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FUNCTIONS SATISFYING POINCARÉ'S MULTIPLICATION FORMULA

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Abstract

Poincaré proved existence of meromorphic function solution of a certain kind of system of multiplication formulae and claimed that the class of those functions is new. In this paper we exemplify Poincaré's meromorphic functions being truly new in a rigorous sense. We prove that those functions cannot be expressed rationally (nor algebraically) by solutions of linear difference equations, the exponential function e^x, the trigonometric functions $\cos x$ and $\sin x$, the Weierstrass function $\wp(x)$ and any other functions satisfying first order algebraic difference equations, where the transforming operator of the difference equations is one sending $y(x)$ to $y(2x)$, not to $y(x+1)$.

1. Introduction

In his paper [8] Poincaré studied meromorphic functions which satisfy a system of difference equations,

$$
\begin{cases}
\varphi_1(mx) = R_1(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)),\\
\varphi_2(mx) = R_2(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)),\\
\dots\\
\varphi_n(mx) = R_n(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)),
\end{cases}
\tag{1}
$$

where $R_1, \dots, R_n$ are rational functions over $\mathbb{C}$ and $m \in \mathbb{C}$ satisfies $|m| > 1$. He proved existence of meromorphic function solution of the system (1) under a certain condition and claimed that the class of these functions is new. Our question is whether the system (1) can define "new functions". In Section 5 we answer it by introducing an example which defines "new functions" in the sense explained later.

In this paper we study systems of difference equations of birational form such as

$$
y_2 y = \frac{A(y_1)}{B(y_1)}
\tag{2}
$$

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and

$$(3)\qquad \begin{cases} y_1 y = \dfrac{A(z)}{B(z)}, \\ z_1 z = \dfrac{C(y_1)}{D(y_1)}, \end{cases}$$

where A, B, C and D are polynomials over $\mathbb{C}(x)$, A and B are relatively prime and C and D are relatively prime. The symbols y_1 and z_1 denote the first transforms of y and z respectively such as $y(x+1)$, $y(mx)$ or $y(x^2)$ for $y(x)$, and y_2 denotes the second transform of y.

It would be worth to mention q-Painlevé equations which were presented by B. Grammaticos, A. Ramani and other authors (see [2, 9, 10, 11, 12]). According to the paper [12] by H. Sakai, 8 of 11 q-Painlevé equations have the form (2) or (3) with $\max\{\deg A, \deg B, \deg C, \deg D\} \leq 2$. For example,

$$q\text{-P}(A_7)\colon\ y_2 y = \frac{a(1-y_1)}{y_1^2}$$

and

$$q\text{-P}(A_3)\colon \begin{cases} y_1 y = b_7 b_8 \dfrac{(z-b_1)(z-b_2)}{(z-b_3)(z-b_4)}, \\ z_1 z = b_3 b_4 \dfrac{(y_1 - q b_5)(y_1 - q b_6)}{(y_1 - b_7)(y_1 - b_8)}, \end{cases} \qquad q = \frac{b_1 b_2 b_7 b_8}{b_3 b_4 b_5 b_6},$$

where $y = y(t)$, $y_1 = y(qt)$, $y_2 = y(q^2 t)$, $z = z(t)$, $z_1 = z(qt)$ and a and b_i's are parameters.

In Sections 3 and 4 we respectively show that the equation (2) is irreducible if $\max\{\deg A, \deg B\} > 2$ and that the system (3) is irreducible if

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} > 4.$$

Here the irreducibility is in the sense of the decomposable extension defined in the Definition 2 and implies that each transcendental function component of any vector solution cannot be expressed rationally (nor algebraically) by solutions of linear difference equations and solutions of first order algebraic difference equations. The precise explanation of the decomposable extension will be seen in Section 2 (cf. [4, 5, 6, 7]).

In Section 5 we introduce the theory of Poincaré and study the following system of difference equations,

$$(4)\qquad \begin{cases} y(mx) = C(y(x), z(x)), \quad C(Y, Z) = \dfrac{A(Z)}{Y + \alpha} - \alpha, \\ z(mx) = D(y(x), z(x)), \quad D(Y, Z) = \dfrac{B(C(Y, Z))}{Z + \beta} - \beta, \end{cases}$$

where $m \in \mathbb{C}$ with $|m| > 1$, $\alpha, \beta \in \mathbb{C}^\times$ and A and B are non-zero polynomials over $\mathbb{C}$ such that $\deg A \cdot \deg B > 4$, $A(0) = \alpha^2$, $B(0) = \beta^2$ and

$$\frac{A'(0)B'(0)}{\alpha\beta} = \frac{(m+1)^2}{m}.$$

A' and B' are the derivatives of A and B respectively. This system is derived from

$$\begin{cases}(y(mx)+\alpha)(y(x)+\alpha) = A(z(x)),\\ (z(mx)+\beta)(z(x)+\beta) = B(y(mx)).\end{cases}$$

For the system (4), we obtain a meromorphic function solution by the method of Poincaré. We show that the solution cannot be contained in any decomposable extension. For example, in the case $m = 2$, this implies that the solution cannot be expressed rationally (nor algebraically) by solutions of linear difference equations, the exponential function e^x, the trigonometric functions $\cos x$ and $\sin x$, the Weierstrass function $\wp(x)$ and any other functions satisfying first order algebraic difference equations. We note that the above functions respectively satisfy the following equations which can be regarded as first order algebraic difference equations,

$$\mathrm{e}^{2x} = (\mathrm{e}^x)^2,$$

$$\cos 2x = 2(\cos x)^2 - 1,$$

$$(\sin 2x)^2 = -4(\sin x)^4 + 4(\sin x)^2,$$

$$\wp(2x) = \frac{1}{16} \cdot \frac{16\wp(x)^4 + 8g_2\wp(x)^2 + 32g_3\wp(x) + g_2^2}{4\wp(x)^3 - g_2\wp(x) - g_3}.$$

NOTATION. Throughout the paper every field is of characteristic zero. When K is a field and τ is an isomorphism of K into itself, namely an injective endomorphism, the pair $\mathcal{K} = (K, \tau)$ is called a difference field. We call τ the (transforming) operator and K the underlying field. For a difference field $\mathcal{K}$, K often denotes its underlying field. For $a \in K$, an element $\tau^n a \in K$, $n \in \mathbb{Z}$, is called the n-th transform of a and is frequently denoted by a_n if it exists. If $\tau K = K$, we say that $\mathcal{K}$ is inversive. If $K/\tau K$ is algebraic, we say that $\mathcal{K}$ is almost inversive. For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$, $\mathcal{K}'/\mathcal{K}$ is called a difference field extension if K'/K is a field extension and $\tau'|_K = \tau$. In this case we say that $\mathcal{K}'$ is a difference overfield of $\mathcal{K}$ or $\mathcal{K}$ is a difference subfield of $\mathcal{K}'$. For brevity we sometimes use (K, τ') instead of $(K, \tau'|_K)$. We define a difference intermediate field in the proper way. Let $\mathcal{K}$ be a difference field, $\mathcal{L} = (L, \tau)$ a difference overfield of $\mathcal{K}$ and B a subset of L. The difference subfield $\mathcal{K}\langle B\rangle_{\mathcal{L}}$ of $\mathcal{L}$ is defined to be the difference field $(K(B, \tau B, \tau^2 B, \dots), \tau)$ and is denoted by $\mathcal{K}\langle B\rangle$ for brevity. A solution of a system of difference equations over $\mathcal{K}$ is defined to be a tuple of elements of some difference overfield of $\mathcal{K}$ which satisfies the equations (cf. the books [1, 3]).

2. Preliminaries

We begin this section by introducing the decomposable extension.

Lemma 1. *Let $\mathcal{L} = (L, \tau)$ be a difference field and $\overline{L}$ an algebraic closure of L. Then there is an isomorphism $\overline{\tau}$ of $\overline{L}$ into itself such that $\overline{\mathcal{L}} = (\overline{L}, \overline{\tau})$ is a difference overfield of $\mathcal{L}$. We call $\overline{\mathcal{L}}$ an algebraic closure of $\mathcal{L}$. If there is an almost inversive difference subfield $\mathcal{K} = (K, \tau|_K)$ of $\mathcal{L}$ such that* $\operatorname{tr.deg} L/K < \infty$, *then $\overline{\mathcal{L}}$ is inversive.*

Proof. There exists an isomorphism $\overline{\tau}$ of $\overline{L}$ onto $\overline{\tau L} \subset \overline{L}$ satisfying $\overline{\tau}|_L = \tau$ (cf. the book [13], Chapter II, §14, Theorem 33). Suppose that there is an almost inversive difference subfield $\mathcal{K} = (K, \tau|_K)$ of $\mathcal{L}$ such that $\operatorname{tr.deg} L/K < \infty$. Since $\operatorname{tr.deg} L/K = \operatorname{tr.deg} \tau L/\tau K$, the extension $L/\tau L$ is algebraic, which implies $\overline{\tau L} = \overline{L}$. Therefore $\overline{\tau}\overline{L} = \overline{L}$. □

The following is the definition of the decomposable extension. Although it is different from the one in the preceding paper [6], they are equivalent.

DEFINITION 2 (decomposable extension). Let $\mathcal{K}$ be a difference field, and $\mathcal{L}$ an algebraically closed difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$. We define decomposable extensions by induction on $\operatorname{tr.deg} \mathcal{L}/\mathcal{K}$.

(i) If $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \leq 1$, then $\mathcal{L}/\mathcal{K}$ is decomposable.

(ii) When $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \geq 2$, $\mathcal{L}/\mathcal{K}$ is decomposable if there exist difference fields $\mathcal{U}$, $\mathcal{E}$ and $\mathcal{M}$ such that $\mathcal{U}$ is an algebraically closed difference overfield of $\mathcal{L}$, $\mathcal{E}$ is a difference intermediate field of $\mathcal{U}/\mathcal{K}$, $\operatorname{tr.deg} \mathcal{E}/\mathcal{K} < \infty$, $\mathcal{E}$ and $\mathcal{L}$ are free over $\mathcal{K}$, $\mathcal{M}$ is a difference intermediate field of $\mathcal{L}\mathcal{E}/\mathcal{E}$, $\operatorname{tr.deg} \mathcal{L}\mathcal{E}/\mathcal{M} \geq 1$, $\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$ and both $\overline{\mathcal{L}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{L}\mathcal{E}}$ and $\overline{\mathcal{M}}$ are the algebraic closures of $\mathcal{L}\mathcal{E}$ and $\mathcal{M}$ in $\mathcal{U}$ respectively.

REMARK. For any first order algebraic difference equation over a difference field $\mathcal{K}$, each of its solutions, say f, generates a decomposable extension $\overline{\mathcal{K}\langle f\rangle}/\mathcal{K}$, where $\overline{\mathcal{K}\langle f\rangle}$ is any algebraic closure of $\mathcal{K}\langle f\rangle$. In fact, f and its first transform f_1 satisfy $P(f, f_1) = 0$ for some non-zero polynomial $P(X, Y) \in K[X, Y]$, which implies that f_{i+1} is algebraic over $\tau^i K(f_i) \subset K(f_i)$ for all $i \geq 0$. Hence we conclude that

$$\operatorname{tr.deg} \overline{\mathcal{K}\langle f\rangle}/\mathcal{K} = \operatorname{tr.deg} \mathcal{K}\langle f\rangle/\mathcal{K} = \operatorname{tr.deg} \mathcal{K}(f)/\mathcal{K} \leq 1.$$

We also have the following propositions.

Proposition 3 (Corollary 8 in [6]). *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1}y_{n-1} + \cdots + a_0 y = b \tag{5}$$

be a linear difference equation over $\mathcal{K}$, *where* $n \geq 1$, *and* f *a solution of* (5). *Then* $\overline{\mathcal{K}\langle f\rangle}/\mathcal{K}$ *is decomposable for any algebraic closure* $\overline{\mathcal{K}\langle f\rangle}$ *of* $\mathcal{K}\langle f\rangle$.

Proposition 4 (Proposition 3 in [6])**.** *Let* $\mathcal{K}$ *be a difference field, and* $\mathcal{L}/\mathcal{K}$ *and* $\mathcal{N}/\mathcal{L}$ *be decomposable extensions. Then* $\mathcal{N}/\mathcal{K}$ *is decomposable.*

Putting together the facts mentioned above, we find that, for any tower of difference field extensions generated by solutions of linear difference equations and solutions of first order algebraic difference equations, its algebraic closure is a decomposable extension.

The following two lemmas are used in proofs of irreducibility. The former is a modification of the corresponding lemma concerned only with single equations in the paper [6]. The latter is what we use in the following sections.

Lemma 5. *Let* $\mathcal{K}$ *be a difference field,* $\mathcal{D}$ *a decomposable extension of* $\mathcal{K}$ *and* $B \subset D$. *Suppose that for any difference overfield* $\mathcal{L}$ *of* $\mathcal{K}$ *of finite transcendence degree and for any difference overfield* $\mathcal{U}$ *of* $\mathcal{L}$ *such that* $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$, *the following holds,*

$$\text{tr.deg}\, \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \textit{any } f \in B \textit{ is algebraic over } L.$$

Then any $f \in B$ *is algebraic over* K.

Proof. The proof is almost the same as the proof of Lemma 9 in [6]. □

Lemma 6. *Let* $\mathcal{K}$ *be an almost inversive difference field,* $\mathcal{D}$ *a decomposable extension of* $\mathcal{K}$ *and* $B \subset D$. *Suppose that for any inversive difference overfield* $\mathcal{L}$ *of* $\mathcal{K}$ *and for any difference overfield* $\mathcal{U}$ *of* $\mathcal{L}$ *with* $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$, *the following holds,*

$$\text{tr.deg}\, \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \textit{any } f \in B \textit{ is algebraic over } L.$$

Then any $f \in B$ *is algebraic over* K.

Proof. Let $\mathcal{L}$ be a difference overfield of $\mathcal{K}$ of finite transcendence degree and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$.

We show

$$\text{tr.deg}\, \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \text{any } f \in B \text{ is algebraic over } L.$$

Suppose $\text{tr.deg}\, \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1$. Let $\overline{\mathcal{U}}$ an algebraic closure of $\mathcal{U}$ and $\overline{\mathcal{L}}$ the algebraic closure of $\mathcal{L}$ in $\overline{\mathcal{U}}$. Note that $\overline{\mathcal{L}}$ is inversive. We find

$$\text{tr.deg}\, \overline{\mathcal{L}}\langle B\rangle_{\overline{\mathcal{U}}}/\overline{\mathcal{L}} = \text{tr.deg}\, \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1,$$

which implies that any $f \in B$ is algebraic over $\overline{L}$. Therefore any $f \in B$ is algebraic over L.

By Lemma 5 we conclude that any $f \in B$ is algebraic over K. □

We also need the following.

Lemma 7. *Let L be a field, $m, n \in \mathbb{Z}_{\geq 1}$ and $A, B, P, R, R', S, S' \in L[X] \setminus \{0\}$ polynomials over L such that A and B are relatively prime,*

$$\max\{\deg R, \deg R', \deg S, \deg S'\} \leq n,$$

$A^m R = PS$ and $B^m R' = PS'$. Then $\deg A \leq 2n/m$ and $\deg B \leq 2n/m$.

Proof. It is sufficient to prove that $\deg A \leq 2n/m$. For polynomials $C, D \in L[X] \setminus \{0\}$ we let (C, D) denote the monic greatest common divisor of C and D. Put $C = (A^m, S)$. From $A^m R = PS$ we obtain

$$(A^m/C)R = P(S/C), \quad A^m/C, S/C \in L[X].$$

Since A^m/C and S/C are relatively prime, we find $(A^m/C) \mid P$, which implies

$$\begin{aligned} \deg(A^m, P) &\geq \deg \frac{A^m}{C} = m \deg A - \deg C \\ &\geq m \deg A - \deg S \geq m \deg A - n. \end{aligned}$$

We obtain $(A^m, P) \mid B^m R'$ from $B^m R' = PS'$ and $(A^m, P) \mid P$. Since (A^m, P) and B^m are relatively prime, we find $(A^m, P) \mid R'$, which implies

$$\deg(A^m, P) \leq \deg R' \leq n.$$

Therefore we conclude that $\deg A \leq 2n/m$. □

3. Single equation of birational form

In this section we study irreducibility of the single equation,

$$y_2 y = \frac{A(y_1)}{B(y_1)},$$

where A and B are polynomials.

Proposition 8. *Let $\mathcal{L} = (L, \tau)$ be an inversive difference field and f a solution of the equation over $\mathcal{L}$,*

$$B(y_1)y_2y = A(y_1),$$

where $A, B \in L[X] \setminus \{0\}$ are polynomials over L such that A and B are relatively prime and $\max\{\deg A, \deg B\} > 2$. *Then it follows that*

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Proof. To obtain $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \neq 1$ we assume $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$. Then f_i is transcendental over L for any $i \geq 0$. Since it follows that $\operatorname{tr.deg} L(f, f_1)/L = 1$, there exists an irreducible polynomial F over L,

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} a_{ij} Y^i Y_1^j \in L[Y, Y_1] \setminus \{0\}, \quad a_{ij} \in L,$$

such that $F(f, f_1) = 0$, $n_0 = \deg_Y F \geq 1$, $n_1 = \deg_{Y_1} F \geq 1$ and $a_{n_0 n_1} \in \{0, 1\}$. Put

$$F_1 = (YB(Y_1))^{n_1} F^*\left(Y_1, \frac{A(Y_1)}{YB(Y_1)}\right),$$
$$F_0 = (Y_1B(Y))^{n_0} F\left(\frac{A(Y)}{Y_1B(Y)}, Y\right),$$

where $F^* = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \tau(a_{ij}) Y^i Y_1^j$. It is seen that $F_1, F_0 \in L[Y, Y_1] \setminus \{0\}$. We find

$$F_1(f, f_1) = (fB(f_1))^{n_1} F^*\left(f_1, \frac{A(f_1)}{fB(f_1)}\right)$$
$$= (fB(f_1))^{n_1} F^*(f_1, f_2) = 0$$

and

$$F_0(f_1, f_2) = (f_2B(f_1))^{n_0} F\left(\frac{A(f_1)}{f_2B(f_1)}, f_1\right)$$
$$= (f_2B(f_1))^{n_0} F(f, f_1) = 0,$$

which imply $F \mid F_1$ and $F^* \mid F_0$. Therefore we obtain

$$n_0 = \deg_Y F \leq \deg_Y F_1 \leq n_1 = \deg_{Y_1} F^* \leq \deg_{Y_1} F_0 \leq n_0,$$

and so $n_0 = n_1$. Put $n = n_0 = n_1 \geq 1$.

Let $P \in L[Y, Y_1] \setminus \{0\}$ be a polynomial satisfying $F_1 = PF$. We find $P \in L[Y_1]$ by $\deg_Y P = \deg_Y F_1 - \deg_Y F = 0$. We have

$$\begin{aligned} F_1 &= (YB(Y_1))^n \sum_{i=0}^{n} \sum_{j=0}^{n} \tau(a_{ij}) Y_1^i \left(\frac{A(Y_1)}{YB(Y_1)} \right)^j \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n} \tau(a_{ij}) Y_1^i (YB(Y_1))^{n-j} A(Y_1)^j \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n} \tau(a_{i,n-j}) Y_1^i A(Y_1)^{n-j} B(Y_1)^j Y^j \\ &= \sum_{j=0}^{n} \left\{ A(Y_1)^{n-j} B(Y_1)^j \sum_{i=0}^{n} \tau(a_{i,n-j}) Y_1^i \right\} Y^j \end{aligned}$$

and

$$PF = P \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} Y^i Y_1^j = P \sum_{j=0}^{n} \sum_{i=0}^{n} a_{ji} Y^j Y_1^i = \sum_{j=0}^{n} \left\{ P \sum_{i=0}^{n} a_{ji} Y_1^i \right\} Y^j.$$

From $F_1 = PF$ we obtain

$$A(Y_1)^n \sum_{i=0}^{n} \tau(a_{in}) Y_1^i = P \sum_{i=0}^{n} a_{0i} Y_1^i \quad (\neq 0), \tag{6}$$

$$B(Y_1)^n \sum_{i=0}^{n} \tau(a_{i0}) Y_1^i = P \sum_{i=0}^{n} a_{ni} Y_1^i \quad (\neq 0). \tag{7}$$

By Lemma 7 we find $\deg A \leq 2$ and $\deg B \leq 2$, which imply

$$\max\{\deg A, \deg B\} \leq 2,$$

a contradiction. Therefore we conclude that $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} \neq 1$, which yields

$$\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L,$$

the required. □

Theorem 9. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{N}$ a decomposable extension of $\mathcal{K}$ and $f \in N$ a solution in $\mathcal{N}$ of the equation over $\mathcal{K}$,*

$$B(y_1) y_2 y = A(y_1),$$

where $A, B \in K[X] \setminus \{0\}$ *are polynomials over* K *such that* A *and* B *are relatively prime and* $\max\{\deg A, \deg B\} > 2$. *Then* f *is algebraic over* K.

Proof. Let $\mathcal{L}$ be an inversive difference overfield of $\mathcal{K}$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle f\rangle_{\mathcal{N}} \subset \mathcal{U}$. Then by Proposition 8 we obtain

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Therefore we find that f is algebraic over K by Lemma 6. □

4. System of two equations of birational form

In this section we study irreducibility of the system of equations,

$$\begin{cases} y_1 y = \dfrac{A(z)}{B(z)}, \\ z_1 z = \dfrac{C(y_1)}{D(y_1)}, \end{cases}$$

where A, B, C and D are polynomials.

Lemma 10. *Let* $\mathcal{L} = (L, \tau)$ *be an inversive difference field and* $(y, z) = (f, g)$ *a solution of the system of equations over* $\mathcal{L}$,

$$\begin{cases} B(z)y_1 y = A(z), \\ D(y_1)z_1 z = C(y_1), \end{cases}$$

where $A, B, C, D \in L[X]\setminus\{0\}$ *are polynomials over* L *such that* A *and* B *are relatively prime,* C *and* D *relatively prime,* $\deg AB \geq 1$ *and* $\deg CD \geq 1$. *Then*

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = \operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L} = \operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L}.$$

If we suppose $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1$ *then we find that there are polynomials over* L *with indeterminates* Y *and* Z,

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} Y^i Z^j \in L[Y, Z] \setminus \{0\}, \quad \alpha_{ij} \in L,$$

$$G = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i Z^j \in L[Y, Z] \setminus \{0\}, \quad \beta_{ij} \in L,$$

$P \in L[Z] \setminus \{0\}$ and $Q \in L[Y] \setminus \{0\}$ such that $F(f, g) = G(f_1, g) = 0$, both F and G are irreducible,

$$n_0 = \deg_Y F = \deg_Y G \geq 1,$$
$$n_1 = \deg_Z F = \deg_Z G \geq 1,$$
$$\alpha_{n_0 n_1}, \beta_{n_0 n_1} \in \{0, 1\},$$

$$\sum_{i=0}^{n_0} \left\{ A(Z)^{n_0 - i} B(Z)^i \sum_{j=0}^{n_1} \alpha_{n_0 - i, j} Z^j \right\} Y^i = \sum_{i=0}^{n_0} \left\{ P \sum_{j=0}^{n_1} \beta_{ij} Z^j \right\} Y^i \tag{8}$$

and

$$\sum_{j=0}^{n_1} \left\{ C(Y)^{n_1 - j} D(Y)^j \sum_{i=0}^{n_0} \beta_{i, n_1 - j} Y^i \right\} Z^j = \sum_{j=0}^{n_1} \left\{ Q \sum_{i=0}^{n_0} \tau(\alpha_{ij}) Y^i \right\} Z^j. \tag{9}$$

Proof. (1) Firstly, we prove

$$\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = \operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L}.$$

g is a zero of the polynomial $f_1 f B(X) - A(X) \in L(f, f_1)[X]$ because $B(g) f_1 f = A(g)$. If we assume $f_1 f B(X) - A(X) = 0$ then we find that $A(X)$ and $B(X)$ has a common divisor in $L(f, f_1)[X]$, a contradiction. Therefore we have $f_1 f B(X) - A(X) \neq 0$, which implies that g is algebraic over $L(f, f_1)$. We obtain the required from

$$\begin{aligned} \operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L} &= \operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L}\langle f \rangle + \operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} \\ &= \operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L}. \end{aligned}$$

(2) Secondly, we prove

$$\operatorname{tr.deg} \mathcal{L}\langle g \rangle / \mathcal{L} = \operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L}.$$

f_1 is a zero of the polynomial $g_1 g D(X) - C(X) \in L(g, g_1)[X]$ because $D(f_1) g_1 g = C(f_1)$. If we assume $g_1 g D(X) - C(X) = 0$ then we find that $C(X)$ and $D(X)$ has a common divisor in $L(g, g_1)[X]$, a contradiction. Therefore we have $g_1 g D(X) - C(X) \neq 0$, which implies that f_1 is algebraic over $L(g, g_1)$.

We may suppose that f is transcendental over L because we have

$$\operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L} = \operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L}\langle g \rangle + \operatorname{tr.deg} \mathcal{L}\langle g \rangle / \mathcal{L}.$$

Since $\mathcal{L}$ is inversive, we find that f_1 is also transcendental over L, which implies that g is transcendental over L. Then from $B(g) f_1 f = A(g)$ we obtain

$$f = \frac{A(g)}{B(g) f_1} \in L(f_1, g),$$

and so

$$\begin{aligned}\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} &= \operatorname{tr.deg} \mathcal{L}\langle f_1, g\rangle/\mathcal{L}\\ &= \operatorname{tr.deg} \mathcal{L}\langle f_1, g\rangle/\mathcal{L}\langle g\rangle + \operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L}\\ &= \operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L},\end{aligned}$$

which yields the required.

(3) Finally we suppose $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1$. By (1) and (2) we find that f_i and g_i are transcendental over L for all $i \geq 0$, where note that $\mathcal{L}$ is inversive. Since it follows that $\operatorname{tr.deg} L(f, g)/L = 1$, we find that there exists an irreducible polynomial $F \in L[Y, Z] \setminus \{0\}$ over L,

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} Y^i Z^j, \quad \alpha_{ij} \in L,$$

such that $F(f, g) = 0$, $n_0 = \deg_Y F \geq 1$, $n_1 = \deg_Z F \geq 1$, and $\alpha_{n_0 n_1} \in \{0, 1\}$. By $\operatorname{tr.deg} L(f_1, g)/L = 1$ there exists an irreducible polynomial $G \in L[Y, Z] \setminus \{0\}$,

$$G = \sum_{i=0}^{n_2} \sum_{j=0}^{n_3} \beta_{ij} Y^i Z^j, \quad \beta_{ij} \in L,$$

such that $G(f_1, g) = 0$, $n_2 = \deg_Y G$, $n_3 = \deg_Z G$ and $\beta_{n_2 n_3} \in \{0, 1\}$.

For any $P = \sum_i p_i X^i \in L[X]$ we define P^* as $P^* = \sum_i \tau(p_i) X^i$, and for any $P = \sum_{i,j} p_{ij} Y^i Z^j$, we define P^* as $P^* = \sum_{i,j} \tau(p_{ij}) Y^i Z^j$. Put

$$\begin{aligned}F_1 &= \{Z D(Y)\}^{n_1} F^*\left(Y, \frac{C(Y)}{Z D(Y)}\right) \in L[Y, Z] \setminus \{0\},\\ G_1 &= \{Y B^*(Z)\}^{n_2} G^*\left(\frac{A^*(Z)}{Y B^*(Z)}, Z\right) \in L[Y, Z] \setminus \{0\}.\end{aligned}$$

Then we have

$$\begin{aligned}F_1(f_1, g) &= \{g D(f_1)\}^{n_1} F^*\left(f_1, \frac{C(f_1)}{g D(f_1)}\right)\\ &= \{g D(f_1)\}^{n_1} F^*(f_1, g_1) = 0\end{aligned}$$

and

$$\begin{aligned}G_1(f_1, g_1) &= \{f_1 B^*(g_1)\}^{n_2} G^*\left(\frac{A^*(g_1)}{f_1 B^*(g_1)}, g_1\right)\\ &= \{f_1 B^*(g_1)\}^{n_2} G^*(f_2, g_1) = 0,\end{aligned}$$

which imply $G \mid F_1$ and $F^* \mid G_1$ respectively. Put

$$F_0 = \{YB(Z)\}^{n_0} F\left(\frac{A(Z)}{YB(Z)}, Z\right) \in L[Y, Z] \setminus \{0\},$$

$$G_0 = \{ZD(Y)\}^{n_3} G\left(Y, \frac{C(Y)}{ZD(Y)}\right) \in L[Y, Z] \setminus \{0\}.$$

Then we have

$$F_0(f_1, g) = \{f_1 B(g)\}^{n_0} F\left(\frac{A(g)}{f_1 B(g)}, g\right) = \{f_1 B(g)\}^{n_0} F(f, g) = 0$$

and

$$G_0(f_1, g_1) = \{g_1 D(f_1)\}^{n_3} G\left(f_1, \frac{C(f_1)}{g_1 D(f_1)}\right) = \{g_1 D(f_1)\}^{n_3} G(f_1, g) = 0,$$

which imply $G \mid F_0$ and $F^* \mid G_0$ respectively. Therefore we find $n_0 = n_2$ and $n_1 = n_3$ by

$$n_0 = \deg_Y F^* \le \deg_Y G_1 \le n_2 = \deg_Y G \le \deg_Y F_0 \le n_0$$

and

$$n_1 = \deg_Z F^* \le \deg_Z G_0 \le n_3 = \deg_Z G \le \deg_Z F_1 \le n_1.$$

Let $P, Q \in L[Y, Z] \setminus \{0\}$ be polynomials such that $F_0 = PG$ and $G_0 = QF^*$. Since we have

$$\deg_Y P = \deg_Y F_0 - \deg_Y G = 0,$$

and

$$\deg_Z Q = \deg_Z G_0 - \deg_Z F^* = 0,$$

we obtain $P \in L[Z]$ and $Q \in L[Y]$. Calculate F_0 and PG as follows,

$$\begin{aligned} F_0 &= \{YB(Z)\}^{n_0} \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} \left(\frac{A(Z)}{YB(Z)}\right)^i Z^j \\ &= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} A(Z)^i Z^j (YB(Z))^{n_0-i} \\ &= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{n_0-i,j} A(Z)^{n_0-i} Z^j (YB(Z))^i \\ &= \sum_{i=0}^{n_0} \left\{ A(Z)^{n_0-i} B(Z)^i \sum_{j=0}^{n_1} \alpha_{n_0-i,j} Z^j \right\} Y^i, \end{aligned}$$

$$PG = P \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i Z^j = \sum_{i=0}^{n_0} \left\{ P \sum_{j=0}^{n_1} \beta_{ij} Z^j \right\} Y^i.$$

Then we obtain the equation (8). To obtain the equation (9) we calculate G_0 and QF^* as follows,

$$\begin{aligned}
G_0 &= \{ZD(Y)\}^{n_1} \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i \left(\frac{C(Y)}{ZD(Y)} \right)^j \\
&= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i C(Y)^j (ZD(Y))^{n_1-j} \\
&= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{i,n_1-j} Y^i C(Y)^{n_1-j} (ZD(Y))^j \\
&= \sum_{j=0}^{n_1} \left\{ C(Y)^{n_1-j} D(Y)^j \sum_{i=0}^{n_0} \beta_{i,n_1-j} Y^i \right\} Z^j, \\
QF^* &= Q \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \tau(\alpha_{ij}) Y^i Z^j = \sum_{j=0}^{n_1} \left\{ Q \sum_{i=0}^{n_0} \tau(\alpha_{ij}) Y^i \right\} Z^j. \qquad \square
\end{aligned}$$

Proposition 11. *Let $\mathcal{L} = (L, \tau)$ be an inversive difference field and $(y, z) = (f, g)$ be a solution of the system of equations over $\mathcal{L}$,*

$$\begin{cases} B(z) y_1 y = A(z), \\ D(y_1) z_1 z = C(y_1), \end{cases}$$

where $A, B, C, D \in L[X] \setminus \{0\}$ are polynomials over L such that A and B are relatively prime, C and D relatively prime and

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} > 4.$$

Then it follows that

$$\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle / \mathcal{L} \le 1 \Rightarrow f \text{ and } g \text{ are algebraic over } L.$$

Proof. To obtain $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle / \mathcal{L} \neq 1$ we assume $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle / \mathcal{L} = 1$. By Lemma 10 there exist polynomials $F, G, P, Q \in L[Y, Z] \setminus \{0\}$ over L such that

$$\begin{aligned}
F &= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} Y^i Z^j, \quad \alpha_{ij} \in L, \\
G &= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i Z^j, \quad \beta_{ij} \in L,
\end{aligned}$$

$P \in L[Z],\ Q \in L[Y],\ F(f, g) = G(f_1, g) = 0,\ F$ and G are irreducible,

$$n_0 = \deg_Y F = \deg_Y G \geq 1,$$
$$n_1 = \deg_Z F = \deg_Z G \geq 1,$$
$$\alpha_{n_0 n_1}, \beta_{n_0 n_1} \in \{0, 1\},$$

$$\sum_{i=0}^{n_0} \left\{ A(Z)^{n_0 - i} B(Z)^i \sum_{j=0}^{n_1} \alpha_{n_0 - i, j} Z^j \right\} Y^i = \sum_{i=0}^{n_0} \left\{ P \sum_{j=0}^{n_1} \beta_{ij} Z^j \right\} Y^i \tag{10}$$

and

$$\sum_{j=0}^{n_1} \left\{ C(Y)^{n_1 - j} D(Y)^j \sum_{i=0}^{n_0} \beta_{i, n_1 - j} Y^i \right\} Z^j = \sum_{j=0}^{n_1} \left\{ Q \sum_{i=0}^{n_0} \tau(\alpha_{ij}) Y^i \right\} Z^j. \tag{11}$$

From the equation (10) we obtain the following two equations,

$$A(Z)^{n_0} \sum_{j=0}^{n_1} \alpha_{n_0 j} Z^j = P \sum_{j=0}^{n_1} \beta_{0j} Z^j \quad (\neq 0), \tag{12}$$

$$B(Z)^{n_0} \sum_{j=0}^{n_1} \alpha_{0j} Z^j = P \sum_{j=0}^{n_1} \beta_{n_0 j} Z^j \quad (\neq 0). \tag{13}$$

From the equation (11) we obtain the following two equations,

$$C(Y)^{n_1} \sum_{i=0}^{n_0} \beta_{i n_1} Y^i = Q \sum_{i=0}^{n_0} \tau(\alpha_{i0}) Y^i \quad (\neq 0), \tag{14}$$

$$D(Y)^{n_1} \sum_{i=0}^{n_0} \beta_{i0} Y^i = Q \sum_{i=0}^{n_0} \tau(\alpha_{i n_1}) Y^i \quad (\neq 0). \tag{15}$$

By Lemma 7 we find that

$$\deg A \leq \frac{2n_1}{n_0}, \quad \deg B \leq \frac{2n_1}{n_0},$$
$$\deg C \leq \frac{2n_0}{n_1}, \quad \deg D \leq \frac{2n_0}{n_1},$$

which imply

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} \leq \frac{2n_1}{n_0} \cdot \frac{2n_0}{n_1} = 4,$$

a contradiction. Therefore we conclude $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle / \mathcal{L} \neq 1$, which yields

$$\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle / \mathcal{L} \leq 1 \Rightarrow f \text{ and } g \text{ are algebraic over } L,$$

the required. □

Theorem 12. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{N}$ a decomposable extension of $\mathcal{K}$ and $(y, z) = (f, g)$ a solution in $\mathcal{N}$ of the system of equations over $\mathcal{K}$,*

$$\begin{cases} B(z)y_1y = A(z), \\ D(y_1)z_1z = C(y_1), \end{cases}$$

where $A, B, C, D \in K[X]\backslash\{0\}$ are polynomials over K such that A and B are relatively prime, C and D relatively prime and

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} > 4.$$

Then f and g are algebraic over K.

Proof. Let $\mathcal{L}$ be an inversive difference overfield of $\mathcal{K}$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle f, g\rangle_{\mathcal{N}} \subset \mathcal{U}$. By Proposition 11 we obtain

$$\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle_{\mathcal{U}}/\mathcal{L} \le 1 \Rightarrow f \text{ and } g \text{ are algebraic over } L.$$

Therefore by Lemma 6 we conclude that f and g are algebraic over K. □

5. "New" functions of Poincaré

In this section we exemplify Poincaré's functions being truly new in a rigorous sense. First of all, we introduce the theory of Poincaré. He studied the following system of difference equations [8],

$$\begin{cases} \varphi_1(mx) = R_1(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)), \\ \varphi_2(mx) = R_2(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)), \\ \cdots \\ \varphi_n(mx) = R_n(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)), \end{cases} \tag{16}$$

where $R_1, R_2, \dots, R_n \in \mathbb{C}(X_1, X_2, \dots, X_n)$ are rational functions over $\mathbb{C}$ and $m \in \mathbb{C}$ satisfies $|m| > 1$. We additionally suppose $R_i(0, 0, \dots, 0) = 0$ for all i and put

$$\beta_{ik} = \frac{\partial R_i}{\partial X_k}(0, 0, \dots, 0).$$

We define

$$F(s) = \begin{pmatrix} \beta_{11} - s & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} - s & \dots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} - s \end{pmatrix}$$

and suppose that $\det F(m) = 0$ and $\det F(m^p) \neq 0$ for all $p \in \mathbb{Z}_{\geq 2}$. We take $(\alpha_{11}, \alpha_{21}, \dots, \alpha_{n1}) \in \mathbb{C}^n \setminus \{0\}$ such that

$$F(m)\begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{n1} \end{pmatrix} = 0$$

and inductively define $\alpha_{1p}, \alpha_{2p}, \dots, \alpha_{np}$ for all $p \geq 2$ by

$$\begin{pmatrix} \alpha_{1p} \\ \alpha_{2p} \\ \vdots \\ \alpha_{np} \end{pmatrix} = -F(m^p)^{-1}\begin{pmatrix} \gamma_{1p} \\ \gamma_{2p} \\ \vdots \\ \gamma_{np} \end{pmatrix},$$

where γ_{ip} is the coefficient of x^p of

$$R_i\left(\sum_{k=1}^{p-1} \alpha_{1k}x^k, \dots, \sum_{k=1}^{p-1} \alpha_{nk}x^k\right).$$

Put $\varphi_i^0 = \sum_{k=1}^{\infty} \alpha_{ik}x^k$ for all $i = 1, 2, \dots, n$. Then the following theorem holds.

Theorem 13 (Poincaré [8]). *$\varphi_1^0, \varphi_2^0, \dots, \varphi_n^0$ satisfy the system* (16) *as formal power series and converge. Moreover there exist meromorphic functions over $\mathbb{C}$, $\varphi_1, \varphi_2, \dots, \varphi_n$, which satisfy the system* (16) *and their power series representations at* 0 *coincide with $\varphi_1^0, \varphi_2^0, \dots, \varphi_n^0$ respectively.*

From here, we deal with the following system of equations,

$$\begin{cases} y(mx) = C(y(x), z(x)), \quad C(Y, Z) = \dfrac{A(Z)}{Y + \alpha} - \alpha, \\ z(mx) = D(y(x), z(x)), \quad D(Y, Z) = \dfrac{B(C(Y, Z))}{Z + \beta} - \beta, \end{cases} \tag{17}$$

where $m \in \mathbb{C}$ with $|m| > 1$, $\alpha, \beta \in \mathbb{C}^\times$ and A and B are non-zero polynomials over $\mathbb{C}$ such that $\deg A \cdot \deg B > 4$, $A(0) = \alpha^2$, $B(0) = \beta^2$ and

$$\frac{A'(0)B'(0)}{\alpha\beta} = \frac{(m+1)^2}{m}.$$

A' and B' are the derivatives of A and B respectively. Any solution satisfies

$$\begin{cases} (y(mx) + \alpha)(y(x) + \alpha) = A(z(x)), \\ (z(mx) + \beta)(z(x) + \beta) = B(y(mx)). \end{cases} \tag{18}$$

Proposition 14. *The system* (17) *satisfies Poincaré's conditions.*

Proof. We find $C(0,0) = D(0,0) = 0$,

$$\begin{aligned} \frac{\partial C}{\partial Y}(Y,Z) &= -\frac{A(Z)}{(Y+\alpha)^2}, & \frac{\partial C}{\partial Y}(0,0) &= -1, \\ \frac{\partial C}{\partial Z}(Y,Z) &= \frac{A'(Z)}{Y+\alpha}, & \frac{\partial C}{\partial Z}(0,0) &= \frac{A'(0)}{\alpha}, \\ \frac{\partial D}{\partial Y}(Y,Z) &= \frac{B'(C(Y,Z))(\partial C/\partial Y)(Y,Z)}{Z+\beta}, & \frac{\partial D}{\partial Y}(0,0) &= -\frac{B'(0)}{\beta}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial D}{\partial Z}(Y,Z) &= \frac{B'(C(Y,Z))(\partial C/\partial Z)(Y,Z)(Z+\beta) - B(C(Y,Z))}{(Z+\beta)^2}, \\ \frac{\partial D}{\partial Z}(0,0) &= \frac{A'(0)B'(0)}{\alpha\beta} - 1. \end{aligned}$$

Hence we have

$$F(s) = \begin{pmatrix} -1-s & \dfrac{A'(0)}{\alpha} \\ -\dfrac{B'(0)}{\beta} & \dfrac{A'(0)B'(0)}{\alpha\beta} - 1 - s \end{pmatrix}.$$

Its determinant is

$$\begin{aligned} \det F(s) &= (s+1)^2 - \frac{A'(0)B'(0)}{\alpha\beta}(s+1) + \frac{A'(0)B'(0)}{\alpha\beta} \\ &= (s+1)^2 - \frac{(m+1)^2}{m}s \\ &= (s+1)^2 - \left(m + 2 + \frac{1}{m}\right)s \\ &= s^2 - \left(m + \frac{1}{m}\right)s + 1 \\ &= (s-m)(s-m^{-1}), \end{aligned}$$

and so $\det F(m) = 0$. Since we supposed $|m| > 1$, we find that $\det F(m^p) \neq 0$ for all $p \in \mathbb{Z}_{\geq 2}$. □

Therefore, by Theorem 13, there exist meromorphic functions over $\mathbb{C}$, $y(x)$ and $z(x)$, which satisfy the system (17), $y(0) = z(0) = 0$ and $(y'(0), z'(0)) \neq (0,0)$. For this solution $(y(x), z(x))$, we find the following.

Theorem 15. *Let $\mathcal{K} = (\mathbb{C}(x), x \mapsto mx)$. $\mathcal{K}\langle y, z\rangle$ cannot be contained in any decomposable extension of $\mathcal{K}$.*

Proof. To use the results in Section 4, we put

$$f(x) = y(x) + \alpha, \quad g(x) = z(x) + \beta.$$

Since $(y(x), z(x))$ satisfies the system (18), we obtain

$$\begin{cases} f(mx)f(x) = A(g(x) - \beta), \\ g(mx)g(x) = B(f(mx) - \alpha). \end{cases}$$

By Theorem 12 we find that for any decomposable extension $\mathcal{N}$ of $\mathcal{K}$, $\mathcal{K}\langle f, g\rangle \subset \mathcal{N}$ implies that f and g are algebraic over $\mathbb{C}(x)$. Therefore, for any decomposable extension $\mathcal{N}$ of $\mathcal{K}$, $\mathcal{K}\langle y, z\rangle \subset \mathcal{N}$ implies that y and z are algebraic over $\mathbb{C}(x)$.

Assume $\mathcal{K}\langle y, z\rangle \subset \mathcal{N}$ for some decomposable extension $\mathcal{N}$ of $\mathcal{K}$. Then y and z are algebraic over $\mathbb{C}(x)$. Since y and z are meromorphic functions, it follows that y and z are rational functions, namely $y, z \in \mathbb{C}(x)$. Express y and z as

$$y = \frac{P}{Q}, \quad z = \frac{R}{S}, \quad P, Q, R, S \in \mathbb{C}[x],$$

where Q and S are non-zero monic polynomials, P and Q are relatively prime and R and S are relatively prime. From the system (18) we obtain

$$\begin{cases} (P(mx) + \alpha Q(mx))(P(x) + \alpha Q(x))S(x)^{\deg A} \\ \quad = Q(mx)Q(x)S(x)^{\deg A} A\left(\dfrac{R(x)}{S(x)}\right), \\ (R(mx) + \beta S(mx))(R(x) + \beta S(x))Q(mx)^{\deg B} \\ \quad = S(mx)S(x)Q(mx)^{\deg B} B\left(\dfrac{P(mx)}{Q(mx)}\right). \end{cases}$$

Since $S(x)^{\deg A}$ and $S(x)^{\deg A} A(R(x)/S(x))$ are relatively prime, $S(x)^{\deg A}$ divides $Q(mx)Q(x)$, which yields

$$\deg A \cdot \deg S \le 2 \deg Q. \tag{19}$$

Since $Q(mx)^{\deg B}$ and $Q(mx)^{\deg B} B(P(mx)/Q(mx))$ are relatively prime, $Q(mx)^{\deg B}$ divides $S(mx)S(x)$, which yields

$$\deg B \cdot \deg Q \le 2 \deg S. \tag{20}$$

Hence we obtain

$$\deg A \cdot \deg B \cdot \deg Q \cdot \deg S \le 4 \deg Q \cdot \deg S.$$

Recalling $\deg A \cdot \deg B > 4$, we find $\deg Q \cdot \deg S = 0$. If $\deg Q = 0$ then we obtain $\deg S = 0$ by (19). If $\deg S = 0$ then we obtain $\deg Q = 0$ by (20). Therefore we conclude that $\deg Q = \deg S = 0$. Since we supposed that Q and S are monic, this implies $Q = S = 1$.

Now we have

$$\begin{cases} (P(mx) + \alpha)(P(x) + \alpha) = A(R(x)), \\ (R(mx) + \beta)(R(x) + \beta) = B(P(mx)). \end{cases} \tag{21}$$

Note that $y(x) = P(x)$, $z(x) = R(x)$, $y(0) = z(0) = 0$ and $(y'(0), z'(0)) \neq (0, 0)$. If we assume $P = 0$, then we obtain $R \neq 0$, $\deg R \geq 1$ and

$$(R(mx) + \beta)(R(x) + \beta) = B(0) = \beta^2 \in \mathbb{C},$$

which yield a contradiction. Hence we find $P \neq 0$, and so $\deg P \geq 1$. If we assume $R = 0$, then we obtain

$$(P(mx) + \alpha)(P(x) + \alpha) = A(0) = \alpha^2 \in \mathbb{C},$$

a contradiction. Hence we find $R \neq 0$, and so $\deg R \geq 1$. By the equations (21) we obtain $2 \deg P = \deg A \cdot \deg R$ and $2 \deg R = \deg B \cdot \deg P$, which imply

$$\deg A \cdot \deg B = \frac{2 \deg P}{\deg R} \cdot \frac{2 \deg R}{\deg P} = 4,$$

a contradiction. Therefore we conclude that for any decomposable extension $\mathcal{N}$ of $\mathcal{K}$, $\mathcal{K}\langle y, z\rangle$ is not contained in $\mathcal{N}$, the required. □

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