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# [a,b]-factorization of a graph 

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## Abstract

Let $a$ and $b$ be integers such that $0 \leq a \leq b$. Then $a$ graph $G$ is called an $[a, b]-g r a p h$ if $a \leq d_{G}(x) \leq b$ for every $x \in V(G)$, and an [a,b]-factor of a graph is defined to be its spanning subgraph $F$ such that $a \leq d_{F}(x) \leq b$ for every vertex $x$, where $d_{G}(x)$ and $d_{F}(x)$ denote the degrees of $x$ in $G$ and $F$, respectively. If the edges of a graph can be decomposed into [a,b]-factors, then we say that the graph is [a,b]-factorable. We prove the following two theorems: (i) a graph $G$ is [2a,2b]-factorable if and only if $G$ is a [2am,2bm]-graph for some integer $m$, and (ii) every [ $8 \mathrm{~m}+2 \mathrm{k}, 10 \mathrm{~m}+2 \mathrm{k}$ ]graph is [1,2]-factorable.

## 1. Introduction

We deal with finite graphs which may have multiple edges but have no loops. A graph without multiple edges is called a simple graph. All notation and definitions not given here can be found in [4].

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and $H$ be a subgraph of $G$. For a vertex $x$ of $H$, we denote the degree of $x$ in $H$ by $d_{H}(x)$, in partucular, the degree of a vertex $y$ of $G$ is denoted by $d_{G}(y)$. Let $a$ and $b$ be integers such that $0 \leq a \leq b$. Then a graph $G$ is called an $[a, b]$-graph if $a \leq d_{G}(x) \leq b$ for every $x \in V(G)$, and an $[a, b]-$ subgraph can be defined similarly. A spanning [a,b]-subgraph is called an [a,b]-factor. Then, if $F$ is an $[a, b]$ factor of a graph $G$, then $a \leq d_{F}(x) \leq b$ for $a l l$. $x \in V(G)$.

If the edges of a graph $G$ can be decomposed into [a,b]-factors $F_{1}, \ldots, F_{n}$ of $G$, then the union $F_{1} \cup \ldots \cup F_{n}$ is called an $[a, b]-$ factorization of $G$ and $G$ itself is said to be [a,b]-factorable.

We usually call an [r,r]-graph an r-regular graph. Similarly, an [r,r]-factor, an [r,r]-factorization and an [r,r]-factorable graph are called an r-factor, an r-factorization and an r-factorable graph, respectively.

In 1891 Petersen [13],[4,Theorem 8.8] obtained the following theorem.

Theorem 1.1 A graph $G$ is 2-factorable if and only if $G$ is a 2 m -regular graph for some positive integer m .

Recently, Akiyama [1] proved that every r-regular graph is [2,3]-factorable, where $r \geq 2$. This is the first contribution toward [a,b]-factorization with $a<b$. Era [6] proved that if $r \geq 2 k^{2}$, then every r-regular simple graph is [k,k+1]-factorable. We now give our theorems.

Theorem 1.2 Let $0 \leq a \leq b$. Then a graph $G$ is [2a,2b]-factorable if and only if $G$ is a [2am, 2 bm ]-graph for some positive integer $m$.

This thoerem is an extension of Theorem 1.1.
Theorem 1.3 Let $m \geq 1$ and $k \geq 0$. Then every $[8 m+2 k, 10 m+2 k]-$ graph is [1,2]-factorable.

As a corollary of this theorem, we can obtain the next result. Corollary 1.4 (1) If $r \geq 8 \mathrm{~m}$, then every $[\mathrm{r}, \mathrm{r}+2 \mathrm{~m}-1$ ]-graph is [1,2]-factorable.
(2) Every connected $[r, r+1]$-graph is [1,2]-factorable, where $r \geq 1$.

Note that a [2am,2bm]-graph can be decomposed into: m [2a,2b]factors, and a $[8 m+2 k, 10 m+2 k]$-graph can be decompsed into $6 m+k$ [1,2]-factors. But the number of [a,b]-factors in an [a,b]-factorization of a graph is not uniquely determined. For example, a 4 m -regular graph can be decomposed into $k$ [1,2]-factors for every $k, 2 m \leq k \leq 3 m$ (see Theorem 1.1 and Lemma 4.1). It is clear that the union of an odd cycle and a cubic graph, which is a [2,3]-graph with two components,
is not [1,2]-factorable. So the connectivity of a graph in (2) of Corollary 1.4 is necessary. Moreover, we show that there exists a [6,8]-graph which is not [1,2]-factorable (Remark 4.3).

We next mention two factor theorems on which our proof will heavily depend. One is Lovász's ( $g, f$ )-factor theorem (see Lemma 2.2), which plays an important role throughout this article, and the other is Theorem 2.1, which is proved by making use of Lovász's ( $\mathrm{g}, \mathrm{f}$ )factor theorem. By Theorem 2.1, not on1y can we prove many known theorems on r-factors due to Baebler, Gallai, Petersen and others, but also we can obtain some new results on $[a, b]-f a c t o r s$, for instance, Theorem 1.2 is an easy consequence of it.

Let us finally note a survey article [2], in which many results related to our theorems can be found.

## 2. Factor theorem

We begin by introducing some new notation and definitions. For a finite set $X$, we denote by $|X|$ the number of elements in $X$. Let $G$ be a graph, and $g$ and $f$ be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every $x \in V(G)$. Then, a ( $g, f$ )-factor of $G$ is a spanning subgraph $F$ of $G$ satisfying
$g(x) \leq d_{F}(x) \leq f(x)$ for all $x \in V(G)$. For a subset $S$ of $V(G)$, we write G-S for the subgraph of $G$ obtained from $G$ by deleting the vertices in $S$ togehter with their incident edges. If $S$ and $T$ are disjoint subsets of $V(G)$, then $e(S, T)$ denotes the number of edges of $G$ joining $S$ and $T$.

In this section we shall prove the following theorem and give some its corollaries.

Theorem 2.1 Let $G$ be an $n$-edge-connected graph ( $n \geq 1$ ), $\theta$ be a real number such that $0 \leq \theta \leq 1$, and $g$ and $f$ be two integervalued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. If one of $\{(1 a),(1 b)\}$, (2) and one of $\{(3 a),(3 b),(3 c),(3 d),(3 e),(3 f)\}$ hold, then $G$ has a ( $g, f$ )-factor.
(1a) $g(x) \leq \theta d_{G}(x) \leq f(x)$ for all $x \in V(G)$.
(1b) $\quad \varepsilon=\sum_{x \in V(G)}\left[\max \left\{0, g(x)-\theta d_{G}(x)\right\}+\max \left\{0, \theta d_{G}(x)-f(x)\right\}\right]<1$.
(2) $G$ has at least one vertex $v$ such that $g(v)<f(v)$; or $g(x)=f(x)$ for all $x \in V(G)$ and $\sum_{x \in V(G)} f(x) \equiv 0 \quad(\bmod 2)$.
(3a) $n \theta \geq 1$ and $n(1-\theta) \geq 1$.
(3b) $\left\{d_{G}(x) \mid g(x)=f(x), x \in V(G)\right\}$ and $\{f(x) \mid g(x)=f(x), x \in V(G)\}$
both consist of even numbers.
(3c) $\left\{d_{G}(x) \mid g(x)=f(x), x \in V(G)\right\}$ consists of even numbers, $n$ is odd, $(n+1) \theta \geq 1$ and $(n+1)(1-\theta) \geq 1$.
(3d) $\{f(x) \mid g(x)=f(x), x \in V(G)\}$ consists of even numbers and $m(1-\theta) \geq 1$, where $m \in\{n, n+1\}$ and $m \equiv 1(\bmod 2)$.
(3e) $\left\{d_{G}(x) \mid g(x)=f(x), x \in V(G)\right\}$ and $\{f(x) \mid g(x)=f(x), x \in V(G)\}$
both consist of odd numbers and $m \theta \geq 1$, where $m \in\{n, n+1\}$ and $m \equiv 1(\bmod 2)$. (3f) $g(x)<f(x)$ for every $x \in V(G)$ (see [8]).

Note that similar necessary conditions for a graph to have a ( $\mathrm{g}, \mathrm{f}$ )-factor which contains p given edges but has no q given edges are obtained in [9]. In order to prove the above theorem we need the next ( $g, f$ )-factor theorem due to Lovász, to which Tutte [16] gave a short proof.

Lemma 2.2 (Lovász [12], [16, Theorem 7.2]) Let $G$ be a graph and $g$ and $f$ be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then $G$ has a $(g, f)$-factor if and only if

$$
\begin{equation*}
\delta(S, T)=\sum_{t \in T}\left\{d_{G}(t)-g(t)\right\}+\sum_{s \in S} f(s)-e(S, T)-h(S, T) \geq 0 \tag{2.1}
\end{equation*}
$$

for all disjoint subsets $S$ and $T$ of $V(G)$, where $h(S, T)$ denotes the number of components $C$ of $G-(S u T)$ such that $g(x)=f(x)$ for all $x \in V(C)$ and $e(T, V(C))+\sum_{x \in V(G)} f(x) \equiv 1(\bmod 2)$.

Note that the condition $0 \leq g(x) \leq f(x) \leq d_{G}(x)$ in [12] and [16] can be replaced by $g(x) \leq f(x)$ as above ([10], [15]).

Proof of Theorem 2.1 We shall prove that two functions $g$ and $f$ in Theorem 2.1 satisfy the condition (2.1) in Lemma 2.2. It is obvious that (la) implies (1b). Hence we may assume (1b) holds. Let $S, T \subset V(G)$ such that $S \cap T=\phi$. Assume first SuT\# . Let
$\left\{C_{1}, \ldots, C_{r}\right\}$ be the set of components of $G-(S U T)$ which satisfy the conditions on $h(S, T)$, where $r=h(S, T)$. By (1b) of Theorem 2.1, we have

$$
\begin{align*}
\delta(S, T) \geq(1-\theta) & \sum_{t \in T} d_{G}(t)+\theta \sum_{s \in S} d_{G}(s)-\sum_{t \in T} \max \left\{0, g(t)-\theta d_{G}(t)\right\} \\
& \quad \sum_{S \in S} \max \left\{0, \theta d_{G}(s)-f(s)\right\}-e(S, T)-r  \tag{2.2}\\
\geq & (1-\theta)\left\{e(T, S)+\sum_{i=1}^{r} e\left(T, V\left(C_{i}\right)\right)\right\}+\theta\left\{e(S, T)+\sum_{i=1}^{r} e\left(S, V\left(C_{i}\right)\right)\right\} \\
& \quad-\varepsilon-e(S, T)-r \\
= & \sum_{i}\left\{(1-\theta) e\left(T, V\left(C_{i}\right)\right)+\theta e\left(S, V\left(C_{i}\right)\right)-1\right\}-\varepsilon . \tag{2.3}
\end{align*}
$$

Since $\delta(S, T)$ is an integer and $\varepsilon<1$, it suffices to show that $\delta(S, T) \geq-\varepsilon$. If (3f) holds, then $r=0$ and so $\delta(S, T) \geq-\varepsilon$. Hence we may assume that $G$ satisfies (2) and one of $\{(3 a),(3 b),(3 c),(3 d),(3 e)\}$. Take any $C \in\left\{C_{1}, \ldots, C_{r}\right\}$, and put

$$
\Delta(C)=(1-\theta) e(T, V(C))+\theta e(S, V(C))-1
$$

We prove that $\Delta(C) \geq 0$. If $\{f(x) \mid g(x)=f(x), x \in V(G)\}$ consists of even numbers, then

$$
1 \equiv e(T, V(C))+\sum_{x \in V(C)} f(x) \equiv e(T, V(C)) \quad(\bmod 2),
$$

in particular, $e(T, V(C)) \geq 1$. Similarly, if $\{f(x) \mid g(x)=f(x), x \in V(G)\}$ consists of odd numbers, then we have $1 \equiv e(T, V(C))+|V(C)|(\bmod 2)$. Suppose $\left\{d_{G}(x) \mid g(x)=f(x), x \in V(G)\right\}$ consists of even numbers. Then

$$
\begin{gather*}
0 \equiv \sum_{x \in V(C)} d_{G}(x)=2|E(C)|+e(V(C), S U T) \\
\equiv e(\operatorname{SUT}, V(C)) \quad(\bmod 2) \tag{2.4}
\end{gather*}
$$

Thus $e(\operatorname{SuT}, V(C)) \equiv 0(\bmod 2)$. If $\left\{d_{G}(x) \mid g(x)=f(x), x \in V(G)\right\}$ consists of odd numbers, then we have $|V(C)| \equiv e(S u T, V(C))(\bmod 2)$. We consider three cases.

Case 1. $e(T, V(C)) \geq 1$ and $e(S, V(C)) \geq 1$. It follows immediately from $0 \leq \theta \leq 1$ that $\Delta(C) \geq 0$.

Case 2. $e(T, V(C))=0$. We first note that $e(S, V(C))=e(S U T, V(C))$ $\geq \mathrm{n}$ since $G$ is n-edge-connected. By the fact mentioned above, $\{f(x) \mid g(x)=f(x), x \in V(G)\}$ is not a set of even numbers, and so neither (3b) nor (3d) occurs. If $G$ satisfies (3a), then $\Delta(C) \geq \theta n-1 \geq 0$ as $e(S, V(C)) \geq n$. Suppose $G$ satisfies (3c). Then we have $e(S, V(C))$ $\geq n+1$. Hence $\Delta(C) \geq \theta(n+1)-1 \geq 0$. We finally assume that $G$ satisfies (3e). Then it follows from the fact mentioned above that $1 \equiv e(S, V(C))$ (mod 2). If $n$ is odd, then $m=n$ and so $\Delta(C) \geq \theta n-1=\theta m-1 \geq 0$. If $n$ is even, then $e(S, V(C)) \geq n+1$ and $m=n+1$. Hence $\Delta(C) \geq \theta(n+1)-1$ $=\theta m-1 \geq 0$.

Case 3. $e(S, V(C))=0 . \quad$ Note that $e(T, V(C))=e(S \cup T, V(C)) \geq n$. If $G$ satisfies (3a), then $\Delta(C) \geq(1-\theta) n-1 \geq 0$. If $\quad\left\{d_{G}(x) \mid g(x)=f(x)\right.$, $X \in V(G)\}$ consists of even numbers, then $e(T, V(C)) \equiv 0(\bmod 2)$. On the other hand, if $\{f(x) \mid g(x)=f(x), x \in V(G)\}$ consists of even numbers, then $e(T, V(C)) \equiv 1(\bmod 2)$. Hence (3b) does not occur. If (3c) holds, then $e(T, V(C)) \geq n+1$ and so $\Delta(C) \geq(1-\theta)(n+1)-1 \geq 0$. Suppose $G$ satisfies (3d). It is easy to show that we may assume $n$ is even. Since $e(T, V(C)) \equiv 1(\bmod 2)$, we have $e(T, V(C)) \geq n+1$, and thus $\Delta(C) \geq(1-\theta)(n+1)-1=(1-\theta) m-1 \geq 0$. Finally we suppose that
$G$ satisfies (3e). Then $1=e(T, V(C))+|V(C)|(\bmod 2)$ and $|V(C)|$ $\equiv e(T, V(C))(\bmod 2)$, a contradiction. Therefore, (3e) does not occur.

Let $S=T=\phi$ and assume $\delta(\phi, \phi)<0$. Then $h(\phi, \phi)>0$. Since $G$ is connected, it follows from Lemma 2.2 that $g(x)=f(x)$ for all $x \in V(G)$ and $\sum f(x) \equiv 1$ (mod 2), which contradicts (2). Therefore $\delta(\phi, \phi)=0$. Consequently, the proof of the theorem is complete.

We now give some results on factors which can be obtained by Theorem 2.1.

Corollary 2.3 Let $2 \leq b$ and $1 \leq a \leq b \leq 2 a$. Then every 2-edgeconnected $[a, b]$-graph $G$ has $a[1,2]$-factor $F$ such that $d_{F}(x)=2$ if $d_{G}(x)=b$. In particular, every 2-edge-connected r-regular graph has a 2 -factor, where $r \geq 2$ (Baebler [3]).

Proof We may assume $b \geq 3$. Put $\theta=2 / b$ and define two functions $g$ and $f$ on $V(G)$ by

$$
g(x)=\left\{\begin{array}{ll}
2 & \text { if } d_{G}(x)=b \\
1 & \text { otherwise, }
\end{array} \quad \text { and } \quad f(x)=2 \text { for all } x \in V(G) .\right.
$$

Then $\theta, g, f$ and $n=2$ satisfy (1a), (2) and (3d) of Theorem 2.1 . Hence $G$ has a ( $\mathrm{g}, \mathrm{f})$-factor, which is a desired [1,2]-factor.

Corollary 2.4 Let $G$ be a (r-1)-edge-connected [r,2r]-graph with at least one vertex of degree greater than $r$, where $r \geq 1$. Then $G$ has a $[1,2]$-factor $F$ such that $d_{F}(x)=1$ if $d_{G}(x)=r$.

Proof Set $\theta=1 / r$, and define two functions $g$ and $f$ on
$V(G)$ as follows :
$g(x)=1$ for all $x \in V(G)$, and $f(x)= \begin{cases}1 & \text { if } d_{G}(x)=r, \\ 2 & \text { otherwise. }\end{cases}$
Then $\theta, g, f$ and $n=r-1$ satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1 according as the parity of $r$. Hence $G$ has a (g,f)factor, which is a desired [1,2]-factor.

Proposition 2.5 ((1): Petersen [13] (r=3) and Baebler [3] (r¥4); and (2): Little, Grant and Holton [11]) Let $G$ be an ( $\mathrm{r}-1$ )-edgeconnected r-regular graph. Then
(1) if $G$ has an even number of vertices, then $G$ has a 1 -factor; and
(2) if $G$ has an odd number of vertices, then $G-v$ has $a$ 1-factor for any vertex $v$ of $G$.

Proof We prove only (2) since (1) can be proved similarly. Put $\theta=1 / r$, and define two functions $g$ and $f$ on $V(G)$ as
$g(x)=f(x)=1$ for all $x \in V(G) \backslash\{v\}, g(v)=0$ and $f(v)=1$,
where $v$ is a given vertex of $G$. Then $\theta, g, f$ and $n=r-1$ satisfy (1a), (2) and (3c) or (3e) of Theorem 2.1. Therefore, $G$ has a ( $g, f$ )-factor $F$. We can easily see that $d_{F}(v)=0$. Hence (2) follows.

Proposition 2.6 ((1), (2):Gallai [7]; and (3): Bollobas, Saito and Wormald [5]) The following statements hold.
(1) An n-edge-connected $2 r$-regular graph with an even number of vertices has a ( $2 k+1$ )-factor for every $2 k+1,2 r / n \leq 2 k+1 \leq 2 r(n-1) / n$.
(2) An n-edge-connected ( $2 \mathrm{r}+1$ )-regular graph $G$ has a 2 k factor for every $2 k, 0 \leq 2 k \leq(2 r+1)(n-1) / n$. In particular, $G$ has a $(2 \mathrm{~m}+1)$-factor for every $2 \mathrm{~m}+1,(2 \mathrm{r}+1) / \mathrm{n} \leq 2 \mathrm{~m}+1 \leq 2 \mathrm{r}+1$.
(3) A 2 n -edge-connected ( $2 \mathrm{r}+1$ )-regular graph $G$ has a $2 \mathrm{k}-$ factor for every $2 k, 0 \leq 2 k \leq(2 r+1)(2 n) /(2 n+1)$. In particular, $G$ has a ( $2 m+1$ )-factor for every $2 m+1,(2 r+1) /(2 n+1) \leq 2 m+1 \leq 2 r+1$.

Proof We prove only (3) since (1) and (2) can be proved similarly. Set $\theta=2 \mathrm{k} /(2 \mathrm{r}+1)$, and define two functions $g$ and f on $V(G)$ by $g(x)=f(x)=2 k$ for all $x \in V(G)$. Then $\theta, g, f$ and $2 n$ satisfy (1a), (2) and (3d) of Theorem 2.1. Therefore $G$ has a ( $g, f$ )factor, which is a 2 k -factor of $G$. Let $F$ be a $2 k$-factor of $G$. Then $G-E(F)$ is a $(2 r+1-2 k)$-factor of $G$, and so $G$ has a $(2 m+1)-$ factor for every $2 \mathrm{~m}+1,(2 \mathrm{r}+1) /(2 \mathrm{n}+1) \leq 2 \mathrm{~m}+1 \leq 2 \mathrm{r}+1$. Note that the latter can be proved independetly by using (3e) of Theorem 2.1.

## 3. Proof of Theorem 1.2

We shall prove Theorem 1.2 by using Theorem 2.1.
Proof of Theorem 1.2 Let $G$ be a [2a,2b]-factorable graph. Then $G$ can be decomposed into $m$ [2a,2b]-factors for some positive integer $m$. It is clear that $G$ is a [2am,2bm]-graph.

Conversely, suppose that $G$ is a [2am,2bm]-graph. We prove that $G$ can be decomposed into $m$ [2a,2b]-factors by induction on m. Without loss of generality, we may assume $G$ is connected. Put $\theta=1 / \mathrm{m}$, and define two functions g and f on $\mathrm{V}(\mathrm{G})$ as follows :
$f(x)=f(x)=2 a \quad$ if $\quad d_{G}(x)=2 a m$,
$g(x) \leq \theta d_{G}(x) \leq f(x)$ with $f(x)-g(x)=1$ if $2 a m<d_{G}(x)<2 b m$, and $g(x)=f(x)=2 b$ if $\quad d_{G}(x)=2 b m$.

Then, $\theta, \mathrm{g}, \mathrm{f}$ and $\mathrm{n}=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Therefore, $G$ has a ( $g, f$ )-factor F. For any vertex $x$ of $G$ with $2 \mathrm{am}<\mathrm{d}_{\mathrm{G}}(\mathrm{x})<2 \mathrm{bm}$, we have
$2 a<\theta d_{G}(x)<2 b$ and $2 a(m-1)<(1-\theta) d_{G}(x)<2 b(m-1)$.
Hence $F$ is $a[2 a, 2 b]$-factor, and $G-E(F)$ is $a[2 a(m-1), 2 b(m-1)]-$
factor. Consequently, the theorem follows by induction.
4. Proof of Theorem 1.3

In this section we shall prove the following four statements :
(i) every $[8 m+2 k, 10 m+2 k]-g r a p h$ is [1,2]-factorable (Theorem 1.3),
(ii) if $r \geq 8 m$, then every [ $r, r+2 m-1]$-graph is [1,2]-factorable
(Corollary 1.4), (iii) every connected [r,r+1]-graph is [1,2]factorable (Corollary 1.4), and (iv) there exists a [6,8]-graph which
is not [1,2]-factorable (Remark 4.3).
We first prove Theorem 1.3 under the assumption that the following lemma holds.

Lemma 4.1 Let $G$ be a [4,6]-graph with at most one vertex of degree 6. Then $G$ can be decomposed into three [1,2]-factors.

We begin with the next lemma.
Lemma 4.2 Every [8,10]-graph can be decomposed into six [1,2]-factors.

Proof Let $G$ be a [8,10]-graph. Without loss of generality, we may assume $G$ is connected. If $G$ has vertices of degree 10 , then choose any vertex $w$ of degree 10. Set $\theta=1 / 2$, and define two functions $g$ and $f$ on $V(G)$ by

$$
g(x)=\left\{\begin{array}{ll}
4 & \text { if } 8 \leq d_{G}(x) \leq 9 \\
5 & \text { otherwise, }
\end{array} \quad \text { and } \quad f(x)=\left\{\begin{array}{lll}
4 & \text { if } & d_{G}(x)=8 \\
5 & \text { if } & 9 \leq d_{G}(x) \leq 10 \\
6 & \text { if } & x=w
\end{array} \text { and } x \neq w\right.\right.
$$

Then $\theta, g, f$ and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence $G$ has a (g,f)-factor $F$. It follows that $F$ is a $[4,6]$-graph with at most one vertex of degree 6 and $G-E(F)$ is a $[4,5]-g r a p h$, and we conclude by Lemma 4.1 that $G$ can be decomposed into six [1,2]-factors.

## Proof of Theorem 1.3 It follows from Theorem 1.1 and

Lemma 4.2 that every [8m,10m]-graph can be decomposed into 6 m [1,2]factors. We now prove by induction on $k$ that every [ $8 \mathrm{~m}+2 \mathrm{k}, 10 \mathrm{~m}+2 \mathrm{k}]-$ graph can be decomposed into $6 \mathrm{~m}+\mathrm{k}$ [1,2]-factors. Let $G$ be a
[ $8 \mathrm{~m}+2 \mathrm{k}, 10 \mathrm{~m}+2 \mathrm{k}]$-graph with $\mathrm{m} \geq 1$ and $\mathrm{k} \geq 1$. We may assume $G$ is connected. Put $\theta=2 /(10 m+2 k)$ and define two functions $g$ and $f$ on $V(G)$ by

$$
g(x)=\left\{\begin{array}{ll}
2 & \text { if } d_{G}(x)=10 m+2 k \\
1 & \text { otherwise, }
\end{array} \text { and } f(x)=2 \text { for all } x \in V(G)\right.
$$

Then $\theta, g, f$ and $n=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence $G$ has a ( $\mathrm{g}, \mathrm{f}$ )-factor $F$, which is a [1, 2]-factor. Since $G-E(F)$ is a $[8 m+2(k-1), 10 m+2(k-1)]$-graph, we conclude by the induction hypothesis that $G$ is decomposed into $6 m+k \quad[1,2]$-factors.

Proof of Corollary 1.4 We first prove (1). Let $H$ be an [ $\mathrm{r}, \mathrm{r}+2 \mathrm{~m}-1$ ]-graph with $\mathrm{r} \geq 8 \mathrm{~m}$. Then there exist integers $k$ and $t$ such that $r=8 m+2 k+t, 0 \leq k$ and $0 \leq t \leq 1$. It is immediate that $8 m+2 k$ $\leq \mathrm{r}$ and $\mathrm{r}+2 \mathrm{~m}-1 \leq 10 \mathrm{~m}+2 \mathrm{k}$. Hence F is $\mathrm{a}[8 \mathrm{~m}+2 \mathrm{k}, 10 \mathrm{~m}+2 \mathrm{k}]-\mathrm{graph}$, and so H is [1,2]-factorable by Theorem 1.3.

We next prove (2). We first show that every $[2 k-1,2 k]-g r a p h ~ i s$ [1,2]-factorable. Let $G$ be a [2k-1,2k]-graph. Then it follows from Theorem 2.1 that $G$ has a $[1,2]$-factor $F$ such that $d_{F}(x)=2$ if $\quad d_{G}(x)=2 k$ (see Proof of Theorem 1.3). Since $G-E(F)$ is a [2k-3,2k-2]-graph, we have by induction that $G$ is [1,2]-factorable. By the statement (1) and the result given above, it suffices to show that if $r=2,4$ or 6 , then a connected $[r, r+1]$-graph is [1,2]-factorable. It follows from Lemma 4.5, which will given later, that every connected $[2,3]$-graph is [1,2]-factorable. By Lemma 4.1, every [4,5]-graph is


Let $H$ be a connected $[6,7]$-graph. Since a 6-regular graph is 2-factorable, we may assume that $H$ has at least one vertex of degree 7. We show that $H$ can be decomposed into two [3,4]-factors, which implies that $H$ can be decomposed into four [1,2]-factors. Put $\theta=1 / 2$ and define two functions $g$ and $f$ on $V(H)$ by

$$
g(x)=3 \text { for all } x \in V(H), \text { and } f(x)= \begin{cases}3 & \text { if } d_{H}(x)=6 \\ 4 & \text { otherwise }\end{cases}
$$

Then $\theta, g, f$ and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence $H$ has $a(g, f)$-factor $F^{\prime}$. It is clear that both $F^{\prime}$ and $H-E\left(F^{\prime}\right)$ are $[3,4]$-factors of $H$. Therefore $H$ is [1,2]-factorable.

It is convenient to introduce a new definiton. For a set $\{a, b$, $c, \ldots\}$ of integers, a graph $G$ is called an $\{a, b, c, \ldots\}$-graph if $d_{G}(x) \in\{a, b, c, \ldots\}$ for every $x \in V(G)$. The union of graphs $H$ and $K$ is a graph $G$ such that $V(G)=V(H) \cup V(K)$ and $E(G)=E(H) \cup E(K)$.

Remark 4.3 The following three statements hold :
(1) A connected $\{6,8\}$-graph having exactly one vertex of degree 6 cannot be decomposed into four [1,2]-factors.
(2) The 6-regular graph with three vertices, in which every pair of vertices are joined by exactly three multiple edges, cannot be decomposed into five or more [1,2]-factors.
(3) The union of a connected $\{6,8\}$-graph with one vertex of degree 6 and the 6-regular graph given in (2) is not [1,2]-factorable.

Proof We first prove (1). Suppose that a connected $\{6,8\}$ -
graph $G$ with one vertex $v$ of degree 6 has a [1,2]-factorization $F_{1} \cup F_{2} \cup F_{3} \cup F_{4}$. Then it follows for some $F_{i}$ that $d_{F_{i}}(v)=1$ and $d_{F_{i}}(x)$ $=2$ if $x \neq v$, a contradiction. Statement (2) is immediate. Statement (3) is an easy consequence of (1) and (2).

In order to prove Lemma 4.1 , we shall give some lemmas.
Lemma 4.4 Every [0,4]-graph can be decomposed into two [0,2]-factors.

Proof Let $G$ be a connected [0,4]-graph. Then $G$ is a [1,4]-graph. We define $\theta=1 / 2$ and two functions $g$ and $f$ on V(G) by

$$
g(x)=\left\{\begin{array}{lll}
0 & \text { if } & d_{G}(x)=1 \\
1 & \text { if } & 2 \leq d_{G}(x) \leq 3 \\
2 & \text { if } & d_{G}(x)=4,
\end{array} \quad \text { and } \quad f(x)= \begin{cases}1 & \text { if } \\
d_{G}(x)=1 \\
2 & \text { otherwise }\end{cases}\right.
$$

Then $\theta, g, f$ and $n=1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence $G$ has $a(g, f)$-factor $F$, and thus the lemma holds since $F$ and $G-E(F)$ are both $[0,2]$-factors of $G$.

Lemma 4.5 Let $G$ be a connected [2,4]-graph with at least one vertex of degree 3 . Then $G$ can be decomposed into two [1,2]factors.

Proof Set $\theta=1 / 2$, and define two functions $g$ and $f$ on V(G) by

$$
g(x)=\left\{\begin{array}{ll}
1 & \text { if } 2 \leq d_{G}(x) \leq 3 \\
2 & \text { otherwise, }
\end{array} \text { and } f(x)= \begin{cases}1 & \text { if } d_{G}(x)=2 \\
2 & \text { otherwise }\end{cases}\right.
$$

Then $\theta, g, f$ and $n=1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence $G$ has $a(g, f)-f a c t o r ~ F$, and thus $G$ is decomposed into two [1,2]-factors $F$ and $G-E(F)$.

The following lemma, which is a special case of Lemma 4.9 , shows that Lemma 4.1 holds if the graph is 3-edge-connected. Recall that an $\{a, b, c, \ldots\}$-graph satisfies $d_{G}(x) \in\{a, b, c, \ldots\}$ for all $x \in V(G)$.

Lemma 4.6 Let $G$ be a 3-edge-connected $[3,6]$-graph with at most one vertex of degree 6. Then $G$ has a $[0,2]$-factorization $\mathrm{F}_{1} \cup \mathrm{~F}_{2} \cup \mathrm{~F}_{3}$ such that if $\mathrm{d}_{\mathrm{G}}(\mathrm{x}) \geq 4$, then $\mathrm{d}_{\mathrm{F}_{\mathrm{i}}}(\mathrm{x}) \geq 1$ for every $\mathrm{F}_{\mathrm{i}}$.

Proof We first assume that $G$ has at least one vertex of degree 3 or 5 , or $G$ is a $\{4,6\}$-graph with an even number of vertices of degree 4. Let $\theta=1 / 4$ and define two functions $g_{1}$ and $f_{1}$ on V(G) by

$$
g_{1}(x)=\left\{\begin{array}{ll}
0 & \text { if } d_{G}(x)=3 \\
1 & \text { if } 4 \leq d_{G}(x) \leq 5 \\
2 & \text { otherwise }
\end{array} \quad \text { and } f_{1}(x)= \begin{cases}1 & \text { if } 3 \leq d_{G}(x) \leq 4 \\
2 & \text { otherwise }\end{cases}\right.
$$

Then $\theta, g_{1}, f_{1}$ and $n=3$ satisfy ( $1 b ; \varepsilon=0$ or $1 / 2$ ), (2) and (3c) of Theorem 2.1. Hence $G$ has a $\left(g_{1}, f_{1}\right)$-factor $F_{1}$. It is obvious that $G-E\left(F_{1}\right)$ is a $[2,4]$-graph with the property that each vertex of degree 2 in $G-E\left(F_{1}\right)$ has degree 3 in $G$. By Lemma 4.4, $G-E\left(F_{1}\right)$ is decomposed into two $[0,2]-$ factors $F_{2}$ and $F_{3}$. Consequently, $G$ is decomposed
into three $[0,2]$-factors $F_{1}, F_{2}$ and $F_{3}$, which possess the desired property.

We next assume that $G$ is a $\{4,6\}$-graph with an odd number of vertices of degree 4. It suffices to show that $G$ can be decomposed into three $[1,2]$-factors. Suppose $G$ is a 4-regular graph. Then it follows from Proposition 2.5 that $G-v$ has a 1-factor $L_{1}$ for a vertex $v$ of $G$. Let $F_{1}$ be the [1,2]-factor of $G$ obtained from $L_{1}$ by adding an edge of $G-E\left(L_{1}\right)$ incident with $v$. Since $H_{1}=G-E\left(F_{1}\right)$ is a $[2,3]$-graph having exactly one vertex of degree 2 , we have by Lemma 4.5 that $H_{1}$ can be decomposed into two [1,2]-factors $F_{2}$ and $F_{3}$. Therefore, we obtain a required [1,2]-factorization $F_{1} \cup F_{2} \cup F_{3}$ of $G$. Consequently, we may assume that $G$ has exactly one vertex $w$ of degree 6. Set $\theta=1 / 4$ and define two functions $g_{2}$ and $f_{2}$ on $V(G)$ by

$$
g_{2}(x)=f_{2}(x)=1 \text { for all } x \in V(G)
$$

Then $\theta, \mathrm{g}_{2}, \mathrm{f}_{2}$ and $\mathrm{n}=3$ satisfy ( $1 \mathrm{~b} ; \varepsilon=1 / 2$ ), (2) and (3c) of Theorem 2.1. Thus $G$ has a $\left(g_{2}, f_{2}\right)$-factor $L_{2}$. Let $F_{1}$ be the [1,2]-factor of $G$ obtained from $L_{2}$ by adding an edge of $G-E\left(L_{2}\right)$ incident with w. Since $H_{2}=G-E\left(F_{1}\right)$ is a [2,4]-graph having exactly one vertex of degree 4 and one vertex of degree 2 , it follows from Lemma 4.5 that $H_{2}$ can be decomposed into two [1,2]-factors $F_{2}$ and $F_{3}$. Therefore we obtain a desired $[1,2]$-factorization $F_{1} \cup F_{2} \cup F_{3}$ of $G$.

We denote by $x y$ or $y x$ an edge joining two vertices $x$ and $y$. Let $G$ be a graph and $v$ and $w$ be two distinct vertices of $G$.

Then $G+v w$ denotes the graph obtained from $G$ by adding a new edge Vw to $G$, where $G$ may have edges joining $v$ and $w$. The following Lemmas 4.7 and 4.8 will be used in the proof of Lemma 4.9.

Lemma 4.7 Let $G$ be a connected [2,6]-graph which has exactly one vertex $w$ of degree 2 and at most one vertex of degree 6. Suppose that two distinct vertices $u_{1}$ and $u_{2}$ are adjacent to $w$ and $G-w+u_{1} u_{2}$ is a 3-edge-connected graph. Then $G$ has a $[0,2]$-factorization $F_{1} \cup F_{2} \cup F_{3}$ with the property that if $d_{G}(x) \geq 4$, then $d_{F_{i}}(x) \geq 1$ for every $F_{i}$ and $d_{F_{i}}(W) \leq 1$ for every $F_{i}$.

$$
\begin{aligned}
& \text { Proof } \\
& g(x)=\left\{\begin{array}{ll}
0 & \text { if } \quad 2 \leq d_{G}(x) \leq 3 \\
1 & \text { if } 4 \leq d_{G}(x) \leq 5 \\
2 & \text { otherwise }
\end{array} \text { and } f(x)= \begin{cases}1 & \text { if } \quad 2 \leq d_{G}(x) \leq 4 \\
2 & \text { otherwise. }\end{cases} \right.
\end{aligned}
$$

We shall show that $G$ has $a(g, f)$-factor by Lemma 2.2. We denote the vertex of degree 6 , if any, by $v$. Let $S, T \subset V(G)$ such that $S \cap T=\phi$ and SuT尹ф. We write $\left\{C_{1}, \ldots, C_{r}\right\}$ for the set of components of $G-(S U T)$ which satisfy the conditions on $h(S, T)$ in Lemma 2.2 , where $r=h(S, T)$. Then each $C_{i}$ does not contain $w$, and so $e\left(\operatorname{SuT}, V\left(C_{i}\right)\right) \geq 3$. Moreover, we have $e\left(\operatorname{SUT}, V\left(C_{i}\right)\right) \geq 4$ since $e\left(\operatorname{SuT}, V\left(C_{i}\right)\right) \equiv 0(\bmod 2)($ see $(2.4)$ in the proof of Theorem 2.1). We obtain the following inequality by setting $\theta=1 / 4$ in (2.3) in the proof of Theorem 2.1 (Note that (2.3) holds for every graph.).

$$
\delta(S, T) \geq \sum_{i=1}^{r}\left\{\frac{3}{4} e\left(T, V\left(C_{i}\right)\right)+\frac{1}{4} e\left(S, V\left(C_{i}\right)\right)-1\right\}-\varepsilon,
$$

where $\varepsilon=0$ or $1 / 2$ according as $v \notin V(G)$ or $v \in V(G)$. Then

$$
\delta(S, T) \geq \sum_{i}\left\{\frac{1}{4} e\left(\operatorname{SuT}, V\left(C_{i}\right)\right)-1\right\}-\varepsilon \geq-\varepsilon>-1
$$

Since $\delta(\mathrm{S}, \mathrm{T})$ is an integer, we conclude that $\delta(\mathrm{S}, \mathrm{T}) \geq 0$. It is clear that $\delta(\phi, \phi)=0$ as $g(w)<f(w)$. Consequently, $G$ has a (g,f)-factor F. Put $H=G-E(F)$. We consider two cases.

Case 1. $d_{F}(w)=1 . \quad$ By Lemma 4.4, $H$ can be decomposed into two [0,2]-factors $F_{2}$ and $F_{3}$, and it is easy to see that $\left(F_{1}=F\right) \cup F_{2} \cup F_{3}$ is a $[0,2]$-factorization of $G$ with the required property.

Case 2. $\mathrm{d}_{\mathrm{F}}(\mathrm{w})=0$. In this case H is a [2,4]-graph. Let C be any component of $H$. If $C$ does not contain $w$, then we decompose C into two [0,2]-factors. Suppose C contains w. Then C contains $u_{1}$ and $u_{2}$. If $d_{C}\left(u_{1}\right)=d_{G}\left(u_{2}\right)=4$, then we may assume $d_{G}\left(u_{1}\right)=5$, and so $\mathrm{F}^{+w u_{1}}$, where $\mathrm{wu}_{1} \in \mathrm{E}(\mathrm{C})$, is also a $(\mathrm{g}, \mathrm{f})$-factor of $G$. Hence Case 1 occurs, and thus we may assume $d_{C}\left(\mathrm{u}_{1}\right) \leq 3$ or $\mathrm{d}_{\mathrm{C}}\left(\mathrm{u}_{2}\right) \leq 3$. Set $\theta=1 / 2$ and define two functions $\mathrm{g}_{1}$ and $\mathrm{f}_{1}$ on $\mathrm{V}(\mathrm{C})$ by

$$
g_{1}(x)=\left\{\begin{array}{ll}
1 & \text { if } d_{C}(x) \leq 3 \\
2 & \text { otherwise, }
\end{array} \quad \text { and } \quad f_{1}(x)= \begin{cases}1 & \text { if } x=w \\
2 & \text { otherwise }\end{cases}\right.
$$

Then $\theta, g_{1}, f_{1}$ and $n=1$ satisfy (1a), (2) (since $g_{1}\left(u_{1}\right)<f_{1}\left(u_{1}\right)$ or $\left.g_{1}\left(u_{2}\right)<f_{1}\left(u_{2}\right).\right)$ and (3c) of Theorem 2.1. Hence $C$ has a $\left(g_{1}, f_{1}\right)$-factor, and thus $C$ is decomposed into two [1,2]-factors, in each factor of which the degree of $w$ is 1. Therefore, $G$ can be decomposed into three [0,2]-factors with the required property.

Lemma 4.8 Let $G$ be a 3-edge-connected [3,5]-graph having a vertex $w$ of degree 3 . Then $G$ has a [0,2]-factorization $F_{1} \cup F_{2} \cup F_{3}$ with the property that if $d_{G}(x) \geq 4$, then $d_{F_{i}}(x) \geq 1$ for every $F_{i}$ and that $\mathrm{d}_{\mathrm{F}_{\mathrm{i}}}(\mathrm{w})=0$ for some $\mathrm{F}_{\mathbf{i}}$.

Proof Let $g$ and $f$ be functions on $V(G)$ defined by

$$
g(x)=\left\{\begin{array}{ll}
0 & \text { if } d_{G}(x)=3 \\
1 & \text { otherwise, }
\end{array} \quad \text { and } f(x)=\left\{\begin{array}{ll}
0 & \text { if } x=w \\
1 & \text { if } 3 \leq d_{G}(x) \leq 4 \\
2 & \text { otherwise. }
\end{array} \text { and } x \neq w\right.\right.
$$

We shall show that $G$ has $a(g, f)$-factor. Let $S, T \subset V(G)$ such that $S \cap T=\phi$ and $S u T \neq \phi$, and let $\left\{C_{1}, \ldots, C_{r}\right\}$ be the components of $G$ (SUT) which satisfy the conditions on $h(S, T)$, where $r=h(S, T)$. Then we have the following inequality by setting $\theta=1 / 4$ in (2.2).

$$
\delta(S, T) \geq\left(1-\frac{1}{4}\right) \sum_{t \in T} d_{G}(t)+\frac{1}{4} \sum_{s \in S} d_{G}(s)-\varepsilon-e(S, T)-r,
$$

where $\varepsilon=0$ or $3 / 4$ according as $w \notin S$ or $w \in S$. Hence

$$
\begin{aligned}
\delta(S, T) & \geq \sum_{i=1}^{r}\left\{\frac{3}{4} e\left(T, V\left(C_{i}\right)\right)+\frac{1}{4} e\left(S, V\left(C_{i}\right)\right)-1\right\}-\varepsilon \\
& \geq \sum_{i}\left\{\frac{1}{4} e\left(\operatorname{TUS}, V\left(C_{i}\right)\right)-1\right\}-\varepsilon .
\end{aligned}
$$

If $C_{i}$ does not contain $w, ~ t h e n ~ e\left(T \cup S, V\left(C_{i}\right)\right) \equiv 0(\bmod 2)$ (see (2.4)), and so $e\left(T u S, V\left(C_{i}\right)\right) \geq 4$. Therefore

$$
\frac{1}{4} e\left(\operatorname{TUS}, V\left(C_{i}\right)\right)-1 \geq 0
$$

If $C_{i}$ contains $w$, then $\varepsilon=0$ and $e\left(T \cup S, V\left(C_{i}\right)\right) \geq 3$, and so

$$
\frac{1}{4} e\left(T \cup S, V\left(C_{i}\right)\right)-1 \geq-\frac{1}{4} .
$$

Consequently, we obtain $\delta(S, T) \geq-3 / 4$, which implies $\delta(S, T) \geq 0$.

Furthermore, we can show that $\delta(\phi, \phi)=0$ by the fact that $G$ has at least one vertex $x$ with odd degree except $w$, for which $g(x)<f(x)$. Consequently, $G$ has a $(g, f)$-factor $F_{1}$. By Lemma $4.4, G-E\left(F_{1}\right)$ can be decomposed into two $[0,2]$-factors $F_{2}$ and $F_{3}$. Therefore we obtain a desired $[0,2]$-factorization $F_{1} \cup F_{2} \cup F_{3}$ of $G$.

We need some notation and definitions in order to prove Lemma 4.1. A graph having exactly two vertices and one or more edges is called a bond, and we denote the bond with $n$ edges by $B_{n}$ (Fig. 1). Let $v$ be a vertex of a graph $G$ and $w$ be a vertex of the bond $B_{n}$. Then $G+v w+B_{n}$ denotes the graph obtained from $G$ and $B_{n}$ by joining $V$ and $w$ by a new edge $v w$ (Fig. 2).

We shall prove the next lemma instead of Lemma 4.1, which includes Lemma 4.1 as a special case.


Figure 1. The bond $\mathrm{B}_{4^{\circ}} \quad$ Figure 2. $\mathrm{G}+\mathrm{vw}+\mathrm{B}_{3}$.

Lemma 4.9 Let $G$ be a connected [3,6]-graph with at most one vertex of degree 6 . Then $G$ has a $[0,2]$-factorization $F_{1} \cup F_{2} \cup F_{3}$ with the property that

$$
\begin{equation*}
\text { if } d_{G}(x) \geq 4 \text {, then } d_{F_{i}}(x) \geq 1 \text { for every } F_{i} \text {. } \tag{4.1}
\end{equation*}
$$

Proof We prove the lemma by induction on the number of vertices of a graph. Let $G$ be a connected $[3,6]$-graph with at most one vertex of degree 6. By Lemma 4.6, we may assume that $G$ is not 3-edge-connected.


Figure 3. $G$ and $H+v u+B_{3}$.


Figure 4. G and $H-v+x y$.

If $|V(G)|=2$ or 3 , then $G$ must be 3 -edge-connected, which is contrary to the assumption. Hence we may assume $|\mathrm{V}(\mathrm{G})| \geq 4$.

First suppose that $G$ is not 2-edge-connected. Then $G$ has a bridge $e=v w$, where $e \in E(G)$ and $v, w \in V(G)$ (Fig. 3). Let $H$ and $K$ be the components of $G-e$ such that $V \in V(H)$ and $w \in V(K)$ (Fig. 3). If $|V(H)| \geq 3$ and $|V(K)| \geq 3$, then $H^{\prime}=H+v u+B_{3}$ and $K^{\prime}=K+w u+B_{3}$ are both [3,6]-graphs, where $u$ is a vertex of $B_{3}$ (Fig. 3). By the induction hypothesis, $H^{\prime}$ and $K^{\prime}$ can be decomposed into three $[0,2]$-factors with the property (4,1), respectively. It is easy to obtain a desired [0,2]-factorization of $G$ from them. Therefore, we may assume $|V(K)|$ $=2$. Then $K$ is $B_{3}, B_{4}$ or $B_{5}$.

If $d_{H}(v) \geq 3$, then $H$ has a $[0,2]$-factorization with the property (4.1) by induction, and it is easy to obtain a desired [0,2]-factorization of $G$ from it. Hence we may assume $d_{H}(v)=2$. If two distinct vertices $x$ and $y$ of $H$ are adjacent to $v$, then $H-v+x y$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction (Fig. 4). So we can obtain a desired [0,2]-factorization of $G$ from it. We next Suppose that one vertex $x$ and $v$ are joined by two edges in $H$ (Fig. 5). Let $H+B_{3}$ be the graph obtained from $H$ by identifying


Figure 5. G and $\mathrm{H}+\mathrm{B}_{3}$.
$v$ and one of the vertices of $B_{3}$ (Fig. 5). Then, by the induction hypothesis, $\mathrm{H}_{\mathrm{H}} \mathrm{B}_{3}$ has a [0,2]-factorization with the property (4.1), and it is immediate to obtain a desired $[0,2]$-factorization of $G$ from it. Consequently, the proof is complete if $G$ is not 2-edge-connected.

We now deal with the case that $G$ is 2-edge-connected. Since $G$ is not 3-edge-connected, $G$ has a cutset (i.e. a minimal cut) with two edges. We consider three cases.

Case 1. $G$ has a cutset $\left\{e_{1}, e_{2}\right\}$ such that the ends of $e_{1}$ and those of $e_{2}$ are all distinct, where $e_{1}, e_{2} \in E(G)$.

Let $H$ and $K$ be the components of $G-\left\{e_{1}, e_{2}\right\}$, and let $e_{1}=u_{1} W_{1}$ and $e_{2}=\mathrm{u}_{2} \mathrm{w}_{2}$, where $\mathrm{u}_{1}, \mathrm{u}_{2} \in \mathrm{~V}(\mathrm{H}), \mathrm{u}_{1} \neq \mathrm{u}_{2}, \mathrm{w}_{1}, \mathrm{w}_{2} \in \mathrm{~V}(\mathrm{~K})$ and $\mathrm{w}_{1} \neq \mathrm{w}_{2}$. Then $H+u_{1} u_{2}$ and $K+w_{1} W_{2}$ have [0,2]-factorization with property (4.1) by induction. It is easy to obtain a desired [0,2]-factorization of $G$ from them.

Case 2. $G$ has a cutset $\left\{e_{1}, e_{2}\right\}$ such that the ends of $e_{1}$ and those of $e_{2}$ are the same (Fig. 6).


Figure 6.
Let $H$ be an arbitrary component of $G-\left\{e_{1}, e_{2}\right\}$, and $v$ be the end of $e_{1}$ and $e_{2}$ contained in $H$ (Fig. 6). We shall show that $H$ has a $[0,2]$-factorization $F_{1} U F_{2} U F_{3}$ with the property that

$$
\begin{aligned}
& F_{1}+e_{1}, F_{2}+e_{2} \text { and } F_{3} \text { are }[0,2]-f \text { actors of } H+<e_{1}, e_{2}> \\
& \text { and satisfy the condition (4.1) in } H+<e_{1}, e_{2}>,
\end{aligned}
$$

where $H+<e_{1}, e_{2}>$ is the subgraph of $G$ obtained from $H$ by adding
$e_{1}$ and $e_{2}$ together with their common end not contained in $H$. If this statement follows, then we can easily obtain a [0,2]-factorization of $G$ with the property (4.1) from a [0,2]-factorization with the property (4.2) of each component of $G-\left\{e_{1}, e_{2}\right\}$. We now prove the statement.

If $d_{G}(v) \geq 5$, then $d_{H}(v) \geq 3$ and so $H$ has a $[0,2]-f a c t o r i z a t i o n ~$ $F_{1} U F_{2} \cup F_{3}$ with the property (4.1) by induction. Since we may assume $\mathrm{d}_{\mathrm{F}_{1}}(\mathrm{v}) \leq 1$ and $\mathrm{d}_{\mathrm{F}_{2}}(\mathrm{v}) \leq 1$, these factors satisfy the required condition (4.2). If $d_{G}(v)=3$, then $G$ has a bridge, and so Case 1 occurs. Hence we may assume $d_{G}(v)=4$, and thus $d_{H}(v)=2$. If two distinct vertices $x$ and $y$ of $H$ are adjacent to $v$, then $H-v+x y$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction. It is easy to obtain a desired [0,2]-factorization of $H$ from them. We next assume that one vertex $x$ of $H$ and $v$ are joined by two edges (Fig. 7). Let ${ }_{H}+\mathrm{B}_{3}$ be the graph obtained from H and $\mathrm{B}_{3}$ by identifying $v$ and a vertex of $\mathrm{B}_{3}$ (Fig. 7). Then $\mathrm{H}_{\mathrm{H}} \mathrm{B}_{3}$ has a [0,2]-factorization with the property (4.1) by induction, and so we can obtain a desired [0,2]-factorization of $H$ from it. Consequently, each component of $G-\left\{e_{1}, e_{2}\right\}$ has a [0,2]-factorization satisfying the conditions (4.2), and we conclude that the proof of Case 2 is complete.


Figure 7. $G$ and $H+B_{3}$.
Case 3. For every cutset $\left\{e_{1}, e_{2}\right\}$ of $G, e_{1}$ and $e_{2}$ have exactly one common end (Fig. 8).


Figure 8. $G, K+w_{1} W_{2}$ and $K+<e_{1}, e_{2}>$.
Let $\left\{e_{1}, e_{2}\right\}$ be any cutset of $G$. Then we can write $e_{1}=v_{1}$ and $e_{2}=v w_{2}$, where $v, w_{1}, w_{2} \in V(G)$ and $w_{1} \neq w_{2}$. Let $H$ and $K$ be the components of $G-\left\{e_{1}, e_{2}\right\}$ such that $v \in V(H)$ and $w_{1}, w_{2} \in V(K)$. Note that $d_{H}(v) \geq 2$ as $G$ has no bridges. We first prove that if $\left\{e_{1}, e_{2}\right\}$ satisfies one of the following two conditions, then $G$ has $a[0,2]-$ factorization with the property (4.1) :
(i) $\mathrm{K}+\mathrm{w}_{1} \mathrm{~W}_{2}$ is 3-edge-connected (Fig. 8).
(ii) $d_{H}(v)=3$ and $H$ is a 3-edge-connected graph without vertices of degree 6 .

Suppose (i) hold. Then $K+\left\langle e_{1}, e_{2}\right\rangle$ (Fig. 8) can be decomposed into three $[0,2]$-factors which satisfy the conditions in Lemma 4.7. On the other hand, if $d_{H}(v) \geq 3$, then $H$ has a $[0,2]-f a c t o r i z a t i o n ~ w i t h ~$ the property (4.1) by induction, and so we can get a desired [0,2]factorization of $G$. If $d_{H}(v)=2$, then two distinct vertices $x$ and $y$ of $H$ are adjacent to $v$, and so $H-v+x y$ has a [0,2]-factorization with the property (4.1). It is easy to obtain a desired $[0,2]-$ factorization of $G$.

We next suppose that (ii) holds. Then $H$ can be decomposed into three $[0,2]$-factors which satisfy the conditions in Lemma 4.8. It follows that $K^{+w_{1}} W_{2}$ has a $[0,2]$-factorization with the property (4.1) by induction, and thus $G$ has a desired $[0,2]$-factorization.

We shall show that $G$ has a cutset $\left\{e_{1}, e_{2}\right\}$ which satisfies one
of the above conditions (i) and (ii). We can choose a cutset $\left\{e_{1}, e_{2}\right\}$ so that $H$ or $K+w_{1} w_{2}$ is 3-edge-connected. If $d_{H}(v)=2$, then we may assume without loss of generality that $K+w_{1} W_{2}$ is 3-edge-connected. Hence (i) follows.

Suppose $d_{H}(v)=3$. In this case we may assume that $K+w_{1} W_{2}$ is not 3-edge-connected and $H$ contains a unique vertex of $G$ with degree 6 ; (otherwise, $\left\{e_{1}, e_{2}\right\}$ satisfies (i) or (ii)). Let $\left\{f_{1}, f_{2}\right\}$ be any cutset of $\mathrm{K}_{\mathrm{H}} \mathrm{w}_{1} \mathrm{~W}_{2}$. If the ends of $\mathrm{f}_{1}$ and those of $\mathrm{f}_{2}$ are all distinct, then $G$ has such a cutset, which contradicts the assumption of Case 3 . If the ends of $f_{1}$ and those of $f_{2}$ are the same, then $f_{1}$ or $f_{2}$, say $f_{1}$, must be $w_{1} w_{2}$. So it follows that both $\left\{e_{1}, f_{2}\right\}$ and $\left\{e_{2}, f_{2}\right\}$ are cutsets of $G$ and $\mathrm{F}_{2}$ joins $\mathrm{w}_{1}$ and $\mathrm{W}_{2}$. Let T be the component of $G-\left\{e_{1}, f_{2}\right\}$ containing $W_{1}$. If $d_{T}\left(w_{1}\right) \geq 3$, then $T$ has $a[0,2]-$ factorization with the property (4.1) by induction. If $d_{T}\left(w_{1}\right)=2$, then two distinct vertices $t_{1}$ and $t_{2}$ are adjacent to $w_{1}$ in $T$, and thus $T-w_{1}+t_{1} t_{2}$ has a [0,2]-factorization with the property (4.1) by induction. Obviously, the component of $G-\left\{e_{2}, f_{2}\right\}$ containing $W_{2}$ has the same property mentioned above. Furthermore, $H$ also has a [0,2]-factorization with the property (4.1). Therefore, we can obtain a desired [0,2]-factorization of $G$ from them. Consequently, we may assume that for every cutset $\left\{f_{1}, f_{2}\right\}$ of $K+w_{1} W_{2}, f_{1}$ and $f_{2}$ have exactly one common end. Hence we can write $f_{1}=x y_{1}$ and $f_{2}=x y_{2}$, where $\mathrm{x}, \mathrm{y}_{1}, \mathrm{y}_{2} \in \mathrm{~V}\left(\mathrm{~K}+\mathrm{w}_{1} \mathrm{w}_{2}\right)$.

Choose a cutset $\left\{f_{1}=x y_{1}, f_{2}=x y_{2}\right\}$ of $K+w_{1} W_{2}$ so that the component
of $K+w_{1} w_{2}-\left\{f_{1}, f_{2}\right\}$ containing $x$ is 3-edge-connected or the graph obtained from the component of $K+w_{1} W_{2}-\left\{f_{1}, f_{2}\right\}$ containing $y_{1}$ and $y_{2}$ by adding a new edge $y_{1} y_{2}$ is 3-edge-connected. Since $H$ contains the vertex of degree 6 , we can choose such a cutset $\left\{f_{1}, f_{2}\right\}$ so that the 3-edge-connected component (or graph) has no vertices of degree 6. If $W_{1} W_{2} \notin\left\{\mathrm{f}_{1}, \mathrm{f}_{2}\right\}$, then $\left\{\mathrm{f}_{1}, \mathrm{f}_{2}\right\}$ is a cutset of $G$ which satisfies one of (i) and (ii). Hence we may assume $f_{1}=W_{1} W_{2}$ and $f_{2}=W_{1} y_{2}$, where ${ }^{w}{ }_{2} \neq y_{2}$. Then $\left\{e_{2}, y_{2}\right\}$ is a cutset of $G$ which satisfies the condition of Case 1 , a contradiction.

We finally assume $d_{H}(v)=4$ (i.e. $v$ is the vertex of $G$ with degree 6.). If ${ }^{K+w_{1}} W_{2}$ is 3-edge-connected, then we can obtain a $[0,2]$-factorization of $G$ with the property (4.1) by applying Lemma 4.8 to $K+<e_{1}, e_{2}>$. Hence we may assume that $K+w_{1} W_{2}$ is not 3 -edge-connected. In this case we can prove that $G$ has a desired [0,2]-factorization by the same argument in the case of $d_{H}(v)=3$. Consequently, Case 3 is proved.

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