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WEIGHTED PROJECTIVE SPACES AND ITERATED THOM SPACES

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Abstract

For any weight vector χ of positive integers, the weighted projective space $\mathbb{P}(\chi)$ is a projective toric variety, and has orbifold singularities in every case other than standard projective space. Our principal aim is to study the algebraic topology of $\mathbb{P}(\chi)$, paying particular attention to its localisation at individual primes p. We identify certain p-primary weight vectors π for which $\mathbb{P}(\pi)$ is homeomorphic to an iterated Thom space, and discuss how any weighted projective space may be reassembled from its p-primary parts. The resulting Thom isomorphisms provide an alternative to Kawasaki's calculation of the cohomology ring of $\mathbb{P}(\chi)$, and allow us to recover Al Amrani's extension to complex K-theory. Our methods generalise to arbitrary complex oriented cohomology algebras and their dual homology coalgebras, as we demonstrate for complex cobordism theory, the universal example. In particular, we describe a fundamental class that belongs to the complex bordism coalgebra of $\mathbb{P}(\chi)$, and may be interpreted as a resolution of singularities.

1. Introduction

Weighted projective spaces $\mathbb{P}(\chi)$ are defined for every integral weight vector χ, and constitute a family of singular toric varieties on which many hypotheses may be tested. Our central interest is to study their algebraic topology, and identify them up to homeomorphism with spaces whose singularities are more familiar, namely iterated Thom complexes. In the process of working towards this goal, it transpires that the cohomology algebras $E^*(\mathbb{P}(\chi))$, and homology coalgebras $E_*(\mathbb{P}(\chi))$, may be described in terms of iterated Thom isomorphisms for any complex oriented cohomology and homology theories $E^*(-)$ and $E_*(-)$. This provides a fruitful new perspective on algebraic objects of considerable complexity, and we present several explicit examples below.

Until the last few years, literature on the algebraic topology of weighted projective spaces has been sparse, and restricted mainly to work of Kawasaki [14] and Al Amrani [3], [4]. Immediately after the latter, Nishimura–Yoshimura [15] took up the challenge of computing the real K-theory groups $KO^*(\mathbb{P}(\chi))$, whose difficulty is increased by the lack of complex orientation for $KO^*(-)$. More recently, it has become apparent that

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the toric structure of $\mathbb{P}(\chi)$ is particularly important, and our own work [5], [6] has exploited this fact. Most of our current results are independent of the toric framework, but its beauty is sufficiently compelling that we have retained it here for background and motivation.

We present our results in the following order.

Section 2 establishes notation by recalling definitions of $\mathbb{P}(\chi)$ and the associated weighted lens spaces $L(k;\chi)$ in the toric setting. A crucial cofibre sequence of Kawasaki is expressed in terms of normal neighbourhoods over the orbit simplex Δ^n of the canonical torus action, and his resulting computation of the integral cohomology rings $H^*(\mathbb{P}(\chi))$ and $H^*(L(k;\chi))$ are summarised. The important maps $\phi\colon \mathbb{C}P^n \to \mathbb{P}(\chi)$ and $\psi\colon \mathbb{P}(\chi) \to \mathbb{C}P^n$ are described in terms of homogeneous coordinates, and the complex line bundle classified by ψ is identified for future use.

Section 3 introduces the concept of *divisive* weight vectors and their normalisations, and Corollary 3.8 employs an observation of Al Amrani (relating ψ and normal neighbourhoods) to express $\mathbb{P}(\chi)$ as an iterated Thom complex whenever χ is divisive. Fundamental examples are provided by *p-primary* weight vectors π, whose coordinates are powers of a fixed prime p. Archetypal examples are the *p-contents* ${}_p\chi$ of χ, which lead to the p-primary decomposition of χ. Maps $e(\chi/\omega)\colon \mathbb{P}(\omega) \to \mathbb{P}(\chi)$ are introduced as common generalisations of ψ and σ, and determine *extraction* and *inclusion* maps between $\mathbb{P}(\chi)$ and its p-primary parts $\mathbb{P}({}_p\chi)$. The maps e also encode a description of any weighted projective space as the quotient of any other by the action of a certain abelian group.

Section 5 considers complex oriented cohomology theories $E^*(-)$, whose coefficient rings E_* are even dimensional and free of additive torsion. In Theorem 5.8, the complex orientation is exploited to describe $E^*(\mathbb{P}(\chi))$ via iterated Thom isomorphisms and the E-theoretic formal group law, whenever χ is divisive. Illustrative examples are given using integral cohomology and complex K-theory, and the former are shown to recover Kawasaki's calculations in the divisive (and therefore the p-primary) case.

Section 4 addresses the question that immediately arises: is it possible to reassemble the p-primary parts $\mathbb{P}({}_p\chi)$, and recover $\mathbb{P}(\chi)$ up to homeomorphism? Theorem 4.9 answers affirmatively, and offers the surprising addendum that reassembly may be effected in two contrasting ways, which describe $\mathbb{P}(\chi)$ as an iterated limit and an iterated colimit of the $\mathbb{P}({}_p\chi)$s respectively. Both constructions arise over $\mathbb{C}P^n$, utilising the maps $e(1/{}_p\chi)$ and $e({}_p\chi)$. Examples are discussed which show that different reassemblies of the same p-primary parts can produce non-homeomorphic results.

Section 6 introduces cohomological reassembly as a natural analogue of the previous geometry. Theorem 6.10 describes $E^*(\mathbb{P}(\chi))$ as both a limit and a colimit of the p-primary parts $E^*(\mathbb{P}({}_p\chi))$, made explicit in terms of direct sums and iterated tensor products over $E^*(\mathbb{C}P^n)$ respectively. Most memorable is the resulting identification of $E^*(\mathbb{P}(\chi))$ with an intersection of p-primary subalgebras of $E^*(\mathbb{C}P^n)$. Examples are provided in integral cohomology and complex K-theory to show that detailed computation is possible; the former recover Kawasaki's calculations for arbitrary χ, and the latter those of [3].

To date, toric topology has rarely dealt with the homology coalgebra $E_*(X_\Sigma)$ of a toric variety X_Σ. Section 7 redresses this situation, at least for weighted projective space, by studying the universal example $\Omega_*^U(\mathbb{P}(\chi))$ for divisive χ. The relationship between Bott towers and iterated Thom spaces is recalled, and applied in Theorem 7.14 to identify explicit Ω_*^U-generators for the coalgebra $\Omega_*^U(\mathbb{P}(\chi))$. In particular, a rational fundamental class is constructed that may be interpreted in terms of toric desingularisation.

Finally, Section 8 introduces homological reassembly by dualising the results of Section 6. Theorem 8.2 describes the coalgebra $E_*(\mathbb{P}(\chi))$ as both a limit and a colimit of the p-primary parts $E_*(\mathbb{P}({}_p\chi))$ in terms of direct sums and iterated tensor products over $E_*(\mathbb{C}P^n)$ respectively. In particular, $E_*(\mathbb{P}(\chi))$ is identified with an intersection of p-primary subcoalgebras of $E_*(\mathbb{C}P^n)$, and an illustrative examples is given in complex K-theory.

Throughout our work we write S^1 for the circle as a topological space, and $T < \mathbb{C}^1$ for its realisation as the group of unimodular complex numbers with respect to multiplication. For any integer $k > 0$ we write $\mathbb{Z}/k$ for the integers modulo k, and $C_k < T$ for its realisation as the subgroup generated by a primitive k-th root of unity. We interpret the standard simplex Δ^n as the intersection of the positive orthant $\mathbb{R}^{n+1}_{\geqslant}$ with the hyperplane $x_0 + \cdots + x_n = 1$, and denote its boundary by $\partial\Delta^n$. As an abstract simplicial complex, $\partial\Delta^n$ has $\binom{n+1}{k+1}$ faces of dimension k, for $-1 \leq k < n$.

For any generalised cohomology theory, we follow the convention that all homology and cohomology groups $E_*(X)$ and $E^*(X)$ are *reduced* for every space X. The unreduced counterparts are given by adjoining a disjoint basepoint, and considering $E_*(X_+)$ and $E^*(X_+)$. The *coefficient ring* E_* is given by $E_*(S^0) \cong E^{-*}(S^0)$; we identify the homological and cohomological versions without further comment, and interpret $E_*(X_+)$ and $E^*(X_+)$ as E_*-modules and E_*-algebras respectively. We make the important assumption that E_* is even dimensional and free of additive torsion, as holds for integral cohomology $H^*(-)$, complex K-theory $K^*(-)$, and complex cobordism $\Omega_U^*(-)$.

These theories are also *complex oriented* [1, Part II §2], by means of a class x^E in $E^2(\mathbb{C}P^\infty)$ whose restriction to $E^2(\mathbb{C}P^1)$ is a generator. It follows that there exists a canonical isomorphism

$$E^*(\mathbb{C}P^\infty_+) \cong E_*[\![x^E]\!] \tag{1.1}$$

of E_*-algebras, and that complex vector bundles have associated E-theoretic Chern classes. In particular, x^E is the first Chern class $c_1^E(\zeta)$ of the dual Hopf line bundle ζ over $\mathbb{C}P^\infty$. A minor abuse of notation allows x^E to be confused with its restriction to $\mathbb{C}P^n$, and produces an isomorphism

$$E^*(\mathbb{C}P^n_+) \cong E_*[x^E]/((x^E)^{n+1}). \tag{1.2}$$

It is convenient to denote x^E by u in the universal case $\Omega_U^*(\mathbb{C}P^\infty)$, and to write x^H as x in $H^2(\mathbb{C}P^\infty)$.

If the Thom space of ζ is identified with $\mathbb{C}P^\infty$, then x^E may also be interpreted as a Thom class $t^E(\zeta)$, and extended to a universal Thom class $t^E \in E^0(MU)$; thus t^U is represented by the identity map on MU.

2. Weighted projective space

In this section basic notation is established, and the definitions of weighted projective space, weighted lens space, and their associated constructions are recalled. Readers are referred to Al Amrani [3], [4], Kawasaki [14], and the authors' own work [5] for further details.

The *standard action* of the $(n+1)$-dimensional torus T^{n+1} on $\mathbb{C}^{n+1}$ is by coordinatewise multiplication, and restricts to the unit sphere S^{2n+1}. The orbit space of the latter is homeomorphic to the standard simplex $\Delta^n \subset \mathbb{R}^{n+1}_{\geqslant}$, and the quotient map $r\colon S^{2n+1} \to \Delta^n$ is given by $r(z) = (|z_0|^2, \ldots, |z_n|^2)$.

A *weight vector* χ is a sequence $(\chi_0, \ldots, \chi_n)$ of positive integers, and $T\langle\chi\rangle < T^{n+1}$ denotes the subcircle of elements $(t^{\chi_0}, \ldots, t^{\chi_n})$, as t ranges over T. It is convenient to abbreviate the greatest common divisor $\gcd(\chi_0, \ldots, \chi_n)$ and least common multiple $\operatorname{lcm}(\chi_0, \ldots, \chi_n)$ to $g = g(\chi)$ and $l = l(\chi)$ respectively.

DEFINITION 2.1. The *weighted projective space* $\mathbb{P}(\chi)$ is the orbit space of the action of $T\langle\chi\rangle$ on S^{2n+1}; it admits a *canonical action* of the quotient n-torus $T^{n+1}/T\langle\chi\rangle$, with orbit space Δ^n.

The respective quotient maps are

$$S^{2n+1} \xrightarrow{p(\chi)} \mathbb{P}(\chi) \xrightarrow{q(\chi)} \Delta^n, \tag{2.2}$$

whose composition is r. The action of $T\langle\chi\rangle$ is free when $\chi = (d, \ldots, d)$ for any positive integer d, in which case $\mathbb{P}(\chi)$ reduces to $\mathbb{C}P^n$; in general, $\mathbb{P}(\chi)$ has orbifold singularities. Weighted projective spaces provide an important class of singular examples in algebraic and symplectic geometry, although the focus of this article is on their algebraic topology.

It is sometimes convenient to assume that $g(\chi) = 1$, because $T\langle d\chi\rangle$ and $T\langle\chi\rangle$ produce homeomorphic orbit spaces for all d.

DEFINITION 2.3. For any positive integer k, the *weighted lens space* $L(k;\chi)$ is the orbit space of the action of the weighted cyclic subgroup $C_k\langle\chi\rangle < T\langle\chi\rangle$ on S^{2n+1}; it admits *canonical actions* of the quotient circle $T\langle\chi\rangle/C_k\langle\chi\rangle$ with orbit space $\mathbb{P}(\chi)$, and of the $(n+1)$-torus $T^{n+1}/C_k\langle\chi\rangle$ with orbit space Δ^n.

If k is prime to χ_i for $0 \leq i \leq n$, then $L(k;\chi)$ is a standard lens space, and is smooth; otherwise, $C_k\langle\chi\rangle$ fails to act freely, and $L(k;\chi)$ may be singular.

Restricting (2.2) to the hyperplane $z_n = 0$ yields orbit maps

$$S^{2n-1} \xrightarrow{p(\chi')} \mathbb{P}(\chi') \xrightarrow{q(\chi')} \Delta^{n-1}, \tag{2.4}$$

where χ' denotes $(\chi_0, \dots, \chi_{n-1})$ and Δ^{n-1} is the subsimplex $x_n = 0$ of Δ^n. On the other hand, restricting to the cylinder $|z_n|^2 = 1/2$ gives

$$S^{2n-1} \times S^1 \xrightarrow{p_{1/2}} L(\chi_n;\chi') \xrightarrow{q_{1/2}} \Delta_{1/2}, \tag{2.5}$$

where $S^{2n-1} \subset S^{2n+1}$ is the subsphere

$$|z_0|^2 + \cdots + |z_{n-1}|^2 = \frac{1}{2}.$$

Thus $p_{1/2}$ is the orbit map for $T\langle\chi\rangle$, and factors through $L(\chi_n;\chi') \times S^1$ under the actions of $C_{\chi_n}\langle\chi'\rangle$ and $T\langle\chi\rangle/C_{\chi_n}\langle\chi'\rangle$ respectively; the latter is isomorphic to T. Similarly, $q_{1/2}$ is the orbit map for the n-torus $T^n/C_{\chi_n}\langle\chi'\rangle$, whose orbit space $\Delta_{1/2}$ is the $(n-1)$-simplex $x_n = 1/2$. Finally, restricting to the circle $|z_n| = 1$ projects every point $(0, \dots, 0, z_n)$ in S^{2n+1} onto $[0, \dots, 0, 1]$ in $\mathbb{P}(\chi)$, and thence to the vertex $(0, \dots, 0, 1)$ in Δ^n.

Now consider the decomposition of Δ^n into the union of subspaces $N(1/2)$ and $C(1/2)$, specified by $x_n \leq 1/2$ and $x_n \geq 1/2$ respectively; they are homeomorphic to the product $\Delta^{n-1} \times [0, 1/2]$ and the cone $C\Delta_{1/2}$. So $\mathbb{P}(\chi)$ may be expressed as the pushout of

$$N\mathbb{P}(\chi') \xleftarrow{i} L(\chi_n;\chi') \xrightarrow{j} CL(\chi_n;\chi'), \tag{2.6}$$

where $N\mathbb{P}(\chi')$ denotes the neighbourhood $q^{-1}(N(1/2))$ of $\mathbb{P}(\chi')$ in $\mathbb{P}(\chi)$, and $CL(\chi_n;\chi')$ denotes the cone $q^{-1}(C(1/2))$, with basepoint $[0, \dots, 0, 1]$. Equivalently, (2.6) arises by decomposing S^{2n+1} as the pushout of

$$S^{2n-1} \times D^2 \xleftarrow{i} S^{2n-1} \times S^1 \xrightarrow{j} D^{2n} \times S^1,$$

and forming orbit spaces under the action of $T\langle\chi\rangle$. Reparametrising D^{2n} shows that $[0, \dots, 0, 1]$ admits a neighbourhood of the form $\mathbb{C}^n\big/C_{\chi_n}$; repeating at each point $[0, \dots, 0, 1, 0, \dots, 0]$ confirms that $\mathbb{P}(\chi)$ is a complex orbifold.

Diagram (2.6) is cofibrant, and therefore expresses $\mathbb{P}(\chi)$ as the homotopy colimit of the diagram

$$\mathbb{P}(\chi') \xleftarrow{f} L(\chi_n;\chi') \to *, \tag{2.7}$$

where f denotes the orbit map for the circle $T\langle\chi'\rangle/C_{\chi_n}\langle\chi'\rangle$, and $*$ is the point $[0,\ldots,0,1]$. This reinterprets Kawasaki's cofibre sequence [14, p. 245]

$$L(\chi_n;\chi') \xrightarrow{f} \mathbb{P}(\chi') \xrightarrow{g} \mathbb{P}(\chi). \tag{2.8}$$

REMARK 2.9. The category underlying diagram (2.7) may also be construed as $\mathrm{CAT}(\partial\Delta^1)$, whose objects are the faces $\varnothing$, 0 and 1 of $\partial\Delta^1$ and morphisms their inclusions. Iteration on $\mathbb{P}(\chi')$ leads to a description of $\mathbb{P}(\chi)$ as a homotopy colimit over $\mathrm{CAT}(\partial\Delta^n)$, in which the relevant diagram assigns an orbit space $T^n/T^k(\sigma)$ to each $(k-1)$-dimensional face σ of Δ^n; this is precisely the homotopy colimit of [19, Proposition 5.3].

Following Kawasaki, Al Amrani [3, I.1 (b)] defines maps

$$\mathbb{C}P^n \xrightarrow{\phi} \mathbb{P}(\chi) \xrightarrow{\psi} \mathbb{C}P^n \tag{2.10}$$

by $\phi[z_0,\ldots,z_n]=[z_0^{\chi_0},\ldots,z_n^{\chi_n}]$ and $\psi[z_0,\ldots,z_n]=[z_0^{l(\chi)/\chi_0},\ldots,z_n^{l(\chi)/\chi_n}]$. In both cases, the formulae for the homogeneous coordinates of the target values are understood to be normalised. It is sometimes important to make the weights explicit, by writing $\phi(\chi)$ and $\psi(\chi)$ respectively.

Usually, ψ is interpreted as a complex line bundle over $\mathbb{P}(\chi)$, but may equally well be specified by its first Chern class $c_1(\psi)$ in $H^2(\mathbb{P}(\chi);\mathbb{Z})$. The composition $\psi\circ\phi$ has degree $l=l(\chi)$ on $H^2(\mathbb{C}P^n;\mathbb{Z})$, so $\phi^*(\psi)$ is the l-th tensor power ζ^l of the dual Hopf bundle, and $c_1(\phi^*(\psi))=lx$ in $H^2(\mathbb{C}P^n)$, following (1.1).

The rôle of ψ is clarified by identifying the total space $S(\psi)$ of its associated circle bundle.

Proposition 2.11. *The space $S(\psi)$ is a $(2n+1)$-dimensional weighted lens space $L(l;\chi)$.*

Proof. By definition, $S(\psi)$ is the pullback of the diagram

$$\mathbb{P}(\chi) \xrightarrow{\psi} \mathbb{C}P^n \leftarrow S^{2n+1},$$

and is a subspace $X\subset S^{2n+1}\times\mathbb{P}(\chi)$. It contains all pairs $(y,[z])$ that satisfy the equation

$$t(y_0,\ldots,y_n)=(z_0^{l(\chi)/\chi_0},\ldots,z_n^{l(\chi)/\chi_n}) \tag{2.12}$$

in S^{2n+1}, for some $t\in T$. So there exists a map $h\colon L(l(\chi);\chi)\to X$, defined by

$$h[w_0,\ldots,w_n]=((w_0^{l(\chi)/\chi_0},\ldots,w_n^{l(\chi)/\chi_n}),[w_0,\ldots,w_n]);$$

moreover, an inverse to h is given by mapping $(y, [z])$ to the equivalence class of those $(n+1)$-tuples $(z_0, \dots, z_n)$ for which $t = 1$ in (2.12). It follows that h is the required homeomorphism. □

Corollary 2.13. *The circle $T\langle\chi\rangle/C_l\langle\chi\rangle$ acts freely on $L(l;\chi)$, and has orbit space $\mathbb{P}(\chi)$.*

Of course the associated sphere bundle $S(\phi^*(\psi))$ is homeomorphic to the standard lens space $L(l; 1, \dots, 1)$, and is therefore a smooth manifold.

The following natural numbers are associated to χ, and were essentially introduced by Kawasaki; alternative descriptions are recovered in Theorem 6.15.

DEFINITION 2.14. For any subset $J \subseteq [n]$, the integer χ_J is the product $\prod_{j\in J}\chi_j$, and $h_J = h_J(\chi)$ is the quotient $\chi_J/\gcd(\chi_j : j \in J)$; for any $1 \le j \le n$, the integer $l_j = l_j(\chi)$ is $\operatorname{lcm}(h_J : |J| = j)$, and $m_j = m_j(\chi)$ is $l(\chi)^j/l_j$.

Thus $l_1 = l$ and $m_1 = 1$, whereas $l_n = \chi_0\cdots\chi_n/g$ and $m_n = gl^n/\chi_0\cdots\chi_n$.

Kawasaki applies the cofibre sequence (2.8) to identify the integral cohomology ring of $\mathbb{P}(\chi)$ by means of an isomorphism

$$H^*(\mathbb{P}(\chi)_+;\mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_n]/I(\chi), \tag{2.15}$$

where v_j has dimension $2j$ and $I(\chi)$ is the ideal generated by the elements $v_1^j - m_j v_j$ for $1 \le j \le n$. Moreover, v_1 equals $c_1(\psi)$, so $\phi^*(v_1) = lx$ holds in $H^2(\mathbb{C}P^n;\mathbb{Z})$. The same calculations identify the integral cohomology ring of $L(\chi_n;\chi')$ in terms of additive isomorphisms

$$H^j(L(\chi_n;\chi');\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } \ j = 2n-1, \\ \mathbb{Z}/s_k & \text{if } \ j = 2k \text{ for } 1 \le k \le n-1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.16}$$

where $s_k = s_k(\chi)$ is given by l_k/l'_k, and $l'_k := l_k(\chi')$.

REMARK 2.17. These results reveal that the maps ϕ and ψ of (2.10) are mutually inverse rational homotopy equivalences, as their definitions suggest. The rationalisation $\mathbb{P}(\chi)_{\mathbb{Q}}$ is therefore the $2n$-skeleton of an Eilenberg–Mac Lane space $K(\mathbb{Q}, 2)$.

Many of Kawasaki's calculations are recovered in Theorem 6.15.

3. Iterated Thom spaces and *p*-primary parts

In order to follow homotopy theoretical convention and study $\mathbb{P}(\chi)$ prime by prime, it is convenient to introduce certain restrictions on the weights.

If $g(\chi) = 1$, then there exists an isomorphism

$$\mathbb{P}(d\chi_0, \ldots, d\chi_{j-1}, \chi_j, d\chi_{j+1}, \ldots, d\chi_n) \cong \mathbb{P}(\chi)$$

of algebraic varieties for any natural number d such that $\gcd(d, \chi_j) = 1$, and every $0 \leq j \leq n$ [9], [13]. So no generality is lost by insisting that χ is *normalised*, in the sense that

$$\gcd(\chi_0, \ldots, \widehat{\chi_j}, \ldots, \chi_n) = 1 \tag{3.1}$$

for every $0 \leq j \leq n$.

DEFINITIONS 3.2. The weight vector χ is
1. *weakly divisive* if χ_j divides χ_n for every $0 \leq j < n$,
2. *divisive* if χ_{j-1} divides χ_j for every $1 \leq j \leq n$,
3. *p-primary* if every χ_j is a power of a fixed prime p.

If χ is divisive, then $q_j = q_j(\chi)$ denotes the integer χ_j/χ_{j-1}, for $1 \leq j \leq n$; by normalisation, $q_1 = 1$.

So divisive implies weakly divisive, but not conversely. In fact χ is divisive precisely when the reverse sequence $\chi_n, \ldots, \chi_0$ is *well ordered*, in the sense of Nishimura–Yoshimura [15]; then

$$* = \mathbb{P}(\chi_n), \mathbb{P}(\chi_{n-1}, \chi_n) \setminus \mathbb{P}(\chi_n), \ldots, \mathbb{P}(\chi_0, \ldots, \chi_n) \setminus \mathbb{P}(\chi_1, \ldots, \chi_n)$$

is a cell decomposition of $\mathbb{P}(\chi)$ with one cell in every even dimension (compare [6, Remark 3.2] and [15, Proposition 2.3]). This decomposition describes the canonical cells that arise from Corollary 3.8 below.

Lemma 3.3. *Every normalised p-primary weight vector π may be ordered so as to take the form*

$$(1, 1, p^{k_2}, \ldots, p^{k_n}) \tag{3.4}$$

for some sequence $k_2 \leq \cdots \leq k_n$ of exponents.

Proof. Since $g(\pi) = 1$, it follows that $\pi_j = 1$ for some j. Applying (3.1), and reordering if necessary, yields the required form. □

Of course (3.4) is divisive, and identifies k_0 and k_1 as 0.

DEFINITION 3.5. If (X_i) is a sequence of topological spaces for $0 \leq i \leq n$, and (θ_i) a sequence of vector bundles over X_i for $0 \leq i < n$, then X_n is an *n-fold iterated*

Thom space over X_0 whenever X_i is homeomorphic to the Thom space $Th(\theta_{i-1})$ for every $1 \leq i \leq n$.

EXAMPLE 3.6. If ζ_i denotes the dual Hopf line bundle over $\mathbb{C}P^i$ for every $i \geq 0$, then the standard homeomorphisms $\mathbb{C}P^i \cong Th(\zeta_{i-1})$ display $\mathbb{C}P^n$ as an n-fold iterated Thom space over the one-point space $\mathbb{C}P^0$; or, alternatively, as an $(n-1)$-fold iterated Thom space over the 2-sphere $\mathbb{C}P^1$.

In Example 3.6, ζ_i may equally well be replaced by the Hopf bundle itself.

Theorem 3.7. *If χ is weakly divisive, then $\mathbb{P}(\chi)$ is homeomorphic to the Thom space of a complex line bundle over $\mathbb{P}(\chi')$.*

Proof. Al Amrani's proof of [3, I.1 (c)] applies to the line bundle ψ' over $\mathbb{P}(\chi')$, and shows that the unit disc bundle $D(\psi')$ is homeomorphic to the neighbourhood $N\mathbb{P}(\chi')$ of $\mathbb{P}(\chi') \subset \mathbb{P}(\chi)$, as defined in (2.6). The unit sphere bundle $S(\psi')$ is therefore homeomorphic to the weighted lens space $L(\chi_n; \chi')$, and the cofibre sequence (2.8) identifies $\mathbb{P}(\chi)$ with the Thom space $Th(\psi')$. □

Corollary 3.8. *If χ is divisive, then $\mathbb{P}(\chi)$ is homeomorphic to an n-fold iterated Thom space of complex line bundles over the one-point space $*$.*

By analogy with Remark 2.9, an n-fold iterated Thom space may also be expressed as an iterated pushout, and therefore as a homotopy colimit over the category $\mathrm{CAT}(\partial\Delta^n)$.

In order to apply Corollary 3.8 further, the maps ϕ and ψ of (2.10) must be generalised. A suitable context is provided by interpreting weight vectors as elements of the multiplicative monoid $\mathbb{N}^{n+1}$, with identity element $1 = (1, \ldots, 1)$. Given any two such χ and ω, there exists a smallest positive integer $s = s(\chi, \omega)$ such that ω divides $s\chi$. The resulting quotient has coordinates $s\chi_j/\omega_j$ for $0 \leq j \leq n$, and is conveniently denoted by χ/ω; so equations such as

$$\begin{aligned} &s = (s, \ldots, s), \quad \omega(\chi/\omega) = s\chi, \quad (\omega\chi)/\omega = \chi, \\ &\chi = \chi/1, \quad \text{and} \quad 1/\chi = l(\chi)/\chi \end{aligned} \tag{3.9}$$

hold amongst weight vectors.

Every χ may then be expressed as a product of indecomposables. For any $0 \leq j \leq n$ and any prime p, write χ_j as $p^{a(j)}\alpha_j$, where $p^{a(j)}$ denotes the p-content of χ_j and $\gcd(p, \alpha_j) = 1$.

DEFINITION 3.10. The *p-content* of χ is the p-primary weight vector

$$_p\chi := (p^{a(0)}, \ldots, p^{a(n)}),$$

which satisfies $\chi = {}_p\chi\alpha$ in $\mathbb{N}^{n+1}$; the *primary decomposition* of χ is the factorisation $\chi = {}_{p_1}\chi \cdots {}_{p_m}\chi$, as p_i ranges over the prime factors of the χ_j.

If χ is normalised then so is ${}_p\chi$, but the non-decreasing property is *not* hereditary in this sense. It follows from Definition 3.10 that $\alpha = \prod_{p_i \neq p} {}_{p_i}\chi$, and that $l(\chi) = p^{m(a)} l(\alpha)$ where $m(a) := \max_i a(i)$.

REMARK 3.11. Recent results of [6] show that, amongst weighted projective spaces, the homotopy type of $\mathbb{P}(\chi)$ is determined by the *unordered* coordinates of its non-trivial p-contents ${}_p\chi$, for normalised χ. The ${}_p\chi$ may therefore assumed to be non-decreasing, and remultiplied to give a weight vector χ^* for which there exists a homotopy equivalence $\mathbb{P}(\chi) \simeq \mathbb{P}(\chi^*)$. Moreover χ^* is divisive by construction, so Corollary 3.8 implies that every $\mathbb{P}(\chi)$ is homotopy equivalent to an iterated Thom space. If the weights are pairwise coprime then χ^* takes the form $(1, \ldots, 1, c)$ and $\mathbb{P}(\chi^*)$ reduces to $Th(\zeta^c_{n-1})$ over $\mathbb{C}P^{n-1}$, where $c = \prod_i \chi_i$.

DEFINITIONS 3.12. The map $e(\chi/\omega)\colon \mathbb{P}(\omega) \to \mathbb{P}(\chi)$ is given by

$$e(\chi/\omega)([z_0, \ldots, z_n]) = [z_0^{s\chi_0/\omega_0}, \ldots, z_n^{s\chi_n/\omega_n}],$$

where $s = s(\chi, \omega)$, and coordinates are normalised as necessary; the group $C_{\chi/\omega}$ is the product

$$C_{s\chi_0/\omega_0} \times \cdots \times C_{s\chi_n/\omega_n}$$

of cyclic groups, considered as a subgroup of T^{n+1}.

Following (3.9), the cases $e(\phi\chi/\chi)$ and $C_{\phi\chi/\chi}$ reduce to $e(\phi)$ and C_ϕ respectively. By definition, $e(r) = e(r, \ldots, r)$ raises homogeneous coordinates in $\mathbb{P}(\chi)$ to the r-th power, and is therefore known as the *r-th power map* on $\mathbb{P}(\chi)$.

Proposition 3.13. *The map $e(\chi/\omega)$ is the orbit map of the natural action of $C_{\chi/\omega}$ on $\mathbb{P}(\omega)$.*

Proof. Note first that $e(\chi/\omega)([y_0, \ldots, y_n]) = e(\chi/\omega)([z_0, \ldots, z_n])$ holds in $\mathbb{P}(\chi)$ precisely when

$$[y_0, \ldots, y_n] \in \{[\lambda_0 z_0, \ldots, \lambda_n z_n] \colon \lambda_0^{h\omega_0/\chi_0} = \cdots = \lambda_n^{h\omega_n/\chi_n} = 1\}$$

in $\mathbb{P}(\omega)$. Since $e(\chi/\omega)$ is clearly surjective, the result follows. □

Corollary 3.14. *Any weighted projective space arises as the orbit space of any other of the same dimension, under the action of a finite abelian group.*

REMARK 3.15. In the language of Definition 3.12 and Proposition 3.13, Al Amrani's maps $\phi(\chi)$ and $\psi(\chi)$ are given by $e(\chi)$ and $e(1/\chi)$ respectively. Kawasaki [14, p. 243] notes that $\phi(\chi)$ is the orbit map of the action of C_χ on $\mathbb{C}P^n$.

Proposition 3.16. *For any weight vectors ω, σ and χ, the composition*

$$\mathbb{P}(\omega) \xrightarrow{e(\sigma/\omega)} \mathbb{P}(\sigma) \xrightarrow{e(\chi/\sigma)} \mathbb{P}(\chi)$$

factorises as $e(s') \circ e(\chi/\omega) = e(\chi/\omega) \circ e(s')$, where s' denotes the natural number $s(\omega, \sigma)s(\sigma, \chi)/s(\omega, \chi)$.

Proof. It suffices to note that the given composition acts on homogeneous coordinates in $\mathbb{P}(\chi)$ by $z_i \mapsto z_i^{s(\omega,\sigma)s(\sigma,\chi)\chi_i/\omega_i}$, for $0 \leq i \leq n$. □

Proposition 3.16 implies the factorisations

$$e(\chi/\omega) \circ e(\omega/\chi) = e(s'') \quad \text{and} \quad e(\omega/\chi) \circ e(\chi/\omega) = e(s''),$$

where $s'' := s(\omega, \chi)s(\chi, \omega)$. Similarly, $e(\chi) \circ e(1/\omega) = e(l(\omega)/s(\omega, \chi))$ for any weight vectors χ and ω.

DEFINITION 3.17. For any weight vector χ and any prime p, the *p-primary part* of $\mathbb{P}(\chi)$ is the weighted projective space $\mathbb{P}({}_p\chi)$; the canonical maps

$$e({}_p\chi/\chi)\colon \mathbb{P}(\chi) \to \mathbb{P}({}_p\chi) \quad \text{and} \quad e(\chi/{}_p\chi)\colon \mathbb{P}({}_p\chi) \to \mathbb{P}(\chi)$$

are *p-extraction* and *p-insertion* respectively.

In the notation of Definition 3.10, extraction and insertion are given by

$$(3.18)\qquad \begin{aligned} e({}_p\chi/\chi)[z_0, \ldots, z_n] &= [z_0^{l(\alpha)/\alpha_0}, \ldots, z_n^{l(\alpha)/\alpha_n}] \quad \text{and} \\ e(\chi/{}_p\chi)[z_0, \ldots, z_n] &= [z_0^{\alpha_0}, \ldots, z_0^{\alpha_n}], \end{aligned}$$

in terms of homogeneous coordinates. By Proposition 3.16, the compositions $e(\chi/{}_p\chi) \circ e({}_p\chi/\chi)$ and $e({}_p\chi/\chi) \circ e(\chi/{}_p\chi)$ reduce to the appropriate power maps $e(l(\alpha))$.

REMARK 3.19. It follows immediately from Lemma 3.3 and Corollary 3.8 that every p-primary part $\mathbb{P}({}_p\chi)$ is an iterated Thom space over $*$.

EXAMPLE 3.20. The 2-, 3-, and 5-primary parts of $\mathbb{P}(3, 4, 5)$ are $\mathbb{P}(1, 4, 1)$, $\mathbb{P}(3, 1, 1)$, and $\mathbb{P}(1, 1, 5)$ respectively. They form the codomains of the 2-, 3-, and 5-extraction maps, whose values on $[z_0, z_1, z_2]$ are given by

$$[z_0^5, z_1^{15}, z_2^3], \quad [z_0^{20}, z_1^5, z_2^4], \quad \text{and} \quad [z_0^4, z_1^3, z_2^{12}]$$

respectively. The 2-, 3-, and 5-primary parts also form the domains of the 2-, 3-, and 5-insertion maps, whose values on $[z_0, z_1, z_2]$ are given by

$$[z_0^3, z_1, z_2^5], \quad [z_0, z_1^4, z_2^5], \quad \text{and} \quad [z_0^3, z_1^4, z_2]$$

respectively.

4. Geometric reassembly

The problem of reassembling $\mathbb{P}(\chi)$ from its p-primary parts must now be addressed. The solution is best understood in terms of weight vectors σ and σ', and commutative squares of the form

$$\begin{array}{ccc} \mathbb{P}(\sigma\sigma') & \xrightarrow{e(1/\sigma')} & \mathbb{P}(\sigma) \\ {\scriptstyle e(1/\sigma)}\downarrow & & \downarrow{\scriptstyle e(1/\sigma)} \\ \mathbb{P}(\sigma') & \xrightarrow[e(1/\sigma')]{} & \mathbb{C}P^n \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{C}P^n & \xrightarrow{e(\sigma)} & \mathbb{P}(\sigma) \\ {\scriptstyle e(\sigma')}\downarrow & & \downarrow{\scriptstyle e(\sigma')} \\ \mathbb{P}(\sigma') & \xrightarrow[e(\sigma)]{} & \mathbb{P}(\sigma\sigma'); \end{array} \tag{4.1}$$

these may be incorporated into a single generalised square, although restrictions must be imposed upon σ and σ'.

For any weight vector χ, it is convenient to write $Q(\chi) \subset \mathbb{Z}$ for the set of primes that occur non-trivially in its p-primary decomposition.

DEFINITION 4.2. Two weight vectors σ and σ' are *coprime* if $Q(\sigma) \cap Q(\sigma') = \varnothing$; in other words, if $\gcd(\sigma_j, \sigma'_k) = 1$ for every $0 \le j, k \le n$.

In the context of Definition 3.12, this condition is equivalent to the coprimality of $|C_\sigma|$ and $|C_{\sigma'}|$. It also implies the weaker condition, that

$$h(g_0, \ldots, g_n) = ((g_0^{\sigma_0}, \ldots, g_n^{\sigma_n}), (g_0^{\sigma'_0}, \ldots, g_n^{\sigma'_n})) \tag{4.3}$$

defines an isomorphism $h\colon C_{\sigma\sigma'} \to C_{\sigma'} \times C_\sigma$; or equivalently, that $C_{\sigma\sigma'}$ is generated by the subgroups C_σ and $C_{\sigma'}$.

REMARK 4.4. Since $Q(\sigma) = Q(1/\sigma)$, it follows that σ and σ' are coprime if and only if $1/\sigma$ and $1/\sigma'$ are coprime.

Proposition 4.5. *If σ and σ' are coprime, then the left-hand diagram* (4.1) *is a pullback square.*

Proof. The pullback of the diagram $\mathbb{P}(\sigma') \to \mathbb{C}P^n \leftarrow \mathbb{P}(\sigma)$ is the subspace

$$P := \{(z', z)\colon e(1/\sigma')(z') = e(1/\sigma)(z)\} \subset \mathbb{P}(\sigma') \times \mathbb{P}(\sigma),$$

and the canonical map $f\colon \mathbb{P}(\sigma\sigma') \to P$ acts by $f(z) = (e(1/\sigma)(z), e(1/\sigma')(z))$. Since P is Hausdorff, it remains to show that f is bijective.

The maps $e(1/\sigma)$ and $e(1/\sigma')$ are quotients by $C_{1/\sigma}$ and $C_{1/\sigma'}$ respectively, by Proposition 3.13. So $C_{1/\sigma'} \times C_{1/\sigma}$ acts on P, and f is equivariant with respect to the isomorphism h of (4.3). The projection $p\colon P \to \mathbb{P}(\sigma)$ is the quotient by $C_{1/\sigma}$, and is equivariant with respect to the projection $C_{1/\sigma'} \times C_{1/\sigma} \to C_{1/\sigma'}$; the corresponding statement for $p'\colon P \to \mathbb{P}(\sigma')$ holds by analogy. Both $e(1/\sigma)$ and p are surjective, and the former factorises as $p \circ f\colon \mathbb{P}(\sigma\sigma') \to \mathbb{P}(\sigma)$; therefore f is surjective, by equivariance.

To confirm injectivity, choose $x \in \mathbb{P}(\sigma\sigma')$ and let $G < C_{1/\sigma}$ be the isotropy group of $y = f(x)$. Then $c := |f^{-1}(y)| = |Gx|$ divides $|G|$, so c divides $|C_{1/\sigma}|$. Replacing σ by σ' and applying the corresponding reasoning shows that c also divides $|C_{1/\sigma'}|$. But σ and σ' are coprime, so $c = 1$ as sought. □

REMARK 4.6. Since all the maps in the square are algebraic, Proposition 4.5 also holds in the category of complex algebraic varieties.

The following example shows that the square is not generally a pullback.

EXAMPLE 4.7. If $\sigma = (1, 2, 2)$ and $\sigma' = (2, 1, 2)$, then $\mathbb{P}(\sigma) \cong \mathbb{P}(\sigma') \cong \mathbb{C}P^2$ and $e(1/\sigma)$ and $e(1/\sigma')$ are homeomorphisms. So the pullback is also homeomorphic to $\mathbb{C}P^2$, and cannot be $\mathbb{P}(\sigma\sigma') \cong \mathbb{P}(1,1,2)$ because the latter contains a singular point. Proposition 3.13 confirms that the canonical map $f\colon \mathbb{P}(\sigma\sigma') \to \mathbb{C}P^2$ is the quotient by $C_2 \times C_2$.

An example with normalised weights is given by increasing the dimensions, with additional weights 1; thus $\sigma = (1, 1, 2, 2)$ and $\sigma' = (1, 2, 1, 2)$.

Remarkably, weighted projective spaces may also be expressed as pushouts.

Proposition 4.8. *If σ and σ' are coprime, then the right-hand diagram* (4.1) *is a pushout square.*

Proof. The pushout of the diagram $\mathbb{P}(\sigma') \leftarrow \mathbb{C}P^n \to \mathbb{P}(\sigma)$ is the space

$$R = (\mathbb{P}(\sigma') \sqcup \mathbb{P}(\sigma))/\sim,$$

where the equivalence relation is generated by $e(\sigma')(z) \sim e(\sigma)(z)$ for any z in $\mathbb{C}P^n$; the canonical map $g\colon R \to \mathbb{P}(\sigma\sigma')$ is defined by

$$g([e(\sigma')(z)]) = g([e(\sigma)(z)]) = e(\sigma\sigma')(z).$$

Since R is compact, it suffices to show that g is bijective.

Both $e(\sigma)$ and $e(\sigma')$ are surjective, and the natural map $q\colon \mathbb{C}P^n \to R$ is given by $q(z) = [e(\sigma)(z)] = [e(\sigma')(z)]$; so q is surjective. It is also invariant with respect to the action of $C_{\sigma\sigma'}$, since the latter is generated by the subgroups $C_{\sigma'}$ and C_{σ} (via (4.3)), which act trivially on $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma')$ respectively. Hence q induces a surjective map $\bar{q}\colon \mathbb{P}(\sigma'\sigma) \to R$.

The map $e(\sigma'\sigma)\colon \mathbb{C}P^n \to \mathbb{P}(\sigma'\sigma)$ factorises as $g \circ q$, so $g \circ \bar{q} = 1$ on $\mathbb{P}(\sigma'\sigma)$, and $\bar{q}$ is injective. Thus $\bar{q}$ is bijective, and has inverse g, as required. □

In order to state the main reassembly theorem, it is convenient to write $[m]$ for the simplicial complex consisting of m disjoint vertices.

Theorem 4.9. *The weighted projective space $\mathbb{P}(\chi)$ is homeomorphic to the limit of the* $\mathrm{CAT}^{op}[m]$*-diagram*

$$\begin{array}{ccccc} & \iddots & \mathbb{P}({}_{p_i}\chi) & \ddots & \\ & & \big\downarrow{\scriptstyle e(1/{}_{p_i}\chi)} & & \\ \mathbb{P}({}_{p_1}\chi) & \xrightarrow[e(1/{}_{p_1}\chi)]{} & \mathbb{C}P^n & \xleftarrow[e(1/{}_{p_m}\chi)]{} & \mathbb{P}({}_{p_m}\chi), \end{array}$$

and the associated universal maps $\mathbb{P}(\chi) \to \mathbb{P}({}_{p_i}\chi)$ may be identified with $e({}_{p_i}\chi/\chi)$ for every $1 \leq i \leq m$. Similarly, $\mathbb{P}(\chi)$ is also homeomorphic to the colimit of the $\mathrm{CAT}[m]$*-diagram*

$$\begin{array}{ccccc} & \iddots & \mathbb{P}({}_{p_i}\chi) & \ddots & \\ & & \big\uparrow{\scriptstyle e({}_{p_i}\chi)} & & \\ \mathbb{P}({}_{p_1}\chi) & \xleftarrow[e({}_{p_1}\chi)]{} & \mathbb{C}P^n & \xrightarrow[e({}_{p_m}\chi)]{} & \mathbb{P}({}_{p_m}\chi), \end{array}$$

and the associated universal maps $\mathbb{P}({}_{p_i}\chi) \to \mathbb{P}(\chi)$ may be identified with $e(\chi/{}_{p_i}\chi)$ for every $1 \leq i \leq m$.

Proof. Proceed by induction on m, noting that the results are trivial for $m = 1$.

Suppose that $Q(\chi) = \{p_1, \ldots, p_k, p\}$ and $\chi_i = p^{a(i)}\alpha_i$, as in Definition 3.10; so $Q(\alpha) = \{p_1, \ldots, p_k\}$. By the inductive hypotheses, $\mathbb{P}(\alpha)$ is homeomorphic to the pullback of the $\mathbb{P}({}_{p_i}\chi)$ along the maps $e(1/{}_{p_i}\chi)$, and to the pushout of the $\mathbb{P}({}_{p_i}\chi)$ along the maps $e({}_{p_i}\chi)$; also, the universal maps $\mathbb{P}(\alpha) \to \mathbb{P}({}_{p_i}\chi)$ and $\mathbb{P}({}_{p_i}\chi) \to \mathbb{P}(\alpha)$ are given by $e({}_{p_i}\chi/\chi)$ and $e(\chi/{}_{p_i}\chi)$ respectively, for $1 \leq i \leq k$. It therefore remains to prove that $\mathbb{P}(\chi)$ is homeomorphic to the pullback of

$$\mathbb{P}({}_{p}\chi) \xrightarrow{e(1/{}_{p}\chi)} \mathbb{C}P^n \xleftarrow{e(1/\alpha)} \mathbb{P}(\alpha) \tag{4.10}$$

and the pushout of

$$\mathbb{P}({}_{p}\chi) \xleftarrow{e({}_{p}\chi)} \mathbb{C}P^n \xrightarrow{e(\alpha)} \mathbb{P}(\alpha) \tag{4.11}$$

respectively, and to confirm the identity of the associated maps $\mathbb{P}(\chi) \to \mathbb{P}({}_p\chi)$, $\mathbb{P}(\chi) \to \mathbb{P}(\alpha)$, $\mathbb{P}({}_p\chi) \to \mathbb{P}(\chi)$, and $\mathbb{P}(\alpha) \to \mathbb{P}(\chi)$. These follow directly from Proposition 4.5 and Proposition 4.8 respectively, because ${}_p\chi$ and α are coprime. The induction is then complete. □

EXAMPLE 4.12. Theorem 4.9 applies to Example 3.20, and expresses $\mathbb{P}(3, 4, 5)$ as the limit and colimit of the CAT[3]-diagrams

$$\begin{array}{ccccc} & & \mathbb{P}_3 & & \\ & & \big\downarrow{\scriptstyle e_{1/3}} & & \\ \mathbb{P}_2 & \xrightarrow[e_{1/2}]{} & \mathbb{C}P^2 & \xleftarrow[e_{1/5}]{} & \mathbb{P}_5 \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \mathbb{P}_3 & & \\ & & \big\uparrow{\scriptstyle e_{3}} & & \\ \mathbb{P}_2 & \xleftarrow[e_{2}]{} & \mathbb{C}P^2 & \xrightarrow[e_{5}]{} & \mathbb{P}_5 \end{array}$$

respectively, where $\mathbb{P}_2 := \mathbb{P}(1, 4, 1)$, $\mathbb{P}_3 := \mathbb{P}(3, 1, 1)$, and $\mathbb{P}_5 := \mathbb{P}(1, 1, 5)$.

Observe that $\mathbb{P}(1,5,12)$, $\mathbb{P}(1,4,15)$, $\mathbb{P}(1,3,20)$, and $\mathbb{P}(1,1,60)$ may also be obtained by recombining $\mathbb{P}(1, 4, 1)$, $\mathbb{P}(3, 1, 1)$, and $\mathbb{P}(1, 1, 5)$ with permuted coordinates. No two of the four are homeomorphic, as their singularity structure shows; but the results of [6] (as described in Remark 3.11 above) prove that all four are homotopy equivalent to $\mathbb{P}(3, 4, 5)$. The fact that their cohomology rings are isomorphic is noted in [5], and reproven in Theorem 6.15 below; it is also, of course, implicit in [14].

5. Iterated Thom isomorphisms

From this point onwards, $E^*(-)$ denotes a complex oriented cohomology theory, with orientation class x^E. As described in Section 1, the crucial examples are: $H^*(-)$, with the Thom orientation; $K^*(-)$, with the Conner–Floyd orientation; and $\Omega_U^*(-)$, with the universal orientation. In particular, $H^*(X)$ denotes the reduced integral cohomology ring of any space X.

The existence of a Thom class t^E leads to the *Thom isomorphism*, which features in the following standard result.

Proposition 5.1. *For any k-dimensional complex vector bundle θ over X, the E_*-algebra $E^*(Th(\theta))$ is a free module over $E^*(X_+)$ on the single generator $t^E(\theta)$; its multiplicative structure is determined by the relation*

$$(t^E(\theta))^2 = c_k^E(\theta) \cdot t^E(\theta) \tag{5.2}$$

in $E^{4k}(Th(\theta))$.

More explicitly, the Thom isomorphism

$$\cdot t^E(\theta)\colon E^*(X_+) \xrightarrow{\cong} E^{*+2k}(Th(\theta))$$

is given by forming the relative cup product with $t^E(\theta)$, and is induced by the relative diagonal map $\delta\colon Th(\theta) \to X_+ \wedge Th(\theta)$.

Proposition 5.1 applies to Definition 3.5 whenever the bundles θ_i are complex. In this case, the cohomology algebra $E^*(X_i)$ arises from $E^*(X_0)$ by means of i-fold iterated Thom isomorphisms. For example, if $X_0 = *$ and $\dim_{\mathbb{C}} \theta_0 = k$, then the first iteration identifies $E^*(X_1)$ with $E_*[t]/(t^2)$, where t lies in $E^{2k}(X_1)$; this, of course, is because X_1 is homeomorphic to S^{2k}. Further iterations require the Chern classes of the θ_i.

In subsequent applications, the θ_i are line bundles. Over X, the isomorphism classes of such bundles form an abelian group with respect to tensor product, which is isomorphic to $H^2(X)$ under the Chern class $c_1(-)$. So any $w \in H^2(X)$ gives rise to a complex line bundle

$$\lambda = \lambda(w) \quad \text{such that} \quad c_1(\lambda(w)) = w; \tag{5.3}$$

it is unique up to isomorphism, and $\lambda(d_1 w) = \lambda(w)^{d_1}$ for any integer d_1. The Thom class $t(\lambda(w)^{d_1})$ lies in $H^2(Th(\lambda(w)^{d_1}))$, and the second stage of the iteration begins with a cohomology class $d_2 t(\lambda(w)^{d_1})$, for some integer d_2. The corresponding line bundle is $\lambda(w)^{d_1,d_2} := \lambda(t(\lambda(w)^{d_1}))^{d_2}$ over $Th(\lambda(w)^{d_1})$. In this language, the n-th stage identifies the bundle

$$\lambda(w)^{d_1,\dots,d_n} := \lambda(t(\lambda(w)^{d_1,\dots,d_{n-1}}))^{d_n} \quad \text{over} \quad Th(\lambda(w)^{d_1,\dots,d_{n-1}}). \tag{5.4}$$

In other words, $X_n = Th(\lambda(w)^{d_1,\dots,d_n})$ is an n-fold iterated Thom space over $X_0 = X$, for which $\theta_0 = \lambda(w)^{d_1}$, $\theta_1 = \lambda(w)^{d_1,d_2}, \dots, \theta_{n-1} = \lambda(w)^{d_1,\dots,d_n}$.

It is perfectly acceptable to choose $X_0 = *$, in which case $w = 0$ and $\lambda(w)$ is the trivial line bundle $\mathbb{C}$. In this context, Corollary 3.8 may be combined with Al Amrani's proof of [3, I.1 (c)] to provide a homeomorphism

$$\mathbb{P}(\chi) \cong Th(\lambda(0)^{1,q_2,\dots,q_n}) \tag{5.5}$$

for any divisive weight vector χ, where $q_j = \chi_j/\chi_{j-1}$ as in Definition 3.2.

In general, $E^*(X_n)$ may be computed by iterating Proposition 5.1 and exploiting two consequences of the fact that θ has dimension 1. Firstly, the E-theory Thom class satisfies

$$t^E(\theta) = c_1^E(\lambda(t(\theta))) \tag{5.6}$$

in $E^2(Th(\theta))$, which follows directly from the universal example ζ over $\mathbb{C}P^\infty$. Secondly, for any integer r, the equation

$$c_1^E(\theta^r) = [r](c_1^E(\theta)) \tag{5.7}$$

holds in $E^2(X)$, where $[r]$ denotes the *r-series* of the formal group law F_E associated to x^E [12]. Thus

$$[r](u) \equiv ru + \frac{1}{2}r(r-1)a^E u^2 \mod (u^3)$$

in $E^*[\![u]\!]$, where $a^E \in E_2$ is the coefficient of u_1u_2 in $F_E(u_1, u_2)$.

Theorem 5.8. *For any divisive χ, the E_*-algebra $E^*(\mathbb{P}(\chi))$ is isomorphic to*

$$E_*[w_n, w_{n-1}w_n, \ldots, w_1w_2\cdots w_n]/J^E, \tag{5.9}$$

where $w_hw_{h+1}\cdots w_n$ lies in $E^{2(n-h+1)}(\mathbb{P}(\chi))$ for any $h \le i$, and J^E denotes the ideal generated by elements of the form

$$(w_i - [q_i](w_{i-1}))w_i\cdots w_n$$

for $1 \le i \le n$; also $w_0 = 0$.

REMARK 5.10. The elements w_i do not themselves exist in $E^2(\mathbb{P}(\chi))$ for any $i \ne n$, but appear only in monomials divisible by $w_hw_{h+1}\cdots w_n$ for some $h \le i$. Nevertheless, the description provided by (5.9) is notationally convenient, and encodes the product structure by repeated application of the relations in J^E.

Proof of Theorem 5.8. Combine Proposition 5.1 with (5.4), (5.5), (5.6), and (5.7). Then the first stage identifies $E^*(\mathbb{P}(1,1)_+)$ as $E_*[w_1]/(w_1^2)$, where $w_1 := t^E(\lambda(0))$ and $\lambda(0) = \mathbb{C}$ over $*$. The n-th stage identifies $E^*(\mathbb{P}(\chi))$ as a free $E^*(\mathbb{P}(\chi')_+)$-module on the single generator

$$w_n := t^E(\lambda(0)^{1,q_2,\ldots,q_n}),$$

with the relation $w_n^2 = [q_n](w_{n-1})w_n$ of (5.2). □

EXAMPLE 5.11. The formal group law associated to integral cohomology is additive, and its r-series is given by $[r](u) = ru$ in $\mathbb{Z}[\![u]\!]$. So for any p-primary weight vector $\pi = (1, 1, p^{k_2}, \ldots, p^{k_n})$, (5.9) identifies $H^*(\mathbb{P}(\pi))$ with

$$\mathbb{Z}[w_n, w_{n-1}w_n, \ldots, w_1w_2\cdots w_n]/J, \tag{5.12}$$

where J is generated by elements of the form

$$(w_i - p^{k_i-k_{i-1}}w_{i-1})w_i\cdots w_n \tag{5.13}$$

for $1 \le i \le n$, and $w_0 = 0$. In fact (5.12) is isomorphic to $\mathbb{Z}[v_1,\ldots,v_n]/I(\pi)$ of (2.15), where Kawasaki's ideal $I(\pi)$ is generated by the relations

$$v_1^j = m(\pi)_j v_j, \quad \text{where} \quad m(\pi)_j = p^{(j-1)k_n}/p^{k_{n-1}+\cdots+k_{n-j+1}}, \tag{5.14}$$

for $2 \le j \le n$ [14, p. 243]. The isomorphism arises from the bijection of generators $w_{n-j+1} \cdots w_n \leftrightarrow v_j$, by repeated application of (5.13); it is multiplicative because $w_n^j = \prod_{h=1}^{j-1} p^{k_n - k_{n-h}} w_{n-j+1} \cdots w_n$ by induction on j.

REMARKS 5.15. Rationally, Theorem 5.8 states that $E\mathbb{Q}^*(\mathbb{P}(\chi)_+)$ is isomorphic to

$$E\mathbb{Q}_*[w_n]/(w_n^{n+1}).$$

Furthermore, if $\chi = 1$, then $\mathbb{P}(\chi)$ reduces to $\mathbb{C}P^n$, and Theorem 5.8 identifies w_n with x^E, and w_n^j with $w_{n-j+1} \cdots w_n$ for every $j \ge 1$.

These observations illustrate the homotopy equivalences of Remark 2.17.

EXAMPLE 5.16. The 2-primary part of $\mathbb{P}(1, 2, 3, 4)$ is $\mathbb{P}(1, 2, 1, 4)$, which is a 2-fold iterated Thom space over $\mathbb{P}(1, 1) = \mathbb{C}P^1$. So $E^*(\mathbb{P}(1, 1)_+)$ is isomorphic to $E_*[w_1]/(w_1^2)$, as in the proof of Theorem 5.8. Furthermore, $\lambda(0)^{1,2} \cong \zeta_1^2$ over $\mathbb{C}P^1$, and $\mathbb{P}(1, 1, 2)$ is homeomorphic to $Th(\zeta_1^2)$; thus $E^*(\mathbb{P}(1, 1, 2)_+)$ is isomorphic to

$$E_*[w_2,\, w_1w_2]/J_1^E,$$

where $w_2 = t^E(\zeta_1^2) = c_1^E(\lambda(t(\zeta_1^2)))$ by (5.6), and J_1^E is the ideal generated by

$$w_1^2 w_2 \quad \text{and} \quad w_2^2 - 2w_1w_2.$$

Similarly, $\mathbb{P}(1,1,2,4)$ is homeomorphic to $Th(\zeta_1^{2,2})$; so $E^*(\mathbb{P}(1,1,2,4)_+)$ is isomorphic to

$$E_*[w_3,\, w_2w_3,\, w_1w_2w_3]/J_2^E,$$

where $w_3 = t^E(\zeta_1^{2,2}) = c_1^E(\lambda(t(\zeta_1^{2,2})))$, and J_2^E is generated by

$$w_1^2w_2w_3, \quad (w_2^2 - 2w_1w_2)w_3, \quad \text{and} \quad w_3^2 - 2w_2w_3 - a^E w_2^2 w_3.$$

The 3-primary part of $\mathbb{P}(1,2,3,4)$ is $\mathbb{P}(1,1,3,1)$, which is a Thom space over $\mathbb{P}(1,1,1) = \mathbb{C}P^2$. So $E^*(\mathbb{P}(1,1,1)_+)$ is isomorphic to $E_*[w_2]/(w_2^3)$, where $w_2 = t^E(\lambda(0)^{1,1})$ generates $E^2(\mathbb{P}(1,1,1))$ and $w_2^2 = w_1w_2$. Moreover, $\lambda(0)^{1,1,3} \cong \zeta_2^3$ over $\mathbb{C}P^2$, and $\mathbb{P}(1,1,1,3)$ is homeomorphic to $Th(\zeta_2^3)$; so $E^*(\mathbb{P}(1, 1, 1, 3)_+)$ is isomorphic to

$$E^*[w_3,\, w_2w_3,\, w_1w_2w_3]/J_3^E,$$

where $w_3 = t^E(\zeta_2^3) = c_1^E(\lambda(t(\zeta_2^3)))$, and J_3^E is generated by

$$w_1^2w_2w_3, \quad (w_2^2 - w_1w_2)w_3, \quad \text{and} \quad w_3^2 - 3w_2w_3 - 3a^E w_2^2 w_3.$$

The multiplicative formal group law is associated to complex K-theory and the Conner–Floyd orientation. The coefficient ring is $K_* \cong \mathbb{Z}[z, z^{-1}]$, and the element $zx^K \in K^0(\mathbb{C}P^\infty)$ is represented by the virtual Hopf bundle $\zeta - \mathbb{C}$. The r-series is induced by the tensor power map $\zeta \mapsto \zeta^r$, and is therefore given by

$$[r](u) = z^{-1}((1+zu)^r - 1) \tag{5.17}$$

in $K_*[\![u]\!]$, for any integer r. Al Amrani's results of [2] may then be recovered.

EXAMPLE 5.18. Theorem 5.8 and (5.17) combine to show that, for any integer r, the K_*-algebra $K^*(\mathbb{P}(1, \ldots, 1, r)_+)$ is isomorphic to

$$K_*[w_n, w_{n-1}w_n, \ldots, w_1 w_2 \cdots w_n]/J^K, \tag{5.19}$$

where J^K denotes the ideal generated by elements of the form

$$(w_i - w_{i-1})w_i \cdots w_n \quad \text{for} \quad 1 \le i \le n-1,$$

and $(w_n - z^{-1}((1+zw_{n-1})^r - 1))w_n$. The latter is equivalent to

$$w_n^2 = \sum_{s=1}^{r} \binom{r}{s} z^{s-1} w_{n-s} \cdots w_n, \quad \text{where} \quad w_0 = 0.$$

6. Cohomological reassembly

It is now possible to follow the lead of Theorem 4.9 by reassembling the E_*-algebra $E^*(\mathbb{P}(\chi))$ from its constituent components $E^*(\mathbb{P}({}_{p_i}\chi))$.

For any weight vector χ, recall that $Q(\chi) = \{p_1, \ldots, p_m\}$ denotes the primes occurring in χ. The decomposition of Definition 3.10 may then be expressed as $\chi = {}_{p_i}\chi\alpha(i)$ for each $1 \le i \le m$, where $Q(\alpha(i)) = Q(\chi) \setminus \{p_i\}$. It is convenient to write $\mathbb{Z}_\chi$ for the subring $\mathbb{Z}[p_1^{-1}, \ldots, p_m^{-1}] < \mathbb{Q}$.

Proposition 6.1. *The E_*-algebra $E^*(\mathbb{P}(\chi)_+)$ is a free E_*-module, with one generator in each even dimension $\le 2n$.*

Proof. Consider the insertion map $e(\alpha(j))\colon \mathbb{P}({}_{p_j}\chi) \to \mathbb{P}(\chi)$ of (3.17), for some $1 \le j \le m$. By Proposition 3.13, it is the orbit map for the action of the finite group $C_{\alpha(j)}$, whose order is divisible by every p_i such that $i \ne j$. It therefore induces an isomorphism

$$e(\alpha(j))^*\colon H^*(\mathbb{P}(\chi)_+; \mathbb{Z}_{\alpha(j)}) \to H^*(\mathbb{P}({}_{p_j}\chi)_+; \mathbb{Z}_{\alpha(j)}). \tag{6.2}$$

Example 5.11 shows that the graded abelian group $H^*(\mathbb{P}({}_{p_j}\chi)_+)$ is free, with one generator in each even dimension $\le 2n$. So $H^*(\mathbb{P}(\chi)_+)$ contains at most p_i torsion, for

$1 \leq i \neq j \leq m$. Repeating the argument for every $1 \leq j \leq m$ in turn proves that $H^*(\mathbb{P}(\chi)_+)$ is torsion free, and therefore has one generator in each even dimension $\leq 2n$.

Since E_* is also torsion free and even dimensional, the Atiyah–Hirzebruch spectral sequence for $E^*(\mathbb{P}(\chi)_+)$ collapses, and the conclusion follows. □

Corollary 6.3. *For any weight vectors χ and σ, the induced homomorphism*

$$e(\sigma)^* \otimes 1\colon E^*(\mathbb{P}(\sigma\chi)_+) \otimes \mathbb{Z}_\sigma \to E^*(\mathbb{P}(\chi)_+) \otimes \mathbb{Z}_\sigma$$

is an isomorphism of algebras over $E_* \otimes \mathbb{Z}_\sigma$.

Proof. Proposition 6.1 implies that $e(\sigma)^*$ induces an isomorphism of E_2-terms of Atiyah–Hirzebruch spectral sequences, which collapse. It therefore induces the required isomorphism on their limits. □

Proposition 6.1 may, of course, be deduced from Kawasaki's calculations; as proven above, it follows from the theory of Thom spaces. Isomorphism (6.2) confirms that $e(\alpha)\colon \mathbb{P}({}_p\chi) \to \mathbb{P}(\chi)$ and $e(1/\alpha)\colon \mathbb{P}(\chi) \to \mathbb{P}({}_p\chi)$ are mutually inverse p-local homotopy equivalences, for any p in $Q(\chi)$.

The next step is to identify a cohomological version of Proposition 4.5, by applying $E^*(-)$ to the first diagram (4.1).

Proposition 6.4. *If σ and σ' are coprime, then the diagram*

$$\begin{array}{ccc} E^*(\mathbb{P}(\sigma\sigma')_+) & \xleftarrow{e(1/\sigma')^*} & E^*(\mathbb{P}(\sigma)_+) \\ {\scriptstyle e(1/\sigma)^*}\uparrow & & \uparrow{\scriptstyle e(1/\sigma)^*} \\ E^*(\mathbb{P}(\sigma')_+) & \xleftarrow[e(1/\sigma')^*]{} & E^*(\mathbb{C}P^n_+) \end{array} \tag{6.5}$$

is a pushout square; in other words, the canonical homomorphism

$$h\colon E^*(\mathbb{P}(\sigma)_+) \otimes_{E^*(\mathbb{C}P^n_+)} E^*(\mathbb{P}(\sigma')_+) \to E^*(\mathbb{P}(\sigma\sigma')_+)$$

is an isomorphism of E_*-algebras.

Proof. Corollary 6.3 ensures that the horizontal and vertical homomorphisms of (6.5) induce isomorphisms over $E_* \otimes \mathbb{Z}_{\sigma'}$ and $E_* \otimes \mathbb{Z}_\sigma$ respectively. The same is therefore true of the corresponding pushout square. Hence h induces an isomorphism of E_*-algebras over both $E_* \otimes \mathbb{Z}_{\sigma'}$ and $E_* \otimes \mathbb{Z}_\sigma$. But σ and σ' are coprime, so h is an isomorphism. □

The cohomological version of Proposition 4.8 has a similar proof, with arrows reversed.

Proposition 6.6. *If σ and σ' are coprime, then the diagram*

$$(6.7)\qquad \begin{array}{ccc} E^*(\mathbb{P}(\sigma\sigma')_+) & \xrightarrow{e(\sigma)^*} & E^*(\mathbb{P}(\sigma')_+) \\ {\scriptstyle e(\sigma')^*}\downarrow & & \downarrow{\scriptstyle e(\sigma')^*} \\ E^*(\mathbb{P}(\sigma)_+) & \xrightarrow[e(\sigma)^*]{} & E^*(\mathbb{C}P^n_+) \end{array}$$

is a pullback square; in other words, the canonical homomorphism

$$h\colon E^*(\mathbb{P}(\sigma\sigma')_+) \to E^*(\mathbb{P}(\sigma)_+) \times_{E^*(\mathbb{C}P^n_+)} E^*(\mathbb{P}(\sigma')_+)$$

is an isomorphism of E_*-algebras.

REMARK 6.8. Since $e(\sigma)^*$ and $e(\sigma')^*$ are monic, the limit in Proposition 6.6 may be interpreted as an intersection

$$(6.9)\qquad E^*(\mathbb{P}(\sigma)_+) \cap E^*(\mathbb{P}(\sigma')_+) < E^*(\mathbb{C}P^n_+).$$

For p and p'-primary weight vectors π and π', this provides an illuminating description of $E^*(\mathbb{P}(\pi\pi')_+)$ as a subalgebra of $E^*(\mathbb{C}P^n_+)$.

The cohomological version of Theorem 4.9 is now within reach, essentially by applying $E^*(-)$ to the geometrical proof.

Theorem 6.10. *For any weight vector χ, the E_*-algebra $E^*(\mathbb{P}(\chi)_+)$ is isomorphic to the colimit of the* CAT$[m]$*-diagram*

$$\begin{array}{ccccc} & & E^*(\mathbb{P}({}_{p_i}\chi)_+) & & \\ & \cdot^{\cdot^{\cdot}} & \uparrow{\scriptstyle e(1/_{p_i}\chi)^*} & \ddots & \\ E^*(\mathbb{P}({}_{p_1}\chi)_+) & \xleftarrow[e(1/_{p_1}\chi)^*]{} & E^*(\mathbb{C}P^n_+) & \xrightarrow[e(1/_{p_m}\chi)^*]{} & E^*((\mathbb{P}\,{}_{p_m}\chi)_+), \end{array}$$

and the associated universal homomorphisms $E^*(\mathbb{P}({}_{p_i}\chi)_+) \to E^*(\mathbb{P}(\chi)_+)$ may be identified with $e({}_{p_i}\chi/\chi)^*$ for every $1 \leq i \leq m$. Similarly, $E^*((\mathbb{P}(\chi))_+)$ is also isomorphic to the limit of the CAT$^{op}[m]$*-diagram*

$$\begin{array}{ccccc} & & E^*(\mathbb{P}({}_{p_i}\chi)_+) & & \\ & \cdot^{\cdot^{\cdot}} & \downarrow{\scriptstyle e({}_{p_i}\chi)^*} & \ddots & \\ E^*(\mathbb{P}({}_{p_1}\chi)_+) & \xrightarrow[e({}_{p_1}\chi)^*]{} & E^*(\mathbb{C}P^n_+) & \xleftarrow[e({}_{p_m}\chi)^*]{} & E^*(\mathbb{P}({}_{p_m}\chi)_+), \end{array}$$

and the associated universal homomorphisms $E^*(\mathbb{P}(\chi)_+) \to E^*(\mathbb{P}({}_{p_i}\chi)_+)$ *may be identified with* $e(\chi/_{p_i}\chi)^*$ *for every* $1 \le i \le m$.

Proof. Proceed by induction on m, as in the proof of Theorem 4.9. The inductive steps appeal to Propositions 6.4 and 6.6 respectively. □

REMARK 6.11. Theorem 6.10 shows that $E^*(-)$ converts the geometric limits and colimits of Theorem 4.9 into the corresponding algebraic colimits and limits. Although the geometric pullbacks are not of fibrations, the induced algebraic pushouts are those of a collapsed Eilenberg–Moore spectral sequence.

The pullback and pushout descriptions of Theorem 6.10 yield isomorphisms

$$E^*(\mathbb{P}(\chi)_+) \xrightarrow{\cong} E^*(\mathbb{P}({}_{p_1}\chi)_+) \otimes_{E^*(\mathbb{C}P^n_+)} \cdots \otimes_{E^*(\mathbb{C}P^n_+)} E^*(\mathbb{P}({}_{p_m}\chi)_+) \tag{6.12}$$

and

$$E^*(\mathbb{P}({}_{p_1}\chi)_+) \times_{E^*(\mathbb{C}P^n_+)} \cdots \times_{E^*(\mathbb{C}P^n_+)} E^*(\mathbb{P}({}_{p_m}\chi)_+) \xrightarrow{\cong} E^*(\mathbb{P}(\chi)_+) \tag{6.13}$$

respectively; by analogy with (6.9), the latter may be rewritten as

$$E^*(\mathbb{P}({}_{p_1}\chi)_+) \cap \cdots \cap E^*(\mathbb{P}({}_{p_m}\chi)_+) < E^*(\mathbb{C}P^n_+). \tag{6.14}$$

Each of these E_*-algebras has one E_*-generator in each even dimension $\le 2n$, and leads directly back to Kawasaki's original calculations.

Theorem 6.15. *In the case of integral cohomology, the isomorphisms* (6.12), (6.13), *and* (6.14) *identify* $H^*(\mathbb{P}(\chi)_+)$ *with* $\mathbb{Z}[v_1, \dots, v_n]/I(\chi)$, *as in* (2.15).

Proof. For each prime $p_i \in Q(\chi)$, let $v(i)_j \in H^{2j}(\mathbb{P}(\chi)_+)$ denote the image of Kawasaki's generator $1 \otimes \cdots \otimes v_{i,j} \otimes \cdots \otimes 1$ under the isomorphism (6.12), where $v_{i,j} \in H^{2j}(\mathbb{P}({}_{p_i}\chi))$. Example 5.11 shows that $m_{i,j} v_{i,j} = v_{i,1}^j$ lies in the image of $H^{2j}(\mathbb{C}P^n)$ for every i, where

$$m_{i,j} = p_i^{(j-1)k_{i,n}} / p_i^{k_{i,n-1}+\cdots+k_{i,n-j+1}}$$

as in (5.14); thus $m_{1,j} v(1)_j = m_{i,j} v(i)_j$ for every i. The numbers $M_i := \prod_{l \ne i} m_{l,j}$ are coprime, and satisfy $M_i v(h)_j = M_h v(i)_j$ for every h and i; so there exist non-zero integers A_h such that $\sum_h A_h M_h = 1$. Now define the element v_j to be $\sum_h A_h v(h)_j$ in $H^{2j}(\mathbb{P}(\chi))$. For every i, it follows that

$$M_i v_j = \sum_h A_h M_i v(h)_j = \left(\sum_h A_h M_h\right) v(i)_j = v(i)_j,$$

and hence that $H^{2j}(\mathbb{P}(\chi))$ is free abelian, on generator v_j. By construction, $v_1^j = \prod_i m_{i,j} v_j$ for every $1 \leq j \leq n$, as required by (2.15).

Alternatively, the isomorphism (6.14) identifies $v_{i,j} \in H^{2j}(\mathbb{P}({}_{p_i}\chi))$ with the element $p_i^{k_{i,n}+\cdots+k_{i,n-j+1}} x^j \in H^{2j}(\mathbb{C}P^n)$. The intersection of the cyclic groups so generated is therefore infinite cyclic on

$$v_j = \prod_i p_i^{k_{i,n}+\cdots+k_{i,n-j+1}} x^j,$$

and the relation $v_1^j = \prod_i m_{i,j} v_j$ follows again. □

EXAMPLE 6.16. By (6.14), $K^*(\mathbb{P}(3, 4, 5)_+)$ may be identified with

(6.17) $$K^*(\mathbb{P}(1, 4, 1)_+) \cap K^*(\mathbb{P}(3, 1, 1)_+) \cap K^*(\mathbb{P}(1, 1, 5)_+) < K^*(\mathbb{C}P^2_+).$$

Example 5.18 with $n = 2$ shows that $K^*(\mathbb{P}(1, 1, r)_+)$ is isomorphic to

(6.18) $$K_*[w_2, w_1 w_2]/(w_1^2 w_2, w_2^2 - r w_1 w_2),$$

and that (6.14) identifies w_2 with $[r](x^K)$ in $K^2(\mathbb{C}P^2)$ and $w_1 w_2$ with $r(x^K)^2$ in $K^4(\mathbb{C}P^2)$. Substituting $r = 3$, 4, and 5 in turn confirms that $K^*(\mathbb{P}(3, 4, 5))$ has $y_1 = 60x^K + 90z(x^K)^2$ and $y_2 = 60(x^K)^2$ as K_*-generators, in dimensions 2 and 4 respectively. So there is an isomorphism

(6.19) $$K^*(\mathbb{P}(3, 4, 5)_+) \cong K_*[y_1, y_2]/(y_1^2 - 60y_2).$$

Setting $z = 1$ in (6.19) provides an example of Al Amrani's abstract isomorphism [4, Corollary 3.2] between $K^*(\mathbb{P}(\chi))$ and $H^*(\mathbb{P}(\chi))$ for certain χ.

Theorem 6.15 expresses Kawasaki's discovery (which he did not make explicit) that the ring $H^*(\mathbb{P}(\chi)_+)$ depends only on the unordered coordinates of the vectors ${}_p\chi$, as p ranges over $Q(\chi)$. The same holds for the E_*-algebra $E^*(\mathbb{P}(\chi)_+)$.

REMARK 6.20. These facts also follow from [6]. As explained in Remark 3.11, $E^*(\mathbb{P}(\chi))$ may always be described in terms of iterated Thom isomorphisms as $E^*(\mathbb{P}(\chi^*))$. The advantage presented by Theorem 6.10 is that the p-primary parts $E^*(\mathbb{P}({}_p\chi))$ are each computed using Theorem 5.8; since the computations involve the E-theory p^k-series (rather than the r-series for composite r), the technical machinery of Brown-Peterson cohomology theory [16] may then be brought to bear.

7. Homology and fundamental classes

Since Davis and Januszkiewicz's original work [8], toric topology has tended to focus on cohomological calculations to the detriment of their homological counterparts.

For weighted projective spaces, however, the complex bordism coalgebras $\Omega_*^U(\mathbb{P}(\chi))$ are of particular interest, and this section is devoted to understanding $E_*(\mathbb{P}(\chi))$ for any complex oriented homology theory $E_*(-)$.

For $\mathbb{C}P^n$, the complex orientation reveals itself as an isomorphism

$$E_*(\mathbb{C}P^n_+) \xrightarrow{\cong} E_*\langle b_0, b_1, \dots, b_n\rangle \tag{7.1}$$

of free E_*-coalgebras, where b_j has dimension $2j$ and supports the coproduct

$$\delta(b_j) = \sum_{i=0}^{j} b_i \otimes b_{j-i}$$

in $E_*(\mathbb{C}P^n_+) \otimes E_*(\mathbb{C}P^n_+)$; the b_j form the dual E_*-basis to the powers $(x^E)^j$ for $0 \leq j \leq n$, and b_0 is the counit 1.

For notational clarity, two conventions are adopted throughout the remainder of this section. Firstly, b_j is expanded to b_j^E whenever the homology theory needs emphasising; and secondly, following Chapter 1, the universal complex orientation is usually denoted by u in $\Omega_U^2(\mathbb{C}P^n)$.

It is important to clarify the relationship between the b_j and the Poincaré duality isomorphism

$$\cap_\sigma : E^i(\mathbb{C}P^n_+) \xrightarrow{\cong} E_{2n-i}(\mathbb{C}P^n_+), \tag{7.2}$$

defined by cap product with a fundamental class $\sigma \in E_{2n}(\mathbb{C}P^n_+)$. This is best done in the context of the universal example, and has an interesting history.

During the early days of the theory, it was usual to identify $\Omega_*^U(\mathbb{C}P^n_+)$ with the free Ω_*^U-module on generators cp_j, represented by the inclusions $\mathbb{C}P^j \to \mathbb{C}P^n$ for $0 \leq j \leq n$. From this viewpoint, cp_n is the bordism class of the identity map $1_{\mathbb{C}P^n}$, and the most natural choice of fundamental class σ. In particular, iteration of the formula

$$u \cap cp_n = cp_{n-1} \tag{7.3}$$

in $\Omega_*^U(\mathbb{C}P^n_+)$ shows that cp_j is the Poincaré dual of u^{n-j} for every $0 \leq j \leq n$. On the other hand, the cp_j are certainly not Hom dual to the u^j, but may be expanded by

$$cp_j = [\mathbb{C}P^j]1 + [\mathbb{C}P^{j-1}]b_1 + \cdots + [\mathbb{C}P^1]b_{j-1} + b_j \tag{7.4}$$

in terms of the basis (7.1).

Formula (7.4) is originally due to Novikov, and is an immediate consequence of (7.3). It emphasises the fact that b_j lies in the *reduced* group $\Omega_{2j}^U(\mathbb{C}P^n)$ for every $1 \leq j \leq n$, whereas cp_j has obvious non-trivial augmentation. Nevertheless, b_n may be deployed equally well as a fundamental class, and determines an alternative Poincaré duality isomorphism. Since

$$u \cap b_n^U = b_{n-1}^U$$

holds in $\Omega_*^U(\mathbb{C}P^n)$ by definition, b_j is the alternative Poincaré dual of u^{n-j} by analogy with (7.3). Under the Thom orientation $\Omega_*^U(-) \to H_*(-)$, both b_n and cp_n map to the canonical fundamental class b_n^H in $H_{2n}(\mathbb{C}P^n)$. They therefore induce the same Poincaré duality isomorphism in integral homology and cohomology.

The problem arises of identifying geometrical representatives $B_j \to \mathbb{C}P^n$ for the b_j, bearing in mind that the stably complex manifolds B_j must bound when $j \geq 1$, because the b_j are reduced. This was solved in [17], where the B_j are constructed as iterated sphere bundles whose stably complex structures extend over the associated disc bundles. Subsequently, the B_j were identified as *Bott towers* [11], and therefore as non-singular toric varieties with canonical complex structures; the stabilisations of these structures do *not* bound, having non-trivial Chern numbers. The language of iterated sphere bundles is documented in [7, Sections 2 and 3], and used extensively below.

Each Bott tower is determined by a list $(r_1, \ldots, r_n)$ of integral j-vectors r_j. Given any divisive weight vector χ, let $q_j = \chi_j/\chi_{j-1}$ as in Definition 3.2, and define the Bott tower $(B_j(\chi) : 0 \leq j \leq n)$ by choosing

$$r_j = (0, \ldots, 0, q_j).$$

Then $B_0(\chi) = *$, and $B_j(\chi)$ is a $2j$-dimensional stably complex manifold equipped with canonical complex line bundles $\gamma_i = \gamma_i(\chi)$, for $0 \leq i \leq j \leq n$. It is defined inductively as the total space $S(\delta_j(\chi))$ of the 2-sphere bundle of

(7.5) $$\delta_j(\chi) := \mathbb{R} \oplus \gamma_{j-1}^{q_j}$$

over $B_{j-1}(\chi)$, where $\mathbb{R}$ denotes the trivial real line bundle. The unit $1 \in \mathbb{R}$ determines a section i_{j-1} for $\delta_j(\chi)$, which features in the cofibre sequence

(7.6) $$B_{j-1}(\chi) \xrightarrow{i_j} B_j(\chi) \xrightarrow{l_j} Th(\gamma_{j-1}^{q_j}).$$

In terms of the complex orientation x^E, the corresponding Thom class $t^E(\gamma_{j-1}^{q_j})$ generates $E^2(Th(\gamma_{j-1}^{q_j}))$ and pulls back to the generator $v_j = v_j^E$ of $E^2(B_j(\chi))$, as described in [7, Chapter 3]. The inductive description is completed by appealing to (5.3), and letting γ_j be the complex line bundle $\lambda(v_j^H)$.

The stably complex structure on $B_j(\chi)$ is induced from the defining S^2-bundle (7.5), and extends over the 3-disc bundle $D(\delta_j(\chi))$ by Szczarba [18]. It is specified by a canonical isomorphism

(7.7) $$c\colon \tau(B_j(\chi)) \oplus \mathbb{R} \xrightarrow{\cong} \gamma_0^{q_1} \oplus \cdots \oplus \gamma_{j-1}^{q_j} \oplus \mathbb{R}$$

of $SO(2j+1)$-bundles, where $\gamma_0 = \mathbb{C}$.

Recall that (5.5) expresses $\mathbb{P}(\chi)$ as an iterated Thom space over $*$. The sequence $X_0, X_1, \ldots, X_n$ of Thom spaces may be written as

$$*, Th(\lambda(0)^{q_1}), Th(\lambda(0)^{q_1,q_2}), \ldots, Th(\lambda(0)^{q_1,q_2,\ldots,q_n}), \tag{7.8}$$

or equivalently as

$$\mathbb{P}(1), \mathbb{P}(1,\chi_1), \mathbb{P}(1,\chi_1,\chi_2), \ldots, \mathbb{P}(1,\chi_1,\chi_2,\ldots,\chi_n). \tag{7.9}$$

For any $1 \leq j \leq n$, the Thom class $t^E(\lambda(0)^{q_1,\ldots,q_j})$ that arises from (7.8) coincides with the generator w_j in $E^2(\mathbb{P}(1,\chi_1,\ldots,\chi_j))$ that arises from (7.9); it is convenient to denote them both by t_j in $E^2(X_j)$.

Lemma 7.10. *For every $1 \leq j \leq n$, there exists a map $f_{j,n}\colon B_j(\chi) \to X_n$ such that $f^*_{j,n}(t_n) = v_j$ in $E^2(B_j(\chi))$.*

Proof. Proceed by induction on j, with base case $j = 1$.

For any $n \geq 1$, the map $f_{1,n}\colon B_1(\chi) \to X_n$ is necessarily the inclusion of the fibre $S^2 \subset Th(\lambda(0)^{q_1,\ldots,q_n})$, and coincides with the map $\mathbb{P}(\chi_{n-1},\chi_n) \to \mathbb{P}(\chi)$ induced on the final two homogeneous coordinates.

Assume that $f_{j-1,n}$ exists with the required properties, and choose $2 \leq j \leq n$. Thus $1 \leq j-1 \leq n-1$, and it follows that $f_{j-1,n-1}\colon B_{j-1}(\chi) \to X_{n-1}$ satisfies $f^*_{j-1,n-1}(t_{n-1}) = v_{j-1}$ in $E^2(B_{j-1}(\chi))$. Now define $f_{j,n}$ as the composition

$$B_j(\pi) \xrightarrow{l_j} Th(\gamma_{j-1}^{q_j}) \xrightarrow{f'_{j-1,n-1}} X_n \tag{7.11}$$

for $2 \leq j \leq n$, where $\gamma_{j-1} = \lambda(v^H_{j-1})$ and $f'_{j-1,n-1}$ denotes $Th(f_{j-1,n-1})$. Thus $(f'_{j-1,n-1})^*(t_n) = t^E(\gamma_{j-1}^{q_j})$ in $E^2(Th(\gamma_{j-1}^{q_j}))$, and applying l_j^* yields the required equation. □

Every complex orientation induces natural transformations $\Omega^U_*(-) \to E_*(-)$ and $\Omega^*_U(-) \to E^*(-)$, both of which are written x^E_*. They reduce to the identity in the universal case.

DEFINITION 7.12. For any $0 \leq j \leq n$, the bordism class $b_j(\chi)$ is represented by the map $f_{j,n}$ of Lemma 7.10, and lies in $\Omega^U_{2j}(\mathbb{P}(\chi)_+)$; its image $x^E_*(b_j(\chi))$ is also denoted by $b_j(\chi)$ (or $b^E_j(\chi)$ to avoid ambiguity), and lies in $E_{2j}(\mathbb{P}(\chi)_+)$.

The $B_j(\chi)$ bound as stably complex manifolds for every $j > 0$, so the corresponding $b_j(\chi)$ actually belong to the reduced groups $E_{2j}(\mathbb{P}(\chi))$.

According to Theorem 5.8, the elements

$$\{w_{i+1}\cdots w_n\colon 0\le i\le n\} \tag{7.13}$$

form an Ω_*^U-basis for $\Omega_U^*(\mathbb{P}(\chi)_+)$, where the case $i=n$ is interpreted as 1.

Theorem 7.14. *For any divisive χ, the elements $\{b_j(\chi)\colon 0\le j\le n\}$ form a basis for the Ω_*^U-coalgebra $\Omega_*^U(\mathbb{P}(\chi)_+)$; this basis is dual to* (7.13).

Proof. Proceed by induction on n, noting that the result is trivial for $n=0$ and $\chi=(1)$. The inductive assumption is that $\{b_j(\chi')\colon 0\le j\le n-1\}$ and

$$\{w_{i+1}\cdots w_{n-1}\colon 0\le i\le n-1\}$$

are dual Ω_*^U-bases for $\Omega_*^U(\mathbb{P}(\chi')_+)$ and $\Omega_U^*(\mathbb{P}(\chi')_+)$ respectively. In other words, the Kronecker product $\langle w_{i+1}\cdots w_{n-1}, b_j(\chi')\rangle$ evaluates to $\delta_{n-i-1,j}$ in $\Omega^U_{2(j-i)}$, for every $0\le i,\ j\le n-1$.

Now consider (7.11), and the Thom isomorphisms

$$\Phi^*\colon \Omega_U^*((X_{n-1})_+)\to \Omega_U^{*+2}(X_n)\quad\text{and}\quad \Phi_*\colon \Omega^U_{*+2}(X_n)\to \Omega_*^U((X_{n-1})_+)$$

given by the Thom class $w^U(n)\in\Omega_U^2(X_n)$. Then Φ^* satisfies

$$\Phi^*(w_{i+1}\cdots w_{n-1}) = w_{i+1}\cdots w_n \tag{7.15}$$

for $0\le i\le n-1$, and Lemma 7.10 shows that Φ_* satisfies

$$\Phi_*(b_k(\chi)) = b_{k-1}(\chi') \tag{7.16}$$

for $1\le k\le n$. To check that the stably complex structures behave as required by (7.16), appeal must be made to (7.7).

By (7.15) and (7.16), the Kronecker product $\langle w_{i+1}\cdots w_n, b_k(\chi)\rangle$ may be rewritten as

$$\begin{aligned}\langle \Phi^*(w_{i+1}\cdots w_{n-1}), b_k(\chi)\rangle &= \langle w_{i+1}\cdots w_{n-1}, \Phi_*(b_k(\chi))\rangle\\ &= \langle w_{i+1}\cdots w_{n-1}, b_{k-1}(\chi')\rangle\end{aligned}$$

for $0\le i\le n-1$ and $1\le k\le n$. This evaluates to $\delta_{n-i-1,k-1}=\delta_{n-i,k}$ by induction, so it remains only to confirm the cases $i=n$ and $k=0$. They involve $1\in\Omega_U^0((X_n)_+)$ and $1\in\Omega_0^U((X_n)_+)$, and follow immediately. □

Corollary 7.17. *Given any complex oriented homology theory $E_*(-)$, the elements $\{b_j(\chi)\colon 0\le j\le n\}$ form an E_*-basis for $E_*(\mathbb{P}(\chi)_+)$; it is dual to the basis $\{w_{i+1}\cdots w_n\colon 0\le i\le n\}$ for $E^*(\mathbb{P}(\chi)_+)$ given by* Theorem 5.8.

It follows from Corollary 7.17 that the coalgebra structure on $E_*(\mathbb{P}(\chi)_+)$ is determined by expressions of the form

$$\delta(b_j(\chi)) = \sum_{0\leq k+l\leq j} e_{j,k,l} b_k(\chi) \otimes b_l(\chi) \tag{7.18}$$

in $E_*(\mathbb{P}(\chi)_+) \otimes E_*(\mathbb{P}(\chi)_+)$, for $0 \leq j \leq n$. The coefficient $e_{j,k,l}$ lies in $E_{2(j-k-l)}$, and is given by the coefficient of $w_{n-j+1}\cdots w_n$ in the product

$$w_{n-k+1}\cdots w_n \cdot w_{n-l+1}\cdots w_n.$$

EXAMPLE 7.19. If $\chi = (1, 1, 1, r)$, then Example 5.16 shows that $E_*(\mathbb{P}(\chi)_+)$ is freely generated over E_* by elements $\{b_j := b_j(\chi) \colon 0 \leq j \leq 2\}$, where $b_0 = 1$. Applying Theorem 5.8 to (7.18) yields $\delta(1) = 1 \otimes 1$, together with

$$\delta(b_1) = b_1 \otimes 1 + 1 \otimes b_1, \quad \delta(b_2) = b_2 \otimes 1 + rb_1 \otimes b_1 + 1 \otimes b_2,$$

and

$$\delta(b_3) = b_3 \otimes 1 + rb_2 \otimes b_1 + \binom{r}{2} a^E b_1 \otimes b_1 + rb_1 \otimes b_2 + 1 \otimes b_3$$

in $E_*(\mathbb{P}(\chi)_+) \otimes E_*(\mathbb{P}(\chi)_+)$.

In the case of integral homology, the top dimensional group $H_{2n}(\mathbb{P}(\chi))$ is isomorphic to $\mathbb{Z}$, and cap product with the generator $b_n^H(\chi)$ defines Poincaré duality over $\mathbb{Q}$; this does not, of course, extend to $\mathbb{Z}$, symptomising the existence of singularities. Nevertheless, $b_n^H(\chi)$ may still be thought of as a fundamental class, and is the image of the universal $b_n(\chi)$ under x_*^H. In this sense, the representing map $j_{n,n} \colon B_n(\chi) \to \mathbb{P}(\chi)$ may be interpreted as a desingularisation of $\mathbb{P}(\chi)$; it is closely related to the associated toric desingularisation [10, §2.6], as we shall explain in a future note.

8. Homological reassembly

It remains to consider the assembly problem for $E_*(\mathbb{P}(\chi))$, by dualising the results of Section 6. This approach is not strictly necessary, as explained in Remark 6.20; for $E_*(\mathbb{P}(\chi)_+)$ is isomorphic as E_*-coalgebras to $E_*(\mathbb{P}(\chi^*))$, and the latter may be described in terms of iterated Thom isomorphisms. Nevertheless, the homological advantages of proceeding prime by prime are as valid as for cohomology.

Proposition 8.1. *For any weight vector χ, the E_*-coalgebra $E_*(\mathbb{P}(\chi)_+)$ is a free E_*-module, with one generator in each even dimension $\leq 2n$.*

Proof. Because E_* is even dimensional and torsion free, the result follows directly from applying $\mathrm{Hom}_{E_*}(-, E_*)$ to Proposition 6.1. □

A more explicit description is obtained by dualising the free E_*-modules that appear in Theorem 6.10.

Theorem 8.2. *For any weight vector χ, the E_*-coalgebra $E_*(\mathbb{P}(\chi)_+)$ is isomorphic to the limit of the* $\mathrm{CAT}^{op}[m]$*-diagram*

$$\begin{array}{ccccc} & \cdot^{\cdot^{\cdot}} & E_*(\mathbb{P}({}_{p_i}\chi)_+) & \ddots & \\ & & \Big\downarrow{\scriptstyle e(1/{}_{p_i}\chi)_*} & & \\ E_*(\mathbb{P}({}_{p_1}\chi)_+) & \xrightarrow[e(1/{}_{p_1}\chi)_*]{} & E_*(\mathbb{C}P^n_+) & \xleftarrow[e(1/{}_{p_m}\chi)_*]{} & E_*(\mathbb{P}({}_{p_m}\chi)_+); \end{array}$$

the corresponding universal map $E_*(\mathbb{P}(\chi)_+) \to E_*(\mathbb{P}({}_{p_i}\chi)_+)$ may be identified with $e({}_{p_i}\chi/\chi)_*$ for every $1 \le i \le m$. Similarly, $E_*(\mathbb{P}(\chi)_+)$ is also isomorphic to the colimit of the $\mathrm{CAT}[m]$*-diagram*

$$\begin{array}{ccccc} & \cdot^{\cdot^{\cdot}} & E_*(\mathbb{P}({}_{p_i}\chi)_+) & \ddots & \\ & & \Big\uparrow{\scriptstyle e({}_{p_i}\chi)_*} & & \\ E_*(\mathbb{P}({}_{p_1}\chi)_+) & \xleftarrow[e({}_{p_1}\chi)_*]{} & E_*(\mathbb{C}P^n_+) & \xrightarrow[e({}_{p_m}\chi)_*]{} & E_*(\mathbb{P}({}_{p_m}\chi)_+); \end{array}$$

the corresponding universal map $E_*(\mathbb{P}({}_{p_i}\chi)_+) \to E_*(\mathbb{P}(\chi)_+)$ may be identified with $e(\chi/{}_{p_i}\chi)_*$ for every $1 \le i \le m$.

The limit described by Theorem 8.2 is actually the iterated pullback

$$E_*(\mathbb{P}({}_{p_1}\chi)_+) \times_{E_*(\mathbb{C}P^n_+)} \cdots \times_{E_*(\mathbb{C}P^n_+)} E_*(\mathbb{P}({}_{p_m}\chi)_+)$$

of E_*-coalgebras, and the colimit is the iterated pushout

$$E_*(\mathbb{P}({}_{p_1}\chi)_+) \otimes_{E_*(\mathbb{C}P^n_+)} \cdots \otimes_{E_*(\mathbb{C}P^n_+)} E_*(\mathbb{P}({}_{p_m}\chi)_+).$$

By analogy with (6.14), the former may be rewritten as

$$E_*(\mathbb{P}({}_{p_1}\chi)_+) \cap \cdots \cap E_*(\mathbb{P}({}_{p_m}\chi)_+) < E_*(\mathbb{C}P^n_+). \tag{8.3}$$

EXAMPLE 8.4. Expression (8.3) identifies $K_*(\mathbb{P}(3,4,5)_+)$ with

$$K_*(\mathbb{P}(1,4,1)_+) \cap K_*(\mathbb{P}(3,1,1)_+) \cap K_*(\mathbb{P}(1,1,5)_+) < K_*(\mathbb{C}P^2_+). \tag{8.5}$$

Applying Corollary 7.17 with $\pi = (1, 1, r)$ shows that $K_*(\mathbb{P}(\pi)_+)$ is isomorphic to

$$K_*\langle 1, b_1(\pi), b_2(\pi)\rangle, \tag{8.6}$$

and also that (8.3) identifies $b_1(\pi)$ with b_1 in $K_2(\mathbb{C}P^2)$ and $b_2(\pi)$ with rb_2 in $K_4(\mathbb{C}P^2)$. These identifications are compatible with the diagonals

$$\delta(b_1(\pi)) = b_1(\pi) \otimes 1 + 1 \otimes b_1(\pi)$$

and

$$\delta(b_2(\pi)) = b_2(\pi) \otimes 1 + rb_1(\pi) \otimes b_1(\pi) + 1 \otimes b_2(\pi).$$

Substituting $r = 3$, 4, and 5 and applying (8.3) confirms that $K_*(\mathbb{P}(3, 4, 5)_+)$ is isomorphic to the subcoalgebra

$$K_*\langle 1, b_1, 60b_2\rangle < K_*(\mathbb{C}P^2_+), \tag{8.7}$$

as predicted by Proposition 8.1. The resulting coalgebra is K_*-dual to the K_*-algebra description of $K^*(\mathbb{P}(3, 4, 5)_+)$ given by Example 6.16.

Setting $z = 1$ in (8.7) yields an abstract isomorphism of coalgebras between $K_*(\mathbb{P}(\chi))$ and $H_*(\mathbb{P}(\chi))$, and dualises Al Amrani's algebra isomorphism of [4].

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