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NON-EMBEDDABILITY OF CERTAIN CLASSES OF LEVI FLAT MANIFOLDS

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Abstract

On the basis of a result of Barrett [2], we show that members of certain classes of abstract Levi flat manifolds with boundary, whose Levi foliation contains a compact leaf with contracting, flat holonomy, admit no *CR* embedding as a hypersurface of a complex manifold. In particular, it follows that the foliation constructed in [6] is not embeddable.

In [2], Barrett showed that there is no Levi flat submanifold $S \cong S^3$, smoothly embedded in a complex 2-manifold M, such that its foliation is diffeomorphic to Reeb's one. A key ingredient in the proof is a result by Ueda [10], which allows to find an equation for a compact complex curve $C \subset M$ (in a neighborhood of C), provided that its normal bundle satisfies certain triviality conditions.

We show that Barrett's method can be adapted to prove that other classes of Levi flat manifolds, of dimension greater than 3, are non-embeddable as smooth hypersurfaces of a complex manifold. This is due to the fact that the relevant part of Ueda's argument is valid also in dimension greater than 1.

In our situation, we assume the existence of a compact leaf whose holonomy is isomorphic to $\mathbb{Z}$, contracting and flat, as in the case of Reeb's foliation. Moreover, we ask for the holonomy covering of the compact leaf to be (partially) "extendable" at infinity, a technical condition (based on the notion of partial compactification employed in [7]) which can be verified in several examples—as in Reeb's case, and in the case of the examples discussed in Section 2.

The proof of Theorem 1.3 comes as a consequence of [2], [7] and [10]; the purpose of this note is essentially to explain it in detail, and a part of our argument is in fact pointing out why Theorem 3 in [10] applies to our situation. Once a defining function for the compact leaf has been found, the proof becomes a not too difficult application of the maximum principle and of the compactification lemma in [7] (see the end of Section 1). Afterwards, in Section 2 we show how Theorem 1.3 applies to the case of some well-known foliations.

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1. Main result

Statement. Let S be a C^∞ Levi flat $2n+1$-manifold with boundary $C = bS$, and denote by $\mathcal{F}$ its smooth foliation by complex leaves. We will assume that C is a compact complex manifold of dimension n.

In order to state our main result, we need some definitions. First, we give a notion which extends that of partial compactification employed in [7].

DEFINITION 1.1. Let M be a complex manifold of dimension n. We say that M has an *end* E if there exists a sequence $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots$ of connected open subsets such that every $b\overline{\mathcal{U}_j}$ is compact and $\bigcap_i \overline{\mathcal{U}_i} = \emptyset$. Let now X be another n-dimensional complex manifold, and let $\Omega \subset X$ be a proper subdomain such that $b\Omega$ is compact. We say that X *extends* M through its end E if there exists a biholomorphism $\Psi\colon M \to \Omega$ such that $\bigcap_i \overline{\Psi(\mathcal{U}_i)} = b\Omega$. If Ω is dense in X and $b\Omega \cong H$ is a compact k-dimensional complex submanifold of X, with $k < n$, we say—in accordance with the definition given in [7]—that X is a (partial) *holomorphic compactification* of M by H at E-infinity. Assume, now, that $M = L$ is a leaf of a foliation $\mathcal{F}$ as before. Let $E = \{\mathcal{U}_i\}$ be an end of L, and suppose that (with respect to the topology of S) $\bigcap_i \overline{\mathcal{U}_i} = C$. In this situation, we say that L *ends at* C and an extension of L at E-infinity is also said to be at *C-infinity*.

Next, we have to introduce some properties related to the holonomy of the compact curve C.

DEFINITION 1.2. Let $\mathcal{G}$ denote the (germs at 0 of) smooth functions $[0,1) \to [0,1)$ fixing 0 and let the homomorphism $\mathfrak{h}\colon \pi_1(C) \to \mathcal{G}$ be the one-sided holonomy mapping of $\mathcal{F}$ around C. The *holonomy group* is the subgroup of $\mathcal{G}$, isomorphic to $\pi_1(C)/\ker \mathfrak{h}$, given by $\mathfrak{h}(\pi_1(C))$. We say that the holonomy of C is *contracting* if there exists an element d of $\mathfrak{h}(\pi_1(C))$ such that $d(t) < t$ for $t \in [0,1)$. Moreover, we say that the holonomy is *smoothly flat* if for any germ d in the holonomy group we have $d(t) - t = o(t^k)$ for all $k \in \mathbb{N}$. The *holonomy covering* $p\colon \tilde{C} \to C$ is the (regular) covering of C with the property that $p_*(\pi_1(\tilde{C})) = \ker \mathfrak{h}$.

Let S be as above; we regard C as a boundary leaf for $\mathcal{F}$. Our main assumptions will regard the holonomy of this compact leaf:
(A) the (one sided) holonomy group of C is isomorphic to $\mathbb{Z}$; moreover, the holonomy is contracting and smoothly flat.
When (A) is satisfied, the holonomy group of C has a contracting generator d, and $\tilde{C}$ has precisely one end E corresponding to d (see also Remark 1.2).
(B) the holonomy covering $\tilde{C}$ of C extends through E.
Then we have

Theorem 1.3. *With the hypotheses above, there is no smooth embedding of S as a Levi flat hypersurface* (*with boundary*) *of a complex manifold.*

We remark that the hypotheses of Theorem 1.3 regard only the compact leaf C. If the holonomy of C is tangent to identity but not smoothly flat, then there may exist an embedding of S; in fact, in [2] Barrett shows an explicit construction of a (Lipschitz) embedding of S^3, with a foliation that is homeomorphic to Reeb's one but whose toric leaf's holonomy is not C^∞ flat. In our context, since we are dealing with a situation with boundary, it is not difficult to give counterexamples where the holonomy is even of class C^k (see Example 2.1).

Proof. To prove Theorem 1.3, as said before, we follow step by step the method employed by Barrett in [2]. Assume, then, that there is a smooth embedding of S into a complex $(n+1)$-manifold M; we will fix our attention to a neighborhood of the compact leaf C. We claim that

Lemma 1.4. *There exists a holomorphic defining function h for C, defined in a neighborhood of C in M. Moreover, h can be chosen in such a way that $d(\operatorname{Re} h)|_S$ does not vanish in C.*

To prove this lemma, we first give—following [10]—a definition:

DEFINITION 1.5. Let C be a compact complex hypersurface of a complex manifold M, and suppose that the normal bundle of C is holomorphically trivial. Let $\mathfrak{V} = \{V_i\}$ be a small enough covering of a neighborhood of C in M, and let $\mathfrak{U} = \{U_i\} = \{V_i \cap C\}$; then it is easy to see that there exists a system $\{w_i\}$ of local equations of U_i in V_i such that w_i/w_k is well defined and equal to 1 in $U_{ik} = U_i \cap U_k$. Denoting by z_i a suitable set of local coordinates in U_i (such that (z_i, w_i) give coordinates for V_i), this means that for some positive integer ν and $f_{ik} \in \mathcal{O}(U_{ik})$ we have

$$w_k - w_i = f_{ik}(z_i)w_i^{\nu+1} + o(\nu+1)$$

on $V_{ik} = V_i \cap V_k$. In such a case, the system $\{w_i\}$ is said to be *of type* ν. It is readily verified (see again [10]) that f_{ik} is a cocycle in $\mathcal{Z}^1(\mathfrak{U}, \mathcal{O})$, and that it is a coboundary if and only if there exists a system of type $\nu + 1$. C is said to be *of infinite type* if any such system is a coboundary, i.e. there exists a system of type ν for all $\nu \in \mathbb{N}$.

REMARK 1.1. The type in the sense of Ueda defined above has the following geometrical meaning: it is the order of contact along C of the line bundle $[C]$ (generated by C as a divisor) and the trivial extension of the normal bundle to a neighborhood.

By hypothesis, the holonomy of the foliation of S that we are considering along the compact leaf C is trivial to infinite order. As a consequence of this fact, in the

Appendix of [2] the following is proven:

Lemma 1.6. *The normal bundle of C is holomorphically trivial; moreover, C is of infinite type in M.*

The point of the proof of the previous statement lies in the isomorphism (up to any finite order) between the sheaf of functions which are locally constant on the leaves of $\mathcal{F}$ and a particular subsheaf of holomorphic functions of M. This isomorphism in turn depends on a result in [3] about the local (finite order) approximation of Levi flat hypersurfaces by zero sets of pluriharmonic functions, which holds for any dimension.

Lemma 1.4 is then a consequence of Theorem 3 in [10]. Although that theorem is stated only for complex curves—since that is the framework of Ueda's paper—its proof works as well for any compact complex hypersurface of a complex manifold. In fact, the proof involves the construction of a new set of coordinate functions $\{u_i\}$ in V_i which satisfy $u_i = u_j$ on V_{ij}. This is first carried out formally, expressing u_i as a power series in $\{w_i\}$ with coefficients in $\mathcal{O}(U_i)$ in such a way that the relation is satisfied; the construction is possible because of the existence of a system of type ν for all $\nu \in \mathbb{N}$, which (roughly) implies the vanishing of the obstruction to the existence of each successive term of the series. The variables z_i appear only through coefficients of the series in $\mathcal{O}(U_i)$, and the number of coordinates z_i plays no role. The power series in w_i can be so constructed that they are convergent, using a lemma by Kodaira and Spencer [5]. The argument is valid regardless of the dimension of C.

Proof of Theorem 1.3. Let h be the function obtained by Lemma 1.4; $\operatorname{Re} h$ has constant sign in a neighborhood of C in S, we may suppose $\operatorname{Re} h > 0$. For a small enough ε, $\{0 < \operatorname{Re} h < \varepsilon\}$ is a one-sided tubular neighborhood W of C in S. A contradiction will be obtained by considering the behavior of the restriction of h to $L \cap W$, where L is a leaf in $S \setminus C$ whose closure contains C. To this purpose, we first define a notion introduced in [6], [7]:

DEFINITION 1.7. we say that $\mathcal{F}$ is *tame* if the following occurs: define the manifold S' as

$$S' = S \sqcup (C \times [0, 1])/bS \sim C \times \{0\}$$

(i.e. S' extends S by attaching a collar $C \times (0, 1]$ along C), and consider the foliation of S' which agrees with $\mathcal{F}$ on S and with the trivial one (induced by the submersion $C \times [0, 1] \to [0, 1]$) on $C \times [0, 1]$. Moreover, endow the leaves of the foliation of S' contained in S with the complex structure inherited by $\mathcal{F}$, and each leaf contained in $C \times [0, 1]$ with the complex structure of C. Then the foliation obtained is a smooth Levi foliation of S'.

For tame foliations, we can employ a compactification lemma proved in [7]. Consider, on the restriction of $\mathcal{F}$ to a tubular neighborhood of C in S, a second (tame) complex structure J_1 such that the structure induced on C is the same as the original one. Then we can give the following variant of the compactification lemma cited above:

Lemma 1.8. *Let L be a leaf of $\mathcal{F}$ ending in C, and V a small enough tubular neighborhood of C. Let L_1 be the same leaf, but endowed by a complex structure J_1 as above. If L_1 admits an extension at C-infinity by a complex manifold X, then so does L.*

Proof. The proof employed in [7] carries over: in fact, by the same argument the tameness of $\mathcal{F}$ implies that J_1 extends to $b\Omega$ smoothly (as an endomorphism of $T(X)$) and $J_1|_{b\Omega} = J|_{b\Omega}$. Hence J_1 extends smoothly on all of X, and it must be integrable since it is in Ω and $X \setminus \Omega$, so it is a complex structure in X. □

To use the previous lemma, we first need the following standard fact from foliation theory:

Lemma 1.9. *Under the assumption* (A) *of* Theorem 1.3, *there exists a leaf L which has an end in C.*

Proof. We can apply Theorem 1 in [9]. The cases (1) and (2) in the statement of that result do not occur since, respectively, the holonomy of C is (strictly) contracting and the holonomy group is isomorphic to $\mathbb{Z}$. From the description in case (3) then follows that, for a suitable neighborhood V of C, all the leaves of $\mathcal{F}|_V$ have in fact (exactly) one end in C. □

REMARK 1.2. Let $\tilde{C} \xrightarrow{\pi} C$ be the holonomy covering of C; then there exists an open subset $\tilde{V} \subset \tilde{C} \times [0, 1)$, $\tilde{C} \times \{0\} \subset \tilde{V}$, and a covering map (which we still denote by π) from $\tilde{V}$ onto a small tubular neighborhood V of C in S which coincides with the holonomy covering on $\tilde{C} \times \{0\}$ and such that the lift of $\mathcal{F}|_V$ by π coincides with the trivial foliation by $\tilde{C} \times \{t\}$, $t \in [0, 1)$. A generator $\mathcal{T}$ of the deck transformation group for π can be expressed as

$$\mathcal{T}(p, t) = (T(p, t), d(t))$$

with $p \in \tilde{C}$ and $t \in [0, 1)$, where the function $d(t)$ satisfies $d(t) < t$ for $t \in (0, 1)$ and $d(t) - t = o(t^k)$ for all $k \in \mathbb{N}$. As a consequence, the restriction of the action of the deck group to $\tilde{V} \cap (\tilde{C} \times (0, 1))$ does not fix any leaf, hence for each $t > 0$ the covering map π sends $(\tilde{C} \times \{t\})|_{\tilde{V}}$ diffeomorphically to a leaf L_t of $\mathcal{F}|_V$. The end of L_t identified in Lemma 1.9 induces then an end E of $\tilde{C}$, which is the one considered in the assumption (B) of Theorem 1.3.

Next, in order to apply Lemma 1.8 we show—following the reparametrization method of Corollary 3 in [7]—that we can reduce to a tame situation. We use the notation of the previous remark; consider the continuous mapping $\Phi\colon \tilde{C} \times [0,1] \to \tilde{C} \times [0,1]$ defined by

$$\Phi(p,t) = (p, \theta^{-1}(t)), \quad \theta(t) = e^{-1/t}.$$

For any fixed $t \in [0,1]$, the restriction of Φ to $\tilde{C} \times \{t\}$ sends it diffeomorphically to $\tilde{C} \times \{\theta^{-1}(t)\}$. We can thus endow $\tilde{V}$ with the pull-back of the original CR structure of by $\Phi|_{\tilde{V}}$, obtaining a manifold $\tilde{V}'$ with a foliation whose leaves are, by definition, biholomorphic to the leaves of $\tilde{V}$. Now, the quotient V' of $\tilde{V}'$ under the action of the group generated by

$$\Phi \circ \mathcal{T} \circ \Phi^{-1}(p,t) = (T(p, \theta(t)), \theta^{-1} \circ d \circ \theta(t))$$

carries a smooth, tame foliation $\mathcal{F}'$ whose leaves are biholomorphic to the leaves of $\mathcal{F}$. Now, by (B) and Lemma 1.8 we deduce that a leaf L ending in C can be extended at C-infinity. In fact, if we endow each leaf L of V' with the structure L^{pb} (obtained by pulling back the complex structure of C by a suitable submersion, see the corollary of compactification lemma in [7]) and we give to $\tilde{V}'$ the trivial structure, the previously described covering $\tilde{V}' \to V'$ is a biholomorphism along the leaves. It follows that each L_t^{pb}, hence L_t, can be extended.

Consider, then, the biholomorphism $\Psi\colon L \to \Omega$ given by Definition 1.1; we are interested in $g = h \circ \Psi \in \mathcal{O}(\Psi^{-1}(L_W))$, where $L_W = L \cap W$. Since $h|_{L_W}$ converges to zero at the ending corresponding to C, we have that g extends continuously (by 0) to $b\Omega$ and thus to $(X \setminus \Omega) \cup \Psi^{-1}(L_W)$. By Rado's theorem, then, follows that g is holomorphic everywhere, hence Ω is actually dense in X and $b\Omega = H$ is an analytic subset of X. Then $\operatorname{Re} h \circ \Psi$ is a non-constant pluriharmonic function on $\Psi^{-1}(L_W) \cup H$ which assumes minimum in its interior part (on H), a contradiction.

2. Examples

Suspension of a Hopf manifold. Fix coordinates (z, w) in $\mathbb{C}^2$. As classified by Kodaira [4], any *Hopf surface* is a quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the action of

$$H\colon (z,w) \to (\alpha z + \lambda w^m, \beta w)$$

where $m \in \mathbb{N}$ and $\alpha, \beta, \lambda \in \mathbb{C}$ satisfy $(\beta^m - \alpha)\lambda = 0$ and $0 < |\alpha| \le |\beta| < 1$.

Let, now, $\varrho\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing smooth function such that $\varrho(t) < t$ for $t > 0$ and $\varrho(t) - t = o(t^d)$ as $t \to 0$ for all $d \in \mathbb{N}$. Consider $r\colon (\mathbb{C}^2 \setminus \{(0,0)\}) \times \mathbb{R} \to (\mathbb{C}^2 \setminus \{(0,0)\}) \times \mathbb{R}$ defined as

$$r\colon (z,w,t) \to (H(z,w), \varrho(t))$$

and let S be the quotient of $(\mathbb{C}^2 \setminus \{(0,0)\}) \times \mathbb{R}$ by the action of r. Since the action of r preserves the foliation of $(\mathbb{C}^2 \setminus \{(0,0)\}) \times \mathbb{R}$ by $\{t = \text{const.}\}$, S inherits a structure of Levi flat manifold; the leaves are all isomorphic to $\mathbb{C}^2 \setminus \{(0,0)\}$, except a compact leaf C corresponding to $\{t = 0\}$ which is a Hopf surface diffeomorphic to $S^3 \times S^1$. Clearly, since C is compact and non-Kähler, we know a priori that there is no embedding of S as a Levi flat submanifold of either a Stein or a Kähler manifold. By Theorem 1.3 we have that actually

Corollary 2.1. *S does not admit a C^∞ embedding as a Levi flat hypersurface of a complex* 3*-manifold.*

In this case, the holonomy covering of the Hopf surface C coincides with its universal covering $\mathbb{C}^2 \setminus \{(0,0)\}$, which has a partial holomorphic compactification by the $\mathbb{CP}^1$ at infinity (and in fact the non-compact leaves are in turn compactifiable). Moreover, by the choice of ϱ the holonomy of C is contracting and C^∞ flat, so that Theorem 1.3 applies.

On the other hand, in the non-smooth case the embedding is possible:

EXAMPLE 2.1. In fact, one can obtain an embedding in such a way that the holonomy of C is flat up to any fixed order d: let $\varrho(t) = t - t^d$, and define the suspension as above. The resulting S has a real analytic Levi foliation, and as such it can be embedded (see [1]). Notice that, since $\mathbb{C}^2 \setminus \{(0,0)\}$ is compactifiable at both ends, the example also works for $\varrho(t) = t + t^d$.

Partial generalization. Let P be a homogeneous polynomial in $\mathbb{C}^n$, and assume that $V = \{P = 0\}$ is a smooth complex manifold outside the origin, with a smooth closure in $\mathbb{CP}^n$. Choosing $0 < \alpha < 1$ and $\varrho\colon \mathbb{R} \to \mathbb{R}$ as above, we define the suspension

$$S = (V \setminus \{0\}) \times \mathbb{R} / \{(z,t) \sim (\alpha z, \varrho(t))\}.$$

We shall denote by C the compact leaf, corresponding to $\{t = 0\}$, of the foliation of S induced by that of $(V \setminus \{0\}) \times \mathbb{R}$; the other leaves are isomorphic to $V \setminus \{0\}$.

As before, we have

Corollary 2.2. *S does not admit a C^∞ embedding as a Levi flat hypersurface of a complex n-manifold.*

In this case, too, the holonomy covering of C coincides with its universal covering; the partial compactification of $\tilde{C} = V \setminus \{0\}$ is obtained by adding $\overline{V} \cap \mathbb{CP}^{n-1}$, where $\overline{V}$ is the closure in $\mathbb{CP}^n$.

Foliation of a 5-manifold in [6]. In [6] (cf. also [8]) it is constructed a smooth, one-codimensional Levi foliation of a certain real 5-manifold Z, with two compact leaves. Each one of the compact leaves is isomorphic to a principal bundle over an elliptic curve $\mathbb{E}_\omega$ whose fibers are in turn elliptic curves. Since these compact leaves are not Kähler, it is once again clear that this foliation does not admit an embedding as Levi flat submanifold of a Stein or Kähler manifold. In fact

Corollary 2.3. *There is no smooth embedding of Z as a Levi flat hypersurface of a complex manifold whose Levi foliation is diffeomorphic to the one obtained in* [6].

In order to show that Theorem 1.3 applies, we give a brief description of the foliation of Z. This is constructed by gluing two partial ones, defined in certain 5-manifolds with boundary $\mathcal{M}$ and $\mathcal{N}$. The foliation in $\mathcal{N}$ is defined by taking a suitable quotient of $\widetilde{X} = \mathbb{C}^* \times (\mathbb{C} \times [0, \infty) \setminus \{(0, 0)\})$ (whose foliation is the trivial one, induced by the level sets $\{t = t_0\}$ where t is the $[0, \infty)$-coordinate) by two commuting actions T and U. T does not act on the t-coordinate, while U acts by a contracting function $d(t)$ which is tangent to the identity to infinite order. The holonomy of the compact boundary leaf S_λ is thus isomorphic to $\mathbb{Z}$; in a neighborhood of S_λ the foliation is homeomorphic to the product of a disc by a neighborhood of the toric leaf in Reeb's foliation. In particular we have that the holonomy along S_λ is contracting and trivial to infinite order.

In this case the holonomy covering of S_λ does not coincide with its universal covering, but it is isomorphic to the complex manifold W defined as

$$W = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\} : z_1^3 + z_2^3 + z_3^3 = 0\}.$$

Hence, the holonomy covering admits a partial holomorphic compactification by $\overline{W} \cap \mathbb{CP}^2_\infty$, where once again $\overline{W}$ is the closure in $\mathbb{CP}^3$. Thus, the hypotheses of Theorem 1.3 are satisfied, which gives Corollary 2.3.

Alternatively, a more direct proof of the corollary can be achieved in the following way: let h be as in Lemma 1.4. For a leaf L sufficiently close to S_λ and a small enough ε, the intersection

$$L \cap \{0 < \operatorname{Re} h < \varepsilon\}$$

is holomorphically equivalent to $D \times D^\star$, where D is the unit disc and $D^\star$ is an annulus. The restriction of $\operatorname{Re} h$ to $0 \times D^\star$ is a positive harmonic function which vanishes, along with its conjugate, at 0. But then $\operatorname{Re} h$ extend to the whole disc, giving a contradiction by the maximum principle (see also [2]).

REMARK 2.1. Regardless of the validity of assumption (B) in Theorem 1.3, whenever it can be established that an internal leaf of the foliation extends at C-infinity the arguments of Section 1 apply.

REMARK 2.2. One may conjecture that the flatness of the holonomy alone is sufficient to ensure that no embedding exists; the methods used in the paper, though, do not seem sufficient to prove such a result.

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