<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>AFFINE AND DEGENERATE AFFINE BMW ALGEBRAS : THE CENTER</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Daugherty, Zaji; Ram, Arun; Virk, Rahbar</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 51(1) P.257–P.283</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2014-01</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/29192">https://doi.org/10.18910/29192</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/29192</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive: OUKA

http://ir.library.osaka-u.ac.jp/dspace/

Osaka University
AFFINE AND DEGENERATE AFFINE BMW ALGEBRAS:
THE CENTER

ZAJJ DAUGHERTY, ARUN RAM and RAHBAR VIRK

(Received October 2, 2011, revised August 7, 2012)

Abstract

The degenerate affine and affine BMW algebras arise naturally in the context of
Schur–Weyl duality for orthogonal and symplectic Lie algebras and quantum groups,
respectively. Cyclotomic BMW algebras, affine Hecke algebras, cyclotomic Hecke
algebras, and their degenerate versions are quotients. In this paper the theory is uni-
ified by treating the orthogonal and symplectic cases simultaneously; we make an ex-
act parallel between the degenerate affine and affine cases via a new algebra which
takes the role of the affine braid group for the degenerate setting. A main result
of this paper is an identification of the centers of the affine and degenerate affine
BMW algebras in terms of rings of symmetric functions which satisfy a “cancel-
lation property” or “wheel condition” (in the degenerate case, a reformulation of a
result of Nazarov). Miraculously, these same rings also arise in Schubert calculus,
as the cohomology and K-theory of isotropic Grassmannians and symplectic loop
Grassmannians. We also establish new intertwiner-like identities which, when pro-
jected to the center, produce the recursions for central elements given previously by
Nazarov for degenerate affine BMW algebras, and by Beliakova–Blanchet for affine
BMW algebras.

1. Introduction

The degenerate affine BMW algebras $W_k$ and the affine BMW algebras $W_k$ arise
naturally in the context of Schur–Weyl duality and the application of Schur functors to
modules in category $\mathcal{O}$ for orthogonal and symplectic Lie algebras and quantum groups
(using the Schur functors of [41], [1], and [28]). The degenerate algebras $W_k$ were in-
troduced in [27] and the affine versions $W_k$ appeared in [28], following foundational
work of [17]–[19]. The representation theory of $W_k$ and $W_k$ contains the representa-
tion theory of any quotient: in particular, the degenerate cyclotomic BMW algebras
$W_{r,k}$, the cyclotomic BMW algebras $W_{r,k}$, the degenerate affine Hecke algebras $H_k$,
the affine Hecke algebras $H_k$, the degenerate cyclotomic Hecke algebras $H_{r,k}$, and the
cyclotomic Hecke algebras $H_{r,k}$ as quotients. In [31, 34, 35, 2, 8] and other works, the
representation theory of the affine BMW algebra is derived by cellular algebra tech-
niques. As indicated in [28], the Schur–Weyl duality also provides a path to the repre-
sentation theory of the affine BMW algebras as an image of the representation theory
of category $\mathcal{O}$ for orthogonal and symplectic Lie algebras and their quantum groups

2010 Mathematics Subject Classification. 17B37.
in the same way that the affine Hecke algebras arise in Schur–Weyl duality with the enveloping algebra of $gl_n$ and its Drinfeld–Jimbo quantum group.

In the literature, the algebras $W_k$ and $W_k$ have often been treated separately. One of the goals of this paper is to unify the theory. To do this we have begun by adjusting the definitions of the algebras carefully to make the presentations match, relation by relation. In the same way that the affine BMW algebra is a quotient of the group algebra of the affine braid group, we have defined a new algebra, the degenerate affine braid algebra which has the degenerate affine BMW algebra and the degenerate affine Hecke algebras as quotients. We have done this carefully, to ensure that the Schur–Weyl duality framework is completely analogous for both the degenerate affine and the affine cases. We have also added a parameter $\epsilon$ (which takes values $\pm 1$) so that both the orthogonal and symplectic cases can be treated simultaneously. Our new presentations of the algebras $W_k$ and $W_k$ are given in Section 2.

In Section 3 we consider some remarkable recursions for generating central elements in the algebras $W_k$ and $W_k$. These recursions were given by Nazarov [27] in the degenerate case, and then extended to the affine BMW algebra by Beliakova–Blanchet [4]. Another proof in the affine cyclotomic case appears in [35, Lemma 4.21] and, in the degenerate case, in [2, Lemma 4.15]. In all of these proofs, the recursion is obtained by a rather mysterious and tedious computation. We show that there is an “intertwiner” like identity in the full algebra which, when “projected to the center” produces the Nazarov recursions. Our approach provides new insight into where these recursions are coming from. Moreover, the proof is exactly analogous in both the degenerate and the affine cases, and includes the parameter $\epsilon$, so that both the orthogonal and symplectic cases are treated simultaneously.

In Section 4 we identify the center of the degenerate and affine BMW algebras. In the degenerate case this has been done in [27]. Nazarov stated that the center of the degenerate affine BMW algebra is the subring of the ring of symmetric functions generated by the odd power sums. We identify the ring in a different way, as the subring of symmetric functions with the Q-cancellation property, in the language of Pragacz [29]. This is a fascinating ring. Pragacz identifies it as the cohomology ring of orthogonal and symplectic Grassmannians; the same ring appears again as the cohomology of the loop Grassmannian for the symplectic group in [24, 22]; and references for the relationship of this ring to the projective representation theory of the symmetric group, the BKP hierarchy of differential equations, representations of Lie superalgebras, and twisted Gelfand pairs are found in [25, Chapter II §8]. For the affine BMW algebra, the Q-cancellation property can be generalized well to provide a suitable description of the center. From our perspective, one would expect that the ring which appears as the center of the affine BMW algebra should also appear as the K-theory of the orthogonal and symplectic Grassmannians and as the K-theory of the loop Grassmannian for the symplectic group, but we are not aware that these identifications have yet been made in the literature.
The recent paper [31] classifies the irreducible representations of $W_k$ by multisegments, and the recent paper [6] adds to this program of study by setting up commuting actions between the algebras $W_k$ and $W_k$ and the enveloping algebras of orthogonal and symplectic Lie algebras and their quantum groups, showing how the central elements which arise in the Nazarov recursions coincide with central elements studied in Baumann [3], and providing an approach to admissibility conditions by providing “universal admissible parameters” in an appropriate ground ring (arising naturally, from Schur–Weyl duality, as the center of the enveloping algebra, or quantum group). We would also like to mention the recent paper of A. Sartori [36] which establishes similar results in the case of the degenerate affine walled Brauer algebra and the recent work of M. Ehrig and C. Stroppel [7] which studies these algebras in the context of categorification.

2. Affine and degenerate affine BMW algebras

In this section, we define the affine Birman–Murakami–Wenzl (BMW) algebra $W_k$ and its degenerate version $W_k$. We have adjusted the definitions to unify the theory. In particular, in Section 2.2, we define a new algebra, the degenerate affine braid algebra $B_k$, which has the degenerate affine BMW algebras $W_k$ and the degenerate affine Hecke algebras $H_k$ as quotients. The motivation for the definition of $B_k$ is that the affine BMW algebras $W_k$ and the affine Hecke algebras $H_k$ are quotients of the group algebra of affine braid group $C B_k$.

The definition of the degenerate affine braid algebra $B_k$ also makes the Schur–Weyl duality framework completely analogous in both the affine and degenerate affine cases. Both $B_k$ and $C B_k$ are designed to act on tensor space of the form $M \otimes V^{\otimes k}$. In the degenerate affine case this is an action commuting with a complex semisimple Lie algebra $\mathfrak{g}$, and in the affine case this is an action commuting with the Drinfeld–Jimbo quantum group $U_q \mathfrak{g}$. The degenerate affine and affine BMW algebras arise when $\mathfrak{g}$ is $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ and $V$ is the first fundamental representation and the degenerate affine and affine Hecke algebras arise when $\mathfrak{g}$ is $\mathfrak{gl}_n$ or $\mathfrak{sl}_n$ and $V$ is the first fundamental representation. In the case when $M$ is the trivial representation and $\mathfrak{g}$ is $\mathfrak{so}_n$, the “Jucys–Murphy” elements $y_1, \ldots, y_k$ in $B_k$ become the “Jucys–Murphy” elements for the Brauer algebras used in [27] and, in the case that $\mathfrak{g} = \mathfrak{gl}_n$, these become the classical Jucys–Murphy elements in the group algebra of the symmetric group. The Schur–Weyl duality actions are explained in [6].

2.1. The affine BMW algebra $W_k$. The affine braid group $B_k$ is the group given by generators $T_1, T_2, \ldots, T_{k-1}$ and $X^{e_1}$, with relations

\begin{align}
T_i T_j &= T_j T_i, & \text{if} & \quad j \neq i \pm 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & \text{for} & \quad i = 1, 2, \ldots, k - 2, \\
X^{e_1} T_i X^{e_1} T_i &= T_i X^{e_1} T_i X^{e_1}, \\
X^{e_1} T_i &= T_i X^{e_1}, & \text{for} & \quad i = 2, 3, \ldots, k - 1.
\end{align}
Let $C$ be a commutative ring and let $CB_k$ be the group algebra of the affine braid group. Fix constants

\[ q, z \in C \quad \text{and} \quad Z^{(l)}_0 \in C, \quad \text{for} \quad l \in \mathbb{Z}, \]

with $q$ and $z$ invertible. Let $Y_i = zX^{e_i}$ so that

\[ (2.5) \quad Y_1 = zX^{e_1}, \quad Y_i = T_{i-1}Y_{i-1}T_{i-1}, \quad \text{and} \quad Y_iY_j = Y_jY_i, \quad \text{for} \quad 1 \leq i, j \leq k. \]

In the affine braid group

\[ (2.6) \quad T_iY_iY_{i+1} = Y_iY_{i+1}T_i. \]

Assume that $q - q^{-1}$ is invertible in $C$ and define $E_i$ in the group algebra of the affine braid group by

\[ (2.7) \quad T_iY_i = Y_{i+1}T_i - (q - q^{-1})Y_{i+1}(1 - E_i). \]

The affine BMW algebra $W_k$ is the quotient of the group algebra $CB_k$ of the affine braid group $B_k$ by the relations

\[ (2.8) \quad E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\mp 1}E_i, \quad E_iT_{i-1}^{\pm 1}E_i = E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i, \]

\[ (2.9) \quad E_iY_i^{(l)}E_1 = Z^{(l)}_0E_1, \quad E_iY_iY_{i+1} = E_i = Y_iY_{i+1}E_i. \]

The affine Hecke algebra $H_k$ is the affine BMW algebra $W_k$ with the additional relations

\[ (2.10) \quad E_i = 0, \quad \text{for} \quad i = 1, \ldots, k - 1. \]

Fix $b_1, \ldots, b_r \in C$. The cyclotomic BMW algebra $W_{r,k}(b_1, \ldots, b_r)$ is the affine BMW algebra $W_k$ with the additional relation

\[ (2.11) \quad (Y_1 - b_1) \cdots (Y_1 - b_r) = 0. \]

The cyclotomic Hecke algebra $H_{r,k}(b_1, \ldots, b_r)$ is the affine Hecke algebra $H_k$ with the additional relation (2.11).

Since the composite map $C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \to CB_k \to W_k \to H_k$ is injective and the last two maps are surjections, it follows that the Laurent polynomial ring $C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ is a subalgebra of $CB_k$ and $W_k$.

**Proposition 2.1.** The affine BMW algebra $W_k$ coincides with the one defined in [28].

Proof. In [28] the affine BMW algebra is defined as the quotient of the group algebra of the affine braid group by the relations in (2.8) which are [28, (6.3b) and
(6.3c)], the first relation in (2.9) which is [28, (6.3d)], the second relation in (2.9) for \( i = 1 \) which is [28, (6.3e)], and the second relation in (2.15) below where, in [28], the element \( E_i \) is defined by the first equation in (2.13) below.

Working in \( W_k \), since \( Y_{i+1}^{-1}(T_i Y_i)Y_{i+1} = Y_{i+1}^{-1}Y_i Y_{i+1}T_i = Y_i T_i \), conjugating (2.7) by \( Y_{i+1}^{-1} \) gives

\[ Y_i T_i = T_i Y_{i+1} - (q - q^{-1})(1 - E_i) Y_{i+1} \tag{2.12} \]

Left multiplying (2.7) by \( Y_{i+1}^{-1} \) and using the second identity in (2.5) shows that (2.7) is equivalent to \( T_i - T_i^{-1} = (q - q^{-1})(1 - E_i) \), so that

\[ E_i = 1 - \frac{T_i - T_i^{-1}}{q - q^{-1}} \quad \text{and} \quad T_i T_{i+1} E_i T_{i+1}^{-1} T_i^{-1} = E_{i+1} \tag{2.13} \]

Thus the \( E_i \) in \( W_k \) coincides with the \( E_i \) used in [28].

Multiply the second relation in (2.13) on the left and the right by \( E_i \), and then use the relations in (2.8) to get

\[ E_i E_{i+1} E_i = E_i T_i T_{i+1} E_i T_{i+1}^{-1} T_i^{-1} = E_i E_{i+1} E_i T_{i+1}^{-1} E_i = z E_i T_{i+1}^{-1} E_i = E_i, \]

so that

\[ E_i E_{i+1} E_i = E_i, \quad \text{and} \quad E_i^2 = \left(1 + \frac{z - z^{-1}}{q - q^{-1}}\right) E_i \tag{2.14} \]

is obtained by multiplying the first equation in (2.13) by \( E_i \) and using (2.8). As one can construct representations on which \( E_i \) acts non-trivially, the first relation in (2.9) implies

\[ Z_0^{(0)} = 1 + \frac{z - z^{-1}}{q - q^{-1}} \quad \text{and} \quad (T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0, \tag{2.15} \]

since \( (T_i - z^{-1})(T_i + q^{-1})(T_i - q) T_i^{-1} = (T_i - z^{-1})(T_i^2 - (q - q^{-1}) T_i - 1) T_i^{-1} = (T_i - z^{-1})(T_i - T_i^{-1} - (q - q^{-1})) = (T_i - z^{-1})(q - q^{-1})(-E_i) = -(z^{-1} - z^{-1})(q - q^{-1}) E_i = 0. \]

This shows that the relation [28, (6.3a)] follows from the relations in \( W_k \).

To complete the proof let us show that the relations in \( W_k \) follow from [28, (6.3a-e)]. Given the equivalence of the definitions of \( E_i \) as established in (2.13), and the coincidences of the relations [28, (6.3b-e)] with the relations in (2.8) and (2.9) it only remains to show that the second set of relations in (2.9), for \( i > 1 \) follow from [28, (6.3a-e)]. But this is established by [28, (6.12)] and the identity

\[ E_i Y_i Y_{i+1} = \left(1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}\right) Y_i Y_{i+1} = Y_i Y_{i+1} \left(1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}\right) = Y_i Y_{i+1} E_i, \]

which follows from (2.6). \( \square \)
The relations

\begin{align*}
(2.16) \quad E_{i+1}E_i &= E_{i+1}T_iT_{i+1}, \quad E_iE_{i+1} = T_iT_{i+1}^{-1}E_{i+1}, \\
(2.17) \quad T_iE_{i+1}E_i &= T_i^{-1}E_i, \quad \text{and} \quad E_iE_{i+1}T_i = T_iE_{i+1}T_{i+1}^{-1},
\end{align*}

are consequences of (2.8), and the second relation in (2.13).

2.2. The degenerate affine braid algebra \( \mathcal{B}_k \). Let \( C \) be a commutative ring, and let \( S_k \) denote the symmetric group on \( \{1, \ldots, k\} \). For \( i \in \{1, \ldots, k\} \), write \( s_i \) for the transposition in \( S_k \) that switches \( i \) and \( i + 1 \). The degenerate affine braid algebra is the algebra \( \mathcal{B}_k \) over \( C \) generated by

\begin{equation}
(2.18) \quad t_u \quad (u \in S_k), \quad \kappa_0, \kappa_1, \quad \text{and} \quad y_1, \ldots, y_k,
\end{equation}

with relations

\begin{align*}
(2.19) \quad t_u t_v &= t_v t_u, \quad y_i y_j = y_j y_i, \quad \kappa_0 \kappa_1 = \kappa_1 \kappa_0, \quad \kappa_0 y_i = y_i \kappa_0, \quad \kappa_1 y_i = y_i \kappa_1, \\
(2.20) \quad \kappa_0 t_j &= t_j \kappa_0, \quad \kappa_1 t_j \kappa_1 t_j = t_j \kappa_1 t_j \kappa_1, \quad \text{and} \quad \kappa_1 t_j = t_j \kappa_1, \quad \text{for} \quad j \neq 1, \\
(2.21) \quad t_u (y_i + y_{i+1}) &= (y_i + y_i y_{i+1}) t_u, \quad \text{and} \quad y_i t_j = t_j y_i, \quad \text{for} \quad j \neq i, i + 1, \\
(2.22) \quad \kappa_1 t_{i+1} y_i t_{i+1} &= t_{i+1} y_i t_{i+1} \kappa_1,
\end{align*}

and

\begin{equation}
(2.23) \quad t_{i+1} t_i y_{i+1} t_i t_{i+1} t_i = y_{i+1} t_i t_{i+1} t_i t_{i+1} t_i,
\end{equation}

where \( y_{i+1} = y_i - t_i y_i t_i \) for \( i = 1, \ldots, k - 2 \).

In the degenerate affine braid algebra \( \mathcal{B}_k \) let \( c_0 = \kappa_0 \) and

\begin{equation}
(2.24) \quad c_j = \kappa_0 + 2(y_1 + \cdots + y_j),
\end{equation}

so that \( y_j = \frac{1}{2}(c_j - c_{j-1}) \), for \( j = 1, \ldots, k \).

Then \( c_0, \ldots, c_k \) commute with each other, commute with \( \kappa_1 \), and the relations (2.21) are equivalent to

\begin{equation}
(2.25) \quad t_i c_j = c_j t_i, \quad \text{for} \quad j \neq i.
\end{equation}

**Theorem 2.2.** The degenerate affine braid algebra \( \mathcal{B}_k \) has another presentation by generators

\begin{equation}
(2.26) \quad t_u, \quad \text{for} \quad u \in S_k, \quad \kappa_0, \ldots, \kappa_k \quad \text{and} \quad \gamma_{i,j},
\end{equation}

for \( 0 \leq i, j \leq k \) with \( i \neq j \).
and relations

\[(2.27) \quad t_u t_v = t_{uv}, \quad t_w \kappa_i t_{w^{-1}} = \kappa_{w(i)}, \quad t_w y_{i,j} t_{w^{-1}} = y_{w(i),w(j)},\]

\[(2.28) \quad \kappa_i \kappa_j = \kappa_j \kappa_i, \quad \kappa_i \gamma_{l,m} = \gamma_{l,m} \kappa_i, \]

\[(2.29) \quad \gamma_{i,j} = \gamma_{j,i}, \quad \gamma_{p,r} \gamma_{m} = \gamma_{m} \gamma_{p,r}, \quad \text{and} \quad \gamma_{i,j}(\gamma_{l,r} + \gamma_{j,r}) = (\gamma_{l,r} + \gamma_{j,r})\gamma_{i,j},\]

for \(p \neq l \) and \( p \neq m \) and \( r \neq l \) and \( r \neq m \) and \( i \neq j, i \neq r \) and \( j \neq r \).

The commutation relations between the \( \kappa_i \) and the \( \gamma_{i,j} \) can be rewritten in the form

\[(2.30) \quad [\kappa_r, \gamma_{l,m}] = 0, \quad [\gamma_{i,j}, \gamma_{l,m}] = 0, \quad \text{and} \quad [\gamma_{i,j}, \gamma_{l,m}] = [\gamma_{l,m}, \gamma_{j,m}],\]

for all \( r \) and all \( i \neq l \) and \( i \neq m \) and \( j \neq l \) and \( j \neq m \).

Proof of Theorem 2.2. The generators in (2.26) are written in terms of the generators in (2.18) by the formulas

\[(2.31) \quad \kappa_0 = \kappa_0, \quad \kappa_1 = \kappa_1, \quad t_w = t_w, \]

\[(2.32) \quad \gamma_{0,1} = y_1 - \frac{1}{2} \kappa_1, \quad \text{and} \quad \gamma_{j,j+1} = y_{j+1} - t_j y_j t_j, \quad \text{for} \quad j = 1, \ldots, k - 1,\]

and

\[(2.33) \quad \kappa_m = t_u \kappa_1 t_u^{-1}, \quad \gamma_{0,m} = t_u \gamma_{0,1} t_u^{-1} \quad \text{and} \quad \gamma_{i,j} = t_v \gamma_{1,2} t_v^{-1},\]

for \( u, v \in S_k \) such that \( u(1) = m, \ v(1) = i \) and \( v(2) = j \).

The generators in (2.18) are written in terms of the generators in (2.26) by the formulas

\[(2.34) \quad \kappa_0 = \kappa_0, \quad \kappa_1 = \kappa_1, \quad t_w = t_w, \quad \text{and} \quad y_j = \frac{1}{2} \kappa_j + \sum_{0 \leq l < j} \gamma_{l,j}.\]

Let us show that relations in (2.19)–(2.22) follow from the relations in (2.27)–(2.29).

(a) The relation \( t_u t_v = t_{uv} \) in (2.19) is the first relation in (2.27).

(b) The relation \( \gamma_i y_j = y_j \gamma_i \) in (2.19): Assume that \( i < j \). Using the relations in (2.28) and (2.29),

\[
[y_i, y_j] = \left[ \frac{1}{2} \kappa_i + \sum_{l<i} \gamma_{l,i}, \frac{1}{2} \kappa_j + \sum_{m<j} \gamma_{m,j} \right] = \left[ \sum_{l<i} \gamma_{l,i}, \sum_{m<j} \gamma_{m,j} \right] = 0.
\]
(c) The relation $\kappa_0\kappa_1 = \kappa_1\kappa_0$ in (2.19) is part of the first relation in (2.28), and the relations $\kappa_0y_i = y_i\kappa_0$ and $\kappa_1y_i = y_i\kappa_1$ in (2.19) follow from the relations $\kappa_i\kappa_j = \kappa_j\kappa_i$ and $\kappa_iy_{i,m} = y_{i,m}\kappa_i$ in (2.28).

(d) The relations $\kappa_0t_j = t_j\kappa_0$ and $\kappa_1t_j = t_j\kappa_1$ for $j \neq 1$ from (2.20) follow from the relation $t_w\kappa_i t_{w^{-1}} = \kappa_w(i)$ in (2.27), and the relation $\kappa_1t_{s_j}\kappa_1t_{s_1} = t_j\kappa_1t_{s_j}\kappa_1$ from (2.20) follows from $\kappa_1\kappa_2 = \kappa_2\kappa_1$, which is part of the first relation in (2.28).

(e) The relations in (2.21) and (2.23) all follow from the relations $t_w\kappa_i t_{w^{-1}} = \kappa_w(i)$ and $t_w\gamma_i, j, l_{w^{-1}} = \gamma_w(i), w(j)$ in (2.27).

(f) By second relation in (2.28) and the (already established) second relation in (2.20)

$$[\kappa_1, t_{s_1}y_1t_{s_1}] = \left[\kappa_1, t_{s_1}\left(y_1 - \frac{1}{2}\kappa_1\right)t_{s_1} + \frac{1}{2}t_{s_1}\kappa_1t_{s_1}\right] = 0,$$

which establishes (2.22).

To complete the proof let us show that the relations of (2.27)–(2.29) follow from the relations in (2.19)–(2.22).

(a) The relation $t_wt_j = t_{s_j}$ in (2.27) is the first relation in (2.19).

(b) The relations $t_w\kappa_i t_{w^{-1}} = \kappa_w(i)$ in (2.27) follow from the first and last relations in (2.20) (and force the definition of $\kappa_m$ in (2.33)).

(c) Since $\gamma_{0,1} = y_1 - (1/2)\kappa_1$, the relations $t_w\gamma_{0, j} t_{w^{-1}} = \gamma_{0, w(j)}$ in (2.28) follow from the last relation in each of (2.20) and (2.21) (and force the definition of $\gamma_{0, m}$ in (2.33)).

(d) Since $\gamma_{1,2} = y_2 - t_{s_j}y_1t_{s_1}$, the first relation in (2.21) gives that $\gamma_{2,1} = \gamma_{1,2}$ since

$$t_{s_1}\gamma_{1,2}t_{s_1} = \gamma_{1,2} = (t_{s_1}y_2t_{s_1} - y_1) - y_2 + t_{s_1}y_1t_{s_1} = t_{s_1}(y_1 + y_2)t_{s_1} - (y_1 + y_2) = 0.$$  

The relations $t_w\gamma_{1,2} t_{w^{-1}} = \gamma_{w(1), w(2)}$ in (2.27) then follow from (2.35) and the last relation in (2.21) (and force the definitions $\gamma_{i, j} = t_{0}\gamma_{1,2} t_{0, i}$ in (2.33)).

(e) The third relation in (2.19) is $\kappa_0\kappa_1 = \kappa_1\kappa_0$ and the second relation in (2.20) gives $\kappa_1\kappa_2 = \kappa_2\kappa_1$. The relations $\kappa_i\kappa_j = \kappa_j\kappa_i$ in (2.28) then follow from the second set of relations in (2.27).

(f) The second relation in (2.20) gives $[\kappa_1, \kappa_2] = 0$. Multiplying (2.22) on the left and right by $t_{s_1}$ gives $[\gamma_1, \kappa_2] = [\gamma_1, t_{s_1}\kappa_1t_{s_1}] = 0$. Using these and the relations in (2.19),

$$[\kappa_1, \gamma_{0,2} + \gamma_{1,2}] = \left[\kappa_1, \left(y_2 - \frac{1}{2}\kappa_2 - \gamma_{1,2}\right) + \gamma_{1,2}\right] = -\left[\kappa_1, \frac{1}{2}\kappa_2\right] = 0,$$

and

$$[\gamma_{0,1}, \gamma_{0,2} + \gamma_{1,2}] = \left[\gamma_{0,1}, y_2 - \frac{1}{2}\kappa_1 - \frac{1}{2}\kappa_2\right] = \frac{1}{4}[\kappa_1, \kappa_2] = 0,$$
so that

\[
[y_{0,1}, \gamma_2] = \gamma_{0,1} - 2y_2 \gamma_{0,2} + \gamma_{1,2}] = [\gamma_{0,1}, 2y_2]
\]

\[
= y_1 - \frac{1}{2} \kappa_1, 2y_2 = -[\kappa_1, y_2] = 0.
\]

Conjugating the last relation by \(t_s\) gives

\[
[k_1, y_{0,2}] = 0,
\]

and thus \([k_1, \gamma_{1,2}] = 0,\)

by (2.36). By the third and fourth relations in (2.19),

\[
[k_0, \gamma_{0,1}] = \kappa_0, y_1 - \frac{1}{2} \kappa_1 = 0,
\]

and \([k_1, \gamma_{0,1}] = \kappa_1, y_1 - \frac{1}{2} \kappa_1 = 0.\)

By the relations in (2.20) and (2.19),

\[
[k_0, \gamma_{1,2}] = \kappa_0, y_2 - t_1, y_1 t_1 = 0\]

and \([k_1, \gamma_{2,3}] = \kappa_1, y_3 - t_2, y_2 t_2 = 0.\)

Putting these together with the (already established) relations in (2.27) provides the second set of relations in (2.28).

(g) From the commutativity of the \(y_i\) and the second relation in (2.21)

\[
y_{1,2} y_{3,4} = (y_2 - t_1, y_1 t_1)(y_4 - t_3, y_3 t_3) = (y_4 - t_3, y_3 t_3)(y_2 - t_1, y_1 t_1) = y_{3,4} y_{1,2}.
\]

By the last relation in (2.19) and the last relation in (2.20),

\[
[y_{0,1}, \gamma_{2,3}] = \gamma_1 - \frac{1}{2} \kappa_1, y_3 - t_2, y_2 t_2 = 0.
\]

Together with the (already established) relations in (2.27), we obtain the first set of relations in (2.29).

(h) Conjugating (2.37) by \(t_2, t_3, t_2\) gives \([\gamma_{0,2}, y_{0,3} + \gamma_{2,3}] = 0,\) and this and the (already established) relations in (2.28) and the first set of relations in (2.29) provide

\[
0 = [y_2, y_3] = \left[\frac{1}{2} \kappa_2 + y_{0,2} + \gamma_{1,2}, \frac{1}{2} \kappa_3 + y_{0,3} + \gamma_{1,3} + \gamma_{2,3}\right]
\]

\[
= [y_{0,2} + \gamma_{1,2}, y_{0,3} + \gamma_{1,3} + \gamma_{2,3}] = [\gamma_{1,2}, y_{0,3} + \gamma_{1,3} + \gamma_{2,3}] = [\gamma_{1,2}, \gamma_{1,3} + \gamma_{2,3}].
\]

Note also that

\[
[y_{1,2}, y_{1,0} + y_{2,0}] = [y_{1,2}, y_{0,1} + y_{0,2}] = -[y_{0,1}, \gamma_{1,2}] + [\gamma_{1,2}, y_{0,2}]
\]

\[
= [y_{0,1}, \gamma_{0,2}] + [\gamma_{1,2}, y_{0,2}] = t_1, y_{0,2} + y_{1,2}, \gamma_{0,1} t_{s_1} = 0,
\]

by (two applications of) (2.37). The last set of relations in (2.29) now follow from the last set of relations in (2.27).
By the first formula in (2.24) and the last formula in (2.34),

\[(2.38)\]

\[c_j = \sum_{i=0}^{j} \kappa_i + 2 \sum_{0 \leq j < m \leq j} \gamma_{i,m}.\]

2.3. The degenerate affine BMW algebra \(W_k\). Let \(C\) be a commutative ring and let \(B_k\) be the degenerate affine braid algebra over \(C\) as defined in Section 2.2. Define \(e_i\) in the degenerate affine braid algebra by

\[(2.39)\]

\[t_i y_i = y_{i+1} t_i - (1 - e_i), \quad \text{for} \quad i = 1, 2, \ldots, k - 1,\]

so that, with \(y_{i,i+1}\) as in (2.23),

\[(2.40)\]

\[y_{i,i+1} t_i = 1 - e_i.\]

Fix constants

\[\epsilon = \pm 1 \quad \text{and} \quad z_0^{(l)} \in C, \quad \text{for} \quad l \in \mathbb{Z}_{\geq 0}.\]

The degenerate affine Birman–Wenzl–Murakami (BMW) algebra \(W_k\) (with parameters \(\epsilon\) and \(z_0^{(l)}\)) is the quotient of the degenerate affine braid algebra \(B_k\) by the relations

\[(2.41)\]

\[e_i t_i = t_i e_i = \epsilon e_i, \quad e_i t_{i+1} e_i = e_i t_{i+1} e_i = \epsilon e_i,\]

\[(2.42)\]

\[e_1 y_1 e_1 = z_0^{(l)} e_1, \quad e_i (y_i + y_{i+1}) = 0 = (y_i + y_{i+1}) e_i.\]

The degenerate affine Hecke algebra \(H_k\) is the quotient of \(W_k\) by the relations

\[(2.43)\]

\[e_i = 0, \quad \text{for} \quad i = 1, \ldots, k - 1.\]

Fix \(b_1, \ldots, b_r \in C\). The degenerate cyclotomic BMW algebra \(W_{r,k}(b_1, \ldots, b_r)\) is the degenerate affine BMW algebra with the additional relation

\[(2.44)\]

\[(y_1 - b_1) \cdots (y_1 - b_r) = 0.\]

The degenerate cyclotomic Hecke algebra \(H_{r,k}(b_1, \ldots, b_r)\) is the degenerate affine Hecke algebra \(H_k\) with the additional relation (2.44).

Since the composite map \(C[y_1, \ldots, y_k] \to B_k \to W_k \to H_k\) is injective (see [20, Theorem 3.2.2]) and the last two maps are surjections, it follows that the polynomial ring \(C[y_1, \ldots, y_k]\) is a subalgebra of \(B_k\) and \(W_k\).

**Proposition 2.3.** Let \(C = \mathbb{C}, \kappa_0, \kappa_1 \in \mathbb{C}\) and \(\epsilon = 1\). Then the degenerate affine BMW algebra \(W_k\) coincides with the one defined in [27].
Proof. In [27], the degenerate affine BMW algebra is defined with the first two relations in (2.19) and the second set of relations in (2.21), which are [27, (4.1)] and the first relations in [27, (1.2) and (1.3)], the relations in (2.39) which are [27, (4.2)], the relations in (2.42) which are [27, (4.3) and (4.4)], the first relations in (2.41) which is the third set of relations in [27, (1.2)], the relations in (2.48) below which are the last two relations in [27, (1.3)] and the second relation in [27, (1.2)], the relations in (2.50) below which are [27, (1.4)], and the relations

\[ e_i t_{s_j} = t_{s_j} e_i \quad \text{and} \quad e_i e_j = e_j e_i, \quad \text{for} \quad |j - i| > 1, \]

which are the second and third relations in [27, (1.5)].

Working in \( \mathcal{W}_k \) and conjugating (2.39) by \( t_s \) and using the first relation in (2.41) gives

\[ y_i t_{s_i} = t_{s_i} y_{i+1} - (1 - e_i). \]

Then, by (2.40) and (2.23),

\[ y_{i+1} = t_{s_i} - e e_i, \quad \text{and} \quad e_{i+1} = t_{s_i} t_{s_{i+1}} e_i t_{s_{i+1}} t_{s_i}. \]

Multiply the second relation in (2.47) on the left and the right by \( e_i \), and then use the relations in (2.41) to get

\[ e_i e_{i+1} e_i = e_i t_{s_i} t_{s_{i+1}} e_i t_{s_i} e_i = e_i t_{s_i} e_i t_{s_{i+1}} e_i = e_i, \]

so that

\[ e_i e_{i+1} e_i = e_i. \]

Note that \( e_i^2 = z_i^{(0)} e_i \)

is, for \( i = 1 \), a special case of the first identity in (2.42) and then, for general \( i \), follows from the second identity in (2.47). The relations

\[ e_{i+1} e_i = e_{i+1} t_{s_i} t_{s_{i+1}}, \quad e_i e_{i+1} = t_{s_i} e_i t_{s_{i+1}} e_{i+1}, \]

result from (2.41) and the second relation in (2.47). The relations in (2.45) follow from (2.39) the first two relations in (2.19) and the last relations in (2.21). Thus the relations in the definition of the degenerate affine BMW algebra in [27] follow from the defining relations of \( \mathcal{W}_k \).

To complete the proof we must show that the first relations in (2.21), the relations in (2.23), and the second relations in (2.41) follow from the defining relations used in [27]. Because of the assumption that \( \kappa_0, \kappa_1 \in \mathbb{C} \) the other relations in (2.19)--(2.23) are automatic.
(a) Multiplying the first relation in (2.50) on the left by \(e_i\) and using the first relations in (2.41) and the first relations in (2.48) provides part of the second relations in (2.41) and the other part is obtained similarly by multiplying the second relations in (2.50) on the right by \(e_{i+1}\).
(b) Conjugating (2.39) by \(t_k\) produces (2.46) and then adding (2.39) and (2.46) produces the first relations in (2.21).
(c) Using (2.50),
\[
e_{i+1} = t_k t_k (e_{i+1} t_k) t_k = t_k t_k e_i e_i t_{h_i+1} t_{h_i} = t_k t_{h_i+1} e_i t_{h_i+1} t_{h_i},
\]
which, with (2.39), gives the relations in (2.23).

3. Identities in affine and degenerate affine BMW algebras

In [27], Nazarov defined some naturally occurring central elements in the degenerate affine BMW algebra \(\mathcal{W}_k\) and proved a remarkable recursion for them. This recursion was generalized to analogous central elements in the affine BMW algebra \(W_k\) by Beliakova–Blanchet [4]. In both cases, the recursion was accomplished with an involved computation. In this section, we provide a new proof of the Nazarov and Beliakova–Blanchet recursions by lifting them out of the center, to intertwiner-like identities in \(\mathcal{W}_k\) and \(W_k\) (Propositions 3.1 and 3.3). These intertwiner-like identities for the degenerate affine and affine BMW algebras are reminiscent of the intertwiner identities for the degenerate affine and affine Hecke algebras found, for example, in [21, Proposition 2.5 (c)] and [30, Proposition 2.14 (c)], respectively. The central element recursions of [27] and [4] are then obtained by multiplying the intertwiner-like identities by the projectors \(e_k\) and \(E_k\), respectively. We shall not include our new proofs of Proposition 3.1 and Theorem 3.2 here since, given our parallel setup of the degenerate affine and the affine BMW algebras in Section 2, the proof is exactly parallel to the proofs of Proposition 3.3 and Theorem 3.4.

3.1. The degenerate affine case. Let \(\mathcal{W}_k\) be the degenerate affine BMW algebra as defined in (2.41)–(2.42) and let \(1 \leq i < k - 1\). Let \(u\) be a variable and let

\[
(3.1) \quad u^+_i = \frac{1}{u - y_i} \quad \text{and} \quad u^-_i = \frac{1}{u + y_i}.
\]

**Proposition 3.1.** In the degenerate affine BMW algebra \(\mathcal{W}_{i+1}\),

\[
(3.2) \quad \left( e_i \frac{1}{1 - y_{i+1}} - t_i - \frac{1}{2u - (y_i + y_{i+1})} \right) \left( e_i \frac{1}{1 - y_i} + t_i - \frac{1}{2u - (y_i + y_{i+1})} \right) = \frac{-2u - (y_i + y_{i+1}) + 1)(2u - (y_i + y_{i+1}) - 1)}{(2u - (y_i + y_{i+1}))^2}.
\]
and
\[
\left( u_{i+1}^+ + t_i - e_i \frac{1}{2u - (y_i + y_{i+1})} \right) - u_i^+ \left( u_{i+1}^+ + t_i - e_i \frac{1}{2u - (y_i + y_{i+1})} \right) u_i^+ \\
= \left( t_i u_i^+ t_i - e_i \frac{1}{2u - (y_i + y_{i+1})} \right) - u_{i+1}^+ \left( e_i u_i^+ e_i + e e_i - e_i \frac{1}{2u - (y_i + y_{i+1})} \right) u_{i+1}^+ .
\]

(3.3)

The identities (3.4) and (3.5) of the following theorem are [27, Lemma 2.5], and [27, Lemma 3.8], respectively.

**Theorem 3.2** ([27]). Let $W_k$ be the degenerate affine BMW algebra as defined in (2.41)–(2.42) and let $1 \leq i < k - 1$. Let $z_0(u) = \sum_{\ell \in \mathbb{Z}} z_0^{(\ell)} u^{-\ell}$. Then

\[
(3.4) \quad \left( e_i u_i^+ - \frac{1}{2u} \right) \left( e_i u_i^+ + \frac{1}{2u} \right) e_i = - \left( \epsilon + \frac{1}{2u} \right) \left( \epsilon - \frac{1}{2u} \right) e_i,
\]

and

\[
(3.5) \quad \left( e_{i+1} u_{i+1}^+ + \epsilon - \frac{1}{2u} \right) e_{i+1} = \left( \frac{z_0(u)}{u} + \epsilon - \frac{1}{2u} \right) \prod_{j=1}^{i} \frac{(u + y_j - 1)(u + y_j + 1)(u - y_j)^2}{(u + y_j)^2(u - y_j + 1)(u - y_j - 1)} e_{i+1}.
\]

3.2. The affine case. Let $W_k$ be the affine BMW algebra as defined in (2.8)–(2.9) and let $1 \leq i < k - 1$. Let $u$ be a variable,

\[
U_i^+ = \frac{Y_i}{u - Y_i}, \quad \text{and note that} \quad U_i^+ U_{i+1}^+ = \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} (U_i^+ + U_{i+1}^+ + 1).
\]

By the definition of $E_i$ in (2.7),

\[
(u - Y_{i+1}) T_i = T_i (u - Y_i) - (q - q^{-1}) Y_{i+1} (1 - E_i),
\]

and, by (2.12),

\[
(u - Y_i) T_i = T_i (u - Y_{i+1}) + (q - q^{-1}) (1 - E_i) Y_{i+1},
\]

so that

\[
(3.7) \quad T_i \frac{1}{u - Y_i} = \frac{1}{u - Y_{i+1}} T_i - (q - q^{-1}) \frac{Y_{i+1}}{u - Y_{i+1}} (1 - E_i) \frac{1}{u - Y_i}.
\]
and

\[(3.8) \quad T_i \frac{1}{u - Y_{i+1}} = \frac{1}{u - Y_i} T_i + (q - q^{-1}) \frac{1}{u - Y_i} (1 - E_i) \frac{Y_i}{u - Y_{i+1}}.\]

The relations

\[(3.9) \quad T_i U_i^+ = U_{i+1}^+ T_i^{-1} - (q - q^{-1}) U_{i+1}^+ (1 - E_i) U_i^+ \]
\[= U_{i+1}^+ (T_i^{-1} - (q - q^{-1})(1 - E_i) U_i^+),\]

and

\[(3.10) \quad T_i^{-1} U_i^+ = U_i^+ T_i - (q - q^{-1}) U_i^+ E_i U_{i+1}^+ + (q - q^{-1}) U_i^+ U_{i+1}^+ \]
\[= U_i^+ (T_i + (q - q^{-1})(1 - E_i) U_{i+1}^+),\]

are obtained by multiplying (3.7) and (3.8) on the right (resp. left) by \(Y_i\) and using the relation \(T_i Y_i = Y_{i+1} T_i^{-1}\).

Taking the coefficient of \(u^{-(l+1)}\) on each side of (3.7) and (3.8) gives

\[(3.11) \quad T_i Y_i^l = Y_{i+1}^l T_i - (q - q^{-1})(Y_{i+1}^l(1 - E_i) + Y_{i+1}^{l-1}(1 - E_i) Y_i + \cdots + Y_i Y_{i+1}(1 - E_i))/l^l) + Y_i Y_{i+1}^{l-1},\]
\[(3.12) \quad T_i Y_i^l = Y_i^l T_i + (q - q^{-1})(Y_i^l(1 - E_i) Y_{i+1} + Y_i^{l-1}(1 - E_i) Y_{i+1}^2 + \cdots + (1 - E_i) Y_{i+1}^l),\]

respectively, for \(l \in \mathbb{Z}_{\geq 0}\). Therefore,

\[(3.13) \quad T_i Y_i^l = Y_{i+1}^l T_i + (q - q^{-1})(Y_{i+1}^l(1 - E_i) Y_i^l + \cdots + (1 - E_i) Y_i^{l-1}) + Y_i Y_{i+1}^{l-1},\]
\[(3.14) \quad T_i Y_i^l = Y_i^l T_i + (q - q^{-1})(Y_i^l(1 - E_i) + \cdots + Y_i^{l-1}(1 - E_i) Y_i^{l-1}(1 - E_i) Y_{i+1}^{l-1}).\]

**Proposition 3.3.** Let \(Q = q - q^{-1}\). Then, in the affine BMW algebra \(W_{i+1}\),

\[(3.15) \quad \left(\frac{E_i}{u - Y_{i+1}} - \frac{T_i}{u - Y_i} + \frac{Y_i}{u - Y_{i+1}}\right) \left(\frac{Y_i}{u - Y_i} + \frac{Y_i}{u - Y_{i+1}}\right) = \frac{-(u^2 - q^2 Y_i Y_{i+1})(u^2 - q^{-2} Y_i Y_{i+1})}{Q^2(u^2 - Y_i Y_{i+1})^2},\]

and

\[(3.16) \quad \left(\frac{U_i^+ + \frac{T_i}{Q} - E_i Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}\right) = \left(\frac{T_i U_i^+ T_i^{-1} + \frac{T_i}{Q} - E_i Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}\right) U_i^+ \]
\[= Q^2 U_i^+ \left(\frac{E_i}{u^2 - Y_i Y_{i+1}} + \frac{Y_i}{Q} - E_i Y_i Y_{i+1}\right) U_i^+ + 1.\]
Proof. Putting (3.6) into (3.9) says that if
\[ A = \frac{T_i}{Q} + \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \]
and
\[ B = E_i U_i^+ + \frac{T_i^{-1}}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \]
then
\[ A U_i^+ = U_{i+1}^+ B - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}. \]
Next,
\[ A E_i = E_i A \]
follows from (2.8) and (2.9). So
\[ \left( E_i \frac{Y_{i+1}}{u - Y_{i+1}} - \frac{T_i}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) \left( E_i \frac{Y_i}{u - Y_i} + \frac{T_i^{-1}}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) \]
\[ = E_i(U_{i+1}^+ B) - AB = E_i \left( A U_i^+ + \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) - AB \]
\[ = A(E_i U_i^+ - B) + E_i \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \]
\[ = -\left( \frac{T_i}{Q} + \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) \left( \frac{T_i^{-1}}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) + E_i \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}, \]
and, by (2.13), multiplying out the right hand side gives (3.15).

Rewrite \( T_i^{-1} U_{i+1}^+ = U_i^+ T_i^{-1} + Q U_i^+(1 - E_i)(U_{i+1}^+ + 1) \) as
\[ T_i^{-1} U_{i+1}^+ - Q(U_{i+1}^+ + 1)U_i^+ = U_i^+ T_i^{-1} - Q U_i^+ E_i(U_{i+1}^+ + 1), \]
and multiply on the left by \( T_i \) to get
\[ U_{i+1}^+ - QT_i(U_{i+1}^+ + 1)U_i^+ = T_i U_i^+ T_i^{-1} - QT_i U_i^+ E_i(U_{i+1}^+ + 1). \]
Then, since \( T_i = T_i^{-1} + Q(1 - E_i) \), equations (3.10) and (3.9) imply
\[ T_i(U_{i+1}^+ + 1) = Q(U_i^+ + 1) \left( \frac{T_i}{Q} + (1 - E_i) U_{i+1}^+ \right) \]
and
\[ T_i U_i^+ = Q U_{i+1}^+ \left( \frac{T_i^{-1}}{Q} - (1 - E_i) U_i^+ \right). \]
and so (3.17) is

\[
U_{i+1}^+ - Q^2(U_i^+ + 1)\left(\frac{T_i}{Q} + (1 - E_i)U_{i+1}^+\right)U_i^+ \\
= T_iU_i^+T_i^{-1} - Q^2U_i^+\left(\frac{T_i^{-1}}{Q} - (1 - E_i)U_i^+\right)E_i(U_{i+1}^+ + 1).
\]

(3.18)

Using (3.6) and adding

\[
\frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} - Q^2\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}(U_i^+ + 1)E_i(U_{i+1}^+ + 1)
\]
to each side of (3.18) gives

\[
U_{i+1}^+ + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} - Q^2(U_i^+ + 1)\left(U_{i+1}^+ + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)U_i^+ \\
= T_iU_i^+T_i^{-1} + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} \\
- Q^2U_i^+\left(E_iU_i^+ + \frac{T_i^{-1}}{Q} - \frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)E_i(U_{i+1}^+ + 1) \\
= T_iU_i^+T_i^{-1} + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}} \\
- Q^2U_i^+\left(E_iU_i^+E_i + \frac{E_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2 - Y_iY_{i+1}}\right)(U_i^+ + 1),
\]

completing the proof of (3.16). \(\square\)

Let \(Z_0^+\) and \(Z_0^-\) be the generating functions

\[
Z_0^+ = \sum_{l \in \mathbb{Z}_{>0}} Z_{0}^{(l)} u^{-l} \quad \text{and} \quad Z_0^- = \sum_{l \in \mathbb{Z}_{>0}} Z_{0}^{(l)} u^{-l}.
\]

If

\[
U_i^- = \frac{Y_i^{-1}}{u - Y_i^{-1}} \quad \text{then} \quad E_iU_i^+ = E_iU_i^- \quad \text{and} \quad U_{i+1}^+E_i = U_i^-E_i,
\]

by the second identity in (2.9). The first identity in (2.9) is equivalent to

\[
E_iU_i^+E_i = (Z_0^+ - Z_0^{(0)})E_i.
\]

In the following theorem, the identity (3.20) is equivalent to [13, Lemma 2.8, parts (2) and (3)] or [14, Lemma 2.6(4)] and the identity (3.21) is equivalent to the identity found in [4, Lemma 7.4].
Theorem 3.4 ([4, 13, 14]). Let $W_k$ be the affine BMW algebra as defined in (2.8)–(2.9) and let $1 \leq i < k - 1$. Then

\[
\left( E_i U_i^- - \frac{z^{-1}}{q - q^{-1}} - \frac{1}{u^2 - 1} \right) \left( E_i U_i^+ + \frac{z}{q - q^{-1}} - \frac{1}{u^2 - 1} \right) E_i \\
= \frac{(u^2 - q^2)(u^2 - q^{-2})}{(u^2 - 1)^2(q - q^{-1})^2} E_i,
\]

and

\[
\left( E_{i+1} U_{i+1}^+ + \frac{z}{q - q^{-1}} - \frac{1}{u^2 - 1} \right) E_{i+1} \\
= \left( Z_0^+ + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1} \right) \left( \prod_{j=1}^{i} \left( \frac{(u - Y_j)^2(u - q^{-2}Y_j^{-1})(u - q^2Y_j^{-1})}{(u - Y_j^{-1})^2(u - q^2Y_j)(u - q^{-2}Y_j)} \right) \right) E_{i+1}.
\]

Proof. Multiply (3.15) on the right by $E_i$ and use $Z_{i-1}^{(0)} = 1 + (z - z^{-1})/(q - q^{-1})$ to get (3.20).

Multiplying (3.16) on the left and right by $E_{i+1}$ and using the relations in (2.8), (2.9), (2.14), and

\[
E_{i+1} T_i U_i^+ T_i^{-1} E_{i+1} = E_{i+1} T_i T_{i+1} U_i^+ T_{i+1}^{-1} T_i^{-1} E_{i+1} = E_{i+1} E_i U_i^+ E_i E_{i+1},
\]
gives

\[
\left( E_{i+1} U_{i+1}^+ + \frac{z}{Q} - \frac{1}{u^2 - 1} \right) E_{i+1} \left( 1 - Q^2(U_i^+ + 1)U_i^+ \right) \\
= E_{i+1} \left( E_i U_i^+ + \frac{z}{Q} - \frac{1}{u^2 - 1} \right) E_i E_{i+1} \\
- Q^2 U_i^- E_{i+1} \left( E_i U_i^+ + \frac{z}{Q} - \frac{1}{u^2 - 1} \right) E_i E_{i+1} (U_i^- + 1) \\
= \left( 1 - Q^2 L_{U_i^-} R_{U_i^-+1} \right) \left( E_{i+1} \left( E_i U_i^+ + \frac{z}{Q} - \frac{1}{u^2 - 1} \right) E_i E_{i+1} \right)
\]

where $L_{U_i^-}$ is the operator of left multiplication by $U_i^-$ and $R_{U_i^-+1}$ is the operator of right multiplication by $U_i^- + 1$. Then, by induction,

\[
\left( E_{i+1} U_{i+1}^+ + \frac{z}{Q} - \frac{1}{u^2 - 1} \right) E_{i+1} \prod_{j=1}^{i} \left( 1 - Q^2 U_j^+ (U_j^+ + 1) \right) \\
= \left( \prod_{j=1}^{i} \left( 1 - Q^2 L_{U_j^-} R_{U_j^-+1} \right) \right) \left( E_{i+1} E_i \cdots E_2 \left( E_1 U_1^+ + \frac{z}{Q} - \frac{1}{u^2 - 1} \right) E_1 E_2 \cdots E_i E_{i+1} \right)
\]
is given by generators $e_i$. power sums. $y_i$ (1.2) – (1.5) (where our $e_i$
were formulated an alternate description of $Z$ in the degenerate case, bras of symmetric functions with a “cancellation property” (in the language of [29]). In the degenerate case, both centers arise as algebras of symmetric functions with a “cancellation property” (in the language of [29]) or “wheel condition” (in the language of [9]). In the degenerate case, $Z(V_k)$ is the ring of symmetric functions in $y_1, \ldots, y_k$ with the $Q$-cancellation property of Pragacz. By [29, Theorem 2.11 (Q)], this is the same ring as the ring generated by the odd power sums, which is the way that Nazarov [27] identified $Z(V_k)$.

The cancellation property in the case of $W_k$ is analogous, exhibiting the center of the affine BMW algebra $Z(W_k)$ as a subalgebra of the ring of symmetric Laurent polynomials. At the end of this section, in an attempt to make the theory for the affine BMW algebra completely analogous to that for the degenerate affine BMW algebra, we have formulated an alternate description of $Z(W_k)$ as a ring generated by “negative” power sums.

4.1. Bases of $V_k$ and $W_k$. The Brauer algebra, depending on a parameter $x$, is given by generators $e_i, \ldots, e_{k+1}$ and $s_1, \ldots, s_{k-1}$ and relations as given in [27, (1.2) – (1.5)] (where our $e_i$ is denoted $\bar{s}_i$ and our $x$ is denoted $N$). The Brauer algebra
also has a diagrammatic presentation (see [5]) with basis

\[(4.1) \quad D_k = \{\text{diagrams on } k \text{ dots}\},\]

where a \textit{(Brauer) diagram} on \(k\) dots is a graph with \(k\) dots in the top row, \(k\) dots in the bottom row and \(k\) edges pairing the dots. We label the vertices of the top row, left to right, with 1, 2, \ldots, \(k\) and the vertices in the bottom row, left to right, with 1', 2', \ldots, \(k'\) so that, for example,

\[(4.2) \quad d = \text{\begin{tabular}{c}
 1
  \hline
  2 & 3 & 4
  \\
  \hline
  5 & 6 & 7
\end{tabular}} = (13)(21')(45)(66')(2'7')(3'5')\]

is a Brauer diagram on 7 dots. Setting

\[x = z_0^{(0)} \quad \text{and} \quad s_i = \epsilon t_i,\]

realizes the Brauer algebra as a subalgebra of the degenerate affine BMW algebra \(\mathcal{W}_k\).

The Brauer algebra is also the quotient of \(\mathcal{W}_k\) by \(y_1 = 0\) and, hence, can be viewed as the degenerate cyclotomic BMW algebra \(\mathcal{W}_{1,k}(0)\).

**Theorem 4.1** ([27, 2]). Let \(\mathcal{W}_k\) be the degenerate affine BMW algebra and let \(\mathcal{W}_{r,k}(b_1, \ldots, b_r)\) be the degenerate cyclotomic BMW algebra as defined in (2.41)–(2.42) and (2.43), respectively. For \(n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}\) and a diagram \(d\) on \(k\) dots let

\[d^{n_1,\ldots,n_k} = y_1^{n_1} \cdots y_0 y_i y_{i+1} \cdots y_k,\]

where, in the lexicographic ordering of the edges \((i_1, j_1), \ldots, (i_k, j_k)\) of \(d\), \(i_1, \ldots, i_l\) are in the top row of \(d\) and \(i_{l+1}, \ldots, i_k\) are in the bottom row of \(d\). Let \(D_k\) be the set of diagrams on \(k\) dots, as in (4.1).

(a) If \(\kappa_0, \kappa_1 \in \mathbb{C}\) and

\[(4.3) \quad \left(z_0(-u) - \left(\frac{1}{2} + \epsilon u\right)\right)\left(z_0(u) - \left(\frac{1}{2} - \epsilon u\right)\right) = \left(\frac{1}{2} - \epsilon u\right)\left(\frac{1}{2} + \epsilon u\right)\]

then \(\{d^{n_1,\ldots,n_l} \mid d \in D_k, \quad n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}\}\) is a \(\mathbb{C}\)-basis of \(\mathcal{W}_k\).

(b) If \(\kappa_0, \kappa_1 \in \mathbb{C}\), (4.3) holds, and

\[(4.4) \quad \left(z_0(u) + u - \frac{1}{2}\right) = \left(u - \frac{1}{2}(-1)^r\right)\left(\prod_{i=1}^r \frac{u + b_i}{u - b_i}\right)\]

then \(\{d^{n_1,\ldots,n_l} \mid d \in D_k, \quad 0 \leq n_1, \ldots, n_k \leq r - 1\}\) is a \(\mathbb{C}\)-basis of \(\mathcal{W}_{r,k}(b_1, \ldots, b_r)\).

Part (a) of Theorem 4.1 is [27, Theorem 4.6] (see also [2, Theorem 2.12]) and part (b) is [2, Proposition 2.15 and Theorem 5.5].
Theorem 4.2 ([14, 39]). Let $W_k$ be the affine BMW algebra and let $W_{r,k}(b_1, \ldots, b_r)$ be the cyclotomic BMW algebra as defined in Section 2.1. Let $d \in D_k$ be a Brauer diagram, where $D_k$ is as in (4.1). Choose a minimal length expression of $d$ as a product of $e_1, \ldots, e_{k-1}, s_1, \ldots, s_{k-1}$,

$$d = a_1 \cdots a_i, \quad a_i \in \{e_1, \ldots, e_{k-1}, s_1, \ldots, s_{k-1}\},$$

such that the number of $s_i$ in this product is the number of crossings in $d$. For each $a_i$ which is in $\{s_1, \ldots, s_{k-1}\}$ fix a choice of sign $\epsilon_j = \pm 1$ and set

$$T_d = A_1 \cdots A_j, \quad \text{where} \quad A_j = \begin{cases} E_i, & \text{if } a_j = e_i, \\ T_j, & \text{if } a_j = s_i. \end{cases}$$

For $n_1, \ldots, n_k \in \mathbb{Z}$, let

$$T_d^{n_1, \ldots, n_k} = Y_{i_1}^{n_1} \cdots Y_{i_l}^{n_l} T_d Y_{i_{l+1}}^{n_{l+1}} \cdots Y_{i_k}^{n_k},$$

where, in the lexicographic ordering of the edges $(i_1, j_1), \ldots, (i_k, j_k)$ of $d$, $i_1, \ldots, i_l$ are in the top row of $d$ and $i_{l+1}, \ldots, i_k$ are in the bottom row of $d$.

(a) If

$$\left(Z_0^+ - \frac{z}{q - q^{-1}} - \frac{u^2}{u^2 - 1}\right) \left(Z_0^+ + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1}\right) = \frac{(u^2 - q^2)(u^2 - q^{-2})}{(u^2 - 1)^2(q - q^{-1})^2}$$

(4.5)

then $\{T_d^{n_1, \ldots, n_k} \mid d \in D_k, \ n_1, \ldots, n_k \in \mathbb{Z}\}$ is a C-basis of $W_k$.

(b) If (4.5) holds and

$$Z_0^+ + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1} = \left(\frac{z}{q - q^{-1}} + \frac{uz}{u^2 - 1}\right) \prod_{j=1}^r \frac{u - b_j^{-1}}{u - b_j}$$

(4.6)

then $\{T_d^{n_1, \ldots, n_k} \mid d \in D_k, \ 0 \leq n_1, \ldots, n_k \leq r - 1\}$ is a C-basis of $W_{r,k}(b_1, \ldots, b_r)$.

Part (a) of Theorem 4.2 is [14, Theorem 2.25] and part (b) is [14, Theorem 5.5] and [39, Theorem 8.1]. We refer to these references for proof, remarking only that one key point in showing that $\{T_d^{n_1, \ldots, n_k} \mid d \in D_k, \ n_1, \ldots, n_k \in \mathbb{Z}\}$ spans $W_k$ is that if $(i, j)$ is a top-to-bottom edge in $d$ then

$$Y_i T_d = T_d Y_j + \text{(terms with fewer crossings)},$$

(4.7)

and, if $(i, j)$ is a top-to-top edge in $d$ then

$$Y_i T_d = Y_j^{-1} T_d + \text{(terms with fewer crossings)}.$$  

(4.8)
4.2. The center of $\mathcal{W}_k$. The degenerate affine BMW algebra is the algebra $\mathcal{W}_k$ over $C$ defined in Section 2.3 and the polynomial ring $C[y_1, \ldots, y_k]$ is a subalgebra of $\mathcal{W}_k$. The symmetric group $S_k$ acts on $C[y_1, \ldots, y_k]$ by permuting the variables. A classical fact (see, for example, [20, Theorem 3.3.1]) is that the center of the degenerate affine BMW algebra $\mathcal{H}_k$ is the ring of symmetric functions

$$Z(\mathcal{H}_k) = C[y_1, \ldots, y_k]^{S_k} = \{ f \in C[y_1, \ldots, y_k] \mid wf = f, \text{ for } w \in S_k \}.$$ 

Theorem 4.3 gives an analogous characterization of the center of the degenerate affine BMW algebra. We shall not include the proof here since, given our parallel setup of the degenerate affine BMW algebras and the affine BMW algebras in Section 2, the proof is exactly parallel to the proof of Theorem 4.4.

**Theorem 4.3.** The center of the degenerate affine BMW algebra $\mathcal{W}_k$ is

$$\mathcal{R}_k = \{ f \in C[y_1, \ldots, y_k]^{S_k} \mid f(y_1, -y_1, y_3, \ldots, y_k) = f(0, 0, y_3, \ldots, y_k) \}.$$ 

The power sum symmetric functions $p_i$ are given by

$$p_i = y_1^i + y_2^i + \cdots + y_k^i, \text{ for } i \in \mathbb{Z}_{>0}.$$ 

The Hall–Littlewood polynomials (see [25, Chapter III (2.1)]) are given by

$$P_{\lambda}(y; t) = P_{\lambda}(y_1, \ldots, y_k; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_k} w \left( y_1^{\lambda_1} \cdots y_k^{\lambda_k} \prod_{1 \leq i < j \leq k} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where $v_\lambda(t)$ is a normalizing constant (a polynomial in $t$) so that the coefficient of $y_1^{\lambda_1} \cdots y_k^{\lambda_k}$ in $P_{\lambda}(y; t)$ is equal to 1. The Schur $Q$-functions (see [25, Chapter III (8.7)]) are

$$Q_\lambda = \begin{cases} 0, & \text{if } \lambda \text{ is not strict,} \\ 2^{l(\lambda)} P_{\lambda}(y; -1), & \text{if } \lambda \text{ is strict,} \end{cases}$$

where $l(\lambda)$ is the number of (nonzero) parts of $\lambda$ and the partition $\lambda$ is strict if all its (nonzero) parts are distinct. Let $\mathcal{R}_k$ be as in Theorem 4.3. Then (see [27, Corollary 4.10], [29, Theorem 2.11 (Q)] and [25, Chapter III §8])

$$(4.9) \quad \mathcal{R}_k = C[p_1, p_3, p_5, \ldots] = \text{span}\{Q_\lambda \mid \lambda \text{ is strict}\}.$$ 

More generally, let $r \in \mathbb{Z}_{\geq 0}$ and let $\zeta$ be a primitive $r$th root of unity. Define

$$\mathcal{R}_{r,k} = \{ f \in C[\zeta][y_1, \ldots, y_k]^{S_k} \mid f(y_1, \zeta y_1, \ldots, \zeta^{r-1} y_1, y_{r+1}, \ldots, y_k) = f(0, 0, \ldots, 0, y_{r+1}, \ldots, y_k) \}.$$
Then
\begin{equation}
\mathcal{R}_{r,k} \otimes_{\mathbb{Z}[\zeta]} \mathbb{Q}(\zeta) = \mathbb{Q}(\zeta)[p_i \mid i \neq 0 \text{ mod } r],
\end{equation}
and
\begin{equation}
\mathcal{R}_{r,k} \text{ has } \mathbb{Z}[\zeta]-\text{basis } \{ P_{\lambda}(y; \zeta) \mid m_i(\lambda) < r \text{ and } \lambda_1 \leq k \},
\end{equation}
where \( m_i(\lambda) \) is the number parts of size \( i \) in \( \lambda \). The ring \( \mathcal{R}_{r,k} \) is studied in [26], [23], [25, Chapter III Example 5.7 and Example 7.7], [37], [9], and others. The proofs of (4.10) and (4.11) follow from [25, Chapter III Example 7.7], [37, Lemma 2.2 and Proposition 3.5], and the arguments in the proofs of [9, Lemma 3.2 and Proposition 3.5].

4.3. The center of \( W_k \). The affine BMW algebra is the algebra \( W_k \) over \( C \) defined in Section 2.1 and the ring of Laurent polynomials \( C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \) is a subalgebra of \( W_k \). The symmetric group \( S_k \) acts on \( C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \) by permuting the variables. A classical fact (see, for example, [16, Proposition 2.1]) is that the center of the affine Hecke algebra \( H_k \) is the ring of symmetric functions,
\[ Z(H_k) = C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} = \{ f \in C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \mid wf = f, \text{ for } w \in S_k \}. \]

Theorem 4.4 is a characterization of the center of the affine BMW algebra.

**Theorem 4.4.** *The center of the affine BMW algebra \( W_k \) is*
\[ R_k = \{ f \in C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} \mid f(Y_1, Y_1^{-1}, Y_3, \ldots, Y_k) = f(1, 1, Y_3, \ldots, Y_k) \}. \]

**Proof.** STEP 1: \( f \in W_k \) commutes with all \( Y_i \iff f \in C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \): Assume \( f \in W_k \) and write
\[ f = \sum c_d^{n_1, \ldots, n_k} T_d^{n_1, \ldots, n_k}, \]
in terms of the basis in Theorem 4.2. Let \( d \in D_k \) with the maximal number of crossings such that \( c_d^{n_1, \ldots, n_k} \neq 0 \) and, using the notation after (4.2), suppose there is an edge \((i, j)\) of \( d \) such that \( j \neq i' \). Then, by (4.7) and (4.8),
\[ \text{the coefficient of } Y_i T_d^{n_1, \ldots, n_k} \text{ in } Y_i f \text{ is } c_d^{n_1, \ldots, n_k} \]
and
\[ \text{the coefficient of } Y_i T_d^{n_1, \ldots, n_k} \text{ in } f Y_i \text{ is } 0. \]
If \( Y_i f = f Y_i \) it follows that there is no such edge, and so \( d = 1 \) (and therefore \( T_d = 1 \)). Thus \( f \in C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \). Conversely, if \( f \in C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \), then \( Y_i f = f Y_i \).
Step 2: \( f \in C[Y_{11}^{\pm 1}, \ldots, Y_{k}^{\pm 1}] \) commutes with all \( T_i \) if and only if \( f \in C[Y_{11}^{\pm 1}, \ldots, Y_{k}^{\pm 1}] \) and write
\[
f = \sum_{a,b \in \mathbb{Z}} Y_1^a Y_2^b f_{a,b}, \quad \text{where} \quad f_{a,b} \in C[Y_{11}^{\pm 1}, \ldots, Y_{k}^{\pm 1}].
\]
Then \( f(1, 1, Y_3, \ldots, Y_k) = \sum_{a,b \in \mathbb{Z}} f_{a,b} \) and
\[
f(Y_1, Y_{11}^{-1}, Y_3, \ldots, Y_k) = \sum_{a,b \in \mathbb{Z}} Y_1^{a-b} f_{a,b} = \sum_{i \in \mathbb{Z}} Y_i \left( \sum_{b \in \mathbb{Z}} f_{i+b,b} \right).
\]
By direct computation using (3.12) and (3.14),
\[
T_1 Y_1^a Y_2^b = Y_1^a Y_2^b T_1 Y_{21}^{-a} = s_1(Y_1^a Y_2^b)T_1 + (q - q^{-1}) Y_1^a Y_2^b - s_1(Y_1^a Y_2^b) \frac{1}{1 - Y_1 Y_2^{-1}} + \mathcal{E}_{b-a},
\]
where
\[
\mathcal{E}_l = \begin{cases} 
-(q - q^{-1}) \sum_{r=1}^{l} Y_1^{l-r} E_1 Y_1^{-r}, & \text{if } l > 0, \\
(q - q^{-1}) \sum_{r=1}^{l} Y_1^{l+r-1} E_1 Y_1^{-r-1}, & \text{if } l < 0, \\
0, & \text{if } l = 0.
\end{cases}
\]
It follows that
\[
T_1 f = (s_1 f) T_1 + (q - q^{-1}) f \frac{f - s_1 f}{1 - Y_1 Y_2^{-1}} + \sum_{i \in \mathbb{Z}, l} \mathcal{E}_l \left( \sum_{b \in \mathbb{Z}} f_{i+b,b} \right).
\]
Thus, if \( f(Y_1, Y_{11}^{-1}, Y_3, \ldots, Y_k) = f(1, 1, Y_3, \ldots, Y_k) \) then, by (4.12),
\[
\sum_{b \in \mathbb{Z}} f_{i+b,b} = 0, \quad \text{for } l \neq 0.
\]
Hence, if \( f \in C[Y_{11}^{\pm 1}, \ldots, Y_{k}^{\pm 1}] \) and \( f(Y_1, Y_{11}^{-1}, Y_3, \ldots, Y_k) = f(1, 1, Y_3, \ldots, Y_k) \) then \( s_1 f = f \) and (4.14) holds so that, by (4.13), \( T_1 f = f T_1 \). Similarly, \( f \) commutes with all \( T_i \).

Conversely, if \( f \in C[Y_{11}^{\pm 1}, \ldots, Y_{k}^{\pm 1}] \) and \( T_i f = f T_i \) then
\[
s_i f = f \quad \text{and} \quad \sum_{b \in \mathbb{Z}} f_{i+b,b} = 0, \quad \text{for } l \neq 0,
\]
so that \( f \in C[Y_{11}^{\pm 1}, \ldots, Y_{k}^{\pm 1}] \) and \( f(Y_1, Y_{11}^{-1}, Y_3, \ldots, Y_k) = f(1, 1, Y_3, \ldots, Y_k) \).

It follows from (2.7) that \( R_k = Z(W_k) \). \qed
The symmetric group $S_k$ acts on $\mathbb{Z}^k$ by permuting the factors. The ring

$$C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k}$$

has basis \{ $m_\lambda | \lambda \in \mathbb{Z}^k$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ \},

where

$$m_\lambda = \sum_{\mu \in S_\lambda} Y_1^{\mu_1} \cdots Y_k^{\mu_k}.$$

The elementary symmetric functions are

$$e_r = m(1^r, 0^{r-1}) \quad \text{and} \quad e_{-r} = m(0^{r-1}, (-1)^r), \quad \text{for} \quad r = 0, 1, \ldots, k,$$

and the power sum symmetric functions are

$$p_r = m(r, 0^{r-1}) \quad \text{and} \quad p_{-r} = m(0^{r-1}, (-1)^r), \quad \text{for} \quad r \in \mathbb{Z}_{>0}.$$

The Newton identities (see [25, Chapter I (2.11')]) say

$$le_l = \sum_{r=1}^{l} (-1)^{r-1} p_r e_{l-r} \quad \text{and} \quad le_{-l} = \sum_{r=1}^{l} (-1)^{r-1} p_{-r} e_{l-(-r)},$$

where the second equation is obtained from the first by replacing $Y_i$ with $Y_i^{-1}$. For $l \in \mathbb{Z}$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$,

$$e_1^l m_\lambda = m_{\lambda + (l^k)}, \quad \text{where} \quad \lambda + (l^k) = (\lambda_1 + l, \ldots, \lambda_k + l).$$

In particular,

$$e_{-r} = e_k^{r-1} e_{k-r}, \quad \text{for} \quad r = 0, \ldots, k.$$

Define

$$p_i^+ = p_i + p_{-i} \quad \text{and} \quad p_i^- = p_i - p_{-i}, \quad \text{for} \quad i \in \mathbb{Z}_{>0}.$$

The consequence of (4.16) and (4.15) is that

$$C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} = \mathbb{C}[e_1^\pm, e_1, \ldots, e_{k-1}]
= \mathbb{C}[e_1^\pm][e_1, e_2, \ldots, e_{[k/2]}, e_k e_{-([k-1]/2)}, \ldots, e_k e_{-2}, e_k e_{-1}]
= \mathbb{C}[e_1^\pm][e_1, e_2, \ldots, e_{[k/2]}, e_{-([k-1]/2)}, \ldots, e_{-2}, e_{-1}]
= \mathbb{C}[e_1^\pm][p_1, p_2, \ldots, p_{[k/2]}, p_{-([k-1]/2)}, \ldots, p_{-2}, p_{-1}]
= \mathbb{C}[e_1^\pm][p_1^+, p_2^+, \ldots, p_{[k/2]}^+, p_{-([k-1]/2)}^+, \ldots, p_2^-, p_1^-].$$
For \( \nu \in \mathbb{Z}^k \) with \( \nu_1 \geq \cdots \geq \nu_l > 0 \) define
\[
p^+_\nu = p^+_\nu_1 \cdots p^+_\nu_l \quad \text{and} \quad p^-_\nu = p^-_\nu_1 \cdots p^-_\nu_l.
\]
Then
\[
\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^S_k \quad \text{has basis} \quad \left\{ e^l p^+_\lambda p^-_\mu \right\} \quad \text{for} \quad l \in \mathbb{Z}, l(\lambda) \leq \left\lfloor \frac{k}{2} \right\rfloor, l(\mu) \leq \left\lfloor \frac{k-1}{2} \right\rfloor.
\]
In analogy with (4.9) we expect that if \( R_k \) is as in Theorem 4.4 then
\[
R_k = \mathbb{C}[e_k^{\pm 1}][p^+_1, p^-_1, \ldots].
\]

**Acknowledgements.** Significant work on this paper was done while the authors were in residence at the Mathematical Sciences Research Institute (MSRI) in 2008, and the writing was completed when A. Ram was in residence at the Hausdorff Institute for Mathematics (HIM) in 2011. We thank MSRI and HIM for hospitality, support, and a wonderful working environment during these stays. This research has been partially supported by the National Science Foundation (DMS-0353038) and the Australian Research Council (DP-0986774). We thank S. Fomin for providing the reference [29] and Fred Goodman for providing the reference [4], many informative discussions, much help in processing the theory and for correcting many of our errors. We thank J. Enyang for his helpful comments on the manuscript. Finally, many thanks to the referee for a critical reading and the discovery of an omission in our original definition of the degenerate affine braid algebra.

**References**


Zaij Daugherty  
Department of Mathematics  
Dartmouth College  
Hanover, NH 03755  
U.S.A.  
e-mail: zajj.b.daugherty@dartmouth.edu

Arun Ram  
Department of Mathematics and Statistics  
University of Melbourne  
Parkville VIC 3010  
Australia  
e-mail: aram@unimelb.edu.au

Rahbar Virk  
Department of Mathematics  
University of Colorado  
Campus Box 395  
Boulder, Colorado 80309  
U.S.A.  
e-mail: rahbar.virk@colorado.edu