



Title	ON THE S^1 -FIBRED NILBOTT TOWER
Author(s)	Nakayama, Mayumi
Citation	Osaka Journal of Mathematics. 2014, 51(1), p. 67-87
Version Type	VoR
URL	https://doi.org/10.18910/29187
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

ON THE S^1 -FIBRED NILBOTT TOWER

MAYUMI NAKAYAMA

(Received January 6, 2012, revised May 1, 2012)

Abstract

We shall introduce a notion of S^1 -fibred nilBott tower. It is an iterated S^1 -bundle whose top space is called an S^1 -fibred nilBott manifold and the S^1 -bundle of each stage realizes a *Seifert construction*. The S^1 -fibred nilBott tower is a generalization of *real Bott tower* from the viewpoint of fibration. In this note we shall prove that any S^1 -fibred nilBott manifold is *diffeomorphic* to an infranilmanifold. According to the group extension of each stage, there are two classes of S^1 -fibred nilBott manifolds which is defined as *finite type* or *infinite type*. We discuss their properties.

Contents

1. Introduction	67
2. Seifert construction	69
3. S^1 -fibred nilBott tower	70
3.1. Proof of Theorem 1.2.	70
3.2. Torus actions on S^1 -fibred nilBott manifolds.	74
4. 3-dimensional S^1 -fibred nilBott towers	77
4.1. 3-dimensional S^1 -fibred nilBott manifolds of finite type.	77
4.2. 3-dimensional S^1 -fibred nilBott manifolds of infinite type.	77
5. Realization	80
5.1. Realization of S^1 -fibration over a Klein bottle K	80
5.2. Realization of S^1 -fibration over T^2	85
6. Halperin–Carlsson conjecture	86
References	87

1. Introduction

Let M be a closed aspherical manifold which is the top space of an iterated S^1 -bundle over a point:

$$(1.1) \quad M = M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_1 \rightarrow \{\text{pt}\}.$$

Suppose X is the universal covering of M and each X_i is the universal covering of M_i and put $\pi_1(M_i) = \pi_i$ ($i = 1, \dots, n - 1$) and $\pi_1(M) = \pi$.

DEFINITION 1.1. An S^1 -fibred nilBott tower is a sequence (1.1) which satisfies I, II and III below. The top space M is said to be an S^1 -fibred nilBott manifold (of depth n).

- I. Each M_i is a fiber space over M_{i-1} with fiber S^1 .
- II. For the group extension

$$(1.2) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \rightarrow \pi_{i-1} \rightarrow 1$$

associated to the fiber space I, there is an equivariant principal bundle:

$$(1.3) \quad \mathbb{R} \rightarrow X_i \xrightarrow{p_i} X_{i-1}.$$

- III. Each π_i normalizes \mathbb{R} .

The purpose of this paper is to prove the following results.

Theorem 1.2. *Suppose that M is an S^1 -fibred nilBott manifold.*

- (i) *If every cocycle of $H_\phi^2(\pi_{i-1}, \mathbb{Z})$ which represents a group extension (1.2) is of finite order, then M is diffeomorphic to a Riemannian flat manifold.*
- (ii) *If there exists a cocycle of $H_\phi^2(\pi_{i-1}, \mathbb{Z})$ which represents a group extension (1.2) is of infinite order, then M is diffeomorphic to an infranilmanifold. In addition, M cannot be diffeomorphic to any Riemannian flat manifold.*

As a consequence, we have the following classification. (See Proposition 4.1 and Proposition 4.2.)

Proposition 1.3. *The 3-dimensional S^1 -fibred nilBott manifolds of finite type are those of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.*

Proposition 1.4. *Any 3-dimensional S^1 -fibred nilBott manifold of infinite type is either a Heisenberg nilmanifold $\mathbb{N}/\Delta(k)$ or an Heisenberg infranilmanifold $\mathbb{N}/\Gamma(k)$.*

Real Bott manifolds consist of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_3$ among these $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.

(Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [13]. We quote the notations $\mathcal{G}_i, \mathcal{B}_i$ there.)

Masuda and Lee [8] have also proved the above results.

By (1.2) of Definition 1.1, a 3-dimensional S^1 -fibred nilBott manifold M gives a group extension:

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow Q \rightarrow 1$$

where Q is the fundamental group of a Klein Bottle K or a torus T^2 . Then this group extension gives a 2-cocycle in the group cohomology $H_\phi^2(Q, \mathbb{Z})$ with a homomorphism $\phi: Q \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm 1\}$. Conversely we have shown

Theorem 1.5. *Every cocycle of $H_\phi^2(Q, \mathbb{Z})$ can be realized as a diffeomorphism class of an S^1 -fibred nilBott manifold.*

2. Seifert construction

We shall explain the Seifert construction briefly. It is a tool to construct a closed aspherical manifold for a given extension. Let

$$(2.1) \quad 1 \rightarrow \Delta \rightarrow \pi \xrightarrow{\nu} Q \rightarrow 1$$

be a group extension and $\phi: Q \rightarrow \text{Aut}(\Delta)$ a conjugation function defined by a section $s: Q \rightarrow \pi$ for the projection ν . Define $f: Q \times Q \rightarrow \Delta$ by $s(\alpha)s(\beta) = f(\alpha, \beta)s(\alpha\beta)$. Then f defines the group π which is the product $\Delta \times Q$ with the group law:

$$(2.2) \quad (n, \alpha)(m, \beta) = (n \cdot \phi(\alpha)(m) \cdot f(\alpha, \beta), \alpha\beta).$$

($\forall n, m \in \Delta, \forall \alpha, \beta \in Q$) (cf. [10] for example).

Suppose Δ is a torsionfree finitely generated nilpotent group. By the Mal'cev's *unique existence* theorem, there is a simply connected nilpotent Lie group \mathcal{N} containing Δ as a discrete uniform subgroup. (See [12] for example.) Moreover if Q acts smoothly and properly discontinuously on a contractible smooth manifold W such that the quotient space W/Q is compact, then there is a map $\lambda: Q \rightarrow \text{Map}(W, \mathcal{N})$ whose images consist of smooth maps of W into \mathcal{N} satisfying:

$$(2.3) \quad f(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}) \cdot \lambda(\alpha) \cdot \lambda(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q),$$

here $\bar{\phi}: Q \rightarrow \text{Aut}(\mathcal{N})$ is the unique extension of ϕ by Mal'cev's *unique existence* property. We simply write $f = \delta^1 \lambda$ for (2.3). And an action of π on $\mathcal{N} \times W$ is obtained by

$$(2.4) \quad (n, \alpha)(x, w) = (n \cdot \bar{\phi}(\alpha)(x) \cdot \lambda(\alpha)(\alpha w), \alpha w).$$

This action $(\pi, \mathcal{N} \times W)$ is said to be a Seifert construction. (See [5] for details.)

In particular, when Q is a finite group F and $W = \{pt\}$ it follows $\text{Map}(W, \mathcal{N}) = \mathcal{N}$ for which there is a smooth map $\chi: F \rightarrow \mathcal{N}$ satisfying $f = \delta^1 \chi$:

$$(2.5) \quad f(\alpha, \beta) = \bar{\phi}(\alpha)(\chi(\beta)) \cdot \chi(\alpha) \cdot \chi(\alpha\beta)^{-1} \quad (\alpha, \beta \in F).$$

Let $E(\mathcal{N})$ be a semidirect product $\mathcal{N} \rtimes \mathcal{K}$ with \mathcal{K} be a maximal compact subgroup of $\text{Aut}(\mathcal{N})$. And we can define a discrete faithful representation $\rho: \pi \rightarrow E(\mathcal{N})$ by

$$(2.6) \quad \rho((n, \alpha)) = (n \cdot \chi(\alpha), \mu(\chi(\alpha)^{-1}) \circ \bar{\phi}(\alpha)),$$

(here μ is a conjugation map). Then the action of π on \mathcal{N} is defined by

$$(2.7) \quad (n, \alpha)(x) = \rho((n, \alpha))(x) = n \cdot \bar{\phi}(\alpha)(x) \cdot \chi(\alpha).$$

Note that the action (π, \mathcal{N}) is a Seifert construction and if π is torsionfree \mathcal{N}/π is an infranilmanifold (cf. [5] or [10]).

3. S^1 -fibred nilBott tower

In this section we shall give a proof of Theorem 1.2 of Introduction and apply our theorem to torus actions.

3.1. Proof of Theorem 1.2. Suppose that

$$(3.1) \quad M = M_n \xrightarrow{S^1} M_{n-1} \xrightarrow{S^1} \cdots \xrightarrow{S^1} M_1 \xrightarrow{S^1} \{\text{pt}\}$$

is an S^1 -fibred nilBott tower. By the definition, there is a group extension of the fiber space;

$$(3.2) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \rightarrow \pi_{i-1} \rightarrow 1$$

for any i . The conjugate by each element of π_i defines a homomorphism $\phi: \pi_{i-1} \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm 1\}$. With this action, \mathbb{Z} is a π_{i-1} -module so that the group cohomology $H_\phi^*(\pi_{i-1}, \mathbb{Z})$ is defined. Then the above group extension (3.2) represents a 2-cocycle in $H_\phi^2(\pi_{i-1}, \mathbb{Z})$ (cf. [10]).

Proof of Theorem 1.2. Given a group extension (3.2), we suppose by induction that there exists a torsionfree finitely generated nilpotent normal subgroup Δ_{i-1} of finite index in π_{i-1} such that the induced extension $\tilde{\Delta}_i$ is a central extension:

$$(3.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} \longrightarrow 1. \end{array}$$

It is easy to see that $\tilde{\Delta}_i$ is a torsionfree finitely generated normal nilpotent subgroup of finite index in π_i . Then π_i is a virtually nilpotent subgroup, i.e. $1 \rightarrow \tilde{\Delta}_i \rightarrow \pi_i \rightarrow F_i \rightarrow 1$ where $F_i = \pi_i/\tilde{\Delta}_i$ is a finite group. Let \tilde{N}_i, N_{i-1} be a nilpotent Lie group containing $\tilde{\Delta}_i, \Delta_{i-1}$ as a discrete cocompact subgroup respectively. Let $A(\tilde{N}_i) = \tilde{N}_i \rtimes \text{Aut}(\tilde{N}_i)$ be the affine group. If \tilde{K}_i is a maximal compact subgroup of $\text{Aut}(\tilde{N}_i)$, then the subgroup $E(\tilde{N}_i) = \tilde{N}_i \rtimes \tilde{K}_i$ is called the euclidean group of \tilde{N}_i . Then there exists a faithful homomorphism (see (2.6)):

$$(3.4) \quad \rho_i: \pi_i \rightarrow E(\tilde{N}_i)$$

for which $\rho_i|_{\tilde{\Delta}_i} = \text{id}$ and the quotient $\tilde{N}_i/\rho_i(\pi_i)$ is an infranilmanifold. The explicit

formula is given by the following

$$(3.5) \quad \rho_i((n, \alpha)) = (n \cdot \chi(\alpha), \mu(\chi(\alpha)^{-1}) \circ \bar{\phi}(\alpha))$$

for $n \in \tilde{\Delta}_i$, $\alpha \in F_i$ where $\chi: F_i \rightarrow \tilde{N}_i$, $\bar{\phi}: F_i \rightarrow \text{Aut}(\tilde{N}_i)$. As $\tilde{\Delta}_i \leq \tilde{N}_i$, there is a 1-dimensional vector space \mathbb{R} containing \mathbb{Z} as a discrete uniform subgroup which has a central group extension (cf. [12]):

$$1 \rightarrow \mathbb{R} \rightarrow \tilde{N}_i \rightarrow N_{i-1} \rightarrow 1$$

where $N_{i-1} = \tilde{N}_i/\mathbb{R}$ is a simply connected nilpotent Lie group. As $\mathbb{Z} \leq \mathbb{R} \cap \tilde{\Delta}_i$ is discrete cocompact in \mathbb{R} and $\mathbb{R} \cap \tilde{\Delta}_i/\mathbb{Z} \rightarrow \tilde{\Delta}_i/\mathbb{Z} \cong \Delta_{i-1}$ is an inclusion, noting that Δ_{i-1} is torsionfree, it follows that $\mathbb{R} \cap \tilde{\Delta}_i = \mathbb{Z}$. We obtain the commutative diagram in which the vertical maps are inclusions:

$$(3.6) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{N}_i & \longrightarrow & N_{i-1} & \longrightarrow & 1. \end{array}$$

On the other hand, (3.4) induces the following group extension:

$$(3.7) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} & \longrightarrow & 1 \\ & & \parallel & & \downarrow \rho_i & & \downarrow \hat{\rho}_i & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \rho_i(\pi_i) & \longrightarrow & \hat{\rho}_i(\pi_{i-1}) & \longrightarrow & 1. \end{array}$$

Since $\tilde{\Delta}_i$ and \tilde{N}_i centralizes \mathbb{Z} and \mathbb{R} respectively, $\hat{\rho}_i$ is a homomorphism from π_{i-1} into $E(N_{i-1})$. The explicit formula is given by the following:

$$(3.8) \quad \hat{\rho}_i((\bar{n}, \alpha)) = (\bar{n} \cdot \bar{\chi}(\alpha), \mu(\bar{\chi}(\alpha)^{-1}) \circ \hat{\phi}(\alpha))$$

for $\bar{n} \in \Delta_{i-1}$, $\alpha \in F_i$ where $\bar{\chi} = p_i \circ \chi: F_i \rightarrow N_{i-1}$, $\hat{\phi}: F_i \rightarrow \text{Aut}(N_{i-1})$;

$$\hat{\phi}(\alpha)(\bar{x}) = \overline{\bar{\phi}(\alpha)(x)}.$$

Using (1.3) and Mal'cev's unique extension property (compare [12]), it is easy to check that the above $\hat{\phi}: F_i \rightarrow \text{Aut}(N_{i-1})$ is a well-defined homomorphism. Thus we obtain an equivariant fibration:

$$(3.9) \quad (\mathbb{Z}, \mathbb{R}) \rightarrow (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{v_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).$$

Suppose by induction that (π_{i-1}, X_{i-1}) is equivariantly diffeomorphic to the infranil-action $(\hat{\rho}_i(\pi_{i-1}), N_{i-1})$ as above. We have two Seifert fibrations from (1.3):

$$(\mathbb{Z}, \mathbb{R}) \rightarrow (\pi_i, X_i) \xrightarrow{p_i} (\pi_{i-1}, X_{i-1})$$

and (3.9):

$$(\mathbb{Z}, \mathbb{R}) \rightarrow (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{v_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).$$

As $\rho_i: \pi_i \rightarrow \rho_i(\pi_i)$ is isomorphic such that $\rho_i|_{\mathbb{Z}} = \text{id}$, the Seifert rigidity implies that (π_i, X_i) is equivariantly diffeomorphic to $(\rho_i(\pi_i), \tilde{N}_i)$. This shows the induction step. If $M = X/\pi$, then (π, X) is equivariantly diffeomorphic to an infranil-action $(\rho(\pi), \tilde{N})$ for which $\rho: \pi \rightarrow E(\tilde{N})$ is a faithful representation.

We have shown that M is diffeomorphic to an infranilmanifold $\tilde{N}/\rho(\pi)$. According to Cases I, II (stated in Theorem 1.2), we prove that \tilde{N} is isomorphic to a vector space for Case I or \tilde{N} is a nilpotent Lie group but not a vector space for Case II respectively.

CASE I. As every cocycle of $H_\phi^2(\pi_{i-1}, \mathbb{Z})$ representing a group extension (3.2) is finite, the cocycle in $H^2(\Delta_{i-1}, \mathbb{Z})$ for the induced extension of (3.3) that $1 \rightarrow \mathbb{Z} \rightarrow \tilde{\Delta}_i \rightarrow \Delta_{i-1} \rightarrow 1$ is also finite. By induction, suppose that Δ_{i-1} is isomorphic to a free abelian group \mathbb{Z}^{i-1} . Then the cocycle in $H^2(\mathbb{Z}^{i-1}, \mathbb{Z})$ is zero, so $\tilde{\Delta}_i$ is isomorphic to a free abelian group \mathbb{Z}^i . Hence the nilpotent Lie group N_i is isomorphic to the vector space \mathbb{R}^i . This shows the induction step. In particular, π_i is isomorphic to a Bieberbach group $\rho_i(\pi_i) \leq E(\mathbb{R}^i)$. As a consequence X/π is diffeomorphic to a Riemannian flat manifold $\mathbb{R}^n/\rho(\pi)$.

CASE II. Suppose that π_{i-1} is virtually free abelian until $i-1$ and the cocycle $[f] \in H_\phi^2(\pi_{i-1}, \mathbb{Z})$ representing a group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_i \rightarrow \pi_{i-1} \rightarrow 1$ is of infinite order in $H_\phi^2(\pi_{i-1}, \mathbb{Z})$. Note that π_{i-1} contains a torsionfree normal free abelian subgroup \mathbb{Z}^{i-1} . As in (3.3), there is a central group extension of $\tilde{\Delta}_i$:

$$(3.10) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow i \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \mathbb{Z}^{i-1} \longrightarrow 1 \end{array}$$

where $[\pi_{i-1}: \mathbb{Z}^{i-1}] < \infty$. Recall that there is a transfer homomorphism $\tau: H^2(\mathbb{Z}^{i-1}, \mathbb{Z}) \rightarrow H_\phi^2(\pi_{i-1}, \mathbb{Z})$ such that $\tau \circ i^* = [\pi_{i-1}: \mathbb{Z}^{i-1}]: H_\phi^2(\pi_{i-1}, \mathbb{Z}) \rightarrow H_\phi^2(\pi_{i-1}, \mathbb{Z})$, see [1, (9.5) Proposition p. 82] for example. The restriction $i^*[f]$ gives the bottom extension sequence of (3.10). If $i^*[f] = 0 \in H^2(\mathbb{Z}^2, \mathbb{Z})$, then $0 = \tau \circ i^*[f] = [\pi_{i-1}: \mathbb{Z}^{i-1}][f] \in H_\phi^2(\pi_{i-1}, \mathbb{Z})$. So $i^*[f] \neq 0$. Therefore $\tilde{\Delta}_i$ (respectively \tilde{N}_i) is not abelian (respectively not isomorphic to a vector space). As a consequence, \tilde{N} is a simply connected (non-abelian) nilpotent Lie group. \square

In order to study S^1 -fibred nilBott manifolds further, we introduce the following definition:

DEFINITION 3.1. If an S^1 -fibred nilBott manifold M satisfies Case I (respectively Case II) of Theorem 1.2, then M is said to be an S^1 -fibred nilBott manifold of finite type (respectively of infinite type).

Apparently there is no inter between finite type and infinite type. And S^1 -fibred nilBott manifolds are of finite type until dimension 2.

REMARK 3.2. Let M be an S^1 -fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 1.2). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension

$$(3.11) \quad 1 \rightarrow \mathbb{Z}^n \rightarrow \rho(\pi) \rightarrow H \rightarrow 1$$

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and H is the holonomy group of $\rho(\pi)$. We may identify $\rho(\pi)$ with π whenever π is torsionfree.

Proposition 3.3. *Suppose M is an S^1 -fibred nilBott manifold of finite type. Then the holonomy group of π is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $O(n)$ ($0 \leq s \leq n$).*

Proof. Let M be an S^1 -fibred nilBott manifold of finite type. Recall an equivariant fibration:

$$(\mathbb{Z}, \mathbb{R}) \rightarrow (\pi_i, \tilde{N}_i) \xrightarrow{P_i} (\pi_{i-1}, N_{i-1}).$$

If f is a cocycle in $H_\phi^2(\pi_{i-1}, \mathbb{Z})$ for Case I representing (3.2), then there exists a map $\lambda: \pi_{i-1} \rightarrow \mathbb{R}$ such that

$$(3.12) \quad f(\alpha, \beta) = \bar{\phi}(\alpha)(\lambda(\beta)) + \lambda(\alpha) - \lambda(\alpha\beta) \quad (\alpha, \beta \in \pi_{i-1})$$

(see [3]). Moreover let $(n, \alpha) \in \pi_i$ and $(x, w) \in \tilde{N}_i = \mathbb{R} \times N_{i-1}$, then the action of π_i is given by

$$(3.13) \quad (n, \alpha)(x, w) = (n + \bar{\phi}(\alpha)(x) + \lambda(\alpha), \alpha w)$$

($n \in \mathbb{Z}$, $\alpha \in \pi_{i-1}$). See (2.4). As we have shown in Case I of Theorem 1.2, N_{i-1}/π_{i-1} is a Riemannian flat manifold $\mathbb{R}^{i-1}/\pi_{i-1}$, we may assume that

$$\alpha w = b_\alpha + A_\alpha w \quad (w \in \mathbb{R}^{i-1})$$

$(b_\alpha \in \mathbb{R}^i, A_\alpha \in \mathrm{O}(i-1))$ in the above action of (3.13). Then the above action (3.13) has the formula:

$$(3.14) \quad (n, \alpha) \begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} n + \lambda(\alpha) \\ b_\alpha \end{pmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & \mathbf{0} \\ \mathbf{0} & A_\alpha \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\begin{bmatrix} x \\ w \end{bmatrix} \in \tilde{N}_i = \mathbb{R} \times \mathbb{R}^{i-1} = \mathbb{R}^i$. Suppose inductively that $\{A_\alpha \mid \alpha \in \pi_{i-1}\} \leq (\mathbb{Z}_2)^{i-1}$. Here

$$(3.15) \quad (\mathbb{Z}_2)^{i-1} = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \right\} \leq \mathrm{O}(i-1).$$

Since $\bar{\phi}(\pi_{i-1}) \leq \{\pm 1\}$, the holonomy group H_i of π_i is isomorphic to $(\mathbb{Z}_2)^s$, ($0 \leq s \leq i$). This proves the induction step. \square

3.2. Torus actions on S^1 -fibred nilBott manifolds. Given an effective T^k -action on a closed aspherical manifold M , define an orbit map $ev: T^k \rightarrow M$ by $ev(t) = tx$ ($\exists x \in M$). Then ev induces a homomorphism of the fundamental groups $ev_*: \pi_1(T^k) \rightarrow \pi_1(M)$ which is known to be injective by Conner and Raymond [3]. But $ev_*: H_1(T^k) \rightarrow H_1(M)$ is not necessarily injective.

DEFINITION 3.4. When $ev_*: H_1(T^k) \rightarrow H_1(M)$ is injective, we call that the T^k -action is homologically injective.

Corollary 3.5. *Each S^1 -fibred nilBott manifold of finite type M_i admits a homologically injective T^k -action where $k = \mathrm{Rank} H_1(M_i)$. Moreover, the action is maximal, i.e. $k = \mathrm{Rank} C(\pi_i)$.*

Proof. We suppose by induction that there is a *homologically injective* maximal T^{k-1} -action on $M_{i-1} = T^{i-1}/H_{i-1}$ such that $k-1 = \mathrm{Rank} H_1(M_{i-1}) = \mathrm{Rank} C(\pi_{i-1})$ ($k-1 > 0$). Since π_i, π_{i-1} are Bieberbach groups, there are two group extensions

$$\begin{aligned} 1 &\rightarrow \mathbb{Z}^i \rightarrow \pi_i \xrightarrow{h_i} H_i \rightarrow 1, \\ 1 &\rightarrow \mathbb{Z}^{i-1} \rightarrow \pi_{i-1} \xrightarrow{h_{i-1}} H_{i-1} \rightarrow 1 \end{aligned}$$

where H_i, H_{i-1} are holonomy groups of π_i, π_{i-1} respectively and $\mathbb{Z}^i = \pi_i \cap \mathbb{R}^i, \mathbb{Z}^{i-1} =$

$\pi_{i-1} \cap \mathbb{R}^{i-1}$. We have a following diagram

$$(3.16) \quad \begin{array}{ccccccc} & & & 1 & & 1 & \\ & & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^i & \longrightarrow & \mathbb{Z}^{i-1} \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} \longrightarrow 1 \\ & & & & \downarrow h_i & & \downarrow h_{i-1} \\ & & & & H_i & \xlongequal{\quad} & H_{i-1} \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

Let $p: \mathbb{R}^i = \mathbb{R} \times \mathbb{R}^{i-1} \rightarrow T^i = S^1 \times T^{i-1}$ be the canonical projection such that $\text{Ker } p = \mathbb{Z}^i = \pi_i \cap \mathbb{R}^i$. By Proposition 3.3, $H_i = (\mathbb{Z}_2)^s$ for some s ($1 \leq s \leq i$). The action (π_i, \mathbb{R}^i) induces an isometric action (H_i, T^i) from (3.14). We may represent the action as follows:

$$(3.17) \quad \hat{\alpha} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{pmatrix} = \begin{pmatrix} t_{\hat{\alpha}} \cdot \psi(\hat{\alpha})(z_1) \\ z_2' \\ \vdots \\ z_i' \end{pmatrix}$$

here $\hat{\alpha} = h_i((n, \alpha)) \in H_i$, $t_{\hat{\alpha}} = p(n + \lambda(\alpha)) \in S^1$, and $\psi: H_i \rightarrow \{\pm 1\}$ is defined by

$$(3.18) \quad \psi(\hat{\alpha})(z_1) = \begin{cases} z_1 & \text{if } \bar{\phi}(\alpha) = 1, \\ \bar{z}_1 & \text{if } \bar{\phi}(\alpha) = -1. \end{cases}$$

Note that $(t_{\hat{\alpha}})^2 = p(n + \lambda(\alpha))p(n + \lambda(\alpha)) = p(2n + 2\lambda(\alpha))$. By (3.14) if $\bar{\phi}(\alpha) = 1$, then

$$(3.19) \quad (n, \alpha)^2 \begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} 2n + 2\lambda(\alpha) \\ b_\alpha + A_\alpha w \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix}.$$

Since $2n + 2\lambda(\alpha) \in \mathbb{Z}$, $(t_{\hat{\alpha}})^2 = 1$ i.e. $t_{\hat{\alpha}} = \pm 1$.

If $\psi(\hat{\alpha}) = 1$ for all $\hat{\alpha}$, it follows from (3.17) that the left translation of S^1 on $T^i = S^1 \times T^{i-1}$ induces an S^1 -action on $M_i = T^i/H_i$ so that T^k -action on $M_i = T^i/H_i$ follows

$$(3.20) \quad \begin{pmatrix} t \\ t' \end{pmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix} = \begin{bmatrix} t \cdot z_1 \\ t' \cdot \begin{pmatrix} z_2 \\ \vdots \\ z_i \end{pmatrix} \end{bmatrix}$$

where $(t, t') \in S^1 \times T^{k-1}$, $[z_1, \dots, z_i] \in M_i = T^i/H_i$. On the other hand, if there is an element $\hat{\alpha}$ of H_i which $\psi(\hat{\alpha})(z) = \bar{z}$, then M_i admits a T^{k-1} -action by the induction hypothesis. The group extension (3.11) gives rise to a group extension:

$$(3.21) \quad 1 \rightarrow \mathbb{Z}/[\pi_i, \pi_i] \cap \mathbb{Z} \rightarrow \pi_i/[\pi_i, \pi_i] \xrightarrow{v_i} \pi_{i-1}/[\pi_{i-1}, \pi_{i-1}] \rightarrow 1.$$

As in the proof of Proposition 3.3, $[(0, \alpha), (n, 1)] = ((\phi(\alpha) - 1)(n), 1)$. It follows that $[\pi_i, \pi_i] \cap \mathbb{Z} = \{1\}$ or $[\pi_i, \pi_i] \cap \mathbb{Z} = 2\mathbb{Z}$ according to whether $H_i = \text{Ker } \psi$ or not. So (3.21) becomes

$$(3.22) \quad 1 \rightarrow \mathbb{Z} \rightarrow H_1(M_i) \xrightarrow{v_i} H_1(M_{i-1}) \rightarrow 1,$$

or

$$(3.23) \quad 1 \rightarrow \mathbb{Z}_2 \rightarrow H_1(M_i) \xrightarrow{v_i} H_1(M_{i-1}) \rightarrow 1.$$

For (3.22), it follows $k = \text{Rank } H_1(M_i)$ for which M_i admits a homologically injective T^k -action as above. For (3.23), $k - 1 = \text{Rank } H_1(M_i)$ and M_i admits a homologically injective T^{k-1} -action by the induction hypothesis.

Now we show the action is maximal. Suppose $\psi(\hat{\alpha}) = 1$ for all $\hat{\alpha}$. Noting that the group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_i \xrightarrow{p_i} \pi_{i-1} \rightarrow 1$ is a central extension, we obtain a group extension:

$$1 \rightarrow \mathbb{Z} \rightarrow C(\pi_i) \xrightarrow{p_i} p_i(C(\pi_i)) \rightarrow 1.$$

On the other hand, since M_i admits the above T^k -action, $\mathbb{Z}^k \subset C(\pi_i)$. Let $\text{Rank } C(\pi_i) = k + l$, ($l = 0, 1, 2, \dots$), then $\mathbb{Z}^{k+l-1} \subset p_i(C(\pi_i))$. By the induction hypothesis, $k - 1 = \text{Rank } C(\pi_{i-1}) \geq \text{Rank } p_i(C(\pi_i))$. Therefore $l = 0$ that is $k = \text{Rank } C(\pi_i)$.

Assume that there exists an element $\hat{\alpha} \in H_i$ such that $\psi(\hat{\alpha})(z) = \bar{z}$. It is easy to check that $\mathbb{Z} \cap C(\pi_i) = \{1\}$, i.e. $C(\pi_i) \leq C(\pi_{i-1})$ and since M_i admits T^{k-1} -action, $\mathbb{Z}^{k-1} \leq C(\pi_i)$. By the induction hypothesis, $k - 1 = \text{Rank } C(\pi_i)$. Hence in each case the torus action is maximal. \square

4. 3-dimensional S^1 -fibred nilBott towers

By the definition of S^1 -fibred nilBott manifold M_n , M_2 is either a torus T^2 or a Klein bottle K so that M_2 is a Riemannian flat manifold.

4.1. 3-dimensional S^1 -fibred nilBott manifolds of finite type. Any 3-dimensional S^1 -fibred nilBott manifold M_3 of finite type is a Riemannian flat manifold. It is known that there are just 10-isomorphism classes $\mathcal{G}_1, \dots, \mathcal{G}_6, \mathcal{B}_1, \dots, \mathcal{B}_4$ of 3-dimensional Riemannian flat manifolds. (Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [13].) In particular, for Riemannian flat 3-manifolds corresponding to \mathcal{B}_2 and \mathcal{B}_4 , we have shown that there are two S^1 -fibred nilBott towers: $\mathcal{B}_2 \rightarrow K \rightarrow S^1 \rightarrow \{\text{pt}\}$ and $\mathcal{B}_4 \rightarrow K \rightarrow S^1 \rightarrow \{\text{pt}\}$ in [10]. Remark that every real Bott manifold is an S^1 -fibred nilBott manifold of finite type and \mathcal{B}_2 and \mathcal{B}_4 are not real Bott manifolds. And the following proposition has been proved. See [10] for details.

Proposition 4.1. *The 3-dimensional S^1 -fibred nilBott manifolds of finite type are those of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.*

4.2. 3-dimensional S^1 -fibred nilBott manifolds of infinite type. Any 3-dimensional S^1 -fibred nilBott manifold M_3 of infinite type is an infranil-Heisenberg manifold. The 3-dimensional simply connected nilpotent Lie group N_3 is isomorphic to the Heisenberg Lie group N which is the product $\mathbb{R} \times \mathbb{C}$ with group law:

$$(x, z) \cdot (y, w) = (x + y - \text{Im } \bar{z}w, z + w).$$

Then a maximal compact Lie subgroup of $\text{Aut}(N)$ is $U(1) \rtimes \langle \tau \rangle$ which acts on N

$$(4.1) \quad \begin{aligned} e^{i\theta}(x, z) &= (x, e^{i\theta}z), \quad (e^{i\theta} \in U(1)), \\ \tau(x, z) &= (-x, \bar{z}). \end{aligned}$$

A 3-dimensional compact infranilmanifold is obtained as a quotient N/Γ where Γ is a torsionfree discrete uniform subgroup of $E(N) = N \rtimes (U(1) \rtimes \langle \tau \rangle)$. See [4].

Let

$$S^1 \rightarrow M_3 \rightarrow M_2$$

be an S^1 -fibred nilBott manifold of infinite type which has a group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_3 \rightarrow \pi_2 \rightarrow 1$. As before this group extension contains a central group extension $1 \rightarrow \mathbb{Z} \rightarrow \tilde{\Delta}_3 \rightarrow \Delta_2 \rightarrow 1$. Since $\mathbb{R} \subset N$ is the center, this induces the commutative diagram of central extensions (cf. (3.16)):

$$(4.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_3 & \longrightarrow & \Delta_2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & N & \longrightarrow & \mathbb{C} \longrightarrow 1. \end{array}$$

Using this, we obtain an embedding:

$$(4.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow \hat{\rho} & & \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & E(\mathbb{N}) & \longrightarrow & \mathbb{C} \rtimes (U(1) \rtimes \langle \tau \rangle) & \longrightarrow & 1. \end{array}$$

Note that $\mathbb{C} \rtimes (U(1) \rtimes \langle \tau \rangle) = \mathbb{R}^2 \rtimes O(2) = E(2)$. Since $\mathbb{R} \cap \pi_3 = \mathbb{Z}$ from (4.3), $\hat{\rho}(\pi_2)$ is a Bieberbach group in $E(2)$ so that $\mathbb{R}^2/\hat{\rho}(\pi_2)$ is either T^2 or K .

Define $L: E(\mathbb{N}) \rightarrow U(1) \rtimes \langle \tau \rangle$ to be the canonical projection.

CASE (i). Suppose $L(\pi_3) = \{1\}$. Then $\hat{\rho}(\pi_2) \leq \mathbb{C}$. So we may assume $\pi_3 = \tilde{\Delta}_3$ from (4.2). For each $k \in \mathbb{Z}$, we introduce the nilpotent group $\Delta(k)$ which is a subgroup of \mathbb{N} generated by

$$c = (2k, 0), \quad a = (0, k), \quad b = (0, k\mathbf{i}).$$

Put $Z = \langle c \rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that

$$(4.4) \quad [a, b] = c^{-k}.$$

Then $\tilde{\Delta}_3 \leq \mathbb{N}$ is isomorphic to $\Delta(k)$ for some $k \in \mathbb{Z}$. Since \mathbb{R} is the center of \mathbb{N} , we have a principal bundle

$$S^1 = \mathbb{R}/Z \rightarrow \mathbb{N}/\Delta(k) \rightarrow \mathbb{C}/\mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [9] for example.)

CASE (ii). Suppose that the holonomy group of π_3 is nontrivial. Then we note that $L(\pi_3) = \mathbb{Z}_2 \leq U(1) \rtimes \langle \tau \rangle$, but not in $U(1)$. By (3.16) $L(\pi_3) = L(\pi_2)$, first remark that $L(\pi_2)$ is not contained in $U(1)$. For this, suppose that (b, A) is an element of $\pi_2 \leq \mathbb{R}^2 \rtimes O(2)$. Then for any $x \in \mathbb{R}^2$, $(b, A)x \neq x$, because the action of π_2 on \mathbb{R}^2 is free. Therefore $\det(A - I) = 0$. This implies that if $A \in SO(2) = U(1)$, then $A = I$. So $L(\pi_2) = L(\pi_3)$ is not contained in $U(1)$.

Suppose that there exists an element $g \in \pi_3$ such that $L(g) = (e^{i\theta}, \tau) \in U(1) \rtimes \langle \tau \rangle$. Noting (4.1), it follows $L(g)^2 = 1$. Then $L(\pi_3) = (U(1) \cap L(\pi_3)) \cdot \langle L(g) \rangle$. Let $\pi'_3 = L^{-1}(U(1) \cap L(\pi_3)) \leq \pi_3$ which has the commutative diagram:

$$(4.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \xrightarrow{p_3} & \pi_2 & \longrightarrow & 1 \\ & & \parallel & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi'_3 & \longrightarrow & \pi'_2 & \longrightarrow & 1. \end{array}$$

Here $\pi'_2 = p_3(\pi'_3)$. Since π'_2 also acts on \mathbb{R}^2 freely, it follows $L(\pi'_2) = L(\pi'_3) = U(1) \cap L(\pi_3) = \{1\}$. Hence $L(\pi_2) = L(\pi_3) = \mathbb{Z}_2 = \langle L(g) \rangle$. In particular M_2 is the Klein bottle K .

Let $n = (x, 0)$ be a generator of $\mathbb{Z} \leq \mathbf{N}$. Choose $h \in \pi_3$ with $L(h) = 1$ such that the subgroup $\langle p_3(g), p_3(h) \rangle$ is the fundamental group of K . It has a relation $p_3(g)p_3(h)p_3(g)^{-1} = p_3(h)^{-1}$. Then $\langle n, g, h \rangle$ is isomorphic to π_3 . In particular, those generators satisfy

$$(4.6) \quad \begin{aligned} ghg^{-1} &= n^k h^{-1} \quad (\exists k \in \mathbb{Z}), \\ gng^{-1} &= L(g)n = \tau n = n^{-1}, \quad ahnh^{-1} = L(h)n = n. \end{aligned}$$

On the other hand, fix a non-zero integer k . Let $\Gamma(k)$ be a subgroup of $\mathbf{E}(\mathbf{N})$ generated by

$$(4.7) \quad n = ((k, 0), I), \quad \alpha = \left(\left(0, \frac{k}{2} \right), \tau \right), \quad \beta = ((0, ki), I),$$

where $(a, x) \in \mathbf{N} = \mathbf{R} \times \mathbb{C} \leq \mathbf{E}(\mathbf{N})$.

Note that $\alpha^2 = ((0, k), I)$. Then it is easily checked that

$$(4.8) \quad \alpha\beta\alpha^{-1} = n^k\beta^{-1}, \quad \alpha n\alpha^{-1} = n^{-1}, \quad \beta n\beta^{-1} = n,$$

$$(4.9) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{E}(\mathbf{N}) & \longrightarrow & \mathbb{C} \rtimes (\mathbf{U}(1) \rtimes \langle \tau \rangle) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \langle n \rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle \hat{\alpha}, \hat{\beta} \rangle \longrightarrow 1. \end{array}$$

Then the subgroup generated by $\hat{\alpha}^2, \hat{\beta}$ is isomorphic to the subgroup of translations of \mathbb{R}^2 ; $t_1 = \begin{bmatrix} k \\ 0 \end{bmatrix}$, $t_2 = \begin{bmatrix} 0 \\ k \end{bmatrix}$. Let $T^2 = \mathbb{R}^2 / \langle t_1, t_2 \rangle$. Then it is easy to see that the quotient $\gamma = [\hat{\alpha}]$ of order 2 acts on T^2 as

$$(4.10) \quad \gamma(z_1, z_2) = (-z_1, \bar{z}_2).$$

As a consequence, $\mathbb{R}^2 / \langle \hat{\alpha}, \hat{\beta} \rangle = T^2 / \langle \gamma \rangle$ turns out to be K . So $M_3 = \mathbf{N} / \Gamma(k)$ is an S^1 -fibred nilBott manifold:

$$S^1 \rightarrow \mathbf{N} / \Gamma(k) \rightarrow K$$

where $S^1 = \mathbf{R} / \langle n \rangle$ is the fiber (but not an action).

Compared (4.6) with $\Gamma(k)$, π_3 is isomorphic to $\Gamma(k)$ with the following commutative arrows of isomorphisms:

$$(4.11) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \langle n \rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle \hat{\alpha}, \hat{\beta} \rangle \longrightarrow 1. \end{array}$$

As both (π_3, X_3) and $(\Gamma(k), \mathbf{N})$ are Seifert actions, the isomorphism of (4.11) implies that they are equivariantly diffeomorphic, i.e. $M_3 = X_3/\pi_3 \cong \mathbf{N}/\Gamma(k)$. This shows the following.

Proposition 4.2. *A 3-dimensional S^1 -fibred nilBott manifold M_3 of infinite type is either a Heisenberg nilmanifold $\mathbf{N}/\Delta(k)$ or a Heisenberg infranilmanifold $\mathbf{N}/\Gamma(k)$.*

5. Realization

5.1. Realization of S^1 -fibration over a Klein bottle K . Let Q be a fundamental group of a Klein Bottle K , then Q has a presentation:

$$(5.1) \quad \{g, h \mid ghg^{-1} = h^{-1}\}.$$

A group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi \rightarrow Q \rightarrow 1$ for any 3-dimensional S^1 -fibred nilBott manifold over K represents a 2-cocycle in $H_\phi^2(Q, \mathbb{Z})$ for some representation ϕ . Conversely, given any representation $\phi: Q \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm 1\}$, we shall prove that any element of $H_\phi^2(Q, \mathbb{Z})$ can be realized as an S^1 -fibred nilBott manifold.

We must consider following cases of a representation ϕ :

CASE 1. $\phi(g) = 1, \phi(h) = 1$.

CASE 2. $\phi(g) = 1, \phi(h) = -1$.

CASE 3. $\phi(g) = -1, \phi(h) = 1$.

CASE 4. $\phi(g) = -1, \phi(h) = -1$.

Suppose ϕ_i ($i = 1, 2, 3, 4$) is the representation ϕ for Case i . Any element of $H_{\phi_i}^2(Q, \mathbb{Z})$ gives rise to a group extension

$$1 \rightarrow \mathbb{Z} \rightarrow \pi \xrightarrow{p} Q \rightarrow 1.$$

Then π is generated by \tilde{g}, \tilde{h}, n such that $\langle n \rangle = \mathbb{Z}$ and $p(\tilde{g}) = g, p(\tilde{h}) = h$. There exists $k \in \mathbb{Z}$ which satisfies

$$(5.2) \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Put $\pi = {}_i\pi(k)$ for each $k \in \mathbb{Z}$ and $[f_k]$ denotes the 2-cocycle of $H_{\phi_i}^2(Q, \mathbb{Z})$ representing ${}_i\pi(k)$. Note that $[f_0] = 0$.

CASE 1: Since ϕ_1 is trivial, $H_{\phi_1}^2(Q, \mathbb{Z}) = H^2(Q, \mathbb{Z}) \approx H^2(K, \mathbb{Z}) \approx \mathbb{Z}_2$, and the group ${}_1\pi(k)$ satisfies the following presentation:

$$(5.3) \quad \tilde{g}n\tilde{g}^{-1} = n, \quad \tilde{h}n\tilde{h}^{-1} = n, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Lemma 5.1. *The groups ${}_1\pi(0), {}_1\pi(1)$ are isomorphic to $\mathcal{B}_1, \mathcal{B}_2$ respectively.*

Proof. First we shall discuss ${}_1\pi(0)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(0)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = n$ and $t_3 = \tilde{h}$. Remark that a group generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(0)$. Using the relation (5.3),

$$\begin{aligned}\varepsilon^2 &= t_1, \\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g} \tilde{h} \tilde{g}^{-1} = \tilde{h}^{-1} = t_2^{-1}, \\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g} n \tilde{g}^{-1} = n = t_3.\end{aligned}$$

Compared these relations with those of \mathcal{B}_1 , ${}_1\pi(0)$ is isomorphic to \mathcal{B}_1 (due to the Wolf's notation [13]).

Second, we shall discuss ${}_1\pi(1)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(1)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{g}^{-2}n$ and $t_3 = \tilde{h}$. A group generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(1)$. By using the relation (5.3),

$$\begin{aligned}\varepsilon^2 &= t_1, \\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g} \tilde{g}^{-2} n \tilde{g}^{-1} = \tilde{g}^{-1} n \tilde{g}^{-1} = \tilde{g}^{-2} n = t_2, \\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g} \tilde{h} \tilde{g}^{-1} = \tilde{g}^2 \tilde{g}^{-2} n \tilde{h}^{-1} = t_1 t_2 t_3^{-1}.\end{aligned}$$

This implies that ${}_1\pi(1)$ is isomorphic to \mathcal{B}_2 . (See [13].) □

For arbitrary $k \in \mathbb{Z}$, we have the following.

Proposition 5.2. *The group extension ${}_1\pi(k)$ is isomorphic to \mathcal{B}_1 , or \mathcal{B}_2 in accordance with k is even or odd.*

Proof. Take $[f_1] \in H_{\phi_1}^2(Q, \mathbb{Z}) \approx \mathbb{Z}_2$ by Lemma 5.1, then

$$\begin{aligned}(5.4) \quad n &= \tilde{g} \tilde{h} \tilde{g}^{-1} \tilde{h} = (0, g)(0, h)(-f_1(g^{-1}, g), g^{-1})(0, h) \\ &= f_1(g, h) - f_1(g^{-1}, g) + f_1(gh, g^{-1}) + f_1(h^{-1}, h),\end{aligned}$$

and so

$$(5.5) \quad n^k = k f_1(g, h) - k f_1(g^{-1}, g) + k f_1(gh, g^{-1}) + k f_1(h^{-1}, h).$$

Since $[k f_1] \in H_{\phi_1}^2(Q, \mathbb{Z})$, we can construct a group H_k which is represented by $(k f_1, \phi_1)$. Then H_k is generated by the elements n and $g' = (0, g)$, $h' = (0, h)$ satisfying that

$$(n, \alpha)(m, \beta) = (n + \phi_1(\alpha)(m) + k f_1(\alpha, \beta), \alpha\beta) \quad (\forall n, m \in \mathbb{Z}, \forall \alpha, \beta \in Q).$$

It follows

$$\begin{aligned}
g'h'g'^{-1}h' &= (0, g)(0, h)(-kf_1(g^{-1}, g), g^{-1})(0, h) \\
&= kf_1(g, h) - kf_1(g^{-1}, g) + kf_1(gh, g^{-1}) + kf_1(h^{-1}, h) \\
&= n^k \quad (\text{from (5.5)}).
\end{aligned}$$

Thus we obtain $g'h'g'^{-1} = n^k h'^{-1}$. In view of (5.2), a correspondence $g' \mapsto \tilde{g}$, $h' \mapsto \tilde{h}$ gives an isomorphism Ψ of the group extensions:

$$(5.6) \quad \begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_k & \longrightarrow & Q \longrightarrow 1 \\
& & \text{id} \downarrow & & \Psi \downarrow & & \text{id} \downarrow \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_1\pi(k) & \longrightarrow & Q \longrightarrow 1.
\end{array}$$

If we recall that $[f_k]$ (resp. $[k \cdot f_1]$) represents ${}_1\pi(k)$ (resp. H_k), then it follows $[f_k] = k \cdot [f_1]$. Noting that $[f_1]$ is a two torsion element, the result follows. \square

CASE 2: Let $\phi_2(g) = 1$, $\phi_2(h) = -1$, then ${}_2\pi(k)$ has the following presentation.

$$(5.7) \quad \tilde{g}n\tilde{g}^{-1} = n, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1},$$

for some $k \in \mathbb{Z}$.

Proposition 5.3. *The groups ${}_2\pi(0)$, ${}_2\pi(1)$ are isomorphic to \mathcal{B}_3 , \mathcal{B}_4 respectively.*

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_2\pi(0)$ be as before. Put $\alpha = \tilde{h}\tilde{g}$, $\varepsilon = \tilde{h}^{-1}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}^{-2}$ and $t_3 = n$. Note that the group generated by $\alpha, \varepsilon, t_1, t_2, t_3$ coincides with ${}_2\pi(0)$. Using the relation (5.7),

$$\begin{aligned}
\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}\tilde{h}^{-1}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1, \\
\varepsilon^2 &= t_2, \\
\varepsilon\alpha\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h} = \tilde{h}^{-1}\tilde{g} = t_2\alpha, \\
\alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1}, \\
\alpha t_3\alpha^{-1} &= \tilde{h}\tilde{g}n\tilde{g}^{-1}\tilde{h}^{-1} = n^{-1} = t_3^{-1}, \\
\varepsilon t_1\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{g}^2\tilde{h} = \tilde{h}^{-1}\tilde{g}\tilde{h}^{-1}\tilde{g} = \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1, \\
\varepsilon t_3\varepsilon^{-1} &= \tilde{h}^{-1}n\tilde{h} = n^{-1} = t_3^{-1}.
\end{aligned}$$

Since these relations correspond to those of \mathcal{B}_3 (cf. [13]), ${}_2\pi(0)$ is isomorphic to \mathcal{B}_3 .

Let $\tilde{g}, \tilde{h}, n \in {}_2\pi(1)$ be as above. Put $\alpha = \tilde{h}\tilde{g}$, $\varepsilon = n^{-1}\tilde{h}^{-1}$, $t_1 = n^{-1}\tilde{g}^2$, $t_2 = \tilde{h}^{-2}$, and $t_3 = n^{-1}$. Using the relation (5.7), we obtain the following presentation:

$$\begin{aligned}\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}n\tilde{h}^{-1}\tilde{g}\tilde{g} = n^{-1}\tilde{g}^2 = t_1, \\ \varepsilon^2 &= t_2, \\ \varepsilon\alpha\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h}n = \tilde{h}^{-1}\tilde{g}n = t_2t_3\alpha, \\ \alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1}, \\ \alpha t_3\alpha^{-1} &= \tilde{h}\tilde{g}n^{-1}\tilde{g}^{-1}\tilde{h}^{-1} = n = t_3^{-1}, \\ \varepsilon t_1\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{g}^2\tilde{h}n = n^{-1}\tilde{g}^2 = t_1, \\ \varepsilon t_3\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{h}n = n = t_3^{-1}.\end{aligned}$$

This implies that ${}_2\pi(1)$ is isomorphic to \mathcal{B}_4 . (See [13]). \square

Proposition 5.4. $H_{\phi_2}^2(Q, \mathbb{Z})$ is isomorphic to \mathbb{Z}_2 .

Proof. We first show that $H_{\phi_2}^2(Q, \mathbb{Z})$ is a 2-torsion group. Let Q' be the subgroup of Q generated by $g, h^2 \in Q$ satisfying that $gh^2g^{-1} = (ghg^{-1})^2 = h^{-2}$. We have a commutative diagram:

$$(5.8) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_2\pi(k) & \xrightarrow{p} & Q & \longrightarrow & 1 \\ & & \parallel & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \xrightarrow{p} & Q' & \longrightarrow & 1 \end{array}$$

where π' is the subgroup of ${}_2\pi(k)$ generated by $n, \tilde{g}, \tilde{h}^2$. Note that

$$\tilde{g}\tilde{h}^2\tilde{g}^{-1} = n^k\tilde{h}^{-1}n^k\tilde{h}^{-1} = \tilde{h}^{-2}.$$

Since the subgroup $\langle \tilde{g}, \tilde{h}^2 \rangle$ of π' maps isomorphically onto Q' and a restriction $\phi_2|_{Q'} = \text{id}$, it follows $\pi' = \mathbb{Z} \times Q'$. This shows that the restriction homomorphism $\iota^*: H_{\phi_2}^2(Q, \mathbb{Z}) \rightarrow H^2(Q', \mathbb{Z})$ is the zero map, equivalently $\iota^*[f_k] = 0$. Using the transfer homomorphism $\tau: H^2(Q', \mathbb{Z}) \rightarrow H_{\phi_2}^2(Q, \mathbb{Z})$ and by the property $\tau \circ \iota^*([f]) = [Q: Q'] [f] = 2[f]$ ($\forall [f] \in H_{\phi_2}^2(Q, \mathbb{Z})$), we obtain $2[f] = 0$.

Let $[f_k]$ be a 2-cocycle of ${}_2\pi(k)$. Similarly as in the proof of Proposition 5.2 we obtain

$$(5.9) \quad [f_k] = k \cdot [f_1].$$

As a consequence, $H_{\phi_2}^2(Q, \mathbb{Z})$ is isomorphic to \mathbb{Z}_2 . \square

The following is obvious using Proposition 5.3 and Proposition 5.4.

Corollary 5.5. *The group extension ${}_2\pi(k)$ is isomorphic to \mathcal{B}_3 or \mathcal{B}_4 in accordance with k is even or odd.*

CASE 3: The group ${}_3\pi(k)$ has the following presentation for some $k \in \mathbb{Z}$;

$$(5.10) \quad \tilde{g}n\tilde{g}^{-1} = n^{-1}, \quad \tilde{h}n\tilde{h}^{-1} = n, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Lemma 5.6. *The groups ${}_3\pi(0)$, ${}_3\pi(k)$ are isomorphic to \mathcal{G}_2 , $\Gamma(k)$ respectively. (cf. (4.7).)*

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_3\pi(0)$ be as before. Put $\alpha = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}$ and $t_3 = n$. Note that the group generated by α, t_1, t_2, t_3 coincides with ${}_3\pi(0)$. By using the relation (5.10), it is easy to check that:

$$\begin{aligned} \alpha^2 &= t_1, \\ \alpha t_2 \alpha^{-1} &= t_2^{-1}, \\ \alpha t_3 \alpha^{-1} &= t_3^{-1}. \end{aligned}$$

And so ${}_3\pi(0)$ is isomorphic to \mathcal{G}_2 . (See [13].)

Suppose $\tilde{g}, \tilde{h}, n \in {}_3\pi(k)$ ($k \neq 0$). By the relations (4.6) and (5.10), ${}_3\pi(k)$ is isomorphic to $\Gamma(k)$ (cf. (4.7)). \square

Proposition 5.7. *$H_{\phi_3}^2(G, \mathbb{Z})$ is isomorphic to \mathbb{Z} .*

Proof. From Theorem 1.2 and Lemma 5.6, $\Gamma(k)$ represents the torsionfree element $[f_k]$ in $H_{\phi_3}^2(G, \mathbb{Z})$. Moreover as in the proof of Proposition 5.2, we can show that $[f_k] = k \cdot [f_1]$. Therefore $H_{\phi_3}^2(G, \mathbb{Z})$ is isomorphic to \mathbb{Z} . \square

CASE 4. The group ${}_4\pi(k)$ has the following presentation.

$$(5.11) \quad \tilde{g}n\tilde{g}^{-1} = n^{-1}, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Put $\alpha = \tilde{g}\tilde{h}$. It is easy to check that

$$(5.12) \quad \alpha n \alpha^{-1} = n, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \alpha \tilde{h} \alpha^{-1} = n^k \tilde{h}^{-1}.$$

In view of (5.7), this implies that ${}_4\pi(k)$ is isomorphic to ${}_2\pi(k)$.

We have shown that any element of $H_{\phi_1}^2(Q, \mathbb{Z})$ can be realized an S^1 -fibred nilBott manifold M_3 , and obtain the following table:

		Case 1	Case 2 and 4	Case 3
	$H_\phi^2(Q, \mathbb{Z})$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
$\pi_1(M_3)$	$[f] = 0$	\mathcal{B}_1	\mathcal{B}_3	\mathcal{G}_2
	$[f] \neq 0$: torsion	\mathcal{B}_2	\mathcal{B}_4	—
	$[f]$: torsionfree	—	—	$\Gamma(k)$

5.2. Realization of S^1 -fibration over T^2 . Let \mathbb{Z}^2 be the fundamental group of a torus T^2 generated by α, β . Given a representation $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z} = \{\pm 1\}$, we shall show that any element of $H_\phi^2(\mathbb{Z}^2, \mathbb{Z})$ can be realized as an S^1 -fibred nilBott manifold.

We must consider following cases of a representation ϕ :

CASE 5. $\phi(\alpha) = 1, \phi(\beta) = 1$.

CASE 6. $\phi(\alpha) = 1, \phi(\beta) = -1$.

CASE 7. $\phi(\alpha) = -1, \phi(\beta) = -1$.

Suppose ϕ_i ($i = 5, 6, 7$) is the representation ϕ for Case i. Any element of $H_{\phi_i}^2(\mathbb{Z}^2, \mathbb{Z})$ gives rise to a group extension

$$1 \rightarrow \mathbb{Z} \rightarrow \pi \xrightarrow{p} \mathbb{Z}^2 \rightarrow 1.$$

Then π is generated by $\tilde{\alpha}, \tilde{\beta}, m$ such that $\langle m \rangle = \mathbb{Z}$ and $p(\tilde{\alpha}) = \alpha, p(\tilde{\beta}) = \beta$. There exists $k \in \mathbb{Z}$ which satisfies

$$(5.13) \quad \tilde{\alpha} \tilde{\beta} \tilde{\alpha}^{-1} = m^k \tilde{\beta}.$$

Put $\pi = {}_i\pi(k)$ for each $k \in \mathbb{Z}$ and $[f_k]$ denotes the 2-cocycle of $H_{\phi_i}^2(\mathbb{Z}^2, \mathbb{Z})$ representing ${}_i\pi(k)$. Note that $[f_0] = 0$.

CASE 5: The group ${}_5\pi(k)$ has the following presentation.

$$(5.14) \quad \tilde{\alpha} m \tilde{\alpha}^{-1} = m, \quad \tilde{\beta} m \tilde{\beta}^{-1} = m, \quad \tilde{\alpha} \tilde{\beta} \tilde{\alpha}^{-1} = m^k \tilde{\beta},$$

for some $k \in \mathbb{Z}$. Compared these relations with (4.4),

Proposition 5.8. *The groups ${}_5\pi(0), {}_5\pi(k)$ are isomorphic to $\pi_1(T^3), \pi_1(\Delta(-k))$ respectively.*

Similarly as in the proof of Proposition 5.7, we obtain

Proposition 5.9. *$H_{\phi_5}^2(\mathbb{Z}^2, \mathbb{Z})$ is isomorphic to \mathbb{Z} .*

CASE 6: The group ${}_6\pi(k)$ has the following presentation.

$$(5.15) \quad \tilde{\alpha} m \tilde{\alpha}^{-1} = m, \quad \tilde{\beta} m \tilde{\beta}^{-1} = m^{-1}, \quad \tilde{\alpha} \tilde{\beta} \tilde{\alpha}^{-1} = m^k \tilde{\beta},$$

for some $k \in \mathbb{Z}$.

Proposition 5.10. *The groups ${}_6\pi(0)$, ${}_6\pi(1)$ are isomorphic to \mathcal{B}_1 , \mathcal{B}_2 respectively.*

Proof. First let $k = 0$. Put $m = \tilde{h}$, $\tilde{\alpha} = n$, $\tilde{\beta} = \tilde{g}$, then we can check easily that ${}_6\pi(0)$ is isomorphic to ${}_1\pi(0)$. So ${}_6\pi(0)$ is isomorphic to \mathcal{B}_1 .

Suppose $k = 1$. Put $m = n$, $\tilde{\alpha} = \tilde{g}$, $m^{-1}\tilde{\beta} = \tilde{h}$, then it is easy to check that ${}_6\pi(1)$ is isomorphic to \mathcal{B}_2 . \square

Moreover similarly as in the proof of Proposition 5.4, we obtain

Proposition 5.11. *$H_{\phi_6}^2(\mathbb{Z}^2, \mathbb{Z})$ is isomorphic to \mathbb{Z}_2 .*

CASE 7: The group ${}_7\pi(k)$ has the following presentation.

$$(5.16) \quad \tilde{\alpha}m^{-1}\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m^{-1}, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta},$$

for some $k \in \mathbb{Z}$. Then it is easy to check that ${}_7\pi(k)$ is isomorphic to ${}_6\pi(k)$ if we put $g = \tilde{\alpha}\tilde{\beta}$.

We have shown that any element of $H_{\phi}^2(\mathbb{Z}^2, \mathbb{Z})$ can be realized an S^1 -fibred nilBott manifold M_3 , and we obtain the following table:

		Case 5	Case 6 and 7
	$H_{\phi}^2(\mathbb{Z}^2, \mathbb{Z})$	\mathbb{Z}	\mathbb{Z}_2
$\pi_1(M_3)$	$[f] = 0$	\mathcal{G}_1	\mathcal{B}_1
	$[f] \neq 0$: torsion	—	\mathcal{B}_2
	$[f]$: torsionfree	$\Delta(k)$	—

6. Halperin–Carlsson conjecture

Theorem 6.1 (Halperin–Carlsson conjecture [11]). *Let T^s be an arbitrary effective action on an m -dimensional S^1 -fibred nilBott manifold M of finite type. Then*

$$(6.1) \quad {}_sC_j \leq b_j \quad (= \text{the } j\text{-th Betti number of } M).$$

In particular $2^s \leq \sum_{j=0}^m \text{Rank } H_j(M)$.

Proof. By Corollary 3.5, M admits a homologically injective T^k -action where $k = \text{Rank } C(\pi)$ where $\pi = \pi_1(M)$. Then we have shown in [6] that any homologically injective T^k -actions on any closed aspherical manifold satisfies that

$${}_kC_j \leq b_j \quad (= \text{the } j\text{-th Betti number of } M).$$

It follows from the result of Conner–Raymond [3] that there is an injective homomorphism $1 \rightarrow \mathbb{Z}^s \rightarrow C(\pi)$. This shows that $s \leq k$ so we obtain

$$(6.2) \quad {}_sC_j \leq b_j. \quad \square$$

REMARK 6.2. This result is obtained when M_i is a real Bott manifold by Masuda, Choi and Oum [2].

References

- [1] K.S. Brown: Cohomology of Groups, Graduate Texts in Mathematics **87**, Springer, New York, 1982.
- [2] S. Choi, M. Masuda and S. Oum: *Classification of real Bott manifolds and acyclic digraphs*, preprint, arXiv:1006.4658v1, math.AT (2010).
- [3] P.E. Conner and F. Raymond: *Actions of compact Lie groups on aspherical manifolds*; in Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, IL, 1970, 227–264.
- [4] K. Dekimpe: Almost-Bieberbach Groups: Affine and Polynomial Structures, Lecture Notes in Mathematics **1639**, Springer, Berlin, 1996.
- [5] Y. Kamishima, K.B. Lee and F. Raymond: The Seifert construction and its applications to infranilmanifolds, Quart. J. Math. Oxford Ser. (2) **34** (1983), 433–452.
- [6] Y. Kamishima and M. Nakayama: *On the nil-Bott tower of aspherical manifolds*, in preparation.
- [7] K.B. Lee and F. Raymond: *Geometric realization of group extensions by the Seifert construction*; in Contributions to Group Theory, Contemp. Math. **33**, Amer. Math. Soc., Providence, RI, 1984, 353–411.
- [8] J.B. Lee and M. Masuda: *Topology of iterated S^1 -bundles*, preprint, arXiv:1108.0293, math.AT (2011).
- [9] J. Milnor: *On the 3-dimensional Brieskorn manifolds $M(p, q, r)$* ; in Knots, Groups, and 3-Manifolds (Papers dedicated to the memory of R.H. Fox), Ann. of Math. Studies **84**, Princeton Univ. Press, Princeton, NJ, 1975, 175–225.
- [10] M. Nakayama: *Seifert construction for nilpotent groups and application to an S^1 -fibred nil-Bott Tower*, to appear in RIMS Kokyuroku Bessatsu.
- [11] V. Puppe: *Multiplicative aspects of the Halperin–Carlsson conjecture*, Georgian Math. J. **16** (2009), 369–379.
- [12] M.S. Raghunathan: Discrete Subgroups of Lie Groups, Ergebnisse Math. Grenzgebiete **68**, Springer, New York, 1972.
- [13] J.A. Wolf: Spaces of Constant Curvature, McGraw-Hill, New York, 1967.

Department of Mathematics and Information of Sciences
 Tokyo Metropolitan University
 Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397
 Japan
 e-mail: nakayama-mayumi@ed.tmu.ac.jp