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TRILINEAR FORMS AND CHERN CLASSES OF CALABI–YAU THREEFOLDS

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Abstract

Let X be a Calabi–Yau threefold and μ the symmetric trilinear form on the second cohomology group $H^2(X, \mathbb{Z})$ defined by the cup product. We investigate the interplay between the Chern classes $c_2(X)$, $c_3(X)$ and the trilinear form μ, and demonstrate some numerical relations between them. When the cubic form $\mu(x, x, x)$ has a linear factor over $\mathbb{R}$, some properties of the linear form and the residual quadratic form are also obtained.

1. Introduction

This paper is concerned with the interplay of the symmetric trilinear form μ on the second cohomology group $H^2(X, \mathbb{Z})$ and the Chern classes $c_2(X)$, $c_3(X)$ of a Calabi–Yau threefold X. It is an open problem whether or not the number of topological types of Calabi–Yau threefolds is bounded and the original motivation of this work was to investigate topological types of Calabi–Yau threefolds via the trilinear form μ on $H^2(X, \mathbb{Z})$. The role that the trilinear form μ plays in the geography of 6-manifolds is indeed prominent as C.T.C. Wall proved the following celebrated theorem by using surgery methods and homotopy information associated with these surgeries.

Theorem 1.1 (C.T.C. Wall [14]). *Diffeomorphism classes of simply-connected, spin, oriented, closed* 6*-manifolds* X *with torsion-free cohomology correspond bijectively to isomorphism classes of systems of invariants consisting of*
(1) *free Abelian groups* $H^2(X, \mathbb{Z})$ *and* $H^3(X, \mathbb{Z})$,
(2) *a symmetric trilinear from* μ: $H^2(X,\mathbb{Z})^{\otimes 3} \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}$ *defined by* $\mu(x,y,z) := x \cup y \cup z$,
(3) *a linear map* p_1: $H^2(X,\mathbb{Z}) \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}$ *defined by* $p_1(x) := p_1(X) \cup x$, *where* $p_1(X) \in H^4(X, \mathbb{Z})$ *is the first Pontrjagin class of* X,
subject to: *for any* $x, y \in H = H^2(X, \mathbb{Z})$,

$$\mu(x, x, y) + \mu(x, y, y) \equiv 0 \pmod 2, \quad 4\mu(x, x, x) - p_1(x) \equiv 0 \pmod{24}.$$

The isomorphism $H^6(X, \mathbb{Z}) \cong \mathbb{Z}$ *above is given by pairing the cohomology class with the fundamental class* $[X]$ *with natural orientation.*

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At present the classification of trilinear forms, which is as difficult as that of diffeomorphism classes of 6-manifolds, is unknown. In the light of the essential role of the K3 lattice in the study of K3 surfaces, we would like to propose the following question: *what kind of trilinear forms μ occur on Calabi–Yau threefolds*? The quantized version of the trilinear forms, known as Gromov–Witten invariants or A-model Yukawa couplings, are also of interest to both mathematicians and physicists. One advantage of working with complex threefolds is that we can reduce our questions to the theory of complex surfaces by considering linear systems of divisors. Furthermore, for Calabi–Yau threefolds X, the second Chern class $c_2(X)$ and the Kähler cone $\mathcal{K}_X$ turn out to encode important information about μ (see [16, 18] for details). One purpose of this paper is to take the first step towards an investigation on how the Calabi–Yau structure affects the trilinear form μ and the Chern classes of the underlying manifold.

It is worth mentioning some relevant work from elsewhere. Let (X, H) be a polarized Calabi–Yau threefold. A bound for the value $c_2(X) \cup H$ in terms of the triple intersection H^3 is well-known (see for example [17]) and hence there are only finitely many possible Hilbert polynomials

$$\chi(X, \mathcal{O}_X(nH)) = \frac{H^3}{6}n^3 + \frac{c_2(X) \cup H}{12}n$$

for such (X, H). By the footnote below and standard Hilbert scheme theory, we know that the Calabi–Yau threefold X belongs to a finite number of families. This implies that once we fix a positive integer $n \in \mathbb{N}$, there are only finitely many diffeomorphism classes of polarized Calabi–Yau threefolds (X, H) with $H^3 = n$, and in particular only finitely many possibilities for the Chern classes $c_2(X)$ and $c_3(X)$ of X. Explicit bounds on the Euler characteristic $c_3(X)$ in terms of H^3 for certain types of Calabi–Yau threefolds are given in [6, 1]; the idea of this article is to record the following simple explicit result which holds in general, and which may be useful for both mathematicians and physicists.

Theorem 1.2. *Let (X, H) be a very amply polarized Calabi–Yau threefold, i.e. $x = H$ is a very ample divisor on X. Then the following inequality holds*:

$$-36\mu(x, x, x) - 80 \leq \frac{c_3(X)}{2} = h^{1,1}(X) - h^{2,1}(X) \leq 6\mu(x, x, x) + 40.$$

Moreover, the above inequality can be sharpened by replacing the left hand side by -80, -180 *and right hand side by* 28, 54 *when* $\mu(x, x, x) = 1, 3$ *respectively*[1].

In the last section, we study the cubic form $\mu(x, x, x)\colon H^2(X, \mathbb{Z}) \to \mathbb{Z}$ for a Kähler threefold X, assuming that $\mu(x, x, x)$ has a linear factor over $\mathbb{R}$. Some properties of

[1] It is shown by K. Oguiso and T. Peternell [11] that we can always pass from an ample divisor H on a Calabi–Yau threefold to a very ample one $10H$.

the linear form and the residual quadratic form on $H^2(X, \mathbb{R})$ are obtained; possible signatures of the residual quadratic form are determined under a certain condition (for example X is a Calabi–Yau threefold).

2. Bound for $c_2(X) \cup H$

In this section, we collect some properties of the trilinear form and the second Chern classes of a Calabi–Yau threefold. We will always work over the field of complex numbers $\mathbb{C}$.

Let X be a smooth Kähler threefold. Throughout this paper, we write $c_i(X) := c_i(TX)$ the i-th Chern class of the tangent bundle TX. Kähler classes constitute an open cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$, called the Kähler cone. The closure $\overline{\mathcal{K}_X}$ then consists of nef classes and hence is called the nef cone. The second Chern class $c_2(X) \in H^4(X, \mathbb{Z})$ defines a linear function on $H^2(X, \mathbb{R})$. Under the assumption that X is minimal (for instance a Calabi–Yau threefold), results of Y. Miyaoka [8] imply that for any nef class $x \in \overline{\mathcal{K}_X}$, we have $c_2(X) \cup x \geq 0$.

Let X be a smooth complex threefold. We define a symmetric trilinear form μ: $H^2(X,\mathbb{Z})^{\otimes 3} \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}$ by setting $\mu(x,y,z) := x \cup y \cup z$ for $x,y,z \in H^2(X,\mathbb{Z})$. By small abuse of notation we also use μ for its scalar extension.

DEFINITION 2.1. A Calabi–Yau threefold X is a complex projective smooth threefold with trivial canonical bundle K_X such that $H^1(X, \mathcal{O}_X) = 0$.

For a Calabi–Yau threefold X, the exponential exact sequence gives an identification $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times) \cong H^2(X, \mathbb{Z})$. The divisor class $[D]$ is then identified with the first Chern class $c_1(\mathcal{O}_X(D))$ of the associated line bundle $\mathcal{O}_X(D)$. In the following we freely use this identification.

The Hirzebruch–Riemann–Roch theorem for a Calabi–Yau threefold X states that

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{6}\mu(x, x, x) + \frac{1}{12}c_2(X) \cup x$$

for any $x = D \in H^2(X, \mathbb{Z})$. Therefore

$$2\mu(x, x, x) + c_2(X) \cup x \equiv 0 \pmod{12}.$$

In particular, $c_2(X) \cup x$ is an even integer for any $x \in H^2(X, \mathbb{Z})$. In the case when the cohomology is torsion-free, this also follows from the fact $p_1(X) = -2c_2(X)$ and Wall's Theorem 1.1. The role played by $p_1(X)$ in his theorem is replaced by $c_2(X)$ for Calabi–Yau threefolds.

For a compact complex surface S, the geometric genus $p_g(S)$ is defined by $p_g(S) := \dim_{\mathbb{C}} H^0(S, \Omega_S^2)$. The basic strategy we take in the following is to reduce the question on Calabi–Yau threefolds to compact complex surface theory by considering linear systems of divisors.

Proposition 2.2. *Let X be a Calabi–Yau threefold.*
(1) *For any ample $x = H \in \mathcal{K}_X \cap H^2(X, \mathbb{Z})$ with $|H|$ free and $\dim_{\mathbb{C}} |H| \geq 2$, the following inequalities hold.*

$$\frac{1}{2} c_2(X) \cup x \leq 2\mu(x, x, x) + C$$

where $C = 18$ when $\mu(x, x, x)$ even and $C = 15$ otherwise.
(2) *If furthermore the canonical map $\Phi_{|K_H|} \colon H \to \mathbb{P}^{|K_H|}$ (which is given by the restriction of the map $\Phi_{|H|}$ to H) is birational onto its image, the following inequality holds.*

$$\frac{1}{2} c_2(X) \cup x \leq \mu(x, x, x) + 20.$$

(3) *If furthermore the image of the canonical map in (2) is generically an intersection of quadrics, the following inequality holds.*

$$c_2(X) \cup x \leq \mu(x, x, x) + 48.$$

Proof. (1) By Bertini's theorem, a general member of the complete linear system $|H|$ is irreducible and gives us a smooth compact complex surface $S \subset X$. Applying the Hirzebruch–Riemann–Roch theorem and the Kodaira vanishing theorem to the ample line bundle $\mathcal{O}_X(H)$, we can readily show that the geometric genus

$$p_g(S) = \frac{1}{6}\mu(x, x, x) + \frac{1}{12} c_2(X) \cup x - 1.$$

Since K_S is ample, the surface S is a minimal surface of general type. Then the Noether's inequality $(1/2)K_S^2 \geq p_g(S) - 2$ yields the desired two equalities depending on the parity of $K_S^2 = \mu(x, x, x)$.

(2) The proof is almost identical to the first case. Since the surface S obtained above is a minimal canonical surface, i.e. the canonical map $\Phi_{|K_S|} \colon S \to \mathbb{P}^{|K_S|}$ is birational onto its image, the Castelnuovo inequality for minimal canonical surfaces $K_S^2 \geq 3p_g(S) - 7$ yields the inequality.

(3) We say that an irreducible variety $S \subset \mathbb{P}^{p_g - 1}$ is generically an intersection of quadrics if S is one component of the intersection of all quadrics through S. In this case, M. Reid [12] improved the above inequality to $K_S^2 \geq 4p_g(S) + q(S) - 12$. The irregularity $q(S) := \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S) = 0$ in our case. □

If $x \in \mathcal{K}_X$ is very ample, the conditions in Proposition 2.2 (1) and (2) are automatically satisfied. The first two inequalities are optimal in the sense that equalities hold for the complete intersection Calabi–Yau threefolds $\mathbb{P}_{(1^4,4)} \cap (8)$ and $\mathbb{P}^4 \cap (5)$.

It is worth noting that polarized Calabi–Yau threefolds (X, H) with Δ-genus $\Delta(X, H) \leq 2$ are classified by K. Oguiso [10] and it is observed by the second author [17] that the inequality $c_2(X) \cup H \leq 10H^3$ holds for those with $\Delta(X, H) > 2$.

R. Schimmrigk's experimental observation [13] however conjectures the existence of a better linear upper bound of $c_2(X)$ for Calabi–Yau hypersurfaces in weighted projective spaces.

Proposition 2.3. *The surface S in the proof of* Proposition 2.2 *is a minimal surface of general type with non-positive second Segre class $s_2(S)$. $s_2(S)$ is negative if and only if $c_2(X)$ is not identically zero.*

Proof. Let $i\colon S \hookrightarrow X$ be the inclusion and we identify $H^4(S,\mathbb{Z}) \cong \mathbb{Z}$. A simple computation shows $c_1(S) = -i^*(x)$ and $c_2(S) = \mu(x, x, x) + c_2(X) \cup x$. Since $x \in \mathcal{K}_X$,

$$s_2(S) = c_1(S)^2 - c_2(S) = -c_2(X) \cup x \leq 0$$

by the result of Y. Miyaoka [8]. The second claim follows from the fact that $\mathcal{K}_X \subset H^2(X,\mathbb{R})$ is an open cone. □

If X is a Calabi–Yau threefold and the linear form $c_2(X)$ is identically zero, it is well known that X is the quotient of an Abelian threefold by a finite group acting freely on it.

3. Bound for $c_3(X)$

In this section, we apply to smooth projective threefolds the Fulton–Lazarsfeld theory for nef vector bundles developed by J.P. Demailly, T. Peternell and M. Schneider [2]. This gives us several inequalities among Chern classes and cup products of certain cohomology classes. When X is a Calabi–Yau threefold, these inequalities simplify and provide us with effective bounds for the Chern classes.

Recall that a vector bundle E on a complex manifold X is called nef if the Serre line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the projectivized bundle $\mathbb{P}(E)$ is nef.

Theorem 3.1 (J.P. Demailly, T. Peternell, M. Schneider [2])**.** *Let E be a nef vector bundle over a complex manifold X equipped with a Kähler class $\omega_X \in \mathcal{K}_X$. Then for any Schur polynomial P_λ of degree $2r$ and any complex submanifold Y of dimension d, we have*

$$\int_Y P_\lambda(c(E)) \wedge \omega_X^{d-r} \geq 0.$$

Here we let $\deg c_i(E) = 2i$ for $0 \leq i \leq \operatorname{rank} E$ and the Schur polynomial $P_\lambda(c(E))$ of degree $2r$ is defined by

$$P_\lambda(c(E)) := \det(c_{\lambda_i - i + j}(E))$$

for each partition $\lambda := (\lambda_1, \lambda_2, \ldots) \dashv r$ of a non-negative integer $r \leq \dim Y$ with $\lambda_k \geq \lambda_{k+1}$ for all $k \in \mathbb{N}$.

EXAMPLE 3.2 ([7], p. 118). Let X be a complex threefold and E a vector bundle of rank $E = 3$, then

$$P_{(1)}(c(E)) = c_1(E), \quad P_{(2)}(c(E)) = c_2(E), \quad P_{(1,1)}(c(E)) = c_1(E)^2 - c_2(E),$$
$$P_{(3)}(c(E)) = c_3(E), \quad P_{(2,1)}(c(E)) = c_1(E) \cup c_2(E) - c_3(E),$$
$$P_{(1,1,1)}(c(E)) = c_1(E)^3 - 2c_1(E) \cup c_2(E) + c_3(E).$$

Proposition 3.3. *Let X be a smooth projective threefold, $x, y \in \mathcal{K}_X \cap H^2(X, \mathbb{Z})$ and assume x is very ample, then the following inequalities hold.*

(1) $8\mu(x, x, x) + 2c_2(X) \cup x \geq 4\mu(c_1(X), x, x) + c_3(X)$,
(2) $64\mu(x, x, x) + 4\mu(c_1(X), c_1(X), x) + 4c_2(X) \cup x + c_3(X)$
$\geq 32\mu(c_1(X), x, x) + c_1(X) \cup c_2(X)$,
(3) $80\mu(x, x, x) + 10\mu(c_1(X), c_1(X), x) + 2c_1(X) \cup c_2(X)$
$\geq 40\mu(c_1(X), x, x) + \mu(c_1(X), c_1(X), c_1(X)) + 10c_2(X) \cup x + c_3(X)$,
(4) $12\mu(x, x, y) + c_2(X) \cup y \geq 4\mu(c_1(X), x, y)$,
(5) $24\mu(x, x, y) + \mu(c_1(X), c_1(X), y) \geq 8\mu(c_1(X), x, y) + c_2(X) \cup y$,
(6) $6\mu(x, y, y) \geq \mu(c_1(X), y, y)$.

Proof. The very ample divisor $x = H$ gives us an embedding $\Phi_{|H|} : X \to \mathbb{P}(V)$, where $V := H^0(X, \mathcal{O}_X(H))$. Using the Euler sequence and the Koszul complex, we obtain the following exact sequence of sheaves

$$0 \to \Omega^{k+1}_{\mathbb{P}(V)} \to \bigwedge^{k+1} V \otimes \mathcal{O}_{\mathbb{P}(V)}((-k-1)H) \to \Omega^k_{\mathbb{P}(V)} \to 0$$

for each $1 \leq k \leq \dim_{\mathbb{C}} V - 1$. We see that $\Omega_{\mathbb{P}(V)}(2H)$ is a quotient of $\mathcal{O}_{\mathbb{P}(V)}^{\oplus\binom{\dim_{\mathbb{C}} V}{2}}$. The vector bundle $\Omega_X(2H)$ is then generated by global sections because it is a quotient of the globally generated vector bundle $\Omega_{\mathbb{P}(V)}|_X(2H)$. We hence conclude that $\Omega_X(2H)$ is a nef vector bundle. Applying Theorem 3.1 (or rather the inequalities derived using the above example) to our nef vector bundle $\Omega_X(2H)$, straightforward computation shows the desired inequalities. $\square$

The above result (with appropriate modification) certainly carries over to complex manifolds of dimension other than 3.

Corollary 3.4. *Let X be a Calabi–Yau threefold, $x, y \in \mathcal{K}_X \cap H^2(X, \mathbb{Z})$ and assume x is very ample, then the following inequalities hold.*

(1) $8\mu(x, x, x) + 2c_2(X) \cup x \geq c_3(X)$,
(2) $64\mu(x, x, x) + 4c_2(X) \cup x + c_3(X) \geq 0$,
(3) $80\mu(x, x, x) \geq 10c_2(X) \cup x + c_3(X)$,
(4) $24\mu(x, x, y) \geq c_2(X) \cup y$.

In recent literature there has been some interest in finding practical bounds for topological invariants of Calabi–Yau threefolds. As is mentioned in the introduction, the standard Hilbert scheme theory assures that possible Chern classes of a polarized Calabi–Yau threefold (X, H) are in principle bounded once we fix a triple intersection number $H^3 = n \in \mathbb{N}$, but now that we have effective bounds for the Chern classes (with a bit of extra data for the second Chern class $c_2(X)$) as follows. Recall first that it is shown by K. Oguiso and T. Peternell [11] that we can always pass from an ample divisor H on a Calabi–Yau threefold to a very ample one $10H$. Then the last inequality in Corollary 3.4 says that once we know the trilinear form μ on the ample cone $\mathcal{K}_X$ there are only finitely many possibilities for the linear function $c_2(X)\colon H^2(X, \mathbb{Z}) \to \mathbb{Z}$. We shall now give a simple explicit formula to give a range of the Euler characteristic $c_3(X)$ of a Calabi–Yau threefold X.

Theorem 1.2 *Let (X,H) be a very amply polarized Calabi–Yau threefold, i.e. $x = H$ is a very ample divisor on X. Then the following inequality holds*:

$$-36\mu(x, x, x) - 80 \le \frac{c_3(X)}{2} = h^{1,1}(X) - h^{2,1}(X) \le 6\mu(x, x, x) + 40.$$

Moreover, the above inequality can be sharpened by replacing the left hand side by -80, -180 *and right hand side by* 28, 54 *when* $\mu(x, x, x) = 1, 3$ *respectively.*

Proof. This is readily proved by combining Proposition 2.2 (1), (2) and Corollary 3.4 (1), (2), (4). □

The smallest and largest known Euler characteristics $c_3(X)$ of a Calabi–Yau threefold X are -960 and 960 respectively. Our formula may replace the question of finding a range of $c_3(X)$ by that of estimating the value $\mu(x, x, x)$ for an ample class $x \in \mathcal{K}_X \cap H^2(X, \mathbb{Z})$.

4. Quadratic forms associated with special cubic forms

In this section we further study the cubic form $\mu(x, x, x)\colon H^2(X, \mathbb{Z}) \to \mathbb{Z}$ for a Kähler threefold X, assuming that $\mu(x, x, x)$ has a linear factor over $\mathbb{R}$. We will see that the linear factor and the residual quadratic form are not independent. Possible signatures of the residual quadratic form are also determined under a certain condition. If the second Betti number $b_2(X) > 3$, the residual quadratic form may endow the second cohomology $H^2(X, \mathbb{Z})$ mod torsion with a lattice structure.

We start with fixing our notation. Let $\xi\colon V \to \mathbb{R}$ be a real quadratic form. Once we fix a basis of the $\mathbb{R}$-vector space V, ξ may be represented as $\xi(x) = x^t A_\xi x$ for some symmetric matrix A_ξ. The signature of a quadratic form ξ is a triple (s_+, s_0, s_-) where s_0 is the number of zero eigenvalues of A_ξ and s_+ (s_-) is the number of positive (negative) eigenvalues of A_ξ. A_ξ also defines a linear map $A_\xi\colon V \to V^\vee$ (or a symmetric bilinear form $A_\xi\colon V^{\otimes 2} \to \mathbb{R}$). The quadratic form ξ is called (non-)degenerate if $\dim_{\mathbb{R}} \operatorname{Ker}(A_\xi) > 0$ $(= 0)$. We say that ξ is definite if it is non-degenerate and either s_+ or s_- is zero, and indefinite otherwise.

Let X be a Kähler threefold and assume that its cubic form $\mu(x, x, x)$ factors as $\mu(x, x, x) = \nu(x)\xi(x)$, where ν is linear and ξ is quadratic map $H^2(X, \mathbb{R}) \to \mathbb{R}$. We can always choose the linear form ν so that it is positive on the Kähler cone $\mathcal{K}_X$. It is proven (see the proof of Lemma 4.3 in [15]) that there exists a non-zero point on the quadric

$$Q_\xi := \{x \in H^2(X, \mathbb{R}) \mid \xi(x) = 0\}$$

and hence ξ is indefinite provided that the irregularity $q(X) = 0$ and the second Betti number $b_2(X) > 3$.

Proposition 4.1. *Let X be a Kähler threefold. Assume that the trilinear form $\mu(x,x,x)$ decomposes as $\nu(x)\xi(x)$ over $\mathbb{R}$ (if the quadratic form is not a product of linear forms, then we may work over $\mathbb{Q}$) and the linear form ν is positive on the Kähler cone $\mathcal{K}_X$. Then the following hold.*
(1) $\dim_{\mathbb{R}} \operatorname{Ker}(A_\xi) \leq 1$. *If ξ is a degenerate quadratic form, its restriction $\xi|_{H_\nu}$ to the hyperplane*

$$H_\nu := \{x \in H^2(X, \mathbb{R}) \mid \nu(x) = 0\}$$

is non-degenerate.
(2) *If the irregularity $q(X) = 0$ (for example a Calabi–Yau threefold), then the signature of ξ is either* $(2, 0, b_2(X) - 2)$, $(1, 1, b_2(X) - 2)$ *or* $(1, 0, b_2(X) - 1)$.
(3) *The above three signatures are realized by some Calabi–Yau threefolds with* $b_2(X) = 2$.

Proof. (1) Let $\omega_X \in \mathcal{K}_X$ be a Kähler class. The Hard Lefschetz theorem states that the map $H^2(X,\mathbb{R}) \to H^4(X,\mathbb{R})$ defined by $\alpha \mapsto \omega_X \cup \alpha$ is an isomorphism. Hence the cubic form $\mu(x, x, x)$ depends on exactly $b_2(X)$ variables. Then the quadratic form ξ must depend on at least $b_2(X) - 1$ variables and thus we have $\dim_{\mathbb{R}}(\operatorname{Ker}(A_\xi)) \leq 1$. Assume next that the quadratic form ξ is degenerate. Then the linear form ν is not the zero form on $\operatorname{Ker}(A_\xi)$ (otherwise $\mu(x, x, x)$ depends on less than $b_2(X)$ variables). The restriction $\xi|_{H_\nu}$ is non-degenerate because $H^2(X, \mathbb{R}) = H_\nu \oplus \operatorname{Ker}(A_\xi)$ as a $\mathbb{R}$-vector space.

(2) Let $L_1 \in \mathcal{K}_X \cap H^2(X,\mathbb{R})$ be an ample class such that $\mu(L_1, L_1, L_1) = 1$. Since the Kähler cone $\mathcal{K}_X \subset H^2(X, \mathbb{R})$ is an open cone, X is projective by the Kodaira embedding theorem. Then the Hodge index theorem states that the symmetric bilinear form

$$b_{\mu,L_1} := \mu(L_1, *, **)\colon H^2(X, \mathbb{R})^{\otimes 2} \cong (NS(X) \otimes \mathbb{R})^{\otimes 2} \to \mathbb{R}$$

has signature $(1, 0, b_2(X) - 1)$, where $NS(X)$ is the Neron–Severi group of X. Note that $\dim_{\mathbb{R}}(L_1^\perp \cap H_\nu) \geq b_2(X) - 2$, where $L_1^\perp$ denotes the orthogonal space to L_1 with respect to the non-degenerate bilinear form b_{μ,L_1}. We then have two cases; the first is when $\dim_{\mathbb{R}}(L_1^{\perp} \cap H_\nu) = b_2(X) - 1$ (i.e. $L_1^{\perp} = H_\nu$). In this case we can write down a basis $L_2, \ldots, L_{b_2(X)}$ for the subspace H_ν which diagonalizes the quadratic form $b_{\mu,L_1}|_{H_\nu}$, and hence (noting that $L_1 \notin H_\nu$) the Gramian matrix of b_{μ,L_1} with respect to the basis $L_1, \ldots, L_{b_2(X)}$ of $H^2(X, \mathbb{R})$ is

$$A_{b_{\mu,L_1}} := (b_{\mu,L_1}(L_i, L_j)) = \mathrm{diag}(1, -1, \ldots, -1).$$

If $\dim_{\mathbb{R}}(L_1^{\perp} \cap H_\nu) = b_2(X) - 2$, then we can write down a basis $L_2, \ldots, L_{b_2(X)-1}$ for the subspace $L_1^{\perp} \cap H_\nu$ which diagonalizes the quadratic form $b_{\mu,L_1}|_{L_1^{\perp} \cap H_\nu}$, and then extend it to a basis $L_2, \ldots, L_{b_2(X)}$ of H_ν. Thus in both cases $L_1, \ldots, L_{b_2(X)}$ is a basis for $H^2(X, \mathbb{R})$; the corresponding matrix $A_{b_{\mu,L_1}}$ will not be diagonal in this second case, but the first $(b_2(X) - 1)$-principal minor is, with one $+1$ and $b_2(X) - 2$ entries -1 on the diagonal.

Let us define a new basis $\{M_i\}_{i=1}^{b_2(X)}$ of $H^2(X, \mathbb{R})$ by setting $M_i = L_i$ for $1 \le i \le b_2(X) - 1$ and

$$M_{b_2(X)} = L_{b_2(X)} + \sum_{i=2}^{b_2(X)} b_{\mu,L_1}(L_i, L_{b_2(X)}) L_i \in H_\nu.$$

Let $x = \sum_{i=1}^{b_2(X)} a_i M_i$. Then the hyperplane H_ν is defined by the equation $a_1 = 0$ and the Kähler cone $\mathcal{K}_X$ lies on the side where $a_1 > 0$ by the assumption on ν. Therefore we have

$$\mu(x, x, x) = a_1 \left(a_1^2 - \sum_{i=2}^{b_2(X)-1} a_i^2 + C a_1 a_{b_2(X)} + D a_{b_2(X)}^2 \right)$$

for some (explicit) constants $C, D \in \mathbb{R}$. Since the quadratic form is positive on the the Kähler cone $\mathcal{K}_X$, there must be at least one positive eigenvalue and hence possible signatures are $(2, 0, b_2(X) - 2)$, $(1, 1, b_2(X) - 2)$ and $(1, 0, b_2(X) - 1)$.

(3) Consider a Calabi–Yau threefold $X_7^{II}(1,1,1,2,2)^2_{-186}$ from p. 575 [5] given as a resolution of a degree 7 hypersurface in the weighted projective space $\mathbb{P}_{(1,1,1,2,2)}$. Its cubic form is given by

$$a_1(14a_1^2 + 21a_1a_2 + 9a_2^2),$$

whose quadratic form has signature $(2, 0, 0)$. The cubic form of a hypersurface Calabi–Yau threefold $(\mathbb{P}^3 \times \mathbb{P}^1) \cap (4, 2)$ is

$$2a_1^3 + 12a_1^2 a_2,$$

whose quadratic form has signature either $(1, 0, 1)$ or $(1, 1, 0)$, depending on its decomposition. □

The restriction $\xi|_{H_\nu}$ may be degenerate if ξ is non-degenerate. The cubic form of the above Calabi–Yau threefold $(\mathbb{P}^3 \times \mathbb{P}^1) \cap (4,2)$ gives an example of such phenomenon. Let $\nu(a) = 2a_1$ and $\xi(a) = a_1(a_1 + 6a_2)$. Then ξ is hyperbolic and non-degenerate, but its restriction to H_ν is trivial.

Let X be a Kähler threefold. If $b_2(X) > 3$, the cubic form μ cannot consist of three linear factors over $\mathbb{R}$ and hence if μ contains a linear factor it must be rational (see also the comment after Lemma 4.2 [15]). Hence an appropriate scalar multiple of ξ endows the second cohomology $H^2(X, \mathbb{Z})$ mod torsion with a lattice structure.

EXAMPLE 4.2 (Enriques Calabi–Yau threefold [3, 4]). Let X be a generic K3 surface with an Enriques involution ι_S. Let E be an elliptic curve and -1_E the negation. Then we can define a new involution ι of $S \times E$ by $\iota := (\iota_S, -1_E)$. The free quotient

$$X := (S \times E)/\langle \iota \rangle$$

is a Calabi–Yau threefold with $b_2(X) = 11$. The cubic form $\mu(x, x, x)$ of X has a linear factor (which, we assume, is positive on the Kähler cone $\mathcal{K}_X$) and the residual quadratic form ξ has signature (1,1,9). More precisely, the lattice structure on $H^2(X,\mathbb{Z})$ mod torsion associated with appropriate ξ is given by

$$U \oplus E_8(-1) \oplus \langle 0 \rangle,$$

where U is the hyperbolic lattice, $E_8(-1)$ is the root lattice of type E_8 multiplied by -1 and $\langle 0 \rangle$ is a trivial lattice of rank 1.

Proposition 4.3. *Let G be a finite group acting on a Kähler threefold X and $\phi\colon G \to \mathrm{GL}(H^2(X, \mathbb{Z}))$ the induced representation. Assume that the trilinear form decomposes $\mu(x, x, x) = \nu(x)\xi(x)$ as above. Then the image of $\phi\colon G \to \mathrm{GL}(H^2(X, \mathbb{Z}))$ lies in the orthogonal group $\mathrm{O}(\xi)$ associated with the quadratic form ξ.*

Proof. Since the cubic form $\mu\colon H^2(X,\mathbb{R}) \to \mathbb{R}$ is invariant under G, it is enough to show that the linear form ν is invariant under G. There exists $x \in \mathcal{K}_X$ such that $\mathbb{R}x$ is a trivial representation of G (by averaging a Kähler class over G) and then the representation ϕ is a direct sum of two subrepresentations $\mathbb{R}x \oplus H_\nu$. Since ν is a linear form, this shows the invariance of ν under G. □

This proposition may be useful to study group actions on the cohomology group $H^2(X, \mathbb{Z})$.

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References

[1] M.-C. Chang and H. Kim: *The Euler number of certain primitive Calabi–Yau threefolds*, Math. Proc. Cambridge Philos. Soc. **128** (2000), 79–86.

[2] J.-P. Demailly, T. Peternell and M. Schneider: *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. **3** (1994), 295–345.

[3] K. Hashimoto and A. Kanazawa: *Calabi–Yau Threefolds of Type K* (I): *Classification*, preprint.

[4] K. Hashimoto and A. Kanazawa: *Calabi–Yau Threefolds of Type K* (II): *Mirror Symmetry*, preprint.

[5] S. Hosono, B.H. Lian and S.-T. Yau: *GKZ-generalized hypergeometric systems in mirror symmetry of Calabi–Yau hypersurfaces*, Comm. Math. Phys. **182** (1996), 535–577.

[6] B. Hunt: *A bound on the Euler number for certain Calabi–Yau* 3-*folds*, J. Reine Angew. Math. **411** (1990), 137–170.

[7] R. Lazarsfeld: Positivity in Algebraic Geometry, II, Springer, Berlin, 2004.

[8] Y. Miyaoka: *The Chern classes and Kodaira dimension of a minimal variety*; in Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987, 449–476.

[9] Ch. Okonek and A. Van de Ven: *Cubic forms and complex* 3-*folds*, Enseign. Math. (2) **41** (1995), 297–333.

[10] K. Oguiso: *On polarized Calabi–Yau* 3-*folds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **38** (1991), 395–429.

[11] K. Oguiso and T. Peternell: *On polarized canonical Calabi–Yau threefolds*, Math. Ann. **301** (1995), 237–248.

[12] M. Reid: *Quadrics through a canonical surface*; in Algebraic Geometry (L'Aquila, 1988), Lecture Notes in Math. **1417**, Springer, Berlin, 1990, 191–213.

[13] R. Schimmrigk: *Scaling behavior on the space of Calabi–Yau manifolds*; in Mirror Symmetry, II, AMS/IP Stud. Adv. Math. **1**, Amer. Math. Soc., Providence, RI, 1997, 443–453.

[14] C.T.C. Wall: *Classification problems in differential topology* V. *On certain* 6-*manifolds*, Inv. Math. **1** (1966), 355-374.

[15] P.M.H. Wilson: *Calabi–Yau manifolds with large Picard number*, Invent. Math. **98** (1989), 139–155.

[16] P.M.H. Wilson: *The Kähler cone on Calabi–Yau threefolds*, Invent. Math. **114** (1993), 231–233.

[17] P.M.H. Wilson: *Minimal models of Calabi–Yau threefolds*; in Classification of Algebraic Varieties (L'Aquila, 1992), Contemp. Math. **162**, Amer. Math. Soc., Providence, RI, 1994, 403–410.

[18] P.M.H. Wilson: *The role of c_2 in Calabi–Yau classification—a preliminary survey*; in Mirror Symmetry, II, AMS/IP Stud. Adv. Math. **1**, Amer. Math. Soc., Providence, RI, 1997, 381–392.

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