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# INVARIANT MATSUMOTO METRICS ON HOMOGENEOUS SPACES

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**Abstract**

In this paper we consider invariant Matsumoto metrics which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces, and then we give the flag curvature formula of them. Also we study the special cases of naturally reductive spaces and bi-invariant metrics. We end the article by giving some examples of geodesically complete Matsumoto spaces.

## 1. Introduction

In the last decade the study of invariant Finsler structures on Lie groups and homogeneous spaces has been extended. Lie groups and homogeneous spaces equipped with invariant Finsler metrics are best spaces for finding spaces with some curvature properties. Some curvature properties of these manifolds have been studied in [4], [5], [6], [11], [12] and [13].

An important family of Finsler metrics is the family of $(\alpha, \beta)$-metrics. These metrics are introduced by M. Matsumoto (see [9]). The interesting and important examples of $(\alpha, \beta)$-metrics are Randers metric $\alpha + \beta$, Kropina metric $\alpha^2/\beta$, and Matsumoto metric $\alpha^2/(\alpha - \beta)$, where $\alpha(x, y) = \sqrt{g_{ij}(x)y^i y^j}$, $\beta(x, y) = b_i(x)y^i$, and $g$ and $b$ are a Riemannian metric and a 1-form respectively as follows:

$$g = g_{ij}\, dx^i \otimes dx^j, \tag{1.1}$$

$$b = b_i\, dx^i. \tag{1.2}$$

In the Matsumoto metric, the 1-form $b = b_i\, dx^i$ was originally to be induced by the Earth's gravity (see [1] or [8]).

In a natural way, the Riemannian metric $g$ induces an inner product on any cotangent space $T_x^* M$ such that $\langle dx^i(x), dx^j(x)\rangle = g^{ij}(x)$. The induced inner product on $T_x^* M$ induces a linear isomorphism between $T_x^* M$ and $T_x M$ (for more details see [5]). Then the 1-form $b$ corresponds to a vector field $\tilde{X}$ on $M$ such that

$$g(y, \tilde{X}(x)) = \beta(x, y). \tag{1.3}$$

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Therefore we can write the Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ as follows:

$$F(x, y) = \frac{\alpha(x, y)^2}{\alpha(x, y) - g(\tilde{X}(x), y)}. \tag{1.4}$$

One of the fundamental quantities which is associated with a Finsler space is flag curvature. Flag curvature is computed by the following formula:

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y) \,.\, g_Y(U, U) - g_Y^2(Y, U)}, \tag{1.5}$$

where $g_Y(U,V) = (1/2)(\partial^2/(\partial s \partial t))(F^2(Y+sU+tV))|_{s=t=0}$, $P = \mathrm{span}\{U,Y\}$, $R(U,Y)Y = \nabla_U \nabla_Y Y - \nabla_Y \nabla_U Y - \nabla_{[U,Y]} Y$ and $\nabla$ is the Chern connection induced by $F$ (see [2] and [14].).

In this paper we consider invariant Matsumoto metrics which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces then we give the flag curvature formula of them. Also we study the special cases of naturally reductive spaces and bi-invariant metrics. We end the article by giving some examples of geodesically complete Matsumoto spaces.

## 2. Flag curvature of invariant Matsumoto metrics on homogeneous spaces

Let $G$ be a compact Lie group, $H$ a closed subgroup, and $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively. Suppose that $g_0$ is a bi-invariant Riemannian metric on $G$, then the tangent space of the homogeneous space $G/H$ is given by the orthogonal compliment $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $g_0$. Each invariant metric $g$ on $G/H$ is determined by its restriction to $\mathfrak{m}$. The arising $\mathrm{Ad}_H$-invariant inner product from $g$ on $\mathfrak{m}$ can extend to an $\mathrm{Ad}_H$-invariant inner product on $\mathfrak{g}$ by taking $g_0$ for the components in $\mathfrak{h}$. In this way the invariant metric $g$ on $G/H$ determines a unique left invariant metric on $G$ that we also denote by $g$. The values of $g_0$ and $g$ at the identity are inner products on $\mathfrak{g}$ and we determine them by $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle$ respectively. The inner product $\langle \cdot, \cdot \rangle$ determines a positive definite endomorphism $\phi$ of $\mathfrak{g}$ such that $\langle X, Y \rangle = \langle \phi X, Y \rangle_0$ for all $X, Y \in \mathfrak{g}$.

**Theorem 2.1.** *Let $G$, $H$, $\mathfrak{g}$, $\mathfrak{h}$, $g$, $g_0$ and $\phi$ be as above. Assume that $\tilde{X}$ is an invariant vector field on $G/H$ which is parallel with respect to $g$ and $\sqrt{g(\tilde{X}, \tilde{X})} < 1/2$ and $X := \tilde{X}_H$. Suppose that $F = \alpha^2/(\alpha - \beta)$ is the Matsumoto metric induced by $g$ and $\tilde{X}$. Assume that $(P, Y)$ is a flag in $T_H(G/H)$ such that $\{Y, U\}$ is an orthonormal basis of $P$ with respect to $\langle \cdot, \cdot \rangle$. Then the flag curvature of the flag $(P, Y)$ in $T_H(G/H)$ is given by*

$$K(P, Y) = \frac{(1 - \langle Y, X \rangle)^2 \{B(1 - \langle Y, X \rangle)(1 - 2\langle Y, X \rangle) + 3A\langle U, X \rangle\}}{(1 - \langle Y, X \rangle)(1 - 2\langle Y, X \rangle) + 2\langle U, X \rangle^2}, \tag{2.1}$$

*where*

$$
\begin{aligned}
A &= \langle R(U, Y)Y, X\rangle \\
&= -\frac{1}{4}(\langle[\phi U, Y] + [U, \phi Y], [Y, X]\rangle_0 + \langle[U, Y], [\phi Y, X] + [Y, \phi X]\rangle_0) \\
&\quad - \frac{3}{4}\langle[Y, U], [Y, X]_{\mathfrak{m}}\rangle - \frac{1}{2}\langle[U, \phi X] + [X, \phi U], \phi^{-1}([Y, \phi Y])\rangle_0 \\
&\quad + \frac{1}{4}\langle[U, \phi Y] + [Y, \phi U], \phi^{-1}([Y, \phi X] + [X, \phi Y])\rangle_0,
\end{aligned} \tag{2.2}
$$

*and*

$$
\begin{aligned}
B &= \langle R(U, Y)Y, U\rangle \\
&= -\frac{1}{2}\langle[\phi U, Y] + [U, \phi Y], [Y, U]\rangle_0 \\
&\quad - \frac{3}{4}\langle[Y, U], [Y, U]_{\mathfrak{m}}\rangle - \langle[U, \phi U], \phi^{-1}([Y, \phi Y])\rangle_0 \\
&\quad + \frac{1}{4}\langle[U, \phi Y] + [Y, \phi U], \phi^{-1}([Y, \phi U] + [U, \phi Y])\rangle_0.
\end{aligned} \tag{2.3}
$$

Proof. From the assumption, $\tilde{X}$ is parallel with respect to $g$, and therefore the Chern connection of $F$ coincides on the Levi-Civita connection of $g$ (see [1]). So the Finsler metric $F$ and the Riemannian metric $g$ have the same curvature tensor. We show it by $R$.

By using the definition of $g_Y(U, V)$ and some computations for $F$ we have:

$$
\begin{aligned}
g_Y(U, V) &= \frac{1}{(\sqrt{g(Y, Y)} - g(Y, X))^2}\{4g(Y, U)g(Y, V) + 2g(Y, Y)g(U, V)\} \\
&\quad + \frac{1}{(\sqrt{g(Y, Y)} - g(Y, X))^4} \\
&\qquad \times \{-4g(Y, Y)g(Y, U)g(Y, V) \\
&\qquad\quad + g(Y, Y)^{3/2}(g(Y, V)g(U, X) + g(Y, U)g(V, X)) \\
&\qquad\quad + g(Y, Y)^2(3g(U, X)g(V, X) - g(U, V)) \\
&\qquad\quad + \sqrt{g(Y, Y)}g(Y, X)(7g(Y, U)g(Y, V) + g(Y, Y)g(U, V)) \\
&\qquad\quad - 4g(Y, Y)g(Y, X)(g(Y, V)g(U, X) + g(Y, U)g(V, X))\}.
\end{aligned} \tag{2.4}
$$

Now by using the above formula and the fact that $\{Y, U\}$ is an orthonormal basis for

$P$ with respect to $g$, we have

$$
\begin{aligned}
& g_Y(R(U,Y)Y,U) \\
& = \frac{2\langle R(U,Y)Y,U\rangle}{(1-\langle Y,X\rangle)^2} \\
& \quad + \frac{1}{(1-\langle Y,X\rangle)^4}\{\langle R(U,Y)Y,Y\rangle\langle U,X\rangle \\
& \qquad + 3\langle R(U,Y)Y,X\rangle\langle U,X\rangle - \langle R(U,Y)Y,U\rangle \\
& \qquad + \langle Y,X\rangle\langle R(U,Y)Y,U\rangle \\
& \qquad - 4\langle Y,X\rangle\langle R(U,Y)Y,Y\rangle\langle U,X\rangle\}
\end{aligned}
\tag{2.5}
$$

and

$$
g_Y(Y,Y)\,.\,g_Y(U,U) - g_Y^2(U,Y) = \frac{2}{(1-\langle Y,X\rangle)^4} + \frac{2\langle U,X\rangle^2 + \langle Y,X\rangle - 1}{(1-\langle Y,X\rangle)^6}. \tag{2.6}
$$

We can obtain the relations (2.2) and (2.3) by using Püttmann's formula (see [10].).

Substituting the relations (2.5) and (2.6) in the equation (1.5), we complete the proof. $\square$

REMARK 2.2. A homogeneous space $M = G/H$ with a $G$-invariant indefinite Riemannian metric $g$ is said to be naturally reductive if it admits an $\mathrm{ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying the condition

$$
B(X,[Z,Y]_{\mathfrak{m}}) + B([Z,X]_{\mathfrak{m}},Y) = 0 \quad \text{for} \quad X,Y,Z \in \mathfrak{m}, \tag{2.7}
$$

where $B$ is the bilinear form on $\mathfrak{m}$ induced by $\mathfrak{g}$ and $[\ ,\ ]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (for more details see [7]). In this case the relation (2.1) for the flag curvature reduces to a simpler equation, because in the case of naturally reductive homogeneous space we have (see [7])

$$
R(U,Y)Y = \frac{1}{4}[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[U,Y]_{\mathfrak{h}}]. \tag{2.8}
$$

Now we consider the case when the invariant Matsumoto metric is defined by a bi-invariant Riemannian metric on a Lie group.

**Theorem 2.3.** *Let $G$ be a Lie group and $g$ be a bi-invariant Riemannian metric on $G$. Assume that $\tilde{X}$ is a left invariant vector field on $G$ which is parallel with respect to $g$ and $\sqrt{g(\tilde{X},\tilde{X})} < 1/2$ and $X := \tilde{X}_H$. Suppose that $F = \alpha^2/(\alpha-\beta)$ is the Matsumoto metric induced by $g$ and $\tilde{X}$, also let $(P,Y)$ be a flag in $T_eG$ such that*

*$\{Y, U\}$ be an orthonormal basis of $P$ with respect to $\langle\cdot,\cdot\rangle$. Then the flag curvature of the flag $(P, Y)$ in $T_eG$ is given by*

(2.9)

$$K(P,Y) = \frac{-(1-\langle Y, X\rangle)^2}{4(1-\langle Y, X\rangle)(1-2\langle Y, X\rangle)+8\langle U, X\rangle^2} \times \{\langle[[U,Y],Y],U\rangle(1-\langle Y, X\rangle)(1-2\langle Y, X\rangle)+3\langle[[U,Y],Y],X\rangle\langle U, X\rangle\}.$$

Proof. $g$ is bi-invariant therefore we have $R(U,Y)Y = -(1/4)[[U,Y],Y]$. Now by using Theorem 2.1, the proof is completed. □

## 3. Some examples of geodesically complete Matsumoto spaces

In this section we give some examples of geodesically complete Matsumoto spaces. We begin with a definition from [3].

DEFINITION 3.1. The Riemannian manifold $(M, g)$ is said to be homogeneous if the group of isometries of $M$ acts transitively on $M$.

**Theorem 3.2.** *Suppose that $(M, g)$ is a homogeneous Riemannian manifold. Let $F$ be a Matsumoto metric of Berwald type defined by $g$ and a* 1*-form $b$. Then $(M, F)$ is geodesically complete. Moreover if $M$ is connected then $(M, F)$ is complete.*

Proof. The Chern connection of $F$ and the Levi-Civita connection of $g$ coincide and therefore their geodesics coincide too. On the other hand $(M, g)$ is a homogeneous Riemannian manifold, hence $(M, g)$ is geodesically complete (see [3] p. 185). Therefore $(M, F)$ is geodesically complete. If $M$ is connected then by using Hofp–Rinow theorem for Finsler manifolds, $(M, F)$ is complete. □

**Corollary 3.3.** *Let $G$ be a Lie group and $g$ be a left invariant Riemannian metric on $G$. Also suppose that $X$ is a parallel vector field with respect to the Levi-Civita connection of $g$ such that $\sqrt{g(X,X)} < 1/2$. Then the Matsumoto metric defined by $g$, $X$ and the relation* (1.4) *is geodesically complete.*

Now we consider an abelian Lie group equipped with a left invariant Riemannian metric. We know that this space is flat. In this case we have the following theorem,

**Theorem 3.4.** *Let $G$ be an abelian Lie group equipped with a left invariant Riemannian metric $g$, and let $\mathfrak{g}$ be the Lie algebra of $G$. Suppose that $X \in \mathfrak{g}$ is a left invariant vector field with $\sqrt{g(X,X)} < 1/2$. Then the Matsumoto metric $F$ defined by the formula* (1.4) *is a flat geodesically complete locally Minkowskian metric on $G$.*

Proof. Assume that $U, V, W \in \mathfrak{g}$, now by using the Koszul's formula and the fact that $G$ is abelian we have $\nabla_Y X = 0$, for any $Y \in \mathfrak{g}$. Hence $X$ is parallel with respect to $\nabla$ and $F$ is of Berwald type. Also the curvature tensor $R = 0$ of $g$ coincides on the curvature tensor of $F$ and therefore the flag curvature of $F$ is zero. $F$ is a flat Berwald metric therefore by Proposition 10.5.1 (p. 275) of [2], $F$ is locally Minkowskian. □

EXAMPLE 3.5 ($E(2)$ group of rigid motions of Euclidean 2-space). We consider the Lie group $E(2)$ as follows:

$$E(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, a, b, \theta \in \mathbb{R} \right\}. \tag{3.1}$$

The Lie algebra of $E(2)$ is of the form

$$\mathfrak{e}(2) = \operatorname{span}\left\{ x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \tag{3.2}$$

where

$$[x, y] = 0, \quad [y, z] = x, \quad [z, x] = y. \tag{3.3}$$

Now let $g$ be the left invariant Riemannian metric induced by the following inner product,

$$\langle x, x\rangle = \langle y, y\rangle = \langle z, z\rangle = \lambda^2, \quad \langle x, y\rangle = \langle y, z\rangle = \langle z, x\rangle = 0, \quad \lambda > 0. \tag{3.4}$$

In [13] we showed that the left invariant vector fields which are parallel with respect to the Levi-Civita connection of this space are of the form $U = uz$. Also we proved that $R = 0$. Assume that $\sqrt{\langle U, U\rangle} < 1/2$, in other words let $0 < |u| < 1/(2\lambda)$. Hence, the left invariant Matsumoto metric $F$ defined by $g$ and $U$ with formula (1.4) is of Berwald type. Also since $F$ is of Berwald type therefore the curvature tensor of $F$ and $g$ coincide and $F$ is of zero constant flag curvature. Hence $F$ is locally Minkowskian.

EXAMPLE 3.6. Another example of flat geodesically complete locally Minkowskian Matsumoto spaces is described as follows.

Let $\mathfrak{g} = \operatorname{span}\{x, y, z\}$ be a Lie algebra such that

$$[x, y] = \alpha y + \alpha z, \quad [y, z] = 2\alpha x, \quad [z, x] = \alpha y + \alpha z, \quad \alpha \in \mathbb{R}. \tag{3.5}$$

Also consider the inner product described by (3.4) on $\mathfrak{g}$.

Suppose that $G$ is a Lie group with Lie algebra $\mathfrak{g}$, and $g$ is the left invariant Riemannian metric induced by the above inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $G$.

A direct computation shows that $R = 0$, therefore $(G, g)$ is a flat Riemannian manifold. Also in [13] we proved that vector fields which are parallel with respect to the

Levi-Civita connection of $(G, g)$ are of the form $U = uy - uz$. Now suppose that $\sqrt{2}|u|\lambda = \sqrt{\langle U, U\rangle} < 1/2$ or equivalently let $0 < |u| < 1/(2\sqrt{2}\lambda)$. Therefore the invariant Matsumoto metric $F$ defined by $g$ and $U$ is a flat geodesically complete locally Minkowskian metric on $G$. Also if $G$ is connected, $(G, F)$ is complete.

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