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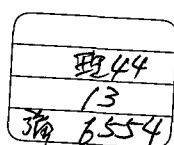
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Abstract

Let  $a$  and  $b$  be integers such that  $0 \leq a \leq b$ . Then a graph  $G$  is called an  $[a,b]$ -graph if  $a \leq d_G(x) \leq b$  for every  $x \in V(G)$ , and an  $[a,b]$ -factor of a graph is defined to be its spanning subgraph  $F$  such that  $a \leq d_F(x) \leq b$  for every vertex  $x$ , where  $d_G(x)$  and  $d_F(x)$  denote the degrees of  $x$  in  $G$  and  $F$ , respectively. If the edges of a graph can be decomposed into  $[a,b]$ -factors, then we say that the graph is  $[a,b]$ -factorable. We prove the following two theorems:  
(i) a graph  $G$  is  $[2a,2b]$ -factorable if and only if  $G$  is a  $[2am,2bm]$ -graph for some integer  $m$ , and (ii) every  $[8m+2k,10m+2k]$ -graph is  $[1,2]$ -factorable.

## 1. Introduction

We deal with finite graphs which may have multiple edges but have no loops. A graph without multiple edges is called a simple graph. All notation and definitions not given here can be found in [4].

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ , and  $H$  be a subgraph of  $G$ . For a vertex  $x$  of  $H$ , we denote the degree of  $x$  in  $H$  by  $d_H(x)$ , in particular, the degree of a vertex  $y$  of  $G$  is denoted by  $d_G(y)$ . Let  $a$  and  $b$  be integers such that  $0 \leq a \leq b$ . Then a graph  $G$  is called an  $[a,b]$ -graph if  $a \leq d_G(x) \leq b$  for every  $x \in V(G)$ , and an  $[a,b]$ -subgraph can be defined similarly. A spanning  $[a,b]$ -subgraph is called an  $[a,b]$ -factor. Then, if  $F$  is an  $[a,b]$ -factor of a graph  $G$ , then  $a \leq d_F(x) \leq b$  for all  $x \in V(G)$ .

If the edges of a graph  $G$  can be decomposed into  $[a,b]$ -factors  $F_1, \dots, F_n$  of  $G$ , then the union  $F_1 \cup \dots \cup F_n$  is called an  $[a,b]$ -factorization of  $G$  and  $G$  itself is said to be  $[a,b]$ -factorable.

We usually call an  $[r,r]$ -graph an  $r$ -regular graph. Similarly, an  $[r,r]$ -factor, an  $[r,r]$ -factorization and an  $[r,r]$ -factorable graph are called an  $r$ -factor, an  $r$ -factorization and an  $r$ -factorable graph, respectively.

In 1891 Petersen [13], [4, Theorem 8.8] obtained the following theorem.

Theorem 1.1 A graph  $G$  is 2-factorable if and only if  $G$  is a  $2m$ -regular graph for some positive integer  $m$ .

Recently, Akiyama [1] proved that every  $r$ -regular graph is  $[2,3]$ -factorable, where  $r \geq 2$ . This is the first contribution toward  $[a,b]$ -factorization with  $a < b$ . Era [6] proved that if  $r \geq 2k^2$ , then every  $r$ -regular simple graph is  $[k,k+1]$ -factorable. We now give our theorems.

Theorem 1.2     Let  $0 \leq a \leq b$ . Then a graph  $G$  is  $[2a,2b]$ -factorable if and only if  $G$  is a  $[2am,2bm]$ -graph for some positive integer  $m$ .

This theorem is an extension of Theorem 1.1.

Theorem 1.3     Let  $m \geq 1$  and  $k \geq 0$ . Then every  $[8m+2k, 10m+2k]$ -graph is  $[1,2]$ -factorable.

As a corollary of this theorem, we can obtain the next result.

Corollary 1.4     (1) If  $r \geq 8m$ , then every  $[r, r+2m-1]$ -graph is  $[1,2]$ -factorable.

(2) Every connected  $[r, r+1]$ -graph is  $[1,2]$ -factorable, where  $r \geq 1$ .

Note that a  $[2am, 2bm]$ -graph can be decomposed into  $m$   $[2a, 2b]$ -factors, and a  $[8m+2k, 10m+2k]$ -graph can be decomposed into  $6m+k$   $[1,2]$ -factors. But the number of  $[a,b]$ -factors in an  $[a,b]$ -factorization of a graph is not uniquely determined. For example, a  $4m$ -regular graph can be decomposed into  $k$   $[1,2]$ -factors for every  $k$ ,  $2m \leq k \leq 3m$  (see Theorem 1.1 and Lemma 4.1). It is clear that the union of an odd cycle and a cubic graph, which is a  $[2,3]$ -graph with two components,

is not  $[1,2]$ -factorable. So the connectivity of a graph in (2) of Corollary 1.4 is necessary. Moreover, we show that there exists a  $[6,8]$ -graph which is not  $[1,2]$ -factorable (Remark 4.3).

We next mention two factor theorems on which our proof will heavily depend. One is Lovász's  $(g,f)$ -factor theorem (see Lemma 2.2), which plays an important role throughout this article, and the other is Theorem 2.1, which is proved by making use of Lovász's  $(g,f)$ -factor theorem. By Theorem 2.1, not only can we prove many known theorems on  $r$ -factors due to Baebler, Gallai, Petersen and others, but also we can obtain some new results on  $[a,b]$ -factors, for instance, Theorem 1.2 is an easy consequence of it.

Let us finally note a survey article [2], in which many results related to our theorems can be found.

## 2. Factor theorem

We begin by introducing some new notation and definitions. For a finite set  $X$ , we denote by  $|X|$  the number of elements in  $X$ . Let  $G$  be a graph, and  $g$  and  $f$  be two integer-valued functions defined on  $V(G)$  such that  $g(x) \leq f(x)$  for every  $x \in V(G)$ . Then, a  $(g,f)$ -factor of  $G$  is a spanning subgraph  $F$  of  $G$  satisfying

$g(x) \leq d_F(x) \leq f(x)$  for all  $x \in V(G)$ . For a subset  $S$  of  $V(G)$ , we write  $G-S$  for the subgraph of  $G$  obtained from  $G$  by deleting the vertices in  $S$  together with their incident edges. If  $S$  and  $T$  are disjoint subsets of  $V(G)$ , then  $e(S, T)$  denotes the number of edges of  $G$  joining  $S$  and  $T$ .

In this section we shall prove the following theorem and give some its corollaries.

Theorem 2.1 Let  $G$  be an  $n$ -edge-connected graph ( $n \geq 1$ ),  $\theta$  be a real number such that  $0 \leq \theta \leq 1$ , and  $g$  and  $f$  be two integer-valued functions defined on  $V(G)$  such that  $g(x) \leq f(x)$  for all  $x \in V(G)$ . If one of  $\{(1a), (1b)\}$ , (2) and one of  $\{(3a), (3b), (3c), (3d), (3e), (3f)\}$  hold, then  $G$  has a  $(g, f)$ -factor.

$$(1a) \quad g(x) \leq \theta d_G(x) \leq f(x) \quad \text{for all } x \in V(G).$$

$$(1b) \quad \epsilon = \sum_{x \in V(G)} [\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}] < 1.$$

(2)  $G$  has at least one vertex  $v$  such that  $g(v) < f(v)$ ; or  $g(x) = f(x)$  for all  $x \in V(G)$  and  $\sum_{x \in V(G)} f(x) \equiv 0 \pmod{2}$ .

$$(3a) \quad n\theta \geq 1 \quad \text{and} \quad n(1-\theta) \geq 1.$$

(3b)  $\{d_G(x) \mid g(x) = f(x), x \in V(G)\}$  and  $\{f(x) \mid g(x) = f(x), x \in V(G)\}$  both consist of even numbers.

(3c)  $\{d_G(x) \mid g(x) = f(x), x \in V(G)\}$  consists of even numbers,  $n$  is odd,  $(n+1)\theta \geq 1$  and  $(n+1)(1-\theta) \geq 1$ .

(3d)  $\{f(x) \mid g(x) = f(x), x \in V(G)\}$  consists of even numbers and  $m(1-\theta) \geq 1$ , where  $m \in \{n, n+1\}$  and  $m \equiv 1 \pmod{2}$ .

$$(3e) \quad \{d_G(x) \mid g(x) = f(x), x \in V(G)\} \quad \text{and} \quad \{f(x) \mid g(x) = f(x), x \in V(G)\}$$

both consist of odd numbers and  $m \geq 1$ , where  $m \in \{n, n+1\}$  and  $m \equiv 1 \pmod{2}$ .

(3f)  $g(x) < f(x)$  for every  $x \in V(G)$  (see [8]).

Note that similar necessary conditions for a graph to have a  $(g, f)$ -factor which contains  $p$  given edges but has no  $q$  given edges are obtained in [9]. In order to prove the above theorem we need the next  $(g, f)$ -factor theorem due to Lovász, to which Tutte [16] gave a short proof.

Lemma 2.2 (Lovász [12], [16, Theorem 7.2]) Let  $G$  be a graph and  $g$  and  $f$  be integer-valued functions defined on  $V(G)$  such that  $g(x) \leq f(x)$  for all  $x \in V(G)$ . Then  $G$  has a  $(g, f)$ -factor if and only if

$$\delta(S, T) = \sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e(S, T) - h(S, T) \geq 0 \quad (2.1)$$

for all disjoint subsets  $S$  and  $T$  of  $V(G)$ , where  $h(S, T)$  denotes the number of components  $C$  of  $G - (S \cup T)$  such that  $g(x) = f(x)$  for all  $x \in V(C)$  and  $e(T, V(C)) + \sum_{x \in V(G)} f(x) \equiv 1 \pmod{2}$ .

Note that the condition  $0 \leq g(x) \leq f(x) \leq d_G(x)$  in [12] and [16] can be replaced by  $g(x) \leq f(x)$  as above ([10], [15]).

Proof of Theorem 2.1 We shall prove that two functions  $g$  and  $f$  in Theorem 2.1 satisfy the condition (2.1) in Lemma 2.2. It is obvious that (1a) implies (1b). Hence we may assume (1b) holds. Let  $S, T \subset V(G)$  such that  $S \cap T = \emptyset$ . Assume first  $S \cup T \neq \emptyset$ . Let



$\{C_1, \dots, C_r\}$  be the set of components of  $G-(SuT)$  which satisfy the conditions on  $h(S, T)$ , where  $r=h(S, T)$ . By (1b) of Theorem 2.1, we have

$$\begin{aligned} \delta(S, T) &\geq (1-\theta) \sum_{t \in T} d_G(t) + \theta \sum_{s \in S} d_G(s) - \sum_{t \in T} \max\{0, g(t) - \theta d_G(t)\} \\ &\quad - \sum_{s \in S} \max\{0, \theta d_G(s) - f(s)\} - e(S, T) - r \end{aligned} \quad (2.2)$$

$$\begin{aligned} &\geq (1-\theta)\{e(T, S) + \sum_{i=1}^r e(T, V(C_i))\} + \theta\{e(S, T) + \sum_{i=1}^r e(S, V(C_i))\} \\ &\quad - \varepsilon - e(S, T) - r \\ &= \sum_i \{(1-\theta)e(T, V(C_i)) + \theta e(S, V(C_i)) - 1\} - \varepsilon. \end{aligned} \quad (2.3)$$

Since  $\delta(S, T)$  is an integer and  $\varepsilon < 1$ , it suffices to show that  $\delta(S, T) \geq -\varepsilon$ .

If (3f) holds, then  $r=0$  and so  $\delta(S, T) \geq -\varepsilon$ . Hence we may assume that  $G$  satisfies (2) and one of  $\{(3a), (3b), (3c), (3d), (3e)\}$ . Take any  $C \in \{C_1, \dots, C_r\}$ , and put

$$\Delta(C) = (1-\theta)e(T, V(C)) + \theta e(S, V(C)) - 1.$$

We prove that  $\Delta(C) \geq 0$ . If  $\{f(x) \mid g(x)=f(x), x \in V(G)\}$  consists of even numbers, then

$$1 \equiv e(T, V(C)) + \sum_{x \in V(C)} f(x) \equiv e(T, V(C)) \pmod{2},$$

in particular,  $e(T, V(C)) \geq 1$ . Similarly, if  $\{f(x) \mid g(x)=f(x), x \in V(G)\}$  consists of odd numbers, then we have  $1 \equiv e(T, V(C)) + |V(C)| \pmod{2}$ .

Suppose  $\{d_G(x) \mid g(x)=f(x), x \in V(G)\}$  consists of even numbers. Then

$$\begin{aligned} 0 &\equiv \sum_{x \in V(C)} d_G(x) = 2|E(C)| + e(V(C), SuT) \\ &\equiv e(SuT, V(C)) \pmod{2} \end{aligned} \quad (2.4)$$

Thus  $e(SuT, V(C)) \equiv 0 \pmod{2}$ . If  $\{d_G(x) \mid g(x)=f(x), x \in V(G)\}$  consists of odd numbers, then we have  $|V(C)| \equiv e(SuT, V(C)) \pmod{2}$ .

We consider three cases.

Case 1.  $e(T, V(C)) \geq 1$  and  $e(S, V(C)) \geq 1$ . It follows immediately from  $0 \leq \theta \leq 1$  that  $\Delta(C) \geq 0$ .

Case 2.  $e(T, V(C)) = 0$ . We first note that  $e(S, V(C)) = e(SuT, V(C)) \geq n$  since  $G$  is  $n$ -edge-connected. By the fact mentioned above,  $\{f(x) \mid g(x)=f(x), x \in V(G)\}$  is not a set of even numbers, and so neither (3b) nor (3d) occurs. If  $G$  satisfies (3a), then  $\Delta(C) \geq \theta n - 1 \geq 0$  as  $e(S, V(C)) \geq n$ . Suppose  $G$  satisfies (3c). Then we have  $e(S, V(C)) \geq n+1$ . Hence  $\Delta(C) \geq \theta(n+1) - 1 \geq 0$ . We finally assume that  $G$  satisfies (3e). Then it follows from the fact mentioned above that  $1 \equiv e(S, V(C)) \pmod{2}$ . If  $n$  is odd, then  $m=n$  and so  $\Delta(C) \geq \theta n - 1 = \theta m - 1 \geq 0$ . If  $n$  is even, then  $e(S, V(C)) \geq n+1$  and  $m=n+1$ . Hence  $\Delta(C) \geq \theta(n+1) - 1 = \theta m - 1 \geq 0$ .

Case 3.  $e(S, V(C)) = 0$ . Note that  $e(T, V(C)) = e(SuT, V(C)) \geq n$ . If  $G$  satisfies (3a), then  $\Delta(C) \geq (1-\theta)n - 1 \geq 0$ . If  $\{d_G(x) \mid g(x)=f(x), x \in V(G)\}$  consists of even numbers, then  $e(T, V(C)) \equiv 0 \pmod{2}$ . On the other hand, if  $\{f(x) \mid g(x)=f(x), x \in V(G)\}$  consists of even numbers, then  $e(T, V(C)) \equiv 1 \pmod{2}$ . Hence (3b) does not occur. If (3c) holds, then  $e(T, V(C)) \geq n+1$  and so  $\Delta(C) \geq (1-\theta)(n+1) - 1 \geq 0$ . Suppose  $G$  satisfies (3d). It is easy to show that we may assume  $n$  is even. Since  $e(T, V(C)) \equiv 1 \pmod{2}$ , we have  $e(T, V(C)) \geq n+1$ , and thus  $\Delta(C) \geq (1-\theta)(n+1) - 1 = (1-\theta)m - 1 \geq 0$ . Finally we suppose that

$G$  satisfies (3e). Then  $1 \equiv e(T, V(C)) + |V(C)| \pmod{2}$  and  $|V(C)| \equiv e(T, V(C)) \pmod{2}$ , a contradiction. Therefore, (3e) does not occur.

Let  $S=T=\phi$  and assume  $\delta(\phi, \phi) < 0$ . Then  $h(\phi, \phi) > 0$ . Since  $G$  is connected, it follows from Lemma 2.2 that  $g(x) = f(x)$  for all  $x \in V(G)$  and  $\sum f(x) \equiv 1 \pmod{2}$ , which contradicts (2). Therefore  $\delta(\phi, \phi) = 0$ . Consequently, the proof of the theorem is complete. ■

We now give some results on factors which can be obtained by Theorem 2.1.

Corollary 2.3 Let  $2 \leq b$  and  $1 \leq a \leq b \leq 2a$ . Then every 2-edge-connected  $[a, b]$ -graph  $G$  has a  $[1, 2]$ -factor  $F$  such that  $d_F(x) = 2$  if  $d_G(x) = b$ . In particular, every 2-edge-connected  $r$ -regular graph has a 2-factor, where  $r \geq 2$  (Baebler [3]).

Proof We may assume  $b \geq 3$ . Put  $\theta = 2/b$  and define two functions  $g$  and  $f$  on  $V(G)$  by

$$g(x) = \begin{cases} 2 & \text{if } d_G(x) = b \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = 2 \text{ for all } x \in V(G).$$

Then  $\theta, g, f$  and  $n=2$  satisfy (1a), (2) and (3d) of Theorem 2.1.

Hence  $G$  has a  $(g, f)$ -factor, which is a desired  $[1, 2]$ -factor. ■

Corollary 2.4 Let  $G$  be a  $(r-1)$ -edge-connected  $[r, 2r]$ -graph with at least one vertex of degree greater than  $r$ , where  $r \geq 1$ .

Then  $G$  has a  $[1, 2]$ -factor  $F$  such that  $d_F(x) = 1$  if  $d_G(x) = r$ .

Proof Set  $\theta = 1/r$ , and define two functions  $g$  and  $f$  on

$V(G)$  as follows :

$$g(x)=1 \text{ for all } x \in V(G), \text{ and } f(x) = \begin{cases} 1 & \text{if } d_G(x)=r, \\ 2 & \text{otherwise.} \end{cases}$$

Then  $\theta, g, f$  and  $n=r-1$  satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1 according as the parity of  $r$ . Hence  $G$  has a  $(g,f)$ -factor, which is a desired  $[1,2]$ -factor. ■

Proposition 2.5 ((1): Petersen [13] ( $r=3$ ) and Baebler [3] ( $r \geq 4$ ); and (2): Little, Grant and Holton [11]) Let  $G$  be an  $(r-1)$ -edge-connected  $r$ -regular graph. Then

- (1) if  $G$  has an even number of vertices, then  $G$  has a 1-factor; and
- (2) if  $G$  has an odd number of vertices, then  $G-v$  has a 1-factor for any vertex  $v$  of  $G$ .

Proof We prove only (2) since (1) can be proved similarly. Put  $\theta=1/r$ , and define two functions  $g$  and  $f$  on  $V(G)$  as  $g(x)=f(x)=1$  for all  $x \in V(G) \setminus \{v\}$ ,  $g(v)=0$  and  $f(v)=1$ , where  $v$  is a given vertex of  $G$ . Then  $\theta, g, f$  and  $n=r-1$  satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1. Therefore,  $G$  has a  $(g,f)$ -factor  $F$ . We can easily see that  $d_F(v)=0$ . Hence (2) follows. ■

Proposition 2.6 ((1),(2): Gallai [7]; and (3): Bollobas, Saito and Wormald [5]) The following statements hold.

- (1) An  $n$ -edge-connected  $2r$ -regular graph with an even number of vertices has a  $(2k+1)$ -factor for every  $2k+1, 2r/n \leq 2k+1 \leq 2r(n-1)/n$ .

(2) An  $n$ -edge-connected  $(2r+1)$ -regular graph  $G$  has a  $2k$ -factor for every  $2k$ ,  $0 \leq 2k \leq (2r+1)(n-1)/n$ . In particular,  $G$  has a  $(2m+1)$ -factor for every  $2m+1$ ,  $(2r+1)/n \leq 2m+1 \leq 2r+1$ .

(3) A  $2n$ -edge-connected  $(2r+1)$ -regular graph  $G$  has a  $2k$ -factor for every  $2k$ ,  $0 \leq 2k \leq (2r+1)(2n)/(2n+1)$ . In particular,  $G$  has a  $(2m+1)$ -factor for every  $2m+1$ ,  $(2r+1)/(2n+1) \leq 2m+1 \leq 2r+1$ .

Proof We prove only (3) since (1) and (2) can be proved similarly. Set  $\theta = 2k/(2r+1)$ , and define two functions  $g$  and  $f$  on  $V(G)$  by  $g(x) = f(x) = 2k$  for all  $x \in V(G)$ . Then  $\theta$ ,  $g$ ,  $f$  and  $2n$  satisfy (1a), (2) and (3d) of Theorem 2.1. Therefore  $G$  has a  $(g, f)$ -factor, which is a  $2k$ -factor of  $G$ . Let  $F$  be a  $2k$ -factor of  $G$ . Then  $G - E(F)$  is a  $(2r+1-2k)$ -factor of  $G$ , and so  $G$  has a  $(2m+1)$ -factor for every  $2m+1$ ,  $(2r+1)/(2n+1) \leq 2m+1 \leq 2r+1$ . Note that the latter can be proved independently by using (3e) of Theorem 2.1. ■

### 3. Proof of Theorem 1.2

We shall prove Theorem 1.2 by using Theorem 2.1.

Proof of Theorem 1.2 Let  $G$  be a  $[2a, 2b]$ -factorable graph. Then  $G$  can be decomposed into  $m$   $[2a, 2b]$ -factors for some positive integer  $m$ . It is clear that  $G$  is a  $[2am, 2bm]$ -graph.

Conversely, suppose that  $G$  is a  $[2am, 2bm]$ -graph. We prove that  $G$  can be decomposed into  $m$   $[2a, 2b]$ -factors by induction on  $m$ . Without loss of generality, we may assume  $G$  is connected. Put  $\theta = 1/m$ , and define two functions  $g$  and  $f$  on  $V(G)$  as follows :

$$\begin{aligned} f(x) &= f(x) = 2a \quad \text{if} \quad d_G(x) = 2am, \\ g(x) &\leq \theta d_G(x) \leq f(x) \quad \text{with} \quad f(x) - g(x) = 1 \quad \text{if} \quad 2am < d_G(x) < 2bm, \text{ and} \\ g(x) &= f(x) = 2b \quad \text{if} \quad d_G(x) = 2bm. \end{aligned}$$

Then,  $\theta$ ,  $g$ ,  $f$  and  $n=1$  satisfy (1a), (2) and (3b) of Theorem 2.1. Therefore,  $G$  has a  $(g, f)$ -factor  $F$ . For any vertex  $x$  of  $G$  with  $2am < d_G(x) < 2bm$ , we have

$$2a < \theta d_G(x) < 2b \quad \text{and} \quad 2a(m-1) < (1-\theta)d_G(x) < 2b(m-1).$$

Hence  $F$  is a  $[2a, 2b]$ -factor, and  $G - E(F)$  is a  $[2a(m-1), 2b(m-1)]$ -factor. Consequently, the theorem follows by induction. ■

#### 4. Proof of Theorem 1.3

In this section we shall prove the following four statements :

- (i) every  $[8m+2k, 10m+2k]$ -graph is  $[1, 2]$ -factorable (Theorem 1.3),
- (ii) if  $r \geq 8m$ , then every  $[r, r+2m-1]$ -graph is  $[1, 2]$ -factorable (Corollary 1.4),
- (iii) every connected  $[r, r+1]$ -graph is  $[1, 2]$ -factorable (Corollary 1.4), and
- (iv) there exists a  $[6, 8]$ -graph which

is not  $[1,2]$ -factorable (Remark 4.3).

We first prove Theorem 1.3 under the assumption that the following lemma holds.

Lemma 4.1 Let  $G$  be a  $[4,6]$ -graph with at most one vertex of degree 6. Then  $G$  can be decomposed into three  $[1,2]$ -factors.

We begin with the next lemma.

Lemma 4.2 Every  $[8,10]$ -graph can be decomposed into six  $[1,2]$ -factors.

Proof Let  $G$  be a  $[8,10]$ -graph. Without loss of generality, we may assume  $G$  is connected. If  $G$  has vertices of degree 10, then choose any vertex  $w$  of degree 10. Set  $\theta=1/2$ , and define two functions  $g$  and  $f$  on  $V(G)$  by

$$g(x) = \begin{cases} 4 & \text{if } 8 \leq d_G(x) \leq 9 \\ 5 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 4 & \text{if } d_G(x)=8 \\ 5 & \text{if } 9 \leq d_G(x) \leq 10 \text{ and } x \neq w \\ 6 & \text{if } x=w. \end{cases}$$

Then  $\theta$ ,  $g$ ,  $f$  and  $n=1$  satisfy (1a), (2) and (3c) of Theorem 2.1. Hence  $G$  has a  $(g,f)$ -factor  $F$ . It follows that  $F$  is a  $[4,6]$ -graph with at most one vertex of degree 6 and  $G-E(F)$  is a  $[4,5]$ -graph, and we conclude by Lemma 4.1 that  $G$  can be decomposed into six  $[1,2]$ -factors. ■

Proof of Theorem 1.3 It follows from Theorem 1.1 and Lemma 4.2 that every  $[8m,10m]$ -graph can be decomposed into  $6m$   $[1,2]$ -factors. We now prove by induction on  $k$  that every  $[8m+2k,10m+2k]$ -graph can be decomposed into  $6m+k$   $[1,2]$ -factors. Let  $G$  be a

$[8m+2k, 10m+2k]$ -graph with  $m \geq 1$  and  $k \geq 1$ . We may assume  $G$  is connected.

Put  $\theta = 2/(10m+2k)$  and define two functions  $g$  and  $f$  on  $V(G)$  by

$$g(x) = \begin{cases} 2 & \text{if } d_G(x) = 10m+2k \\ 1 & \text{otherwise,} \end{cases} \quad \text{and } f(x) = 2 \text{ for all } x \in V(G).$$

Then  $\theta$ ,  $g$ ,  $f$  and  $n=1$  satisfy (1a), (2) and (3b) of Theorem 2.1.

Hence  $G$  has a  $(g, f)$ -factor  $F$ , which is a  $[1, 2]$ -factor. Since

$G - E(F)$  is a  $[8m+2(k-1), 10m+2(k-1)]$ -graph, we conclude by the induction

hypothesis that  $G$  is decomposed into  $6m+k$   $[1, 2]$ -factors. ■

#### Proof of Corollary 1.4

We first prove (1). Let  $H$  be an  $[r, r+2m-1]$ -graph with  $r \geq 8m$ . Then there exist integers  $k$  and  $t$  such that  $r = 8m+2k+t$ ,  $0 \leq k$  and  $0 \leq t \leq 1$ . It is immediate that  $8m+2k \leq r$  and  $r+2m-1 \leq 10m+2k$ . Hence  $F$  is a  $[8m+2k, 10m+2k]$ -graph, and so  $H$  is  $[1, 2]$ -factorable by Theorem 1.3.

We next prove (2). We first show that every  $[2k-1, 2k]$ -graph is  $[1, 2]$ -factorable. Let  $G$  be a  $[2k-1, 2k]$ -graph. Then it follows from Theorem 2.1 that  $G$  has a  $[1, 2]$ -factor  $F$  such that  $d_F(x) = 2$  if  $d_G(x) = 2k$  (see Proof of Theorem 1.3). Since  $G - E(F)$  is a  $[2k-3, 2k-2]$ -graph, we have by induction that  $G$  is  $[1, 2]$ -factorable. By the statement (1) and the result given above, it suffices to show that if  $r = 2, 4$  or  $6$ , then a connected  $[r, r+1]$ -graph is  $[1, 2]$ -factorable. It follows from Lemma 4.5, which will be given later, that every connected  $[2, 3]$ -graph is  $[1, 2]$ -factorable. By Lemma 4.1, every  $[4, 5]$ -graph is  $[1, 2]$ -factorable. Hence we may restrict ourselves to the case of  $r = 6$ .



Let  $H$  be a connected  $[6,7]$ -graph. Since a 6-regular graph is 2-factorable, we may assume that  $H$  has at least one vertex of degree 7. We show that  $H$  can be decomposed into two  $[3,4]$ -factors, which implies that  $H$  can be decomposed into four  $[1,2]$ -factors. Put  $\theta=1/2$  and define two functions  $g$  and  $f$  on  $V(H)$  by

$$g(x)=3 \text{ for all } x \in V(H), \text{ and } f(x) = \begin{cases} 3 & \text{if } d_H(x)=6 \\ 4 & \text{otherwise.} \end{cases}$$

Then  $\theta, g, f$  and  $n=1$  satisfy (1a), (2) and (3c) of Theorem 2.1.

Hence  $H$  has a  $(g,f)$ -factor  $F'$ . It is clear that both  $F'$  and  $H-E(F')$  are  $[3,4]$ -factors of  $H$ . Therefore  $H$  is  $[1,2]$ -factorable. ■

It is convenient to introduce a new definition. For a set  $\{a,b,c,\dots\}$  of integers, a graph  $G$  is called an  $\{a,b,c,\dots\}$ -graph if  $d_G(x) \in \{a,b,c,\dots\}$  for every  $x \in V(G)$ . The union of graphs  $H$  and  $K$  is a graph  $G$  such that  $V(G)=V(H) \cup V(K)$  and  $E(G)=E(H) \cup E(K)$ .

Remark 4.3 The following three statements hold :

(1) A connected  $\{6,8\}$ -graph having exactly one vertex of degree 6 cannot be decomposed into four  $[1,2]$ -factors.

(2) The 6-regular graph with three vertices, in which every pair of vertices are joined by exactly three multiple edges, cannot be decomposed into five or more  $[1,2]$ -factors.

(3) The union of a connected  $\{6,8\}$ -graph with one vertex of degree 6 and the 6-regular graph given in (2) is not  $[1,2]$ -factorable.

Proof We first prove (1). Suppose that a connected  $\{6,8\}$ -

graph  $G$  with one vertex  $v$  of degree 6 has a  $[1,2]$ -factorization  $F_1 \cup F_2 \cup F_3 \cup F_4$ . Then it follows for some  $F_i$  that  $d_{F_i}(v)=1$  and  $d_{F_i}(x)=2$  if  $x \neq v$ , a contradiction. Statement (2) is immediate. Statement (3) is an easy consequence of (1) and (2). ■

In order to prove Lemma 4.1, we shall give some lemmas.

Lemma 4.4 Every  $[0,4]$ -graph can be decomposed into two  $[0,2]$ -factors.

Proof Let  $G$  be a connected  $[0,4]$ -graph. Then  $G$  is a  $[1,4]$ -graph. We define  $\theta=1/2$  and two functions  $g$  and  $f$  on  $V(G)$  by

$$g(x) = \begin{cases} 0 & \text{if } d_G(x)=1 \\ 1 & \text{if } 2 \leq d_G(x) \leq 3 \\ 2 & \text{if } d_G(x)=4, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x)=1 \\ 2 & \text{otherwise.} \end{cases}$$

Then  $\theta, g, f$  and  $n=1$  satisfy (1a), (2) and (3b) of Theorem 2.1. Hence  $G$  has a  $(g,f)$ -factor  $F$ , and thus the lemma holds since  $F$  and  $G-E(F)$  are both  $[0,2]$ -factors of  $G$ .

Lemma 4.5 Let  $G$  be a connected  $[2,4]$ -graph with at least one vertex of degree 3. Then  $G$  can be decomposed into two  $[1,2]$ -factors.

Proof Set  $\theta=1/2$ , and define two functions  $g$  and  $f$  on  $V(G)$  by

$$g(x) = \begin{cases} 1 & \text{if } 2 \leq d_G(x) \leq 3 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x) = 2 \\ 2 & \text{otherwise.} \end{cases}$$

Then  $\theta$ ,  $g$ ,  $f$  and  $n=1$  satisfy (1a), (2) and (3c) of Theorem 2.1.

Hence  $G$  has a  $(g,f)$ -factor  $F$ , and thus  $G$  is decomposed into two  $[1,2]$ -factors  $F$  and  $G-E(F)$ . ■

The following lemma, which is a special case of Lemma 4.9, shows that Lemma 4.1 holds if the graph is 3-edge-connected. Recall that an  $\{a,b,c,\dots\}$ -graph satisfies  $d_G(x) \in \{a,b,c,\dots\}$  for all  $x \in V(G)$ .

Lemma 4.6 Let  $G$  be a 3-edge-connected  $[3,6]$ -graph with at most one vertex of degree 6. Then  $G$  has a  $[0,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  such that if  $d_G(x) \geq 4$ , then  $d_{F_i}(x) \geq 1$  for every  $F_i$ .

Proof We first assume that  $G$  has at least one vertex of degree 3 or 5, or  $G$  is a  $\{4,6\}$ -graph with an even number of vertices of degree 4. Let  $\theta=1/4$  and define two functions  $g_1$  and  $f_1$  on  $V(G)$  by

$$g_1(x) = \begin{cases} 0 & \text{if } d_G(x)=3 \\ 1 & \text{if } 4 \leq d_G(x) \leq 5 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 1 & \text{if } 3 \leq d_G(x) \leq 4 \\ 2 & \text{otherwise.} \end{cases}$$

Then  $\theta$ ,  $g_1$ ,  $f_1$  and  $n=3$  satisfy (1b;  $\epsilon=0$  or  $1/2$ ), (2) and (3c) of Theorem 2.1. Hence  $G$  has a  $(g_1, f_1)$ -factor  $F_1$ . It is obvious that  $G-E(F_1)$  is a  $[2,4]$ -graph with the property that each vertex of degree 2 in  $G-E(F_1)$  has degree 3 in  $G$ . By Lemma 4.4,  $G-E(F_1)$  is decomposed into two  $[0,2]$ -factors  $F_2$  and  $F_3$ . Consequently,  $G$  is decomposed

into three  $[0,2]$ -factors  $F_1$ ,  $F_2$  and  $F_3$ , which possess the desired property.

We next assume that  $G$  is a  $\{4,6\}$ -graph with an odd number of vertices of degree 4. It suffices to show that  $G$  can be decomposed into three  $[1,2]$ -factors. Suppose  $G$  is a 4-regular graph. Then it follows from Proposition 2.5 that  $G-v$  has a 1-factor  $L_1$  for a vertex  $v$  of  $G$ . Let  $F_1$  be the  $[1,2]$ -factor of  $G$  obtained from  $L_1$  by adding an edge of  $G-E(L_1)$  incident with  $v$ . Since  $H_1=G-E(F_1)$  is a  $[2,3]$ -graph having exactly one vertex of degree 2, we have by Lemma 4.5 that  $H_1$  can be decomposed into two  $[1,2]$ -factors  $F_2$  and  $F_3$ . Therefore, we obtain a required  $[1,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  of  $G$ . Consequently, we may assume that  $G$  has exactly one vertex  $w$  of degree 6. Set  $\theta=1/4$  and define two functions  $g_2$  and  $f_2$  on  $V(G)$  by

$$g_2(x)=f_2(x)=1 \text{ for all } x \in V(G).$$

Then  $\theta$ ,  $g_2$ ,  $f_2$  and  $n=3$  satisfy (1b;  $\epsilon=1/2$ ), (2) and (3c) of Theorem 2.1. Thus  $G$  has a  $(g_2, f_2)$ -factor  $L_2$ . Let  $F_1$  be the  $[1,2]$ -factor of  $G$  obtained from  $L_2$  by adding an edge of  $G-E(L_2)$  incident with  $w$ . Since  $H_2=G-E(F_1)$  is a  $[2,4]$ -graph having exactly one vertex of degree 4 and one vertex of degree 2, it follows from Lemma 4.5 that  $H_2$  can be decomposed into two  $[1,2]$ -factors  $F_2$  and  $F_3$ . Therefore we obtain a desired  $[1,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  of  $G$ . ■

We denote by  $xy$  or  $yx$  an edge joining two vertices  $x$  and  $y$ . Let  $G$  be a graph and  $v$  and  $w$  be two distinct vertices of  $G$ .

Then  $G+vw$  denotes the graph obtained from  $G$  by adding a new edge  $vw$  to  $G$ , where  $G$  may have edges joining  $v$  and  $w$ . The following Lemmas 4.7 and 4.8 will be used in the proof of Lemma 4.9.

Lemma 4.7 Let  $G$  be a connected  $[2,6]$ -graph which has exactly one vertex  $w$  of degree 2 and at most one vertex of degree 6. Suppose that two distinct vertices  $u_1$  and  $u_2$  are adjacent to  $w$  and  $G-w+u_1u_2$  is a 3-edge-connected graph. Then  $G$  has a  $[0,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  with the property that if  $d_G(x) \geq 4$ , then  $d_{F_i}(x) \geq 1$  for every  $F_i$  and  $d_{F_i}(w) \leq 1$  for every  $F_i$ .

Proof Let us define two functions  $g$  and  $f$  on  $V(G)$  by

$$g(x) = \begin{cases} 0 & \text{if } 2 \leq d_G(x) \leq 3 \\ 1 & \text{if } 4 \leq d_G(x) \leq 5 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } 2 \leq d_G(x) \leq 4 \\ 2 & \text{otherwise.} \end{cases}$$

We shall show that  $G$  has a  $(g,f)$ -factor by Lemma 2.2. We denote the vertex of degree 6, if any, by  $v$ . Let  $S, T \subset V(G)$  such that  $S \cap T = \emptyset$  and  $S \cup T \neq \emptyset$ . We write  $\{C_1, \dots, C_r\}$  for the set of components of  $G - (S \cup T)$  which satisfy the conditions on  $h(S, T)$  in Lemma 2.2, where  $r = h(S, T)$ . Then each  $C_i$  does not contain  $w$ , and so  $e(S \cup T, V(C_i)) \geq 3$ . Moreover, we have  $e(S \cup T, V(C_i)) \geq 4$  since  $e(S \cup T, V(C_i)) \equiv 0 \pmod{2}$  (see (2.4) in the proof of Theorem 2.1). We obtain the following inequality by setting  $\theta = 1/4$  in (2.3) in the proof of Theorem 2.1 (Note that (2.3) holds for every graph.).

$$\delta(S, T) \geq \sum_{i=1}^r \left\{ \frac{3}{4} e(T, V(C_i)) + \frac{1}{4} e(S, V(C_i)) - 1 \right\} - \epsilon,$$

where  $\varepsilon=0$  or  $1/2$  according as  $v \notin V(G)$  or  $v \in V(G)$ . Then

$$\delta(S,T) \geq \sum_i \left\{ \frac{1}{4} e(S \cup T, V(C_i)) - 1 \right\} - \varepsilon \geq -\varepsilon > -1.$$

Since  $\delta(S,T)$  is an integer, we conclude that  $\delta(S,T) \geq 0$ . It is clear that  $\delta(\phi, \phi) = 0$  as  $g(w) < f(w)$ . Consequently,  $G$  has a  $(g, f)$ -factor  $F$ . Put  $H = G - E(F)$ . We consider two cases.

Case 1.  $d_F(w) = 1$ . By Lemma 4.4,  $H$  can be decomposed into two  $[0, 2]$ -factors  $F_2$  and  $F_3$ , and it is easy to see that  $(F_1 = F) \cup F_2 \cup F_3$  is a  $[0, 2]$ -factorization of  $G$  with the required property.

Case 2.  $d_F(w) = 0$ . In this case  $H$  is a  $[2, 4]$ -graph. Let  $C$  be any component of  $H$ . If  $C$  does not contain  $w$ , then we decompose  $C$  into two  $[0, 2]$ -factors. Suppose  $C$  contains  $w$ . Then  $C$  contains  $u_1$  and  $u_2$ . If  $d_C(u_1) = d_C(u_2) = 4$ , then we may assume  $d_G(u_1) = 5$ , and so  $F + wu_1$ , where  $wu_1 \in E(C)$ , is also a  $(g, f)$ -factor of  $G$ . Hence Case 1 occurs, and thus we may assume  $d_C(u_1) \leq 3$  or  $d_C(u_2) \leq 3$ . Set  $\theta = 1/2$  and define two functions  $g_1$  and  $f_1$  on  $V(C)$  by

$$g_1(x) = \begin{cases} 1 & \text{if } d_C(x) \leq 3 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 1 & \text{if } x = w \\ 2 & \text{otherwise.} \end{cases}$$

Then  $\theta, g_1, f_1$  and  $n=1$  satisfy (1a), (2) (since  $g_1(u_1) < f_1(u_1)$  or  $g_1(u_2) < f_1(u_2)$ .) and (3c) of Theorem 2.1. Hence  $C$  has a  $(g_1, f_1)$ -factor, and thus  $C$  is decomposed into two  $[1, 2]$ -factors, in each factor of which the degree of  $w$  is 1. Therefore,  $G$  can be decomposed into three  $[0, 2]$ -factors with the required property. ■

Lemma 4.8 Let  $G$  be a 3-edge-connected  $[3,5]$ -graph having a vertex  $w$  of degree 3. Then  $G$  has a  $[0,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  with the property that if  $d_G(x) \geq 4$ , then  $d_{F_i}(x) \geq 1$  for every  $F_i$  and that  $d_{F_i}(w) = 0$  for some  $F_i$ .

Proof Let  $g$  and  $f$  be functions on  $V(G)$  defined by

$$g(x) = \begin{cases} 0 & \text{if } d_G(x) = 3 \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x = w \\ 1 & \text{if } 3 \leq d_G(x) \leq 4 \text{ and } x \neq w \\ 2 & \text{otherwise.} \end{cases}$$

We shall show that  $G$  has a  $(g, f)$ -factor. Let  $S, T \subset V(G)$  such that  $S \cap T = \emptyset$  and  $S \cup T \neq \emptyset$ , and let  $\{C_1, \dots, C_r\}$  be the components of  $G - (S \cup T)$  which satisfy the conditions on  $h(S, T)$ , where  $r = h(S, T)$ . Then we have the following inequality by setting  $\theta = 1/4$  in (2.2).

$$\delta(S, T) \geq (1 - \frac{1}{4}) \sum_{t \in T} d_G(t) + \frac{1}{4} \sum_{s \in S} d_G(s) - \epsilon - e(S, T) - r,$$

where  $\epsilon = 0$  or  $3/4$  according as  $w \notin S$  or  $w \in S$ . Hence

$$\begin{aligned} \delta(S, T) &\geq \sum_{i=1}^r \left\{ \frac{3}{4} e(T, V(C_i)) + \frac{1}{4} e(S, V(C_i)) - 1 \right\} - \epsilon \\ &\geq \sum_{i=1}^r \left\{ \frac{1}{4} e(T \cup S, V(C_i)) - 1 \right\} - \epsilon. \end{aligned}$$

If  $C_i$  does not contain  $w$ , then  $e(T \cup S, V(C_i)) \equiv 0 \pmod{2}$  (see (2.4)), and so  $e(T \cup S, V(C_i)) \geq 4$ . Therefore

$$\frac{1}{4} e(T \cup S, V(C_i)) - 1 \geq 0.$$

If  $C_i$  contains  $w$ , then  $\epsilon = 0$  and  $e(T \cup S, V(C_i)) \geq 3$ , and so

$$\frac{1}{4} e(T \cup S, V(C_i)) - 1 \geq -\frac{1}{4}.$$

Consequently, we obtain  $\delta(S, T) \geq -3/4$ , which implies  $\delta(S, T) \geq 0$ .

Furthermore, we can show that  $\delta(\phi, \phi) = 0$  by the fact that  $G$  has at least one vertex  $x$  with odd degree except  $w$ , for which  $g(x) < f(x)$ . Consequently,  $G$  has a  $(g, f)$ -factor  $F_1$ . By Lemma 4.4,  $G - E(F_1)$  can be decomposed into two  $[0, 2]$ -factors  $F_2$  and  $F_3$ . Therefore we obtain a desired  $[0, 2]$ -factorization  $F_1 \cup F_2 \cup F_3$  of  $G$ . ■

We need some notation and definitions in order to prove Lemma 4.1. A graph having exactly two vertices and one or more edges is called a bond, and we denote the bond with  $n$  edges by  $B_n$  (Fig. 1). Let  $v$  be a vertex of a graph  $G$  and  $w$  be a vertex of the bond  $B_n$ . Then  $G + vw + B_n$  denotes the graph obtained from  $G$  and  $B_n$  by joining  $v$  and  $w$  by a new edge  $vw$  (Fig. 2).

We shall prove the next lemma instead of Lemma 4.1, which includes Lemma 4.1 as a special case.



Figure 1. The bond  $B_4$ .

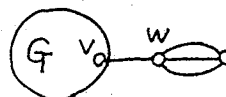


Figure 2.  $G + vw + B_3$ .

Lemma 4.9 Let  $G$  be a connected  $[3, 6]$ -graph with at most one vertex of degree 6. Then  $G$  has a  $[0, 2]$ -factorization  $F_1 \cup F_2 \cup F_3$  with the property that

$$\text{if } d_G(x) \geq 4, \text{ then } d_{F_i}(x) \geq 1 \text{ for every } F_i. \quad (4.1)$$

Proof We prove the lemma by induction on the number of vertices of a graph. Let  $G$  be a connected  $[3, 6]$ -graph with at most one vertex of degree 6. By Lemma 4.6, we may assume that  $G$  is not 3-edge-connected.





Figure 3.  $G$  and  $H+vu+B_3$ .

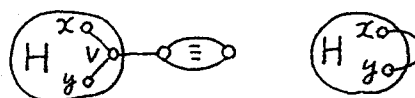


Figure 4.  $G$  and  $H-v+xy$ .

If  $|V(G)|=2$  or  $3$ , then  $G$  must be 3-edge-connected, which is contrary to the assumption. Hence we may assume  $|V(G)|\geq 4$ .

First suppose that  $G$  is not 2-edge-connected. Then  $G$  has a bridge  $e=vw$ , where  $e\in E(G)$  and  $v, w\in V(G)$  (Fig. 3). Let  $H$  and  $K$  be the components of  $G-e$  such that  $v\in V(H)$  and  $w\in V(K)$  (Fig. 3). If  $|V(H)|\geq 3$  and  $|V(K)|\geq 3$ , then  $H'=H+vu+B_3$  and  $K'=K+wu+B_3$  are both  $[3,6]$ -graphs, where  $u$  is a vertex of  $B_3$  (Fig. 3). By the induction hypothesis,  $H'$  and  $K'$  can be decomposed into three  $[0,2]$ -factors with the property (4.1), respectively. It is easy to obtain a desired  $[0,2]$ -factorization of  $G$  from them. Therefore, we may assume  $|V(K)|=2$ . Then  $K$  is  $B_3, B_4$  or  $B_5$ .

If  $d_H(v)\geq 3$ , then  $H$  has a  $[0,2]$ -factorization with the property (4.1) by induction, and it is easy to obtain a desired  $[0,2]$ -factorization of  $G$  from it. Hence we may assume  $d_H(v)=2$ . If two distinct vertices  $x$  and  $y$  of  $H$  are adjacent to  $v$ , then  $H-v+xy$  can be decomposed into three  $[0,2]$ -factors with the property (4.1) by induction (Fig. 4). So we can obtain a desired  $[0,2]$ -factorization of  $G$  from it. We next suppose that one vertex  $x$  and  $v$  are joined by two edges in  $H$  (Fig. 5). Let  $H+B_3$  be the graph obtained from  $H$  by identifying

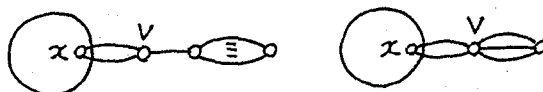


Figure 5.  $G$  and  $H+B_3$ .

$v$  and one of the vertices of  $B_3$  (Fig. 5). Then, by the induction hypothesis,  $H+B_3$  has a  $[0,2]$ -factorization with the property (4.1), and it is immediate to obtain a desired  $[0,2]$ -factorization of  $G$  from it. Consequently, the proof is complete if  $G$  is not 2-edge-connected.

We now deal with the case that  $G$  is 2-edge-connected. Since  $G$  is not 3-edge-connected,  $G$  has a cutset (i.e. a minimal cut) with two edges. We consider three cases.

Case 1.  $G$  has a cutset  $\{e_1, e_2\}$  such that the ends of  $e_1$  and those of  $e_2$  are all distinct, where  $e_1, e_2 \in E(G)$ .

Let  $H$  and  $K$  be the components of  $G - \{e_1, e_2\}$ , and let  $e_1 = u_1 w_1$  and  $e_2 = u_2 w_2$ , where  $u_1, u_2 \in V(H)$ ,  $u_1 \neq u_2$ ,  $w_1, w_2 \in V(K)$  and  $w_1 \neq w_2$ . Then  $H + u_1 u_2$  and  $K + w_1 w_2$  have  $[0,2]$ -factorization with property (4.1) by induction. It is easy to obtain a desired  $[0,2]$ -factorization of  $G$  from them.

Case 2.  $G$  has a cutset  $\{e_1, e_2\}$  such that the ends of  $e_1$  and those of  $e_2$  are the same (Fig. 6).

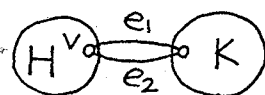


Figure 6.

Let  $H$  be an arbitrary component of  $G - \{e_1, e_2\}$ , and  $v$  be the end of  $e_1$  and  $e_2$  contained in  $H$  (Fig. 6). We shall show that  $H$  has a  $[0,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  with the property that

$$\begin{aligned} F_1 + e_1, F_2 + e_2 \text{ and } F_3 \text{ are } [0,2]\text{-factors of } H + \langle e_1, e_2 \rangle \\ \text{and satisfy the condition (4.1) in } H + \langle e_1, e_2 \rangle, \end{aligned} \quad (4.2)$$

where  $H + \langle e_1, e_2 \rangle$  is the subgraph of  $G$  obtained from  $H$  by adding

$e_1$  and  $e_2$  together with their common end not contained in  $H$ . If this statement follows, then we can easily obtain a  $[0,2]$ -factorization of  $G$  with the property (4.1) from a  $[0,2]$ -factorization with the property (4.2) of each component of  $G - \{e_1, e_2\}$ . We now prove the statement.

If  $d_G(v) \geq 5$ , then  $d_H(v) \geq 3$  and so  $H$  has a  $[0,2]$ -factorization  $F_1 \cup F_2 \cup F_3$  with the property (4.1) by induction. Since we may assume  $d_{F_1}(v) \leq 1$  and  $d_{F_2}(v) \leq 1$ , these factors satisfy the required condition (4.2). If  $d_G(v) = 3$ , then  $G$  has a bridge, and so Case 1 occurs.

Hence we may assume  $d_G(v) = 4$ , and thus  $d_H(v) = 2$ . If two distinct vertices  $x$  and  $y$  of  $H$  are adjacent to  $v$ , then  $H - v + xy$  can be decomposed into three  $[0,2]$ -factors with the property (4.1) by induction. It is easy to obtain a desired  $[0,2]$ -factorization of  $H$  from them. We next assume that one vertex  $x$  of  $H$  and  $v$  are joined by two edges (Fig. 7). Let  $H + B_3$  be the graph obtained from  $H$  and  $B_3$  by identifying  $v$  and a vertex of  $B_3$  (Fig. 7). Then  $H + B_3$  has a  $[0,2]$ -factorization with the property (4.1) by induction, and so we can obtain a desired  $[0,2]$ -factorization of  $H$  from it. Consequently, each component of  $G - \{e_1, e_2\}$  has a  $[0,2]$ -factorization satisfying the conditions (4.2), and we conclude that the proof of Case 2 is complete.



Figure 7.  $G$  and  $H + B_3$ .

Case 3. For every cutset  $\{e_1, e_2\}$  of  $G$ ,  $e_1$  and  $e_2$  have exactly one common end (Fig. 8).

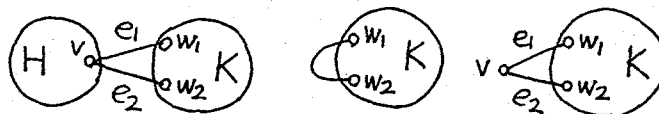


Figure 8.  $G$ ,  $K + w_1w_2$  and  $K + \langle e_1, e_2 \rangle$ .

Let  $\{e_1, e_2\}$  be any cutset of  $G$ . Then we can write  $e_1 = vw_1$  and  $e_2 = vw_2$ , where  $v, w_1, w_2 \in V(G)$  and  $w_1 \neq w_2$ . Let  $H$  and  $K$  be the components of  $G - \{e_1, e_2\}$  such that  $v \in V(H)$  and  $w_1, w_2 \in V(K)$ . Note that  $d_H(v) \geq 2$  as  $G$  has no bridges. We first prove that if  $\{e_1, e_2\}$  satisfies one of the following two conditions, then  $G$  has a  $[0, 2]$ -factorization with the property (4.1) :

- (i)  $K + w_1w_2$  is 3-edge-connected (Fig. 8).
- (ii)  $d_H(v) = 3$  and  $H$  is a 3-edge-connected graph without vertices of degree 6.

Suppose (i) hold. Then  $K + \langle e_1, e_2 \rangle$  (Fig. 8) can be decomposed into three  $[0, 2]$ -factors which satisfy the conditions in Lemma 4.7. On the other hand, if  $d_H(v) \geq 3$ , then  $H$  has a  $[0, 2]$ -factorization with the property (4.1) by induction, and so we can get a desired  $[0, 2]$ -factorization of  $G$ . If  $d_H(v) = 2$ , then two distinct vertices  $x$  and  $y$  of  $H$  are adjacent to  $v$ , and so  $H - v + xy$  has a  $[0, 2]$ -factorization with the property (4.1). It is easy to obtain a desired  $[0, 2]$ -factorization of  $G$ .

We next suppose that (ii) holds. Then  $H$  can be decomposed into three  $[0, 2]$ -factors which satisfy the conditions in Lemma 4.8. It follows that  $K + w_1w_2$  has a  $[0, 2]$ -factorization with the property (4.1) by induction, and thus  $G$  has a desired  $[0, 2]$ -factorization.

We shall show that  $G$  has a cutset  $\{e_1, e_2\}$  which satisfies one

of the above conditions (i) and (ii). We can choose a cutset  $\{e_1, e_2\}$  so that  $H$  or  $K+w_1w_2$  is 3-edge-connected. If  $d_H(v)=2$ , then we may assume without loss of generality that  $K+w_1w_2$  is 3-edge-connected. Hence (i) follows.

Suppose  $d_H(v)=3$ . In this case we may assume that  $K+w_1w_2$  is not 3-edge-connected and  $H$  contains a unique vertex of  $G$  with degree 6 ; (otherwise,  $\{e_1, e_2\}$  satisfies (i) or (ii) ). Let  $\{f_1, f_2\}$  be any cutset of  $K+w_1w_2$ . If the ends of  $f_1$  and those of  $f_2$  are all distinct, then  $G$  has such a cutset, which contradicts the assumption of Case 3. If the ends of  $f_1$  and those of  $f_2$  are the same, then  $f_1$  or  $f_2$ , say  $f_1$ , must be  $w_1w_2$ . So it follows that both  $\{e_1, f_2\}$  and  $\{e_2, f_2\}$  are cutsets of  $G$  and  $f_2$  joins  $w_1$  and  $w_2$ . Let  $T$  be the component of  $G-\{e_1, f_2\}$  containing  $w_1$ . If  $d_T(w_1) \geq 3$ , then  $T$  has a  $[0,2]$ -factorization with the property (4.1) by induction. If  $d_T(w_1)=2$ , then two distinct vertices  $t_1$  and  $t_2$  are adjacent to  $w_1$  in  $T$ , and thus  $T-w_1+t_1t_2$  has a  $[0,2]$ -factorization with the property (4.1) by induction. Obviously, the component of  $G-\{e_2, f_2\}$  containing  $w_2$  has the same property mentioned above. Furthermore,  $H$  also has a  $[0,2]$ -factorization with the property (4.1). Therefore, we can obtain a desired  $[0,2]$ -factorization of  $G$  from them. Consequently, we may assume that for every cutset  $\{f_1, f_2\}$  of  $K+w_1w_2$ ,  $f_1$  and  $f_2$  have exactly one common end. Hence we can write  $f_1=xy_1$  and  $f_2=xy_2$ , where  $x, y_1, y_2 \in V(K+w_1w_2)$ .

Choose a cutset  $\{f_1=xy_1, f_2=xy_2\}$  of  $K+w_1w_2$  so that the component

of  $K+w_1w_2-\{f_1,f_2\}$  containing  $x$  is 3-edge-connected or the graph obtained from the component of  $K+w_1w_2-\{f_1,f_2\}$  containing  $y_1$  and  $y_2$  by adding a new edge  $y_1y_2$  is 3-edge-connected. Since  $H$  contains the vertex of degree 6, we can choose such a cutset  $\{f_1,f_2\}$  so that the 3-edge-connected component (or graph) has no vertices of degree 6. If  $w_1w_2 \notin \{f_1,f_2\}$ , then  $\{f_1,f_2\}$  is a cutset of  $G$  which satisfies one of (i) and (ii). Hence we may assume  $f_1=w_1w_2$  and  $f_2=w_1y_2$ , where  $w_2 \neq y_2$ . Then  $\{e_2,y_2\}$  is a cutset of  $G$  which satisfies the condition of Case 1, a contradiction.

We finally assume  $d_H(v)=4$  (i.e.  $v$  is the vertex of  $G$  with degree 6.). If  $K+w_1w_2$  is 3-edge-connected, then we can obtain a  $[0,2]$ -factorization of  $G$  with the property (4.1) by applying Lemma 4.8 to  $K+\langle e_1,e_2 \rangle$ . Hence we may assume that  $K+w_1w_2$  is not 3-edge-connected. In this case we can prove that  $G$  has a desired  $[0,2]$ -factorization by the same argument in the case of  $d_H(v)=3$ . Consequently, Case 3 is proved.

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