \beta$ , where  $R_i$  is a representation of  $G$  which affords the character  $\chi_i$ .

Proof. It suffices to show that  $\hat{X}e_i=0$ . Suppose  $\hat{X}e_i \neq 0$ . Since  $\hat{X}e_i$  is a  $G$ -invariant element of  $C[G]e_i$ ,  $\overline{\chi}_i \chi_i^{\sigma^{-1}}$  contains a principal character. Therefore  $[\overline{\chi}_i \chi_i^{\sigma^{-1}}, 1]_G = [\chi_i^{\sigma^{-1}}, \chi_i]_G \neq 0$ . This contradicts the choice of  $\chi_i$ . Thus (2.7) is proved.



(2.8). The number of orbits of the  $G$ -set  $G$  coincides with  $\beta$ .

Proof. Let  $\chi$  be the character afforded by  $G$ -module  $C[G]$ .

By (2.6),  $\chi = \sum_{i=1}^Y \overline{\chi_i} \chi_i^{\sigma^{-1}}$ . Hence we have  $[\chi, 1]_G = \beta$ . On the other hand,

$[\chi, 1]_G$  coincides with the number of orbits of the  $G$ -set  $G$  (see [3] (5.15)). This proved (2.8).

In the remainder of this section, let  $Q$  be the field of rational numbers and let  $E = Q(\zeta)$ , where  $\zeta$  is a primitive  $m$ -th root of unity.

(2.9). Let  $\lambda$  be an algebraic integer in  $E$ . Suppose  $\lambda \overline{\lambda} = 1$ .

Then: the following hold.

(i)  $\lambda^{2m} = 1$ , if  $m$  is odd.

(ii)  $\lambda^m = 1$ , if  $m$  is even.

Proof. Considering the order of the Galois group of  $E$  over  $Q$ , (2.9) is immediate from Problem (3.2) of [3].

(2.10). Let  $\lambda_1, \dots, \lambda_s$  be algebraic integers in  $E$ . Suppose

$\sum_{i=1}^s \lambda_i \overline{\lambda_i} = 1$ . Then there exists some  $j$  such that  $\lambda_j \overline{\lambda_j} = 1$  and  $\lambda_i = 0$

for all  $i$  distinct from  $j$ .

Proof. See [7], 405-406.

(2.11). Let  $r_1$  be a rational integer relatively prime to  $m$  and let  $s$  be any integer. Then there exists an integer  $r_2$  relatively prime to  $s$  with  $r_1 \equiv r_2 \pmod{m}$ . (This is well-known and the proof is omitted.)

### 3. On quasi-co-prime automorphisms

(3.1). Let  $\sigma_1 = \sigma, \sigma_2, \dots, \sigma_t$  be the conjugates of  $\sigma$  in  $\Gamma$ .

Then the following conditions are equivalent:

(i)  $\sigma$  is a quasi-co-prime automorphism of  $G$ .

(ii)  $G\sigma = \bigcup_{i=1}^t C_G(\sigma_i)\sigma_i$ .

(iii)  $\bigcup_{i=1}^t C_G(\sigma_i)\sigma_i$  is a disjoint sum.

Proof. ((i)→(ii)) Take an element  $g$  of  $G$ . Then  $g^{-1} = x\sigma^{-1}h_1x^{-1}$  for some  $h_1$  and some element  $x$  of  $G$ . It follows that  $g\sigma = xh_1^{-1}(x^{-1})\sigma^{-1}\sigma = xh_1^{-1}ox^{-1}$ , which implies  $g\sigma \in \bigcup_{i=1}^t C_G(\sigma_i)\sigma_i$ . Since the converse inclusion is obvious, we get (ii).

((ii)→(iii)) Since  $|G\sigma| = |G|$  and  $|C_G(\sigma_i)\sigma_i| = |H|$ ,  $\bigcup_{i=1}^t C_G(\sigma_i)\sigma_i$  is forced to be a disjoint sum. Hence we have (iii).

((iii)→(ii)) By the assumption,  $|\bigcup_{i=1}^t C_G(\sigma_i)\sigma_i| = |G\sigma|$ , thus (ii) is obvious.

((ii)→(i)) Take an element  $g$  of  $G$ . Then, by (ii),  $g^{-1}\sigma = x(h_j^{-1}\sigma)x^{-1}$  for some  $h_j$  and some element  $x$  of  $G$ . But then  $g = (xh_j^{-1}(x^{-1})\sigma^{-1})^{-1} = x\sigma^{-1}h_jx^{-1} \in X(h_j)$ , which yields  $G \subseteq \bigcup_{i=1}^t X(h_i)$ . Hence

we have (i). This completes the proof of (3.1).

(3.2). Let  $\sigma$  be a quasi-co-prime automorphism of  $G$ .

Then we have the following.

(i)  $G = \bigcup_{i=1}^{\alpha} X(h_i)$  is a disjoint sum.

(ii) For any generator  $\sigma^m$  of  $\langle \sigma \rangle$ , where  $m$  is an integer relatively prime to  $n$ ,  $\sigma^m$  is also a quasi-co-prime automorphism of  $G$ .

(iii)  $H$  controls fusion in  $H$  with respect to  $G$  (that is for any two elements  $x$  and  $y$  of  $H$ , they are conjugate in  $G$  if and only if they are conjugate in  $H$ ).

(iv) Let  $x$  be in the center of  $H$ . Then  $\sigma x$  is also a quasi-co-prime automorphism of  $G$ . (where the automorphism  $\sigma x$  is the restriction of the inner automorphism of  $\Gamma$  by  $\sigma x$  to  $G$ ).

Proof. (i) Let  $K_i = \{g \in G \mid g^{-1}h_i g^\sigma = h_i\}$ ,  $i=1, \dots, \alpha$ . Then  $K_i$  is a subgroup of  $G$  which contains  $C_H(h_i)$ . Hence  $|K_i| \geq |C_H(h_i)|$ , and it follows  $|X(h_i)| \leq t|H:C_H(h_i)|$ . But then  $|G| \leq \sum_{i=1}^{\alpha} |X(h_i)| \leq t \sum_{i=1}^{\alpha} |H:C_H(h_i)| = |G|$ . This implies  $|X(h_i)| = t|H:C_H(h_i)|$  and the sum  $\bigcup_{i=1}^{\alpha} X(h_i)$  is disjoint.

(ii) Since  $m$  is relatively prime to  $n$ , there exists an integer  $r$  such that  $rm \equiv 1 \pmod{n}$ . Then by (2.11), there exists an integer  $s$  relatively prime to  $|H|$  such that  $s \equiv r \pmod{n}$ . Define a mapping  $f$  of  $G\sigma^m$  into  $G\sigma$  by  $f(x) = x^s$  for each element  $x$  of  $G\sigma^m$ . Let  $h\sigma^m$  be an element of  $H\sigma^m$ . Then  $f(h\sigma^m) = h^s \sigma^{ms} = h^s \sigma$ . Hence we conclude that  $f$  maps  $H\sigma^m$  onto  $H\sigma$ . Similarly we have  $f(C_G(\sigma_i)\sigma_i^m) = C_G(\sigma_i)\sigma_i$ ,  $i=1, \dots, t$ . Therefore  $C_G(\sigma_1)\sigma_1^m, C_G(\sigma_2)\sigma_2^m, \dots, C_G(\sigma_t)\sigma_t^m$  are mutually disjoint. Thus (ii) follows from (3.1).

(iii) By (2.2) and (2.8), the number of  $\sigma$ -invariant conjugacy classes of  $G$  is  $\alpha$ . Hence it suffices to show that  $K \cap H \neq \emptyset$  for any  $\sigma$ -invariant conjugacy class  $K$  of  $G$ . Let  $K$  be a  $\sigma$ -invariant conjugacy class of  $G$ . Then there exists an element  $g$  of  $K$  with  $C_\Gamma(g) \ni h\sigma$  for some  $h \in H$ . Hence  $h\sigma = h^g \sigma^g$ . But then, since  $G\sigma = \bigcup_{i=1}^t C_G(\sigma_i)\sigma_i$  is a disjoint sum, we conclude  $h = h^g$  and  $\sigma = \sigma^g$ . Therefore  $g \in H$ . Hence we have  $K \cap H \neq \emptyset$ . Thus (iii) is verified.

(iv) Let  $x$  be in the center of  $H$ . Put  $\sigma' = \sigma x$ . Let  $\sigma_1 = \sigma, \sigma_2, \dots, \sigma_t$  be the conjugates of  $\sigma$  in  $\Gamma$  and let  $\sigma'_1 = \sigma', \sigma'_2, \dots, \sigma'_t$  be the conjugates of  $\sigma'$  in  $\Gamma$ . Rearranging the subindices if necessary, we may assume that  $\sigma'_i \sigma_i^{-1}$  is contained in the center of  $C_G(\sigma_i)$ ,  $i=1, 2, \dots, t$ . Then it is easily checked that  $C_G(\sigma'_i)\sigma'_i = C_G(\sigma_i)\sigma_i$ ,  $i=1, 2, \dots, t$ . Hence we conclude that  $G\sigma' = G\sigma = \bigcup_{i=1}^t C_G(\sigma_i)\sigma_i = \bigcup_{i=1}^t C_G(\sigma'_i)\sigma'_i$ . Thus (iv) is obtained from (3.1). This completes the proof of (3.2).

(3.3). Suppose  $n$  and  $|H|$  are relatively prime. Then  $\sigma$  is a quasi-co-prime automorphism of  $G$ . Consequently, a co-prime automorphism is a quasi-co-prime automorphism.

Proof. Let  $\sigma_1, \sigma_2, \dots, \sigma_t$  be the conjugates of  $\sigma$  in  $\Gamma$ . Suppose  $C_G(\sigma_i)\sigma_i \cap C_G(\sigma_j)\sigma_j \neq \emptyset$ . Take an element  $g$  of  $C_G(\sigma_i)\sigma_i \cap C_G(\sigma_j)\sigma_j$ . Then  $g = h\sigma_i = h'\sigma_j$  for some  $h \in C_G(\sigma_i)$  and some  $h' \in C_G(\sigma_j)$ . By the assumption,  $n$  and the order of  $h$  are relatively prime, the same holds for  $n$  and the order of  $h'$ . Since such a decomposition of  $g$  is unique, we conclude  $\sigma_i = \sigma_j$ . Hence,  $\bigcup_{k=1}^t C_G(\sigma_k)\sigma_k$  is a disjoint sum.

Therefore (3.3) follows from (3.1).

(3.4). Let  $\sigma$  be a quasi-co-prime automorphism of  $G$ . Suppose  $\sigma$  is an inner automorphism of  $G$ . Then  $\sigma=1$ . Consequently, a non-trivial quasi-co-prime automorphism is an outer automorphism.

Proof. Let  $K_i = \{g \in G \mid g^{-1}h_i g^\sigma = h_i\}$ ,  $i=1,2,\dots,\alpha$ . Then by the argument in the proof of (3.2)(i),  $K_i = C_H(h_i)$ ,  $i=1,2,\dots,\alpha$ . Suppose  $\sigma$  coincides with the inner automorphism by an element  $x$  of  $G$ . Since  $x$  is contained in the center of  $H$ ,  $x=h_j$  for some  $j$ . Then  $g^{-1}h_j g^\sigma = g^{-1}h_j x^{-1} g x = x$  for each element  $g$  of  $G$ . Hence we have  $X(h_j) = \{x\}$ . Therefore  $G = K_j = C_H(x)$ . Thus (3.4) is proved.

Remark (3.5). (ii) of (3.2) is not valid for all integers. Let  $\sigma_1$  be a co-prime automorphism of  $G$ . Suppose  $G$  has a trivial center and  $C_G(\sigma_1)$  is nilpotent (indeed there exists such an example). Let  $x$  be an element of the center of  $C_G(\sigma_1)$  distinct from the identity. Set  $\sigma = \sigma_1 x$ . Then by (3.2)(iv) and (3.3),  $\sigma$  is a quasi-co-prime automorphism of  $G$ . Take an integer  $m$  such that  $\sigma^m = x$ . By (3.4),  $\sigma^m$  is not a quasi-co-prime automorphism of  $G$ .

#### 4. Proof of Theorem A

(4.1). Suppose  $\sigma$  is a quasi-co-prime automorphism of  $G$ .

Then we have the following.

(i) There exists the Glauberman correspondence  $\pi$  with respect to  $\sigma$ .

(ii) Let  $\chi^*$  be an extension of  $\chi \in \text{Irr}_\sigma(G)$  to  $\Gamma$ . Set  $\psi(\chi^*, \sigma) = \lambda\pi(\chi)$ . Then  $\lambda$  is a  $2n$ -th root of unity if  $n$  is odd and a  $n$ -th root of unity if  $n$  is even.

Proof. (i) Let  $\chi \in \text{Irr}_\sigma(G)$  and let  $\chi^*$  be an extension of  $\chi$  to  $\Gamma$ . Set  $\chi^*|_{H \times \langle \sigma \rangle} = \sum_{\theta \in \text{Irr}(H)} \theta \times \Lambda_\theta$ , where  $\Lambda_\theta$  is a character of  $\langle \sigma \rangle$  for each  $\theta \in \text{Irr}(H)$ . Then  $\psi(\chi^*, \sigma) = \sum_{\theta \in \text{Irr}(H)} \Lambda_\theta(\sigma)\theta$ . By (2.1)(ii) and (3.1), we have  $[\psi(\chi^*, \sigma), \psi(\chi^*, \sigma)]_H = 1$ . Hence  $\sum_{\theta \in \text{Irr}(H)} \Lambda_\theta(\sigma) \overline{\Lambda_\theta(\sigma)} = 1$ .

Since  $\Lambda_\theta(\sigma)$  is an algebraic integer in  $\mathbb{Q}(\zeta)$  for each  $\theta \in \text{Irr}(H)$ , where  $\zeta$  is a primitive  $n$ -th root of unity, we conclude from (2.10) that  $\Lambda_\theta(\sigma) \overline{\Lambda_\theta(\sigma)} = 1$  for some uniquely determined  $\theta \in \text{Irr}(H)$  and  $\Lambda_{\theta'}(\sigma) = 0$  for any  $\theta' \in \text{Irr}(H)$  distinct from  $\theta$ . Take another extension  $\mu_j \chi^*$  of  $\chi$ . Then  $\psi(\mu_j \chi^*, \sigma) = \mu_j(\sigma) \Lambda_\theta(\sigma) \theta$ . Thus  $\theta$  is determined independently of the choice of an extension of  $\chi$  to  $\Gamma$ . Therefore we can define a mapping  $\pi$  of  $\text{Irr}_\sigma(G)$  into  $\text{Irr}(H)$  by  $\pi(\chi) = \theta$ . Now we shall show that  $\pi$  is the Glauberman correspondence with respect to  $\sigma$ . By the above argument, it suffices to prove that  $\pi$  is a bijection. Let  $\chi_i \neq \chi_j$ ,  $1 \leq i, j \leq \alpha$ . Set  $\pi(\chi_i) = \theta_i$  and  $\pi(\chi_j) = \theta_j$ .

Let  $\chi_i^*$  and  $\chi_j^*$  be extensions of  $\chi_i$  and  $\chi_j$  respectively. Then there exist non-zero scalars  $\lambda_i$  and  $\lambda_j$  such that  $\psi(\chi_i^*, \sigma) = \lambda_i \theta_i$  and

$\psi(\chi_j^*, \sigma) = \lambda_j \theta_j''$ . By (2.1)(iii),  $\sum_{g \in G\sigma} \chi_i^*(g) \overline{\chi_j^*(g)} = 0$ , it follows that

$[\lambda_i \theta_i, \lambda_j \theta_j'']_H = \lambda_i \bar{\lambda}_j [\theta_i, \theta_j'']_H = 0$ . Hence we have  $\theta_i \neq \theta_j''$ . Since  $\alpha = \beta$ , we conclude  $\pi$  is a bijection of  $\text{Irr}_\sigma(G)$  onto  $\text{Irr}(H)$ , which proves (i).

(ii) Set  $\psi(\chi^*, \sigma) = \lambda \theta$ . Then by the above argument,  $\lambda$  is an algebraic integer in  $Q(\zeta)$  with  $\lambda \bar{\lambda} = 1$ . Hence by (2.9), (ii) is immediate. This completes the proof of (4.1).

(4.2). Suppose there exists the Glauberman correspondence  $\pi_1$  with respect to  $\sigma^{-1}$ . Then  $\sigma$  is a quasi-co-prime automorphism.

Proof. By the assumption, it follows that  $\alpha = \beta$ . Then by (2.8), it suffices to show that  $X(h_1), X(h_2), \dots, X(h_\alpha)$  are mutually distinct subsets of  $G$ . Suppose to the contrary that  $X(h_i) = X(h_j)$  for some distinct  $i$  and  $j$ . Then we can choose some  $\theta \in \text{Irr}(H)$  such that  $\theta(h_i) \neq \theta(h_j)$ . Let  $\chi \in \text{Irr}_\sigma(G)$  be such that  $\pi_1(\chi) = \theta$ . Let  $\chi^*$  be an extension of  $\chi$  and let  $R^*$  be a representation of  $\Gamma$  which affords  $\chi^*$ . Set  $\psi(\chi^*, \sigma^{-1}) = \lambda \theta$ , where  $\lambda$  is a non-zero scalar. Put  $|X(h_i)| = |X(h_j)| = s$ . Considering  $R^*(g^{-1} h_i g^\sigma \sigma^{-1}) = R^*(g^{-1} h_i \sigma^{-1} g)$ , we get  $\chi^*(\widehat{X(h_i)} \sigma^{-1}) = s \chi^*(h_i \sigma^{-1}) = s \lambda \theta(h_i)$ . Similarly,  $\chi^*(\widehat{X(h_j)} \sigma^{-1}) = s \lambda \theta(h_j)$ . Since  $\theta(h_i) \neq \theta(h_j)$ , we obtain  $\chi^*(\widehat{X(h_i)} \sigma^{-1}) \neq \chi^*(\widehat{X(h_j)} \sigma^{-1})$ . This contradicts the choice of  $X(h_i)$  and  $X(h_j)$ . Hence we proved (4.2).

(4.3). Suppose there exists the Glauberman correspondence with respect to  $\sigma$ . Then  $\sigma$  is a quasi-co-prime automorphism.

Proof. By (4.2),  $\sigma^{-1}$  is a quasi-co-prime automorphism. Then by (3.2)(ii),  $\sigma$  is a quasi-co-prime automorphism, which proves (4.3).

By (4.1) and (4.3), the proof of Theorem A is completed.

## 5. Proof of Theorem B

In this section  $\sigma$  is a quasi-co-prime automorphism of  $G$  and  $\pi$  is the Glauberman correspondence with respect to  $\sigma$ . Arranging the subindices, we may assume  $\pi(\chi_i) = \theta_i$ ,  $i=1, \dots, \alpha$ .

(5.1). There exists a unique extension  $\chi_i^*$  of  $\chi_i$  to  $\Gamma$  such that  $\chi_i^*(\sigma) = \epsilon_i \theta_i(1)$ , where  $\epsilon_i \in \{\pm 1\}$  if  $n$  is odd and  $\epsilon_i = 1$  if  $n$  is even,  $i=1, 2, \dots, \alpha$ .

Proof. Let  $\chi^*$  be an extension of  $\chi_i$ . Then by (4.1)(ii),  $\psi(\chi^*, \sigma) = \lambda \theta_i$ , where  $\lambda$  is a  $2n$ -th root of unity if  $n$  is odd and a  $n$ -th root of unity if  $n$  is even. Assume  $n$  is odd, then there exists some  $\mu_j \in \text{Hom}(\Gamma/G, C^X)$  with  $\mu_j(\sigma)\lambda \in \{\pm 1\}$ . Set  $\chi_i^* = \mu_j \chi^*$ . Then  $\chi_i^*(\sigma) = \epsilon_i \theta_i(1)$ , where  $\epsilon_i = \mu_j(\sigma)\lambda \in \{\pm 1\}$ . Assume  $n$  is even, then there exists some  $\mu_k \in \text{Hom}(\Gamma/G, C^X)$  with  $\mu_k(\sigma)\lambda = 1$ . Set  $\chi_i^* = \mu_k \chi^*$ . Then  $\chi_i^*(\sigma) = \theta_i(1)$ . Thus there exists an extension  $\chi_i^*$  of  $\chi_i$  which satisfies the condition of (5.1),  $i=1, 2, \dots, \alpha$ . Since  $\mu_s \chi_i^*(\sigma) = \mu_s(\sigma) \chi_i^*(\sigma) = \mu_s(\sigma) \epsilon_i \theta_i(1)$ , it can be easily checked that the above  $\chi_i^*$  is



uniquely determined. This proves (5.1).

In the rest of this section let  $\chi_i^*$  be the canonical extension of  $\chi_i$  to  $\Gamma$  (see Theorem B (i)) and set  $\chi_i^*(\sigma) = \epsilon_i \theta_i(1)$ ,  $i=1,2,\dots,\alpha$ .

(5.2). Let  $m$  be a rational integer relatively prime to  $n$ . Then  $\chi_i^*(\sigma^m) = \chi_i^*(\sigma)$ ,  $i=1,2,\dots,\alpha$ .

Proof. Let  $\zeta$  be a primitive  $n$ -th root of unity and let  $E=Q(\zeta)$ . Then there exists some  $\tau \in \text{Gal}(E/Q)$  with  $\zeta^\tau = \zeta^m$ . Hence  $\chi_i^*(\sigma)^\tau = \chi_i^*(\sigma^m)$ . But then, since  $\chi_i^*(\sigma) \in Q$ , we have  $\chi_i^*(\sigma^m) = \chi_i^*(\sigma)^\tau = \chi_i^*(\sigma)$ . Thus (5.2) is proved.

Remark (5.3). Let  $m$  be a rational integer relatively prime to  $n$ . By (3.2)(ii),  $\sigma^m$  is a quasi-co-prime automorphism of  $G$ . Hence by Theorem A, there exists the Glauberman correspondence  $\pi'$  with respect to  $\sigma^m$ . By (5.2), the canonical extensions and  $\epsilon_i$ 's are independent of the choice of a generator of  $\langle \sigma \rangle$ . Furthermore,  $\pi'$  coincides with  $\pi$  (see the following assertion).

(5.4). Let  $\pi'$  be as above. Then  $\pi' = \pi$ .

Proof. Let  $s=n|H|$ . Let  $\zeta_1$  be a primitive  $s$ -th root of unity and let  $\zeta_2$  be a primitive  $n$ -th root of unity. Set  $E=Q(\zeta_1)$  and  $F=Q(\zeta_2)$ . Let  $\tau_2 \in \text{Gal}(F/Q)$  be such that  $\zeta_2^{\tau_2} = \zeta_2^m$ . Then there exists  $\tau_1 \in \text{Gal}(E/Q)$  whose restriction to  $F$  coincides with  $\tau_2$ . Set  $\zeta_1^{\tau_1} = \zeta_1^r$ . Then  $r$  and  $s$  are relatively prime and  $r \equiv m \pmod{n}$ .

Since  $\varepsilon_i(\theta_i(h))^{\tau^{-1}} = (\varepsilon_i \theta_i(h))^{\tau^{-1}} = (\chi_i^*(h\sigma))^{\tau^{-1}} = \chi_i^*(h^r \sigma^r) = \chi_i^*(h^r \sigma^m)$   
 $= \varepsilon_i \pi'(\chi_i)(h^r) = \varepsilon_i(\pi'(\chi_i)(h))^{\tau^{-1}}$ , we have  $\theta_i(h) = \pi'(\chi_i)(h)$  for each  
element  $h$  of  $H$ . It follows that  $\theta_i = \pi'(\chi_i)$ ,  $i=1,2,\dots,\alpha$ . Hence  
we obtain  $\pi' = \pi$ , which proves (5.4).

Note that (i) and (ii) of Theorem B are proved by (5.1),  
(5.2) and (5.4).

(5.5). Suppose  $n$  is a power of a prime  $p$ . Then

$$[\chi_i|_H, \theta_j]_H \equiv \delta_{ij} \varepsilon_i \pmod{p}, \quad 1 \leq i, j \leq \alpha.$$

Proof. Let  $\chi_i^*|_{H \times \langle \sigma \rangle} = \sum_{k=1}^{\alpha} \theta_k \times \Lambda_{\theta_k}$ , where  $\Lambda_{\theta_k}$  is a character  
of  $\langle \sigma \rangle$ . Assume  $j \neq i$ . Since  $\Lambda_{\theta_j}(\sigma^m) = 0$  for any  $\sigma^m \in \langle \sigma \rangle - \langle \sigma^p \rangle$ , we  
get  $\Lambda_{\theta_j}(1) \equiv 0 \pmod{p}$  from Problem 2.16 of [3]. Considering  
 $\chi_i^*|_H = \sum_{k=1}^{\alpha} \Lambda_{\theta_k}(1) \theta_k$ , it follows that  $[\chi_i^*|_H, \theta_j]_H \equiv 0 \pmod{p}$  for  
any  $j$  distinct from  $i$ . Therefore, to prove (5.5), it suffices to  
show that  $[\chi_i^*|_H, \theta_i]_H \equiv \varepsilon_i \pmod{p}$ ,  $i=1,2,\dots,\alpha$ . By (5.2),  $\Lambda_{\theta_i}$  takes  
the constant value  $\varepsilon_i$  on  $\langle \sigma \rangle - \langle \sigma^p \rangle$ . Thus  $\Lambda_{\theta_i} - \varepsilon_i 1_{\langle \sigma \rangle}$  vanishes  
on  $\langle \sigma \rangle - \langle \sigma^p \rangle$ . Hence again from Problem 2.16 of [3], we have  
 $\Lambda_{\theta_i}(1) - \varepsilon_i \equiv 0 \pmod{p}$ , which implies  $\Lambda_{\theta_i}(1) \equiv \varepsilon_i \pmod{p}$ . This  
proved (5.5).

(The present form of the proof of (5.5) is due to H.Nagao,  
which is simpler than the original version.)

(5.6). The degree  $n_i$  divides  $t\theta_i(1)$ ,  $i=1,2,\dots,\alpha$ .

Proof. Since  $|\Gamma:C_\Gamma(\sigma)|\chi_i^*(\sigma)/n_i$  is an algebraic integer and  $\chi_i^*(\sigma)=\epsilon_i\theta_i(1)$ ,  $i=1,2,\dots,\alpha$ , we conclude (5.6).

(5.7). Let  $R_i^*$  be a representation of  $\Gamma$  which affords the character  $\chi_i^*$ ,  $i=1,2,\dots,\alpha$ . Set  $R_i^*|_G=R_i$ . Then  $R_i(\hat{X})=\epsilon_i t\theta_i(1)/n_i R_i^*(\sigma)$ . Consequently,  $\chi_i(\hat{X})$  is a positive rational integer,  $i=1,2,\dots,\alpha$ .

Proof. Let  $G=Hg_1 + Hg_2 + \dots + Hg_t$ . Since  $\hat{X}\sigma^{-1} = \sum_{j=1}^t g_j^{-1} g_j^\sigma \sigma^{-1} = \sum_{j=1}^t g_j^{-1} \sigma^{-1} g_j$ ,  $\hat{X}\sigma^{-1}$  is contained in the center of  $C[\Gamma]$ . Hence there exist complex numbers  $v_i$ ,  $i=1,2,\dots,\alpha$ , such that  $R_i^*(\hat{X}\sigma^{-1}) = v_i R_i^*(1)$ . But then  $t\chi_i^*(\sigma^{-1}) = v_i n_i$ . It follows that  $v_i = \epsilon_i t\theta_i(1)/n_i$ . Therefore we have  $R_i(\hat{X}) = v_i R_i^*(\sigma) = \epsilon_i t\theta_i(1)/n_i R_i^*(\sigma)$ ,  $i=1,2,\dots,\alpha$ . Moreover,  $\chi_i(\hat{X}) = \epsilon_i t\theta_i(1)/n_i \chi_i^*(\sigma) = t(\theta_i(1))^2/n_i$ , which is a positive rational integer by (5.6). Thus (5.7) is verified.

By (5.5), (5.6) and (5.7), (iii), (iv) and (v) of Theorem B are proved. Thus the proof of Theorem B is completed.

Remark (5.8).  $Q(\theta_i) = Q(\chi_i)$ ,  $i=1, \dots, \alpha$ , can be proved similarly as in Theorem 5 (c) of [1].

(5.9). Suppose one of the following conditions holds;

(i)  $n$  is relatively prime to  $|H|$ .

(ii)  $n$  is a prime.

Then  $\langle \sigma^2 \rangle$  is contained in the kernel of  $\det \chi_i^*$ ,  $i=1, \dots, \alpha$ .

Consequently, if  $n$  is odd, then  $\det \chi_i^*(\sigma) = 1$ ,  $i=1, \dots, \alpha$ .

Proof. First assume (i). Then it is easy to show that  $\det \chi_i^*(\sigma) = \pm 1$  by Remark (5.8), which implies (5.9). Next assume (ii). Let  $\zeta$  be a primitive  $n$ -th root of unity. Set  $\chi_i^*|_{\langle \sigma \rangle} = m_1 \lambda_1 + \dots + m_n \lambda_n$ , where  $\lambda_i$  is an irreducible character with  $\lambda_i(\sigma) = \zeta^{i-1}$ ,  $i=1, \dots, n$ , and  $m_i$ 's are non-negative rational integers. By (5.2),  $\chi_i^*|_{\langle \sigma \rangle}$  is rational valued. Hence we have  $m_2 = \dots = m_n$ . Then  $\det \chi_i^*(\sigma) = (\zeta \zeta^2 \dots \zeta^{n-1})^{m_2} = \pm 1$  ( $-1$  occurs only in the case  $n=2$  and  $m_2$  is odd), which implies (5.9). Furthermore, if  $n$  is odd, then  $\langle \sigma^2 \rangle = \langle \sigma \rangle$ . This proved (5.9).

Remark (5.10). Let  $V$  be a two dimensional vector space over  $GF(5)$  and  $N \langle \sigma \rangle$  be a subgroup of  $GL(2,5)$  isomorphic to  $\Sigma_3$  (the symmetric group on three letters), where  $N$  is a cyclic group of order 3 and  $\sigma^2 = 1$ . Set  $G = VN$  be a semi-direct product of  $V$  by  $N$  with respect to the natural action. Then  $\langle \sigma \rangle$  acts on  $G$  in the natural manner. Let  $\chi \in \text{Irr}_0(G)$  be such that  $\chi(1) = 3$  and let  $\chi^*$  be the canonical extension of  $\chi$ . Since  $\chi^*(\sigma) = 1$ ,  $\chi^*|_{\langle \sigma \rangle} = 2\mu_1 + \mu_2$ ,

where  $\mu_2(\sigma)=-1$ . Hence we have  $\det \chi^*(\sigma)=-1$ . In this case, the canonical extension  $\chi^{**}$  of  $\chi$  in the sense of Glauberman [1] satisfies  $\chi^{**}(\sigma)=-1$ .

Finally we shall prove Corollary C. If  $i>\alpha$ , then by (2.7),  $\chi_i(\hat{X})=0$ . If  $i\leq\alpha$ , then by (5.7),  $\chi_i(\hat{X})\neq 0$ . Hence Corollary C is obtained.

Appendix. After finishing this work, the author was acquainted with T.Shintani's paper [7] by Prof. N.Kawanaka. Indeed the same argument as in the proof of (4.1) can be found in [7], 405-406. Also the similar result to (2.8) was verified in the proof of Lemma 2-7 of [7].

## References

- [1] G.Glauberman: Correspondences of characters for relatively prime operator groups, *Canad. J. Math.* 20 (1968), 1465-1488.
- [2] D.Gorenstein: *Finite groups*, Harper and Row, New York, 1968.
- [3] I.M.Isaacs: *Character theory of finite groups*, Academic Press, New York, 1976.
- [4] N.Kawanaka: Liftings of irreducible characters of finite classical groups I, *J. Fac. Sci. Univ. Tokyo*, 28 (1982), 851-861.
- [5] N.Kawanaka: Liftings of irreducible characters of finite classical groups II, preprint.
- [6] H.Nagao: Some correspondences in the representation theory of finite groups, to appear in *Proc. Int. Math. Conf.*, Singapore, 1981.
- [7] T.Shintani: Two remarks on irreducible characters of finite general linear groups, *J. Math. Soc. Japan*, 28 (1976), 396-414.
- [8] M.Suzuki: *Group theory I*, Springer-Verlag, Berlin and New York, 1982.

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