[a,b]-factorization of a graph

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Abstract

Let a and b be integers such that $0 \le a \le b$. Then a graph G is called an $[a,b]$-graph if $a \le d_G(x) \le b$ for every $x \in V(G)$, and an $[a,b]$-factor of a graph is defined to be its spanning subgraph F such that $a \le d_F(x) \le b$ for every vertex x, where $d_G(x)$ and $d_F(x)$ denote the degrees of x in G and F, respectively. If the edges of a graph can be decomposed into $[a,b]$-factors, then we say that the graph is $[a,b]$-factorable. We prove the following two theorems: (i) a graph G is $[2a,2b]$-factorable if and only if G is a $[2am,2bm]$-graph for some integer m, and (ii) every $[8m+2k,10m+2k]$-graph is $[1,2]$-factorable.

1. Introduction

We deal with finite graphs which may have multiple edges but have no loops. A graph without multiple edges is called a simple graph. All notation and definitions not given here can be found in [4].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and H be a subgraph of G. For a vertex x of H, we denote the degree of x in H by $d_H(x)$, in partucular, the degree of a vertex y of G is denoted by $d_G(y)$. Let a and b be integers such that $0 \leq a \leq b$. Then a graph G is called an $[a,b]$-graph if $a \leq d_G(x) \leq b$ for every $x \in V(G)$, and an $[a,b]$-subgraph can be defined similarly. A spanning $[a,b]$-subgraph is called an $[a,b]$-factor. Then, if F is an $[a,b]$-factor of a graph G, then $a \leq d_F(x) \leq b$ for all $x \in V(G)$.

If the edges of a graph G can be decomposed into $[a,b]$-factors $F_1, \ldots, F_n$ of G, then the union $F_1 \cup \ldots \cup F_n$ is called an $[a,b]$-factorization of G and G itself is said to be $[a,b]$-factorable.

We usually call an $[r,r]$-graph an r-regular graph. Similarly, an $[r,r]$-factor, an $[r,r]$-factorization and an $[r,r]$-factorable graph are called an r-factor, an r-factorization and an r-factorable graph, respectively.

In 1891 Petersen [13],[4,Theorem 8.8] obtained the following theorem.

Theorem 1.1 A graph G is 2-factorable if and only if G is a $2m$-regular graph for some positive integer m.

Recently, Akiyama [1] proved that every r-regular graph is [2,3]-factorable, where $r \geq 2$. This is the first contribution toward $[a,b]$-factorization with $a<b$. Era [6] proved that if $r \geq 2k^2$, then every r-regular simple graph is $[k,k+1]$-factorable. We now give our theorems.

Theorem 1.2 Let $0 \leq a \leq b$. Then a graph G is $[2a,2b]$-factorable if and only if G is a $[2am,2bm]$-graph for some positive integer m.

This thoerem is an extension of Theorem 1.1.

Theorem 1.3 Let $m \geq 1$ and $k \geq 0$. Then every $[8m+2k,10m+2k]$-graph is [1,2]-factorable.

As a corollary of this theorem, we can obtain the next result.

Corollary 1.4 (1) If $r \geq 8m$, then every $[r,r+2m-1]$-graph is [1,2]-factorable.

(2) Every connected $[r,r+1]$-graph is [1,2]-factorable, where $r \geq 1$.

Note that a $[2am,2bm]$-graph can be decomposed into m $[2a,2b]$-factors, and a $[8m+2k,10m+2k]$-graph can be decompsed into $6m+k$ [1,2]-factors. But the number of $[a,b]$-factors in an $[a,b]$-factorization of a graph is not uniquely determined. For example, a $4m$-regular graph can be decomposed into k [1,2]-factors for every k, $2m \leq k \leq 3m$ (see Theorem 1.1 and Lemma 4.1). It is clear that the union of an odd cycle and a cubic graph, which is a [2,3]-graph with two components, is not [1,2]-factorable. So the connectivity of a graph in (2) of Corollary 1.4 is necessary. Moreover, we show that there exists a [6,8]-graph which is not [1,2]-factorable (Remark 4.3).

We next mention two factor theorems on which our proof will heavily depend. One is Lovász's (g,f)-factor theorem (see Lemma 2.2), which plays an important role throughout this article, and the other is Theorem 2.1, which is proved by making use of Lovász's (g,f)-factor theorem. By Theorem 2.1, not only can we prove many known theorems on r-factors due to Baebler, Gallai, Petersen and others, but also we can obtain some new results on [a,b]-factors, for instance, Theorem 1.2 is an easy consequence of it.

Let us finally note a survey article [2], in which many results related to our theorems can be found.

2. Factor theorem

We begin by introducing some new notation and definitions. For a finite set X, we denote by $|X|$ the number of elements in X. Let G be a graph, and g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every $x \in V(G)$. Then, a (g,f)-<u>factor</u> of G is a spanning subgraph F of G satisfying

$g(x) \le d_F(x) \le f(x)$ for all $x \in V(G)$. For a subset S of $V(G)$, we write $G-S$ for the subgraph of G obtained from G by deleting the vertices in S togehter with their incident edges. If S and T are disjoint subsets of $V(G)$, then $e(S,T)$ denotes the number of edges of G joining S and T.

In this section we shall prove the following theorem and give some its corollaries.

Theorem 2.1 Let G be an n-edge-connected graph ($n \ge 1$), θ be a real number such that $0 \le \theta \le 1$, and g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for all $x \in V(G)$. If one of {(1a),(1b)}, (2) and one of {(3a),(3b),(3c),(3d),(3e),(3f)} hold, then G has a (g,f)-factor.

(1a) $g(x) \le \theta d_G(x) \le f(x)$ for all $x \in V(G)$.

(1b) $\varepsilon = \sum_{x \in V(G)} [\max\{0, g(x)-\theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}] < 1$.

(2) G has at least one vertex v such that $g(v) < f(v)$; or $g(x) = f(x)$ for all $x \in V(G)$ and $\sum_{x \in V(G)} f(x) \equiv 0 \pmod 2$.

(3a) $n\theta \ge 1$ and $n(1-\theta) \ge 1$.

(3b) $\{d_G(x) \mid g(x)=f(x),\ x \in V(G)\}$ and $\{f(x) \mid g(x)=f(x),\ x \in V(G)\}$ both consist of even numbers.

(3c) $\{d_G(x) \mid g(x)=f(x),\ x \in V(G)\}$ consists of even numbers, n is odd, $(n+1)\theta \ge 1$ and $(n+1)(1-\theta) \ge 1$.

(3d) $\{f(x) \mid g(x)=f(x),\ x \in V(G)\}$ consists of even numbers and $m(1-\theta) \ge 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.

(3e) $\{d_G(x) \mid g(x)=f(x),\ x \in V(G)\}$ and $\{f(x) \mid g(x)=f(x),\ x \in V(G)\}$

both consist of odd numbers and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.

(3f) $g(x) < f(x)$ for every $x \in V(G)$ (see [8]).

Note that similar necessary conditions for a graph to have a (g,f)-factor which contains p given edges but has no q given edges are obtained in [9]. In order to prove the above theorem we need the next (g,f)-factor theorem due to Lovász, to which Tutte [16] gave a short proof.

Lemma 2.2 (Lovász [12], [16, Theorem 7.2]) Let G be a graph and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a (g,f)-factor if and only if

$$\delta(S,T) = \sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e(S,T) - h(S,T) \geq 0 \qquad (2.1)$$

for all disjoint subsets S and T of $V(G)$, where $h(S,T)$ denotes the number of components C of $G-(S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e(T,V(C)) + \sum_{x \in V(G)} f(x) \equiv 1 \pmod 2$.

Note that the condition $0 \leq g(x) \leq f(x) \leq d_G(x)$ in [12] and [16] can be replaced by $g(x) \leq f(x)$ as above ([10],[15]).

Proof of Theorem 2.1 We shall prove that two functions g and f in Theorem 2.1 satisfy the condition (2.1) in Lemma 2.2. It is obvious that (1a) implies (1b). Hence we may assume (1b) holds. Let $S, T \subset V(G)$ such that $S \cap T = \phi$. Assume first $S \cup T \neq \phi$. Let

$\{C_1,\dots,C_r\}$ be the set of components of $G-(S\cup T)$ which satisfy the conditions on $h(S,T)$, where $r=h(S,T)$. By (1b) of Theorem 2.1, we have

$$\delta(S,T) \ge (1-\theta)\sum_{t\in T} d_G(t) + \theta\sum_{s\in S} d_G(s) - \sum_{t\in T}\max\{0,g(t)-\theta d_G(t)\}$$
$$- \sum_{s\in S}\max\{0,\theta d_G(s)-f(s)\} - e(S,T) - r \qquad (2.2)$$
$$\ge (1-\theta)\{e(T,S) + \sum_{i=1}^{r} e(T,V(C_i))\} + \theta\{e(S,T) + \sum_{i=1}^{r} e(S,V(C_i))\}$$
$$- \varepsilon - e(S,T) - r$$
$$= \sum_{i}\{(1-\theta)e(T,V(C_i)) + \theta e(S,V(C_i)) - 1\} - \varepsilon. \qquad (2.3)$$

Since $\delta(S,T)$ is an integer and $\varepsilon<1$, it suffices to show that $\delta(S,T)\ge-\varepsilon$.

If (3f) holds, then $r=0$ and so $\delta(S,T)\ge-\varepsilon$. Hence we may assume that G satisfies (2) and one of $\{(3a),(3b),(3c),(3d),(3e)\}$. Take any $C\in\{C_1,\dots,C_r\}$, and put

$$\Delta(C) = (1-\theta)e(T,V(C)) + \theta e(S,V(C)) - 1.$$

We prove that $\Delta(C)\ge0$. If $\{f(x) \mid g(x)=f(x),\ x\in V(G)\}$ consists of even numbers, then

$$1 \equiv e(T,V(C)) + \sum_{x\in V(C)} f(x) \equiv e(T,V(C)) \pmod 2,$$

in particular, $e(T,V(C))\ge1$. Similarly, if $\{f(x) \mid g(x)=f(x),\ x\in V(G)\}$ consists of odd numbers, then we have $1\equiv e(T,V(C))+|V(C)| \pmod 2$. Suppose $\{d_G(x) \mid g(x)=f(x),\ x\in V(G)\}$ consists of even numbers. Then

$$0 \equiv \sum_{x\in V(C)} d_G(x) = 2|E(C)| + e(V(C),S\cup T)$$
$$\equiv e(S\cup T,V(C)) \pmod 2 \qquad (2.4)$$

Thus $e(S\cup T,V(C))\equiv 0 \pmod 2$. If $\{d_G(x) \mid g(x)=f(x),\ x\in V(G)\}$ consists of odd numbers, then we have $|V(C)|\equiv e(S\cup T,V(C)) \pmod 2$. We consider three cases.

Case 1. $e(T,V(C))\geq 1$ and $e(S,V(C))\geq 1$. It follows immediately from $0\leq\theta\leq 1$ that $\Delta(C)\geq 0$.

Case 2. $e(T,V(C))=0$. We first note that $e(S,V(C))=e(S\cup T,V(C))$ $\geq n$ since G is n-edge-connected. By the fact mentioned above, $\{f(x) \mid g(x)=f(x),\ x\in V(G)\}$ is not a set of even numbers, and so neither (3b) nor (3d) occurs. If G satisfies (3a), then $\Delta(C)\geq\theta n-1\geq 0$ as $e(S,V(C))\geq n$. Suppose G satisfies (3c). Then we have $e(S,V(C))$ $\geq n+1$. Hence $\Delta(C)\geq\theta(n+1)-1\geq 0$. We finally assume that G satisfies (3e). Then it follows from the fact mentioned above that $1\equiv e(S,V(C))$ $\pmod 2$. If n is odd, then $m=n$ and so $\Delta(C)\geq\theta n-1=\theta m-1\geq 0$. If n is even, then $e(S,V(C))\geq n+1$ and $m=n+1$. Hence $\Delta(C)\geq\theta(n+1)-1$ $=\theta m-1\geq 0$.

Case 3. $e(S,V(C))=0$. Note that $e(T,V(C))=e(S\cup T,V(C))\geq n$. If G satisfies (3a), then $\Delta(C)\geq(1-\theta)n-1\geq 0$. If $\{d_G(x) \mid g(x)=f(x),$ $x\in V(G)\}$ consists of even numbers, then $e(T,V(C))\equiv 0 \pmod 2$. On the other hand, if $\{f(x) \mid g(x)=f(x),\ x\in V(G)\}$ consists of even numbers, then $e(T,V(C))\equiv 1 \pmod 2$. Hence (3b) does not occur. If (3c) holds, then $e(T,V(C))\geq n+1$ and so $\Delta(C)\geq(1-\theta)(n+1)-1\geq 0$. Suppose G satisfies (3d). It is easy to show that we may assume n is even. Since $e(T,V(C))\equiv 1 \pmod 2$, we have $e(T,V(C))\geq n+1$, and thus $\Delta(C)\geq(1-\theta)(n+1)-1=(1-\theta)m-1\geq 0$. Finally we suppose that

G satisfies (3e). Then $1 \equiv e(T,V(C))+|V(C)|$ (mod 2) and $|V(C)| \equiv e(T,V(C))$ (mod 2), a contradiction. Therefore, (3e) does not occur.

Let $S=T=\phi$ and assume $\delta(\phi,\phi)<0$. Then $h(\phi,\phi)>0$. Since G is connected, it follows from Lemma 2.2 that $g(x)=f(x)$ for all $x \in V(G)$ and $\sum f(x) \equiv 1$ (mod 2), which contradicts (2). Therefore $\delta(\phi,\phi)=0$. Consequently, the proof of the theorem is complete. ∎

We now give some results on factors which can be obtained by Theorem 2.1.

Corollary 2.3 Let $2 \le b$ and $1 \le a \le b \le 2a$. Then every 2-edge-connected $[a,b]$-graph G has a $[1,2]$-factor F such that $d_F(x)=2$ if $d_G(x)=b$. In particular, every 2-edge-connected r-regular graph has a 2-factor, where $r \ge 2$ (Baebler [3]).

Proof We may assume $b \ge 3$. Put $\theta=2/b$ and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 2 & \text{if } d_G(x)=b \\ 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f(x)=2 \text{ for all } x \in V(G).$$

Then θ, g, f and $n=2$ satisfy (1a), (2) and (3d) of Theorem 2.1. Hence G has a (g,f)-factor, which is a desired $[1,2]$-factor. ∎

Corollary 2.4 Let G be a $(r-1)$-edge-connected $[r,2r]$-graph with at least one vertex of degree greater than r, where $r \ge 1$. Then G has a $[1,2]$-factor F such that $d_F(x)=1$ if $d_G(x)=r$.

Proof Set $\theta=1/r$, and define two functions g and f on

$V(G)$ as follows :

$$g(x)=1 \text{ for all } x\in V(G), \text{ and } \quad f(x)=\begin{cases} 1 & \text{if } d_G(x)=r, \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n=r-1$ satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1 according as the parity of r. Hence G has a (g,f)-factor, which is a desired $[1,2]$-factor. ∎

<u>Proposition 2.5</u> ((1): Petersen [13]($r=3$) and Baebler [3]($r\geq 4$); and (2): Little, Grant and Holton [11]) Let G be an $(r-1)$-edge-connected r-regular graph. Then

(1) if G has an even number of vertices, then G has a 1-factor; and

(2) if G has an odd number of vertices, then $G-v$ has a 1-factor for any vertex v of G.

<u>Proof</u> We prove only (2) since (1) can be proved similarly. Put $\theta=1/r$, and define two functions g and f on $V(G)$ as

$$g(x)=f(x)=1 \text{ for all } x\in V(G)\setminus\{v\}, \quad g(v)=0 \text{ and } f(v)=1,$$

where v is a given vertex of G. Then θ, g, f and $n=r-1$ satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1. Therefore, G has a (g,f)-factor F. We can easily see that $d_F(v)=0$. Hence (2) follows. ∎

<u>Proposition 2.6</u> ((1),(2):Gallai [7]; and (3): Bollobas, Saito and Wormald [5]) The following statements hold.

(1) An n-edge-connected $2r$-regular graph with an even number of vertices has a $(2k+1)$-factor for every $2k+1$, $2r/n\leq 2k+1\leq 2r(n-1)/n$.

(2) An n-edge-connected $(2r+1)$-regular graph G has a $2k$-factor for every $2k$, $0 \le 2k \le (2r+1)(n-1)/n$. In particular, G has a $(2m+1)$-factor for every $2m+1$, $(2r+1)/n \le 2m+1 \le 2r+1$.

(3) A 2n-edge-connected $(2r+1)$-regular graph G has a $2k$-factor for every $2k$, $0 \le 2k \le (2r+1)(2n)/(2n+1)$. In particular, G has a $(2m+1)$-factor for every $2m+1$, $(2r+1)/(2n+1) \le 2m+1 \le 2r+1$.

Proof We prove only (3) since (1) and (2) can be proved similarly. Set $\theta=2k/(2r+1)$, and define two functions g and f on $V(G)$ by $g(x)=f(x)=2k$ for all $x \in V(G)$. Then θ, g, f and $2n$ satisfy (1a), (2) and (3d) of Theorem 2.1. Therefore G has a (g,f)-factor, which is a $2k$-factor of G. Let F be a $2k$-factor of G. Then $G-E(F)$ is a $(2r+1-2k)$-factor of G, and so G has a $(2m+1)$-factor for every $2m+1$, $(2r+1)/(2n+1) \le 2m+1 \le 2r+1$. Note that the latter can be proved independetly by using (3e) of Theorem 2.1. ∎

3. Proof of Theorem 1.2

We shall prove Theorem 1.2 by using Theorem 2.1.

Proof of Theorem 1.2 Let G be a $[2a,2b]$-factorable graph. Then G can be decomposed into m $[2a,2b]$-factors for some positive integer m. It is clear that G is a $[2am,2bm]$-graph.

Conversely, suppose that G is a $[2am,2bm]$-graph. We prove that G can be decomposed into m $[2a,2b]$-factors by induction on m. Without loss of generality, we may assume G is connected. Put $\theta=1/m$, and define two functions g and f on $V(G)$ as follows :

$f(x)=f(x)=2a$ if $d_G(x)=2am$,

$g(x)\le\theta d_G(x)\le f(x)$ with $f(x)-g(x)=1$ if $2am<d_G(x)<2bm$, and

$g(x)=f(x)=2b$ if $d_G(x)=2bm$.

Then, θ, g, f and $n=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Therefore, G has a (g,f)-factor F. For any vertex x of G with $2am<d_G(x)<2bm$, we have

$$2a<\theta d_G(x)<2b \quad \text{and} \quad 2a(m-1)<(1-\theta)d_G(x)<2b(m-1).$$

Hence F is a $[2a,2b]$-factor, and $G-E(F)$ is a $[2a(m-1),2b(m-1)]$-factor. Consequently, the theorem follows by induction. ∎

4. Proof of Theorem 1.3

In this section we shall prove the following four statements : (i) every $[8m+2k,10m+2k]$-graph is $[1,2]$-factorable (Theorem 1.3), (ii) if $r\ge 8m$, then every $[r,r+2m-1]$-graph is $[1,2]$-factorable (Corollary 1.4), (iii) every connected $[r,r+1]$-graph is $[1,2]$-factorable (Corollary 1.4), and (iv) there exists a $[6,8]$-graph which is not [1,2]-factorable (Remark 4.3).

We first prove Theorem 1.3 under the assumption that the following lemma holds.

Lemma 4.1 Let G be a [4,6]-graph with at most one vertex of degree 6. Then G can be decomposed into three [1,2]-factors.

We begin with the next lemma.

Lemma 4.2 Every [8,10]-graph can be decomposed into six [1,2]-factors.

Proof Let G be a [8,10]-graph. Without loss of generality, we may assume G is connected. If G has vertices of degree 10, then choose any vertex w of degree 10. Set $\theta=1/2$, and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 4 & \text{if } 8 \le d_G(x) \le 9 \\ 5 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 4 & \text{if } d_G(x)=8 \\ 5 & \text{if } 9 \le d_G(x) \le 10 \text{ and } x \ne w \\ 6 & \text{if } x=w. \end{cases}$$

Then θ, g, f and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence G has a (g,f)-factor F. It follows that F is a [4,6]-graph with at most one vertex of degree 6 and $G-E(F)$ is a [4,5]-graph, and we conclude by Lemma 4.1 that G can be decomposed into six [1,2]-factors. ∎

Proof of Theorem 1.3 It follows from Theorem 1.1 and Lemma 4.2 that every $[8m,10m]$-graph can be decomposed into $6m$ [1,2]-factors. We now prove by induction on k that every $[8m+2k,10m+2k]$-graph can be decomposed into $6m+k$ [1,2]-factors. Let G be a

$[8m+2k, 10m+2k]$-graph with $m \geq 1$ and $k \geq 1$. We may assume G is connected. Put $\theta = 2/(10m+2k)$ and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 2 & \text{if } d_G(x) = 10m+2k \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = 2 \quad \text{for all } x \in V(G).$$

Then θ, g, f and $n=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence G has a (g,f)-factor F, which is a $[1,2]$-factor. Since $G-E(F)$ is a $[8m+2(k-1), 10m+2(k-1)]$-graph, we conclude by the induction hypothesis that G is decomposed into $6m+k$ $[1,2]$-factors. ∎

Proof of Corollary 1.4 We first prove (1). Let H be an $[r, r+2m-1]$-graph with $r \geq 8m$. Then there exist integers k and t such that $r = 8m+2k+t$, $0 \leq k$ and $0 \leq t \leq 1$. It is immediate that $8m+2k \leq r$ and $r+2m-1 \leq 10m+2k$. Hence F is a $[8m+2k, 10m+2k]$-graph, and so H is $[1,2]$-factorable by Theorem 1.3.

We next prove (2). We first show that every $[2k-1, 2k]$-graph is $[1,2]$-factorable. Let G be a $[2k-1, 2k]$-graph. Then it follows from Theorem 2.1 that G has a $[1,2]$-factor F such that $d_F(x)=2$ if $d_G(x)=2k$ (see Proof of Theorem 1.3). Since $G-E(F)$ is a $[2k-3, 2k-2]$-graph, we have by induction that G is $[1,2]$-factorable. By the statement (1) and the result given above, it suffices to show that if $r=2$, 4 or 6, then a connected $[r, r+1]$-graph is $[1,2]$-factorable. It follows from Lemma 4.5, which will given later, that every connected $[2,3]$-graph is $[1,2]$-factorable. By Lemma 4.1, every $[4,5]$-graph is $[1,2]$-factorable. Hence we may restrict ourselves to the case of $r=6$.

Let H be a connected $[6,7]$-graph. Since a 6-regular graph is 2-factorable, we may assume that H has at least one vertex of degree 7. We show that H can be decomposed into two $[3,4]$-factors, which implies that H can be decomposed into four $[1,2]$-factors. Put $\theta=1/2$ and define two functions g and f on $V(H)$ by

$$g(x)=3 \quad \text{for all} \quad x \in V(H), \text{ and } \quad f(x) = \begin{cases} 3 & \text{if } \; d_H(x)=6 \\ 4 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence H has a (g,f)-factor F'. It is clear that both F' and $H-E(F')$ are $[3,4]$-factors of H. Therefore H is $[1,2]$-factorable. ∎

It is convenient to introduce a new definiton. For a set $\{a,b,c,\ldots\}$ of integers, a graph G is called an $\{a,b,c,\ldots\}$-<u>graph</u> if $d_G(x) \in \{a,b,c,\ldots\}$ for every $x \in V(G)$. The <u>union</u> of graphs H and K is a graph G such that $V(G)=V(H)\cup V(K)$ and $E(G)=E(H)\cup E(K)$.

<u>Remark 4.3</u> The following three statements hold :

(1) A connected $\{6,8\}$-graph having exactly one vertex of degree 6 cannot be decomposed into four $[1,2]$-factors.

(2) The 6-regular graph with three vertices, in which every pair of vertices are joined by exactly three multiple edges, cannot be decomposed into five or more $[1,2]$-factors.

(3) The union of a connected $\{6,8\}$-graph with one vertex of degree 6 and the 6-regular graph given in (2) is not $[1,2]$-factorable.

<u>Proof</u> We first prove (1). Suppose that a connected $\{6,8\}$-

graph G with one vertex v of degree 6 has a $[1,2]$-factorization $F_1 \cup F_2 \cup F_3 \cup F_4$. Then it follows for some F_i that $d_{F_i}(v)=1$ and $d_{F_i}(x)=2$ if $x \neq v$, a contradiction. Statement (2) is immediate. Statement (3) is an easy consequence of (1) and (2). ∎

In order to prove Lemma 4.1, we shall give some lemmas.

Lemma 4.4 Every $[0,4]$-graph can be decomposed into two $[0,2]$-factors.

Proof Let G be a connected $[0,4]$-graph. Then G is a $[1,4]$-graph. We define $\theta=1/2$ and two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 0 & \text{if } d_G(x)=1 \\ 1 & \text{if } 2 \le d_G(x) \le 3 \\ 2 & \text{if } d_G(x)=4, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x)=1 \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence G has a (g,f)-factor F, and thus the lemma holds since F and $G-E(F)$ are both $[0,2]$-factors of G.

Lemma 4.5 Let G be a connected $[2,4]$-graph with at least one vertex of degree 3. Then G can be decomposed into two $[1,2]$-factors.

Proof Set $\theta=1/2$, and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 1 & \text{if } 2 \le d_G(x) \le 3 \\ 2 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 1 & \text{if } d_G(x) = 2 \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence G has a (g,f)-factor F, and thus G is decomposed into two $[1,2]$-factors F and $G-E(F)$. ∎

The following lemma, which is a special case of Lemma 4.9, shows that Lemma 4.1 holds if the graph is 3-edge-connected. Recall that an $\{a,b,c,\ldots\}$-graph satisfies $d_G(x) \in \{a,b,c,\ldots\}$ for all $x \in V(G)$.

Lemma 4.6 Let G be a 3-edge-connected $[3,6]$-graph with at most one vertex of degree 6. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ such that if $d_G(x) \ge 4$, then $d_{F_i}(x) \ge 1$ for every F_i.

Proof We first assume that G has at least one vertex of degree 3 or 5, or G is a $\{4,6\}$-graph with an even number of vertices of degree 4. Let $\theta = 1/4$ and define two functions g_1 and f_1 on $V(G)$ by

$$g_1(x) = \begin{cases} 0 & \text{if } d_G(x) = 3 \\ 1 & \text{if } 4 \le d_G(x) \le 5 \\ 2 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f_1(x) = \begin{cases} 1 & \text{if } 3 \le d_G(x) \le 4 \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g_1, f_1 and $n=3$ satisfy (1b; $\varepsilon=0$ or $1/2$), (2) and (3c) of Theorem 2.1. Hence G has a (g_1,f_1)-factor F_1. It is obvious that $G-E(F_1)$ is a $[2,4]$-graph with the property that each vertex of degree 2 in $G-E(F_1)$ has degree 3 in G. By Lemma 4.4, $G-E(F_1)$ is decomposed into two $[0,2]$-factors F_2 and F_3. Consequently, G is decomposed into three $[0,2]$-factors F_1, F_2 and F_3, which possess the desired property.

We next assume that G is a $\{4,6\}$-graph with an odd number of vertices of degree 4. It suffices to show that G can be decomposed into three $[1,2]$-factors. Suppose G is a 4-regular graph. Then it follows from Proposition 2.5 that $G-v$ has a 1-factor L_1 for a vertex v of G. Let F_1 be the $[1,2]$-factor of G obtained from L_1 by adding an edge of $G-E(L_1)$ incident with v. Since $H_1=G-E(F_1)$ is a $[2,3]$-graph having exactly one vertex of degree 2, we have by Lemma 4.5 that H_1 can be decomposed into two $[1,2]$-factors F_2 and F_3. Therefore, we obtain a required $[1,2]$-factorization $F_1 \cup F_2 \cup F_3$ of G. Consequently, we may assume that G has exactly one vertex w of degree 6. Set $\theta=1/4$ and define two functions g_2 and f_2 on $V(G)$ by

$$g_2(x)=f_2(x)=1 \quad \text{for all} \quad x \in V(G).$$

Then θ, g_2, f_2 and $n=3$ satisfy (1b;$\varepsilon=1/2$), (2) and (3c) of Theorem 2.1. Thus G has a (g_2,f_2)-factor L_2. Let F_1 be the $[1,2]$-factor of G obtained from L_2 by adding an edge of $G-E(L_2)$ incident with w. Since $H_2=G-E(F_1)$ is a $[2,4]$-graph having exactly one vertex of degree 4 and one vertex of degree 2, it follows from Lemma 4.5 that H_2 can be decomposed into two $[1,2]$-factors F_2 and F_3. Therefore we obtain a desired $[1,2]$-factorization $F_1 \cup F_2 \cup F_3$ of G. ∎

We denote by xy or yx an edge joining two vertices x and y. Let G be a graph and v and w be two distinct vertices of G.

Then $G+vw$ denotes the graph obtained from G by adding a new edge vw to G, where G may have edges joining v and w. The following Lemmas 4.7 and 4.8 will be used in the proof of Lemma 4.9.

Lemma 4.7 Let G be a connected $[2,6]$-graph which has exactly one vertex w of degree 2 and at most one vertex of degree 6. Suppose that two distinct vertices u_1 and u_2 are adjacent to w and $G-w+u_1u_2$ is a 3-edge-connected graph. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that if $d_G(x) \geq 4$, then $d_{F_i}(x) \geq 1$ for every F_i and $d_{F_i}(w) \leq 1$ for every F_i.

Proof Let us defint two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 0 & \text{if } 2 \leq d_G(x) \leq 3 \\ 1 & \text{if } 4 \leq d_G(x) \leq 5 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } 2 \leq d_G(x) \leq 4 \\ 2 & \text{otherwise.} \end{cases}$$

We shall show that G has a (g,f)-factor by Lemma 2.2. We denote the vertex of degree 6, if any, by v. Let S, $T \subset V(G)$ such that $S \cap T = \phi$ and $S \cup T \neq \phi$. We write $\{C_1, \ldots, C_r\}$ for the set of components of $G-(S \cup T)$ which satisfy the conditions on $h(S,T)$ in Lemma 2.2, where $r=h(S,T)$. Then each C_i does not contain w, and so $e(S \cup T, V(C_i)) \geq 3$. Moreover, we have $e(S \cup T, V(C_i)) \geq 4$ since $e(S \cup T, V(C_i)) \equiv 0 \pmod 2$ (see (2.4) in the proof of Theorem 2.1). We obtain the following inequality by setting $\theta = 1/4$ in (2.3) in the proof of Theorem 2.1 (Note that (2.3) holds for every graph.).

$$\delta(S,T) \geq \sum_{i=1}^{r} \{\frac{3}{4} e(T, V(C_i)) + \frac{1}{4} e(S, V(C_i)) - 1\} - \varepsilon,$$

where $\varepsilon=0$ or $1/2$ according as $v\notin V(G)$ or $v\in V(G)$. Then

$$\delta(S,T) \geq \sum_i \{\frac{1}{4} e(S\cup T,V(C_i)) -1\} - \varepsilon \geq - \varepsilon > -1.$$

Since $\delta(S,T)$ is an integer, we conclude that $\delta(S,T)\geq 0$. It is clear that $\delta(\phi,\phi)=0$ as $g(w)<f(w)$. Consequently, G has a (g,f)-factor F. Put $H=G-E(F)$. We consider two cases.

Case 1. $d_F(w)=1$. By Lemma 4.4, H can be decomposed into two $[0,2]$-factors F_2 and F_3, and it is easy to see that $(F_1=F)\cup F_2\cup F_3$ is a $[0,2]$-factorization of G with the required property.

Case 2. $d_F(w)=0$. In this case H is a $[2,4]$-graph. Let C be any component of H. If C does not contain w, then we decompose C into two $[0,2]$-factors. Suppose C contains w. Then C contains u_1 and u_2. If $d_C(u_1)=d_G(u_2)=4$, then we may assume $d_G(u_1)=5$, and so $F+wu_1$, where $wu_1\in E(C)$, is also a (g,f)-factor of G. Hence Case 1 occurs, and thus we may assume $d_C(u_1)\leq 3$ or $d_C(u_2)\leq 3$. Set $\theta=1/2$ and define two functions g_1 and f_1 on $V(C)$ by

$$g_1(x) = \begin{cases} 1 & \text{if } d_C(x)\leq 3 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 1 & \text{if } x=w \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g_1, f_1 and $n=1$ satisfy (1a), (2) (since $g_1(u_1)<f_1(u_1)$ or $g_1(u_2)<f_1(u_2)$.) and (3c) of Theorem 2.1. Hence C has a (g_1,f_1)-factor, and thus C is decomposed into two $[1,2]$-factors, in each factor of which the degree of w is 1. Therefore, G can be decomposed into three $[0,2]$-factors with the required property. ∎

Lemma 4.8 Let G be a 3-edge-connected $[3,5]$-graph having a vertex w of degree 3. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that if $d_G(x) \geq 4$, then $d_{F_i}(x) \geq 1$ for every F_i and that $d_{F_i}(w)=0$ for some F_i.

Proof Let g and f be functions on $V(G)$ defined by

$$g(x) = \begin{cases} 0 & \text{if } d_G(x)=3 \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x=w \\ 1 & \text{if } 3 \leq d_G(x) \leq 4 \text{ and } x \neq w \\ 2 & \text{otherwise.} \end{cases}$$

We shall show that G has a (g,f)-factor. Let $S, T \subset V(G)$ such that $S \cap T = \phi$ and $S \cup T \neq \phi$, and let $\{C_1, \ldots, C_r\}$ be the components of $G-(S \cup T)$ which satisfy the conditions on $h(S,T)$, where $r=h(S,T)$. Then we have the following inequality by setting $\theta=1/4$ in (2.2).

$$\delta(S,T) \geq (1-\frac{1}{4}) \sum_{t \in T} d_G(t) + \frac{1}{4} \sum_{s \in S} d_G(s) - \varepsilon - e(S,T) - r,$$

where $\varepsilon=0$ or $3/4$ according as $w \notin S$ or $w \in S$. Hence

$$\begin{aligned} \delta(S,T) &\geq \sum_{i=1}^{r} \{ \frac{3}{4} e(T,V(C_i)) + \frac{1}{4} e(S,V(C_i)) -1\} - \varepsilon \\ &\geq \sum_{i} \{ \frac{1}{4} e(T \cup S,V(C_i)) - 1\} - \varepsilon. \end{aligned}$$

If C_i does not contain w, then $e(T \cup S,V(C_i)) \equiv 0 \pmod 2$ (see (2.4)), and so $e(T \cup S,V(C_i)) \geq 4$. Therefore

$$\frac{1}{4} e(T \cup S,V(C_i)) - 1 \geq 0.$$

If C_i contains w, then $\varepsilon=0$ and $e(T \cup S,V(C_i)) \geq 3$, and so

$$\frac{1}{4} e(T \cup S,V(C_i)) -1 \geq - \frac{1}{4} .$$

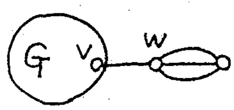
Consequently, we obtain $\delta(S,T) \geq -3/4$, which implies $\delta(S,T) \geq 0$.

Furthermore, we can show that $\delta(\phi,\phi)=0$ by the fact that G has at least one vertex x with odd degree except w, for which $g(x)<f(x)$. Consequently, G has a (g,f)-factor F_1. By Lemma 4.4, $G-E(F_1)$ can be decomposed into two $[0,2]$-factors F_2 and F_3. Therefore we obtain a desired $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ of G. ∎

We need some notation and definitions in order to prove Lemma 4.1. A graph having exactly two vertices and one or more edges is called a <u>bond</u>, and we denote the bond with n edges by B_n (Fig. 1). Let v be a vertex of a graph G and w be a vertex of the bond B_n. Then $G+vw+B_n$ denotes the graph obtained from G and B_n by joining v and w by a new edge vw (Fig. 2).

We shall prove the next lemma instead of Lemma 4.1, which includes Lemma 4.1 as a special case.

<u>Figure 1.</u> The bond B_4.



<u>Figure 2.</u> $G+vw+B_3$.

<u>Lemma 4.9</u> Let G be a connected $[3,6]$-graph with at most one vertex of degree 6. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that

$$\text{if } d_G(x)\geq 4, \text{ then } d_{F_i}(x)\geq 1 \text{ for every } F_i. \tag{4.1}$$

<u>Proof</u> We prove the lemma by induction on the number of vertices of a graph. Let G be a connected $[3,6]$-graph with at most one vertex of degree 6. By Lemma 4.6, we may assume that G is not 3-edge-connected.

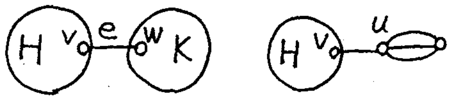


Figure 3. G and $H+vu+B_3$.

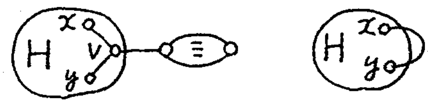


Figure 4. G and $H-v+xy$.

If $|V(G)|=2$ or 3, then G must be 3-edge-connected, which is contrary to the assumption. Hence we may assume $|V(G)|\geq 4$.

First suppose that G is not 2-edge-connected. Then G has a bridge $e=vw$, where $e\in E(G)$ and v, $w\in V(G)$ (Fig. 3). Let H and K be the components of $G-e$ such that $v\in V(H)$ and $w\in V(K)$ (Fig. 3). If $|V(H)|\geq 3$ and $|V(K)|\geq 3$, then $H'=H+vu+B_3$ and $K'=K+wu+B_3$ are both $[3,6]$-graphs, where u is a vertex of B_3 (Fig. 3). By the induction hypothesis, H' and K' can be decomposed into three $[0,2]$-factors with the property (4.1), respectively. It is easy to obtain a desired $[0,2]$-factorization of G from them. Therefore, we may assume $|V(K)|$ $=2$. Then K is B_3, B_4 or B_5.

If $d_H(v)\geq 3$, then H has a $[0,2]$-factorization with the property (4.1) by induction, and it is easy to obtain a desired $[0,2]$-factorization of G from it. Hence we may assume $d_H(v)=2$. If two distinct vertices x and y of H are adjacent to v, then $H-v+xy$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction (Fig. 4). So we can obtain a desired $[0,2]$-factorization of G from it. We next suppose that one vertex x and v are joined by two edges in H (Fig. 5). Let $H+B_3$ be the graph obtained from H by identifying

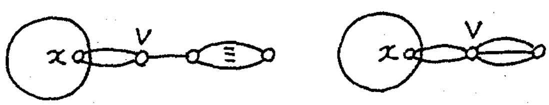


Figure 5. G and $H+B_3$.

v and one of the vertices of B_3 (Fig. 5). Then, by the induction hypothesis, $H+B_3$ has a [0,2]-factorization with the property (4.1), and it is immediate to obtain a desired [0,2]-factorization of G from it. Consequently, the proof is complete if G is not 2-edge-connected.

We now deal with the case that G is 2-edge-connected. Since G is not 3-edge-connected, G has a cutset (i.e. a minimal cut) with two edges. We consider three cases.

Case 1. G has a cutset $\{e_1,e_2\}$ such that the ends of e_1 and those of e_2 are all distinct, where $e_1,e_2 \in E(G)$.

Let H and K be the components of $G-\{e_1,e_2\}$, and let $e_1=u_1w_1$ and $e_2=u_2w_2$, where $u_1,u_2 \in V(H)$, $u_1 \neq u_2$, $w_1,w_2 \in V(K)$ and $w_1 \neq w_2$. Then $H+u_1u_2$ and $K+w_1w_2$ have [0,2]-factorization with property (4.1) by induction. It is easy to obtain a desired [0,2]-factorization of G from them.

Case 2. G has a cutset $\{e_1,e_2\}$ such that the ends of e_1 and those of e_2 are the same (Fig. 6).

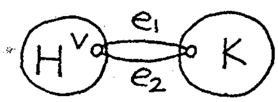


Figure 6.

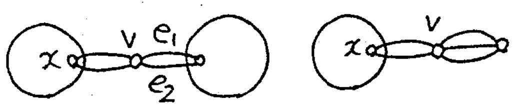
Let H be an arbitrary component of $G-\{e_1,e_2\}$, and v be the end of e_1 and e_2 contained in H (Fig. 6). We shall show that H has a [0,2]-factorization $F_1 \cup F_2 \cup F_3$ with the property that

$$\begin{array}{l} F_1+e_1,\ F_2+e_2 \text{ and } F_3 \text{ are [0,2]-factors of } H+\langle e_1,e_2\rangle \\ \text{and satisfy the condition (4.1) in } H+\langle e_1,e_2\rangle, \end{array} \qquad (4.2)$$

where $H+\langle e_1,e_2\rangle$ is the subgraph of G obtained from H by adding

e_1 and e_2 together with their common end not contained in H. If this statement follows, then we can easily obtain a $[0,2]$-factorization of G with the property (4.1) from a $[0,2]$-factorization with the property (4.2) of each component of $G-\{e_1,e_2\}$. We now prove the statement.

If $d_G(v)\geq 5$, then $d_H(v)\geq 3$ and so H has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property (4.1) by induction. Since we may assume $d_{F_1}(v)\leq 1$ and $d_{F_2}(v)\leq 1$, these factors satisfy the required condition (4.2). If $d_G(v)=3$, then G has a bridge, and so Case 1 occurs. Hence we may assume $d_G(v)=4$, and thus $d_H(v)=2$. If two distinct vertices x and y of H are adjacent to v, then $H-v+xy$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction. It is easy to obtain a desired $[0,2]$-factorization of H from them. We next assume that one vertex x of H and v are joined by two edges (Fig. 7). Let $H+B_3$ be the graph obtained from H and B_3 by identifying v and a vertex of B_3 (Fig. 7). Then $H+B_3$ has a $[0,2]$-factorization with the property (4.1) by induction, and so we can obtain a desired $[0,2]$-factorization of H from it. Consequently, each component of $G-\{e_1,e_2\}$ has a $[0,2]$-factorization satisfying the conditions (4.2), and we conclude that the proof of Case 2 is complete.



<u>Figure 7.</u> G and $H+B_3$.

<u>Case 3.</u> <u>For every cutset</u> $\{e_1,e_2\}$ <u>of G</u>, e_1 <u>and</u> e_2 <u>have exactly one common end</u> (Fig. 8).

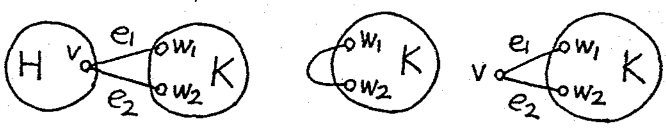


Figure 8. G, $K+w_1w_2$ and $K+\langle e_1,e_2\rangle$.

Let $\{e_1,e_2\}$ be any cutset of G. Then we can write $e_1=vw_1$ and $e_2=vw_2$, where $v,w_1,w_2\in V(G)$ and $w_1\neq w_2$. Let H and K be the components of $G-\{e_1,e_2\}$ such that $v\in V(H)$ and $w_1,w_2\in V(K)$. Note that $d_H(v)\geq 2$ as G has no bridges. We first prove that if $\{e_1,e_2\}$ satisfies one of the following two conditions, then G has a $[0,2]$-factorization with the property (4.1) :

(i) $K+w_1w_2$ is 3-edge-connected (Fig. 8).

(ii) $d_H(v)=3$ and H is a 3-edge-connected graph without vertices of degree 6.

Suppose (i) hold. Then $K+\langle e_1,e_2\rangle$ (Fig. 8) can be decomposed into three $[0,2]$-factors which satisfy the conditions in Lemma 4.7. On the other hand, if $d_H(v)\geq 3$, then H has a $[0,2]$-factorization with the property (4.1) by induction, and so we can get a desired $[0,2]$-factorization of G. If $d_H(v)=2$, then two distinct vertices x and y of H are adjacent to v, and so $H-v+xy$ has a $[0,2]$-factorization with the property (4.1). It is easy to obtain a desired $[0,2]$-factorization of G.

We next suppose that (ii) holds. Then H can be decomposed into three $[0,2]$-factors which satisfy the conditions in Lemma 4.8. It follows that $K+w_1w_2$ has a $[0,2]$-factorization with the property (4.1) by induction, and thus G has a desired $[0,2]$-factorization.

We shall show that G has a cutset $\{e_1,e_2\}$ which satisfies one of the above conditions (i) and (ii). We can choose a cutset $\{e_1,e_2\}$ so that H or $K+w_1w_2$ is 3-edge-connected. If $d_H(v)=2$, then we may assume without loss of generality that $K+w_1w_2$ is 3-edge-connected. Hence (i) follows.

Suppose $d_H(v)=3$. In this case we may assume that $K+w_1w_2$ is not 3-edge-connected and H contains a unique vertex of G with degree 6 ; (otherwise, $\{e_1,e_2\}$ satisfies (i) or (ii)). Let $\{f_1,f_2\}$ be any cutset of $K+w_1w_2$. If the ends of f_1 and those of f_2 are all distinct, then G has such a cutset, which contradicts the assumption of Case 3. If the ends of f_1 and those of f_2 are the same, then f_1 or f_2, say f_1, must be w_1w_2. So it follows that both $\{e_1,f_2\}$ and $\{e_2,f_2\}$ are cutsets of G and f_2 joins w_1 and w_2. Let T be the component of $G-\{e_1,f_2\}$ containing w_1. If $d_T(w_1)\geq 3$, then T has a [0,2]-factorization with the property (4.1) by induction. If $d_T(w_1)=2$, then two distinct vertices t_1 and t_2 are adjacent to w_1 in T, and thus $T-w_1+t_1t_2$ has a [0,2]-factorization with the property (4.1) by induction. Obviously, the component of $G-\{e_2,f_2\}$ containing w_2 has the same property mentioned above. Furthermore, H also has a [0,2]-factorization with the property (4.1). Therefore, we can obtain a desired [0,2]-factorization of G from them. Consequently, we may assume that for every cutset $\{f_1,f_2\}$ of $K+w_1w_2$, f_1 and f_2 have exactly one common end. Hence we can write $f_1=xy_1$ and $f_2=xy_2$, where $x,y_1,y_2\in V(K+w_1w_2)$.

Choose a cutset $\{f_1=xy_1,\ f_2=xy_2\}$ of $K+w_1w_2$ so that the component of $K+w_1w_2-\{f_1,f_2\}$ containing x is 3-edge-connected or the graph obtained from the component of $K+w_1w_2-\{f_1,f_2\}$ containing y_1 and y_2 by adding a new edge y_1y_2 is 3-edge-connected. Since H contains the vertex of degree 6, we can choose such a cutset $\{f_1,f_2\}$ so that the 3-edge-connected component (or graph) has no vertices of degree 6. If $w_1w_2\notin\{f_1,f_2\}$, then $\{f_1,f_2\}$ is a cutset of G which satisfies one of (i) and (ii). Hence we may assume $f_1=w_1w_2$ and $f_2=w_1y_2$, where $w_2\neq y_2$. Then $\{e_2,y_2\}$ is a cutset of G which satisfies the condition of Case 1, a contradiction.

We finally assume $d_H(v)=4$ (i.e. v is the vertex of G with degree 6.). If $K+w_1w_2$ is 3-edge-connected, then we can obtain a [0,2]-factorization of G with the property (4.1) by applying Lemma 4.8 to $K+\langle e_1,e_2\rangle$. Hence we may assume that $K+w_1w_2$ is not 3-edge-connected. In this case we can prove that G has a desired [0,2]-factorization by the same argument in the case of $d_H(v)=3$. Consequently, Case 3 is proved.

Acknowledgments

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