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Osaka University
[a,b]-factorization of a graph

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Abstract

Let \( a \) and \( b \) be integers such that \( 0 \leq a \leq b \). Then a graph \( G \) is called an \([a,b]\)-graph if \( a \leq d_G(x) \leq b \) for every \( x \in \mathcal{V}(G) \), and an \([a,b]\)-factor of a graph is defined to be its spanning subgraph \( F \) such that \( a \leq d_F(x) \leq b \) for every vertex \( x \), where \( d_G(x) \) and \( d_F(x) \) denote the degrees of \( x \) in \( G \) and \( F \), respectively. If the edges of a graph can be decomposed into \([a,b]\)-factors, then we say that the graph is \([a,b]\)-factorable. We prove the following two theorems: 

(i) a graph \( G \) is \([2a,2b]\)-factorable if and only if \( G \) is a \([2am,2bm]\)-graph for some integer \( m \), and (ii) every \([8m+2k,10m+2k]\)-graph is \([1,2]\)-factorable.
1. Introduction

We deal with finite graphs which may have multiple edges but have no loops. A graph without multiple edges is called a simple graph. All notation and definitions not given here can be found in [4].

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and $H$ be a subgraph of $G$. For a vertex $x$ of $H$, we denote the degree of $x$ in $H$ by $d_H(x)$, in particular, the degree of a vertex $y$ of $G$ is denoted by $d_G(y)$. Let $a$ and $b$ be integers such that $0 \leq a \leq b$. Then a graph $G$ is called an $[a,b]$-graph if $a \leq d_G(x) \leq b$ for every $x \in V(G)$, and an $[a,b]$-subgraph can be defined similarly. A spanning $[a,b]$-subgraph is called an $[a,b]$-factor. Then, if $F$ is an $[a,b]$-factor of a graph $G$, then $a \leq d_F(x) \leq b$ for all $x \in V(G)$.

If the edges of a graph $G$ can be decomposed into $[a,b]$-factors $F_1, \ldots, F_n$ of $G$, then the union $F_1 \cup \cdots \cup F_n$ is called an $[a,b]$-factorization of $G$ and $G$ itself is said to be $[a,b]$-factorable.

We usually call an $[r,r]$-graph an $r$-regular graph. Similarly, an $[r,r]$-factor, an $[r,r]$-factorization and an $[r,r]$-factorable graph are called an $r$-factor, an $r$-factorization and an $r$-factorable graph, respectively.

In 1891 Petersen [13],[4,Theorem 8.8] obtained the following theorem.

Theorem 1.1 A graph $G$ is 2-fatorable if and only if $G$ is a $2m$-regular graph for some positive integer $m$. 
Recently, Akiyama [1] proved that every r-regular graph is [2,3]-factorable, where \( r \geq 2 \). This is the first contribution toward \([a,b]\)-factorization with \( a < b \). Era [6] proved that if \( r \geq 2k^2 \), then every r-regular simple graph is \([k,k+1]\)-factorable. We now give our theorems.

**Theorem 1.2** Let \( 0 \leq a \leq b \). Then a graph \( G \) is \([2a,2b]\)-factorable if and only if \( G \) is a \([2am,2bm]\)-graph for some positive integer \( m \).

This theorem is an extension of Theorem 1.1.

**Theorem 1.3** Let \( m \geq 1 \) and \( k \geq 0 \). Then every \([8m+2k,10m+2k]\)-graph is \([1,2]\)-factorable.

As a corollary of this theorem, we can obtain the next result.

**Corollary 1.4** (1) If \( r \geq 8m \), then every \([r,r+2m-1]\)-graph is \([1,2]\)-factorable.

(2) Every connected \([r,r+1]\)-graph is \([1,2]\)-factorable, where \( r \geq 1 \).

Note that a \([2am,2bm]\)-graph can be decomposed into \( m \) \([2a,2b]\)-factors, and a \([8m+2k,10m+2k]\)-graph can be decomposed into \( 6m+k \) \([1,2]\)-factors. But the number of \([a,b]\)-factors in an \([a,b]\)-factorization of a graph is not uniquely determined. For example, a \( 4m \)-regular graph can be decomposed into \( k \) \([1,2]\)-factors for every \( k \), \( 2m \leq k \leq 3m \) (see Theorem 1.1 and Lemma 4.1). It is clear that the union of an odd cycle and a cubic graph, which is a \([2,3]\)-graph with two components,
is not \([1,2]\)-factorable. So the connectivity of a graph in (2) of Corollary 1.4 is necessary. Moreover, we show that there exists a \([6,8]\)-graph which is not \([1,2]\)-factorable (Remark 4.3).

We next mention two factor theorems on which our proof will heavily depend. One is Lovász's \((g,f)\)-factor theorem (see Lemma 2.2), which plays an important role throughout this article, and the other is Theorem 2.1, which is proved by making use of Lovász's \((g,f)\)-factor theorem. By Theorem 2.1, not only can we prove many known theorems on \(r\)-factors due to Baebler, Gallai, Petersen and others, but also we can obtain some new results on \([a,b]\)-factors, for instance, Theorem 1.2 is an easy consequence of it.

Let us finally note a survey article [2], in which many results related to our theorems can be found.

2. Factor theorem

We begin by introducing some new notation and definitions. For a finite set \(X\), we denote by \(|X|\) the number of elements in \(X\). Let \(G\) be a graph, and \(g\) and \(f\) be two integer-valued functions defined on \(V(G)\) such that \(g(x) \leq f(x)\) for every \(x \in V(G)\). Then, a \((g,f)\)-factor of \(G\) is a spanning subgraph \(F\) of \(G\) satisfying
g(x) ≤ d_G(x) ≤ f(x) for all x ∈ V(G). For a subset S of V(G), we write G−S for the subgraph of G obtained from G by deleting the vertices in S together with their incident edges. If S and T are disjoint subsets of V(G), then e(S,T) denotes the number of edges of G joining S and T.

In this section we shall prove the following theorem and give some its corollaries.

**Theorem 2.1** Let G be an n-edge-connected graph (n ≥ 1), θ be a real number such that 0 ≤ θ ≤ 1, and g and f be two integer-valued functions defined on V(G) such that g(x) ≤ f(x) for all x ∈ V(G). If one of \{(1a),(1b)\}, (2) and one of \{(3a),(3b),(3c),(3d),(3e),(3f)\} hold, then G has a (g,f)-factor.

(1a) g(x) ≤ θd_G(x) ≤ f(x) for all x ∈ V(G).

(1b) ε = \sum_{x ∈ V(G)}[^{\max\{0, g(x) - θ d_G(x)\}} + ^{\max\{0, θ d_G(x) - f(x)\}}] < 1.

(2) G has at least one vertex v such that g(v) < t(v); or g(x) = f(x) for all x ∈ V(G) and \sum_{x ∈ V(G)} f(x) ≡ 0 (mod 2).

(3a) n ≥ 1 and n(1−θ) ≥ 1.

(3b) \{d_G(x) | g(x) = f(x), x ∈ V(G)\} and \{f(x) | g(x) = f(x), x ∈ V(G)\} both consist of even numbers.

(3c) \{d_G(x) | g(x) = f(x), x ∈ V(G)\} consists of even numbers, n is odd, (n+1)θ ≥ 1 and (n+1)(1−θ) ≥ 1.

(3d) \{f(x) | g(x) = f(x), x ∈ V(G)\} consists of even numbers and m(1−θ) ≥ 1, where m ∈ \{n,n+1\} and m ≡ 1 (mod 2).

(3e) \{d_G(x) | g(x) = f(x), x ∈ V(G)\} and \{f(x) | g(x) = f(x), x ∈ V(G)\}
both consist of odd numbers and \( m \geq 1 \), where \( m \in \{n,n+1\} \) and \( m \equiv 1 \pmod{2} \).

(3f) \( g(x) < f(x) \) for every \( x \in V(G) \) (see [8]).

Note that similar necessary conditions for a graph to have a 
\((g,f)\)-factor which contains \( p \) given edges but has no \( q \) given edges
are obtained in [9]. In order to prove the above theorem we need
the next \((g,f)\)-factor theorem due to Lovász, to which Tutte [16]
gave a short proof.

**Lemma 2.2** (Lovász [12], [16, Theorem 7.2]) Let \( G \) be a graph
and \( g \) and \( f \) be integer-valued functions defined on \( V(G) \) such
that \( g(x) \leq f(x) \) for all \( x \in V(G) \). Then \( G \) has a \((g,f)\)-factor if
and only if

\[
\delta(S,T) = \sum_{t \in T} \{ d_g(t) - g(t) \} + \sum_{s \in S} f(s) - e(S,T) - h(S,T) \geq 0 \quad (2.1)
\]

for all disjoint subsets \( S \) and \( T \) of \( V(G) \), where \( h(S,T) \) denotes
the number of components \( C \) of \( G-(S \cup T) \) such that \( g(x) = f(x) \) for
all \( x \in V(C) \) and \( e(T,V(C)) + \sum_{x \in V(G)} f(x) \equiv 1 \pmod{2} \).

Note that the condition \( 0 \leq g(x) \leq f(x) \leq d_g(x) \) in [12] and [16] can
be replaced by \( g(x) \leq f(x) \) as above ([10],[15]).

**Proof of Theorem 2.1** We shall prove that two functions \( g \)
and \( f \) in Theorem 2.1 satisfy the condition (2.1) in Lemma 2.2.
It is obvious that (1a) implies (1b). Hence we may assume (1b)
holds. Let \( S, T \subseteq V(G) \) such that \( S \cap T = \emptyset \). Assume first \( S \cup T \neq \emptyset \). Let
\{C_1, \ldots, C_r\} be the set of components of \( G-(S\cup T) \) which satisfy the conditions on \( h(S, T) \), where \( r=h(S, T) \). By (1b) of Theorem 2.1, we have

\[
\delta(S, T) \geq (1-\theta) \sum_{t \in T} d_G(t) + \theta \sum_{s \in S} d_G(s) - \sum_{t \in T} \max\{0, g(t)-\theta d_G(t)\} - \sum_{s \in S} \max\{0, \theta d_G(s)-f(s)\} - e(S, T) - r \tag{2.2}
\]

\[
\geq (1-\theta)\{e(T, S) + \sum_{i=1}^r e(T, V(C_i))\} + \theta\{e(S, T) + \sum_{i=1}^r e(S, V(C_i))\} - \epsilon - e(S, T) - r
\]

\[
= \sum_{i=1}^r \{-(1-\theta)e(T, V(C_i)) + \theta e(S, V(C_i)) - 1\} - \epsilon. \tag{2.3}
\]

Since \( \delta(S, T) \) is an integer and \( \epsilon<1 \), it suffices to show that \( \delta(S, T) \geq -\epsilon \).

If (3f) holds, then \( r=0 \) and so \( \delta(S, T) \geq -\epsilon \). Hence we may assume that \( G \) satisfies (2) and one of \( \{(3a), (3b), (3c), (3d), (3e)\} \).

Take any \( C \in \{C_1, \ldots, C_r\} \), and put

\[
\Delta(C) = (1-\theta)e(T, V(C)) + \theta e(S, V(C)) - 1.
\]

We prove that \( \Delta(C) \geq 0 \). If \( \{f(x) | g(x)=f(x), x \in V(G)\} \) consists of even numbers, then

\[
1 \equiv e(T, V(C)) + \sum_{x \in V(C)} f(x) \equiv e(T, V(C)) \pmod{2},
\]

in particular, \( e(T, V(C)) \equiv 1 \). Similarly, if \( \{f(x) | g(x)=f(x), x \in V(G)\} \)

consists of odd numbers, then we have \( 1 \equiv e(T, V(C)) + |V(C)| \pmod{2} \).

Suppose \( \{d_G(x) | g(x)=f(x), x \in V(G)\} \) consists of even numbers. Then

\[
0 \equiv \sum_{x \in V(C)} d_G(x) = 2|E(C)| + e(V(C), S \cup T)
\]

\[
\equiv e(S \cup T, V(C)) \pmod{2} \tag{2.4}
\]
Thus \( e(S \cup T, V(C)) \equiv 0 \pmod{2} \). If \( \{d_G(x) \mid g(x) = f(x), x \in V(G)\} \) consists of odd numbers, then we have \( |V(C)| \equiv e(S \cup T, V(C)) \pmod{2} \).

We consider three cases.

**Case 1.** \( e(T, V(C)) \geq 1 \) and \( e(S, V(C)) \geq 1 \). It follows immediately from \( 0 \leq \theta \leq 1 \) that \( \Delta(C) \geq 0 \).

**Case 2.** \( e(T, V(C)) = 0 \). We first note that \( e(S, V(C)) = e(S \cup T, V(C)) \geq n \) since \( G \) is \( n \)-edge-connected. By the fact mentioned above, \( \{f(x) \mid g(x) = f(x), x \in V(G)\} \) is not a set of even numbers, and so neither (3b) nor (3d) occurs. If \( G \) satisfies (3a), then \( \Delta(C) \geq 0n-1 \geq 0 \) as \( e(S, V(C)) \geq n \). Suppose \( G \) satisfies (3c). Then we have \( e(S, V(C)) \geq n+1 \). Hence \( \Delta(C) \geq \theta(n+1)-1 \geq 0 \). We finally assume that \( G \) satisfies (3e). Then it follows from the fact mentioned above that \( 1 \equiv e(S, V(C)) \pmod{2} \). If \( n \) is odd, then \( m = n \) and so \( \Delta(C) \geq \theta(n+1) = \theta m - 1 \geq 0 \). If \( n \) is even, then \( e(S, V(C)) \geq n+1 \) and \( m = n+1 \). Hence \( \Delta(C) \geq \theta(n+1)-1 = \theta m - 1 \geq 0 \).

**Case 3.** \( e(S, V(C)) = 0 \). Note that \( e(T, V(C)) = e(S \cup T, V(C)) \geq n \). If \( G \) satisfies (3a), then \( \Delta(C) \geq (1-\theta)n-1 \geq 0 \). If \( \{d_G(x) \mid g(x) = f(x), x \in V(G)\} \) consists of even numbers, then \( e(T, V(C)) \equiv 0 \pmod{2} \). On the other hand, if \( \{f(x) \mid g(x) = f(x), x \in V(G)\} \) consists of even numbers, then \( e(T, V(C)) \equiv 1 \pmod{2} \). Hence (3b) does not occur. If (3c) holds, then \( e(T, V(C)) \geq n+1 \) and so \( \Delta(C) \geq (1-\theta)(n+1)-1 \geq 0 \).

Suppose \( G \) satisfies (3d). It is easy to show that we may assume \( n \) is even. Since \( e(T, V(C)) \equiv 1 \pmod{2} \), we have \( e(T, V(C)) \geq n+1 \), and thus \( \Delta(C) \geq (1-\theta)(n+1)-1 = (1-\theta) m - 1 \geq 0 \). Finally we suppose that
G satisfies (3e). Then \(1 \equiv e(T,V(C)) + |V(C)| \pmod{2}\) and \(|V(C)| \equiv e(T,V(C)) \pmod{2}\), a contradiction. Therefore, (3e) does not occur.

Let \(S = T = \emptyset\) and assume \(\delta(\phi, \phi) < 0\). Then \(h(\phi, \phi) > 0\). Since \(G\) is connected, it follows from Lemma 2.2 that \(g(x) = f(x)\) for all \(x \in V(G)\) and \(\sum f(x) \equiv 1 \pmod{2}\), which contradicts (2). Therefore \(\delta(\phi, \phi) = 0\). Consequently, the proof of the theorem is complete.

We now give some results on factors which can be obtained by Theorem 2.1.

**Corollary 2.3** Let \(2 \leq b\) and \(1 \leq a \leq b \leq 2a\). Then every 2-edge-connected \([a, b]\)-graph \(G\) has a \([1, 2]\)-factor \(F\) such that \(d_F(x) = 2\) if \(d_G(x) = b\). In particular, every 2-edge-connected \(r\)-regular graph has a 2-factor, where \(r \geq 2\) (Baebler [3]).

**Proof** We may assume \(b \geq 3\). Put \(\theta = 2/b\) and define two functions \(g\) and \(f\) on \(V(G)\) by

\[
g(x) = \begin{cases} 2 & \text{if } d_G(x) = b \\ 1 & \text{otherwise} \end{cases} \quad \text{and } f(x) = 2 \text{ for all } x \in V(G).
\]

Then \(\theta, g, f\) and \(n=2\) satisfy (1a), (2) and (3d) of Theorem 2.1. Hence \(G\) has a \((g,f)\)-factor, which is a desired \([1, 2]\)-factor.

**Corollary 2.4** Let \(G\) be a \((r-1)\)-edge-connected \([r, 2r]\)-graph with at least one vertex of degree greater than \(r\), where \(r \geq 1\). Then \(G\) has a \([1, 2]\)-factor \(F\) such that \(d_F(x) = 1\) if \(d_G(x) = r\).

**Proof** Set \(\theta = 1/r\), and define two functions \(g\) and \(f\) on
V(G) as follows:
\[ g(x) = 1 \text{ for all } x \in V(G), \quad f(x) = \begin{cases} 
1 & \text{if } d_G(x) = r, \\
2 & \text{otherwise}.
\end{cases} \]

Then \( \theta, g, f \) and \( n = r - 1 \) satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1 according as the parity of \( r \). Hence \( G \) has a \((g,f)\)-factor, which is a desired \([1,2]\)-factor. \( \square \)

**Proposition 2.5** ((1): Petersen [13](\( r = 3 \)) and Baebler [3](\( r \geq 4 \)); and (2): Little, Grant and Holton [11]) Let \( G \) be an \((r-1)\)-edge-connected \( r \)-regular graph. Then

1. (1) If \( G \) has an even number of vertices, then \( G \) has a \( 1 \)-factor; and
2. (2) If \( G \) has an odd number of vertices, then \( G-v \) has a \( 1 \)-factor for any vertex \( v \) of \( G \).

**Proof** We prove only (2) since (1) can be proved similarly. Put \( \theta = 1/r \), and define two functions \( g \) and \( f \) on \( V(G) \) as
\[ g(x) = f(x) = 1 \text{ for all } x \in V(G) \setminus \{v\}, \quad g(v) = 0 \text{ and } f(v) = 1, \]
where \( v \) is a given vertex of \( G \). Then \( \theta, g, f \) and \( n = r - 1 \) satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1. Therefore, \( G \) has a \((g,f)\)-factor \( F \). We can easily see that \( d_F(v) = 0 \). Hence (2) follows. \( \square \)

**Proposition 2.6** ((1),(2): Gallai [7]; and (3): Bollobas, Saito and Wormald [5]) The following statements hold.

1. (1) An \( n \)-edge-connected \( 2r \)-regular graph with an even number of vertices has a \((2k+1)\)-factor for every \( 2k+1, 2r/n \leq 2k+1 \leq 2r(n-1)/n. \)
(2) An n-edge-connected \((2r+1)\)-regular graph \(G\) has a \(2k\)-factor for every \(2k, 0 \leq 2k \leq (2r+1)(n-1)/n\). In particular, \(G\) has a \((2m+1)\)-factor for every \(2m+1, (2r+1)/n \leq 2m+1 \leq 2r+1\).

(3) A \(2n\)-edge-connected \((2r+1)\)-regular graph \(G\) has a \(2k\)-factor for every \(2k, 0 \leq 2k \leq (2r+1)(2n)/(2n+1)\). In particular, \(G\) has a \((2m+1)\)-factor for every \(2m+1, (2r+1)/(2n+1) \leq 2m+1 \leq 2r+1\).

Proof We prove only (3) since (1) and (2) can be proved similarly. Set \(\theta = 2k/(2r+1)\), and define two functions \(g\) and \(f\) on \(V(G)\) by \(g(x) = f(x) = 2k\) for all \(x \in V(G)\). Then \(\theta, g, f\) and \(2n\) satisfy (1a), (2) and (3d) of Theorem 2.1. Therefore \(G\) has a \((g,f)\)-factor, which is a \(2k\)-factor of \(G\). Let \(F\) be a \(2k\)-factor of \(G\). Then \(G-E(F)\) is a \((2r+1-2k)\)-factor of \(G\), and so \(G\) has a \((2m+1)\)-factor for every \(2m+1, (2r+1)/(2n+1) \leq 2m+1 \leq 2r+1\). Note that the latter can be proved independently by using (3e) of Theorem 2.1.

3. Proof of Theorem 1.2

We shall prove Theorem 1.2 by using Theorem 2.1.

Proof of Theorem 1.2 Let \(G\) be a \([2a,2b]\)-factorable graph. Then \(G\) can be decomposed into \(m\) \([2a,2b]\)-factors for some positive integer \(m\). It is clear that \(G\) is a \([2am,2bm]\)-graph.
Conversely, suppose that \( G \) is a \([2am,2bm]\)-graph. We prove that \( G \) can be decomposed into \( m \) \([2a,2b]\)-factors by induction on \( m \). Without loss of generality, we may assume \( G \) is connected. Put \( \theta = 1/m \), and define two functions \( g \) and \( f \) on \( V(G) \) as follows:

\[
f(x) = f(x) = 2a \text{ if } d_G(x) = 2am, \\
g(x) = g(x) = 2a \text{ if } 0 < d_G(x) < f(x) \text{ with } f(x) - g(x) = 1 \text{ if } 2am < d_G(x) < 2bm, \text{ and} \\
g(x) = f(x) = 2b \text{ if } d_G(x) = 2bm.
\]

Then, \( \theta, g, f \) and \( n=1 \) satisfy (1a), (2), and (3b) of Theorem 2.1. Therefore, \( G \) has a \((g,f)\)-factor \( F \). For any vertex \( x \) of \( G \) with \( 2am < d_G(x) < 2bm \), we have

\[
2a < \theta d_G(x) < 2b \text{ and } 2a(m-1) < (1-\theta)d_G(x) < 2b(m-1).
\]

Hence \( F \) is a \([2a,2b]\)-factor, and \( G-E(F) \) is a \([2a(m-1),2b(m-1)]\)-factor. Consequently, the theorem follows by induction. \( \blacksquare \)

4. Proof of Theorem 1.3

In this section we shall prove the following four statements:

(i) every \([8m+2k,10m+2k]\)-graph is \([1,2]\)-factorable (Theorem 1.3),
(ii) if \( r \geq 8m \), then every \([r,r+2m-1]\)-graph is \([1,2]\)-factorable (Corollary 1.4),
(iii) every connected \([r,r+1]\)-graph is \([1,2]\)-factorable (Corollary 1.4), and
(iv) there exists a \([6,8]\)-graph which
is not \([1,2]\)-factorable (Remark 4.3). We first prove Theorem 1.3 under the assumption that the following lemma holds.

\textbf{Lemma 4.1} Let \(G\) be a \([4,6]\)-graph with at most one vertex of degree 6. Then \(G\) can be decomposed into three \([1,2]\)-factors.

We begin with the next lemma.

\textbf{Lemma 4.2} Every \([8,10]\)-graph can be decomposed into six \([1,2]\)-factors.

\textbf{Proof} Let \(G\) be a \([8,10]\)-graph. Without loss of generality, we may assume \(G\) is connected. If \(G\) has vertices of degree 10, then choose any vertex \(w\) of degree 10. Set \(\theta=1/2\), and define two functions \(g\) and \(f\) on \(V(G)\) by

\[
g(x) = \begin{cases} 
4 & \text{if } 8 \leq d_G(x) < 9 \\
5 & \text{otherwise,}
\end{cases} \quad \text{and} \quad f(x) = \begin{cases} 
4 & \text{if } d_G(x) = 8 \\
5 & \text{if } 9 \leq d_G(x) \leq 10 \text{ and } x \neq w \\
6 & \text{if } x = w.
\end{cases}
\]

Then \(\theta, g, f\) and \(n=1\) satisfy (1a), (2) and (3c) of Theorem 2.1. Hence \(G\) has a \((g,f)\)-factor \(F\). It follows that \(F\) is a \([4,6]\)-graph with at most one vertex of degree 6 and \(G-E(F)\) is a \([4,5]\)-graph, and we conclude by Lemma 4.1 that \(G\) can be decomposed into six \([1,2]\)-factors. \(\blacksquare\)

\textbf{Proof of Theorem 1.3} It follows from Theorem 1.1 and Lemma 4.2 that every \([8m,10m]\)-graph can be decomposed into \(6m\) \([1,2]\)-factors. We now prove by induction on \(k\) that every \([8m+2k,10m+2k]\)-graph can be decomposed into \(6m+k\) \([1,2]\)-factors. Let \(G\) be a
A \([8m+2k, 10m+2k]\)-graph with \(m \geq 1\) and \(k \geq 1\). We may assume \(G\) is connected. Put \(\theta = 2/(10m+2k)\) and define two functions \(g\) and \(f\) on \(V(G)\) by
\[
g(x) = \begin{cases} 
2 & \text{if } d_G(x) = 10m+2k \\
1 & \text{otherwise,}
\end{cases}
\]
and \(f(x) = 2\) for all \(x \in V(G)\).

Then \(\theta\), \(g\), \(f\) and \(n = 1\) satisfy (1a), (2) and (3b) of Theorem 2.1. Hence \(G\) has a \((g,f)\)-factor \(F\), which is a \([1, 2]\)-factor. Since \(G-E(F)\) is a \([8m+2(k-1), 10m+2(k-1)]\)-graph, we conclude by the induction hypothesis that \(G\) is decomposed into \(6m+k\) \([1, 2]\)-factors. \(\blacksquare\)

**Proof of Corollary 1.4** We first prove (1). Let \(H\) be an \([r, r+2m-1]\)-graph with \(r \geq 8m\). Then there exist integers \(k\) and \(t\) such that \(r = 8m+2k+t\), \(0 \leq k\) and \(0 \leq t \leq 1\). It is immediate that \(8m+2k\) \(\leq r\) and \(r+2m-1 \leq 10m+2k\). Hence \(F\) is a \([8m+2k, 10m+2k]\)-graph, and so \(H\) is \([1, 2]\)-factorable by Theorem 1.3.

We next prove (2). We first show that every \([2k-1, 2k]\)-graph is \([1, 2]\)-factorable. Let \(G\) be a \([2k-1, 2k]\)-graph. Then it follows from Theorem 2.1 that \(G\) has a \([1, 2]\)-factor \(F\) such that \(d_F(x) = 2\) if \(d_G(x) = 2k\) (see Proof of Theorem 1.3). Since \(G-E(F)\) is a \([2k-3, 2k-2]\)-graph, we have by induction that \(G\) is \([1, 2]\)-factorable.

By the statement (1) and the result given above, it suffices to show that if \(r = 2, 4, 6\), then a connected \([r, r+1]\)-graph is \([1, 2]\)-factorable. It follows from Lemma 4.5, which will be given later, that every connected \([2, 3]\)-graph is \([1, 2]\)-factorable. By Lemma 4.1, every \([4, 5]\)-graph is \([1, 2]\)-factorable. Hence we may restrict ourselves to the case of \(r = 6\).
Let $H$ be a connected $[6,7]$-graph. Since a 6-regular graph is 2-factorable, we may assume that $H$ has at least one vertex of degree 7. We show that $H$ can be decomposed into two $[3,4]$-factors, which implies that $H$ can be decomposed into four $[1,2]$-factors. Put $\theta=1/2$ and define two functions $g$ and $f$ on $V(H)$ by

$$
\begin{align*}
g(x)&=3 \quad \text{for all } x \in V(H), \\
f(x)&=\begin{cases} 3 & \text{if } d_H(x)=6 \\ 4 & \text{otherwise.}
\end{cases}
\end{align*}
$$

Then $\theta$, $g$, $f$ and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence $H$ has a $(g,f)$-factor $F'$. It is clear that both $F'$ and $H-E(F')$ are $[3,4]$-factors of $H$. Therefore $H$ is $[1,2]$-factorable. 

It is convenient to introduce a new definition. For a set $\{a,b,c,\ldots\}$ of integers, a graph $G$ is called an $\{a,b,c,\ldots\}$-graph if $d_G(x) \in \{a,b,c,\ldots\}$ for every $x \in V(G)$. The union of graphs $H$ and $K$ is a graph $G$ such that $V(G)=V(H)\cup V(K)$ and $E(G)=E(H)\cup E(K)$.

Remark 4.3 The following three statements hold:

(1) A connected $[6,8]$-graph having exactly one vertex of degree 6 cannot be decomposed into four $[1,2]$-factors.

(2) The 6-regular graph with three vertices, in which every pair of vertices are joined by exactly three multiple edges, cannot be decomposed into five or more $[1,2]$-factors.

(3) The union of a connected $[6,8]$-graph with one vertex of degree 6 and the 6-regular graph given in (2) is not $[1,2]$-factorable.

Proof We first prove (1). Suppose that a connected $[6,8]$-
graph $G$ with one vertex $v$ of degree 6 has a $[1,2]$-factorization $F_1 \cup F_2 \cup F_3 \cup F_4$. Then it follows for some $F_i$ that $d_{F_i}(v) = 1$ and $d_{F_i}(x) = 2$ if $x \neq v$, a contradiction. Statement (2) is immediate. Statement (3) is an easy consequence of (1) and (2).

In order to prove Lemma 4.1, we shall give some lemmas.

**Lemma 4.4** Every $[0,4]$-graph can be decomposed into two $[0,2]$-factors.

**Proof** Let $G$ be a connected $[0,4]$-graph. Then $G$ is a $[1,4]$-graph. We define $\theta = 1/2$ and two functions $g$ and $f$ on $V(G)$ by

$$g(x) = \begin{cases} 0 & \text{if } d_G(x) = 1 \\ 1 & \text{if } 2 \leq d_G(x) \leq 3 \\ 2 & \text{if } d_G(x) = 4, \end{cases}$$

$$f(x) = \begin{cases} 1 & \text{if } d_G(x) = 1 \\ 2 & \text{otherwise}. \end{cases}$$

Then $\theta$, $g$, $f$ and $n=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence $G$ has a $(g,f)$-factor $F$, and thus the lemma holds since $F$ and $G-E(F)$ are both $[0,2]$-factors of $G$.

**Lemma 4.5** Let $G$ be a connected $[2,4]$-graph with at least one vertex of degree 3. Then $G$ can be decomposed into two $[1,2]$-factors.

**Proof** Set $\theta = 1/2$, and define two functions $g$ and $f$ on $V(G)$ by
Then $\theta$, $g$, $f$ and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence $G$ has a $(g,f)$-factor $F$, and thus $G$ is decomposed into two $[1,2]$-factors $F$ and $G-E(F)$. □

The following lemma, which is a special case of Lemma 4.9, shows that Lemma 4.1 holds if the graph is 3-edge-connected. Recall that an $\{a,b,c,...\}$-graph satisfies $d_G(x)\in\{a,b,c,...\}$ for all $x\in V(G)$.

Lemma 4.6 Let $G$ be a 3-edge-connected $[3,6]$-graph with at most one vertex of degree 6. Then $G$ has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ such that if $d_G(x) \geq 4$, then $d_{F_i}(x) \geq 1$ for every $F_i$.

Proof We first assume that $G$ has at least one vertex of degree 3 or 5, or $G$ is a $\{4,6\}$-graph with an even number of vertices of degree 4. Let $\theta=1/4$ and define two functions $g_1$ and $f_1$ on $V(G)$ by

$$g_1(x) = \begin{cases} 
0 & \text{if } d_G(x) = 3 \\
1 & \text{if } 4 \leq d_G(x) \leq 5
\end{cases}$$

and

$$f_1(x) = \begin{cases} 
1 & \text{if } 3 \leq d_G(x) \leq 4 \\
2 & \text{otherwise}.
\end{cases}$$

Then $\theta$, $g_1$, $f_1$, and $n=3$ satisfy (1b; $\varepsilon=0$ or $1/2$), (2) and (3c) of Theorem 2.1. Hence $G$ has a $(g_1,f_1)$-factor $F_1$. It is obvious that $G-E(F_1)$ is a $[2,4]$-graph with the property that each vertex of degree 2 in $G-E(F_1)$ has degree 3 in $G$. By Lemma 4.4, $G-E(F_1)$ is decomposed into two $[0,2]$-factors $F_2$ and $F_3$. Consequently, $G$ is decomposed.
into three \([0,2]\)-factors \(F_1, F_2\) and \(F_3\), which possess the desired property.

We next assume that \(G\) is a \([4,6]\)-graph with an odd number of vertices of degree 4. It suffices to show that \(G\) can be decomposed into three \([1,2]\)-factors. Suppose \(G\) is a 4-regular graph. Then it follows from Proposition 2.5 that \(G-v\) has a 1-factor \(L_1\) for a vertex \(v\) of \(G\). Let \(F_1\) be the \([1,2]\)-factor of \(G\) obtained from \(L_1\) by adding an edge of \(G-E(L_1)\) incident with \(v\). Since \(H_1=G-E(F_1)\) is a \([2,3]\)-graph having exactly one vertex of degree 2, we have by Lemma 4.5 that \(H_1\) can be decomposed into two \([1,2]\)-factors \(F_2\) and \(F_3\). Therefore, we obtain a desired \([1,2]\)-factorization \(F_1 \cup F_2 \cup F_3\) of \(G\).

Consequently, we may assume that \(G\) has exactly one vertex \(w\) of degree 6. Set \(\theta=1/4\) and define two functions \(g_2\) and \(f_2\) on \(V(G)\) by

\[
g_2(x) = f_2(x) = 1 \text{ for all } x \in V(G).
\]

Then \(\theta, g_2, f_2\) and \(n=3\) satisfy (1b; \(e=1/2\)), (2) and (3c) of Theorem 2.1. Thus \(G\) has a \((g_2, f_2)\)-factor \(L_2\). Let \(F_1\) be the \([1,2]\)-factor of \(G\) obtained from \(L_2\) by adding an edge of \(G-E(L_2)\) incident with \(w\). Since \(H_2=G-E(F_1)\) is a \([2,4]\)-graph having exactly one vertex of degree 4 and one vertex of degree 2, it follows from Lemma 4.5 that \(H_2\) can be decomposed into two \([1,2]\)-factors \(F_2\) and \(F_3\). Therefore we obtain a desired \([1,2]\)-factorization \(F_1 \cup F_2 \cup F_3\) of \(G\). 

We denote by \(xy\) or \(yx\) an edge joining two vertices \(x\) and \(y\). Let \(G\) be a graph and \(v\) and \(w\) be two distinct vertices of \(G\).
Then $G+vw$ denotes the graph obtained from $G$ by adding a new edge $vw$ to $G$, where $G$ may have edges joining $v$ and $w$. The following Lemmas 4.7 and 4.8 will be used in the proof of Lemma 4.9.

**Lemma 4.7** Let $G$ be a connected $[2,6]$-graph which has exactly one vertex $w$ of degree 2 and at most one vertex of degree 6. Suppose that two distinct vertices $u_1$ and $u_2$ are adjacent to $w$ and $G-w+u_1u_2$ is a 3-edge-connected graph. Then $G$ has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that if $d_G(x) \geq 4$, then $d_{F_i}(x) \geq 1$ for every $F_i$ and $d_{F_i}(w) \leq 1$ for every $F_i$.

**Proof** Let us define two functions $g$ and $f$ on $V(G)$ by

$$g(x) = \begin{cases} 0 & \text{if } 2 \leq d_G(x) \leq 3 \\ 1 & \text{if } 4 \leq d_G(x) \leq 5 \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } 2 \leq d_G(x) \leq 4 \\ 2 & \text{otherwise}. \end{cases}$$

We shall show that $G$ has a $(g,f)$-factor by Lemma 2.2. We denote the vertex of degree 6, if any, by $v$. Let $S$, $T \subseteq V(G)$ such that $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$. We write $\{C_1, \ldots, C_r\}$ for the set of components of $G-(S \cup T)$ which satisfy the conditions on $h(S,T)$ in Lemma 2.2, where $r = h(S,T)$. Then each $C_i$ does not contain $w$, and so $e(S \cup T, V(C_i)) \geq 3$. Moreover, we have $e(S \cup T, V(C_i)) \geq 4$ since $e(S \cup T, V(C_i)) \equiv 0 \pmod{2}$ (see (2.4) in the proof of Theorem 2.1). We obtain the following inequality by setting $\theta = 1/4$ in (2.3) in the proof of Theorem 2.1 (Note that (2.3) holds for every graph.).

$$\delta(S,T) \geq \frac{1}{r} \sum_{i=1}^{r} \left( \frac{3}{4} e(T,V(C_i)) + \frac{1}{4} e(S,V(C_i)) - 1 \right) - \varepsilon,$$
where \( e = 0 \) or \( 1/2 \) according as \( v \notin V(G) \) or \( v \in V(G) \). Then
\[
\delta(S,T) \geq \sum_{i} \left\{ \frac{1}{4} e(S \cup T, V(C_i)) - 1 \right\} - \varepsilon \geq -\varepsilon > -1.
\]

Since \( \delta(S,T) \) is an integer, we conclude that \( \delta(S,T) \geq 0 \). It is clear that \( \delta(\phi, \phi) = 0 \) as \( g(w) < f(w) \). Consequently, \( G \) has a \((g,f)\)-factor \( F \).

Put \( H = G - E(F) \). We consider two cases.

**Case 1.** \( d_F(w) = 1 \). By Lemma 4.4, \( H \) can be decomposed into two \([0,2]\)-factors \( F_2 \) and \( F_3 \), and it is easy to see that \( (F_1 = F) \cup F_2 \cup F_3 \) is a \([0,2]\)-factorization of \( G \) with the required property.

**Case 2.** \( d_F(w) = 0 \). In this case \( H \) is a \([2,4]\)-graph. Let \( C \) be any component of \( H \). If \( C \) does not contain \( w \), then we decompose \( C \) into two \([0,2]\)-factors. Suppose \( C \) contains \( w \). Then \( C \) contains \( u_1 \) and \( u_2 \). If \( d_C(u_1) = d_C(u_2) = 4 \), then we may assume \( d_G(u_1) = 5 \), and so \( F + wu_1 \), where \( wu_1 \in E(C) \), is also a \((g,f)\)-factor of \( G \). Hence Case 1 occurs, and thus we may assume \( d_C(u_1) \leq 3 \) or \( d_C(u_2) \leq 3 \). Set \( \delta = 1/2 \) and define two functions \( g_1 \) and \( f_1 \) on \( V(C) \) by
\[
g_1(x) = \begin{cases} 
1 & \text{if } d_C(x) \leq 3 \\
2 & \text{otherwise,} 
\end{cases}
\]
and
\[
f_1(x) = \begin{cases} 
1 & \text{if } x = w \\
2 & \text{otherwise.} 
\end{cases}
\]

Then \( \delta, g_1, f_1 \) and \( n = 1 \) satisfy (1a), (2) (since \( g_1(u_1) < f_1(u_1) \) or \( g_1(u_2) < f_1(u_2) \)) and (3c) of Theorem 2.1. Hence \( C \) has a \((g_1,f_1)\)-factor, and thus \( C \) is decomposed into two \([1,2]\)-factors, in each factor of which the degree of \( w \) is 1. Therefore, \( G \) can be decomposed into three \([0,2]\)-factors with the required property. \( \blacksquare \)
Lemma 4.8 Let $G$ be a 3-edge-connected $[3,5]$-graph having a vertex $w$ of degree 3. Then $G$ has a $[0,2]$-factorization $F_1 u F_2 u F_3$ with the property that if $d_G(x) \geq 4$, then $d_{F_i}(x) \geq 1$ for every $F_i$ and that $d_{F_i}(w) = 0$ for some $F_i$.

Proof Let $g$ and $f$ be functions on $V(G)$ defined by

$$
g(x) = \begin{cases} 
0 & \text{if } d_G(x) = 3 \\
1 & \text{otherwise},
\end{cases} \quad \text{and} \quad f(x) = \begin{cases} 
0 & \text{if } x = w \\
1 & \text{if } 3 \leq d_G(x) \leq 4 \text{ and } x \neq w \\
2 & \text{otherwise}.
\end{cases}
$$

We shall show that $G$ has a $(g,f)$-factor. Let $S,T \subseteq V(G)$ such that $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$, and let $\{C_1, \ldots, C_r\}$ be the components of $G-(S \cup T)$ which satisfy the conditions on $h(S,T)$, where $r = h(S,T)$. Then we have the following inequality by setting $\theta = 1/4$ in (2.2).

$$
\delta(S,T) \geq (1-\frac{1}{4}) \sum_{t \in T} d_G(t) + \frac{1}{4} \sum_{s \in S} d_G(s) - \varepsilon - e(S,T) - r,
$$

where $\varepsilon = 0$ or $3/4$ according as $w \notin S$ or $w \in S$. Hence

$$
\delta(S,T) \geq \sum_{i=1}^{r} \left\{ \frac{3}{4} e(T,V(C_i)) + \frac{1}{4} e(S,V(C_i)) - 1 \right\} - \varepsilon
\geq \sum_{i=1}^{r} \left\{ \frac{1}{4} e(TuS,V(C_i)) - 1 \right\} - \varepsilon.
$$

If $C_i$ does not contain $w$, then $e(TuS,V(C_i)) \equiv 0 \pmod{2}$ (see (2.4)), and so $e(TuS,V(C_i)) \geq 4$. Therefore

$$
\frac{1}{4} e(TuS,V(C_i)) - 1 \geq 0.
$$

If $C_i$ contains $w$, then $\varepsilon = 0$ and $e(TuS,V(C_i)) \geq 3$, and so

$$
\frac{1}{4} e(TuS,V(C_i)) - 1 \geq -\frac{1}{4}.
$$

Consequently, we obtain $\delta(S,T) \geq -3/4$, which implies $\delta(S,T) \geq 0$. 

Furthermore, we can show that $\delta(\phi, \phi) = 0$ by the fact that $G$ has at least one vertex $x$ with odd degree except $w$, for which $g(x) < f(x)$. Consequently, $G$ has a $(g,f)$-factor $F_1$. By Lemma 4.4, $G - E(F_1)$ can be decomposed into two $[0,2]$-factors $F_2$ and $F_3$. Therefore we obtain a desired $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ of $G$. \]

We need some notation and definitions in order to prove Lemma 4.1. A graph having exactly two vertices and one or more edges is called a bond, and we denote the bond with $n$ edges by $B_n$ (Fig. 1). Let $v$ be a vertex of a graph $G$ and $w$ be a vertex of the bond $B_n$. Then $G + vw + B_n$ denotes the graph obtained from $G$ and $B_n$ by joining $v$ and $w$ by a new edge $vw$ (Fig. 2).

We shall prove the next lemma instead of Lemma 4.1, which includes Lemma 4.1 as a special case.

**Figure 1.** The bond $E_4$. **Figure 2.** $G + vw + B_3$.

**Lemma 4.9** Let $G$ be a connected $[3,6]$-graph with at most one vertex of degree 6. Then $G$ has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that

$$\text{if } d_G(x) \geq 4, \text{ then } d_{F_1}(x) \geq 1 \text{ for every } F_1.$$  \hspace{1cm} (4.1)

**Proof** We prove the lemma by induction on the number of vertices of a graph. Let $G$ be a connected $[3,6]$-graph with at most one vertex of degree 6. By Lemma 4.6, we may assume that $G$ is not 3-edge-connected.
If $|V(G)|=2$ or 3, then $G$ must be 3-edge-connected, which is contrary to the assumption. Hence we may assume $|V(G)| \geq 4$.

First suppose that $G$ is not 2-edge-connected. Then $G$ has a bridge $e=vw$, where $e \in E(G)$ and $v, w \in V(G)$ (Fig. 3). Let $H$ and $K$ be the components of $G-e$ such that $v \in V(H)$ and $w \in V(K)$ (Fig. 3). If $|V(H)| \geq 3$ and $|V(K)| \geq 3$, then $H'=H+vw+B_3$ and $K'=K+uw+B_3$ are both $[3,6]$-graphs, where $u$ is a vertex of $B_3$ (Fig. 3). By the induction hypothesis, $H'$ and $K'$ can be decomposed into three $[0,2]$-factors with the property (4.1), respectively. It is easy to obtain a desired $[0,2]$-factorization of $G$ from them. Therefore, we may assume $|V(K)| = 2$. Then $K$ is $B_3$, $B_4$, or $B_5$.

If $d_H(v) \geq 3$, then $H$ has a $[0,2]$-factorization with the property (4.1) by induction, and it is easy to obtain a desired $[0,2]$-factorization of $G$ from it. Hence we may assume $d_H(v) = 2$. If two distinct vertices $x$ and $y$ of $H$ are adjacent to $v$, then $H-v+xy$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction (Fig. 4). So we can obtain a desired $[0,2]$-factorization of $G$ from it. We next suppose that one vertex $x$ and $v$ are joined by two edges in $H$ (Fig. 5). Let $H+B_3$ be the graph obtained from $H$ by identifying $x$. 

Figure 3. $G$ and $H+vw+B_3$.

Figure 4. $G$ and $H-v+xy$.

Figure 5. $G$ and $H+B_3$. 
v and one of the vertices of $B_3$ (Fig. 5). Then, by the induction hypothesis, $H+B_3$ has a $[0,2]$-factorization with the property (4.1), and it is immediate to obtain a desired $[0,2]$-factorization of $G$ from it. Consequently, the proof is complete if $G$ is not 2-edge-connected.

We now deal with the case that $G$ is 2-edge-connected. Since $G$ is not 3-edge-connected, $G$ has a cutset (i.e. a minimal cut) with two edges. We consider three cases.

**Case 1.** $G$ has a cutset $\{e_1,e_2\}$ such that the ends of $e_1$ and those of $e_2$ are all distinct, where $e_1,e_2 \in E(G)$.

Let $H$ and $K$ be the components of $G-\{e_1,e_2\}$, and let $e_1=u_1w_1$ and $e_2=u_2w_2$, where $u_1,u_2 \in V(H)$, $u_1 \neq u_2$, $w_1,w_2 \in V(K)$ and $w_1 \neq w_2$. Then $H+u_1u_2$ and $K+w_1w_2$ have $[0,2]$-factorization with property (4.1) by induction. It is easy to obtain a desired $[0,2]$-factorization of $G$ from them.

**Case 2.** $G$ has a cutset $\{e_1,e_2\}$ such that the ends of $e_1$ and those of $e_2$ are the same (Fig. 6).

![Figure 6.](image)

Let $H$ be an arbitrary component of $G-\{e_1,e_2\}$, and $v$ be the end of $e_1$ and $e_2$ contained in $H$ (Fig. 6). We shall show that $H$ has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that $F_1+e_1$, $F_2+e_2$, and $F_3$ are $[0,2]$-factors of $H+\langle e_1,e_2 \rangle$, and satisfy the condition (4.1) in $H+\langle e_1,e_2 \rangle$,

\[(4.2)\]

where $H+\langle e_1,e_2 \rangle$ is the subgraph of $G$ obtained from $H$ by adding
$e_1$ and $e_2$ together with their common end not contained in $H$. If this statement follows, then we can easily obtain a $[0,2]$-factorization of $G$ with the property (4.1) from a $[0,2]$-factorization with the property (4.2) of each component of $G-e_1,e_2$. We now prove the statement.

If $d_G(v) \geq 5$, then $d_H(v) \geq 3$ and so $H$ has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property (4.1) by induction. Since we may assume $d_{F_1}(v) \leq 1$ and $d_{F_2}(v) \leq 1$, these factors satisfy the required condition (4.2). If $d_G(v) = 3$, then $G$ has a bridge, and so Case 1 occurs. Hence we may assume $d_G(v) = 4$, and thus $d_H(v) = 2$. If two distinct vertices $x$ and $y$ of $H$ are adjacent to $v$, then $H-v+xy$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction. It is easy to obtain a desired $[0,2]$-factorization of $H$ from them. We next assume that one vertex $x$ of $H$ and $v$ are joined by two edges (Fig. 7). Let $H+B_3$ be the graph obtained from $H$ and $B_3$ by identifying $v$ and a vertex of $B_3$ (Fig. 7). Then $H+B_3$ has a $[0,2]$-factorization with the property (4.1) by induction, and so we can obtain a desired $[0,2]$-factorization of $H$ from it. Consequently, each component of $G-e_1,e_2$ has a $[0,2]$-factorization satisfying the conditions (4.2), and we conclude that the proof of Case 2 is complete.

Case 3. For every cutset $\{e_1,e_2\}$ of $G$, $e_1$ and $e_2$ have exactly one common end (Fig. 8).
Let \( \{e_1, e_2\} \) be any cutset of \( G \). Then we can write \( e_1 = vw_1 \) and \( e_2 = vw_2 \), where \( v, w_1, w_2 \in V(G) \) and \( w_1 \neq w_2 \). Let \( H \) and \( K \) be the components of \( G-\{e_1, e_2\} \) such that \( v \in V(H) \) and \( w_1, w_2 \in V(K) \). Note that \( d_H(v) \geq 2 \) as \( G \) has no bridges. We first prove that if \( \{e_1, e_2\} \) satisfies one of the following two conditions, then \( G \) has a \([0,2]\)-factorization with the property (4.1):

(i) \( K + w_1 w_2 \) is 3-edge-connected (Fig. 8).

(ii) \( d_H(v) = 3 \) and \( H \) is a 3-edge-connected graph without vertices of degree 6.

Suppose (i) hold. Then \( K + \langle e_1, e_2 \rangle \) (Fig. 8) can be decomposed into three \([0,2]\)-factors which satisfy the conditions in Lemma 4.7. On the other hand, if \( d_H(v) \geq 3 \), then \( H \) has a \([0,2]\)-factorization with the property (4.1) by induction, and so we can get a desired \([0,2]\)-factorization of \( G \). If \( d_H(v) = 2 \), then two distinct vertices \( x \) and \( y \) of \( H \) are adjacent to \( v \), and so \( H-v+xy \) has a \([0,2]\)-factorization with the property (4.1). It is easy to obtain a desired \([0,2]\)-factorization of \( G \).

We next suppose that (ii) holds. Then \( H \) can be decomposed into three \([0,2]\)-factors which satisfy the conditions in Lemma 4.8. It follows that \( K + w_1 w_2 \) has a \([0,2]\)-factorization with the property (4.1) by induction, and thus \( G \) has a desired \([0,2]\)-factorization.

We shall show that \( G \) has a cutset \( \{e_1, e_2\} \) which satisfies one
of the above conditions (i) and (ii). We can choose a cutset \{e_1, e_2\}
so that \(H\) or \(K+w_{1}w_{2}\) is 3-edge-connected. If \(d_H(v) = 2\), then we
may assume without loss of generality that \(K+w_{1}w_{2}\) is 3-edge-connected.
Hence (i) follows.

Suppose \(d_H(v) = 3\). In this case we may assume that \(K+w_{1}w_{2}\) is not
3-edge-connected and \(H\) contains a unique vertex of \(G\) with degree 6;
(otherwise, \{e_1, e_2\} satisfies (i) or (ii)). Let \{f_1, f_2\} be any
cutset of \(K+w_{1}w_{2}\). If the ends of \(f_1\) and those of \(f_2\) are all distinct,
then \(G\) has such a cutset, which contradicts the assumption of Case 3.
If the ends of \(f_1\) and those of \(f_2\) are the same, then \(f_1\) or \(f_2\),
say \(f_1\), must be \(w_1w_2\). So it follows that both \{e_1, f_2\} and \{e_2, f_2\}
are cutsets of \(G\) and \(f_2\) joins \(w_1\) and \(w_2\). Let \(T\) be the component
of \(G-\{e_1, f_2\}\) containing \(w_1\). If \(d_T(w_1) \geq 3\), then \(T\) has a \([0,2]\)-
factorization with the property (4.1) by induction. If \(d_T(w_1) = 2\),
then two distinct vertices \(t_1\) and \(t_2\) are adjacent to \(w_1\) in \(T\),
and thus \(T-w_1+t_1t_2\) has a \([0,2]\)-factorization with the property (4.1)
by induction. Obviously, the component of \(G-\{e_2, f_2\}\) containing \(w_2\)
has the same property mentioned above. Furthermore, \(H\) also has a
\([0,2]\)-factorization with the property (4.1). Therefore, we can obtain
a desired \([0,2]\)-factorization of \(G\) from them. Consequently, we may
assume that for every cutset \{f_1, f_2\} of \(K+w_{1}w_{2}\), \(f_1\) and \(f_2\) have
exactly one common end. Hence we can write \(f_1 = xy_1\) and \(f_2 = xy_2\), where
\(x, y_1, y_2 \in V(K+w_{1}w_{2})\).

Choose a cutset \{f_1 = xy_1, f_2 = xy_2\} of \(K+w_{1}w_{2}\) so that the component
of $K+w_1w_2-{f_1,f_2}$ containing $x$ is 3-edge-connected or the graph obtained from the component of $K+w_1w_2-{f_1,f_2}$ containing $y_1$ and $y_2$ by adding a new edge $y_1y_2$ is 3-edge-connected. Since $H$ contains the vertex of degree 6, we can choose such a cutset $\{f_1,f_2\}$ so that the 3-edge-connected component (or graph) has no vertices of degree 6. If $w_1w_2'\notin\{f_1,f_2\}$, then $\{f_1,f_2\}$ is a cutset of $G$ which satisfies one of (i) and (ii). Hence we may assume $f_1=w_1w_2$ and $f_2=w_1y_2$, where $w_2\neq y_2$. Then $\{e_2,y_2\}$ is a cutset of $G$ which satisfies the condition of Case 1, a contradiction.

We finally assume $d_H(v)=4$ (i.e. $v$ is the vertex of $G$ with degree 6). If $K+w_1w_2$ is 3-edge-connected, then we can obtain a $[0,2]$-factorization of $G$ with the property (4.1) by applying Lemma 4.8 to $K+<e_1,e_2>$. Hence we may assume that $K+w_1w_2$ is not 3-edge-connected. In this case we can prove that $G$ has a desired $[0,2]$-factorization by the same argument in the case of $d_H(v)=3$. Consequently, Case 3 is proved.

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References


