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AFFINE AND DEGENERATE AFFINE BMW ALGEBRAS: THE CENTER

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Abstract

The degenerate affine and affine BMW algebras arise naturally in the context of Schur–Weyl duality for orthogonal and symplectic Lie algebras and quantum groups, respectively. Cyclotomic BMW algebras, affine Hecke algebras, cyclotomic Hecke algebras, and their degenerate versions are quotients. In this paper the theory is unified by treating the orthogonal and symplectic cases simultaneously; we make an exact parallel between the degenerate affine and affine cases via a new algebra which takes the role of the affine braid group for the degenerate setting. A main result of this paper is an identification of the centers of the affine and degenerate affine BMW algebras in terms of rings of symmetric functions which satisfy a "cancellation property" or "wheel condition" (in the degenerate case, a reformulation of a result of Nazarov). Miraculously, these same rings also arise in Schubert calculus, as the cohomology and K-theory of isotropic Grassmannians and symplectic loop Grassmannians. We also establish new intertwiner-like identities which, when projected to the center, produce the recursions for central elements given previously by Nazarov for degenerate affine BMW algebras, and by Beliakova–Blanchet for affine BMW algebras.

1. Introduction

The degenerate affine BMW algebras $\mathcal{W}_k$ and the affine BMW algebras W_k arise naturally in the context of Schur–Weyl duality and the application of Schur functors to modules in category $\mathcal{O}$ for orthogonal and symplectic Lie algebras and quantum groups (using the Schur functors of [41], [1], and [28]). The degenerate algebras $\mathcal{W}_k$ were introduced in [27] and the affine versions W_k appeared in [28], following foundational work of [17]–[19]. The representation theory of $\mathcal{W}_k$ and W_k contains the representation theory of any quotient: in particular, the degenerate cyclotomic BMW algebras $\mathcal{W}_{r,k}$, the cyclotomic BMW algebras $W_{r,k}$, the degenerate affine Hecke algebras $\mathcal{H}_k$, the affine Hecke algebras H_k, the degenerate cyclotomic Hecke algebras $\mathcal{H}_{r,k}$, and the cyclotomic Hecke algebras $H_{r,k}$ as quotients. In [31, 34, 35, 2, 8] and other works, the representation theory of the affine BMW algebra is derived by cellular algebra techniques. As indicated in [28], the Schur–Weyl duality also provides a path to the representation theory of the affine BMW algebras as an image of the representation theory of category $\mathcal{O}$ for orthogonal and symplectic Lie algebras and their quantum groups

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in the same way that the affine Hecke algebras arise in Schur–Weyl duality with the enveloping algebra of $\mathfrak{gl}_n$ and its Drinfeld–Jimbo quantum group.

In the literature, the algebras $\mathcal{W}_k$ and W_k have often been treated separately. One of the goals of this paper is to unify the theory. To do this we have begun by adjusting the definitions of the algebras carefully to make the presentations match, relation by relation. In the same way that the affine BMW algebra is a quotient of the group algebra of the affine braid group, we have defined a new algebra, the degenerate affine braid algebra which has the degenerate affine BMW algebra and the degenerate affine Hecke algebras as quotients. We have done this carefully, to ensure that the Schur–Weyl duality framework is completely analogous for both the degenerate affine and the affine cases. We have also added a parameter ϵ (which takes values ± 1) so that both the orthogonal and symplectic cases can be treated simultaneously. Our new presentations of the algebras $\mathcal{W}_k$ and W_k are given in Section 2.

In Section 3 we consider some remarkable recursions for generating central elements in the algebras $\mathcal{W}_k$ and W_k. These recursions were given by Nazarov [27] in the degenerate case, and then extended to the affine BMW algebra by Beliakova–Blanchet [4]. Another proof in the affine cyclotomic case appears in [35, Lemma 4.21] and, in the degenerate case, in [2, Lemma 4.15]. In all of these proofs, the recursion is obtained by a rather mysterious and tedious computation. We show that there is an "intertwiner" like identity in the full algebra which, when "projected to the center" produces the Nazarov recursions. Our approach provides new insight into where these recursions are coming from. Moreover, the proof is exactly analogous in both the degenerate and the affine cases, and includes the parameter ϵ, so that both the orthogonal and symplectic cases are treated simultaneously.

In Section 4 we identify the center of the degenerate and affine BMW algebras. In the degenerate case this has been done in [27]. Nazarov stated that the center of the degenerate affine BMW algebra is the subring of the ring of symmetric functions generated by the odd power sums. We identify the ring in a different way, as the subring of symmetric functions with the Q-cancellation property, in the language of Pragacz [29]. This is a fascinating ring. Pragacz identifies it as the cohomology ring of orthogonal and symplectic Grassmannians; the same ring appears again as the cohomology of the loop Grassmannian for the symplectic group in [24, 22]; and references for the relationship of this ring to the projective representation theory of the symmetric group, the BKP hierarchy of differential equations, representations of Lie superalgebras, and twisted Gelfand pairs are found in [25, Chapter II §8]. For the affine BMW algebra, the Q-cancellation property can be generalized well to provide a suitable description of the center. From our perspective, one would expect that the ring which appears as the center of the affine BMW algebra should also appear as the K-theory of the orthogonal and symplectic Grassmannians and as the K-theory of the loop Grassmannian for the symplectic group, but we are not aware that these identifications have yet been made in the literature.

The recent paper [31] classifies the irreducible representations of W_k by multisegments, and the recent paper [6] adds to this program of study by setting up commuting actions between the algebras $\mathcal{W}_k$ and W_k and the enveloping algebras of orthogonal and symplectic Lie algebras and their quantum groups, showing how the central elements which arise in the Nazarov recursions coincide with central elements studied in Baumann [3], and providing an approach to admissibility conditions by providing "universal admissible parameters" in an appropriate ground ring (arising naturally, from Schur–Weyl duality, as the center of the enveloping algebra, or quantum group). We would also like to mention the recent paper of A. Sartori [36] which establishes similar results in the case of the degenerate affine walled Brauer algebra and the recent work of M. Ehrig and C. Stroppel [7] which studies these algebras in the context of categorification.

2. Affine and degenerate affine BMW algebras

In this section, we define the affine Birman–Murakami–Wenzl (BMW) algebra W_k and its degenerate version $\mathcal{W}_k$. We have adjusted the definitions to unify the theory. In particular, in Section 2.2, we define a new algebra, the degenerate affine braid algebra $\mathcal{B}_k$, which has the degenerate affine BMW algebras $\mathcal{W}_k$ and the degenerate affine Hecke algebras $\mathcal{H}_k$ as quotients. The motivation for the definition of $\mathcal{B}_k$ is that the affine BMW algebras W_k and the affine Hecke algebras H_k are quotients of the group algebra of affine braid group CB_k.

The definition of the degenerate affine braid algebra $\mathcal{B}_k$ also makes the Schur–Weyl duality framework completely analogous in both the affine and degenerate affine cases. Both $\mathcal{B}_k$ and CB_k are designed to act on tensor space of the form $M \otimes V^{\otimes k}$. In the degenerate affine case this is an action commuting with a complex semisimple Lie algebra $\mathfrak{g}$, and in the affine case this is an action commuting with the Drinfeld–Jimbo quantum group $U_q\mathfrak{g}$. The degenerate affine and affine BMW algebras arise when $\mathfrak{g}$ is $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ and V is the first fundamental representation and the degenerate affine and affine Hecke algebras arise when $\mathfrak{g}$ is $\mathfrak{gl}_n$ or $\mathfrak{sl}_n$ and V is the first fundamental representation. In the case when M is the trivial representation and $\mathfrak{g}$ is $\mathfrak{so}_n$, the "Jucys–Murphy" elements $y_1, \ldots, y_k$ in $\mathcal{B}_k$ become the "Jucys–Murphy" elements for the Brauer algebras used in [27] and, in the case that $\mathfrak{g} = \mathfrak{gl}_n$, these become the classical Jucys–Murphy elements in the group algebra of the symmetric group. The Schur–Weyl duality actions are explained in [6].

2.1. The affine BMW algebra W_k. The *affine braid group B_k* is the group given by generators $T_1, T_2, \ldots, T_{k-1}$ and X^{ε_1}, with relations

$$T_iT_j = T_jT_i, \qquad \text{if} \quad j \neq i \pm 1, \tag{2.1}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad \text{for} \quad i = 1, 2, \ldots, k-2, \tag{2.2}$$

$$X^{\varepsilon_1}T_1X^{\varepsilon_1}T_1 = T_1X^{\varepsilon_1}T_1X^{\varepsilon_1}, \tag{2.3}$$

$$X^{\varepsilon_1}T_i = T_iX^{\varepsilon_1}, \qquad \text{for} \quad i = 2, 3, \ldots, k-1. \tag{2.4}$$

Let C be a commutative ring and let CB_k be the group algebra of the affine braid group. Fix constants

$$q, z \in C \quad \text{and} \quad Z_0^{(l)} \in C, \quad \text{for} \quad l \in \mathbb{Z},$$

with q and z invertible. Let $Y_i = zX^{\varepsilon_i}$ so that

$$Y_1 = zX^{\varepsilon_1}, \quad Y_i = T_{i-1}Y_{i-1}T_{i-1}, \quad \text{and} \quad Y_iY_j = Y_jY_i, \quad \text{for} \quad 1 \le i, j \le k. \tag{2.5}$$

In the affine braid group

$$T_iY_iY_{i+1} = Y_iY_{i+1}T_i. \tag{2.6}$$

Assume that $q - q^{-1}$ is invertible in C and define E_i in the group algebra of the affine braid group by

$$T_iY_i = Y_{i+1}T_i - (q - q^{-1})Y_{i+1}(1 - E_i). \tag{2.7}$$

The *affine BMW algebra* W_k is the quotient of the group algebra CB_k of the affine braid group B_k by the relations

$$E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\mp 1}E_i, \quad E_iT_{i-1}^{\pm 1}E_i = E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i, \tag{2.8}$$

$$E_1Y_1^lE_1 = Z_0^{(l)}E_1, \quad E_iY_iY_{i+1} = E_i = Y_iY_{i+1}E_i. \tag{2.9}$$

The *affine Hecke algebra* H_k is the affine BMW algebra W_k with the additional relations

$$E_i = 0, \quad \text{for} \quad i = 1, \dots, k-1. \tag{2.10}$$

Fix $b_1, \dots, b_r \in C$. The *cyclotomic BMW algebra* $W_{r,k}(b_1, \dots, b_r)$ is the affine BMW algebra W_k with the additional relation

$$(Y_1 - b_1) \cdots (Y_1 - b_r) = 0. \tag{2.11}$$

The *cyclotomic Hecke algebra* $H_{r,k}(b_1, \dots, b_r)$ is the affine Hecke algebra H_k with the additional relation (2.11).

Since the composite map $C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}] \to CB_k \to W_k \to H_k$ is injective and the last two maps are surjections, it follows that the Laurent polynomial ring $C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ is a subalgebra of CB_k and W_k.

Proposition 2.1. *The affine BMW algebra W_k coincides with the one defined in* [28].

Proof. In [28] the affine BMW algebra is defined as the quotient of the group algebra of the affine braid group by the relations in (2.8) which are [28, (6.3b) and

(6.3c)], the first relation in (2.9) which is [28, (6.3d)], the second relation in (2.9) for $i = 1$ which is [28, (6.3e)], and the second relation in (2.15) below where, in [28], the element E_i is defined by the first equation in (2.13) below.

Working in W_k, since $Y_{i+1}^{-1}(T_iY_i)Y_{i+1} = Y_{i+1}^{-1}Y_iY_{i+1}T_i = Y_iT_i$, conjugating (2.7) by Y_{i+1}^{-1} gives

$$Y_iT_i = T_iY_{i+1} - (q-q^{-1})(1-E_i)Y_{i+1}. \tag{2.12}$$

Left multiplying (2.7) by Y_{i+1}^{-1} and using the second identity in (2.5) shows that (2.7) is equivalent to $T_i - T_i^{-1} = (q-q^{-1})(1-E_i)$, so that

$$E_i = 1 - \frac{T_i - T_i^{-1}}{q-q^{-1}} \quad \text{and} \quad T_iT_{i+1}E_iT_{i+1}^{-1}T_i^{-1} = E_{i+1}. \tag{2.13}$$

Thus the E_i in W_k coincides with the E_i used in [28].

Multiply the second relation in (2.13) on the left and the right by E_i, and then use the relations in (2.8) to get

$$E_iE_{i+1}E_i = E_iT_iT_{i+1}E_iT_{i+1}^{-1}T_i^{-1}E_i = E_iT_{i+1}E_iT_{i+1}^{-1}E_i = zE_iT_{i+1}^{-1}E_i = E_i,$$

so that

$$E_iE_{i\pm1}E_i = E_i, \quad \text{and} \quad E_i^2 = \left(1 + \frac{z-z^{-1}}{q-q^{-1}}\right)E_i \tag{2.14}$$

is obtained by multiplying the first equation in (2.13) by E_i and using (2.8). As one can construct representations on which E_1 acts non-trivially, the first relation in (2.9) implies

$$Z_0^{(0)} = 1 + \frac{z-z^{-1}}{q-q^{-1}} \quad \text{and} \quad (T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0, \tag{2.15}$$

since $(T_i - z^{-1})(T_i + q^{-1})(T_i - q)T_i^{-1} = (T_i - z^{-1})(T_i^2 - (q-q^{-1})T_i - 1)T_i^{-1} = (T_i - z^{-1})(T_i - T_i^{-1} - (q-q^{-1})) = (T_i - z^{-1})(q-q^{-1})(-E_i) = -(z^{-1} - z^{-1})(q-q^{-1})E_i = 0$. This shows that the relation [28, (6.3a)] follows from the relations in W_k.

To complete the proof let us show that the relations in W_k follow from [28, (6.3a-e)]. Given the equivalence of the definitions of E_i as established in (2.13), and the coincidences of the relations [28, (6.3b-e)] with the relations in (2.8) and (2.9) it only remains to show that the second set of relations in (2.9), for $i > 1$ follow from [28, (6.3a-e)]. But this is established by [28, (6.12)] and the identity

$$E_iY_iY_{i+1} = \left(1 - \frac{T_i - T_i^{-1}}{q-q^{-1}}\right)Y_iY_{i+1} = Y_iY_{i+1}\left(1 - \frac{T_i - T_i^{-1}}{q-q^{-1}}\right) = Y_iY_{i+1}E_i,$$

which follows from (2.6). □

The relations

$$E_{i+1}E_i = E_{i+1}T_iT_{i+1}, \quad E_iE_{i+1} = T_{i+1}^{-1}T_i^{-1}E_{i+1}, \tag{2.16}$$

$$T_iE_{i+1}E_i = T_{i+1}^{-1}E_i, \quad \text{and} \quad E_{i+1}E_iT_{i+1} = E_{i+1}T_i^{-1}, \tag{2.17}$$

are consequences of (2.8), and the second relation in (2.13).

2.2. The degenerate affine braid algebra $\mathcal{B}_k$. Let C be a commutative ring, and let S_k denote the symmetric group on $\{1,\dots,k\}$. For $i \in \{1,\dots,k\}$, write s_i for the transposition in S_k that switches i and $i+1$. The *degenerate affine braid algebra* is the algebra $\mathcal{B}_k$ over C generated by

$$t_u \quad (u \in S_k), \quad \kappa_0, \kappa_1, \quad \text{and} \quad y_1, \dots, y_k, \tag{2.18}$$

with relations

$$t_ut_v = t_{uv}, \quad y_iy_j = y_jy_i, \quad \kappa_0\kappa_1 = \kappa_1\kappa_0, \quad \kappa_0y_i = y_i\kappa_0, \quad \kappa_1y_i = y_i\kappa_1, \tag{2.19}$$

$$\kappa_0t_{s_i} = t_{s_i}\kappa_0, \quad \kappa_1t_{s_1}\kappa_1t_{s_1} = t_{s_1}\kappa_1t_{s_1}\kappa_1, \quad \text{and} \quad \kappa_1t_{s_j} = t_{s_j}\kappa_1, \quad \text{for} \quad j \neq 1, \tag{2.20}$$

$$t_{s_i}(y_i + y_{i+1}) = (y_i + y_{i+1})t_{s_i}, \quad \text{and} \quad y_jt_{s_i} = t_{s_i}y_j, \quad \text{for} \quad j \neq i, i+1, \tag{2.21}$$

$$\kappa_1t_{s_1}y_1t_{s_1} = t_{s_1}y_1t_{s_1}\kappa_1, \tag{2.22}$$

and

$$\begin{gathered} t_{s_i}t_{s_{i+1}}\gamma_{i,i+1}t_{s_{i+1}}t_{s_i} = \gamma_{i+1,i+2}, \\ \text{where} \quad \gamma_{i,i+1} = y_{i+1} - t_{s_i}y_it_{s_i} \quad \text{for} \quad i = 1, \dots, k-2. \end{gathered} \tag{2.23}$$

In the degenerate affine braid algebra $\mathcal{B}_k$ let $c_0 = \kappa_0$ and

$$\begin{gathered} c_j = \kappa_0 + 2(y_1 + \cdots + y_j), \\ \text{so that} \quad y_j = \frac{1}{2}(c_j - c_{j-1}), \quad \text{for} \quad j = 1, \dots, k. \end{gathered} \tag{2.24}$$

Then $c_0, \dots, c_k$ commute with each other, commute with κ_1, and the relations (2.21) are equivalent to

$$t_{s_i}c_j = c_jt_{s_i}, \quad \text{for} \quad j \neq i. \tag{2.25}$$

Theorem 2.2. *The degenerate affine braid algebra $\mathcal{B}_k$ has another presentation by generators*

$$\begin{gathered} t_u, \quad \textit{for} \quad u \in S_k, \quad \kappa_0, \dots, \kappa_k \quad \textit{and} \quad \gamma_{i,j}, \\ \textit{for} \quad 0 \le i, j \le k \quad \textit{with} \quad i \neq j, \end{gathered} \tag{2.26}$$

and relations

$$t_u t_v = t_{uv}, \quad t_w \kappa_i t_{w^{-1}} = \kappa_{w(i)}, \quad t_w \gamma_{i,j} t_{w^{-1}} = \gamma_{w(i),w(j)}, \tag{2.27}$$

$$\kappa_i \kappa_j = \kappa_j \kappa_i, \quad \kappa_i \gamma_{l,m} = \gamma_{l,m} \kappa_i, \tag{2.28}$$

$$\gamma_{i,j} = \gamma_{j,i}, \quad \gamma_{p,r}\gamma_{l,m} = \gamma_{l,m}\gamma_{p,r}, \quad \textit{and} \quad \gamma_{i,j}(\gamma_{i,r} + \gamma_{j,r}) = (\gamma_{i,r} + \gamma_{j,r})\gamma_{i,j}, \tag{2.29}$$

for $p \neq l$ *and* $p \neq m$ *and* $r \neq l$ *and* $r \neq m$ *and* $i \neq j$, $i \neq r$ *and* $j \neq r$.

The commutation relations between the κ_i and the $\gamma_{i,j}$ can be rewritten in the form

$$[\kappa_r, \gamma_{l,m}] = 0, \quad [\gamma_{i,j}, \gamma_{l,m}] = 0, \quad \text{and} \quad [\gamma_{i,j}, \gamma_{i,m}] = [\gamma_{i,m}, \gamma_{j,m}], \tag{2.30}$$

for all r and all $i \neq l$ and $i \neq m$ and $j \neq l$ and $j \neq m$.

Proof of Theorem 2.2. The generators in (2.26) are written in terms of the generators in (2.18) by the formulas

$$\kappa_0 = \kappa_0, \quad \kappa_1 = \kappa_1, \quad t_w = t_w, \tag{2.31}$$

$$\gamma_{0,1} = y_1 - \frac{1}{2}\kappa_1, \quad \text{and} \quad \gamma_{j,j+1} = y_{j+1} - t_{s_j} y_j t_{s_j}, \quad \text{for} \quad j = 1, \dots, k-1, \tag{2.32}$$

and

$$\kappa_m = t_u \kappa_1 t_{u^{-1}}, \quad \gamma_{0,m} = t_u \gamma_{0,1} t_{u^{-1}} \quad \text{and} \quad \gamma_{i,j} = t_v \gamma_{1,2} t_{v^{-1}}, \tag{2.33}$$

for $u, v \in S_k$ such that $u(1) = m$, $v(1) = i$ and $v(2) = j$.

The generators in (2.18) are written in terms of the generators in (2.26) by the formulas

$$\kappa_0 = \kappa_0, \quad \kappa_1 = \kappa_1, \quad t_w = t_w, \quad \text{and} \quad y_j = \frac{1}{2}\kappa_j + \sum_{0 \le l < j} \gamma_{l,j}. \tag{2.34}$$

Let us show that relations in (2.19)–(2.22) follow from the relations in (2.27)–(2.29).
(a) The relation $t_u t_v = t_{uv}$ in (2.19) is the first relation in (2.27).
(b) The relation $y_i y_j = y_j y_i$ in (2.19): Assume that $i < j$. Using the relations in (2.28) and (2.29),

$$\begin{aligned}
[y_i, y_j] &= \left[\frac{1}{2}\kappa_i + \sum_{l<i} \gamma_{l,i}, \frac{1}{2}\kappa_j + \sum_{m<j} \gamma_{m,j}\right] = \left[\sum_{l<i} \gamma_{l,i}, \sum_{m<j} \gamma_{m,j}\right] \\
&= \sum_{l<i} \left[\gamma_{l,i}, \sum_{m<j} \gamma_{m,j}\right] = \sum_{l<i} \left[\gamma_{l,i}, (\gamma_{l,j} + \gamma_{i,j}) + \sum_{\substack{m<j \\ m \neq l, m \neq i}} \gamma_{m,j}\right] = 0.
\end{aligned}$$

(c) The relation $\kappa_0\kappa_1 = \kappa_1\kappa_0$ in (2.19) is part of the first relation in (2.28), and the relations $\kappa_0 y_i = y_i\kappa_0$ and $\kappa_1 y_i = y_i\kappa_1$ in (2.19) follow from the relations $\kappa_i\kappa_j = \kappa_j\kappa_i$ and $\kappa_i\gamma_{l,m} = \gamma_{l,m}\kappa_i$ in (2.28).
(d) The relations $\kappa_0 t_{s_i} = t_{s_i}\kappa_0$ and $\kappa_1 t_{s_j} = t_{s_j}\kappa_1$ for $j \neq 1$ from (2.20) follow from the relation $t_w\kappa_i t_w^{-1} = \kappa_{w(i)}$ in (2.27), and the relation $\kappa_1 t_{s_1}\kappa_1 t_{s_1} = t_{s_1}\kappa_1 t_{s_1}\kappa_1$ from (2.20) follows from $\kappa_1\kappa_2 = \kappa_2\kappa_1$, which is part of the first relation in (2.28).
(e) The relations in (2.21) and (2.23) all follow from the relations $t_w\kappa_i t_{w^{-1}} = \kappa_{w(i)}$ and $t_w\gamma_{i,j}t_{w^{-1}} = \gamma_{w(i),w(j)}$ in (2.27).
(f) By second relation in (2.28) and the (already established) second relation in (2.20)

$$[\kappa_1, t_{s_1}y_1t_{s_1}] = \left[\kappa_1, t_{s_1}\left(y_1 - \frac{1}{2}\kappa_1\right)t_{s_1} + \frac{1}{2}t_{s_1}\kappa_1 t_{s_1}\right] = \left[\kappa_1, \gamma_{02} + \frac{1}{2}t_{s_1}\kappa_1 t_{s_1}\right] = 0,$$

which establishes (2.22).

To complete the proof let us show that the relations of (2.27)–(2.29) follow from the relations in (2.19)–(2.22).
(a) The relation $t_u t_v = t_{uv}$ in (2.27) is the first relation in (2.19).
(b) The relations $t_w\kappa_i t_{w^{-1}} = \kappa_{w(i)}$ in (2.27) follow from the first and last relations in (2.20) (and force the definition of κ_m in (2.33)).
(c) Since $\gamma_{0,1} = y_1 - (1/2)\kappa_1$, the relations $t_w\gamma_{0,j}t_{w^{-1}} = \gamma_{0,w(j)}$ in (2.28) follow from the last relation in each of (2.20) and (2.21) (and force the definition of $\gamma_{0,m}$ in (2.33)).
(d) Since $\gamma_{1,2} = y_2 - t_{s_1}y_1t_{s_1}$, the first relation in (2.21) gives that $\gamma_{2,1} = \gamma_{1,2}$ since

$$t_{s_1}\gamma_{1,2}t_{s_1} - \gamma_{1,2} = (t_{s_1}y_2t_{s_1} - y_1) - y_2 + t_{s_1}y_1t_{s_1} = t_{s_1}(y_1 + y_2)t_{s_1} - (y_1 + y_2) = 0. \tag{2.35}$$

The relations $t_w\gamma_{1,2}t_{w^{-1}} = \gamma_{w(1),w(2)}$ in (2.27) then follow from (2.35) and the last relation in (2.21) (and force the definitions $\gamma_{i,j} = t_v\gamma_{1,2}t_{v^{-1}}$ in (2.33)).
(e) The third relation in (2.19) is $\kappa_0\kappa_1 = \kappa_1\kappa_0$ and the second relation in (2.20) gives $\kappa_1\kappa_2 = \kappa_2\kappa_1$. The relations $\kappa_i\kappa_j = \kappa_j\kappa_i$ in (2.28) then follow from the second set of relations in (2.27).
(f) The second relation in (2.20) gives $[\kappa_1, \kappa_2] = 0$. Multiplying (2.22) on the left and right by t_{s_1} gives $[y_1, \kappa_2] = [y_1, t_{s_1}\kappa_1 t_{s_1}] = 0$. Using these and the relations in (2.19),

$$[\kappa_1, \gamma_{0,2} + \gamma_{1,2}] = \left[\kappa_1, \left(y_2 - \frac{1}{2}\kappa_2 - \gamma_{1,2}\right) + \gamma_{1,2}\right] = -\left[\kappa_1, \frac{1}{2}\kappa_2\right] = 0, \tag{2.36}$$

and

$$[\gamma_{0,1}, \gamma_{0,2} + \gamma_{1,2}] = \left[y_1 - \frac{1}{2}\kappa_1, y_2 - \frac{1}{2}\kappa_2\right] = \frac{1}{4}[\kappa_1, \kappa_2] = 0, \tag{2.37}$$

so that

$$[\gamma_{0,1}, \kappa_2] = [\gamma_{0,1}, 2y_2 - 2(\gamma_{0,2} + \gamma_{1,2})] = [\gamma_{0,1}, 2y_2]$$
$$= \left[y_1 - \frac{1}{2}\kappa_1, 2y_2\right] = -[\kappa_1, y_2] = 0.$$

Conjugating the last relation by t_{s_1} gives

$$[\kappa_1, \gamma_{0,2}] = 0, \quad \text{and thus} \quad [\kappa_1, \gamma_{1,2}] = 0,$$

by (2.36). By the third and fourth relations in (2.19),

$$[\kappa_0, \gamma_{0,1}] = \left[\kappa_0, y_1 - \frac{1}{2}\kappa_1\right] = 0, \quad \text{and} \quad [\kappa_1, \gamma_{0,1}] = \left[\kappa_1, y_1 - \frac{1}{2}\kappa_1\right] = 0.$$

By the relations in (2.20) and (2.19),

$$[\kappa_0, \gamma_{1,2}] = [\kappa_0, y_2 - t_{s_1}y_1t_{s_1}] = 0 \quad \text{and} \quad [\kappa_1, \gamma_{2,3}] = [\kappa_1, y_3 - t_{s_2}y_2t_{s_2}] = 0.$$

Putting these together with the (already established) relations in (2.27) provides the second set of relations in (2.28).

(g) From the commutativity of the y_i and the second relation in (2.21)

$$\gamma_{1,2}\gamma_{3,4} = (y_2 - t_{s_1}y_1t_{s_1})(y_4 - t_{s_3}y_3t_{s_3}) = (y_4 - t_{s_3}y_3t_{s_3})(y_2 - t_{s_1}y_1t_{s_1}) = \gamma_{3,4}\gamma_{1,2}.$$

By the last relation in (2.19) and the last relation in (2.20),

$$[\gamma_{0,1}, \gamma_{2,3}] = \left[y_1 - \frac{1}{2}\kappa_1, y_3 - t_{s_2}y_2t_{s_2}\right] = 0.$$

Together with the (already established) relations in (2.27), we obtain the first set of relations in (2.29).

(h) Conjugating (2.37) by $t_{s_2}t_{s_1}t_{s_2}$ gives $[\gamma_{0,2}, \gamma_{0,3} + \gamma_{2,3}] = 0$, and this and the (already established) relations in (2.28) and the first set of relations in (2.29) provide

$$0 = [y_2, y_3] = \left[\frac{1}{2}\kappa_2 + \gamma_{0,2} + \gamma_{1,2}, \frac{1}{2}\kappa_3 + \gamma_{0,3} + \gamma_{1,3} + \gamma_{2,3}\right]$$
$$= [\gamma_{0,2} + \gamma_{1,2}, \gamma_{0,3} + \gamma_{1,3} + \gamma_{2,3}] = [\gamma_{1,2}, \gamma_{0,3} + \gamma_{1,3} + \gamma_{2,3}] = [\gamma_{1,2}, \gamma_{1,3} + \gamma_{2,3}].$$

Note also that

$$[\gamma_{1,2}, \gamma_{1,0} + \gamma_{2,0}] = [\gamma_{1,2}, \gamma_{0,1} + \gamma_{0,2}] = -[\gamma_{0,1}, \gamma_{1,2}] + [\gamma_{1,2}, \gamma_{0,2}]$$
$$= [\gamma_{0,1}, \gamma_{0,2}] + [\gamma_{1,2}, \gamma_{0,2}] = t_{s_1}[\gamma_{0,2} + \gamma_{1,2}, \gamma_{0,1}]t_{s_1} = 0,$$

by (two applications of) (2.37). The last set of relations in (2.29) now follow from the last set of relations in (2.27). □

By the first formula in (2.24) and the last formula in (2.34),

$$c_j = \sum_{i=0}^{j} \kappa_i + 2 \sum_{0\le l<m\le j} \gamma_{l,m}. \tag{2.38}$$

2.3. The degenerate affine BMW algebra $\mathcal{W}_k$. Let C be a commutative ring and let $\mathcal{B}_k$ be the degenerate affine braid algebra over C as defined in Section 2.2. Define e_i in the degenerate affine braid algebra by

$$t_{s_i} y_i = y_{i+1} t_{s_i} - (1 - e_i), \quad \text{for} \quad i = 1, 2, \ldots, k-1, \tag{2.39}$$

so that, with $\gamma_{i,i+1}$ as in (2.23),

$$\gamma_{i,i+1} t_{s_i} = 1 - e_i. \tag{2.40}$$

Fix constants

$$\epsilon = \pm 1 \quad \text{and} \quad z_0^{(l)} \in C, \quad \text{for} \quad l \in \mathbb{Z}_{\ge 0}.$$

The *degenerate affine Birman–Wenzl–Murakami* (*BMW*) *algebra* $\mathcal{W}_k$ (with parameters ϵ and $z_0^{(l)}$) is the quotient of the degenerate affine braid algebra $\mathcal{B}_k$ by the relations

$$e_i t_{s_i} = t_{s_i} e_i = \epsilon e_i, \quad e_i t_{s_{i-1}} e_i = e_i t_{s_{i+1}} e_i = \epsilon e_i, \tag{2.41}$$

$$e_1 y_1^l e_1 = z_0^{(l)} e_1, \quad e_i(y_i + y_{i+1}) = 0 = (y_i + y_{i+1}) e_i. \tag{2.42}$$

The *degenerate affine Hecke algebra* $\mathcal{H}_k$ is the quotient of $\mathcal{W}_k$ by the relations

$$e_i = 0, \quad \text{for} \quad i = 1, \ldots, k-1. \tag{2.43}$$

Fix $b_1, \ldots, b_r \in C$. The *degenerate cyclotomic BMW algebra* $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$ is the degenerate affine BMW algebra with the additional relation

$$(y_1 - b_1) \cdots (y_1 - b_r) = 0. \tag{2.44}$$

The *degenerate cyclotomic Hecke algebra* $\mathcal{H}_{r,k}(b_1, \ldots, b_r)$ is the degenerate affine Hecke algebra $\mathcal{H}_k$ with the additional relation (2.44).

Since the composite map $C[y_1, \ldots, y_k] \to \mathcal{B}_k \to \mathcal{W}_k \to \mathcal{H}_k$ is injective (see [20, Theorem 3.2.2]) and the last two maps are surjections, it follows that the polynomial ring $C[y_1, \ldots, y_k]$ is a subalgebra of $\mathcal{B}_k$ and $\mathcal{W}_k$.

Proposition 2.3. *Let $C = \mathbb{C}$, $\kappa_0, \kappa_1 \in \mathbb{C}$ and $\epsilon = 1$. Then the degenerate affine BMW algebra $\mathcal{W}_k$ coincides with the one defined in* [27].

Proof. In [27], the degenerate affine BMW algebra is defined with the first two relations in (2.19) and the second set of relations in (2.21), which are [27, (4.1)] and the first relations in [27, (1.2) and (1.3)], the relations in (2.39) which are [27, (4.2)], the relations in (2.42) which are [27, (4.3) and (4.4)], the first relations in (2.41) which is the third set of relations in [27, (1.2)], the relations in (2.48) below which are the last two relations in [27, (1.3)] and the second relation in [27, (1.2)], the relations in (2.50) below which are [27, (1.4)], and the relations

$$e_i t_{s_j} = t_{s_j} e_i \quad \text{and} \quad e_i e_j = e_j e_i, \quad \text{for} \quad |j-i| > 1, \tag{2.45}$$

which are the second and third relations in [27, (1.5)].

Working in $\mathcal{W}_k$ and conjugating (2.39) by t_{s_i} and using the first relation in (2.41) gives

$$y_i t_{s_i} = t_{s_i} y_{i+1} - (1 - e_i). \tag{2.46}$$

Then, by (2.40) and (2.23),

$$\gamma_{i,i+1} = t_{s_i} - \epsilon e_i, \quad \text{and} \quad e_{i+1} = t_{s_i} t_{s_{i+1}} e_i t_{s_{i+1}} t_{s_i}. \tag{2.47}$$

Multiply the second relation in (2.47) on the left and the right by e_i, and then use the relations in (2.41) to get

$$e_i e_{i+1} e_i = e_i t_{s_i} t_{s_{i+1}} e_i t_{s_{i+1}} t_{s_i} e_i = e_i t_{s_{i+1}} e_i t_{s_{i+1}} e_i = \epsilon e_i t_{s_{i+1}} e_i = e_i,$$

so that

$$e_i e_{i\pm 1} e_i = e_i. \quad \text{Note that} \quad e_i^2 = z_0^{(0)} e_i \tag{2.48}$$

is, for $i = 1$, a special case of the first identity in (2.42) and then, for general i, follows from the second identity in (2.47). The relations

$$e_{i+1} e_i = e_{i+1} t_{s_i} t_{s_{i+1}}, \quad e_i e_{i+1} = t_{s_{i+1}} t_{s_i} e_{i+1}, \tag{2.49}$$

$$t_{s_i} e_{i+1} e_i = t_{s_{i+1}} e_i, \quad \text{and} \quad e_{i+1} e_i t_{s_{i+1}} = e_{i+1} t_{s_i} \tag{2.50}$$

result from (2.41) and the second relation in (2.47). The relations in (2.45) follow from (2.39) the first two relations in (2.19) and the last relations in (2.21). Thus the relations in the definition of the degenerate affine BMW algebra in [27] follow from the defining relations of $\mathcal{W}_k$.

To complete the proof we must show that the first relations in (2.21), the relations in (2.23), and the second relations in (2.41) follow from the defining relations used in [27]. Because of the assumption that $\kappa_0, \kappa_1 \in \mathbb{C}$ the other relations in (2.19)–(2.23) are automatic.

(a) Multiplying the first relation in (2.50) on the left by e_i and using the first relations in (2.41) and the first relations in (2.48) provides part of the second relations in (2.41) and the other part is obtained similarly by multiplying the second relations in (2.50) on the right by e_{i+1}.
(b) Conjugating (2.39) by t_{s_i} produces (2.46) and then adding (2.39) and (2.46) produces the first relations in (2.21).
(c) Using (2.50),

$$e_{i+1} = t_{s_i} t_{s_i} (e_{i+1} t_{s_i}) t_{s_i} = t_{s_i} t_{s_i} e_{i+1} e_i t_{s_{i+1}} t_{s_i} = t_{s_i} t_{s_{i+1}} e_i t_{s_{i+1}} t_{s_i}$$

which, with (2.39), gives the relations in (2.23). □

3. Identities in affine and degenerate affine BMW algebras

In [27], Nazarov defined some naturally occurring central elements in the degenerate affine BMW algebra $\mathcal{W}_k$ and proved a remarkable recursion for them. This recursion was generalized to analogous central elements in the affine BMW algebra W_k by Beliakova–Blanchet [4]. In both cases, the recursion was accomplished with an involved computation. In this section, we provide a new proof of the Nazarov and Beliakova–Blanchet recursions by lifting them out of the center, to intertwiner-like identities in $\mathcal{W}_k$ and W_k (Propositions 3.1 and 3.3). These intertwiner-like identities for the degenerate affine and affine BMW algebras are reminiscent of the intertwiner identities for the degenerate affine and affine Hecke algebras found, for example, in [21, Proposition 2.5 (c)] and [30, Proposition 2.14 (c)], respectively. The central element recursions of [27] and [4] are then obtained by multiplying the intertwiner-like identities by the projectors e_k and E_k, respectively. We shall not include our new proofs of Proposition 3.1 and Theorem 3.2 here since, given our parallel setup of the degenerate affine and the affine BMW algebras in Section 2, the proof is exactly parallel to the proofs of Proposition 3.3 and Theorem 3.4.

3.1. The degenerate affine case. Let $\mathcal{W}_k$ be the degenerate affine BMW algebra as defined in (2.41)–(2.42) and let $1 \le i < k-1$. Let u be a variable and let

$$u_i^+ = \frac{1}{u - y_i} \quad \text{and} \quad u_i^- = \frac{1}{u + y_i}. \tag{3.1}$$

Proposition 3.1. *In the degenerate affine BMW algebra $\mathcal{W}_{i+1}$,*

$$\begin{aligned}&\left(e_i \frac{1}{1 - y_{i+1}} - t_{s_i} - \frac{1}{2u - (y_i + y_{i+1})}\right)\left(e_i \frac{1}{1 - y_i} + t_{s_i} - \frac{1}{2u - (y_i + y_{i+1})}\right)\\ &= \frac{-(2u - (y_i + y_{i+1}) + 1)(2u - (y_i + y_{i+1}) - 1)}{(2u - (y_i + y_{i+1}))^2},\end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
&\Big(u_{i+1}^{+} + t_{s_i} - e_i \frac{1}{2u - (y_i + y_{i+1})}\Big) - u_i^{+}\Big(u_{i+1}^{+} + t_{s_i} - e_i \frac{1}{2u - (y_i + y_{i+1})}\Big)u_i^{+} \\
&= \Big(t_{s_i} u_i^{+} t_{s_i} + t_{s_i} - e_i \frac{1}{2u - (y_i + y_{i+1})}\Big) \\
&\quad - u_{i+1}^{+}\Big(e_i u_i^{+} e_i + \epsilon e_i - e_i \frac{1}{2u - (y_i + y_{i+1})}\Big)u_{i+1}^{+}.
\end{aligned} \tag{3.3}
$$

The identities (3.4) and (3.5) of the following theorem are [27, Lemma 2.5], and [27, Lemma 3.8], respectively.

Theorem 3.2 ([27])**.** *Let $\mathcal{W}_k$ be the degenerate affine BMW algebra as defined in* (2.41)–(2.42) *and let $1 \le i < k-1$. Let $z_0(u) = \sum_{l \in \mathbb{Z}_{\ge 0}} z_0^{(l)} u^{-l}$. Then*

$$
\Big(e_i u_i^{-} - \epsilon - \frac{1}{2u}\Big)\Big(e_i u_i^{+} + \epsilon - \frac{1}{2u}\Big)e_i = -\Big(\epsilon + \frac{1}{2u}\Big)\Big(\epsilon - \frac{1}{2u}\Big)e_i, \tag{3.4}
$$

and

$$
\begin{aligned}
&\Big(e_{i+1} u_{i+1}^{+} + \epsilon - \frac{1}{2u}\Big)e_{i+1} \\
&= \Big(\frac{z_0(u)}{u} + \epsilon - \frac{1}{2u}\Big) \prod_{j=1}^{i} \frac{(u + y_j - 1)(u + y_j + 1)(u - y_j)^2}{(u + y_j)^2 (u - y_j + 1)(u - y_j - 1)} e_{i+1}.
\end{aligned} \tag{3.5}
$$

3.2. The affine case. Let W_k be the affine BMW algebra as defined in (2.8)–(2.9) and let $1 \le i < k-1$. Let u be a variable,

$$
U_i^{+} = \frac{Y_i}{u - Y_i}, \quad \text{and note that} \quad U_i^{+} U_{i+1}^{+} = \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}(U_i^{+} + U_{i+1}^{+} + 1). \tag{3.6}
$$

By the definition of E_i in (2.7),

$$
(u - Y_{i+1})T_i = T_i(u - Y_i) - (q - q^{-1})Y_{i+1}(1 - E_i),
$$

and, by (2.12),

$$
(u - Y_i)T_i = T_i(u - Y_{i+1}) + (q - q^{-1})(1 - E_i)Y_{i+1},
$$

so that

$$
T_i \frac{1}{u - Y_i} = \frac{1}{u - Y_{i+1}} T_i - (q - q^{-1}) \frac{Y_{i+1}}{u - Y_{i+1}} (1 - E_i) \frac{1}{u - Y_i}, \tag{3.7}
$$

and

$$T_i\frac{1}{u-Y_{i+1}}=\frac{1}{u-Y_i}T_i+(q-q^{-1})\frac{1}{u-Y_i}(1-E_i)\frac{Y_{i+1}}{u-Y_{i+1}}. \tag{3.8}$$

The relations

$$\begin{aligned} T_iU_i^+ &= U_{i+1}^+T_i^{-1}-(q-q^{-1})U_{i+1}^+(1-E_i)U_i^+ \\ &= U_{i+1}^+(T_i^{-1}-(q-q^{-1})(1-E_i)U_i^+), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} T_i^{-1}U_{i+1}^+ &= U_i^+T_i-(q-q^{-1})U_i^+E_iU_{i+1}^+ +(q-q^{-1})U_{i+1}^+U_i^+ \\ &= U_i^+(T_i+(q-q^{-1})(1-E_i)U_{i+1}^+) \end{aligned} \tag{3.10}$$

are obtained by multiplying (3.7) and (3.8) on the right (resp. left) by Y_i and using the relation $T_iY_i=Y_{i+1}T_i^{-1}$.

Taking the coefficient of $u^{-(l+1)}$ on each side of (3.7) and (3.8) gives

$$\begin{aligned} T_iY_i^l &= Y_{i+1}^lT_i-(q-q^{-1})(Y_{i+1}^l(1-E_i)+Y_{i+1}^{l-1}(1-E_i)Y_i+\cdots \\ &\qquad +Y_{i+1}(1-E_i)Y_i^{l-1}), \end{aligned} \tag{3.11}$$

$$\begin{aligned} T_iY_{i+1}^l &= Y_i^lT_i+(q-q^{-1})(Y_i^{l-1}(1-E_i)Y_{i+1}+Y_i^{l-2}(1-E_i)Y_{i+1}^2+\cdots \\ &\qquad +(1-E_i)Y_{i+1}^l), \end{aligned} \tag{3.12}$$

respectively, for $l\in\mathbb{Z}_{\geq 0}$. Therefore,

$$T_iY_i^{-l}=Y_{i+1}^{-l}T_i+(q-q^{-1})(Y_{i+1}^{-(l-1)}(1-E_i)Y_i^{-1}+\cdots+(1-E_i)Y_i^{-l}), \tag{3.13}$$

$$T_iY_{i+1}^{-l}=Y_i^{-l}T_i-(q-q^{-1})(Y_i^{-l}(1-E_i)+\cdots+Y_i^{-1}(1-E_i)Y_{i+1}^{-(l-1)}). \tag{3.14}$$

Proposition 3.3. *Let $Q=q-q^{-1}$. Then, in the affine BMW algebra W_{i+1},*

$$\begin{aligned} &\left(E_i\frac{Y_{i+1}}{u-Y_{i+1}}-\frac{T_i}{Q}-\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)\left(E_i\frac{Y_i}{u-Y_i}+\frac{T_i^{-1}}{Q}-\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right) \\ &= \frac{-(u^2-q^2Y_iY_{i+1})(u^2-q^{-2}Y_iY_{i+1})}{Q^2(u^2-Y_iY_{i+1})^2}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\left(U_{i+1}^+ +\frac{T_i}{Q}-E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right) \\ &-Q^2(U_i^+ +1)\left(U_{i+1}^+ +\frac{T_i}{Q}-E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)U_i^+ \\ &=\left(T_iU_i^+T_i^{-1}+\frac{T_i}{Q}-E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right) \\ &\quad -Q^2U_{i+1}^+\left(E_iU_i^+E_i+z\frac{E_i}{Q}-E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)(U_{i+1}^+ +1). \end{aligned} \tag{3.16}$$

Proof. Putting (3.6) into (3.9) says that if

$$A = \frac{T_i}{Q} + \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}$$

and

$$B = E_iU_i^+ + \frac{T_i^{-1}}{Q} - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}$$

then

$$AU_i^+ = U_{i+1}^+B - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}.$$

Next,

$$AE_i = E_iA$$

follows from (2.8) and (2.9). So

$$\begin{aligned}
&\left(E_i\frac{Y_{i+1}}{u - Y_{i+1}} - \frac{T_i}{Q} - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)\left(E_i\frac{Y_i}{u - Y_i} + \frac{T_i^{-1}}{Q} - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)\\
&= E_i(U_{i+1}^+B) - AB = E_i\left(AU_i^+ + \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right) - AB\\
&= A(E_iU_i^+ - B) + E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\\
&= -\left(\frac{T_i}{Q} + \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)\left(\frac{T_i^{-1}}{Q} - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right) + E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}},
\end{aligned}$$

and, by (2.13), multiplying out the right hand side gives (3.15).

Rewrite $T_i^{-1}U_{i+1}^+ = U_i^+T_i^{-1} + QU_i^+(1 - E_i)(U_{i+1}^+ + 1)$ as

$$T_i^{-1}U_{i+1}^+ - Q(U_{i+1}^+ + 1)U_i^+ = U_i^+T_i^{-1} - QU_i^+E_i(U_{i+1}^+ + 1),$$

and multiply on the left by T_i to get

$$U_{i+1}^+ - QT_i(U_{i+1}^+ + 1)U_i^+ = T_iU_i^+T_i^{-1} - QT_iU_i^+E_i(U_{i+1}^+ + 1). \tag{3.17}$$

Then, since $T_i = T_i^{-1} + Q(1 - E_i)$, equations (3.10) and (3.9) imply

$$T_i(U_{i+1}^+ + 1) = Q(U_i^+ + 1)\left(\frac{T_i}{Q} + (1 - E_i)U_{i+1}^+\right)$$

and

$$T_iU_i^+ = QU_{i+1}^+\left(\frac{T_i^{-1}}{Q} - (1 - E_i)U_i^+\right),$$

and so (3.17) is

$$
\begin{aligned}
&U_{i+1}^+ - Q^2(U_i^+ + 1)\left(\frac{T_i}{Q} + (1-E_i)U_{i+1}^+\right)U_i^+ \\
&\qquad = T_iU_i^+T_i^{-1} - Q^2U_{i+1}^+\left(\frac{T_i^{-1}}{Q} - (1-E_i)U_i^+\right)E_i(U_{i+1}^+ + 1).
\end{aligned}
\tag{3.18}
$$

Using (3.6) and adding

$$\frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} - Q^2\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}(U_i^+ + 1)E_i(U_{i+1}^+ + 1)$$

to each side of (3.18) gives

$$
\begin{aligned}
&U_{i+1}^+ + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} - Q^2(U_i^+ + 1)\left(U_{i+1}^+ + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)U_i^+ \\
&= T_iU_i^+T_i^{-1} + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} \\
&\quad - Q^2U_{i+1}^+\left(E_iU_i^+ + \frac{T_i^{-1}}{Q} - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)E_i(U_{i+1}^+ + 1) \\
&= T_iU_i^+T_i^{-1} + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} \\
&\quad - Q^2U_{i+1}^+\left(E_iU_i^+E_i + z\frac{E_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)(U_{i+1}^+ + 1),
\end{aligned}
$$

completing the proof of (3.16). □

Let Z_0^+ and Z_0^- be the generating functions

$$Z_0^+ = \sum_{l\in\mathbb{Z}_{\geq 0}} Z_0^{(l)}u^{-l} \quad \text{and} \quad Z_0^- = \sum_{l\in\mathbb{Z}_{\leq 0}} Z_0^{(l)}u^{-l}.$$

If

$$U_i^- = \frac{Y_i^{-1}}{u - Y_i^{-1}} \quad \text{then} \quad E_iU_{i+1}^+ = E_iU_i^- \quad \text{and} \quad U_{i+1}^+E_i = U_i^-E_i, \tag{3.19}$$

by the second identity in (2.9). The first identity in (2.9) is equivalent to

$$E_1U_1^+E_1 = (Z_0^+ - Z_0^{(0)})E_1.$$

In the following theorem, the identity (3.20) is equivalent to [13, Lemma 2.8, parts (2) and (3)] or [14, Lemma 2.6(4)] and the identity (3.21) is equivalent to the identity found in [4, Lemma 7.4].

Theorem 3.4 ([4, 13, 14]). *Let W_k be the affine BMW algebra as defined in* (2.8)–(2.9) *and let* $1 \le i < k-1$. *Then*

$$\begin{aligned}(3.20)\qquad &\left(E_iU_i^- - \frac{z^{-1}}{q-q^{-1}} - \frac{1}{u^2-1}\right)\left(E_iU_i^+ + \frac{z}{q-q^{-1}} - \frac{1}{u^2-1}\right)E_i \\ &= \frac{-(u^2-q^2)(u^2-q^{-2})}{(u^2-1)^2(q-q^{-1})^2}E_i,\end{aligned}$$

and

$$\begin{aligned}(3.21)\qquad &\left(E_{i+1}U_{i+1}^+ + \frac{z}{q-q^{-1}} - \frac{1}{u^2-1}\right)E_{i+1} \\ &= \left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)\left(\prod_{j=1}^{i}\frac{(u-Y_j)^2(u-q^{-2}Y_j^{-1})(u-q^2Y_j^{-1})}{(u-Y_j^{-1})^2(u-q^2Y_j)(u-q^{-2}Y_j)}\right)E_{i+1}.\end{aligned}$$

Proof. Multiply (3.15) on the right by E_i and use $Z_{i-1}^{(0)} = 1 + (z-z^{-1})/(q-q^{-1})$ to ge (3.20).

Multiplying (3.16) on the left and right by E_{i+1} and using the relations in (2.8), (2.9), (2.14), and

$$E_{i+1}T_iU_i^+T_i^{-1}E_{i+1} = E_{i+1}T_iT_{i+1}U_i^+T_{i+1}^{-1}T_i^{-1}E_{i+1} = E_{i+1}E_iU_i^+E_iE_{i+1},$$

gives

$$\begin{aligned}&\left(E_{i+1}U_{i+1}^+ + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_{i+1}(1-Q^2(U_i^+ + 1)U_i^+) \\ &= E_{i+1}\left(E_iU_i^+ + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_iE_{i+1} \\ &\quad - Q^2U_i^-E_{i+1}\left(E_iU_i^+ + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_iE_{i+1}(U_i^- + 1) \\ &= (1-Q^2L_{U_i^-}R_{U_i^-+1})\left(E_{i+1}\left(E_iU_i^+ + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_iE_{i+1}\right)\end{aligned}$$

where $L_{U_i^-}$ is the operator of left multiplication by U_i^- and $R_{U_i^-+1}$ is the operator of right multiplication by $U_i^- + 1$. Then, by induction,

$$\begin{aligned}&\left(E_{i+1}U_{i+1}^+ + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_{i+1}\prod_{j=1}^{i}(1-Q^2U_j^+(U_j^+ + 1)) \\ &= \left(\prod_{j=1}^{i}(1-Q^2L_{U_j^-}R_{U_j^-+1})\right)\left(E_{i+1}E_i\cdots E_2\left(E_1U_1^+ + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_1E_2\cdots E_iE_{i+1}\right)\end{aligned}$$

$$= \prod_{j=1}^{i}(1-Q^2 L_{U_j^-} R_{U_j^-+1})\left(E_{i+1}E_i \cdots E_2\left(Z_0^+ - Z_0^{(0)} + \frac{z}{Q} - \frac{1}{u^2-1}\right)E_1E_2 \cdots E_i E_{i+1}\right)$$

$$= \left(Z_0^+ - Z_0^{(0)} + \frac{z}{Q} - \frac{1}{u^2-1}\right)\left(\prod_{j=1}^{i}(1-Q^2 L_{U_j^-} R_{U_j^-+1})\right)(E_{i+1}E_i \cdots E_2E_1E_2 \cdots E_i E_{i+1})$$

$$= \left(Z_0^+ - Z_0^{(0)} + \frac{z}{Q} - \frac{1}{u^2-1}\right)\prod_{j=1}^{i}(1-Q^2 U_j^-(U_j^- +1))E_{i+1}.$$

So (3.21) follows from

$$\begin{aligned}
&\frac{1-(q-q^{-1})^2 U_j^-(U_j^- + 1)}{1-(q-q^{-1})^2 U_j^+(U_j^+ + 1)} \\
&\quad = \frac{1-(q-q^{-1})^2 Y_j^{-1}/(u-Y_j^{-1})(Y_j^{-1}/(u-Y_j^{-1})+1)}{1-(q-q^{-1})^2 Y_j/(u-Y_j)(Y_j/(u-Y_j)+1)} \\
&\quad = \frac{((u-Y_j^{-1})^2-(q-q^{-1})^2 Y_j^{-1}u)(1/(u-Y_j^{-1})^2)}{((u-Y_j)^2-(q-q^{-1})^2 Y_j u)(1/(u-Y_j)^2)} \\
&\quad = \frac{(u-q^{-2}Y_j^{-1})(u-q^2Y_j^{-1})(u-Y_j)^2}{(u-q^{-2}Y_j)(u-q^2Y_j)(u-Y_j^{-1})^2}
\end{aligned}$$

and $Z_0^{(0)} = 1 + (z-z^{-1})/(q-q^{-1})$. □

4. The center of the affine and degenerate affine BMW algebras

In this section, we identify the center of $\mathcal{W}_k$ and W_k. Both centers arise as algebras of symmetric functions with a "cancellation property" (in the language of [29]) or "wheel condition" (in the language of [9]). In the degenerate case, $Z(\mathcal{W}_k)$ is the ring of symmetric functions in $y_1, \ldots, y_k$ with the Q-cancellation property of Pragacz. By [29, Theorem 2.11 (Q)], this is the same ring as the ring generated by the odd power sums, which is the way that Nazarov [27] identified $Z(\mathcal{W}_k)$.

The cancellation property in the case of W_k is analogous, exhibiting the center of the affine BMW algebra $Z(W_k)$ as a subalgebra of the ring of symmetric Laurent polynomials. At the end of this section, in an attempt to make the theory for the affine BMW algebra completely analogous to that for the degenerate affine BMW algebra, we have formulated an alternate description of $Z(W_k)$ as a ring generated by "negative" power sums.

4.1. Bases of $\mathcal{W}_k$ and W_k. The Brauer algebra, depending on a parameter x, is given by generators $e_1, \ldots, e_{k-1}$ and $s_1, \ldots, s_{k-1}$ and relations as given in [27, (1.2)–(1.5)] (where our e_i is denoted $\bar{s}_i$ and our x is denoted N). The Brauer algebra also has a diagrammatic presentation (see [5]) with basis

$$D_k = \{\text{diagrams on } k \text{ dots}\}, \tag{4.1}$$

where a (*Brauer*) *diagram* on k dots is a graph with k dots in the top row, k dots in the bottom row and k edges pairing the dots. We label the vertices of the top row, left to right, with $1, 2, \ldots, k$ and the vertices in the bottom row, left to right, with $1', 2', \ldots, k'$ so that, for example,

(4.2) $d =$  $= (13)(21')(45)(66')(74')(2'7')(3'5')$

is a Brauer diagram on 7 dots. Setting

$$x = z_0^{(0)} \quad \text{and} \quad s_i = \epsilon t_{s_i}$$

realizes the Brauer algebra as a subalgebra of the degenerate affine BMW algebra $\mathcal{W}_k$. The Brauer algebra is also the quotient of $\mathcal{W}_k$ by $y_1 = 0$ and, hence, can be viewed as the degenerate cyclotomic BMW algebra $\mathcal{W}_{1,k}(0)$.

Theorem 4.1 ([27, 2]). *Let $\mathcal{W}_k$ be the degenerate affine BMW algebra and let $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$ be the degenerate cyclotomic BMW algebra as defined in* (2.41)–(2.42) *and* (2.43), *respectively. For $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}$ and a diagram d on k dots let*

$$d^{n_1,\ldots,n_k} = y_{i_1}^{n_1} \cdots y_{i_l}^{n_l} d y_{i_{l+1}}^{n_{l+1}} \cdots y_{i_k}^{n_k},$$

where, in the lexicographic ordering of the edges $(i_1, j_1), \ldots, (i_k, j_k)$ of d, $i_1, \ldots, i_l$ are in the top row of d and $i_{l+1}, \ldots, i_k$ are in the bottom row of d. Let D_k be the set of diagrams on k dots, as in (4.1).

(a) *If $\kappa_0, \kappa_1 \in C$ and*

$$\left(z_0(-u) - \left(\frac{1}{2} + \epsilon\, u\right)\right)\left(z_0(u) - \left(\frac{1}{2} - \epsilon\, u\right)\right) = \left(\frac{1}{2} - \epsilon\, u\right)\left(\frac{1}{2} + \epsilon\, u\right) \tag{4.3}$$

then $\{d^{n_1,\ldots,n_k} \mid d \in D_k,\ n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}\}$ is a C-basis of $\mathcal{W}_k$.

(b) *If $\kappa_0, \kappa_1 \in C$,* (4.3) *holds, and*

$$\left(z_0(u) + u - \frac{1}{2}\right) = \left(u - \frac{1}{2}(-1)^r\right)\left(\prod_{i=1}^{r} \frac{u + b_i}{u - b_i}\right) \tag{4.4}$$

then $\{d^{n_1,\ldots,n_k} \mid d \in D_k,\ 0 \leq n_1, \ldots, n_k \leq r - 1\}$ is a C-basis of $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$.

Part (a) of Theorem 4.1 is [27, Theorem 4.6] (see also [2, Theorem 2.12]) and part (b) is [2, Proposition 2.15 and Theorem 5.5].

Theorem 4.2 ([14, 39]). *Let W_k be the affine BMW algebra and let $W_{r,k}(b_1, \dots, b_r)$ be the cyclotomic BMW algebra as defined in* Section 2.1. *Let $d \in D_k$ be a Brauer diagram, where D_k is as in* (4.1). *Choose a minimal length expression of d as a product of $e_1, \dots, e_{k-1}, s_1, \dots, s_{k-1}$,*

$$d = a_1 \cdots a_l, \quad a_i \in \{e_1, \dots, e_{k-1}, s_1, \dots, s_{k-1}\},$$

such that the number of s_i in this product is the number of crossings in d. For each a_i which is in $\{s_1, \dots, s_{k-1}\}$ fix a choice of sign $\epsilon_j = \pm 1$ and set

$$T_d = A_1 \cdots A_l, \quad \text{where} \quad A_j = \begin{cases} E_i, & \text{if} \quad a_j = e_i, \\ T_i^{\epsilon_j}, & \text{if} \quad a_j = s_i. \end{cases}$$

For $n_1, \dots, n_k \in \mathbb{Z}$ let

$$T_d^{n_1,\dots,n_k} = Y_{i_1}^{n_1} \cdots Y_{i_l}^{n_l} T_d Y_{i_{l+1}}^{n_{l+1}} \cdots Y_{i_k}^{n_k},$$

where, in the lexicographic ordering of the edges $(i_1, j_1), \dots, (i_k, j_k)$ of d, $i_1, \dots, i_l$ are in the top row of d and $i_{l+1}, \dots, i_k$ are in the bottom row of d.
(a) *If*

$$\left(Z_0^- - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)\left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) = \frac{-(u^2-q^2)(u^2-q^{-2})}{(u^2-1)^2(q-q^{-1})^2} \tag{4.5}$$

then $\{T_d^{n_1,\dots,n_k} \mid d \in D_k,\ n_1, \dots, n_k \in \mathbb{Z}\}$ is a C-basis of W_k.
(b) *If* (4.5) *holds and*

$$Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1} = \left(\frac{z}{q-q^{-1}} + \frac{uz}{u^2-1}\right)\prod_{j=1}^{r} \frac{u-b_j^{-1}}{u-b_j} \tag{4.6}$$

then $\{T_d^{n_1,\dots,n_k} \mid d \in D_k,\ 0 \le n_1, \dots, n_k \le r-1\}$ is a C-basis of $W_{r,k}(b_1, \dots, b_r)$.

Part (a) of Theorem 4.2 is [14, Theorem 2.25] and part (b) is [14, Theorem 5.5] and [39, Theorem 8.1]. We refer to these references for proof, remarking only that one key point in showing that $\{T_d^{n_1,\dots,n_k} \mid d \in D_k,\ n_1, \dots, n_k \in \mathbb{Z}\}$ spans W_k is that if (i, j) is a top-to-bottom edge in d then

$$Y_i T_d = T_d Y_j + (\text{terms with fewer crossings}), \tag{4.7}$$

and, if (i, j) is a top-to-top edge in d then

$$Y_i T_d = Y_j^{-1} T_d + (\text{terms with fewer crossings}). \tag{4.8}$$

4.2. The center of $\mathcal{W}_k$. The degenerate affine BMW algebra is the algebra $\mathcal{W}_k$ over C defined in Section 2.3 and the polynomial ring $C[y_1, \dots, y_k]$ is a subalgebra of $\mathcal{W}_k$. The symmetric group S_k acts on $C[y_1, \dots, y_k]$ by permuting the variables. A classical fact (see, for example, [20, Theorem 3.3.1]) is that the center of the degenerate affine Hecke algebra $\mathcal{H}_k$ is the ring of symmetric functions

$$Z(\mathcal{H}_k) = C[y_1, \dots, y_k]^{S_k} = \{ f \in C[y_1, \dots, y_k] \mid wf = f, \text{ for } w \in S_k\}.$$

Theorem 4.3 gives an analogous characterization of the center of the degenerate affine BMW algebra. We shall not include the proof here since, given our parallel setup of the degenerate affine BMW algebras and the affine BMW algebras in Section 2, the proof is exactly parallel to the proof of Theorem 4.4.

Theorem 4.3. *The center of the degenerate affine BMW algebra $\mathcal{W}_k$ is*

$$\mathcal{R}_k = \{ f \in C[y_1, \dots, y_k]^{S_k} \mid f(y_1, -y_1, y_3, \dots, y_k) = f(0, 0, y_3, \dots, y_k)\}.$$

The *power sum symmetric functions* p_i are given by

$$p_i = y_1^i + y_2^i + \cdots + y_k^i, \quad \text{for} \quad i \in \mathbb{Z}_{>0}.$$

The *Hall–Littlewood polynomials* (see [25, Chapter III (2.1)]) are given by

$$P_\lambda(y; t) = P_\lambda(y_1, \dots, y_k; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_k} w\left(y_1^{\lambda_1} \cdots y_k^{\lambda_k} \prod_{1 \le i < j \le k} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where $v_\lambda(t)$ is a normalizing constant (a polynomial in t) so that the coefficient of $y_1^{\lambda_1} \cdots y_k^{\lambda_k}$ in $P_\lambda(y; t)$ is equal to 1. The *Schur Q-functions* (see [25, Chapter III (8.7)]) are

$$Q_\lambda = \begin{cases} 0, & \text{if } \lambda \text{ is not strict,} \\ 2^{l(\lambda)} P_\lambda(y; -1), & \text{if } \lambda \text{ is strict,} \end{cases}$$

where $l(\lambda)$ is the number of (nonzero) parts of λ and the partition λ is *strict* if all its (nonzero) parts are distinct. Let $\mathcal{R}_k$ be as in Theorem 4.3. Then (see [27, Corollary 4.10], [29, Theorem 2.11 (Q)] and [25, Chapter III §8])

$$\mathcal{R}_k = C[p_1, p_3, p_5, \dots] = C\text{span-}\{Q_\lambda \mid \lambda \text{ is strict}\}. \tag{4.9}$$

More generally, let $r \in \mathbb{Z}_{>0}$ and let ζ be a primitive rth root of unity. Define

$$\mathcal{R}_{r,k} = \{ f \in \mathbb{Z}[\zeta][y_1, \dots, y_k]^{S_k} \mid f(y_1, \zeta y_1, \dots, \zeta^{r-1} y_1, y_{r+1}, \dots, y_k) = f(0, 0, \dots, 0, y_{r+1}, \dots, y_k)\}.$$

Then

$$\mathcal{R}_{r,k} \otimes_{\mathbb{Z}[\zeta]} \mathbb{Q}(\zeta) = \mathbb{Q}(\zeta)[p_i \mid i \neq 0 \bmod r], \tag{4.10}$$

and

$$\mathcal{R}_{r,k} \quad \text{has} \quad \mathbb{Z}[\zeta]\text{-basis} \quad \{P_\lambda(y;\zeta) \mid m_i(\lambda) < r \text{ and } \lambda_1 \leq k\}, \tag{4.11}$$

where $m_i(\lambda)$ is the number parts of size i in λ. The ring $\mathcal{R}_{r,k}$ is studied in [26], [23], [25, Chapter III Example 5.7 and Example 7.7], [37], [9], and others. The proofs of (4.10) and (4.11) follow from [25, Chapter III Example 7.7], [37, Lemma 2.2 and following remarks] and the arguments in the proofs of [9, Lemma 3.2 and Proposition 3.5].

4.3. The center of W_k. The affine BMW algebra is the algebra W_k over C defined in Section 2.1 and the ring of Laurent polynomials $C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ is a subalgebra of W_k. The symmetric group S_k acts on $C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ by permuting the variables. A classical fact (see, for example, [16, Proposition 2.1]) is that the center of the affine Hecke algebra H_k is the ring of symmetric functions,

$$Z(H_k) = C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{S_k} = \{f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}] \mid wf = f, \text{ for } w \in S_k\}.$$

Theorem 4.4 is a characterization of the center of the affine BMW algebra.

Theorem 4.4. *The center of the affine BMW algebra W_k is*

$$R_k = \{f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{S_k} \mid f(Y_1, Y_1^{-1}, Y_3, \dots, Y_k) = f(1, 1, Y_3, \dots, Y_k)\}.$$

Proof. STEP 1: $f \in W_k$ commutes with all $Y_i \Leftrightarrow f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$: Assume $f \in W_k$ and write

$$f = \sum c_d^{n_1,\dots,n_k} T_d^{n_1,\dots,n_k},$$

in terms of the basis in Theorem 4.2. Let $d \in D_k$ with the maximal number of crossings such that $c_d^{n_1,\dots,n_k} \neq 0$ and, using the notation after (4.2), suppose there is an edge (i, j) of d such that $j \neq i'$. Then, by (4.7) and (4.8),

$$\text{the coefficient of} \quad Y_i T_d^{n_1,\dots,n_k} \quad \text{in} \quad Y_i f \quad \text{is} \quad c_d^{n_1,\dots,n_k}$$

and

$$\text{the coefficient of} \quad Y_i T_d^{n_1,\dots,n_k} \quad \text{in} \quad fY_i \quad \text{is} \quad 0.$$

If $Y_i f = fY_i$ it follows that there is no such edge, and so $d = 1$ (and therefore $T_d = 1$). Thus $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$. Conversely, if $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$, then $Y_i f = fY_i$.

STEP 2: $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ commutes with all $T_i \Leftrightarrow f \in R_k$: Assume $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ and write

$$f = \sum_{a,b\in\mathbb{Z}} Y_1^a Y_2^b f_{a,b}, \quad \text{where} \quad f_{a,b} \in C[Y_3^{\pm 1}, \dots, Y_k^{\pm 1}].$$

Then $f(1, 1, Y_3, \dots, Y_k) = \sum_{a,b\in\mathbb{Z}} f_{a,b}$ and

$$f(Y_1, Y_1^{-1}, Y_3, \dots, Y_k) = \sum_{a,b\in\mathbb{Z}} Y_1^{a-b} f_{a,b} = \sum_{l\in\mathbb{Z}} Y_1^l \Big(\sum_{b\in\mathbb{Z}} f_{l+b,b}\Big). \tag{4.12}$$

By direct computation using (3.12) and (3.14),

$$T_1 Y_1^a Y_2^b = Y_1^a Y_2^a T_1 Y_2^{b-a} = s_1(Y_1^a Y_2^b) T_1 + (q - q^{-1}) \frac{Y_1^a Y_2^b - s_1(Y_1^a Y_2^b)}{1 - Y_1 Y_2^{-1}} + \mathcal{E}_{b-a},$$

where

$$\mathcal{E}_l = \begin{cases} -(q - q^{-1}) \displaystyle\sum_{r=1}^{l} Y_1^{l-r} E_1 Y_1^{-r}, & \text{if } l > 0, \\ (q - q^{-1}) \displaystyle\sum_{r=1}^{-l} Y_1^{l+r-1} E_1 Y_1^{r-1}, & \text{if } l < 0, \\ 0, & \text{if } l = 0. \end{cases}$$

It follows that

$$T_1 f = (s_1 f) T_1 + (q - q^{-1}) \frac{f - s_1 f}{1 - Y_1 Y_2^{-1}} + \sum_{l\in\mathbb{Z}_{\neq 0}} \mathcal{E}_l \Big(\sum_{b\in\mathbb{Z}} f_{l+b,b}\Big). \tag{4.13}$$

Thus, if $f(Y_1, Y_1^{-1}, Y_3, \dots, Y_k) = f(1, 1, Y_3, \dots, Y_k)$ then, by (4.12),

$$\sum_{b\in\mathbb{Z}} f_{l+b,b} = 0, \quad \text{for} \quad l \neq 0. \tag{4.14}$$

Hence, if $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{S_k}$ and $f(Y_1, Y_1^{-1}, Y_3, \dots, Y_k) = f(1, 1, Y_3, \dots, Y_k)$ then $s_1 f = f$ and (4.14) holds so that, by (4.13), $T_1 f = f T_1$. Similarly, f commutes with all T_i.

Conversely, if $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ and $T_i f = f T_i$ then

$$s_i f = f \quad \text{and} \quad \sum_{b\in\mathbb{Z}} f_{l+b,b} = 0, \quad \text{for} \quad l \neq 0,$$

so that $f \in C[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{S_k}$ and $f(Y_1, Y_1^{-1}, Y_3, \dots, Y_k) = f(1, 1, Y_3, \dots, Y_k)$.

It follows from (2.7) that $R_k = Z(W_k)$. □

The symmetric group S_k acts on $\mathbb{Z}^k$ by permuting the factors. The ring

$$C[Y_1^{\pm1}, \ldots, Y_k^{\pm1}]^{S_k} \quad \text{has basis} \quad \{m_\lambda \mid \lambda \in \mathbb{Z}^k \text{ with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k\},$$

where

$$m_\lambda = \sum_{\mu \in S_k \lambda} Y_1^{\mu_1} \cdots Y_k^{\mu_k}.$$

The *elementary symmetric functions* are

$$e_r = m_{(1^r, 0^{k-r})} \quad \text{and} \quad e_{-r} = m_{(0^{k-r}, (-1)^r)}, \quad \text{for} \quad r = 0, 1, \ldots, k,$$

and the *power sum symmetric functions* are

$$p_r = m_{(r, 0^{k-1})} \quad \text{and} \quad p_{-r} = m_{(0^{k-1}, -r)}, \quad \text{for} \quad r \in \mathbb{Z}_{>0}.$$

The Newton identities (see [25, Chapter I (2.11′)]) say

$$le_l = \sum_{r=1}^{l} (-1)^{r-1} p_r e_{l-r} \quad \text{and} \quad le_{-l} = \sum_{r=1}^{l} (-1)^{r-1} p_{-r} e_{-(l-r)}, \tag{4.15}$$

where the second equation is obtained from the first by replacing Y_i with Y_i^{-1}. For $l \in \mathbb{Z}$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$,

$$e_k^l m_\lambda = m_{\lambda + (l^k)}, \quad \text{where} \quad \lambda + (l^k) = (\lambda_1 + l, \ldots, \lambda_k + l).$$

In particular,

$$e_{-r} = e_k^{-1} e_{k-r}, \quad \text{for} \quad r = 0, \ldots, k. \tag{4.16}$$

Define

$$p_i^+ = p_i + p_{-i} \quad \text{and} \quad p_i^- = p_i - p_{-i}, \quad \text{for} \quad i \in \mathbb{Z}_{>0}. \tag{4.17}$$

The consequence of (4.16) and (4.15) is that

$$\begin{aligned}
\mathbb{C}[Y_1^{\pm1}, \ldots, Y_k^{\pm1}]^{S_k} &= \mathbb{C}[e_k^{\pm1}, e_1, \ldots, e_{k-1}] \\
&= \mathbb{C}[e_k^{\pm1}][e_1, e_2, \ldots, e_{\lfloor k/2 \rfloor}, e_k e_{-\lfloor (k-1)/2 \rfloor}, \ldots, e_k e_{-2}, e_k e_{-1}] \\
&= \mathbb{C}[e_k^{\pm1}][e_1, e_2, \ldots, e_{\lfloor k/2 \rfloor}, e_{-\lfloor (k-1)/2 \rfloor}, \ldots, e_{-2}, e_{-1}] \\
&= \mathbb{C}[e_k^{\pm1}][p_1, p_2, \ldots, p_{\lfloor k/2 \rfloor}, p_{-\lfloor (k-1)/2 \rfloor}, \ldots, p_{-2}, p_{-1}] \\
&= \mathbb{C}[e_k^{\pm1}][p_1^+, p_2^+, \ldots, p_{\lfloor k/2 \rfloor}^+, p_{\lfloor (k-1)/2 \rfloor}^-, \ldots, p_2^-, p_1^-].
\end{aligned}$$

For $\nu \in \mathbb{Z}^k$ with $\nu_1 \geq \cdots \geq \nu_l > 0$ define

$$p_\nu^+ = p_{\nu_1}^+ \cdots p_{\nu_l}^+ \quad \text{and} \quad p_\nu^- = p_{\nu_1}^- \cdots p_{\nu_l}^-.$$

Then

$$\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} \quad \text{has basis} \quad \left\{ e_k^l p_\lambda^+ p_\mu^- \;\middle|\; l \in \mathbb{Z}, l(\lambda) \leq \left\lfloor \frac{k}{2} \right\rfloor, l(\mu) \leq \left\lfloor \frac{k-1}{2} \right\rfloor \right\}.$$

In analogy with (4.9) we expect that if R_k is as in Theorem 4.4 then

$$R_k = C[e_k^{\pm 1}][p_1^-, p_2^-, \ldots].$$

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